IOE 511; Homework 10

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1 Solution:

a.
$d = c + \left(\frac{1-e^Tc}{n}\right)e$, then the objective function of problem P_d can be simplified as follows:

$$\frac{1}{2}||x-d||_2^2 = \frac{1}{2}||x-c-\left(\frac{1-e^Tc}{n}\right)e||_2^2 = \frac{1}{2}||x-c||_2^2 + \frac{1}{2}\left(\frac{1-e^Tc}{n}\right)^2||e||_2^2 - \left(\frac{1-e^Tc}{n}\right)e^T(x-c)$$

$$\Rightarrow \frac{1}{2}||x-d||_2^2 = \frac{1}{2}||x-c||_2^2 + \frac{1}{2}\left(\frac{1-e^Tc}{n}\right)^2||e||_2^2 - \left(\frac{1-e^Tc}{n}\right)(1-e^Tc)$$

$$\Rightarrow \min_x \frac{1}{2}||x-d||_2^2 = \frac{1}{2}\left(\frac{1-e^Tc}{n}\right)^2||e||_2^2 - \left(\frac{1-e^Tc}{n}\right)(1-e^Tc) + \min_x \frac{1}{2}||x-c||_2^2$$

Based on the above equation and point that their feasible sets are the same, we see that their optimal solution is the same (Though their optimal objective value are different).

b.
For any feasible x, the objective value of problem P_d is non-negative. Since $e^Td = 1 \;\; \& \;\; d \geq 0$, $x = d$ is a feasible solution of this problem. And since the objective value for $x = d$ is zero, it is an optimal solution of the P_d.

c.
Based on theorem 3.1.8, since all constraints P_d are linear, the KKT conditions are necessary to characterize a local optimal solution. Thus, if x^* is an optimal solution of P_d, it must be a KKT point. Then x^* together with (u, ν) must satisfy the following constraints:

1. x^* is feasible $\Rightarrow e^Tx^* = 1 \;\; \& \;\; x^* \geq 0$
2. $x^* - d - u + \nu e = 0$
3. $u \geq 0$
4. $u_i x_i^* = 0 \;\; \forall i$

Then we have:

$$x^* - d - u + \nu e = 0 \Rightarrow e^T(x^* - d - u + \nu e) = 0 \Rightarrow e^Tx^* - e^Td - e^Tu + \nu e^Te = n\nu - e^Tu = 0$$

$$\Rightarrow \nu = \frac{1}{n} e^T u \Rightarrow \nu \geq 0$$

If $d_j < 0$ for some j, then assume $x_j^* > 0$ for these j. Thus we have:

$$u_j x_j^* = 0 \;\;\&\;\; x_j^* > 0 \Rightarrow u_j = 0$$

$$x^* = d + u - \nu e \Rightarrow x_j^* = d_j + u_j - \nu > 0 \Rightarrow d_j - \nu > 0$$

and $d_j - \nu > 0$ is a contradiction because $d_j < 0$ and $-\nu \leq 0$. Thus our initial assumption $x_j^* > 0$ is wrong.

$$\Rightarrow x_j^* = 0$$

d. The Algorithm is:
1. set $d = c + \left(\frac{1-e^T c}{n}\right) e$. If $d \geq 0$, $x^* = d$ is the optimal solution of problem P_c.
2. Otherwise, if $\exists j;\;\; d_j < 0$, set $x_j^* = 0$ for these js.
3. In order to calculate the remaining components of x^*, remove elements of c and x for which $d_j < 0$ (That is because we already obtain these values). Then consider this new vector as a new c, and go to step 1.

Based on this algorithm, in each iteration we achieve the optimum value for at least one element of vector x. Therefore, after at most n iteration we get the answer.

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2 Solution:

$\Rightarrow$ Based on the definition of support function, it is obvious that if $C = D$, $\Rightarrow S_C(y) = S_D(y) \;\; \forall y$.

$\Leftarrow$ For this part, I will show that if $S_C(y) = S_D(y) \;\; \forall y \Rightarrow C \subseteq D \;\; and \;\; D \subseteq C$
proof is based on contradiction:
Let's assume $S_C(y) = S_D(y) \;\; \forall y$, but $C \nsubseteq D \Rightarrow \exists \bar{x} \in C; \bar{x} \notin D$. Since D is a non-empty closed convex set and $\bar{x}$ does not belong to this set, there exists hyperplane $p \neq 0$ and α such that $p^T \bar{x} > \alpha \;\; and \;\; p^T x < \alpha \;\; \forall x \in D$. We can consider this hyperplane p as a y. Then we have:
1. $y^T \bar{x} > \alpha \Rightarrow S_C(y) = sup\{y^T x | x \in C\} \geq y^T \bar{x} > \alpha \Rightarrow S_C(y) > \alpha$
2. $y^T x < \alpha \;\; \forall x \in D \Rightarrow sup\{y^T x | x \in D\} \leq \alpha \Rightarrow S_D(y) \leq \alpha$

Therefore, $S_C(y)$ and $S_D(y)$ can nor be equal and this is a contradiction. Thus our initial assumption $C \nsubseteq D$ was wrong and $C \subseteq D$.

