Hypothesis testing - Evaluation

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For each test, one should pay attention to clearly specify the statistical model and the hypothesis to be tested. Some excerpts from statistical tables are provided at the end, but one may also use **R** *to compute the useful quantiles etc ...*

<u>Exercice 1</u>
During the 2009 Roland Garros final, the recorded speed of Robin Soderling's first serves was on average higher than Roger Federer's. In the following, a statistical test will be build to confirm whether this difference was significant from a statistical point of view.

A comparison test in the Gaussian framework
Let us suppose here that the speed of Robin Soderling's first serve is a random variable with a Gaussian distribution $\mathcal{N}(m_1, \sigma_1^2)$, and that the speed of Roger Federer's first serve is also a random variable with a Gaussian distribution $\mathcal{N}(m_2, \sigma_2^2)$. We shall furthermore consider that two players' serves are independent.

The first 30 values of the speed of the first serve were recorded for both players. They are summarized in the following tables (in km/h).

For R. Soderling :

183	209	204	219	221	189	183	206	216	205	188	181	185	209	178
194	168	194	203	214	199	199	196	198	167	181	212	207	185	217

For R. Federer :

193	202	184	198	178	204	195	203	215	199	222	172	170	188	172
194	190	185	199	165	192	183	187	180	165	176	198	187	187	217

1. Using a statistical test of level 5%, show that one may suppose that $\sigma_1^2 = \sigma_2^2$.
2. Using the previous question, build a statistical test of level 5% of $H_0 : m_1 \leq m_2$ against $H_1 : m_1 > m_2$.
3. Explain the meaning of choosing these hypothesis and give the conclusion of the test.

Justifying the Gaussian framework
One now wishes to validate the Gaussian model in the previous section, using χ^2 non-parametric testing.

1. Using a χ^2 test with a 5% asymptotic level on the classes $]0, 170[$, $[170, 180[$, $[180, 190[$, $[190, 200[$, $[200, 210[$, $[210, 220[$, $[220, +\infty[$, decide whether the speed of R. Soderling's first serves may be indeed considered as a Gaussian distribution $\mathcal{N}(m_1, \sigma_1^2)$ (with unknown parameters m_1 and σ_1^2, but which may be estimated – prove it ! – by $\hat{m}_1 = 197$ and $\hat{\sigma}_1^2 = 221.38$).
2. Same question for the speed of R. Federer's first serves.
3. If one wishes next to test the independence between the speeds of the two players' first serves, which test would be appropriate ? Describe in a few lines the approach to follow for building this test.

Exercice 2

We consider a sample of size 15 of $X = (X_1, ..., X_{15})$ drawn from an unknown probability distribution $\mathbb{P}$, of which we observed one outcome $x = (x_1, ..., x_{15})$ summarized in the following table :

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
1.69	1.33	0.05	0.17	2.41	1	14.41	0.36	6.79	0.50	0.15	0.25	1.64	2.28	7.64

Using these values, we wish to test whether $\mathbb{P}$ is a log-normal distribution with parameters 0 and 1.

We shall recall here that the density of a log-normal distribution with parameters θ and 1, for $\theta \in \mathbb{R}$ is defined by

$$f_\theta(x) = \frac{1}{x\sqrt{2\pi}} \exp^{-\frac{(\ln x - \theta)^2}{2}} \mathbb{1}_{]0,\infty[}(x)$$

Building a uniformly most powerful test

Throughout this section, we shall suppose that $(X_1, ..., X_{15})$ is a sample from a log-normal distribution with parameters θ and 1, where $\theta \in \mathbb{R}$ is unknown.

1. Show that, for each $i = 1, ..., 15$, $\ln X_i$ is distributed according to a Gaussian law, $\mathcal{N}(\theta, 1)$. Which is the probability distribution of $\sum_{i=1}^{15} \ln X_i$?
2. Prove that the statistical model has a monotone likelihood ratio in $\phi(x) = \sum_{i=1}^{15} \ln x_i$.
3. Using the previous, build a uniformly most powerful test ψ_1 of level 2.5% of $H_0 : \theta = 0$ against $H_1 : \theta < 0$.
4. Build a uniformly most powerful test ψ_2 of level 2.5% of $H_0 : \theta = 0$ against $H_2 : \theta > 0$. Is $\psi_1 + \psi_2$ the uniformly most powerful test of level 5% of $H_0 : \theta = 0$ against $H_3 : \theta \neq 0$? Justify your answer.
5. Is is possible to build a uniformly most powerful test of level 5% of $H_0 : \theta = 0$ against $H_3 : \theta \neq 0$? Justify your answer.

