fychang hwk10

Fangyuan Chang

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1 Problem 1

Hello

1.1 (a)

Suppose $f_c(x) = \frac{1}{2}||x-c||_2^2$, $f_d(x) = \frac{1}{2}||x-d||_2^2$, $\alpha = \frac{1-e^Tc}{n}e$.

$$
\begin{aligned}
&\frac{1}{2}||x-c||^2 - \frac{1}{2}||x-d||_2^2 \\
&= \frac{1}{2}(\sum_{i=1}^{n}(x_i-c_i)^2 - \sum_{i=1}^{n}(x_i-d_i)^2) \\
&= \frac{1}{2}(\sum_{i=1}^{n}(x_i-c_i)^2 - \sum_{i=1}^{n}(x_i-c_i-\frac{1-e^Tc}{n}e)^2) \\
&= \frac{1}{2}(\sum_{i=1}^{n}(x_i-c_i)^2 - \sum_{i=1}^{n}(x_i-c_i-\alpha)^2) \\
&= \frac{1}{2}\sum_{i=1}^{n}((x_i-c_i)^2 - (x_i-c_i-\alpha)^2) \\
&= -\frac{1}{2}\sum_{i=1}^{n}\alpha^2 - 2\alpha(x_i-c_i) \\
&= -\frac{1}{2}(n\alpha^2 + 2\alpha e^T(c-x)) \\
&= -\frac{1}{2}(n\alpha^2 + 2\alpha(e^Tc - e^Tx)) \\
&= -\frac{1}{2}(n\alpha^2 + 2\alpha(e^Tc - 1))
\end{aligned}
$$

From the above equation, because α, c, n are all constant, the value of the equation has no relation to x, which means when $f_c(x)$ attains its minimum, its corresponding optimal solution also makes $f_d(x)$ attains its minimum. That is to say, $(P_c), (P_d)$ have the same solutions.

1.2 (b)

- First,we check it whether satisfy all the constraints.

$$
e^Tx^\star = e^Td = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} c_i + \frac{n - n\sum_{i=1}^{n} c_i}{n} = 1,
$$
$$
x^\star = d \geq 0
$$

  Therefore $x^\star = d$ satisfies all constraints.

- Second,$f_d(x)\Big|_{x^\star=d} = 0 \leq f_d(x)$, which is obviously by the property of norm.

Overall, $x^\star = d$ is an optimal solution of (P_d).

1.3 (c)

We will prove this by contradiction.
Suppose $\bar{x}_j > 0$ for $d_j < 0$ considering the related constraint.
This problem is a convex problem, which means KKT condition is a sufficient and necessary condition.
Assume $\bar{x} = [x_1, ...x_j..., x_n]^T$ is an optimal solution for the problem.
Then by KKT condition, for $x \in X, X = \{x | x \geq 0\}$,

$$\nabla f(\bar{x}) + Jh(\bar{x})v = 0$$

$$i.e. \quad 2\bar{x} - 2d + ev = 0$$

$$2\bar{x}_j - 2d_j + v = 0 \Rightarrow v < 0 \quad (1)$$

$$2e^T\bar{x} - 2e^Td + nv = 0$$

Because $e^Td = 1, e^Tx = 1$,

$$2e^T\bar{x} - 2e^Td + nv = 0 \Rightarrow nv = 0 \Rightarrow v = 0 \quad (2)$$

(1) and (2) contradicts with each other, which means the assumption is not true.
Therefore $\bar{x}_j = 0$ for $d_j < 0$.

1.4 (d)

Given a problem (P_c), we can calculate as the following steps:

- First, calculate d by the formula given in the problem. If $d \geq 0$, then its optimal solution is $x^\star = d$, stop; if not, go to the second step.
- Second, eliminate variables and their corresponding row in c. Then return to the first step.

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2 Problem 2

- First, we prove: if $C = D$, then their support functions are equal. This is a trivial conclusion.
- Second, we prove: if $S_C(y) = S_D(y), \forall y \in \mathbb{R}^n$, then $C = D$. We will prove this by contradiction.
  Suppose $C \neq D$. There exists an element $x \in C$ while $x \notin D$.
  Let $S_C(y) = \alpha$. By the definition of $S_C(y)$ we know $y^Tx \leq \alpha$ because $x \in C$.
  Because C, D are closed convex sets, by the assumption, there exists an hyperplane separate x from D. That is to say, $y^Tz > \alpha, \forall z \in D$. Then $S_D(y) > \alpha = S_C(y)$, which contradicts $S_C(y) = S_D(y)$. That is to say, the assumption is not true, which means if $S_C(y) = S_D(y), \forall y \in \mathbb{R}^n$, then $C = D$.