If we follow the same steps as above we will see that $D \subseteq C$. Therefore, $C = D$

3 Solution:

These two problems are different representations of the same problem and so the objective value of both of them is f^*. Then based on weak duality we have: $\nu^* \leq f^*$ and $\bar{\nu}^* \leq f^*$. Therefore, in order to show $\nu^* \leq \bar{\nu}^* \leq f^*$ we just need to show: $\nu^* \leq \bar{\nu}^*$.

The domain of x is different in these two problems and we have: $\bar{X} \subseteq X$. If assume $u \in R^m$ and $\bar{u} \in R^r$ are respectively dual variables of problem P and $\bar{P}$, then we have:

$$L^*(u) = min_{x \in X} f(x) + \sum_{i=1}^{m} u_i g_i(x) \leq min_{x \in \bar{X}} f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

$$= min_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} u_i g_i(x) + \sum_{i=r+1}^{m} u_i g_i(x) \leq min_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} u_i g_i(x) = L^*(\bar{u}) \leq \bar{\nu^*}$$

The last conclusion is because $g_i(x) \leq 0$ and $u_i \geq 0 \Rightarrow \sum_{i=r+1}^{m} u_i g_i(x) \leq 0$.
Since objective value of any feasible solution of problem D is less than or equal the optimal solution of problem $\bar{D}$, the optimal solution of problem D is also less than or equal the optimal solution of problem $\bar{D}$. Therefore:

$$\nu^* \leq \bar{\nu}^* \leq f^*$$

4 Solution:

a.

$$\begin{gathered} \text{P1: } \min_{x \in R^n} f_1(x) \\ \text{s.t. } f_j(x) \leq b_j \ \ \forall j = 2, ..., s \\ g_i(x) \leq 0 \end{gathered}$$

$$\begin{gathered} \text{P2: } \min_{x \in R^n} \sum_{j=1}^{s} w_j f_j(x) \\ \text{s.t. } g_i(x) \leq 0 \end{gathered}$$

b.
Let's assume x^* is an optimal solution of problem P2 with objective $z_2^* = \sum_{j=1}^{s} w_j f_j(x^*)$. Then if for problem P1 we set $b_j = f_j(x^*) \ \ \forall j = 2, ..., s$, then this point will be feasible for P1. That is because (1) x^* is feasible for P2 and so $g_i(x^*) \leq 0 \ \ \forall i = 1, ..., m$, and (2) based on the assigned values for b_js we have: $f_j(x^*) \leq b_j \ \ \forall j = 2, ..., s$.
By contradiction I will show that this point x^* is also an optimal solution of P1.
Proof: Let us assume x^* is not an optimal solution of P1, and so there exists feasible solution $\bar{x}$ such that $f_1(\bar{x}) < f_1(x^*)$. On the other hand, since $\bar{x}$ is feasible for P1, we can make two conclusions about this point:
(1) $g_i(\bar{x}) \leq 0 \ \ \forall i = 1, ..., m$ and so, $\bar{x}$ is also feasible solution for problem P2.
(2) $f_j(\bar{x}) \leq b_j \ \ \forall j = 2, ..., s \Rightarrow f_j(\bar{x}) \leq f_j(x^*) \ \ \forall j = 2, ..., s$.
Then we have:

$$\sum_{j=1}^{s} w_j f_j(\bar{x}) = w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(\bar{x}) \leq w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(x^*)$$

The above result is because $f_j(\bar{x}) \leq f_j(x^*) \ \ \forall j = 2, ..., s$ and $x_j \geq 0 \ \ \forall j = 2, ..., s$. Besides, since $w_1 > 0$ and $f_1(\bar{x}) < f_1(x^*)$, we have:

$$w_1 f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(x^*) < w_1 f_1(x^*) + \sum_{j=2}^{s} w_j f_j(x^*) = z_2^*$$

and this is a contradiction, because we could find a feasible solution for P2 with smaller objective value (thus x^* cannot be optimal). Therefore, our initial assumption was wrong and so: point x^* is also an optimal solution of P1.

