Solving State-Space Models with Exotic Information Set

Yongyang Cai   Yanchen Hu

April 2024

Abstract

Based on the Simulated Certainty-Equivalent (SCEQ) algorithm in Cai and Judd (2023), we develop a multi-step SCEQ algorithm to solving non-linear dynamic stochastic state-space models with exotic information set (EIS). Based on the order of shock revelation, each period is divided into multiple subperiods. To accommodate this, our algorithm extends SCEQ in two dimensions. First, SCEQ is applied in each subperiod. In particular, the simulation step is carried out before entering the next subperiod by fixing the level of variables whose decisions are made up to the current subperiod. Second, in addition to the conventional certainty equivalence assumption, a subperiod version of certainty equivalence is assumed such that the shock values of future subperiods in the current period also take their mean. This multi-step SCEQ algorithm can be realized through commercial nonlinear programming (NLP) methods. It successfully replicates the pattern that decisions of variables in a certain subperiod do not respond in the current period to the shocks revealed in future subperiods. Through error checking, the algorithm is shown to deliver a high level of accuracy.

1 Introduction

When we model macroeconomic questions through dynamic stochastic general equilibrium (DSGE) models, there is an implicit but commonly used assumption that all exogenous shocks in the model hit the economy at the beginning of each period. Following the revelation of the shocks, agents make decisions within the shock period after they observe the levels of these shocks. This assumption implies non-zero instantaneous response to the structural shocks is incurred onto all endogenous variables. However, the assumption is subject to modification when the model parameters are estimated by matching model-implied impulse response functions (IRFs) with the empirically implied ones. One particular example of the latter is the IRFs generated from the VAR methodology.

Vector autoregression (VAR) has been proven as a decent methodology to empirically model the dynamic interaction between variables in an environment subject to stochastic innovations. For example, to study the interaction between the housing market and interest rates, it is desirable to include two groups of variables in the VAR model. The first group consists of variables associated with the housing market, e.g., housing price index, residential investment, housing starts, home mortgage-to-GDP ratio, and various interest rates, such as the Fed funds rate and 30-year mortgage rate. Data values of these variables are capable of providing rich information via the revelation of the dynamic evolution of the entire system and historical structural shocks that drive the fluctuations of each variable in the system. Mathematically, identification of the shock series must depend on necessary identifying assumptions. Intuitively, these assumptions either quantitatively or qualitatively[1] reflect the effect of one variable on another at different horizons. Among all quantitative assumptions, the short-run exclusion restriction, which imposes zero contemporaneous response, is perhaps one of the most commonly used assumptions. When projected onto the VAR-implied IRFs, percentage deviation from the steady state at the first period is zero in both level and standard deviation.

To fit the model IRFs to the VAR-implied IRFs, one must accommodate the timing of the model to the VAR short-run exclusion restriction. Henceforth, it is necessary to break the assumption that all decisions are made after the revelation of shocks. Instead, it is necessary to break one period into multiple subperiods. Allocations of shock revelation and decision making into these subperiods are solely determined by the short-run exclusion restrictions in VAR. In this chapter, and also in the other chapters, such a multiple-subperiod structural of timing in a certain period is defined as the exotic

[1] One example of the qualitative assumptions is sign restrictions.

information set (EIS). To further understand how EIS is accommodated to VAR IRF matching, it is helpful to give an example. In Christiano et al. (1999), based on existing evidence showing that many macro fundamental variables respond to shock in the Fed funds rate in a delayed and hum-shaped pattern, it is reasonable to put the policy rate in the last place in the vector of endogenous variables, as assumed in the VAR analysis in Christiano et al. (2005). This implies that all other endogenous variables in the VAR model have no contemporaneous response to a monetary policy shock, due to the short-run exclusion restriction. To match this assumption, EIS is introduced into their monetary DSGE model. Specifically, it is assumed in every period, that consumption, investment, capital utilization, price and wage setting are all determined prior to the revelation of monetary policy shock. Along this, each period of the model economy is divided into two subperiods. In the first subperiod, no shock is revealed but agents must make some of the decisions while unable to see the contemporaneous level of shock to monetary policy rate. At the beginning of the second subperiod, a monetary policy shock is incurred, and the rest of the decisions are made according to the observed shock.

In this chapter, we develop an algorithm that introduces EIS into the Simulated Certainty Equivalence (SCEQ) Method developed in Cai and Judd (2023). Unlike conventional state-space methods that expand the model up to the first order, SCEQ directly solves the original non-linear model using nonlinear programming (NLP) methodologies under the assumption of certainty equivalence. To insert the EIS timing structure into the benchmark SCEQ algorithm, we develop a multi-step SCEQ algorithm, with each step corresponding to one subperiod of an entire period.

As a variant of SCEQ, the multi-step SCEQ algorithm inherits the advantages of the SCEQ method. First, it is free of curse of dimensionality, as SCEQ generates simulated data and therefore is free of policy function. Second, it is fast. Due to independence among individual simulations, we can take advantage of parallel computing and largely shorten the computational time. Third, it is accurate. Since it solves directly the non-linear equation without truncation of approximation order, the solution will be more accurate than its linear counterpart. Last but not least, multi-step SCEQ can be easily implemented with an optimization solver. Choice of NLP solver is bountiful as many of them have been embedded into specialized numerical software with high computational efficiency.

