IOE 511 Homework 10

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Problem 1

(a) First, (P_c) and (P_d) has the same feasible region. Then, for the objective function, we have

$$
\begin{aligned}
\frac{1}{2}||x-d||_2^2 &= \frac{1}{2}\sum_{i=1}^{n}(x_i-d_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(x_i-c_i-\frac{1-e^Tc}{n}\right)^2 \\
&= \underbrace{\frac{1}{2}\sum_{i=1}^{n}(x_i-c_i)^2}_{(a)} + \underbrace{\frac{(1-e^Tc)^2}{2n}}_{(b)} - \underbrace{\sum_{i=1}^{n}(x_i-c_i)\frac{1-e^Tc}{n}}_{(c)} \\
(a) &= \frac{1}{2}||x-c||_2^2 \\
(c) &= \frac{1-e^Tc}{n}\left(\sum_{i=1}^{N}x_i-\sum_{i=1}^{N}c_i\right) = \frac{1-e^Tc}{n}(1-e^Tc) = \frac{(1-e^Tc)^2}{n} \\
\Rightarrow \frac{1}{2}||x-d||_2^2 &= \frac{1}{2}||x-c||_2^2 - \frac{(1-e^Tc)^2}{2n}
\end{aligned}
$$

which means that the objective functions of two problems only differs in a constant, which does not depend on x. Therefore, (P_c) and (P_d) have the same optimal solutions.

(b) Without constraint, $\frac{1}{2}||x-d||_2^2 \geq 0$. Also, $e^Td = e^Tc + \frac{1-e^Tc}{n}e^Te = e^Tc + 1 - e^Tc = 1$. If $d \geq 0$, then $x^\star = d$ satisfies all constraints, and $\frac{1}{2}||x^\star - d||_2^2 = 0$. Therefore $x^\star = d$ is an optimal solution of (P_d).

(c) $f(x) = \frac{1}{2}||x-d||_2^2, g_i(x) = -x_i, h(x) = e^Tx - 1, \nabla f(x^\star) = x^\star - d, \nabla g_i(x^\star) = -e_i, \nabla h(x^\star) = e$.
Since $\nabla g_i(x^\star), i = 1, \ldots, n$ are linearly independent, and $\nabla h(x^\star)$ has only one vector, KKT condition is necessary for a local opmimum. Also, $f(x), g_i(x), i = 1, \ldots, n$ are convex functions, and $h(x)$ is linear, (P_d) is a convex problem, KKT is sufficient for a global optimum. According to KKT condition, $x^\star$ satisfies

$$
\begin{gathered}
\nabla f(x^\star) + \sum_{i=1}^{n} u_i \nabla g_i(x^\star) + v\nabla h(x) = 0, \\
u_i \geq 0, i = 1, \ldots, n, \\
u_i g_i(x^\star) = 0, i = 1, \ldots, n
\end{gathered}
$$

which implies

$$\begin{cases} x_1^\star - d_1 - u_1 + v = 0 \\ \vdots \\ x_n^\star - d_n - u_n + v = 0 \end{cases}$$

Summing all the equations, along with the fact that $e^T x^\star = 1, e^T d = 1$, we have

$$v = \frac{1}{n} \sum_{i=1}^{n} u_i \geq 0$$

Suppose for some $d_j < 0$ and $x_j^\star > 0$, then $u_j = 0$, and $x_j^\star - d_j - u_j + v > 0$, which contradicts the KKT condition. Therefore, if $d_j < 0$ for some j, then $x_j^\star = 0$ in any optimal solution.

(d) Given c, compute d. If $d \geq 0$, then $x^\star = d$. Otherwise, find all $d_j < 0$, eliminate them, making the corresponding $x_j^\star = 0$ and $d_j = 0$, which forms the new c. Then repeat this process. In every step, if $1 - e^T c < 0$, then at least one $x_j^\star$ can be computed (set 0); if $1 - e^T c \geq 0$, then $d \geq 0$, $x^\star = d$. Thus, (P_c) can be solved in at most n steps.

Problem 2

10 (1) First, if $C = D$, then for all y, if $S_C(y) = y^T x_1, x_1 \in C$, then we can find $x_2 = x_1, x_2 \in D$, such that $S_D(y) = y^T x_2 = S_C(y)$. Therefore, $\forall y \in \mathbb{R}^n, S_C(y) = S_D(y)$.

(2) Then, show that if the support functions are equal, then $C = D$.
First show that $D \subseteq C$. Suppose there exists a point $x_0 \in D, x_0 \notin C$. Since C is closed and convex, x_0 can be strictly separated from C, i.e., there exists $y \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that $y^T x_0 > b$ and $y^T x < b, \forall x \in C$. This means that

$$\sup_{x \in C} y^T x \leq b < y^T x_0 \leq \sup_{x \in D} y^T x$$

which implies that $S_C(y) \neq S_D(y)$. We showed by contradiction that $D \subseteq C$. By reversing the roles of C and D, we can show that $C \subseteq D$. Therefore, $C = D$.
So, $C = D$ if and only if their support functions are equal.

