Hypothesis Testing - Evaluation

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10 de enero de 2020

Excercice 1

A comparison test in the Gaussian framework

Question 1, answer:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : \sigma_1^2 = \sigma_2^2$
$H_1 : \sigma_1^2 \neq \sigma_2^2$

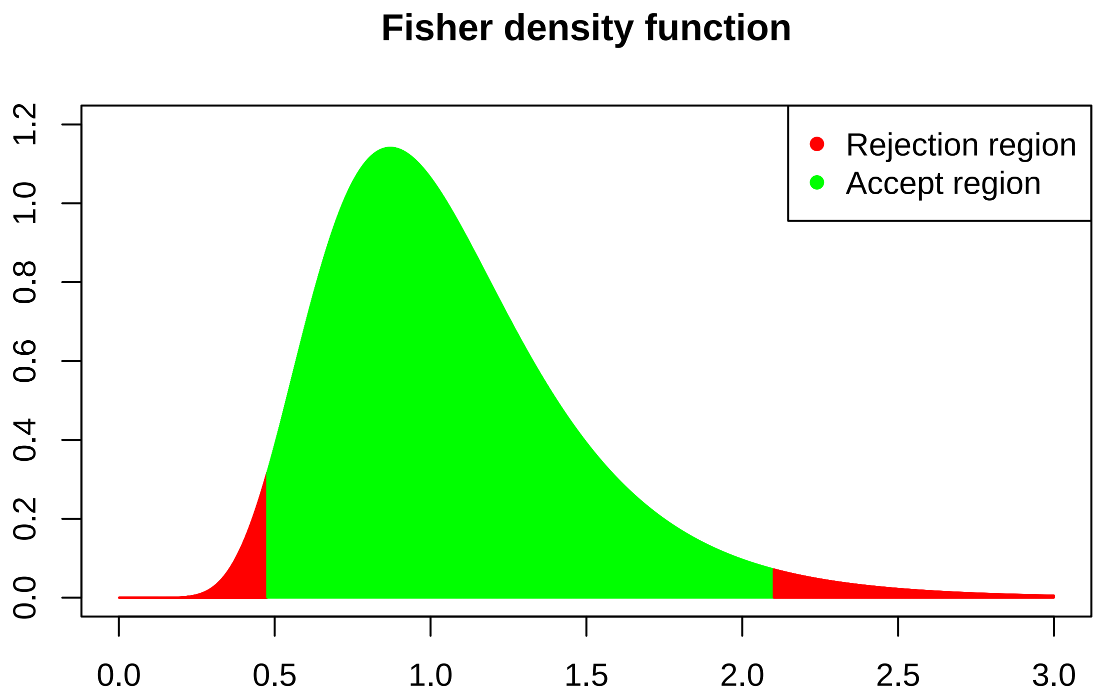
Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T = \frac{S_1^2}{S_2^2} \ \sim \ F(n_1 - 1, n_2 - 1)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W \ = \ \left\{T \ \leq \ F_{\frac{0.05}{2}}(29, 29)\right\} \ \cup \ \left\{T \ \geq \ F_{\left(1-\frac{0.05}{2}\right)}(29, 29)\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W= $\{(0, 0.476) \cup (2.101, \infty)\}$

