Variance-Aware Sparse Linear Bandits

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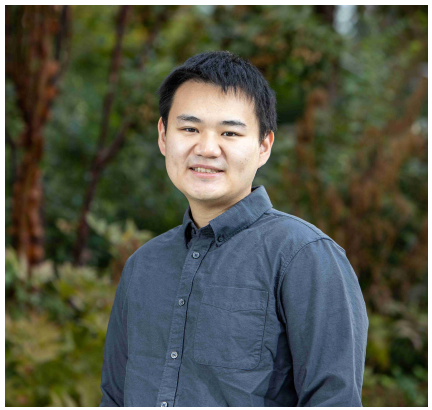
Yan Dai [1] Ruosong Wang [2] Simon S. Du [2]



[1]IIIS, Tsinghua University [2]Paul G. Allen School, University of Washington

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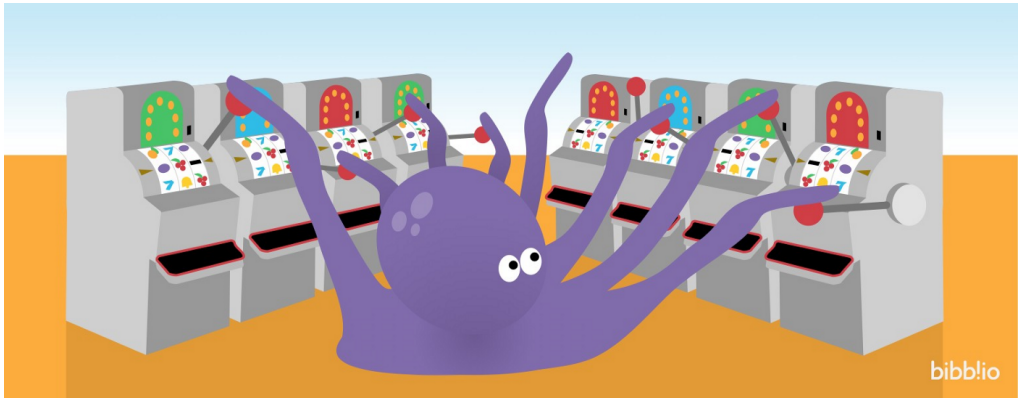

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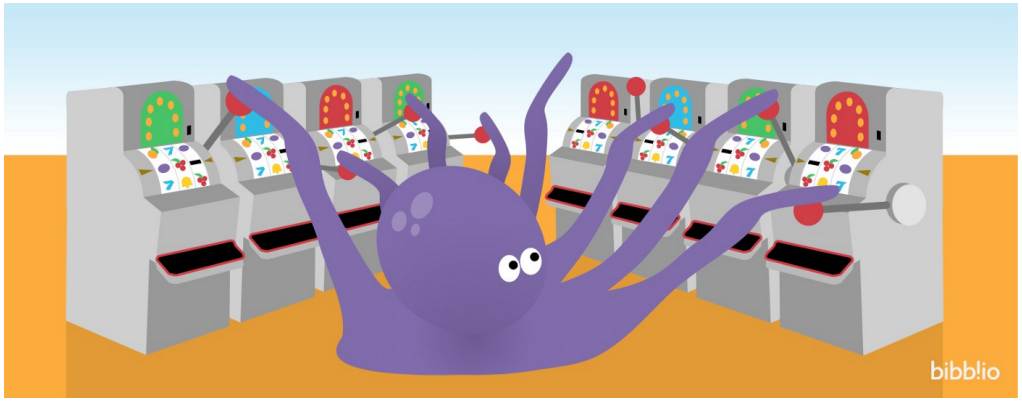