Kolmogorov-Smirnov non-parametric testing

In the following, we wish to build a Kolmogorov-Smirnov test for H_0 : "$\mathbb{P}$ is a log-normal distribution with parameters 0 and 1" against H_1 : "$\mathbb{P}$ is not a lognormal distribution with parameters 0 and 1".

1. Show that the cumulative distribution function F of the log-normal distribution with parameters 0 and 1 writes as

   $$F(t) = \Phi(\ln t)\mathbb{1}_{]0,\infty[}(t) \ ,$$

   where Φ is the cumulative distribution function of the standard Gaussian, $\mathcal{N}(0, 1)$.
2. Give the Kolmogorov-Smirnov test statistic for the present problem, and write this test statistic as a function of Φ and the order statistic associated to $(x_1, ..., x_{15})$.
3. Compute, while justifying it, a Kolmogorov-Smirnov test of level 5%.
4. Which is the conclusion of this test ?

Excerpted values from statistical tables

Gaussian distribution : for several values of $\alpha \in [0,1]$, we give q_α such that $\mathbb{P}(N \le q_\alpha) = \alpha$, where $N \sim \mathcal{N}(0,1)$.

α	0.9	0.95	0.975
q_α	1.282	1.645	1.960

Gaussian distribution : for several values of q, we give $\mathbb{P}(N \le q)$, where $N \sim \mathcal{N}(0,1)$.

q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2
$\mathbb{P}(N \le q)$	0.54	0.58	0.62	0.66	0.69	0.73	0.76	0.79	0.82	0.84	0.86	0.88
q	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4
$\mathbb{P}(N \le q)$	0.90	0.92	0.93	0.95	0.96	0.96	0.97	0.98	0.98	0.99	0.99	0.99

Student distribution : for several values of $\alpha \in [0,1]$ and $n \in \mathbb{N}$, we give $q_{n,\alpha}$ such that $\mathbb{P}(T \le q_{n,\alpha}) = \alpha$, where $T \sim \mathcal{T}(n)$. We recall here also that the Student distribution is symmetric.

α	0.9	0.95	0.975
$q_{29,\alpha}$	1.311	1.699	2.045
$q_{30,\alpha}$	1.310	1.697	2.042
$q_{58,\alpha}$	1.296	1.672	2.002
$q_{59,\alpha}$	1.296	1.671	2.001
$q_{60,\alpha}$	1.296	1.671	2

χ^2-distribution : for several values of $\alpha \in [0,1]$ and $n \in \mathbb{N}$, we give $q_{n,\alpha}$ such that $\mathbb{P}(K \le q_{n,\alpha}) = \alpha$, where $K \sim \chi^2(n)$.

α	0.025	0.05	0.95	0.975
$q_{1,\alpha}$	0.001	0.004	3.841	5.024
$q_{2,\alpha}$	0.051	0.103	5.991	7.378
$q_{3,\alpha}$	0.216	0.352	7.815	9.348
$q_{4,\alpha}$	0.484	0.711	9.488	11.143
$q_{5,\alpha}$	0.831	1.145	11.070	12.833
$q_{6,\alpha}$	1.237	1.635	12.592	14.449
$q_{29,\alpha}$	16.047	17.708	42.557	45.722
$q_{30,\alpha}$	16.791	18.493	43.773	46.979

Fisher distribution : for several values of $\alpha \in [0,1]$ and $n_1, n_2 \in \mathbb{N}$, we give $q_{n_1,n_2,\alpha}$ such that $\mathbb{P}(F \le q_{n_1,n_2,\alpha}) = \alpha$, where $F \sim \mathcal{F}(n_1,n_2)$.

α	0.025	0.05	0.95	0.975
$q_{29,29,\alpha}$	0.476	0.537	1.861	2.101
$q_{30,30,\alpha}$	0.482	0.543	1.841	2.074

Kolmogorov-Smirnov table : for several values of $n \in \mathbb{N}$, we give $q_{0.95}$ such that $\mathbb{P}(\sup_{t\in[0,1]} |F_{U,n}(t) - t| \le q_{0.95}) = 0.95$ (without the term $\sqrt{n}$), where $F_{U,n}$ is the empirical cumulative distribution function associated to a n-sample of a uniform distribution on $[0,1]$.

n	10	15	30	$n > 100$
$q_{0.95}$	0.409	0.338	0.242	$1.358/\sqrt{n}$