Overall, $C = D$ iff their support functions are equal.

3 Problem 3

For (D), suppose $u_i \geq 0, i = 1, ..., m$:

$$L(x, u) = f(x) + \sum_{i=1}^{m} u_i g_i(x), x \in X$$

$$L^\star(u) = \inf_{x \in X} L(x, u) = \inf_{x \in X} (f(x) + \sum_{i=1}^{m} u_i g_i(x))$$

$$v^\star = \sup_{x \in X} L^\star(u)$$

Similarly, for $(\bar{D})$, suppose $u_i \geq 0, i = 1, ...r$:

$$\bar{L}(x,u) = f(x) + \sum_{i=1}^{r} u_i g_i(x), x \in \bar{X}$$

$$\bar{L}^\star(u) = \inf_{x \in \bar{X}} \bar{L}(x,u) = \inf_{x \in \bar{X}} (f(x) + \sum_{i=1}^{r} u_i g_i(x))$$

$$\bar{v}^\star = \sup_{x \in \bar{X}} \bar{L}^\star(u)$$

Build a function as $F(x,u) = f(x) + \sum_{i=1}^{m} u_i g_i(x), x \in \bar{X}$.
On one hand, by the definition of $\bar{X}$, we get to know

$$F(x,u) \leq \bar{L}(x,u), \forall x \in \bar{X}$$

.
Then

$$\inf_{x \in \bar{X}} F(x,u) \leq \inf_{x \in \bar{X}} \bar{L}(x,u) = \bar{L}^\star(u)$$

$$\sup_u \inf_{x \in \bar{X}} F(x,u) \leq \sup_u \bar{L}^\star(u) = \bar{v^\star}$$

On the other hand, according to $X \supseteq \bar{X}$,

$$\inf_{x \in X} L(x,u) = L^\star(u) \leq \inf_{x \in \bar{X}} F(x,u)$$

Then

$$\sup_u L^\star(u) = v^\star \leq \sup_u \inf_{x \in \bar{X}} F(x,u)$$

Overall,

$$v^\star \leq \sup_u \inf_{x \in \bar{X}} F(x,u) \leq \bar{v^\star}$$

Besides, because $u_i \geq 0, i = 1, ...r$, $g_i(x) \leq 0$,

$$\bar{L}(x,u) \leq f(x), \forall x \in \bar{X}$$

$$i.e. \quad \inf \bar{L}(x,u) = \bar{L}^\star(u) \leq f(x)$$

which means

$$\bar{v}^\star = \sup \bar{L}^\star(u) \leq \min f(x) = f^\star$$

In conclusion,

$$v^\star \leq \bar{v}^\star \leq f^\star$$

4 Problem 4

4.1 (a)

- Approach 1:

$$\min f_1(x) \tag{1}$$
$$f_j(x) \leq b_j, j = 2, ...s \tag{2}$$
$$g_i(x) \leq 0 \tag{3}$$
$$x \in X \tag{4}$$

- Approach 2:

$$\min f(x) = \sum_{j=1}^{s} \omega_j f_j(x) \tag{5}$$
$$g_i(x) \le 0 \tag{6}$$
$$x \in X \tag{7}$$

4.2 (b)

Assume $\bar{x}$ is an optimal solution of approach 2. Then we get

$$\sum_{j=1}^{s} \omega_j f_j(x) \ge \sum_{j=1}^{s} \omega_j f_j(\bar{x}) \tag{8}$$
$$g_i(\bar{x}) \le 0 \tag{9}$$
$$\bar{x} \in X \tag{10}$$

If we want Approach 1 to have the same solution with Approach 2, that is to say, (11)-(14) are satisfied.

$$f_1(x) \ge f_1(\bar{x}) \tag{11}$$
$$f_j(\bar{x}) \le b_j, j = 2, ...s \tag{12}$$
$$g_i(\bar{x}) \le 0 \tag{13}$$
$$\bar{x} \in X \tag{14}$$

According to (9)(10), (13)(14) are satisfied.
Then we want to get (11)(12) by (5).
(5) can be rewritten after being divided by ω_1 as:

$$f_1(x) + \sum_{j=2}^{s} \omega_j f_j(x) \ge f_1(\bar{x}) + \sum_{j=2}^{s} \omega_j f_j(\bar{x}) \tag{15}$$

By intuition, we can just let $f_j(\bar{x}) = b_j$.
(15) becomes:

$$f_1(x) + \sum_{j=2}^{s} \omega_j f_j(x) \ge f_1(\bar{x}) + \sum_{j=2}^{s} \omega_j b_j \tag{16}$$

Then according to $f_j(x) \le b_j, \omega_j \ge 0, i = 1, 2...s$,

$$f_1(x) \ge f_1(\bar{x})$$

Overall, (11)-(14) are satisfied.