c.
If constraint qualification is satisfied for both problems, first order necessary conditions for optimality of these problems are:
x^1 is an optimal solution of P1 if there exists multipliers $(u_1^1, ..., u_m^1, u_2^2, ..., u_s^2)$ such that:

$$g_i(x^1) \le 0 \ \ \forall i = 1, ..., m \ \ \& \ \ f_j(x^1) \le b_j \ \ \forall j = 2, ..., s$$

$$\nabla f_1(x^1) + \sum_{i=1}^{m} u_i^1 \nabla g_i(x^1) + \sum_{j=2}^{s} u_j^2 \nabla f_j(x^1) = 0$$

$$u_i^1, u_j^2 \ge 0 \ \ \forall i = 1, ..., m \ \ \& \ \ j = 2, ..., s$$

$$u_i^1 g_i(x^1) = 0 \ \ \forall i = 1, ..., m \ \ \& \ \ u_j^2(f_j(x^1) - b_j) = 0 \ \ \forall j = 2, ..., s$$

x^2 is an optimal solution of P2 if there exists multipliers $(u_1, ..., u_m)$ such that:

$$g_i(x^2) \le 0 \ \ \forall i = 1, ..., m$$

$$\sum_{j=1}^{s} w_j \nabla f_j(x^2) + \sum_{i=1}^{m} u_i \nabla g_i(x^1) = 0$$

$$u_i \ge 0 \ \ \forall i = 1, ..., m$$

$$u_i g_i(x^2) = 0 \ \ \forall i = 1, ..., m$$

d.
Since all functions are convex, and constraint qualification is satisfied, KKT conditions are necessary and sufficient for optimality of both problems. Therefore, assume x^* is an optimal solution of P1, then there must exists multipliers $(u_1^1, ..., u_m^1, u_2^2, ..., u_s^2)$ satisfying KKT conditions. Then in we set weights of problem P2 as: $w_1 = 1$ and $w_i = u_i^2 \ \ \forall i = 2, ..., s$, then we have:
(1) $g_i(x^*) \le 0 \ \ \forall i = 1, ..., m$ based on first line of KKT conditions in P1.
(2) $\sum_{j=1}^{s} w_j \nabla f_j(x^*) + \sum_{i=1}^{m} u_i^1 \nabla g_i(x^*) = \nabla f_1(x^*) + \sum_{i=1}^{m} u_i^1 \nabla g_i(x^*) + \sum_{j=2}^{s} u_j^2 \nabla f_j(x^*) = 0$ based on second line of KKT conditions in P1.
(3) $u_i^1 \ge 0 \ \ \forall i = 1, ..., m$, based on third line of KKT conditions in P1.
(4) $u_i^1 g_i(x^*) = 0 \ \ \forall i = 1, ..., m$, based on forth line of KKT conditions in P1.
Therefore, for x^* together with $(u_1^1, ..., u_m^1)$, KKT conditions hold. Thus x^* is an optimal solution of P2.

5 Solution:

The problem is:

$$\begin{array}{c} \min \sum_{(i,j)\in A} \left(\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij}\right) \\ \text{s.t. } \sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij} \ \ \forall i \in N \end{array}$$

In this problem, objective function and constrains are convex functions, and also constrains are linear, KKT conditions are necessary and sufficient for optimality. Therefore, if x is an optimal solution of P1, then there must exists multipliers $(u_1^1, ..., u_m^1, u_2^2, ..., u_s^2)$ satisfying KKT conditions. Considering $H_i(x) = \sum_{j:(j,i)\in A} x_{ij} - \sum_{j:(i,j)\in A} x_{ij} \Rightarrow \nabla H_i(x) = \begin{cases} -1 & (i,j) \in A \\ 1 & (j,i) \in A \end{cases}$. Then, the KKT conditions for this problem are:

(1) $\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij} \;\; \forall i \in N$
(2) $\sum_{(i,j)\in A} (R_{ij}x_{ij} - t_{ij}) + \sum_{(i,j)\in A}(v_j - v_i) = 0$

Therefore, if there exists current $x_{ij}, (i,j) \in A$ that satisfies law (1) and (2), it will also satisfies KKT conditions. Thus, it will be an optimal solution of our problem. (satisfying the first KKT condition is obvious based on Kirchhoffs law. For second KKT constraint, if based on Ohms law we have $v_i - v_j = R_{ij}x_{ij} - t_{ij}$. then by taking sum on them we have: $\sum_{(i,j)\in A} (R_{ij}x_{ij} - t_{ij}) + \sum_{(i,j)\in A}(v_j - v_i) = 0$ which is the second KKT condition.)

Now we need to show that this point is unique. For that purpose, let us assume there are two current $x_{ij}, (i,j) \in A$ and $\bar{x}_{ij}, (i,j) \in A$ both satisfying Kirchhoffs and Ohms laws. Since both currents satisfy Ohms law, we have:

$$v_i - v_j = R_{ij}x_{ij} - t_{ij} \;\; \forall (i,j) \in A$$

$$v_i - v_j = R_{ij}\bar{x}_{ij} - t_{ij} \;\; \forall (i,j) \in A$$

$$\Rightarrow x_{ij} = \bar{x}_{ij} \;\; \forall (i,j) \in A$$

Thus these two currents are exactly the same, and so, any point satisfying Kirchhoffs and Ohms laws, is the unique solution of the mentioned problem.