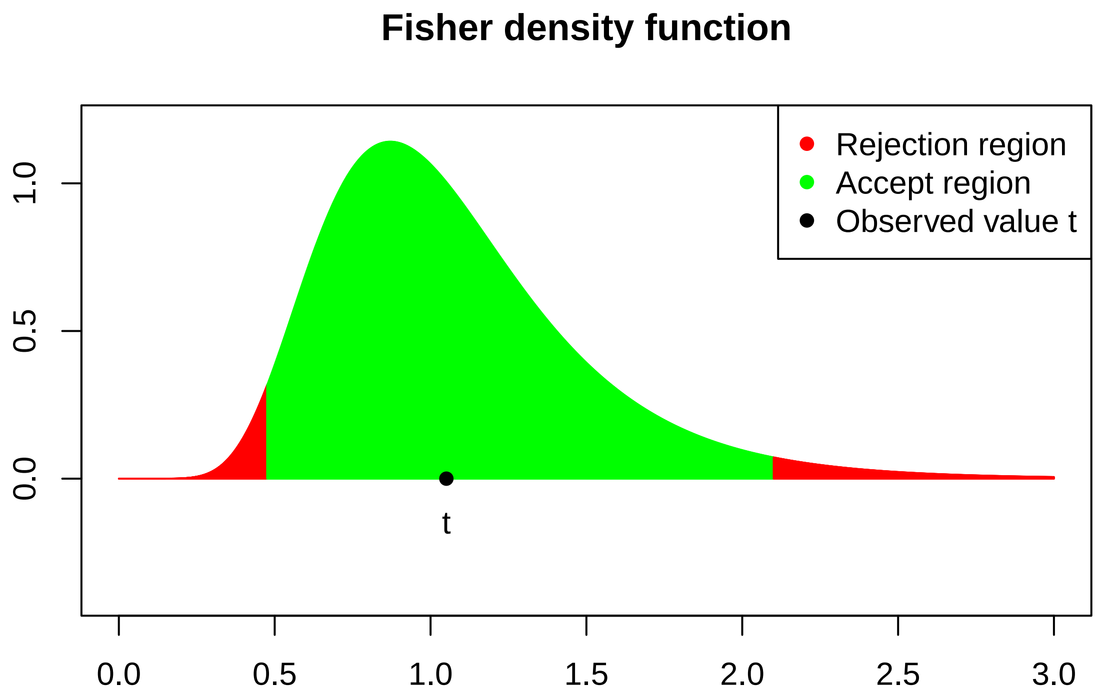


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##
## S1^2 221.379310
## S2^2 210.689655
## t      1.050736
```

Step 8: According to t, decide whether to accept or not H_0.



As t is not in the rejection region W ($t \notin W$), we accept H_0, therefore $\sigma_1^2 = \sigma_2^2$.

Question 2, answer:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : m_1 \leq m_2$
$H_1 : m_1 > m_2$

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T = \frac{\overline{X_1} - \overline{X_2}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

Where:

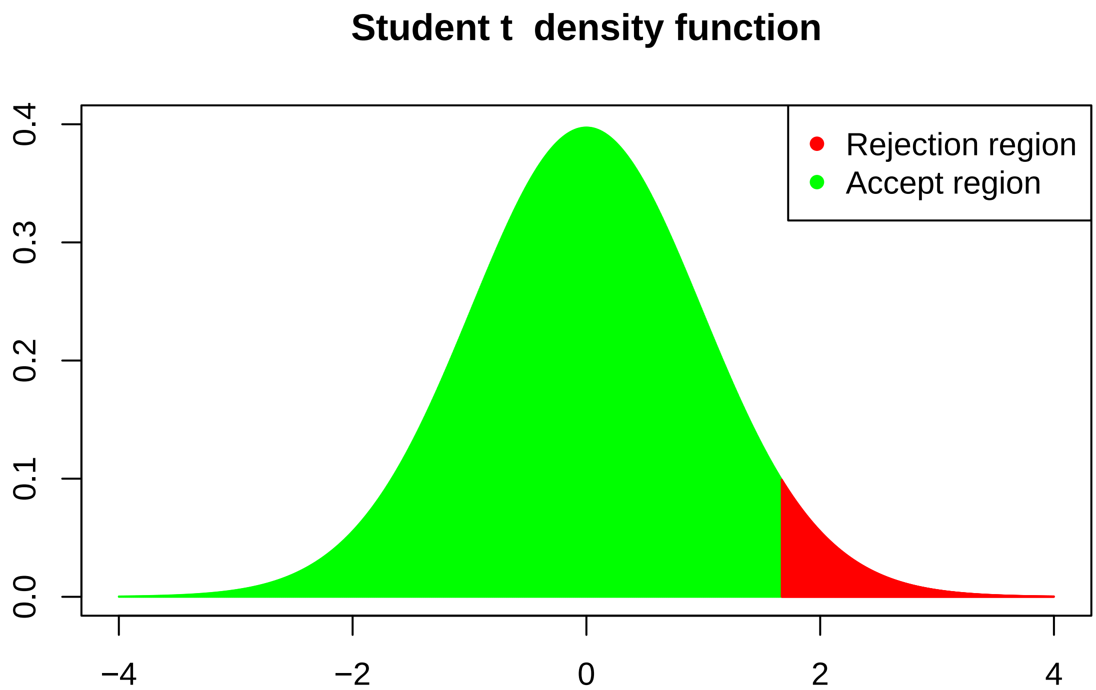
$S_p^2 = \frac{S_1^2(n_1-1) + S_2^2(n_2-1)}{n_1+n_2-2}$

$T \sim t_{\alpha,(n_1+n_2-2)}$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T > t_{\alpha,(n_1+n_2-2)}\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W$= \{(1.672, \infty)\}$

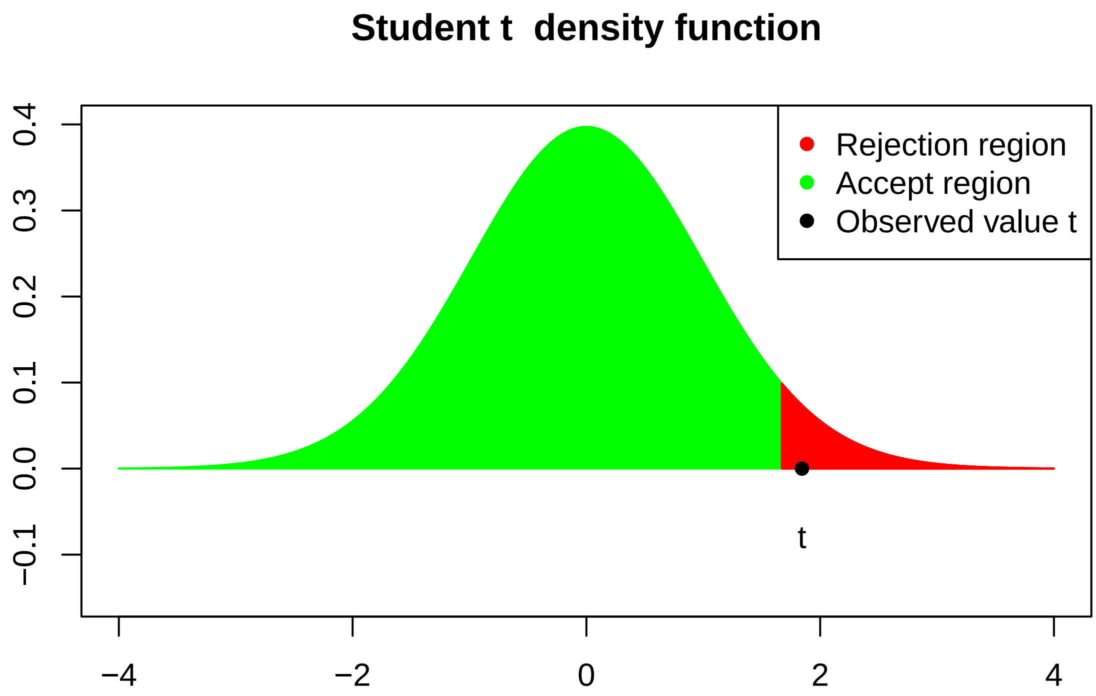


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##
## S1^2 221.379310
## S2^2 210.689655
## Sp^2 216.034483
## t      1.844515
```

Step 8: According to t, decide whether to accept or not H_0.



As t it is in rejection region W $(t \in W)$, we reject H_0 and accept H_1, therefore $m_1 > m_2$.

Question 3, answer:

In he first question it is about checking that the variances in the speed of the first serve are equal, in the second question it is about prove that the speeds of the first serve are different for both players and that the speed of the first servece of Soderling is greater than Federer's speed.

Justifying the Gaussian framework

Question 1, answer:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : P = P_0$
$H_1 : P \neq P_0$

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T^2 = \sum_{k=1}^{m} \frac{\left(N_k(n) - np_0^k\right)^2}{np_0^k}$

