Problem 1

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Part a

$$\min\ 0.5(x-d)^T(x-d) \tag{1}$$
$$s.t\ e^Tx = 1; x \geq 0 \tag{2}$$

Objective function can be re-written as follows

$$0.5(x^Tx - x^Td - d^Tx + d^Td) \tag{3}$$
$$d = c + \frac{1-e^Tc}{n}e \tag{4}$$
$$0.5(x^Tx - 2x^Td + d^Td) \tag{5}$$
$$0.5(x^Tx - 2x^T(c + \frac{1-e^Tc}{n}e) + (c + \frac{1-e^Tc}{n}e)^T(c + \frac{1-e^Tc}{n}e)) \tag{6}$$
$$a = \frac{1-e^Tc}{n} \tag{7}$$
$$0.5(x^Tx - 2x^T(c+ae) + (c+ae)^T(c+e) \tag{8}$$
$$0.5(x^Tx - 2x^Tc - 2x^Tae + c^Tc + c^Te + ae^Tc + ae^Te) \tag{9}$$
$$e^Tx = 1 \tag{10}$$
$$0.5(x^Tx - 2x^Tc - 2a + c^Tc + c^Te + ae^Tc + ae^Te) \tag{11}$$
$$e^Te = n \tag{12}$$
$$0.5(x^Tx - 2x^Tc - 2a + c^Tc + c^Te + ae^Tc + an) \tag{13}$$
$$0.5(x^Tx - 2x^Tc - 2a + c^Tc + c^Te + ae^Tc + 1 - e^Tc) \tag{14}$$
$$0.5(x^Tx - 2x^Tc - 2a + c^Tc + ae^Tc + 1) \tag{15}$$
$$0.5(x^Tx - 2x^Tc + c^Tc + c^Te + ae^Tc + 1 - e^Tc - 2a) \tag{16}$$
$$0.5(x^Tx - x^Tc - c^Tx + c^Tc + c^Te + ae^Tc + 1 - e^Tc - 2a) \tag{17}$$
$$0.5(x^T(x-c) - c^T(x-c) + c^Te + ae^Tc + 1 - e^Tc - 2a) \tag{18}$$
$$0.5((x-c)^T(x-c) + ae^Tc + 1 - 2a) \tag{19}$$
$$0.5((x-c)^T(x-c)) + 0.5(ae^Tc + 1 - 2a) \tag{20}$$

Eq.20 indicates that the objective function of P_d is the sum of a constant and P_c hence they have the same optimal solution.

Part b

Given that $e^Td = 1$ and d≥0 makes d a feasible point. From the definition of norm, it can be understood that the minimum value of the norm is 0. As x=d attains the value of 0, x=d is a optimum solution of the problem P_d

Part c

By KKT conditions

$$\bigtriangledown f_0(x) + \sum v_i \bigtriangledown h_i(x) = 0 \tag{21}$$

$$0.5 \times 2(x - d) + v \times e = 0 \tag{22}$$

$$x = d - v \times e \tag{23}$$

For x to be a feasible solution of the problem it has to be ≥ 0. If d_j is < 0 then the value of x_i becomes infeasible eq.23, as $v_i \geq 0$. Hence $x_j^* = 0$ when ever $d_j < 0$.

incorrect here

Part d

Incorrect alg.

A simple way to compute the solution for P_c would be to check weather $c_i \geq 0$, if it is true set x_i^* to be the value of c_i otherwise set the value of $x_i^* = 0$

Problem 2

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Support function

$$S_c(y) = \sup\{y^T x : x \in C\} \tag{24}$$

- (C=D) $\implies$ ($S_c(y) = S_D(y)$)
  Let us assume that $S_c(y) \neq S_D(y)$. From the definition of support function,

  $$S_c(y) \geq y^T x \forall x \in C \tag{25}$$

  $$S_D(y) \geq y^T x \forall x \in D \tag{26}$$

  These form supporting hyper-planes with the sets C and D (with y , $S_c(y)$ and $S_D(y)$). Given the assumption that $S_c(y) \neq S_D(y)$, it is only possible that $S_c(y) > S_D(y)$ *or* $S_c(y) < S_D(y)$. In either the case, there exists a region where the elements of one set are present and the other are not. For example, when $S_D(y) > S_C(y)$, There could be elements of D which are greater than $S_c(y)$ i.e $y^T x > S_c, x \in D$. But these elements could not be a part of C(from the definition of supporting hyper plane). Thus it can observed that $C \neq D$.

