# IOE 511: Homework 10

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1. (a) Use $\mathcal{F}_c$ and $\mathcal{F}_d$ to denote the feasible region of $(P_c)$ and $(P_d)$. It is clear that $\mathcal{F}_c = \mathcal{F}_d = \{x | e^T x = 1, \text{ and } x \geq 0\}$. The two problems have the same feasible region.
$f_c(x) = \frac{1}{2}\|x - c\|_2^2 = \frac{1}{2}(x-c)^T(x-c)$.
Let $\lambda = \frac{1-e^Tc}{n}$, then $\lambda$ is a scalar only depending on $c$, $d = c + \lambda e$.

$$
\begin{aligned}
f_d(x) &= \frac{1}{2}\|x-d\|_2^2 = \frac{1}{2}(x-d)^T(x-d) \\
&= \frac{1}{2}(x-c-\lambda e)^T(x-c-\lambda e) \\
&= \frac{1}{2}((x-c)^T(x-c) + \lambda^2 e^T e - 2\lambda e^T(x-c)) \\
(\text{Plug in } e^Tx = 1) &= \frac{1}{2}(x-c)^T(x-c) + \frac{1}{2}\lambda^2 e^T e - \lambda(1-e^Tc) \\
&= f_c(x) + \frac{1}{2}\lambda^2 e^T e - \lambda(1-e^Tc)
\end{aligned}
$$

Because $\frac{1}{2}\lambda^2 e^T e - \lambda(1-e^Tc)$ is just a constant, $f_d(x)$ only differs with $f_c(x)$ with a fixed constant.
$\therefore (P_c)$ and $(P_d)$ have the same optimal solution.

(b)

$$
e^T d = e^T c + \frac{e^T e}{n} - \frac{e^T e e^T c}{n} = e^T c + 1 - e^T c = 1
$$

If $d \geq 0$, $d \in \mathcal{F}_d$. And $f_d(d) = \frac{1}{2}\|d-d\|_2^2 = 0$.
Because $f_d(x) = \frac{1}{2}\|x-d\|_2^2 \geq 0 = f_d(d)$ for any $x \in \mathcal{F}_d$, $x^* = d$ is an optimal solution of P(d).

(c) $e/n$ is a Slater point here and $g(x)$ is convex and $\nabla h(x)$ are linearly independent, so KKT conditions are necessary to the optimal solution of $(P_d)$.
Suppose $\bar{x}$ is an optimal solution, then:
$\nabla f(\bar{x}) = \bar{x} - d$, $\nabla g_j(x) = 1$, $\nabla h(x) = e$.
$\nabla f(\bar{x}) + u\nabla g(\bar{x}) + vh(\bar{x}) = 0$ can be written as $\bar{x} - d - u + ve = 0$.
Multiply by $e^T$ and we can solve $v = e^T u/n$.
Other KKT constraints include $e^T\bar{x} = 1, \bar{x} \geq 0$, $u \geq 0$, $g_i(\bar{x})u_i = -\bar{x}_i u_i = 0$.
If $\bar{x}_j \neq 0$ for $d_j < 0$, then $u_j = 0$, $\bar{x}_j = d_j + u_j - v = d_j - v$.
But here $d_j < 0$, $v = e^T u/n \geq 0$, so $\bar{x}_j = d_j - v < 0$, which contradicts the constraint that $\bar{x}_j \geq 0$.
So $\bar{x}_j = 0$ if $d_j < 0$.

(d) Algorithm:
1. Calculate $d = c + \frac{1-e^T c}{n} e$
2. If $d \geq 0$, set $\bar{x} = d$, terminate.
Otherwise, set the values of $\bar{x}_j$ to 0 where $d_j < 0$ for all $j$.
3. Drop the negative dimensions of $d$ in $c$, go to step 1 with new $c$ in lower dimensions.
Every iteration will figure out at least one dimension of $\bar{x}$. So this algoritm will terminate in $n$ steps.