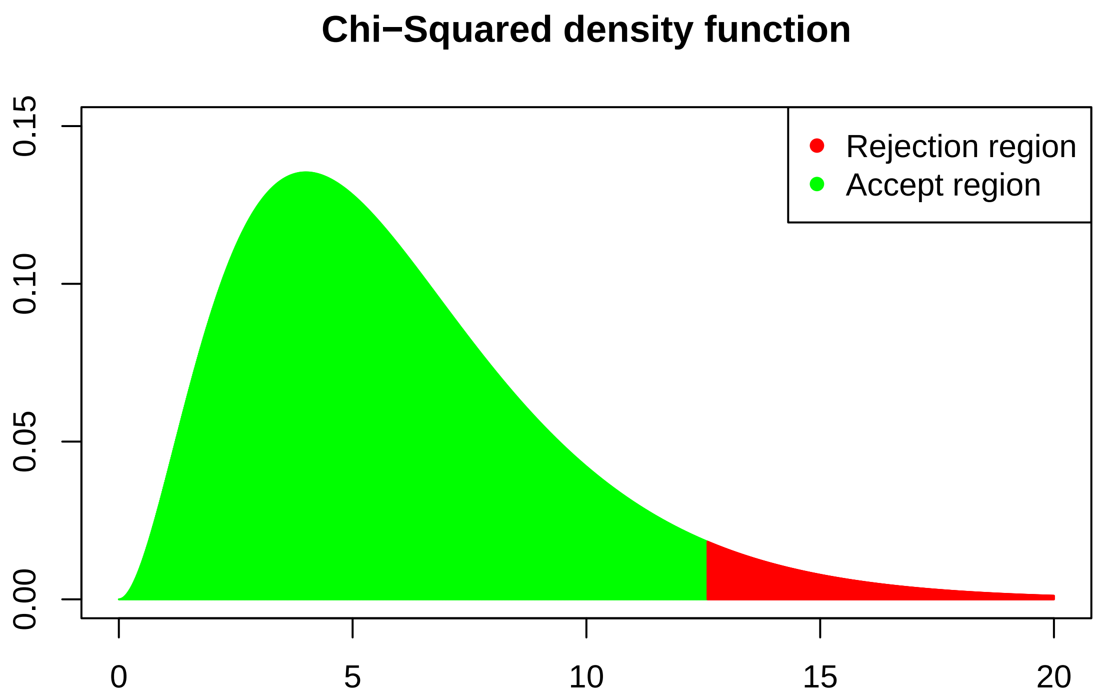
Where:

$T^2 \sim \chi^2\,(m-1)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T > \chi^2\,(m-1)\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W= $\{(12.592, \infty)\}$