SCEQ is a simple method capable of solving medium-to-large-scale nonlinear state-space models. Due to progress in NLP[2], SCEQ can deal with not only continuously differentiable models, but also models with kinks, such as occasionally-binding constraints. To implement SCEQ, one needs an extra assumption that the system converges to a terminal condition after a sufficiently long period. To solve the model at period t given the contemporaneously revealed shocks, one actually solves in practice a sequence of solutions from period t to an assumed terminal period $T > t$. With certainty equivalence assumption, shocks after period t take their mean values. With this deterministic shock profile, a stochastic model is therefore transformed into a deterministic one to be solved by NLP algorithms.

To incorporate EIS into a model, one period in the model needs to be divided into multiple subperiods based on the timing order of revelation of exogenous shocks. Then, each period is solved by SCEQ via multiple steps, with one-to-one correspondence between the steps and the subperiods. When solving the model in each subperiod, the certainty equivalence assumption is repeatedly applied, such that not only future-period shocks but also future-subperiod shocks within the current period are assumed to take their mean values. Similar to the standard SCEQ algorithm, a simulation step must be taken after solving the model in the current subperiod. Prior to moving to the next subperiod, this must be done by fixing the levels of endogenous variables whose decisions are made within the current subperiod.

The rest of this chapter is organized as follows. For the rest of this section, we relate the multi-step SCEQ to the existing literature. Section 2 formerly states the algorithm and makes associated comments. It also builds up a theory on how EIS affects the linear model. Section 3 applies the algorithm to a small-scale RBC model with downward investment rigidity in the form of an occasionally binding constraint. Section 4 concludes. All proofs and derivations are relegated to the appendices.

[2] NLP can be easily implemented through GAMS (The General Algebraic Language). Throughout this chapter, we use CONOPT4 as the NLP algorithm.

1.1 Literature Review

This chapter contributes to the strand of literature that incorporate EIS into DSGE modeling. EIS is derived from the sticky information hypothesis in Mankiw and Reis (2002). Alternative to sticky price, Mankiw and Reis (2002) construct a general equilibrium model where firms keep ignoring the shock information. Durations of information ignorance differ in firms. Sticky information also produces hump-shaped dynamics in both output and inflation, as implied in the sticky-price model. After the pioneering work, sticky-information was applied in later studies[3]. Christiano et al. (2005) simplify the setting of sticky information by assuming a homogeneous duration of inattentiveness. It is assumed that decisions on a subset of all endogenous variables are made without observing the current-period monetary policy shock. A more general version of the algorithm with multiple subperiods and multiple shocks is developed in King and Watson (2002), but only up to the first-order approximation of models. In later years, EIS associated with monetary shock was inherited in more complicated 3-shock models, with two other shocks on TFP and relative price of investment goods[4]. DiCecio (2009) studies a two-sector model with the same three shocks and EIS configuration and illustrates the role of wage stickiness in sectoral co-movement. Dupor et al. (2009) work on a two-shock model with only TFP and monetary shock. Again, EIS is applied on monetary policy shock. They document contradiction in simultaneously matching the model IRFs to the empirical IRFs to the two shocks. However, all the existing literature work with linear models. This paper works as a first attempt to embed EIS into a fully nonlinear environment.

Due to its capability of solving models with OBCs, the multi-step SCEQ algorithm contributes to the set of existing algorithms to solving OBCs. The existing methodologies include the piecewise-linear Occbin algorithm developed in Guerrieri and Iacoviello (2015) and the mixed-integer linear programming (MILP) based method developed in Holden (2023). Both of these two methods are limited to models approximated with first-order expansion and therefore suffer from low solution accuracy. On the other hand, however, these two approaches are closely associated with the SCEQ method, since all three al-

[3] Other papers that embed the sticky-information idea include Mankiw and Reis (2001), Reis (2006a), Reis (2006b), Mankiw and Reis (2006) and Mankiw and Reis (2010).

[4] To name a few, Altig et al. (2011), Christiano et al. (2010) and Christiano et al. (2015)

gorithms are an application of the extended-path method proposed in Fair and Taylor (1983)[5], which in turn is based on the certainty equivalence assumption that can be traced back to Simon (1956) and Theil (1957). As far as we are concerned, this chapter is the first time to propose an algorithm for solving a state-space model with a combination of EIS and OBCs[6]. However, the application of the multi-step idea is not limited to SCEQ. It can be also fused into both the Occbin and the MILP-based algorithms.

2 The Algorithm

To incorporate EIS into a conventional state-space model, assume there is only one exogenous shock $\mathcal{E}_t$ and two subperiods in each period. The first subperiod goes before the revelation of the shock and the second goes after. The two subperiods are solved step by step, and therefore follows a two-step SCEQ algorithm. Such a simple setting does not incur the loss of generality. The two-step SCEQ algorithm can be easily generalized to the case with arbitrary number of shocks and subperiods. The state variables $\mathcal{X}_t$ and action (control) variables $\mathcal{A}_t$ are grouped and ordered based on which subperiod the associated decisions are made, i.e., $\mathcal{X}_t \equiv [x_t', x_{t+1/2}']'$ and $\mathcal{A}_t \equiv [a_t', a_{t+1/2}']'$, where t and $t+1/2$ in the square bracket denotes the first and second subperiod in period t. For the sake of clarity, uppercase letters are used for the set of variables spanning an entire period, and lowercase letters for variables within a single subperiod for the rest of this chapter. Along this, $\mathcal{E}_t \equiv [\varepsilon_t', \varepsilon_{t+1/2}']'$ denotes the vector of exogenous shocks of period t, with ε_t and $\varepsilon_{t+1/2}$ representing respectively vector of shocks revealed at the beginning of the first and second subperiod of period t. As assumed in the majority of macro DSGE models, $\mathcal{E}_t$ are i.i.d. shocks with zero mean and positive variance.