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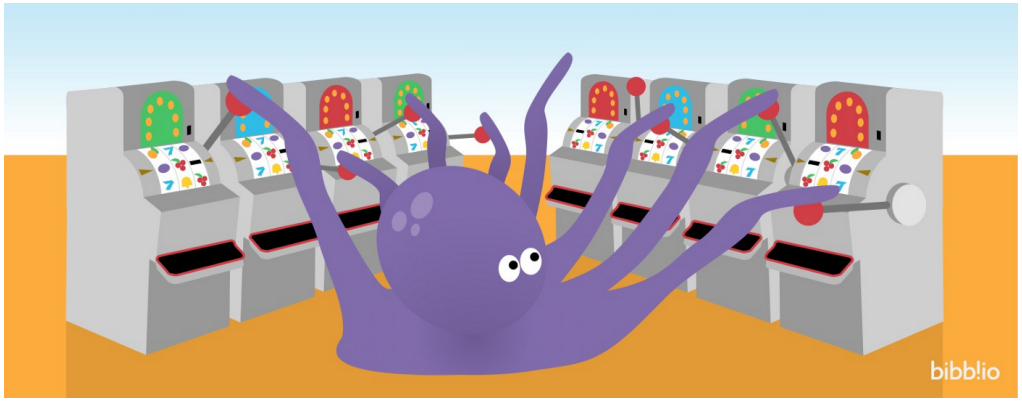

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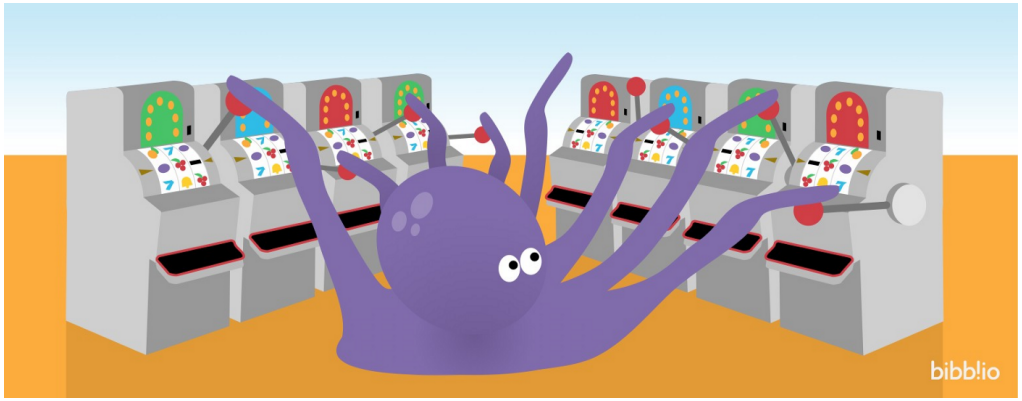

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- For this round, she gains **reward** $r(a_t) = \langle a_t, \theta^* \rangle$ where $\theta^* \in \mathbb{R}^d$ is a *fixed but unknown* parameter.
- She cannot direcly access $r(a_t)$, but only observes noisy feedback $r(a_t) + \eta_t$ where η_t is a zero-mean *random* **noise**. We assume $\eta_t \sim \mathcal{N}(0, \sigma_t^2)$ where σ_t's are *fixed but unknown*.

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Agent's Goal?

Maximize the (expected) total reward

$$\mathbb{E}\left[\sum_{t=1}^{T} r(a_t)\right] = \mathbb{E}\left[\sum_{t=1}^{T} \langle a_t, \theta^* \rangle\right],$$

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or equivalently, minimize the **regret**

$$\begin{aligned}\mathcal{R}_T &\triangleq \max_{a \in \mathbb{S}^{d-1}} \mathbb{E}\left[\sum_{t=1}^{T} \langle a - a_t, \theta^* \rangle\right]. \\ &= \mathbb{E}\left[\sum_{t=1}^{T} \langle \theta^* - a_t, \theta^* \rangle\right].\end{aligned}$$

Sparse Linear Bandit

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- **Known Results on "Worst-Case" ($\sigma_t \equiv 1$) SLB:**
  - Upper Bound: $\widetilde{\mathcal{O}}(\sqrt{sdT})$ [Abbasi-Yadkori et al., 2012].
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  - **Conclusion:** $\widetilde{\mathcal{O}}(\sqrt{sdT})$ is optimal for "worst-case" SLB!

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- **In Between?** Objective of this paper!
  Design an algorithm whose regret is **variance-aware**:

$$\mathcal{R}_T = \widetilde{\mathcal{O}}\left(\mathrm{poly}(s)\sqrt{d\sum_{t=1}^{T}\sigma_t^2} + \mathrm{poly}(s)\right),$$

where $\sigma_t^2 = \mathrm{Var}(\eta_t) \in [0,1]$ is the noise variance (σ_t's are all *unknown*) and $s = \|\theta^*\|_0$ is the sparsity (s is also *unknown*).

