IOE 511/MATH 562
HOMEWORK 10

10

Chien-Yi Tsai (chienyit@umich.edu)

<Question 1>

(a) The objective function of (P_c) is $f_c(x) = \frac{1}{2}(x^Tx - 2c^Tx + c^Tc)$. On the other hand, the objective function of (P_d) is $f_d(x) = \frac{1}{2}(x^Tx - 2c^Tx + c^Tc - 2(\frac{1-e^Tc}{n}) + 2(\frac{1-e^Tc}{n})c^Te + (\frac{1-e^Tc}{n})^2e^Te)$. Observe that $f_d(x) = f_c(x) + a$ where a is a constant, so solving (P_d) is the same as solving (P_c) and they will thus have the same optimal solutions.

(b) First, $x^* = d \geq 0$ is feasible to (P_d):
$e^Td = 1$ and $d \geq 0$.
Second, it is optimal because $f_d(d) = 0$ and it can not further improve since the euclidean norm is always greater or equal to zero.

(c) Want to show that when $d_j < 0$ and $x_j > 0$, there exists a contradiction.
When $d_j < 0$ and $x_j > 0$, by complementarity $x_ju_j = 0$, $u_j = 0$. Moreover, the gradient condition is $x-d+ve-u = 0$, so $x_j - d_j + v - u_j = 0$ and $v < 0$. Multiplying both sides of the gradient condition equation by e, we get $x^Te - d^Te + ve^Te - u^Te = 0$ or $vn - u^Te = 0$. Since $u \geq 0$, $v = \frac{u^Te}{n} \geq 0$, which gets us a contradiction. So $x_j = 0$.

(d) The method is as follows:
Step 1: Solve (P_d) and get d
Step 2: If $d \geq 0$ then d is the solution of P_c, stop. Otherwise, got to Step 3.
Step 3: Eliminate x_j where $d_j < 0$ from (P_d), then go back to Step 1.

<Question 2>

Clearly if $C = D$ the support functions are equal. So we want to show that if the support functions are equal, then $C = D$, by showing that $D \subseteq C$ and $C \subseteq D$.
$D \subseteq C$: Suppose there exists $x_0 \in D$, $x_0 \notin C$. Since C is closed, x_0 can be strictly separated from C, *i.e.*, there exists an $a \neq 0$ with $a^Tx_0 > b$ and $a^Tx < b$ for all $x \in C$. This means that

$$sup_{x \in C} a^Tx \leq b < a^Tx_0 \leq sup_{x \in D} a^Tx$$

which means that $S_C(a) \neq S_D(a)$. By repeating the argument with the roles of C and D reversed, we can show that $C \subseteq D$.

<Question 3>

$L(x,u) = f(x) + \sum_{i=1}^m u_ig_i(x)$, $\bar{L}(x,u) = f(x) + \sum_{i=1}^r u_ig_i(x)$.
Since $L(x,u) = f(x) + \sum_{i=1}^r u_ig_i(x) + \sum_{i=r+1}^m u_ig_i(x)$, so it can be written as $L(x,u) = \bar{L}(x,u) + \sum_{i=r+1}^m u_ig_i(x)$.
By the fact that $u_i \geq 0$ and $g_i \leq 0$, $\sum_{i=r+1}^m u_ig_i(x) \leq 0$, so $L(x,u) \leq \bar{L}(x,u)$. It follows, by the fact that $\bar{X} \subseteq X$ that

$$v^* = inf_{x \in X} L(x,u) \leq inf_{x \in \bar{X}} L(x,u) \leq inf_{x \in \bar{X}} \bar{L}(x,u) = \bar{v}^*$$

And $v^* \leq \bar{v}^* \leq f^*$ follows by weak duality.

<Question 4>

(a) Approach 1:

$$\begin{gathered} \min\ f_1(x) \\ \text{s.t. } f_j(x) \leq b_j,\ j = 2, ..., s \\ g_i(x) \leq 0,\ i = i, ..., m \end{gathered}$$

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Approach 2:

$$\begin{aligned} &\min \textstyle\sum_{j=1}^{s} w_j f_j(x) \\ &\text{s.t. } g_i(x) \leq 0,\ i = 1, ..., m \end{aligned}$$

(b) Denote (P_1) to be the problem in Approach 1, (P_2) to be that in Approach 2.
Suppose x' solves (P_2) and $b_j = f_j(x')$, $j = 2, ..., s$. We want to show there exists contradiction if x' does not solve P_1. Suppose x' is not optimal for P_1, then there exists $x : f_1(x) < f_1(x')$ and is feasible, so $f_j(x) \leq f_j(x')$, $j = 2, ..., s$. It follows that

$$\sum_{j=1}^{s} w_j f_j(x) = w_1 f_1(x) + \sum_{j=2}^{s} w_j f_j(x) \leq w_1 f_1(x) + \sum_{j=2}^{s} w_j f_j(x') \leq w_1 f_1(x') + \sum_{j=2}^{s} w_j f_j(x') = \sum_{j=1}^{s} w_j f_j(x')$$

which is a contradiction since x' is supposed to be optimal for P_2.