Now we formally introduce the 2-subperiod state-space model as follows. Assume that

[5]Fair and Taylor (1983) proposed the extended-path method for linear models. However, due to the certainty equivalence assumption and the necessity to solve a path of solutions that ends at a terminal period, SCEQ can be viewed as a nonlinear version of the canonical extended-path method.

[6]Christiano et al. (2015) also met with the combination of EIS and OBCs when simulating their 3-shock model through the Great Recession periods with the nominal rate hitting the ZLB. Instead of applying a well-built algorithm for OBCs, they simply assume that households expect the nominal rate to bind at ZLB for a fixed number of periods until the rate finally escapes the bound since 2015. This assumption is more ad hoc and cannot be developed into a rigorous algorithm that can be generalized to all models with OBCs.

the model follows the following transition law of state variables at time t[7]

$$\mathcal{X}_t = \mathcal{G}_t(\mathcal{X}_{t-1}, \mathcal{A}_t, \mathcal{E}_t, \mathcal{E}_{t+1}) \tag{2.1}$$

Written in the sense of subperiods, the above transition equations reads

$$\begin{bmatrix} x_t \\ x_{t+1/2} \end{bmatrix} = \begin{bmatrix} g_t(\mathcal{X}_{t-1}, a_t, a_{t+1/2}, \varepsilon_t, \varepsilon_{t+1/2}, \mathcal{E}_{t+1}) \\ g_{t+1/2}(\mathcal{X}_{t-1}, a_t, a_{t+1/2}, \varepsilon_t, \varepsilon_{t+1/2}, \mathcal{E}_{t+1}) \end{bmatrix} \tag{2.2}$$

where $g_t(\cdot)$ and $g_{t+1/2}(\cdot)$ are state transition equations for state functions of the first and second subperiod, respectively. More generally, the model (2.1) can be augmented with a set of inequalities and non-state-transition equalities. Formally speaking, at a given period t, a state space model is characterized by the following equilibrium conditions

$$\begin{cases} \begin{bmatrix} x_s \\ x_{s+1/2} \end{bmatrix} = \begin{bmatrix} g_s(\mathcal{X}_{s-1}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}, \mathcal{E}_{s+1}) \\ g_{s+1/2}(\mathcal{X}_{s-1}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}, \mathcal{E}_{s+1}) \end{bmatrix} \\ 0 \leq \mathcal{F}_s(x_s, x_{s+1/2}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}), \\ 0 = \begin{bmatrix} h_s(\mathcal{X}_{s-1}, x_s, x_{s+1/2}, a_s, a_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, \varepsilon_{s+1/2}, \mathcal{E}_{s+1}) \\ h_{s+1/2}(\mathcal{X}_{s-1}, x_s, x_{s+1/2}, a_s, a_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, \varepsilon_{s+1/2}, \mathcal{E}_{s+1}) \end{bmatrix}, \quad s = t, t+1, t+2, \cdots \end{cases} \tag{2.3}$$

where the set of inequalities represents the feasibility constraints at period s, and the set of equalities $\mathcal{H}_s(\cdot) \equiv [h_s(\cdot)', h_{s+1/2}(\cdot)']'$ is a combination of all equilibrium conditions other than the state transition laws, which could include the Euler equations and market clearing conditions.

Similar to SCEQ, and more generally, the extended-path method, we apply certainty-equivalence assumption to solve the model. The certainty equivalence assumption removes the randomness of the model by setting all future shocks $\mathcal{E}_s$, $s > t$, to their mean

[7]To be consistent with the criterion of subperiod division on the timing of decisions, we also index the timing of state variables based on the decision period, e.g., $\mathcal{X}_t$ means state variables active in period $t+1$ and pre-determined in period t. As a contrast, a more commonly seen form denotes $\mathcal{X}_t$ as the state variables active in period t but pre-determined at period $t-1$.

zero. Along this, the transformed deterministic model reads,

$$\begin{cases} \begin{bmatrix} x_s \\ x_{s+1/2} \end{bmatrix} = \begin{bmatrix} g_s(\mathcal{X}_{s-1}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}, 0) \\ g_{s+1/2}(\mathcal{X}_{s-1}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}, 0) \end{bmatrix} \\ \quad 0 \quad \leq \quad \mathcal{F}_s(x_s, x_{s+1/2}, a_s, a_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}), \\ \quad 0 \quad = \begin{bmatrix} h_s(\mathcal{X}_{s-1}, x_s, x_{s+1/2}, a_s, a_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, \varepsilon_{s+1/2}, 0) \\ h_{s+1/2}(\mathcal{X}_{s-1}, x_s, x_{s+1/2}, a_s, a_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, \varepsilon_{s+1/2}, 0) \end{bmatrix}, \quad s = t, t+1, t+2, \cdots \end{cases} \tag{2.4}$$

In addition to the conventional certainty-equivalence assumption, the intraperiod certainty-equivalence is applied when solving the model within each subperiod. In the two-step setting, when the agent makes decisions during the first subperiod, the shock value of the "future" —— the second subperiod $\varepsilon_{t+1/2} = 0$, is assumed to take its mean value zero. [8].

