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1. (a) We have

$$\begin{aligned}\frac{1}{2}\|x-d\|_2^2 &= \frac{1}{2}\Big\|x - c - \Big(\frac{1-e^Tc}{n}\Big)e\Big\|_2^2 \\ &= \frac{1}{2}\|x-c\|_2^2 - (x-c)^T\Big(\frac{1-e^Tc}{n}e\Big) + \frac{1}{2}\Big\|\frac{1-e^Tc}{n}e\Big\|_2^2 \\ &= \frac{1}{2}\|x-c\|_2^2 - (1-c^Te)\Big(\frac{1-e^Tc}{n}\Big) + \frac{1}{2}\Big\|\frac{1-e^Tc}{n}e\Big\|_2^2,\end{aligned}$$

for all $x \in \mathbb{R}^n$ such that $e^Tx = 1$. Hence (P_c) and (P_d) have the same optimal solutions. $\square$

(b) We have that $\frac{1}{2}\|x-d\|_2^2 \geq 0$ with equality when $x = d$. Morever

$$e^Td = e^Tc + e^T\Big(\frac{1-e^Tc}{n}\Big)e = e^Tc + n\Big(\frac{1-e^Tc}{n}\Big) = 1.$$

Thus if $d \geq 0$, then $x^* = d$ is an optimal solution of (P_d).

(c) Assume $d_j < 0$ and $0 < x_j^*$, for some $j \in [1,n]$, and x^* optimal solution of (P_d). We have $\nabla f(x^*) = (x^* - d)$, $\nabla h(x^*) = e$ and $\nabla g_i(x) = -e_i$, for $i = 1,\ldots,n$. Note that e/n is a Slater point and (P_D) is a convex optimization problem, thus x^* satisfies KKT conditions. That is there exists (u,v) such that $u \geq 0$ and

$$(x^*-d) - \sum_{i\in I(x^*)} u_ie_i + ve = 0$$

By looking at the j-th coordinate, we have $v = d_j - x_j^* < 0$. Now multiply by e^T the above equation

$$0 = e^T(x^*-d) - e^T\sum_{i\in I(x^*)} u_ie_i + e^Tve = -\sum_{i\in I(x^*)} u_i + nv < 0,$$

contradiction. $\square$

(d) Let's say we start with the problem (P_c) in an n-dimensional space. We know that (P_c) and (P_d) have the same optimal solutions. If $d \geq 0$, then $x^* = d$ is the only optimal solution and we are done. Otherwise there exists $j \in [1,n]$ such that $d_j^* < 0$, then we know all optimal solutions of (P_D) have $x_j^* = 0$. Thus let c' be the vector d with the j-coordinate removed. Hence the solutions of (P_D) are exactly the solutions of

$$(P_{c'}) \qquad \begin{array}{ll} \min_x & \frac{1}{2}\|x-c'\|_2^2 \\ \text{s.t.} & e^Tx = 1 \\ & x \geq 0, \end{array}$$

by adding a 0 after the $j-1$th coordinate. Hence we reduced (P_c), a n dimensional problem, to $(P_{c'})$, a $n-1$ dimensional problem. Then we repeat the same argument for $(P_{c'})$. This procedure will take at most n steps. $\square$

2. If $C = D$, then it is clear that their support function are equal. Now assume that their support functions are equal and $C \neq D$. WLOG, there exists $\overline{x} \in C \setminus D$. By Theorem B.3.1, there exists a nonzero vector p and scalar α such that $p^T\overline{x} > \alpha$ and $\alpha > p^Tx$ for all $x \in D$. Thus

$$S_C(p) = \sup\{p^Tx \colon x \in C\} \geq p^T\overline{x} > \alpha \geq \sup\{p^Tx \colon x \in D\} = S_C(p),$$

contradiction. Hence $C = D$. $\square$

3. We have

$$L^*(u_{[n]}) = \inf_{x\in X} f(x) + \sum_{i=1}^m u_ig(i) \leq \inf_{x\in\overline{X}} f(x) + \sum_{i=1}^m u_ig_i(x) \leq \inf_{x\in\overline{X}} f(x) + \sum_{i=1}^r u_ig_i(x) = \overline{L}^*(u_{[r]}),$$

for all $u \geq 0$, because $\overline{X} \subseteq X$ and $g_{r+1}(x) \leq 0, \ldots, g_m(x) \leq 0$, for all $x \in \overline{X}$. Thus

$$v^* = \sup_{u \geq 0} L(u_{[n]}) \leq \sup_{u \geq 0} \overline{L}^*(u_{[r]}) = \overline{v}^*.$$

By weak duality and feasibility, we have $\overline{v}^* \leq f^*$. We conclude that $v^* \leq \overline{v}^* \leq f^*$. $\square$