Problem 3

According to weak duality,

$$\begin{aligned} f^\star &= z^\star = \inf f(x) = \inf \sup \bar{L}(x, u) \\ \bar{v}^\star &= \sup \bar{L}^\star(u) = \sup \inf \bar{L}(x, u) \end{aligned}$$

Therefore, $\bar{v}^\star \le f^\star$. Similarly, $v^\star \le f^\star$.
For the first inequality,

$$\begin{aligned}
L^\star(u) = \inf_{x\in X} L(x,u) &= \inf_{x\in X} f(x) + \sum_{i=1}^{m} u_i g_i(x) \\
&\le \inf_{x\in X} f(x) + \sum_{i=1}^{r} u_i g_i(x) \\
&\le \inf_{x\in \bar{X}} f(x) + \sum_{i=1}^{r} u_i g_i(x) \\
&= \inf_{x\in \bar{X}} \bar{L}(x,u) = \bar{L}^\star(u)
\end{aligned}$$

Let $L^\star(u) = \bar{L}^\star(u) + \xi(u)$, where $\xi(u) \le 0$. Thus,

$$\begin{aligned}
v^\star &= \sup_u L^\star(u) = \sup_u(\bar{L}^\star(u) + \xi(u)) \\
&\le \sup_u \bar{L}^\star(u) + \sup_u \xi(u) \\
&\le \sup_u \bar{L}^\star(u) \\
&= \bar{v}^\star
\end{aligned}$$

Therefore, we have $v^\star \le \bar{v}^\star \le f^\star$.

Problem 4

(a) Approach 1

$$\begin{array}{lll}
(\mathrm{P}_1) & \min_x & f_1(x) \\
& \text{s.t.} & f_j(x) \le b_j, j = 2,\dots,s \\
& & g_i(x) \le 0, i = 1,\dots,m
\end{array}$$

Approach 2

$$\begin{array}{lll}
(\mathrm{P}_2) & \min_x & f(x) = \sum_{j=1}^{s} w_j f_j(x) \\
& \text{s.t.} & w_1 > 0 \\
& & w_j \ge 0, j = 2,\dots,s \\
& & g_i(x) \le 0, i = 1,\dots,m
\end{array}$$

(b) Let $b_j = f_j(x)$. Prove by contradiction that the optimal solution of (P_2) is same as (P_1). Let $\bar{x}$ be the optimal solution of (P_2), assume that $\bar{x}$ is not optimal of (P_1). Then $\exists \hat{x} \in \mathcal{F}_1,$ s.t. $f_1(\hat{x}) < f_1(\bar{x})$. Then $\hat{x} \in \mathcal{F}_2$. Also, since $\hat{x} \in \mathcal{F}_1$, we have $f_j(\hat{x}) \le f_j(\bar{x}), j = 2,\dots,s$. Therefore, $w_j f_j(\hat{x}) \le w_j f_j(\bar{x}), j = 2,\dots,s$ and $w_1 f_1(\hat{x}) < w_1 f_1(\bar{x})$. Sum these inequalities together, we have

$$\sum_{j=1}^{s} w_j f_j(\hat{x}) < \sum_{j=1}^{s} w_j f_j(\bar{x})$$

However, since $\bar{x}$ is the optimal solution of (P_2), and $\hat{x} \in \mathcal{F}_2$ we have

$$\sum_{j=1}^{s} w_j f_j(\hat{x}) \ge \sum_{j=1}^{s} w_j f_j(\bar{x})$$

which is a contradiction. Therefore, we can always find $b_j = f_j(x)$ such that the optimal solution of (P_2) is also the optimal solution of (P_1).

(c) For (P_1):

$$\nabla f_1(\bar{x}) + \sum_{j=2}^{s} u_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0$$
$$u_i \geq 0, u_i g_i(\bar{x}) = 0, i = 1, \ldots, m$$
$$u_j \geq 0, u_j (f_j(\bar{x}) - b_j) = 0, j = 2, \ldots, s$$

For (P_2):

$$\sum_{j=1}^{s} w_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0$$
$$w_1 > 0$$
$$w_j \geq 0, j = 2, \ldots, s$$
$$u_i \geq 0, u_i g_i(\bar{x}) = 0, i = 1, \ldots, m$$

(d) Since all functions are convex and a constraint qualification is satiesfied, for certain $b_j, j = 2, \ldots, s$, we can always find $u_j \geq 0, j = 2, \ldots, s$ that satisfy the KKT conditions of (P_1). Let $w_1 = 1, w_j = u_j, j = 2, \ldots, s$. Then the KKT conditions in (P_2) can also be satisfied, which means that the optimal solution of (P_1) is also the optimal solution of (P_2).

Problem 5

Its obvious that $f(x)$ is convex, $h(x)$ is linear, so this is a convex problem. The KKT condition is

$$\nabla f(\bar{x}) + u \nabla h(\bar{x}) = 0$$

which implies that for any x_{ij},

$$\begin{aligned} &R_{ij} x_{ij} - t_{ij} + u_i(1) + u_j(-1) = 0 \\ \Rightarrow &v_i - v_j + u_i - u_j = 0 \\ \Rightarrow &u_i = v_j, u_j = v_i \text{ is a solution.} \end{aligned}$$

$\bar{x}$ satisfies KKT conditions, which are sufficient for convex problem. So $\bar{x}$ is a global optimum. If x' is also an global optimum, since all constraints are linear, x' must satisfies KKT condition, which is the same as $\bar{x}$, which leads to $x' = \bar{x}$. Therefore, $\bar{x}$ is the unique solution.