To make the model solvable, the original infinite-horizon problem is truncated into a finite-horizon one with a terminal condition. For a stationary model, a commonly used terminal condition is that all endogenous variables return to their steady-state equilibrium levels at some large enough terminal period T, i.e., $(\mathcal{X}_T, \mathcal{A}_T) = (\mathcal{X}_{ss}, a_{ss})$. Without loss of generality, a terminal condition is represented by a terminal policy function $a_{s+\Delta_s} = a^*_{s+\Delta_s}(x_{s+\Delta_s})$ with a given $a^*_{s+\Delta_s}$.

It is now appropriate to introduce the details of the two-step SCEQ algorithm to the stochastic competitive equilibrium (CE) of a two-subperiod state-space model. The algorithm can be easily generated to models with an arbitrary number of subperiods in one period. As each subperiod needs to be solved one by one, the number of steps for solving the model equals the number of subperiods. Algorithm 1 summarizes the two-step SCEQ method for solving stochastic competitive equilibrium (CE) problems with EIS at an arbitrary period s[9].

A few comments are to be left regarding Algorithm 1. First, when compared to Cai

[8] In the two-subperiod model with only one shock, period t shock $\mathcal{E}_t = [0,\ \varepsilon_{t+1/2}]$. The first subperiod must be free of shock, and in the second subperiod the only shock is revealed.

[9] The NLP problem defined in Step 2 has a constant objective function 1. This means to find a feasible solution that satisfy all of the equilibrium conditions in the model.

and Judd (2023), the difference only applies in Step 2. Instead of just one-step optimization in Cai and Judd (2023), the model is solved through NLP with multiple steps. In each step j, the entire model is solved with shocks ε_t^j revealed at the beginning of subperiod j. Second, when solving the model at subperiod j of period s, not only the "conventional" certainty-equivalence (CEQ) assumption is applied with $\varepsilon_t = 0$ for all future periods $t > s$, but also applied is the subperiod version of the CEQ assumption. The latter turns the stochastic model corresponding to the 1^{st} subperiod into a deterministic one by assuming that all shocks to be realized in "future" —— the 2^{nd} subperiod ε_s^2 take their mean value 0. Third, fixing the value x_t^1 and a_t^1 can be viewed as the subperiod-version of the simulation step (Step 3) in the original SCEQ algorithm[10]. Simulation of these variables before moving to the next subperiods ensures that revelation of shocks in the 2^{nd} subperiod has no contemporaneous effect on x_t^1 and a_t^1.

3 Numerical Analysis

In this section, we apply the multi-step SCEQ algorithm to a small-scale RBC model. The model is featured with downward rigidity in investment, which brings kinks into the equilibrium conditions. we first describe the model, including the settings and equilibrium conditions. Next, we compute the error of Euler equations and discuss on the accuracy of model solutions. Finally, we shift to the economic implications of EIS by focusing on how EIS changes the short-run and medium-run dynamics of the model economy.

[10] In the generalized $\mathcal{J}$-subperiod version of EIS, in reference to period $j+1$, the cumulatively fixed endogenous variables in all the previous subperiods $(x_{t+1}^1, x_{t+1}^2, \cdots, x_{t+1}^j)$ and $(a_t^1, a_t^2, \cdots, a_t^j)$ are equivalently pre-determined "state" variables.

[11] Δ_s depends on terminal period T. Two commonly used strategies are used. One is fixing T, such that $\Delta_s = T - s$. The other is fixing Δ_s, such that T is variable.

[12] The competitive equilibrium problem in Cai and Judd (2023) optimizes with an objective function with a constant value. This is equivalent to finding a feasible solution that satisfies all the equilibrium conditions. However, this configuration might lead to numerical instability when subject to large shocks. Here I use the $\mathcal{L}^2$ norm over the distance between the terminal and the steady-state levels. This objective function helps to improve the numerical stability, especially with large shocks, or in large-scale models.

[13] Actually, x_s has already computed in Step 2. In practice, and GAMS in particular, Step 3 is realized via fixing the levels of $\mathcal{X}_s$ at their solution levels in Step 2, before initiating the algorithm for period $s+1$.

Algorithm 1 SCEQ for Stochastic CE Problems with EIS

Step 1. Initialization step. Given the initial state $\mathcal{X}_0$ and a time of interest T^*, choose a time-varying number of periods Δ_s and a time-varying "terminal" policy function $\mathcal{A}^*_{s+\Delta_s}(\mathcal{X}_{s+\Delta_s})$ for each time s[11]. Simulate a sequence of $\mathcal{E}_t$ from $t = 1$ *to* T^*. Proceed to step 2 and 3 for $s = 1, 2, \cdots, T^*$. Since every simulation follows the same algorithm but just unique in shock profile, the index for an individual simulation is omitted here, for the sake of brevity in notations.

Step 2. Optimization step. Solve the following deterministic model starting at time $s \geq t$, given state variable $\mathcal{X}_{t-1}$ and exogenous shock $\mathcal{E}_t$. Step 2 is decomposed into the following three substeps.