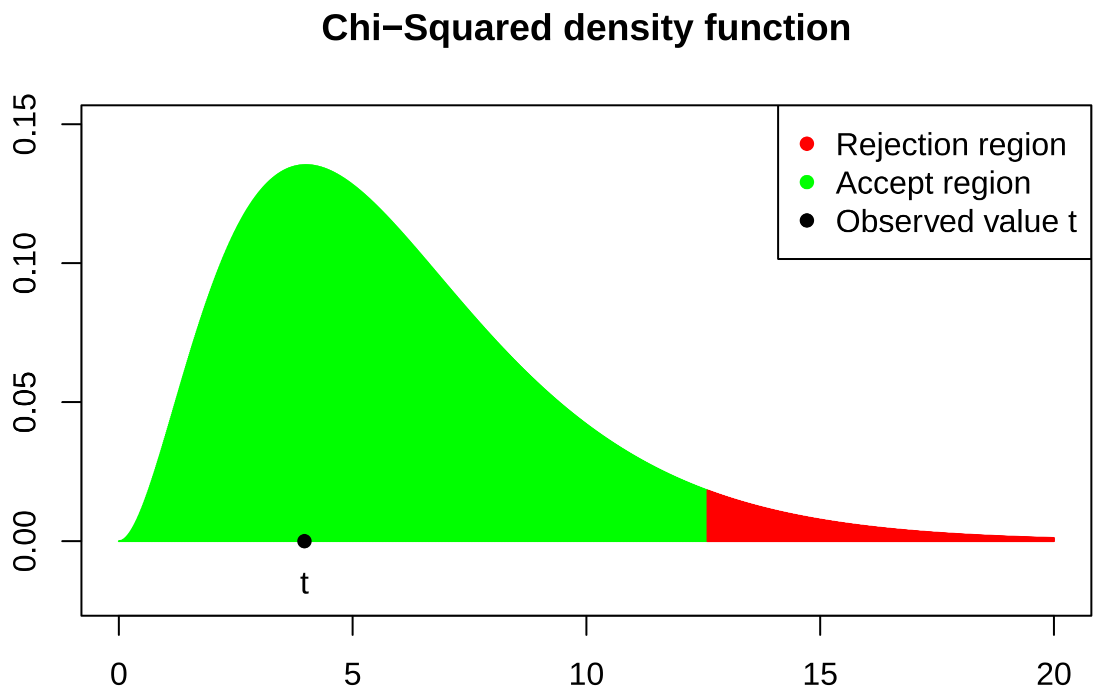


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##     classes Freq        P0k     nP0k       T2
## 1    [0,170)    2 0.03478809 1.043643 0.000000
## 2 [170,180)    1 0.09182181 2.754654 0.000000
## 3 [180,190)    8 0.19240066 5.772020 0.000000
## 4 [190,200)    6 0.26088600 7.826580 0.000000
## 5 [200,210)    7 0.22896946 6.869084 0.000000
## 6 [210,220)    5 0.13006012 3.901804 0.000000
## 7 [220,Inf)    1 0.06107385 1.832216 3.969926
```

Step 8: According to t, decide whether to accept or not H_0.



As t is not in the rejection region W $(t \notin W)$, we accept H_0, therefore $P = P_0$.

We consider that the results are reliable provided that $np_0^k \geq 5$ $\forall k = 1, \ldots, m$, do so the results of this test are not reliable provided.

Question 1.1, answer:

As the previous test are not reliable provided then is nesesary that the partition may be modified by concatenating more of five elements, next we go to work to obtain this objective.

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : P = P_0$
$H_1 : P \neq P_0$

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T^2 = \sum\limits_{k=1}^{m} \frac{\left(N_k(n) - np_0^k\right)^2}{np_0^k}$

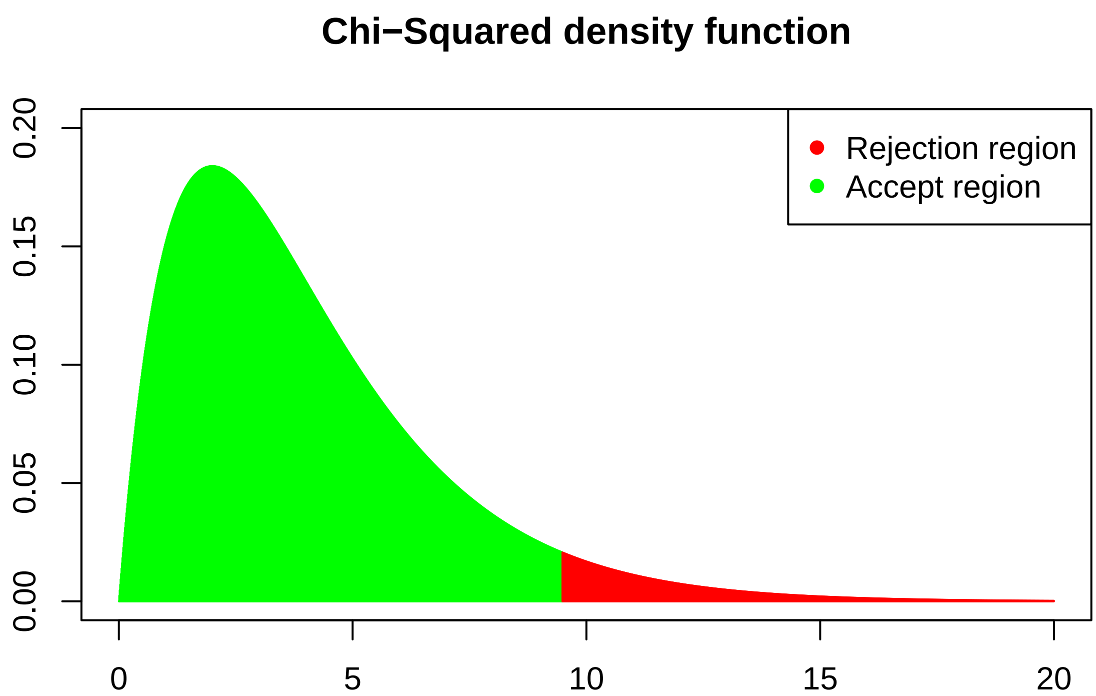
Where:

$T^2 \sim \chi^2 (m-1)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T > \chi^2 (m-1)\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W= $\{(9.488, \infty)\}$

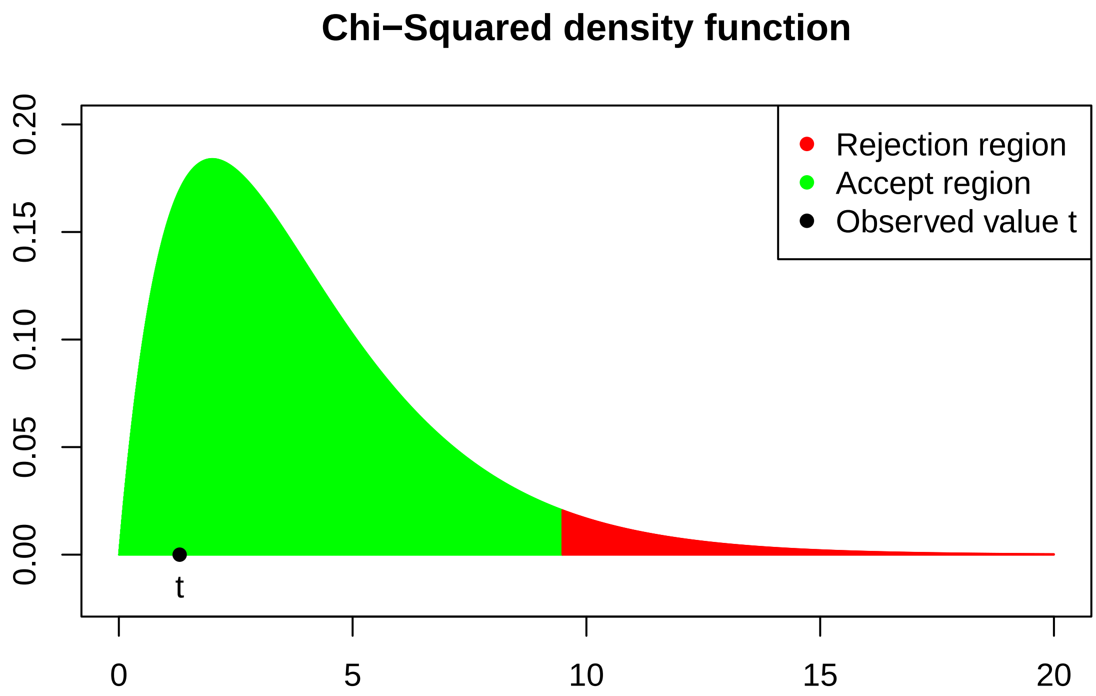


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##     classes Freq       P0k     nP0k       T2
## 1   [0,183)    5 0.1733691 5.201074 0.000000
## 2 [183,191)    6 0.1700102 5.100305 0.000000
## 3 [191,199)    4 0.2100852 6.302555 0.000000
## 4 [199,207)    6 0.1957743 5.873230 0.000000
## 5 [207,Inf)    9 0.2507612 7.522836 1.300476
```

Step 8: According to t, decide whether to accept or not H_0.



As t is not in the rejection region W $(t \notin W)$, we accept H_0, therefore $P = P_0$.

We consider that the results are reliable provided that $np_0^k \geq 5\ \forall k = 1, \ldots, m$, do so the results of this test are reliable provided.

Question 2, answer:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : P = P_0$
$H_1 : P \neq P_0$

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T^2 = \sum\limits_{k=1}^{m} \frac{\left(N_k(n) - np_0^k\right)^2}{np_0^k}$

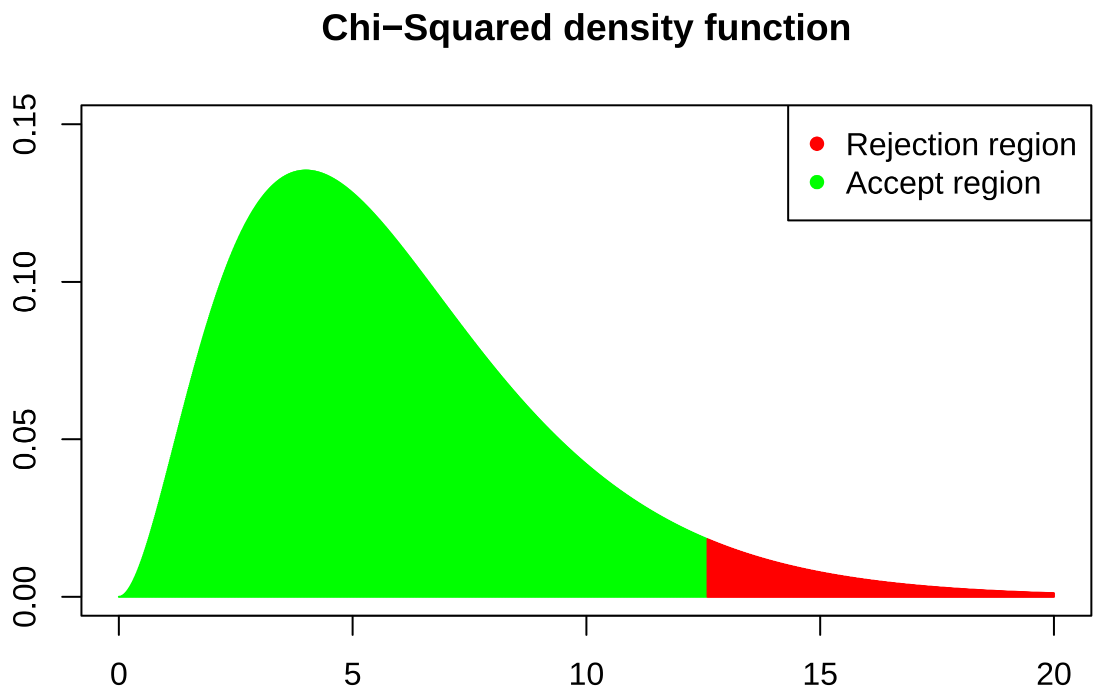
Where:

