IOE511/Math562: Continuous Optimization Methods Homework10

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1 Problem 1

(a) First, these two problems have the same feasible regions $\mathcal{F}$. Suppose $x \in \mathcal{F}$, denote $\theta = \frac{1-e^Tc}{n}$, then $\|x-d\|^2 = \|x-c-\theta e\|^2 = (x-c-\theta e)^T(x-c-\theta e) = \|x-c\|^2 + \theta^2\|e\|^2 - 2\theta e^T(x-c) = \|x-c\|^2 + \theta^2\|e\|^2 - 2\theta(1-e^Tc)$. Since $\theta^2\|e\|^2 - 2\theta(1-e^Tc)$ can be viewed as a constant term for x, the objective functions for the two problems only differ by a constant. Thus, the two optimal problems are the same.

(b) Clearly, $\min \frac{1}{2}\|x-d\|^2 \geq 0$ and the equality holds when $x=d$, so as long as $e^Td=1$, $x^*=d$ is an optimal solution.

(c) Since the constraints are linear, we can apply KKT conditions. So for optimal $x^*, \exists u, v$, s.t., $e^Tx^*=1, x^* \geq 0, x^*-d+ve-u=0, u \geq 0, u_jx_j^*=0, \forall j$. So $e^T(x^*-d)+ve^Te-e^Tu = nv-e^Tu = 0$ as $e^Td=1, e^Tx^*=1, e^Te=n$. So $v=\frac{e^Tu}{n} \geq 0$ Notice that $x_j^* = u_j - v + d_j, \forall j$. If $d_j<0$, $x_j^*>0$, then $u_j=0$, $x_k^*=d_j-v<0$, that's the contradiction, so $x_j^*=0$.

(d)

(1) Let $d = c + \frac{1-e^Tc}{n}e$, if $d \geq 0$, set $x^*=d$ and terminate.

(2) If $d_j<0$, set $x_j^*=0$.

(3) Reduce components of c and x corresponding to negative components of d.

(4) Let c be the vector containing only the remaining components of c, n is the number of remaining components. Go back to step 1.

Since the dimension of the problem may be reduced at every iteration, this algorithm should terminate in at most n iterations.

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2 Problem 2

If $C=D$, then $S_C(y)=S_D(y), \forall y$.

If $S_C(y)=S_D(y), \forall y$, first we show $C \subseteq D$, by contradiction. Suppose $\exists x' \in C$, s.t., $x' \nexists D$, since D is closed and convex, there exists a hyperplane that strictly separates x' from D. i.e.,

$\exists y \in \mathbb{R}^n, \alpha \in \mathbb{R}$, s.t., $y^T x > \alpha$ and $y^T x < \alpha, \forall x \in D$. Then we know $S_D(y) = \sup_{x \in D} y^T x \leq \alpha$, but $S_C(y) = \sup_{x \in C} y^T x \geq y^T x' > \alpha$, and $S_C(y) > S_D(y)$, contradiction. Similarly, we can also show that $D \subseteq C$, so $C = D$.

3 Problem 3

First, $v^* = \sup_{u \geq 0} L^*(u)$, where $u \in \mathbb{R}^n$, $L^*(u) = \inf_{x \in X} f(x) + \sum_{i=1}^{n} u_i g_i(x)$, also $\bar{v}^* = \sup_{\bar{v} \geq 0} \bar{L}^*(\bar{v})$, where $\bar{u} \in \mathbb{R}^r$, $\bar{L}^*(u) = \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x)$. By weak duality, $\bar{v}^* \leq f^*$.

Then, suppose u is a feasible solution for (D), construct $\bar{u}$ by letting $\bar{u}_i = u_i$ for $i = 1, 2, \cdots, r$, then $\bar{u}$ is a feasible solution for $(\bar{D})$. So, $L^*(u) = \inf_{x \in X} f(x) + \sum_{i=1}^{n} u_i g_i(x) \leq L^*(u) = \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{n} u_i g_i(x) \leq \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} u_i g_i(x) = \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x) = \bar{L}^*(u) \leq \bar{v}^*$. The first inequality is because $\bar{X} \subseteq X$, the second inequality is because $u_i \geq 0, g_i(x) \leq 0$. So $\forall u$ feasible for (D), $L^*(u) \leq \bar{v}^*$, so $v^* \leq \bar{v}^*$.