1. Solve the CE solution $\{\{\mathcal{X}^1_s, \mathcal{A}^1_s\}\}_{s\geq t}$ to the whole model within the first subperiod of period s, i.e., the following maximization program[12]

$$\max_{\mathcal{X}^1_s,\mathcal{A}^1_s} \quad ||\mathcal{X}_{s+\Delta_s} - \mathcal{X}_{ss}||_2 + ||\mathcal{A}_{s+\Delta_s} - \mathcal{A}_{ss}||_2$$

subject to

$$\begin{aligned}
\begin{bmatrix} x^1_s \\ x^1_{s+1/2} \end{bmatrix} &= \begin{bmatrix} g_s(\mathcal{X}_{s-1}, a^1_s, a^1_{s+1/2}, \varepsilon_s, 0, 0) \\ g_{s+1/2}(\mathcal{X}_{s-1}, a^1_s, a^1_{s+1/2}, \varepsilon_s, 0, 0) \end{bmatrix} \\
0 &\leq \mathcal{F}_s(x^1_s, x^1_{s+1/2}, a^1_s, a^1_{s+1/2}, \varepsilon_s, 0), \\
0 &= \begin{bmatrix} h_s(\mathcal{X}_{s-1}, x^1_s, x^1_{s+1/2}, a^1_s, a^1_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, 0, 0) \\ h_{s+1/2}(\mathcal{X}_{s-1}, x^1_s, x^1_{s+1/2}, a^1_s, a^1_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, 0, 0) \end{bmatrix}, \quad s = t, t+1, t+2, \cdots
\end{aligned} \tag{2.5}$$

where $\mathcal{X}^1_s \equiv [x^1_s, x^1_{s+1/2}]$ and $\mathcal{A}^1_s \equiv [a^1_s, a^1_{s+1/2}]$.
2. Simulation substep: fix $x^2_s = x^1_s$ and $a^2_s = a^1_s$
3. Solve the CE solution $\{\{\mathcal{X}^2_s, \mathcal{A}^2_s\}\}_{s\geq t}$ to the whole model within the first subperiod of period s, i.e., the following maximization program

$$\max_{\mathcal{X}^2_s,\mathcal{A}^2_s} \quad ||\mathcal{X}_{s+\Delta_s} - \mathcal{X}_{ss}||_2 + ||\mathcal{A}_{s+\Delta_s} - \mathcal{A}_{ss}||_2$$

subject to

$$\begin{aligned}
x^2_{s+1/2} &= g_{s+1/2}(\mathcal{X}_{s-1}, a^2_s, a^2_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}, 0) \\
0 &\leq \mathcal{F}_s(x^2_s, x^2_{s+1/2}, a^2_s, a^2_{s+1/2}, \varepsilon_s, \varepsilon_{s+1/2}), \\
0 &= h_{s+1/2}(\mathcal{X}_{s-1}, x^2_s, x^2_{s+1/2}, a^2_s, a^2_{s+1/2}, \mathcal{A}_{s+1}, \varepsilon_s, \varepsilon_{s+1/2}, 0), \quad s = t, t+1, t+2, \cdots
\end{aligned} \tag{2.6}$$

where $\mathcal{X}^2_s \equiv [x^2_s, x^2_{s+1/2}] = [x^1_s, x^2_{s+1/2}]$ and $\mathcal{A}^2_s \equiv [a^2_s, a^2_{s+1/2}] = [a^1_s, a^2_{s+1/2}]$ based on the second (simulation) substep.

Step 3. Simulation step. Pre-determined state variables are updated as $\mathcal{X}_s = [x^2_s, x^2_{s+1/2}]$[13] before entering the next period.

3.1 The Model

The model economy consists of a representative household, who allocates her income into consumption and capital investment. A central planner maximizes the household's utility

$$\max_{C_t, I_t, K_t} \mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \tag{3.1}$$

subject to the following constraints.

$$C_t + I_t = A_t K_{t-1}^{1-\alpha} \tag{3.2}$$

$$K_t = (1-\delta) K_{t-1} + I_t \tag{3.3}$$

$$I_t \geq \phi I_{ss} \tag{3.4}$$

The three constraints (3.2)-(3.4) can be interpreted as follows. Equation (3.2) is the aggregate resource constraint. The right-hand side of (3.2) shows the aggregate production function, with A_t being the exogenous technology which evolves according to the following AR(1) process

$$\ln A_t = \rho \ln A_{t-1} + \sigma \varepsilon_{At} \tag{3.5}$$

where ρ is persistence of A_t, σ is the standard deviation of technology innovations and $\varepsilon_{At} \sim \mathcal{N}(0,1)$ is an i.i.d. random variables with standard normal distribution. Equation (3.3) is the law of motion of capital stock K_t. Finally, equation (3.4) states that investment cannot fall below a fraction ϕ of the steady-state level of investment. It introduces an OBC into the model[14].

A key difference between this model and the central planner's model in Cai and Judd (2023) is that the planner makes decisions with an exotic information set $\mathcal{E}_0$. In particular, she optimizes investment I_t each period in the subperiod prior to the realization of TFP shock, and on consumption C_t in the subperiod after the shock. When it is applied to equilibrium conditions, each period is divided into two subperiods. Inheriting the notation in Algorithm 1, we use E_t and $E_{t+1/2}$ to denote the subperiod before and after the

[14] When $\phi = 0$, investment is completely irreversible.

TFP shock is revealed, respectively[15].

Let μ_t denote the multiplier for investment constraint. The Euler equation for capital accumulation decision K_t and associated Kuhn-Tucker condition are

$$E_t\big[C_t^{-\gamma}\big] = \beta E_t\big[C_{t+1}^{-\gamma}(1-\delta+\alpha A_{t+1}K_t^{\alpha-1}) + [1-\beta(1-\delta)]\mu_t\big] \tag{3.6}$$

$$E_t\big[\mu_t(I_t - \phi I_{ss})\big] = 0 \tag{3.7}$$

As a reflection of EIS, (3.6), (3.7) and (3.3) are subject to the information set without TFP shock A_t. The endogenous variables corresponding to the three equations are: I_t, μ_t and K_t. Decisions on consumption C_t are subject to the information set containing A_t. Specifically, C_t is determined by aggregate resource constraint (3.2) after A_t is revealed.