- ($S_c(y) = S_D(y)$) $\implies$ (C=D) not clear
  Let us assume that $C \neq D$. Which implies that either C $\subset$ D, D $\subset$ C or $S \bigcap D = \emptyset$. From the above part it can inferred that $S_D(y) > S_C(y)$ when C $\subset$ D and $S_D(y) > S_C(y)$ when D $\subset$ C. In the third case $S \bigcap D = \emptyset$, both the sets cannot be the same side of hyper plane (separating hyper plane theorem). Thus it is observed that $S_c(y) \neq S_D(y)$

Hence proved that (C=D) $\iff$ ($S_c(y) = S_D(y)$)

Problem 3

Expressing both the problems P and $\bar{P}$ as dual functions.

$$L = \min_{x} f(x) + \sum_{1}^{m} u_i g_i(x) \tag{27}$$

$$L^* = \min_{\bar{x}} f(x) + \sum_{1}^{r} u_i g_i(x) \tag{28}$$

As the dual function yields the lower estimate of the primal problem and its best estimate is given Lagrange dual problem. Thus we can safely assume that

$$v^* \leq f^* \tag{29}$$

$$\bar{v}^* \leq f^* \tag{30}$$

$$\begin{aligned} v^* &= \max_{u} \min_{x} f(x) + \sum_{1}^{m} u_i g_i(x) && (31) \\ &= \max_{u} \min_{x} f(x) + \sum_{1}^{r} u_i g_i(x) + \sum_{r}^{m} u_i g_i(x) && (32) \\ \bar{v}^* &= \max_{u} \min_{\bar{x}} f(x) + \sum_{1}^{r} u_i g_i(x) && (33) \end{aligned}$$

Suppose $\bar{x}$ is a feasible point of the primal problem and the optimal value of the primal problem be represented by f^*, $\bar{X} \subset$ X, and if $\bar{x}$ feasible solution of the problem $\bar{P}$ is also a feasible solution of the problem of P. Substituting $\bar{x}$ in the problem, following inferences can be made

$$\begin{aligned} v^* &= \max_{u} \min_{x} f(x) + \sum_{1}^{r} u_i g_i(x) + \sum_{r}^{m} u_i g_i(x) && (34) \\ &= \max_{u} f_0(\bar{x}) + \sum_{1}^{r} u_i g_i(\bar{x}) + \sum_{r}^{m} u_i g_i(\bar{x}) && (35) \\ \bar{v}^* &= \max_{u} \min_{\bar{x}} f(x) + \sum_{1}^{r} u_i g_i(x) && (36) \\ &= \max_{u} f_0(\bar{x}) + \sum_{1}^{r} u_i g_i(\bar{x}) && (37) \end{aligned}$$

As all of the u are positive and $g_i()$ are negative it can be seen that there is an additional term in the eq.35 $\sum_{r}^{m} u_i g_i(\bar{x})$. Hence $v^* \leq \bar{v}^*$ as u ≥ 0. Thus from eq.29, $v^* \leq \bar{v}^* \leq f^*$

Problem 4

Part a

Approach 1

$$\min f_1(x) \tag{38}$$
$$s.t\ f_j(x) - b_j \leq 0\ j = 2, \ldots, s \tag{39}$$
$$g_k(x) \leq 0\ k = 1, \ldots, m \tag{40}$$

Approach 2

$$\min f(x) = \sum_{j=1}^{s} w_j f_j(x)\ w_1 > 0; w_i \geq 0 \tag{41}$$
$$s.t\ g_i(x) \leq 0\ i = 1, \ldots, m \tag{42}$$