$T^2 \sim \chi^2\left(m-1\right)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T > \chi^2\left(m-1\right)\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W$= \{(12.592, \infty)\}$

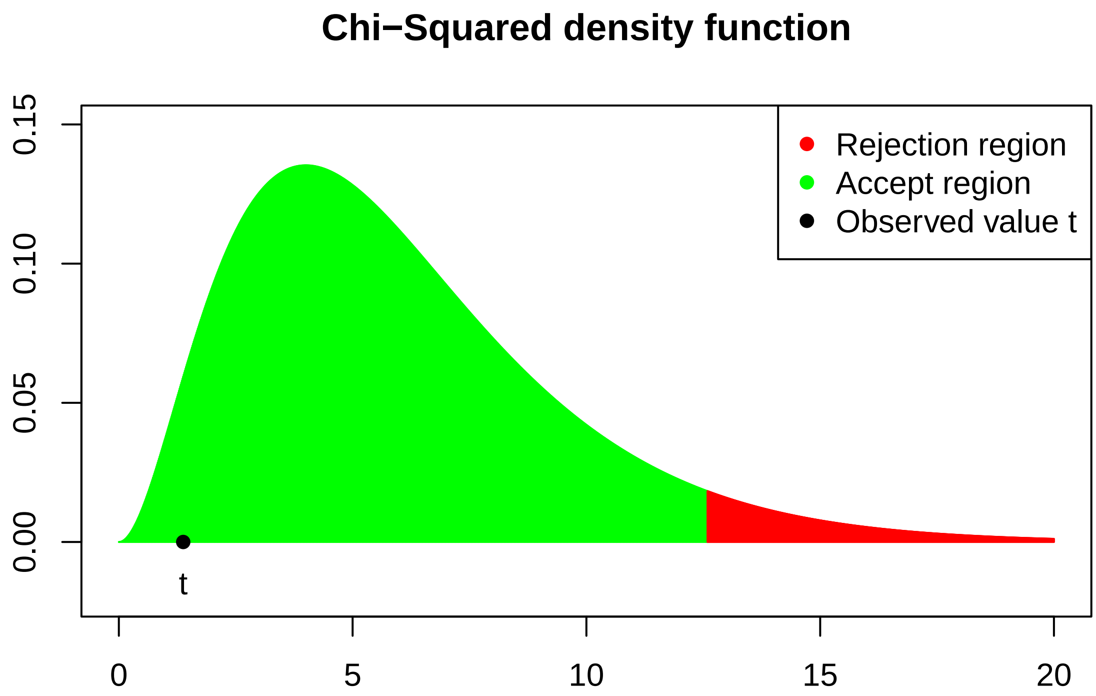


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##      classes Freq        P0k     nP0k       T2
## 1    [0,170)    2 0.08412164 2.523649 0.000000
## 2  [170,180)    5 0.16131037 4.839311 0.000000
## 3  [180,190)    8 0.25456799 7.637040 0.000000
## 4  [190,200)    9 0.25456799 7.637040 0.000000
## 5  [200,210)    3 0.16131037 4.839311 0.000000
## 6  [210,220)    2 0.06474541 1.942362 0.000000
## 7  [220,Inf)    1 0.01937623 0.581287 1.376883
```

Step 8: According to t, decide whether to accept or not H_0.



As t is not in the rejection region W $(t \notin W)$, we accept H_0, therefore $P = P_0$.

We consider that the results are reliable provided that $np_0^k \geq 5$ $\forall k = 1, \ldots, m$, do so the results of this test are not reliable provided.

Question 2.1, answer:

As the previous test are not reliable provided then is nesesary that the partition may be modified by concatenating more of five elements, next we go to work to obtain this objective.

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
$H_0 : P = P_0$
$H_1 : P \neq P_0$

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = 0.05$

Step 3: Select the test statistic, T;
$T^2 = \sum\limits_{k=1}^{m} \frac{\left(N_k(n) - np_0^k\right)^2}{np_0^k}$

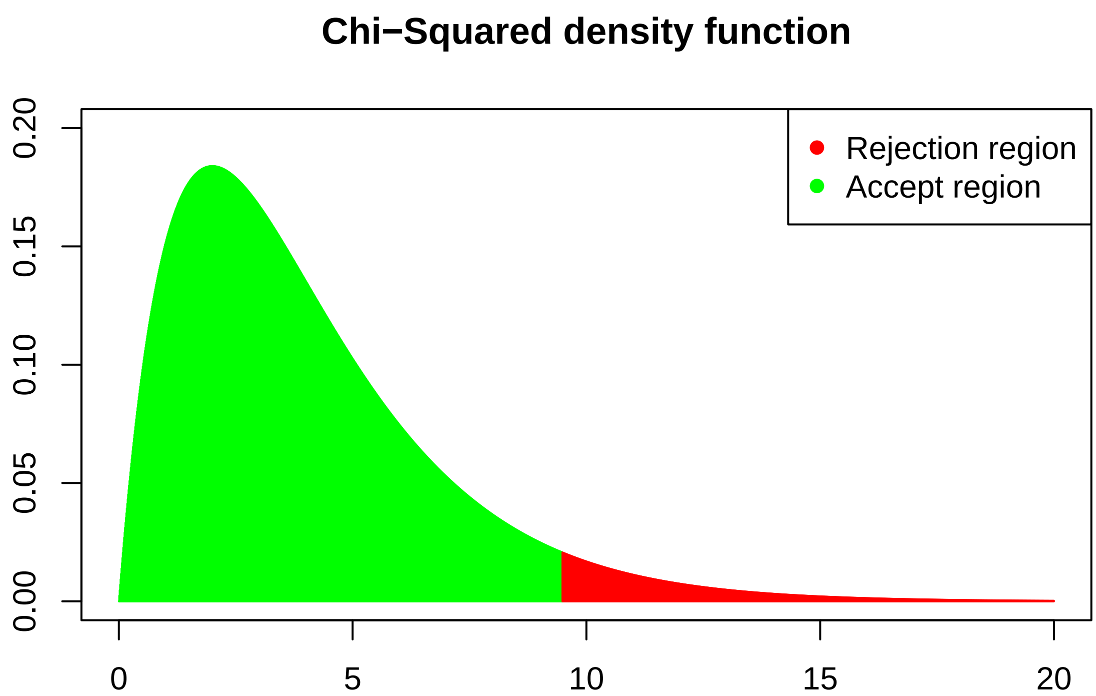
Where:

$T^2 \sim \chi^2 (m-1)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T > \chi^2 (m-1)\right\}$

Step 5: Explicitly compute the rejection region W according to α;

W= $\{(9.488, \infty)\}$

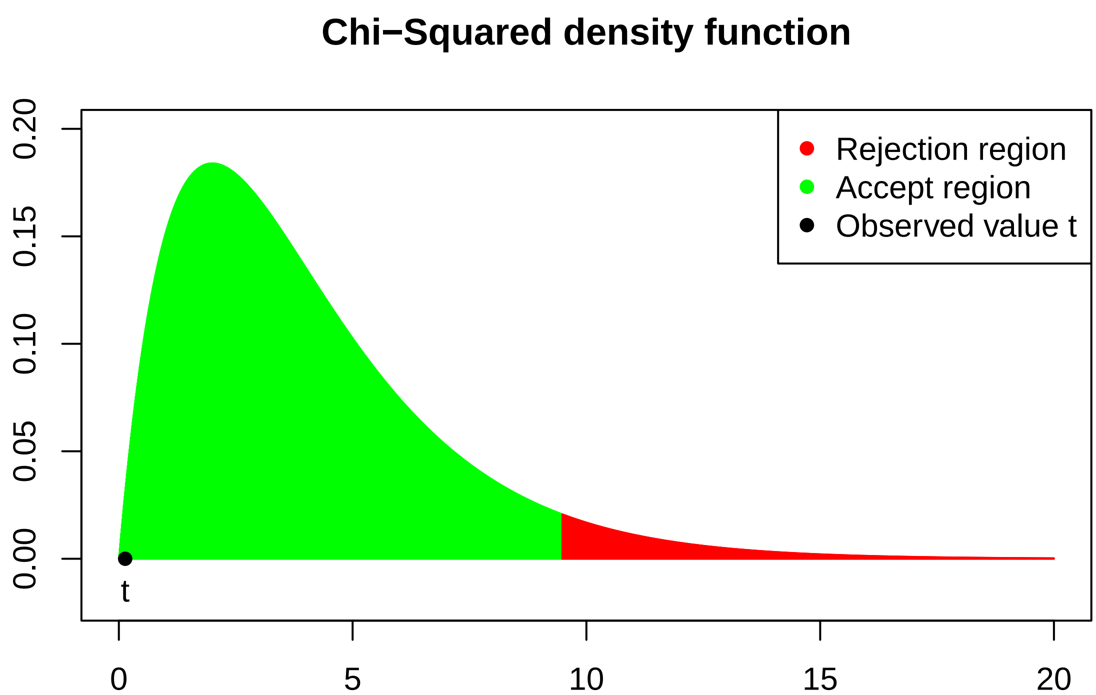


Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