Related Work

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   - $\widetilde{\mathcal{O}}(d^{1.5}\sqrt{\sum \sigma_t^2} + d^2)$ [Kim et al., 2022].
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This paper: convert any LB algorithm $\mathcal{A}$ to SLB algorithm $\mathcal{B}$, s.t.:

$$\text{if } \mathcal{A} \text{ ensures } \mathcal{R}_T^{\mathsf{LB}} = \widetilde{\mathcal{O}}\left(f(d)\sqrt{\sum \sigma_t^2} + g(d)\right) \text{ for some } f, g,$$

$$\text{then } \mathcal{B} \text{ ensures } \mathcal{R}_T^{\mathsf{SLB}} = \widetilde{\mathcal{O}}\left((sf(s) + s\sqrt{d})\sqrt{\sum \sigma_t^2} + sg(s)\right).$$

Classical "Explore-then-Commit" Idea for Worst-Case SLBs

1. **Explore:** Find coordinates with "large enough" magnitudes.
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Regret Analysis: The regret $\mathcal{R}_T = \widetilde{\mathcal{O}}(\sqrt{sdT})$, as:

- **Exploration** causes no more than $N = \widetilde{\mathcal{O}}(\sqrt{sdT})$ regret.
- **Linear bandit** on s coordinates has regret $\widetilde{\mathcal{O}}(s\sqrt{T})$.
- Each **un-explored coordinate** i (which is "small") incurs regret $\leq T\Delta^2 = \sqrt{dT/s}$. There are no more than s such i's.

Generalize to Variance-Aware SLB?

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- **Answer:** Decide the "threshold" Δ adaptively!

Our Idea: "Adaptive" Exploration Threshold

Algorithm "Explore-then-Commit" with Adaptive Threshold

1: **for** $\Delta = \frac{1}{2}, \ldots$ **do**
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- If ignore identified coordinates, they contribute large regret?
- **Solution**: For those with magnitude $> 2\Delta$ (i.e., identified before), always put their estimation $\hat{\theta}_i$ on those coordinates. Use remaining mass $1 - \sum \hat{\theta}_i^2$ to explore remaining ones.

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- The confidence radius $\frac{1}{n}\sqrt{d\sum_{k=1}^{n}\sigma_k^2}$ contains unknown σ_k's!
- Can we use some "empirical" observations to replace σ_k^2?

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Lemma: For common-mean, independent & symmetric $\{X_i\}_{i=1}^n$,

$$\left|\bar{X} - \mu\right| \le \frac{1}{n}\sqrt{2\sum_{i=1}^{n}(X_i - \bar{X})^2 \ln\frac{4}{\delta}} \quad \text{w.p. } 1-\delta,$$

where $n < \infty$ is stopping time, $\mu = \mathbb{E}[X_i]$, and $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

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- Recall LB algorithm can give $\mathcal{O}(\sqrt{T})$ regret: the longer it executes, the closer the action a_t and the parameter θ^* is.

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- Recall LB algorithm can give $\mathcal{O}(\sqrt{T})$ regret: the longer it executes, the closer the action a_t and the parameter θ^* is.
- Can we use this property for an accurate enough estimation?

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"Regret-to-Sample-Complexity": if its per-round regret $< \Delta^2$, i.e.,

$$\mathcal{R}_n^{\mathsf{LB}} = \sum_{k=1}^{n} \langle \theta^* - a_k, \theta^* \rangle \le n\Delta^2 \Longrightarrow \hat{\theta} \triangleq \frac{1}{n} \sum_{k=1}^{n} a_k \text{ satisfies } \langle \theta^* - \hat{\theta}, \theta^* \rangle \le \Delta^2.$$

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So waiting until $\mathcal{R}_n^{\mathsf{LB}} \le n\Delta^2$ can give "good" estimation $\hat{\theta}$!