As a summary of the model, the competitive equilibrium of the economy therefore consists of equations (3.2), (3.3), (3.5), (3.6) and (3.7), which characterize the dynamic evolution of six variables $\{C_t, I_t, A_t, K_t, \mu_t\}$[16]. Equation (3.6) is equivalent to the standard consumption Euler equation when the investment constraint does not bind and $\mu_t = 0$. When the constraint binds and $\mu_t > 0$, the capital decision K_t must be increased. This is consistent with the implication of the KKT condition that I_t cannot be curtailed below its lower bound, which forces the planner to inefficiently overinvest in K_t.

3.2 Impulse Response Functions

As a first glimpse of how EIS affects economic dynamics, we evaluate the impulse response functions (IRFs) of investment and consumption subject to a one-std negative TFP shock. To do so, I simulate the economy with a certainty-equivalent shock profile $[-0.013, 0, \cdots, 0]'$, i.e., a negative shock with the amplitude of one standard deviation of ε_{At}, and 0 afterward. The IRFs are shown in figure 1. In the first row of the figure, we simulate the model without a lower bound of investment. In the first quarter, also the shock period, investment has no response in the model with EIS, which is consistent

[15] For brevity, $E_{t+1/2}$ operator is omitted in model equations, since it is the "canonical" information set with all shocks revealed.

[16] Inheriting notation in the model defined in the last section, we have the state variables $\mathcal{X}_t = [x_t^1, x_t^2]'$, with $x_t^1 = [K_t]$ and $x_t^2 = [A_t]$, and the control/action variables $\mathcal{A}_t = [a_t^1, a_t^2]'$, with $a_t^1 = [I_t, \mu_t]$ and $a_t^2 = [C_t]'$.

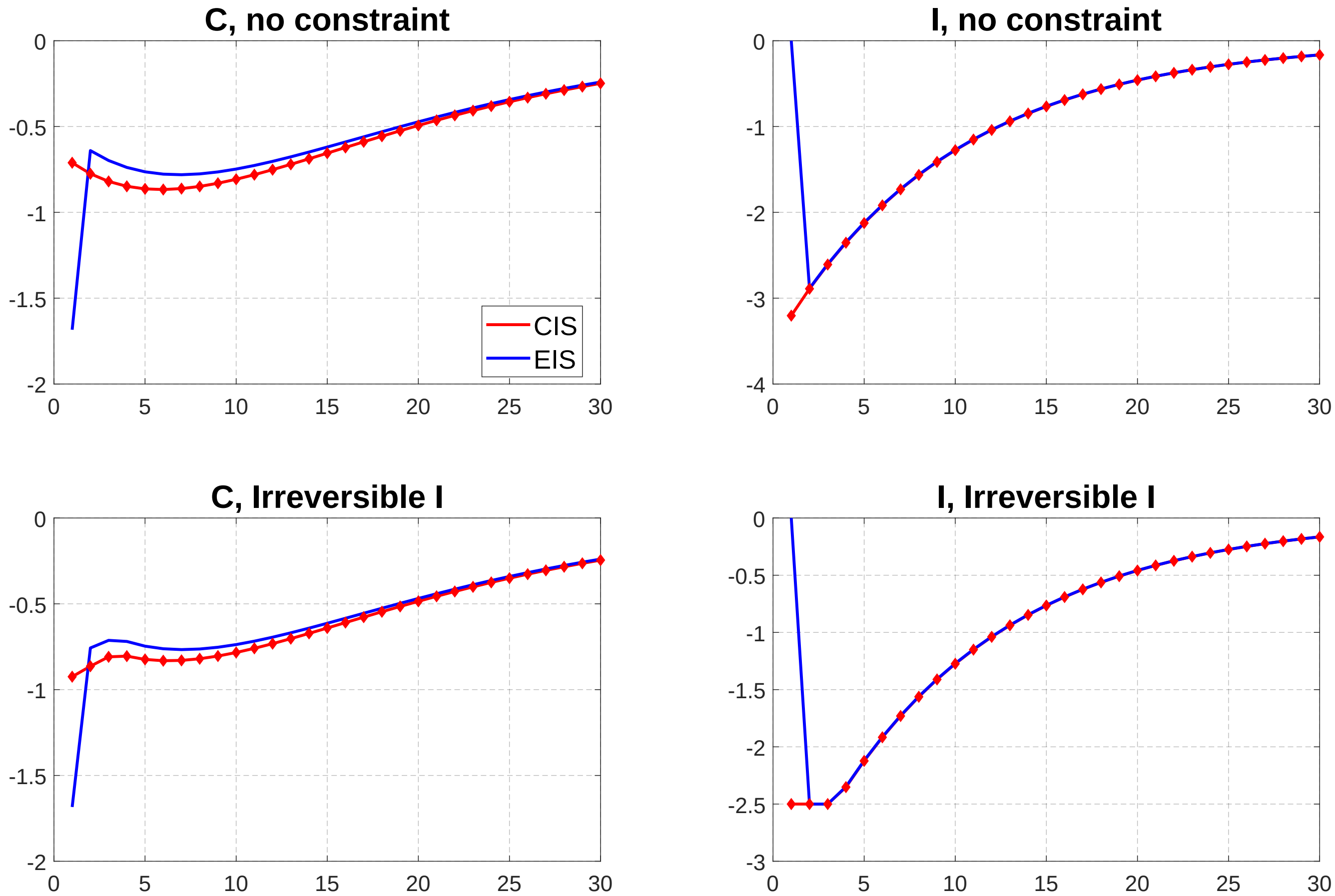

Figure 1: Impulse Response Functions with a one-std negative TFP shock. y-axis: percentage deviations from steady-state level.