(c) For (P_1): If x is optimal to (P_1), then there exists $(u_2, ..., u_s, v_1, ..., v_m)$ which together with x satisfies the following conditions:
primal feasibility:

$$\begin{aligned} g_i(x) &\leq 0, i = 1, ..., m \\ f_j(x) &\leq b_j, j = 2, ..., s \end{aligned}$$

dual feasibility:

$$(u_2, ..., u_s, v_1, ..., v_m) \geq 0$$

complementarity:

$$\begin{aligned} (f_j(x) - b_j)u_j &= 0, j = 2, ..., s \\ g_i(x)v_i &= 0, i = 1, ..., m \end{aligned}$$

gradient condition:

$$\nabla f_1(x) + \sum_{j=2}^{s} \nabla f_j(x) u_j + \sum_{j=1}^{m} \nabla g_j(x) v_j = 0$$

For (P_2): If x is optimal to (P_2), then there exists $(v_1, ..., v_m)$ which together with x satisfies the following conditions:
primal feasibility:

$$g_i(x) \leq 0, i = 1, ..., m$$

dual feasibility:

$$(v_1, ..., v_m) \geq 0$$

complementarity:

$$g_i(x)v_i = 0, i = 1, ..., m$$

gradient condition:

$$\sum_{j=1}^{s} \nabla f_j(x) w_j + \sum_{j=1}^{m} \nabla g_j(x) v_j = 0$$

(d) Since all the functions are convex and a constraint qualification is satisfied, both (P_1) and (P_2) are convex problems. Therefore, their KKT conditions are necessary and sufficient. Suppose x solves (P_1), then x with $(u_2, ..., u_s, v_1, ..., v_m)$ satisfies KKT conditions for (P_1). If the weights in P_2 are set as follows: $w_1 = 1$, $w_j = u_j$, then the x and $(v_1, ..., v_m)$ derived from (P_1) will also satisfies KKT conditions for (P_2). This means that x also solves (P_2).

<Question 5>
Write problem (3) in Lagrangian function:

$$L(x_{ij}, v) = \sum_{(i,j)\in A} (\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij}) + \sum_{i\in N} v_i (\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ij})$$

And use it to formulate the KKT conditions of problem (3):

$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij}$$

$$\nabla_{x_{ij}} L(x_{ij}, v) = R_{ij}x_{ij} - t_{ij} - v_i + v_j = 0$$

It is observed that the first condition corresponds to law (1) while second to law (2). Therefore, a point that satisfies the two laws, or the KKT conditions, will be the global solution of the problem.
Now we need to show the uniqueness of the solution. Suppose x^* is optimal, then we can show that $f(x^*) < f(x)$ for any other feasible point x.
Since f is a strictly convex quadratic function,

$$f(x) = f(x^*) + \nabla f(x^*)^T(x - x^*) + (x - x^*)^T \nabla^2 f(x^*)(x - x^*)$$

Since $\nabla^2 f(x^*) = diag(R_{ij})$ where $(i, j) \in A$ and $x - x^* \neq 0$, the last term is strictly greater than 0.
Since x, x^* are feasible, $x - x^*$ also satisfies

$$\sum_{j:(i,j)\in A} x_{ij} - x^*_{ij} = \sum_{j:(j,i)\in A} x_{ji} - x^*_{ji}$$

And we know that $\frac{\partial f}{\partial x_{ij}} = R_{ij}x_{ij} - t_{ij} = v_i - v_j$. So the middle term can be written as

$$\sum_{(i,j)\in A} (v_i - v_j)(x_{ij} - x^*_{ij}) = \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) - \sum_{(i,j)\in A} v_j(x_{ij} - x^*_{ij}) =$$

$$\sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) - \sum_{(j,i)\in A} v_i(x_{ji} - x^*_{ji}) = \sum_{(j,i)\in A} v_i(x_j - x^*_{ij}) - \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) = 0$$

So $f(x) > f(x^*)$ and the optimal solution is thus unique.