Part b

Considering the objective function in approach 2

$$\min_x f(x) = \sum_{j=1}^{s} w_j f_j(x)\ w_1 > 0; w_i \geq 0 \tag{43}$$
$$s.t\ g_i(x) \leq 0\ i = 1, \ldots, m \tag{44}$$

Expressing the above function as a lagrangian dual

$$g(v) = \min_x f_0(x) + \sum_{i=1}^{p} v_i g_i(x) \tag{45}$$
$$= \min_x \sum_{j=1}^{s} w_j f_j(x) + \sum_{i=1}^{p} v_i g_i(x) \tag{46}$$
$$= \min_x w_1 f_1(x) + \sum_{j=2}^{s} w_j f_j(x) + \sum_{i=1}^{p} v_i g_i(x) \tag{47}$$
$$= \min_x w_1 (f_1(x) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) + \sum_{i=1}^{p} \frac{v_i}{w_1} g_i(x)) \tag{48}$$
$$D \to \max_u w_1 \min_x (f_1(x) + \sum_{j=2}^{s} \frac{w_j}{w_1} f_j(x) + \sum_{i=1}^{p} \frac{v_i}{w_1} g_i(x)) \tag{49}$$
$$s.t\ v \geq 0 \tag{50}$$

Dual of approach 1

$$g(v,u) = \min_x f_0(x) + \sum_{i=1}^{p} v_i g_i(x) \tag{51}$$

$$= \min_x f_1(x) + \sum_{j=2}^{s} v_j(f_j(x) - b_j) + \sum_{i=1}^{p} u_i g_i(x) \tag{52}$$

$$= \min_x f_1(x) + \sum_{j=2}^{s} v_j(f_j(x)) - \sum_{j=1}^{s} v_j b_j + \sum_{i=1}^{p} u_i g_i(x) \tag{53}$$

$$= -\sum_{j=2}^{s} v_j b_j + \min_x(f_1(x)) + \sum_{j=2}^{s} v_j(f_j(x)) + \sum_{i=1}^{p} u_i g_i(x)) \tag{54}$$

$$D \to \max_{u,v}(-\sum_{j=2}^{s} v_j b_j + \min_x(f_1(x)) + \sum_{j=2}^{s} v_j(f_j(x)) + \sum_{i=1}^{p} u_i g_i(x)) \tag{55}$$

$$s.t\ u \geq 0, v \geq 0 \tag{56}$$

From the above dual problems it can be observed that dual of approach 1 can be modified as per the choice of weights in approach 2 so that both have the same optimal solution. For example consider the example case in which, all the weights expect w_1 is chosen to be 0, then the modified problem(approach 2) can be written as follows,

$$D \to \max_u w_1 \min_x(f_1(x) + \sum_{i=1}^{p} \frac{v_i}{w_1} g_i(x)) \tag{57}$$

$$s.t\ v \geq 0 \tag{58}$$

Dual of the approach 1 can be modified to look similar by choosing very high positive values for the b_i. This drives the v_i to 0 as the optimization problem is trying to maximize. The modified problem can be written as follows

$$D \to \max_{u,v=0}(\min_x(f_1(x)) + \sum_{i=1}^{p} u_i g_i(x)) \tag{59}$$

$$s.t\ u \geq 0, v = 0 \tag{60}$$

It can clearly observed that both the problems are a scaled version of each other in this case. Hence the solution to one problem is also the solution of the other.

From the above example it can be clearly seen that the approach 1 can be driven to have the same solution as approach 2(for a set of weights) by choosing a set of b values.

