IOE511 Homework10

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8.5

1 problem 1

1.1 (a)

Let's consider $(x-d)^T(x-d)$

$$\begin{aligned}(x-d)^T(x-d) &= (x-c-\frac{1}{n}(1-e^Tc)e)^T(x-c-\frac{1}{n}(1-e^Tc)e) \\ &= (x-c)^T(x-c) - \frac{2}{n}e^T(1-e^Tc)^T(x-c) + \frac{1}{n^2}e^T(1-e^Tc)^T(1-e^Tc)e \\ &= (x-c)^T(x-c) - \frac{2}{n}x^T(1-e^Tc)e + \frac{2}{n}c^T(1-e^Tc)e + \frac{1}{n^2}e^T(1-e^Tc)^T(1-e^Tc)e\end{aligned}$$

Since $e^Tx = 1$
So

$$(x-d)^T(x-d) = (x-c)^T(x-c) - \frac{2}{n}(1-e^Tc) + \frac{2}{n}c^T(1-e^Tc)e + \frac{1}{n^2}e^T(1-e^Tc)^T(1-e^Tc)e$$

So maximizing $(x-d)^T(x-d)$ is equivalant to maximizing $(x-c)^T(x-c)$.
Sp (P_d) and (P_c) have the same optimal solutions.

1.2 (b)

We know that

$$\begin{aligned}e^Td &= e^Tc + \frac{1}{n}e^T(1-e^Tc)e \\ &= e^Tc + \frac{1}{n}e^Te - \frac{1}{n}e^T(e^Tc)e \\ &= e^Tc + 1 - e^Tc \\ &= 1\end{aligned}$$

For $x^* = d \geq 0$
$e^Tx^* = 1$ and $x^* \geq 0$ are satisfied.

$$f(x) = \frac{1}{2}\|x-d\|_2^2 \geq 0$$

when $x^* = d$, $f(x^*) = 0$
So $x^* = d$ is an optiaml solution of (P_d)

1.3 (c)

$f(c) = \frac{1}{2}\|x-d\|_2^2$ is a convex funtion.
$h(x) = e^T x - 1$ is linear.
$g_i(x) = -x_i \quad i = 1, 2, ..., n$ are linear.
So (P_d) is a convex problem.
$\bar{x} = \frac{1}{n}[1, 1, ..., 1]$ is a Slater point.
So Slater condition is satisfied.
So KKT conditions are necessary for optimality.
x^* satisfies KKT conditions.
List all the KKT condtions:

$$x^* - d - \sum_{i=1}^{n} u_i e_i + v_1 e = 0 \tag{1}$$

$$-x_i^* \leq 0 \tag{2}$$

$$e^T x^* - 1 = \sum_{i=1}^{n} x_i^* - 1 = 0 \tag{3}$$

$$u_i \geq 0 \tag{4}$$

$$-u_i x_i^* = 0 \tag{5}$$

where $e_i = [0, ..., 0, 1, 0, ..., 0]^T$,i.e. ith element is 1, otherwise are 0.
From (1),

$$e^T(x^* - d - \sum_{i=1}^{n} u_i e_i + v_1 e) = 0$$

$$e^T x^* - e^T d - \sum_{i=1}^{n} u_i e^T e_i + v_1 e^T e = 0$$

Since $e^T x^* = 1, e^T d = 1$, we have

$$n v_1 = \sum_{i=1}^{n} u_i \tag{6}$$

For each element of (1)

$$x_i^* - d_i - u_i + v_1 = 0 \tag{7}$$

Let set

$$I = \{i : d_i < 0\}$$

For $j \in I$
Assume that

$$d_j < 0, x_j^* > 0 \tag{8}$$

Then from (5)

$$u_j = 0 \tag{9}$$

Plug in (8)(9) to (7)

$$v_1 < 0 \tag{10}$$

Plug in (10) to (6)

$$\sum_{i=1}^{n} u_i < 0$$

which contradicts to (4)
So the assumption is wrong. For j such that $d_j < 0$, we have

$$x_j^* \leq 0$$

From (2),

$$x_j^* = 0$$

That is, for j such that $d_j < 0$, we have

$$x_j^* = 0$$

in any optimal solution of (P_d)