4 Problem 4

(a) For the first approach, denote the problem as P_1: $\min f_1(x)$, s.t. $f_j(x) \leq b_j, j = 2, \cdots, s$, $g_i(x) \leq 0, i = 1, \cdots, m$.

For the second approach, denote the problem as P_2: $\min \sum_{j=1}^{s} w_j f_j(x)$, s.t. $g_i(x) \leq 0, i = 1, \cdots, m$.

(b) Suppose that x' is optimal for P_2 and $b_j = f_j(x')$ for $j = 2, \cdots, s$. Prove x' is also optimal for P_1 by contradiction. Suppose that x' is not optimal for P_1, i.e., $\exists x$, s.t., $f_1(x) < f_1(x')$. Then x is also feasible for P_2 and we have: $displaystyle \sum_{j=1}^{s} w_j f_j(x) = w_1 f_1(x) + \sum_{j=2}^{s} w_j f_j(x) \leq w_1 f_1(x) + \sum_{j=2}^{s} w_j f_j(x') < w_1 f_1(x') + \sum_{j=2}^{s} w_j f_j(x') = \sum_{j=1}^{s} w_j f_j(x')$. Then x' is not optimal for P_2, that is the contradiction.

(c) Suppose $\bar{x}$ is optimal for P_1, then $\exists(\bar{u}_1, \cdots, \bar{u}_m, v_2, \cdots, v_s)$, s.t., $\nabla f(\bar{x}) + \sum_{j=2}^{s} v_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} \bar{u}_i \nabla g_i(\bar{x}) = 0$, $f_j(\bar{x}) \leq b_j, j = 2, \cdots, s, g_i(\bar{x}) \leq 0, i = 1, \cdots, m$, $(\bar{u}_1, \cdots, \bar{u}_m, v_2, \cdots, v_s) \geq 0$, $v_j(b_j - f_j(\bar{x})) = 0, j = 2, \cdots, s, \bar{u}_i g_i(\bar{x}) = 0, i = 1, \cdots, m$.

If $\tilde{x}$ is optimal for P_2, then $\exists(\tilde{u}_1, \cdots, \tilde{u}_m)$, s.t., $\sum_{j=1}^{s} w_j \nabla f_j(\tilde{x_j}) + \sum_{i=1}^{m} \tilde{u}_i \nabla g_i(\tilde{x}) = 0$, $g_i(\tilde{x}) \leq 0, i = 1, \cdots, m$, $(\tilde{u}_1, \cdots, \tilde{u}_m) \geq 0$, $\tilde{u}_i g_i(x) = 0, i = 1, \cdots, m$.

(d) First, under the conditions of the statement, both P_1 and P_2 are convex problems and KKT conditions are necessary and sufficient. Suppose $\bar{x}$ is optimal for P_1, then $\exists(\bar{u}_1, \cdots, \bar{u}_m, v_2, \cdots, v_s)$ satisfies KKT conditions. Let $w_1 = 1$, $w_j = v_j \geq 0, j = 2, \cdots, s$. Then $\tilde{x}$ and $\tilde{u} = \bar{u}$ satisfy KKT conditions for P_2. Since KKT conditions are sufficient, $\bar{x}$ is also optimal for P_2.

5 Problem 5

First, write the Lagrangian dual of the problem: $L(x, v) = \sum_{(i,j)\in A} (\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij}) + \sum_i v_i (\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ij})$. Since the constraints are linear, we apply KKT conditions, suppose x_{ij} is the optimal solution, then $R_{ij} x_{ij} - t_{ij} - v_j + v_i = 0$ and it is just the Ohm's Law. Also, the objective function is convex, so KKT condition is also sufficient for optimality and x_{ij} is a strict and the unique optimal solution of the problem.