2. (a) If C=D, it is self-explanatory that $S_C(y) = S_D(y)$.
(b) Known condition is that $S_C(y) = S_D(y)$ for any $y$.
The prove will be done by contradiction. Without loss of generality, suppose there is a $\bar{x} \in C$ and $\bar{x} \notin D$.
Because $D$ is a closed convex set, there must be a seperating hyperplane, i.e. there is a $y^*$ and $\alpha$ such that ${y^*}^T \bar{x} > \alpha$ and ${y^*}^T x < \alpha \forall x \in D$. So $S_C(y^*) \geq {y^*}^T \bar{x} > \alpha \geq S_D(y^*)$, contradicting with the assumption that $S_C(y) = S_D(y)$ for any $y$.
$\therefore C = D$.

3. This follows a similar method to prove of weak duality.

(a) Prove $\bar{v}^* \leq f^*$ first.
Formulate $(\bar{D})$, the dual problem of $(\bar{P})$.
The Lagrangian function is:

$$\bar{L}(x, \bar{u}) = f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x)$$

The dual function is:

$$\begin{aligned} \bar{L}^*(\bar{u}) = &\inf f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x) \\ &\text{s.t. } x \in \bar{X} \end{aligned}$$

The dual problem $(\bar{D})$ is:

$$\begin{aligned} \bar{v}^* = &\sup \bar{L}^*(\bar{u}) \\ &\text{s.t. } \bar{u} \geq 0 \end{aligned}$$

If $\bar{x}$ is feasible for $(\bar{P})$, and $\bar{u}$ feasible for $(\bar{D})$, then $x^* \in \bar{X}$, and

$$f(\bar{x}) \geq f(\bar{x}) + \sum_{i=1}^{r} \bar{u}_i g_i(\bar{x}) \geq \inf_{x \in \bar{X}} f(\bar{x}) + \sum_{i=1}^{r} \bar{u}_i g_i(\bar{x}) = \bar{v}^*$$

Therefore $\bar{v}^* \leq f^*$.

(b) Prove $v^* \leq \bar{v}^*$. Formulate $(D)$, the dual problem of $(P)$.
The Lagrangian function is:

$$L(x,u) = f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

The dual function is:

$$L^*(u) = \inf_{x \in X} f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

The dual problem $(\bar{D})$ is:

$$v^* = \sup_{u \geq 0} L^*(u)$$

So the values to be compared are:

$$v^* = \sup_{u \geq 0} \inf_{x \in X} f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

$$\bar{v}^* = \sup_{\bar{u} \geq 0} \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x)$$

For any $\bar{x} \in X$ and $\bar{u} \geq 0$,

$$\begin{aligned} L^*(\bar{u}) &= \inf_{\bar{x} \in X} f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) \\ &\leq \inf_{\bar{x} \in \bar{X}} f(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) \\ &\leq \inf_{\bar{x} \in \bar{X}} f(\bar{x}) + \sum_{i=1}^{r} \bar{u}_i g_i(\bar{x}) = \bar{L}^*(\bar{u}) \end{aligned}$$

The two inequalities above are becarse $\bar{X} \subseteq X$ and $\bar{u} \geq 0, g_i(\bar{x}) \leq 0$.
Therefore $v^* = \sup_{u \geq 0} L^*(u) \leq \sup_{\bar{u} \geq 0} \bar{L}^*(\bar{u}) = \bar{v}^*$.

Therefore $v^* \leq \bar{v}^* \leq f^*$.