Part c

KKT Conditions

$$\bigtriangledown f_0(x) + \sum \lambda_i \bigtriangledown g_i(x) = 0 \tag{61}$$
$$g_i(x) \leq 0 \tag{62}$$
$$\lambda_i \geq 0 \tag{63}$$
$$\lambda_i g_i(x) = 0 \tag{64}$$

First order necessary conditions for the approach 1

$$\bigtriangledown f_1(x^*) + \sum_2^s \lambda_i \bigtriangledown f_i(x^*) + \sum_1^m \bar{\lambda}_i \bigtriangledown g_i(x^*) = 0 \tag{65}$$
$$g_i(x^*) \leq 0 \tag{66}$$
$$f_i(x^*) - b_i \leq 0 \tag{67}$$
$$\lambda_i, \bar{\lambda}_i \geq 0 \tag{68}$$
$$\lambda_i(f_i(x^*) - b_i) = 0 \tag{69}$$
$$\bar{\lambda}_i g_i(x^*) = 0 \tag{70}$$

First order necessary conditions for the approach 2

$$\sum_1^s w_i \bigtriangledown f_i(x^*) + \sum_i^m \bar{\lambda}_i \bigtriangledown g_i(x) = 0 \tag{71}$$
$$\bar{\lambda} \geq 0 \tag{72}$$
$$g_i(x^*) \leq 0 \tag{73}$$
$$\bar{\lambda}_i g_i(x^*) = 0 \tag{74}$$

Part d

For a given values of b_i, if x^* is a solution then it has to satisfy all the KKT conditions for the problem in approach 1. If weights in approach 2 are chosen as the values of the λ (i.e $w_1 = 1$ and $w_i = \lambda_i$ i = 2,...,m)and $\bar{\lambda}$ remains the same as the ones in approach 1. It can observed that both kkt conditions of approach 1 and 2 become equivalent. Hence they share the same optimal solution.

Problem 5

$$\min \sum_{j:(i,j)\in A} (0.5 R_{i,j} x_{i,j}^2 - t_{i,j}) \tag{75}$$
$$s.t \sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{i,j} \ \forall i \in N \tag{76}$$

Applying KKT conditions

$$\bigtriangledown f_0(x^*) + \sum v_i \bigtriangledown h_i(x^*) = 0 \tag{77}$$

$$h(x) = 0 \tag{78}$$

Gradient of the objective function can be represented as follows

$$f(x) = \sum_{j:(i,j)\in A} (0.5 R_{i,j} x_{i,j}^2 - t_{i,j}) \tag{79}$$

$$\bigtriangledown f_0(x) = \begin{bmatrix} R_{1,j}x_{1,j} - t_{1,j} \\ \vdots \\ R_{i,1}x_{i,1} - t_{i,1} \end{bmatrix} \tag{80}$$

Equality constraint can be represented as follows

$$h_j(x) = \sum_{j:(i,j)\in A} x_{i,j} - \sum_{j:(j,i)\in A} x_{i,j} \tag{81}$$

Each constraint represents the difference between total current entering and leaving the node.Given the information that an arc ($x_{i,j}$) is the electric current between 2 nodes. Hence a arc(x_i, j) cannot be in more than 2 constraints(one entering the node and the other leaving the node), thus $x_{i,j}$ can be seen in two constraints, the constraint for node i(leaving) and the constraint for node j(entering).

$$\bigtriangledown f_0(x^*) + \sum v_i \bigtriangledown h_i(x) = 0 \tag{82}$$

$$\begin{bmatrix} R_{1,j}x_{1,j} - t_{1,j} \\ \vdots \\ R_{i,1}x_{i,1} - t_{i,1} \end{bmatrix} + v_i \bigtriangledown h_i(x) + \ldots = 0 \tag{83}$$

$$\begin{bmatrix} \vdots \\ R_{i,j}x_{i,j} - t_{i,j} \\ \vdots \end{bmatrix} + v_i \begin{bmatrix} \vdots \\ -1 \\ \vdots \end{bmatrix} + \ldots + v_j \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix} = 0 \tag{84}$$

$$R_{i,j}x_{i,j} - t_{i,j} - v_i + v_j = 0 \tag{85}$$

$$R_{i,j}x_{i,j} - t_{i,j} = v_i - v_j \tag{86}$$

Thus from eq.78, eq.86 and eq.81 both the ohm and kritchoff's law are proved.