IOE 511: Continuous Optimization Methods; Homework 10

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1.1 a

We can show if $\bar{x}$ is an optimal solution for P_c, which means for all feasible $x, (x-c)^T(x-c) \geq (\bar{x}-c)^T(\bar{x}-c)$. Then we have $x^Tx - \bar{x}^T\bar{x} \geq 2c^T(x-\bar{x})$. Since P_c, P_d have the same feasible region, $x, \bar{x}$ are both feasible for $P_d.(x-d)^T(x-d) - (\bar{x}-d)^T(\bar{x}-d) = (x^Tx - \bar{x}^T\bar{x}) - 2d^T(x-\bar{x}) \geq 2(c-d)^T(x-\bar{x}) = -2\frac{1-e^Tc}{n}e^T(x-\bar{x})$. As $e^Tx = 1, e^T\bar{x} = 1, -2\frac{1-e^Tc}{n}e^T(x-\bar{x}) = 0$. Then $(x-d)^T(x-d) \geq (\bar{x}-d)^T(\bar{x}-d)$, so optimal solutions of P_c are also optimal solutions for P_d. The same, if $\bar{x}$ is an optimal solution for P_d, we'll have $x^Tx - \bar{x}^T\bar{x} \geq 2d^T(x-\bar{x}), (x-c)^T(x-c) - (\bar{x}-c)^T(\bar{x}-c) \geq 2(d-c)^T(x-\bar{x}) = 2\frac{1-e^Tc}{n}e^T(x-\bar{x}) = 0$. So optimal solutions of P_d are also optimal solutions for P_c. Then P_c, P_d have the same optimal solutions.

1.2 b

As $e^Td = 1, d \geq 0$, $x^* = d$ is a feasible solution. As $f(x) \geq 0, f(x^*) = 0$ and x^* is feasible, x^* is an optimal solution.

1.3 c

Suppose the optimal solution has $x_j^* > 0$. Consider a direction f where $e^Tf = 0, f_j < 0, f_i \geq 0, i \neq j$. Then f is a feasible direction. Consider $\nabla f(x^*)^Tf = 2f_j(x_j^* - d_j) + \sum_{i\neq j} 2f_i(x_i^* - d_i)$, as $\sum x_i^* = \sum d_i = 1, x_j^* > 0 > d_j, \sum_{i\neq j}(x_i^* - d_i) < 0$, let $f_i = -\frac{f_j}{n-1}, i \neq j$, then $\nabla f(x^*)^Tf < 0$ and d is a feasible descent direction, which contradicts with x^* is a optimal solution. So $x_j^* = 0$.

1.4 d

Let $d = c + \frac{1-e^Tc}{n}e$, if there exists $d_j < 0$, the corresponding $x_j = 0$. Define the remaining elements of x as x^1 and the remaining elements of d as d^1, n^1 is the length of x^1. Let $d^2 = d^1 + \frac{1-e^Td^1}{n^1}e, e \in \mathbb{R}^{n^1}$. If there exists $d_j^2 < 0$, set the corresponding elements of x^1 and x to 0. Repeat the process until all the elements in d^k after transforming is greater or equal to 0. Then set the corresponding elements of x to d^k.

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If $c = D$, it's obvious that $S_C(y) = S_D(y)$. Now we are going to show if $\forall y, S_C(y) = S_D(y)$, then $C = D$. Suppose $\exists \bar{x} \in C, \bar{x} \notin D$. Then $\exists p, \alpha, \text{s.t.} p^T \bar{x} > \alpha, \forall x \in D, p^T x < \alpha$. As a result $S_D(p) = \sup_{x \in D} p^T x \leq \alpha, S_C(p) \geq p^T \bar{x} > \alpha$, which contradicts with $\forall y, S_C(y) = S_D(y)$. There will also be contradiction if $\exists \bar{x} \in D, \bar{x} \notin C$. So $C = D$.

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According to weak duality, $v^* \leq f^*, \bar{v}^* \leq f^*$. We are going to show $v^* \leq \bar{v}^*. L^*(u) = \inf_{x \in X} f(x) + \sum_{i=1}^m u_i g_i(x), \bar{L}^*(u) = \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x)$. When $x \in \bar{X}$, as $u_i \geq 0, i = 1, \ldots, m, g_i(x) \leq 0, i = r+1, \ldots, m$, then $f(x) + \sum_{i=1}^m u_i g_i(x) \leq f(x) + \sum_{i=1}^r u_i g_i(x)$. As $\bar{X} \subset X, \inf_{x \in X} f(x) + \sum_{i=1}^m u_i g_i(x) \leq \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^m u_i g_i(x) \leq \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^r u_i g_i(x)$. Then $L^*(u) \leq \bar{L}^*(u)$ and $v^* \leq \bar{v}^*$.