```
##      classes Freq       P0k     nP0k        T2
## 1    [0,178)    6 0.2041972 6.125917 0.0000000
## 2  [178,186)    5 0.1872404 5.617211 0.0000000
## 3  [186,194)    7 0.2171248 6.513744 0.0000000
## 4  [194,202)    6 0.1872404 5.617211 0.0000000
## 5  [202,Inf)    6 0.2041972 6.125917 0.1353795
```

Step 8: According to t, decide whether to accept or not H_0.



As t is not in the rejection region W ($t \notin W$), we accept H_0, therefore $P = P_0$.

We consider that the results are reliable provided that $np_0^k \geq 5 \; \forall k = 1, \ldots, m$, do so the results of this test are reliable provided.

Question 3, answer:

In this training we saw three test to prove the independence between the speeds of the first serves of the two players , this test are:

1 - Testing for the correlation coeficient (Pearson test)

2 - χ^2 test for independence

3 - Spearman test

In the Pearson test it is necessary that the speeds of the first servece of the two players is distributed according to a Gaussian distribution, but in the answers of the questions 1 and 2, we prove that the results of the Gaussian-test are not reliable provided.

In the χ^2 test we consider that the results are reliable if $N_{ij} \geq 5, \; \forall i = 1, \ldots, m_1$ and $\forall j = 1, \ldots, m_2$ but:

```
##     Classes Freq_Federer Freq_Soderling
## 1   [0,170)            2              2
## 2 [170,180)            5              1
## 3 [180,190)            8              8
## 4 [190,200)            9              6
## 5 [200,210)            3              7
## 6 [210,220)            2              5
```

