IOE 511 HW 10

Connor Riley

Winter 2017

9

1 Question 1

(a)

$$\begin{array}{lll} (P_c) & \min_x & \frac{1}{2}||x-c||_2^2 \\ & st. & e^\intercal x = 1 \\ & & x \geq 0 \end{array}$$

$$\begin{array}{lll} (P_d) & \min_x & \frac{1}{2}||x-d||_2^2 \\ & st. & e^\intercal x = 1 \\ & & x \geq 0 \end{array}$$

We are given that $d = c + \left(\frac{1-e^\intercal c}{n}\right) e$. Substituting in for d in P_d we get:

$$f_d(x) = \frac{1}{2}(x - c^\intercal)(x - c) - \left(\frac{1 - e^\intercal c}{n}\right)(1 - e^\intercal c) + \frac{1}{2}\left(\left(\frac{1 - e^\intercal c}{n}\right)^2 e^\intercal e\right)$$

$$f_d(x) = f_c(x) - \left(\frac{1 - e^\intercal c}{n}\right)(1 - e^\intercal c) + \frac{1}{2}\left(\left(\frac{1 - e^\intercal c}{n}\right)^2 e^\intercal e\right)$$

Since both (P_d) and (P_c) have the same feasible regions and the objective for (P_d) is the same as the objective for (P_c) with some constant added to it, we will get the same optimal solution for both.

(b) Simple argument in solution .

$$L_d(x, u) = -u + \frac{1}{2}(x^\intercal - d^\intercal)(x - d) + ue^\intercal x$$

$$L_d^*(u) = -u + \inf_x \frac{1}{2}(x^\intercal - d^\intercal)(x - d) + ue^\intercal x$$

$$L_d^*(u) = -u + \inf_x \frac{1}{2}x^\intercal x - d^\intercal x + \frac{1}{2}d^\intercal d + ue^\intercal d$$

We take the derivative of this convex function and set it equal to zero to obtain $x = d - ue$.

$$L_d^*(u) = -u + \frac{1}{2}(d - ue)^\intercal(d - ue) - d^\intercal(d - ue) + \frac{1}{2}d^\intercal d + ue^\intercal(d - ue)$$

This gives us the dual:

$$\begin{array}{lll} (D_d) & \sup_u & -u + ud^\intercal e - \frac{1}{2}u^2 n \\ & st. & ue \leq d \end{array}$$

using $d^\intercal e = 1$ we rewrite the dual as:

$$\begin{array}{lll} (D_d) & \sup_u & -\frac{1}{2}u^2 n \\ & st. & ue \leq d \end{array}$$

Because this function is concave we can take the derivative and set it equal to zero to find the optimal solution. We obtain $u^* = 0$. We know that $x = d - ue$, and therefore we know that $x^* = d$.

should not make the assumption

(c) Suppose we have a $d_j < 0$ and all other $d_j \geq 0$. This would mean that u would have to be less than 0 because $ue \leq d$. Therefore, $u^* = d_j$ and $x_j = d_j - u \rightarrow x_j^* = 0$.

(d) Take the original problem and generate d using the above equation involving c. While d has a negative value in it, do the following: pull out that negative value (at index j) and set $x_j = 0$ then set $c^2 = (d)_{i \neq j}$. Generate d^2 from c^2 and continue. Stop when $d^k \geq 0$ and use this to set the remaining x values corresponding to their indices.

10

2 Questoin 2

We want to prove that $C = D$ if and only if $S_C(y) = S_D(Y)$. First we know that if $C = D$ then $S_C(y) = S_D(y)$ trivially because of the way they are defined. So we only need to prove the second half, that if $S_C(y) = S_D(y)$ then $C = D$. To do this we will show that $C \subseteq D$ and then that $D \subseteq C$.