4. (a) The problem for Approach 1 is:

$$\begin{aligned} \text{(P1)}\ \min\ & f_1(x) \\ \text{s.t.}\ & f_j(x) \leq b_j, j = 2, ..., s \\ & g_i(x) \leq 0, i = 1, ..., m \end{aligned}$$

The problem for Approach 2 is:

$$\begin{aligned}(\text{P2}) \min \ & \sum_{j=1}^{s} w_j f_j(x) \\ \text{s.t. } & g_i(x) \leq 0, i = 1, ..., m\end{aligned}$$

(b) Suppose for given $w_j, j = 1, ..., s$, $\bar{x}$ is an optimal solution for (P2). Let $b_j = f_j(\bar{x}), j = 2, ...s$ be the constraints in (P1), the following will prove $\bar{x}$ is also an optimal solution for (P1).
Suppose $\bar{x}$ is not the optimal solution for (P1). Therefore there exists a feasible $x^*$ for (P1) such that $f_1(x^*) < f_1(\bar{x})$).
Plug $x^*$ into the target function in (P2).

$$\begin{aligned}\sum_{j=1}^{s} w_j f_j(x^*) &= w_1 f_1(x^*) + \sum_{j=2}^{s} w_j f_j(x^*) \\ &< w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(x^*) \\ &\leq w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j b_j \\ &= w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(\bar{x}) \\ &= \sum_{j=1}^{s} w_j f_j(\bar{x})\end{aligned}$$

This result contradicts the assumption that $\bar{x}$ is an optimal solution for (P2), therefore $\bar{x}$ is also an optimal solution for (P1) if $b_j = f_j(\bar{x})$.

(c) Suppose $x_1$ is an optimal solution for (P1). First of all, it must be feasible, which means

$$f_j(x_1) \leq b_j, j = 2, ..., s$$
$$g_i(x_1) \leq 0, i = 1, ..., m$$

$x_1$ should also satisfy first order necessary condition, there exists $(u, v) \geq 0$ such that:

$$\nabla f_1(x_1) + \sum_{j=2}^{s} u_j \nabla f_j(x_1) + \sum_{i=1}^{m} v_i \nabla g_i(x_1) = 0$$
$$u_j(f_j(x_1) - b_j) = 0, j = 2, ..., s; v_i g_i(x_1) = 0, i = 1, ...m$$

Suppose $x_2$ is an optimal solution for (P2). Likewise, $x_2$ must be feasible and satisfy first order necessary condition.
Feasible condions are:

$$g_i(x_2) \leq 0, i = 1, ..., m$$

First order necessary condion will be that there exists $v \geq 0$ such that:

$$\sum_{j=1}^{s} w_j \nabla f_j(x_2) + \sum_{i=1}^{m} v_i \nabla g_i(x_2) = 0$$

$$v_i g_i(x_2) = 0, i = 1, ...m$$

(d) Under given condition, both (P1) and (P2) are convex problems. Therefore KKT is sufficient for optimal solution.
Therefore for any optimal solution for (P1), there exist $(u, v) \geq 0$ satisfying the first order necessary condition. Let $w_1 = 1$ and $w_j = u_j, j = 2, ..., s$, then the KKT condition of (P2) also stands. Therefore an optimal solution for (P1) is also optimal for (P2).

5. Because the constraint here is in fact Kirchhoff's current law, the current satisfying law (1) and (2) is a feasible solution.
The Hessian is a diagonal matrix with resistance $R_{ij} > 0$ on the main diagonal. So $H(x)$ is a positive definite matrix. Considering that all $h(x)$ are linear functions, this is a convex problem.
In this way KKT conditions are sufficient for optimal solution. For current on arc $(i, j)$, KKT conditions require $\nabla f(x) + \sum_k v_k \nabla h_k(x) = 0$.
$\nabla f(x_{ij}) = R_{ij} x_{ij} - t_{ij}$
At node $k$, $h_k(x) = \sum_{m:(k,m)\in A} x_{km} - \sum_{m:(m,k)\in A} x_{mk}$.
$\therefore \nabla h_i(x_{ij}) = -1$, $\nabla h_j(x_{ij}) = 1$. Otherwise $\nabla h_m(x_{ij}) = 0$ for $m \neq i, j$.
$\therefore$KKT conditions gives $R_{ij} x_{ij} - t_{ij} - v_i + v_j = 0$. This suffient condition is also Ohm's law.
$\therefore$Currents satisfying the two laws are the optimal solution of the problem.