with the intuition of EIS that contemporaneous responses of a variable to shocks realized in latter subperiods are zero. In contrast, in the model with CIS, investment falls immediately in response to the negative TFP shock. Due to aggregate resource constraint, consumption also behaves differently across different information structure. With CIS, consumption responds in a hump-shaped pattern, due to risk aversion and consumption smoothing. With EIS, consumption suffers from a rapid slump, followed by a sharp rise and return to the hump-shaped pattern. Since investment has no initial response, consumption fully absorbs the initial negative TFP shock. Later, as investment returns to the normal trajectory, the hump-shaped response is quickly recovered. Note that consumption levels after the first period are slightly higher with EIS than with CIS. The initial no response of investment avoids immediate capital loss. This in turn causes the later periods with larger capital stock and more output. This explains why households benefit from a rapid rebound and a sequence of higher levels of consumption.

The second row of figure 1 shows the IRFs of the model with partially irreversible investment. The patterns of consumption and investment responses are similar to those with fully reversible investments. The initial zero response of investment is also reflected in the model solution. Also, the IRF of investment reflects the partial irreversibility. It remains flat at a constant level when the lower bound is hit. This implies the applicability of the multi-step SCEQ algorithm in non-differentiable models with kinks in the policy function.

3.3 Error Checking

To accommodate the error checking to the EIS configuration, we evaluate a variant of the Euler error that is defined in Cai and Judd (2023)[17]. In Cai and Judd (2023), expectations of the variable levels in the next period are evaluated using Gauss-Hermite quadrature (GHQ) nodes. Due to EIS, there exists uncertainty not only in the next period but also in the current period. Therefore, we need two sets of GHQ nodes[18]. Let λ and ν respectively denote the sets of Lagrangian multipliers associated with state transition equations $\mathcal{G}(\cdot)$ and "other" equalities $\mathcal{H}(\cdot)$, respectively. For an arbitrary simulation i,

[17] Please refer to Equation (A.4) in the online appendix for comparison.

[18] We apply 7 GHQ nodes for each set, so for each period, error checking involves $7^2 = 49$ GHQ nodes.

the general equation for the Euler error $\mathcal{EL}_s^{1,i}$ at the first subperiod of period s and state variable $x_s^{1,i}$ is written as follows:

$$\begin{aligned}\mathcal{EL}_s^{1,i} = \Bigg\|\beta \sum_{j,\ell} w_j w_\ell \frac{1}{\lambda_{s+1/2}^{2,(j)}} \boldsymbol{\nabla}_{x_s^1}\Big[\mathcal{G}_{s+1}\big(\mathcal{X}_s^{(j)}, \mathcal{A}_{s+1}^{(j,\ell)}, \mathcal{E}_s^{(j)}, \mathcal{E}_{s+1}^{(\ell)}\big)\lambda_{s+3/2}^{2,(j,\ell)} \\ + \mathcal{H}_{s+1}\big(\mathcal{X}_{s-1}, \mathcal{X}_s^{(j)}, \mathcal{A}_{s+1}^{(j,\ell)}, \mathcal{E}_s^{(j)}, \mathcal{E}_{s+1}^{(\ell)}\big)\nu_{s+2/3}^{2,(j,\ell)}\Big] - 1\Bigg\|\end{aligned} \tag{3.8}$$

where j and ℓ index the GHQ nodes in period s and $s+1$, respectively, and w_j and w_ℓ denote the weights for the nodes[19]. $\mathcal{X}_s^{(j)}$ is a combination of x_s^1, the state variables whose decisions are made in the first subperiod, and $x_{s+1/2}^{2,(j)}$ in the second subperiod, with the latter depending on the revealed shock level and the GHQ node in period s. Analogously, $\mathcal{A}_{s+1}^{(j,\ell)}$ is a combination of $a_{s+1}^{1,(j)}$, the state variables whose decisions are made in the first subperiod and $a_{s+3/2}^{2,(j,\ell)}$ in the second subperiod, with the latter depending on the revealed shock levels and the GHQ nodes in period s and $s+1$. Notice that $a_{s+1}^{1,(j)}$, the vector of period $s+1$ control variables whose decisions are made in the first subperiod, is again a reflection of accommodation to the EIS configuration.

In the context of the capital Euler equation (3.6), the corresponding Euler error can be approximated using the following equation,

$$\begin{aligned}\mathcal{EL}_s^{K,i} = \Bigg\|\beta \sum_{j,\ell} w_j w_\ell \big(C_{s+1/2}^{(j)}\big)^{\gamma}\Big[\big(1-\delta+\alpha A_{s+1}^{(\ell)} K_s^{\alpha-1}\big)\big(C_{s+3/2}^{(j,\ell)}\big)^{-\gamma} \\ + \big[1-\beta(1-\delta)\big]\mu_{s+2/3}^{(j,\ell)}\Big] - 1\Bigg\|\end{aligned} \tag{3.9}$$

If we have m simulated paths, we can approximate the $\mathcal{L}^\infty$ Euler error of the first T^* periods' solutions for these simulations, defined as

$$\max_{1\le s\le T^*}\big(\max_{1\le i\le m} \mathcal{EL}_s^{K,i}\big) \tag{3.10}$$