```
## 7 [220,Inf)              1                1
```

And sometimes $N_{ij} < 5$.

In the Spearman test all the hypotheses of the test are completed, hence this it is the appropriate test to test the independence between the speeds of the two player's ???rst serves.

Building the Spearman test:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;
H_0 : X and Y are independent $H_1: $ X and Y are not independent

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = \alpha_0$

Step 3: Select the test statistic, T;
$T = S_n = \sum\limits_{i=1}^{n} R_X\left(i\right) R_Y\left(i\right)$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \left\{T < \underset{-}{s}\right\} \cup \left\{T > \bar{s}\right\}$

Step 5: Explicitly compute the rejection region W according to α;

Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;

Step 8: According to t, decide whether to accept or not H_0.

Excercice 2

Building a uniformly most powerful test

Question 1 item a), answer:

$F_{\ln x_i} = P\left(\ln x_i \leq t\right)$

$F_{\ln x_i} = P\left(e^{\ln x_i} \leq e^t\right)$

$F_{\ln x_i} = P\left(x_i \leq e^t\right)$

$F_{\ln x_i} = F_{x_i}\left(e^t\right)$

$\Rightarrow f_{\ln x_i}(t) = \frac{\partial\left[F_{x_i}\left(e^t\right)\right]}{\partial t} = \frac{\partial\left[e^t\right]}{\partial t} * F^{'}_{x_i}\left(e^t\right) = e^t f_{x_i}\left(e^t\right)$

$f_{\ln x_i}(t) = e^t \frac{1}{e^t\sqrt{2\pi}} e^{-\frac{\left(\ln e^t - \theta\right)^2}{2}} I_{]0,\infty[}\left(e^t\right)$

but $I_{]0,\infty[}\left(e^t\right) = 1,\ \forall t \in R$

$\Rightarrow f_{\ln x_i}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-\theta)^2}{2}}$

$\because\ \ln X_i \sim N\left(\theta, 1\right)$

Question 1 item b), answer:

In other order, due that $\prod_{i=1}^{15} X_i \sim Log - N\left(\theta, \sum_{i=1}^{15} \sigma_i^2\right) = Log - N\left(\theta, 15\right)$

$F_{\sum \ln X_i}(t) = P\left(\sum \ln X_i \leq t\right)$

$F_{\sum \ln X_i}(t) = P\left(\ln \prod X_i \leq t\right)$

$F_{\sum \ln X_i}(t) = P\left(e^{\ln \prod X_i} \leq e^t\right)$

$F_{\sum \ln X_i}(t) = P\left(\prod X_i \leq e^t\right)$

$F_{\sum \ln X_i}(t) = F_{\prod X_i}\left(e^t\right)$

Them it is similar to: Question 1 item a) we arrives to:

$\sum_{i=1}^{15} lnX_i \sim N\left(\theta, 15\right)$

Question 2, answer:

Since

$L_\theta\left(\ln X_i\right) = \left(2\pi\right)^{-\frac{15}{2}} e^{\left(-\frac{1}{2}\sum_{i=1}^{15} lnX_i{}^2\right)} e^{\left(-\frac{15\theta^2}{2}\right)} e^{\left(\theta \sum_{i=1}^{15} lnX_i\right)}$

$L_\theta\left(\ln X_i\right) = \left(2\pi\right)^{-\frac{15}{2}} e^{\left(-\frac{1}{2}\sum_{i=1}^{15} lnX_i{}^2\right)} e^{\left(\theta \sum_{i=1}^{15} lnX_i - \frac{15\theta^2}{2}\right)}$

and:

$h\left(\ln X_i\right) = \left(2\pi\right)^{-\frac{15}{2}} e^{\left(-\frac{1}{2}\sum_{i=1}^{15} lnX_i{}^2\right)}$

$g\left(\theta\right) = \theta$ is an increasing function

$T\left(\ln X_i\right) = \sum_{i=1}^{15} lnX_i$

$B\left(\theta\right) = \frac{15\theta^2}{2}$

then:

$L_\theta\left(\ln X_i\right) = h\left(\ln X_i\right) e^{\left[g(\theta)T(\ln X_i) - B(\theta)\right]}$

has a monotone likelihood ratio in $T\left(\ln X_i\right) = \sum_{i=1}^{15} lnX_i$

Question 3, answer:

Since

$H_0 : \theta = 0$

$H_1 : \theta < 0$

and $\alpha = 0.025$

As the parametric model has a monotone likelihood ratio, strictly increasing in $T(\ln X_i)$, an exhaustive statistic, then exists an $UMP(\alpha)$ test, having the following form:

$$\psi_1 = \begin{cases} 1 \text{ if } T(\ln X_i) < k \\ 0 \text{ if } T(\ln X_i) > k \\ c \text{ if } T(\ln X_i) = k \end{cases}$$

Where $T(\ln X_i) = \sum_{i=1}^{15} lnX_i$

Since W$=\{T(\ln X_i) < k\}$ and $P_{H_0}(T \in W) = \alpha$

$\Rightarrow P_{H_0}(T(\ln X_i) < k) = 0.025$ but $T(\ln X_i) = \sum_{i=1}^{15} lnX_i \sim N(\theta, 15)$

$\Rightarrow P_{H_0}(\frac{T(\ln X_i)-\theta_0}{15} < \frac{k-\theta_0}{15}) = 0.025$ as $\frac{T(\ln X_i)-\theta_0}{15} \sim N(0,1)$

$\frac{k-\theta_0}{15} = -1.960$

$\frac{k}{15} = -1.960$

$k = -29.399$

Besides as $T(\ln X_i) \sim N(\theta, 15)$ and Gaussian distribution is continue then $P(T(\ln X_i) = k) = 0$

therefore

$$\psi_1 = \begin{cases} 1 \text{ if } T(\ln X_i) < -29.399 \\ 0 \text{ if } T(\ln X_i) > -29.399 \end{cases}$$

Question 4, answer:

Since

$H_0 : \theta = 0$

$H_1 : \theta > 0$

and $\alpha = 0.025$

As the parametric model has a monotone likelihood ratio, strictly increasing in $T(\ln X_i)$, an exhaustive statistic, then exists an $UMP(\alpha)$ test, having the following form:

$$\psi_2 = \begin{cases} 1 \text{ if } T(\ln X_i) > k \\ 0 \text{ if } T(\ln X_i) < k \\ c \text{ if } T(\ln X_i) = k \end{cases}$$

Where $T(\ln X_i) = \sum_{i=1}^{15} lnX_i$

As W$=\{T(\ln X_i) > k\}$ and $P_{H_0}(T \in W) = \alpha$

$\Rightarrow P_{H_0}(T(\ln X_i) > k) = 0.025$ but $T(\ln X_i) = \sum_{i=1}^{15} lnX_i \sim N(\theta, 15)$

$\Rightarrow P_{H_0}(\frac{T(\ln X_i)-\theta_0}{15} > \frac{k-\theta_0}{15}) = 0.025$

$1 - P_{H_0}(\frac{T(\ln X_i)-\theta_0}{15} < \frac{k-\theta_0}{15}) = 0.025$

$P_{H_0}(\frac{T(\ln X_i)-\theta_0}{15} < \frac{k-\theta_0}{15}) = 0.975$ like $\frac{T(\ln X_i)-\theta_0}{15} \sim N(0,1)$

$\frac{k-\theta_0}{15} = 1.960$

$\frac{k}{15} = 1.960$

$k = 29.399$

Besides, as $T(\ln X_i) \sim N(\theta, 15)$ and Gaussian distribution is continue then $P(T(\ln X_i) = k) = 0$

therefore

$$\psi_2 = \begin{cases} 1 \text{ if } T(\ln X_i) > 29.399 \\ 0 \text{ if } T(\ln X_i) < 29.399 \end{cases}$$

Question 4.1, answer:

$$\psi_1 + \psi_2 = \begin{cases} 1 \text{ if } T(\ln X_i) < -29.399 \text{ or } T(\ln X_i) > 29.399 \\ 0 \text{ if } -29.399 < T(\ln X_i) < 29.399 \\ 0 \text{ if } T(\ln X_i) = 29.399 \\ 0 \text{ if } T(\ln X_i) = -29.399 \end{cases}$$

and $\psi_1 + \psi_2$ is uniformly most powerful among all unbiased tests of level α ,$UMPU(\alpha)$, for the:

Bilateral test

$H_0 : \theta = 0$ against $H_1 : \theta \neq 0$

Question 5, answer:

The Bilateral test $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$ is equivalent to Bilateral test

$H_0 : \theta \in [0, 0]$ against $H_1 : \theta < 0$ or $\theta > 0$

But in $(E, E, P_\theta, \theta \in \Theta)$ a parametric dominated model with respect to a probability measure μ, and $\Theta \in \mathbb{R}$. Also the model has a monotone likelihood ratio, strictly increasing in $T(\ln X_i)$. Then does not exist a $UMP(\alpha)$ test.

Kolmogorov-Smirnov non-parametric testing

Question 1, answer:

As $X \sim Log - N(0, 1)$

$F_x(t) = P(x \leq t)$

$F_x(t) = P(\ln x \leq \ln t) \;\; \forall t \in]0, \infty[$

$F_x(t) = P(\ln x \leq \ln t)\, \mathrm{I}_{]0,\infty[}(t)$

$F_x(t) = F_{\ln x}(\ln t)\, \mathrm{I}_{]0,\infty[}(t)$

$F_x(t) = \Phi(\ln t)\, \mathrm{I}_{]0,\infty[}(t)\,, \text{ because } \ln\mathrm{X} \sim N(0, 1)$

Question 2, answer:

Steps for the construction of test:

Step 1: Select the hypothesis H_0 and H_1;

$H_0 : P$ is a log-normal distribution with parameters 0 and 1

$H_1 : P$ is not a lognormal distribution with parameters 0 and 1

or

$H_0 : F = F_0$

$H_1 : F \neq F_0$

When F_0 is to cumulative distribution function to the log-normal distribution with parameters 0 and 1.

Step 2: Fix the level of the test or the 1st type error equal to α;
$\alpha = \alpha_0$

Step 3: Select the test statistic, T;
$T = D_n = \sqrt{n} sup_{x \in R} \left| F_n(x) - F_0(x) \right| \sim U_n[0,1]$

but in Question 1 we prove that $F_0(x) = \Phi\left(\ln x\right) \mathrm{I}_{]0,\infty[}\left(x\right)$

then

$T = D_n = \sqrt{n} sup_{x \in R} \left| F_n(x) - \Phi\left(\ln x\right) \mathrm{I}_{]0,\infty[}\left(x\right) \right| \sim U_n[0,1]$

Step 4: Determine the form of the rejection region W, depending on the behavior of T under H_1;
$W = \{T > d_{n,1-\alpha_0}\}$

Question 3, answer:

$\alpha = \alpha_0 = 0.05$

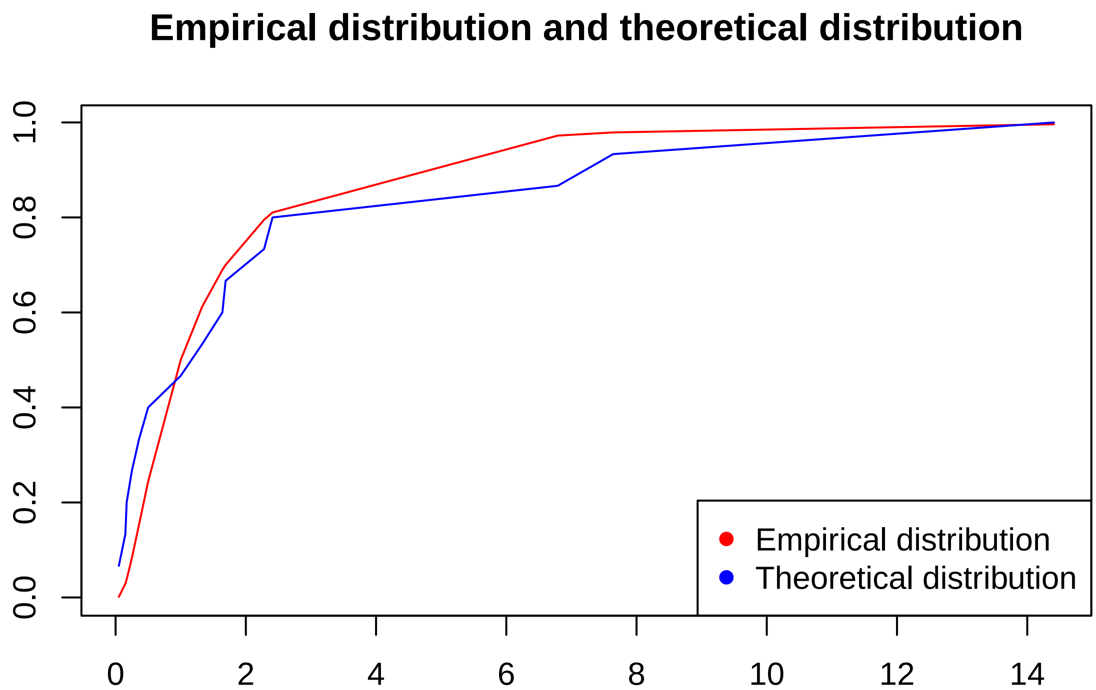
Step 5: Explicitly compute the rejection region W according to α;

W= $\left\{ \left(0.338\sqrt{15}, \infty\right) \right\}$

W= $\{(1.309, \infty)\}$

Step 6: (Optional) Compute the 2nd type error and/or the power of the test;

Step 7: Compute the observed value, t, for the test statistic T;