1.4 (d)

Solving (P_c) is equivalant to solve (P_d)
We first compute

$$d = c + \frac{1}{n}(1 - e^T c)e$$

From (c) we know for any optimal solution,for j such that $d_j < 0$, we have

$$x_j^* = 0$$

$$\begin{aligned} f(x) &= \frac{1}{2}\|x - d\|_2^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}(x_i - d_i)^2 \\ &= \frac{1}{2}\sum_{i\in I} d_i^2 + \frac{1}{2}\sum_{i\notin I}(x_i - d_i)^2 \end{aligned}$$

Since for $i \notin I$,$d_i \geq 0$
So

$$f(x) \geq \frac{1}{2}\sum_{i\in I} d_i^2$$

where x can be built like this:
for j such that $d_j < 0$,

$$x_j^* = 0$$

for j such that $d_j \geq 0$,

$$x_j^* = d_j$$

In conclusion, a simple method for solving (P_c) is
1) Compute

$$d = c + \frac{1}{n}(1 - e^T c)e$$

2)
for j such that $d_j < 0$,

$$x_j^* = 0$$

for j such that $d_j \geq 0$,

How to make sure e^Tx = 1?

$$x_j^* = d_j$$

Such an x^* is the optimal solution of (P_c)

2 Problem 2

• if $C = D$

$$\begin{aligned} S_C(y) = &= \sup\{y^T c : x \in C\} \\ &= \sup\{y^T x : x \in D\} \\ &= S_D(y) \end{aligned}$$

which is trivial.
• if $C \neq D$
which means

$$\exists \bar{x} \in C, \bar{x} \notin D \tag{11}$$

or

$$\exists \bar{x} \in D, \bar{x} \notin C \tag{12}$$

If (11) holds, since D is a convex set, we have

$$\exists p, \alpha, \quad s.t. \quad p^T \bar{x} > \alpha, p^T x < \alpha, \forall x \in D$$

Then

$$\begin{aligned} S_D(p) &= \sup\{p^T x : x \in D\} \leq \alpha \\ S_C(p) &= \sup\{p^T x : x \in C\} \geq p^T \bar{x} > \alpha \end{aligned}$$

So

$$S_D(p) \neq S_C(p)$$

Vise versa, if (12) holds, since C is a convex set,

$$\exists p, \alpha, \quad s.t. \quad p^T \bar{x} > \alpha, p^T x < \alpha, \forall x \in C$$

Then

$$\begin{aligned} S_C(p) &= \sup\{p^T x : x \in C\} \leq \alpha \\ S_D(p) &= \sup\{p^T x : x \in D\} \geq p^T \bar{x} > \alpha \end{aligned}$$

So

$$S_D(p) \neq S_C(p)$$

In conclusion, if $C \neq D$, then $\exists p \in \mathbb{R}^n, S_D(p) \neq S_C(p)$.
Then we have proved that $C = D$ if and only if $\forall y \in \mathbb{R}^n, S_D(y) = S_C(y)$.

3 Problem 3

$$L(x,w) = f(x) + \sum_{i=1}^{m} w_i g_i(x)$$
$$\bar{L}(x,u) = f(x) + \sum_{i=1}^{r} u_i g_i(x)$$

$$L^*(w) = \inf_{x \in X} (f(x) + \sum_{i=1}^{m} w_i g_i(x)) \tag{13}$$
$$\bar{L}^*(u) = \inf_{x \in \bar{X}} (f(x) + \sum_{i=1}^{r} u_i g_i(x)) \tag{14}$$