Suppose $\bar{x} \in C, \bar{x} \notin D$. By Theorem B.3.1 there must exist a nonzero vector p and a scalar α such that $p^\intercal \bar{x} > \alpha$, $p^\intercal x < \alpha\ \forall x \in D$. So we know that $\sup\{p^\intercal x : x \in D\} \leq \alpha < p^\intercal \bar{x} \leq \sup\{p^\intercal x : x \in C\}$. This simplifies to $S_D(p) \leq \alpha < p^\intercal \bar{x} \leq S_C(p)$ so $S_D(p) \neq S_C(p)$. Therefore we have a contradiction and therefore $\bar{x} \in C \rightarrow \bar{x} \in D$. As a result we know that $C \subseteq D$.

Suppose $\bar{x} \in D, \bar{x} \notin C$. By Theorem B.3.1 there must exist a nonzero vector p and a scalar α such that $p^\intercal \bar{x} > \alpha$, $p^\intercal x < \alpha\ \forall x \in C$. So we know that $\sup\{p^\intercal x : x \in C\} \leq \alpha < p^\intercal \bar{x} \leq \sup\{p^\intercal x : x \in D\}$. This simplifies to $S_C(p) \leq \alpha < p^\intercal \bar{x} \leq S_D(p)$ so $S_C(p) \neq S_D(p)$. Therefore we have a contradiction and therefore $\bar{x} \in D \rightarrow \bar{x} \in C$. As a result we know that $D \subseteq C$.

Therefore we can conclude that if $S_C(y) = S_D(y)$ then $C = D$. We have proven both parts of the if and only if statement.

3 Question 3

$$\begin{array}{llll} (P) & \min & f(x) & \\ & st. & g_i(x) \leq 0 & i = 1, \ldots, m \\ & & x \in X. & \end{array}$$

where $X \subset \mathbb{R}^n$.

$$\begin{array}{llll} (\bar{P}) & \min & f(x) & \\ & st. & g_i(x) \leq 0 & i = 1, \ldots, r \\ & & x \in \bar{X}. & \end{array}$$

where $\bar{X} = \{x \in X : g_{r+1} \leq 0, \dots g_m(x) \leq 0\}$

The two duals we obtain from these problems are as follows:

$$\begin{array}{lll} (D) & v^* = \sup & \inf_x f(x) + \sum_{i=1}^m u_i g_i(x) \\ & st. & u_i \geq 0 \ \forall i \end{array}$$

$$\begin{array}{lll} (D) & \bar{v}^* = \sup & \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x) \\ & st. & u_i \geq 0 \ \forall i \end{array}$$

First we know that $f^* \geq \bar{v}^*$ because $(\bar{D})$ is the dual of (P). This can be found by keeping the $r+1-s$ constraints in the domain of (P) and then taking the dual. Therefore, by weak duality, the following inequality holds: $f^* \geq \bar{v}^*$.

Now we just need to show that $v^* \leq \bar{v}^*$.

Because we know that $g_i(x) \leq 0$ and $u_i \geq 0$ we can write:

$$\forall x \in \bar{X} \ f(x) + \sum_{i=1}^r u_i g_i(x) \geq f(x) + \sum_{i=1}^m u_i g_i(x)$$

$$\inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x) \geq \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^m u_i g_i(x)$$

However, we know from a previous homework that increasing the feasible region will never make the objective value worse. In our case, because we are trying to minimize, then increasing the right hand side of our inequality to be on X could only serve to lower its value. As a result, we can write:

$$\inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x) \geq \inf_{x \in X} f(x) + \sum_{i=1}^m u_i g_i(x)$$

$$\sup \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x) \geq \sup \inf_{x \in X} f(x) + \sum_{i=1}^m u_i g_i(x)$$

$$\bar{v}^* \geq v^*$$

Now by the weak duality theorem we get that $v* \leq f*$, therefore, we have $\bar{v}^* \leq v^* \leq f^*$.