4.3 (c)

- Approach 1:

$$f_j(\bar{x}) \le b_j, j = 2, ..., s \tag{17}$$
$$g_i(\bar{x}) \le 0, i = 1, ...m..., m + s - 1 \tag{18}$$
$$u_i \ge 0, i = 1, ...m..., m + s - 1 \tag{19}$$
$$u_i g_i(\bar{x}) = 0, i = 1, ...m..., m + s - 1 \tag{20}$$
$$\nabla f_1(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) + \sum_{i=m+1}^{m+s-1} u_i \nabla f_{i+1-m}(\bar{x}) = 0 \tag{21}$$

- Approach 2:

$$g_i(\bar{x}) \le 0, i = 1, ...m..., m+s-1 \tag{22}$$
$$u_i \ge 0, i = 1, ...m..., m+s-1 \tag{23}$$
$$u_i g_i(\bar{x}) = 0, i = 1, ...m..., m+s-1 \tag{24}$$
$$\sum_{j=1}^{s} \omega_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} v_i \nabla g_i(\bar{x}) = 0 \tag{25}$$

4.4 (d)

By the statement in the problem, we know the two approach problems are convex problem, which means KKT conditions are sufficient and necessary.
Assume $\bar{x}$ is an optimal solution of Approach 1, which means (17)-(21) are true.
Now we want to get (22)-(25) then we can say Approach 2 has the same solution as Approach 1.
Intuitively, (18)-(20) proves (22)-(24).
Then compare (21) with (25):
(25) can be rewritten as :

$$\omega_1 \nabla f_1(\bar{x}) + \sum_{j=2}^{s} \omega_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} v_i \nabla g_i(\bar{x}) = 0 \tag{26}$$

By intuitively, let

$$\omega_1 = 1,$$
$$u_i = v_i, i = 1, , , m,$$
$$u_i = \omega_{i-m+1}, i = m+1, ...m+s-1$$

5 Problem 5

It is trivial that the nonlinear problem is a convex problem, which means KKT conditions are sufficient and necessary.
Its KKT condition is:

$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ji}, \quad \forall i \in N \tag{27}$$
$$\nabla f(\bar{x}) + Jh(\bar{x})^T u = 0 \tag{28}$$

Assume $\forall i \in N$,

$$p_i = \begin{cases} 1, & (i,j) \in A \\ 0, & (i,j) \notin A \end{cases}$$

$$q_i = \begin{cases} 1, & (j,i) \in A \\ 0, & (j,i) \notin A \end{cases}$$

$$x = [..., x_{ij}, ...]^T, (i,j) \in A$$

Now we need to prove we can get (27)(28) by (1)(2) in the problem.
First, it is trivial that (1) $\Rightarrow$ (27).
Then,

$$\nabla f(\bar{x}) = \begin{bmatrix} ... \\ R_{ij}\bar{x}_{ij} - t_{ij} \\ ... \end{bmatrix}, (i,j) \in A \tag{29}$$

(27) can be rewritten as follows:

$$(p_i - q_i)^T x = 0, \forall i \in N$$

$$\nabla Jh(\bar{x}) = \begin{bmatrix} \cdots \\ (p_i - q_i)^T \\ \cdots \end{bmatrix}, \forall i \in N \tag{30}$$

Then some row in (28) can be rewritten as:

$$R_{ij}\bar{x}_{ij} - t_{ij} + \sum_{i=1}^{n} u_i(p_{ir} - q_{ir}) = 0, \tag{31}$$

where p_{ir}, q_{ir} means the element in r_{th} row in p_i, q_i.
Compare (2) in the problem and (31) here, we can find if let

$$v_j = \sum_{i=1}^{n} u_i p_{ir} = u_i$$

$$v_i = \sum_{i=1}^{n} u_i q_{ir} = u_j$$

$(i, j) \in A$ is corresponded to r_{th} row in p_i, q_i.
That is to say (2) $\Rightarrow$ (31).
If (1)(2) are violated, (27)(31) here will not be satisfied, which means it's not KKT condition, and the corresponding solution is not optimal solution.
In conclusion, $x_{ij}, (i, j) \in A$ that satisfies (1)(2) is the unique solution of the programming problem.