Let the feasible set of $(\bar{P})$ be $\bar{F}$. Obviously,

$$\bar{F} \subseteq \bar{X}$$

Since $r < m, u_i \geq 0, g_i(x) \leq 0$

$$\bar{L}(x,u) = f(x) + \sum_{i=1}^{r} u_i g_i(x) \leq f(x)$$

which is satisfied when $x \in \bar{F}$
So

$$\bar{L}^*(u) = \inf_{x \in \bar{X}} \bar{L}(x,u) \leq \inf_{x \in \bar{F}} \bar{L}(x,u) \leq \inf_{x \in \bar{F}} f(x)$$

which is

$$\bar{L}^*(u) \leq \inf_{x \in \bar{F}} f(x)$$

then

$$\bar{v}^* \leq f^*$$

Since $r < m, u_i \geq 0, g_i(x) \leq 0, \bar{X} \subseteq X$, we can always let $w_i = u_i, i = 1, 2, ..., r$ such that

$$\begin{aligned}
\bar{L}^*(u) &= \inf_{x \in \bar{X}} (f(x) + \sum_{i=1}^{r} u_i g_i(x)) \\
&\geq \inf_{x \in \bar{X}} (f(x) + \sum_{i=1}^{r} u_i g_i(x) + \sum_{i=r+1}^{m} w_i g_i(x)) \\
&= \inf_{x \in \bar{X}} (f(x) + \sum_{i=1}^{m} w_i g_i(x)) \\
&\geq \inf_{x \in X} (f(x) + \sum_{i=1}^{m} w_i g_i(x)) \\
&= L^*(w)
\end{aligned}$$

That is,

$$L^*(w) = \bar{L}^*(u) + \xi$$

where

$$\xi \leq 0, w_i = u_i, i = 1, 2, ..., r$$

$$\begin{aligned}
v^* = \sup_{w \geq 0} L^*(w) &= \sup_{u \geq 0, w_i = u_i, i=1,2,...,r} (\bar{L}^*(u) + \xi) \\
&\leq \sup_{u \geq 0, w_i = u_i, i=1,2,...,r} \bar{L}^*(u) + \sup_{u \geq 0} \xi \\
&\leq \sup_{u \geq 0} \bar{L}^*(u) + \sup_{u \geq 0} \xi \\
&\leq \sup_{u \geq 0} \bar{L}^*(u) = \bar{v}^*
\end{aligned}$$

Note that, we loose the restriction $w_i = u_i, i = 1, 2, ..., r$ in the third step of the above derivi-ation.

In conclusion

$$v^* \leq \bar{v}^* \leq f^*$$

4 Problem 4

4.1 (a)

$$\begin{aligned}
(P1) \quad & \min_x f_1(x) \\
s.t. \quad & f_j(x) \leq b_j \qquad j = 2, 3, ..., s \\
& g_i(x) \leq 0 \qquad i = 1, 2, ..., m
\end{aligned}$$

$$\begin{aligned}
(P2) \quad & \min_x \sum_{j=1}^{s} w_j f_j(x) \\
s.t. \quad & g_i(x) \leq 0 \qquad i = 1, 2, ..., m
\end{aligned}$$

4.2 (b)

Let the feasible set of (P1), (P2) are respectively F_1 and F_2.
suppose $\bar{x}$ is the optimal solution of (P2),then $\bar{x} \in F_2$ and

$$g_i(\bar{x}) \leq 0 \quad i = 1, 2, ..., m$$

and

$$\forall x \in F_2 \quad \sum_{j=1}^{s} w_j f_j(x) \geq \sum_{j=1}^{s} w_j f_j(\bar{x})$$

Since $w_1 > 0$, That is

$$\forall x \in F_2 \quad f_1(x) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) \geq f_1(\bar{x}) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(\bar{x})$$

Let

$$b_j = f_j(\bar{x})$$

then

$$\forall x \in F_2 \quad f_1(x) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) \geq f_1(\bar{x}) + \sum_{j=2}^{s} \frac{w_j}{w_1} b_j \tag{15}$$