Similarly, the corresponding $\mathcal{L}^1$ Euler error can be computed using

$$\frac{1}{mT^*}\sum_{1\le s\le T^*, 1\le i\le m}\big(\mathcal{EL}_s^{K,i}\big) \tag{3.11}$$

[19]It must be satisfied that $\sum_{j,\ell} w_j w_\ell = 1$

Table 1 shows the approximated Euler errors across the simulations of the benchmark model using the multi-step SCEQ algorithm. For the CIS case, both $\mathcal{L}^\infty$ and $\mathcal{L}^1$ error are close to the counterparts of the multi-country RBC model in Cai and Judd (2023)[20]. Not surprisingly, the $\mathcal{L}^1$ error for the periods when the irreversible investment constraint binds is slightly larger than the overall $\mathcal{L}^1$ error. However, the former is still in the same order of magnitude with the latter. With EIS, the Euler errors are close to those under the CIS configuration. This qualifies our multi-step SCEQ algorithm as one that delivers a high level of solution accuracy.

Table 1: Euler Errors

	$\mathcal{L}^\infty$	$\mathcal{L}^1$	$\mathcal{L}^1$ binding
CIS	7.2(-3)	2.0(-3)	3.1(-3)
EIS	6.4(-3)	1.7(-3)	2.5(-3)

CIS: canonical information set
$a(-b)$: $a \times 10^{-b}$

4 Concluding Remarks

In this paper, we develop a multi-step SCEQ algorithm for the solution of non-linear DSGE models with EIS. The algorithm relies upon not only the conventional inter-period certainty equivalence but also INTRA-period and INTER-subperiod certainty equivalence. Through inspection of IRFs generated from a neoclassical growth model with partially irreversible investment, we find that the algorithm yields solutions consistent with the key intuition of EIS that a variable has no contemporaneous response to a shock revealed in a subperiod later than its decision subperiod. The algorithm can also handle models with occasionally-binding constraints and kinks in the policy function. Error checking implies that the algorithm is able to solve models with a level of accuracy that is comparable to the original SCEQ algorithm.

[20]The multi-country RBC model consists multiple countries with each country evolve according to the same model as described in this section. It is reported that the Euler errors do not scale up with the number of countries.

References

Altig, David, Lawrence J Christiano, Martin Eichenbaum, and Jesper Linde, "Firm-specific capital, nominal rigidities and the business cycle," *Review of Economic dynamics*, 2011, *14* (2), 225–247.

Cai, Yongyang and Kenneth L Judd, "A simple but powerful simulated certainty equivalent approximation method for dynamic stochastic problems," *Quantitative Economics*, 2023, *14* (2), 651–687.

Christiano, Lawrence J, Martin Eichenbaum, and Charles L Evans, "Monetary policy shocks: What have we learned and to what end?," in "Handbook of Macroeconomics," Vol. 1, Elsevier, 1999, pp. 65–148.

_ , _ , and _ , "Nominal rigidities and the dynamic effects of a shock to monetary policy," *Journal of political Economy*, 2005, *113* (1), 1–45.

_ , Martin S Eichenbaum, and Mathias Trabandt, "Understanding the great recession," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 110–167.

_ , Mathias Trabandt, and Karl Walentin, "DSGE models for monetary policy analysis," in "Handbook of monetary economics," Vol. 3, Elsevier, 2010, pp. 285–367.

DiCecio, Riccardo, "Sticky wages and sectoral labor comovement," *Journal of Economic Dynamics and Control*, 2009, *33* (3), 538–553.

Dupor, Bill, Jing Han, and Yi-Chan Tsai, "What do technology shocks tell us about the New Keynesian paradigm?," *Journal of Monetary Economics*, 2009, *56* (4), 560–569.

Fair, Ray C and John B Taylor, "Solution and maximum likelihood estimation of dynamic nonlinear rationalexpectations models," 1983.

Guerrieri, Luca and Matteo Iacoviello, "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 2015, *70*, 22–38.

Holden, Tom D, "Existence and uniqueness of solutions to dynamic models with occasionally binding constraints," *Review of Economics and Statistics*, 2023, *105* (6), 1481–1499.

King, Robert G and Mark W Watson, "System reduction and solution algorithms for singular linear difference systems under rational expectations," *Computational economics*, 2002, *20*, 57–86.

Mankiw, N Gregory and Ricardo Reis, "Sticky information: A model of monetary nonneutrality and structural slumps," 2001.

_ **and** _ , "Sticky information versus sticky prices: a proposal to replace the New Keynesian Phillips curve," *The Quarterly Journal of Economics*, 2002, *117* (4), 1295–1328.

_ **and** _ , "Pervasive stickiness," *American Economic Review*, 2006, *96* (2), 164–169.

_ **and** _ , "Imperfect information and aggregate supply," *Handbook of monetary economics*, 2010, *3*, 183–229.

Reis, Ricardo, "Inattentive Consumers," *Journal of Monetary Economics*, November 2006, *53* (8), 1761–1800.

_ , "Inattentive Producers," *Review of Economic Studies*, July 2006, *73* (3), 793–821.

Simon, Herbert A, "Dynamic programming under uncertainty with a quadratic criterion function," *Econometrica, Journal of the Econometric Society*, 1956, pp. 74–81.

Theil, Henri, "A note on certainty equivalence in dynamic planning," *Econometrica: Journal of the Econometric Society*, 1957, pp. 346–349.