4 Question 4

(a)

$$\begin{array}{llll} & (I) & \min & f_1(x) \\ g & & st. & f_j(x) \leq b_j \quad j = 2, \dots, s \\ & & & g_i(x) \leq 0 \quad i = 1, \dots, m \end{array}$$

$$\begin{array}{lll} (II) & \min & \sum_{j=1}^s w_j f_j(x) \\ & st. & g_i(x) \leq 0 \qquad i = 1, \dots, m \end{array}$$

(b) We say that x^* is the optimal solution to (II) given fixed w_i values. Next, we set $b_j = f_j(x^*)$. Therefore, we know that x^* must be feasible for formulation (I). Next, we know that multiplying the objective value by a constant will not change the optimal solution (only the optimal value) and therefore, we multiply the objective of formulation (I) by w_1 to obtain the new problem:

$$\begin{array}{lll} (\bar{I}) & \min & w_1 f_1(x) \\ & st. & f_j(x) \leq b_j \quad j = 2, \ldots, s \\ & & g_i(x) \leq 0 \quad\; i = 1, \ldots, m \end{array}$$

Suppose x^* is not optimal for $(\bar{I})$ (or (I) for that matter). We can find a $\bar{x}$ that is optimal for (I), therefore, $w_1 f_(\bar{x}) \leq w_1 f_(x^*)$. Because $\bar{x}$ is feasible for (I) we know it must satisfy all of the constraints of (I) which means that $f_j(x) \leq b_j = f_j(x^*)\ \forall j = 2, \ldots s$. Combining these two inequalities we get the following inequality:

$$w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(\bar{x}) \leq w_1 f_1(x^*) + \sum_{j=2}^{s} w_j f_j(x^*)$$

Because the feasible region of (I) is smaller than the feasible region of (II) we know that $\bar{x}$ is also feasible for (II), and by the above inequality, it is a better solution than x^*! Therefore, we have a contradiction, and x^* must be an optimal solution to (I) if its an optimal solution of (II).

(c) We write the KKT conditions below: For formulation 1:

$$\nabla f_1(x) + \sum_{j=2}^{s} \pi_j \nabla f_j(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) = 0$$

where $u \geq 0$, $\pi \geq 0$, $u_i g_i(x) = 0$ and $\pi_j f_j(x) = 0$.

For formulation 2:

$$\sum_{j=1}^{s} w_j \nabla f_j(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) = 0$$

where $u \geq 0$ and $u_i g_i(x) = 0$

(d) Suppose we have x^* which is the optimal solution to (I) and $u*$ which is the optimal solution to the dual of (I) (all for a particular set of b_j). Set $w_1 = 1$ and $w_j = u_j^*\ j = 2, \ldots, s$. Now we can see that

$$\nabla f_1(x^*) + \sum_{j=2}^{s} \pi_j \nabla f_j(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = \sum_{j=1}^{s} w_j \nabla f_j(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*)$$

Because the problems are convex and x^* satisfies all of the KKT conditions for (II) and we know that KKT conditions are necessary for an optimal solution in this case, then we can conclude that x^* is optimal for (II).

5 Question 5

First, $R_{ij} > 0$ and therefore our objective function is strictly convex. This combined with the fact that our constraints are clearly convex means that the problem is convex and therefore, if we find a solution, that solution will be a unique global solution to the problem. Additionally, the problem being convex means that KKT condition re sufficient for optimality. We find the KKT conditions:

$$L(x, v) = \sum_{\forall (i,j) \in A} \left(\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij} + \sum_{i \in N} v_i \left(\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ij} \right) \right)$$

We know that $\nabla_x L(\hat{x}, v) = 0$ and therefore we know that:

$$\frac{\partial L(\hat{x}, v)}{\partial x_{ij}} = R_{ij} \hat{x}_{ij} - t_{ij} + v_j - v_i = 0$$

Rearranging, we get Ohm's law. Therefore, the solution to this optimization problem $\hat{x}$ satisfies Ohm's law. We already know it satisfies Kirchhoff's law because of the constraints in the optimization problem (which are exactly the same as the law itself). Therefore, $\hat{x}$ is the current that uniquely satisfies the optimization problem.