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4.1 a

Approach 1: $\min f_1(x), \text{s.t.} f_j(x) \leq b_j, j = 2, \ldots, s, g_i(x) \leq 0, i = 1, \ldots, m$.

Approach 2: $\min \sum_{j=1}^s w_j f_j(x), \text{s.t.} g_i(x) \leq 0, i = 1, \ldots, m$.

4.2 b

Suppose $\bar{x}$ is the optimal solution obtained by Approach 2. Then $\forall x, \text{s.t.} g_i(x) \leq 0, \sum w_j (f_j(x) - f_j(\bar{x})) \geq 0$. As $w_1 > 0, f_1(x) - f_1(\bar{x}) + \sum_{j=2}^s \frac{w_j}{w_1}(f_j(x) - f_j(\bar{x})) \geq 0$. Set $b_j = f_j(\bar{x})$. Obviously $\bar{x}$ is feasible for Approach 1. For x feasible for Approach 1, $f_j(x) \leq b_j = f_j(\bar{x}), j = 2, \ldots, s$. Then $\sum_{j=2}^s \frac{w_j}{w_1}(f_j(x) - f_j(\bar{x})) \leq 0$ for feasible x in Approach 1. As a result $f_1(x) \geq f_1(\bar{x})$ and $\bar{x}$ is also a optimal solution for Approach 1.

4.3 c

Approach 1: If $\bar{x}$ is a feasible optimal solution, $\exists u^1 \in \mathbb{R}^{s-1}, u^2 \in \mathbb{R}^m, u^1 \geq 0, u^2 \geq 0, \text{s.t.} \nabla f_1(\bar{x}) + \sum_{j=1}^{s-1} u_j^1 \nabla f_{j+1}(\bar{x}) + \sum_{i=1}^m u_i^2 \nabla g_i(\bar{x}) = 0, u_j^1 f_{j+1}(\bar{x}) = 0, j = 1, \ldots s-1, u_i^2 g_i(\bar{x}) = 0, i = 1, \ldots, m$.

Approach 2: If $\bar{x}$ is a feasible optimal solution, $\exists u \in \mathbb{R}^m, u \geq 0, \text{s.t.} \sum_{j=1}^s w_j \nabla f_j(\bar{x}) + \sum_{i=1}^m u_i \nabla g_i(\bar{x}) = 0, u_i g_i(\bar{x}) = 0, i = 1, \ldots, m$.

4.4 d

Suppose the optimal solution for Approach 1 is $\bar{x}$. As all functions are convex and a constraint qualification is satisfied, for Approach 1, $z^* = v^*$ and the dual problem has a optimal solution u^*.

We dualize the constraints $f_j(x) \leq b_j$ and define $X = \{x : g_i(x) \leq 0, i = 1, \ldots, m\}$. $L(x,u) = f_1(x) + \sum_{j=1}^{s-1} u_j(f_{j+1}(x) - b_{j+1}), L^*(u) = \inf_{x \in X} f_1(x) + \sum_{j=1}^{s-1} u_j(f_{j+1}(x) - b_{j+1})$. Because of strong dual, $f_1(\bar{x}) = L^*(u^*) \leq L(\bar{x}, u^*) = f_1(\bar{x}) + \sum_{j=1}^{s-1} u_j(f_{j+1}(\bar{x}) - b_{j+1}) \leq f_1(\bar{x})$. Then $\bar{x}$ minimizes $L(x, u^*) = f_1(x) + \sum_{j=1}^{s-1} u_j^*(f_{j+1}(x) - b_{j+1})$. It also minimizes $f_1(x) + \sum_{j=1}^{s-1} u_j^* f_{j+1}(x)$ since $u_j^* b_j$ is a constant term not related to x. Then use the weights $w_1 = 1, w_j = u_{j-1}^*, j = 2, \ldots, s, \bar{x}$ also is an optimal solution for Approach 2.

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As $R_{ij} > 0$ and the constraint is linear, the problem is convex. KKT is the necessary and sufficient condition for optimality. According to KKT condition, $\exists u_1, \ldots, u_N, \text{s.t.} \nabla(\sum_{(i,j) \in A}(0.5 R_{ij} x_{ij}^2 - t_{ij} x_{ij})) + \sum_{k=1}^{N} u_k \nabla(\sum_{l:(k,l) \in A} x_{kl} - \sum_{m:(m,k) \in A} x_{mk}) = 0$. We know if $(i,j) \in A, (j,i) \notin A$. Consider the partial derivative with respect to x_{ij}, $R_{ij} x_{ij} - t_{ij} + u_i - u_j = 0$. Let $v_i = -u_i$, the solution must satisfy Ohm's law. As Kirchhoff's law is the constraint, solution satisfies both laws is the unique optimal to the problem.