Note that $\bar{x}$ is also feasible for (P1) because for $j = 2, 3, ..., s$,

$$f_j(\bar{x}) = b_j \leq b_j$$

and

$$f_j(\bar{x}) \leq 0$$

are satisfied.
We know

$$F_1 \subseteq F_2$$

From (15)

$$\forall x \in F_1 \quad f_1(x) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) \geq f_1(\bar{x}) + \sum_{j=2}^{s} \frac{w_j}{w_1} b_j \tag{16}$$

which implies $\forall x \in F_1$

$$f_1(x) \geq f_1(\bar{x}) \tag{17}$$

or

$$\sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) \geq \sum_{j=2}^{s} \frac{w_j}{w_1} b_j \tag{18}$$

At least one of (17)(18) is satisfied.
If (17) holds, then $\bar{x}$ is an optimal solution of (P1).
If (18) holds, then

$$\exists j \in \{2, 3, ..., s\} \quad f_j(x) \geq b_j$$

Since $x \in F_1$, we have $f_j(x) \leq b_j$
So for those j such that $f_j(x) \geq b_j$,

$$f_j(x) = b_j$$

Then

$$\sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) \leq \sum_{j=2}^{s} \frac{w_j}{w_1} b_j$$

Considering (18), we have

$$\sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) = \sum_{j=2}^{s} \frac{w_j}{w_1} b_j$$

Then from (16)

$$f_1(x) \geq f_1(\bar{x})$$

then $\bar{x}$ is an optimal solution of (P1).
In conclusion, if $\bar{x}$ is an optimal solution of (P2), let

$$b_j = f_j(\bar{x}) \quad j = 2, ..., s$$

in this case $\bar{x}$ is also an optimal solution of (P1).

4.3 (c)

List KKT conditions for (P1)

$$
\begin{align}
f_i(x) - b_i \leq 0 \quad & i = 2, 3, ..., s \tag{19} \\
g_i(x) \leq 0 \quad & i = 1, 2, ..., m \tag{20} \\
u_i \geq 0 \quad & i = 1, 2, ..., s + m - 1 \tag{21} \\
u_i g_i(x) = 0 \quad & i = 1, 2, ..., m \tag{22} \\
u_i(f_{i-m+1}(x) - b_{i-m+1}) = 0 \quad & i = m + 1, ..., s + m - 1 \tag{23} \\
\nabla f_1(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) + \sum_{i=m+1}^{s+m-1} u_i \nabla f_{i-m+1}(x) = 0 \quad & i = 1, 2, ..., s + m - 1 \tag{24}
\end{align}
$$

List KKT conditions for (P2)

$$
\begin{align}
g_i(x) \leq 0 \quad & i = 1, 2, ..., m \tag{25} \\
v_i \geq 0 \quad & i = 1, 2, ..., m \tag{26} \\
v_i g_i(x) = 0 \quad & i = 1, 2, ..., m \tag{27} \\
\sum_{i=1}^{s} w_i \nabla f_i(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0 \quad & i = 1, 2, ..., m \tag{28}
\end{align}
$$

4.4 (d)

Since all functions are convex and a constraint qualfication is satisfied,
KKT conditions are sufficient and necessary for optimality for (P1) and (P2).
If an optimal solution x is obtained by Approach 1, i.e. x is an optimal solution of (P1).
Then (19)(20)(21)(22)(23)(24) hold.
Since $w_1 > 0$, let

$$
\begin{aligned}
v_i &= u_i w_1 \qquad & i = 1, 2, ..., m \\
w_i &= u_{i+m-1} w_1 \qquad & i = 2, 3, ..., s
\end{aligned}
$$

(24) holds, then

$$w_1\nabla f_1(x) + \sum_{i=1}^{m} w_1 u_i \nabla g_i(x) + \sum_{i=m+1}^{s+m-1} w_1 u_i \nabla f_{i-m+1}(x) = 0 \tag{29}$$

$$w_1\nabla f_1(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) + \sum_{i=m+1}^{s+m-1} w_{i-m+1} \nabla f_{i-m+1}(x) = 0 \tag{30}$$

That is

$$\sum_{i=1}^{s} w_i \nabla f_i(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0 \quad i = 1, 2, ..., m$$

which means (28) holds.
(22) holds, then $u_i g_i(x) = 0$, then $w_1 u_i g_i(x) = 0$, then $v_i g_i(x) = 0$
which means (27) holds.
(21) holds, then $u_i = w_1 v_i \geq 0$, then $v_i \geq 0$
which means (26) holds.
(20) holds, which is equivalant to (25),
which means (25) holds.
So KKT conditions for (P2) are satisfied.
So x is also an optimal solution of (P2).