4. (a) We have

$$\begin{array}{lll} (P_1) & \min_x & f_1(x) \\ & \text{s.t.} & g_i(x) \leq 0,\ i = [1, m] \\ & & f_j(x) \leq b_j,\ j = [2, s]; \\ (P_2) & \min_x & \sum_{j=1}^s w_j f_j(x) \\ & \text{s.t.} & g_i(x) \leq 0,\ i = [1, m]. \end{array}$$

(b) WLOG, we can assume $w_1 = 1$. Let x^* be a solution of (P_2) and let $b_j = f_j(x^*)$, for $j = [2, \ldots, s]$. We have

$$\begin{array}{ll} \min_x & f_1(x) + \sum_{j=2}^s w_j f_j(x^*) \\ \text{s.t.} & g_i(x) \leq 0,\ i = [1, m] \\ & f_j(x) \leq f_j(x^*),\ j = [2, s]; \end{array} \quad \geq \quad \begin{array}{ll} \min_x & f_1(x) + \sum_{j=2}^s w_j f_j(x) \\ \text{s.t.} & g_i(x) \leq 0,\ i = [1, m] \\ & f_j(x) \leq f_j(x^*),\ j = [2, s]; \end{array} .$$

An optimal solution of the RHS is x^*, thus an optimal solution for LHS is x^*. But note that the LHS has the same optimal solutions as (P_1). $\square$

(c) The first order KKT neccesary conditions for (P_1) are

$$\nabla f_1(x^*) + \sum_{j=2}^{s} u_j \nabla f_j(x^*) + \sum_{i=1}^{m} v_i \nabla g_i = 0$$

$$u_j \geq 0,\ j = 2, \ldots, s; \quad v_i \geq 0,\ i = 1, \ldots, m.$$

$$u_j f_j(x^*) = 0,\ j = 2, \ldots, s; \quad v_i g_i(x^*) = 0,\ i = 1, \ldots, m$$

The first order KKT neccesary conditions for (P_2) are

$$\sum_{j=1}^{s} w_j \nabla f_j(x^*) + \sum_{i=1}^{m} v_i \nabla g_i = 0$$

$$v_i \geq 0,\ i = 1, \ldots, m$$

$$v_i g_i(x^*) = 0,\ i = 1, \ldots, m.$$

$\square$

(d) Let x^* be an optimal solution of (P_1). Since a constraint qualification is satisfied, we have that x^* satisfies the first order KKT necessary conditions. Let $w_1 = 1$ and $w_j = u_j$, for $j = 2, \ldots, s$. Since (P_2) is a convex problem, by Theorem 3.2.1, it follows that x^* is an optimal solution for (P_2). $\square$

5. Let x^* be a current that satisfies law (1) and (2) with voltage drop v^*. Let $\overline{x}$ be a solution to our programming problem. Thus by Theorem 3.1.8, there exists a $\overline{v}$ such that

$$\nabla_{x_{\tilde{i}\tilde{j}}} L(\overline{x}, \overline{v}) = 0$$

$$\Leftrightarrow \nabla_{x_{\tilde{i}\tilde{j}}} \sum_{(i,j) \in A} \Big(\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij}\Big) + \sum_{i \in N} v_i \nabla_{x_{\tilde{i}\tilde{j}}} \Big(\sum_{j:\ (j,i) \in A} x_{ij} - \sum_{j:\ (i,j) \in A)} x_{ij} \Big) = 0$$

$$\Leftrightarrow R_{\tilde{i}\tilde{j}} x_{\tilde{i}\tilde{j}} - t_{\tilde{i}\tilde{j}} - v_{\tilde{i}} + v_{\tilde{j}} = 0$$

for all $(\tilde{i}, \tilde{j}) \in A$. Moreover,

$$\nabla_{x_{i_1 j_1} x_{i_2 j_2}} L(\overline{x}, \overline{v}) = \begin{cases} R_{i_1 j_1}, & \text{if } (i_1, j_1) = (i_2, j_2) \\ 0, & \text{otherwise ,} \end{cases}$$

for all $(i_1, j_1), (i_2, j_2) \in A$. Since $R_{ij} > 0$ for all $(i, j) \in A$, we have that $\nabla_x L(\overline{x}, \overline{v})$ is positive definite. Note that our problem is a convex problem and (x^*, v^*) satisfies the neccesary conditions conditions. By Theorem 3.2.1 and Theorem 3.4.2, we have that x^* is a solution to our problem and it is a strict local minimum. Assume that $\tilde{x}$ was another current that satisfied law (1) and (2) with voltage drop $\tilde{v}$. Thus $\lambda x^* + (1 - \lambda)\tilde{x}$ is a current that satisfies law (1) and (2) with voltage drop $\lambda v^* + (1 - \lambda)\tilde{v}$, for all $\lambda \in [0, 1]$. Hence $\lambda x^* + (1 - \lambda)\tilde{x}$ is an optimal solution for our problem for all $\lambda \in [0, 1]$, contradiction with strict minimality of x^*