Final Algorithm

Algorithm Final Algorithm Using VA LB Algorithm $\mathcal{A}$

1: **for** $\Delta = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ (i.e., halve until T rounds) **do**

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3: Terminate until the "explore" rounds n_Δ^b satisfies

$$2\sqrt{2\sum_{k=1}^{n_\Delta^b}(r_{k,i}-\bar{r}_i)^2 \ln\frac{4}{\delta}} < n_\Delta^b \cdot \frac{\Delta}{4}, \quad \forall i \text{ unidentified},$$

where $r_{k,i}$ is the k-th estimate of θ_i^* and $\bar{r}_i$ is the average of all $r_{k,i}$'s. Then mark all i with $|\bar{r}_i| > \Delta$ as "identified".

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4: Deploy $\mathcal{A}$ on all identified coordinates until the "commit" rounds n_Δ^a satisfies $\mathcal{R}_{n_\Delta^a}^{\mathsf{LB}} < n_\Delta^a \cdot \Delta^2$. Calculate $\hat{\theta}_i$ for all identified coordinates from "Regret-to-Sample-Complexity" conversion.

Analysis Sketch

Recap: For each Δ, n_Δ^b and n_Δ^a are defined as (ignore constants)

$$n_\Delta^b \approx \Delta^{-1}\sqrt{\sum_{k=1}^{n_\Delta^b}(r_{k,i}-\bar{r}_i)^2}, \quad n_\Delta^a \approx \Delta^{-2}\mathcal{R}_{n_\Delta^a}^{\mathsf{LB}}.$$

Analysis Sketch

Recap: For each Δ, n^b_Δ and n^a_Δ are defined as (ignore constants)

$$n^b_\Delta \approx \Delta^{-1}\sqrt{\sum_{k=1}^{n^b_\Delta}(r_{k,i}-\bar{r}_i)^2}, \quad n^a_\Delta \approx \Delta^{-2}\mathcal{R}^{\mathsf{LB}}_{n^a_\Delta}.$$

1. **"Explore" Regret:**
   1. Identified ones contribute regret $n^b_\Delta\langle\theta^*-\hat{\theta},\theta^*\rangle \le n^b_\Delta\cdot\Delta^2$.

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3. **Conclusion:** Total Regret is $\mathcal{O}(\mathbb{E}[\sum_\Delta s\Delta^2(n^b_\Delta+n^a_\Delta)])$.

Analysis Sketch (Cont'd)

Omitting expectations, $\mathcal{R}_T = \mathcal{O}(\sum_\Delta s\Delta^2(n_\Delta^b + n_\Delta^a))$, i.e.,

$$\mathcal{R}_T = \widetilde{\mathcal{O}}(s)\sum_\Delta \Delta^2 \left(\frac{1}{\Delta}\sqrt{\sum_{k=1}^{n_\Delta^b}(r_{k,i} - \overline{r}_i)^2} + \Delta^{-2}\mathcal{R}^{\mathsf{LB}}_{n_\Delta^a} \right).$$

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Plugging in $\mathcal{R}^{\mathsf{LB}}_n = \widetilde{\mathcal{O}}\left(s^{1.5}\sqrt{\sum_{k=1}^{n_\Delta^a}\sigma_k^2} + s^2\right)$ [Kim et al., 2022]

and $\sum_{k=1}^{n_\Delta^b}(r_{k,i}-\bar{r}_i)^2 \approx \sum_{k=1}^{n_\Delta^b}\mathbb{E}[(r_{k,i}-\bar{r}_i)^2] = \sum_{k=1}^{n_\Delta^b}(1+\frac{d}{\Delta^2}\sigma_k^2)$,

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$$\mathcal{R}_T = \widetilde{\mathcal{O}}(s)\sum_\Delta\left(\sqrt{\sum_{k=1}^{n_\Delta^b}(\Delta^2 + d\sigma_k^2)} + s^{1.5}\sqrt{\sum_{k=1}^{n_\Delta^a}\sigma_k^2} + s^2\right)$$

$$= \widetilde{\mathcal{O}}\left((s^{2.5}+s\sqrt{d})\sqrt{\sum_{t=1}^{T}\sigma_t^2} + s^3\right). \qquad \square$$

Thank you for listening!

Questions are more than welcomed.

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