In conclusion, under conditions in this problem, if x is an optimal solution of (P1), then let w_1 be an arbitrary positive number and

$$w_i = u_{i+m-1} w_1 \qquad i = 2, 3, ..., s$$

in this case x is also an optimal solution of (P2).

5 Problem 5

$$f(x) = \sum_{(i,j)\in A} (\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij})$$

is a convex function of x.
where x is a column vector that put each $x_{ij}, (i,j) \in A$ into a sequence.
Give (i,j) in A an index, say r_{ij}. where $r_{ij} \in \{1, 2, ..., |A|\}$. $|A|$ is the amount of element in A.
Then

$$x_{r_{ij}} = x_{ij}$$

Consider the equation restrction

$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij} \quad \forall i \in N \tag{31}$$

Construct two vector $p_i \in \{0,1\}^{|A|}$ and $q_i \in \{0,1\}^{|A|}$.
Let p_i

$$(p_i)_n = \begin{cases} 1 & n = r_{ij} \quad (i,j) \in A \\ 0 & otherwise \end{cases}$$

Let q_j

$$(q_i)_n = \begin{cases} 1 & n = r_{ji} \quad (i,j) \in A \\ 0 & otherwise \end{cases}$$

Then

$$p_i^T x = \sum_{j:(i,j)\in A} x_{ij} \quad q_i^T x = \sum_{j:(j,i)\in A} x_{ij}$$

Equation restrction (31) becomes

$$(p_i - q_i)^T x = 0 \tag{32}$$

Write down the optimization problem again

$$\min_x \quad f(x) = \sum_{(i,j)\in A} (\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij})$$
$$s.t. \quad h_i(x) = (p_i - q_i)^T x = 0 \quad i \in N$$

Note that $f(x)$ is convex and each $h_j(x)$ is linear.
So KKT conditions are necessary and sufficient condition for optimality.
Write down KKT conditions

$$\nabla f(x) + \sum_{i\in N} u_i \nabla h_i(x) = 0 \tag{33}$$

For each element in (33)

$$R_{ij} x_{ij} - t_{ij} + (\sum_{i\in N} u_i (p_i - q_i))_{r_{ij}} = 0 \tag{34}$$

$$\sum_{i\in N} u_i (p_i)_{r_{ij}} = \sum_{i:(i,j)\in A} u_i \qquad j \in N$$
$$\sum_{i\in N} u_i (q_i)_{r_{ij}} = \sum_{i:(j,i)\in A} u_i = \sum_{j:(i,j)\in A} u_j \qquad i \in N$$

Let

$$v_j = \sum_{i:(i,j)\in A} u_i \qquad j \in N \tag{35}$$
$$v_i = \sum_{j:(i,j)\in A} u_j \qquad i \in N \tag{36}$$

Then (34) becomes

$$R_{ij} x_{ij} - t_{ij} = v_i - v_j \tag{37}$$

That is, by constructing v_i, v_j as (35)(36), we can get Ohm's law by KKT conditions.
With the Kirchhoff's law

$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij} \quad \forall i \in N \tag{38}$$

which is also the equation restriction of optimization problem, we can say that if $\{x_{ij}\}$ solve both laws (37)(38), it is the solution of nonlinear programming problem.

For $\{x_{ij}\}$ that does not solve (37)(38), KKT conditions are not satiefied. So it is not the solution of this nonlinear programming problem.

In conclusion, $\{x_{ij}\}$ that solves both laws (37)(38) is the unique solution of this nonlinear porgramming problem.