```
##    Sample         Fn         Iln      Fn_Iln  sqrt15Dn
## 1    0.05 0.06666667 0.001368933 0.065297733 0.0000000
## 2    0.15 0.13333333 0.028906052 0.104427281 0.0000000
## 3    0.17 0.20000000 0.038200861 0.161799139 0.0000000
## 4    0.25 0.26666667 0.082828519 0.183838148 0.0000000
## 5    0.36 0.33333333 0.153472997 0.179860337 0.0000000
## 6    0.50 0.40000000 0.244108596 0.155891404 0.0000000
## 7    1.00 0.46666667 0.500000000 0.033333333 0.0000000
## 8    1.33 0.53333333 0.612246474 0.078913140 0.0000000
## 9    1.64 0.60000000 0.689592723 0.089592723 0.0000000
## 10   1.69 0.66666667 0.700114039 0.033447372 0.0000000
## 11   2.28 0.73333333 0.795080061 0.061746728 0.0000000
## 12   2.41 0.80000000 0.810469228 0.010469228 0.0000000
## 13   6.79 0.86666667 0.972282490 0.105615823 0.0000000
## 14   7.64 0.93333333 0.978993815 0.045660482 0.0000000
## 15  14.41 1.00000000 0.996183906 0.003816094 0.7120021
```

Step 8: According to t, decide whether to accept or not H_0.

As $t = \sqrt{15}D_n = 0.712 < 1.309$ is not in the rejection region W ($t \notin W$) then accept H_0, then P is a log-normal distribution with parameters 0 and 1.

Question 4, answer:

The conclusion of this test is that $X = (X_1, \ldots, X_15) \sim log - N(0, 1)$, and it is cumulative distribution function is: $\Phi(\ln x) \mathrm{I}_{]0,\infty[}(x)$, where Φ is the cumulative distribution function of the standard Gaussian,

$N(0, 1)$.