IOE511 homework10

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8.5

1

(a) We can show that (P_c) and (P_d) have the same optimal solutions by the following derivation.

$$
\begin{aligned}
(x-d)^T(x-d) &= [x - c - (\frac{1-e^Tc}{n})e]^T[x - c - (\frac{1-e^Tc}{n})e] \\
&= (x-c)^T(x-c) - 2(\frac{1-e^Tc}{n})e^T(x-c) + (\frac{1-e^Tc}{n})^2 e^Te \\
&= (x-c)^T(x-c) - 2(\frac{1-e^Tc}{n})(1-e^Tc) + (\frac{1-e^Tc}{n})^2 e^Te
\end{aligned}
$$

The last equality holds because any feasible points x satisfies $e^Tx = 1$. In this way, we can see that the objective function is just $\frac{1}{2}\|x-c\|_2^2$ with a constant, (P_c) and (P_d) must have the same optimal solutions.

(b) This is easy to show because even under unconstrained case, $x^* = d$ is an optimal solution (because x^* will make the non-negetive objective function to be 0) and it satisfies all the constraints. Here we just show the equality constraint (because we have assumed $d \geq 0$ already):

$$
e^Td = e^T(c + (\frac{1-e^Tc}{n})e) = e^Tc + 1 - e^Tc = 1.
$$

Can use KKT since all constrains are linear!

(c) Since $\nabla h(x)$ and $\nabla g_i(x),\ i = 1, ..., n$ are not linearly independent, here we use Fritz John necessary conditions. By this we have

$$
u_0(x-d) - u + ve = 0
$$

Multiplying this by e, we have

$$
u_0(x-d)^Te - u^Te + ve^Te = -u^Te + nv = 0
$$

The first equality is by primal feasibility of equality constraint. Since by dual feasibility we have $u \geq 0$, $v = \frac{1}{n}u^Te \geq 0$.

Assuming for some index j, $d_j < 0$, if $x_j^* > 0$, we will have

$$
u_0(x_j - d_j) - u_j + v = u_0(x_j - d_j) + v = 0
$$

which means $u_0 = 0$ and $v = 0$. However, we notice that $v = 0$ if and only iff $u = 0$, which makes a contradiction to $(u_0, u, v) \neq 0$.

(d) To solve (P_c), we do:

Firstly, transform the problem into (P_d) using the method in the description.

Secondly, identify all the index such that $d_j < 0$, directly make x_j to be 0.

Thirdly, make new x' and d' using the remaining elements of x and d. Since different elements are separate in the expression of objective function, we can formulate a new problem (P_d'):

$$\begin{aligned} \min\ & \frac{1}{2}\|x' - d'\|_2^2 \\ \text{s.t.}\ & e^T x' = 1 \\ & x' \geq 0 \end{aligned}$$

How?

which we can direclty get the solution $x'^* = d'$.

Finally, we rearrange x' and 0's to get the solution of the original problem.

2 By definition obviously if $C = D$ the support functions are equal.

To show the opposite direction, we first show that $D \subseteq C$. Suppose there exists a point $x_0 \in D$ and $x_0 \notin C$. Since C is closed, x_0 can be strictly separated from C, i.e., there exists an $a \neq 0$ with $a^T x_0 > b$ and $a^T x < b$ for all $x \in C$. This means that

$$\sup_{x \in C} a^T x \leq b < a^T x_0 \leq \sup_{x \in D} a^T x,$$

which implies that $S_C(a) \neq S_D(a)$. Since C and D have symmetric position, we can repeat the argument to draw the conclusion that $C = D$.

3 By weak duality, we already know that $v^* \leq f^*$ and $\bar{v}^* \leq f^*$, which means that we just have to show that $v^* \leq \bar{v}^*$.

Assuming $\bar{u}$ is feasible for $D(\bar{x} \geq 0)$, and $\bar{u}_r$ is the first r elements of $\bar{u}$. Thus $\bar{u}_r$ is feasible for $\bar{D}$. The two dual functions can be expressed as

$$L^*(\bar{u}) = \min_{x \in X} f(x) + \bar{u}^T g(x)$$

$$\bar{L}^*(\bar{u}_r) = \min_{x \in \bar{X}} f(x) + \begin{bmatrix} \bar{u}_r \\ 0 \end{bmatrix}^T g(x)$$

Since we have $\bar{X} \subseteq X$ and $\min_{x \in X} f(x) + \bar{u}^T g(x) \leq \min_{x \in \bar{X}} f(x) + \begin{bmatrix} \bar{u}_r \\ 0 \end{bmatrix}^T g(x)$, these give rise to

$$L^*(\bar{u}) \leq \bar{L}^*(\bar{u}_r)$$

which implies that no matter what $\bar{L}^*$ value we achieve using an arbitrary u_r, we can always a smaller or equal L^* by plugging in a feasible $u = \begin{bmatrix} u_r \\ * \end{bmatrix}$. Thus we have proven that $v^* \leq \bar{v}^*$.

4

(a) Approach 1:

$$\begin{aligned} \min\ & f_1(x) \\ \text{s.t. } & f_j(x) \le b_j, \quad j = 2, ..., s \\ & g_i(x) \le 0, \quad i = 1, ..., m \end{aligned}$$

Approach 2:

$$\begin{aligned} \min\ & \sum_{j=1}^{s} w_j f_j(x) \\ \text{s.t. } & g_i(x) \le 0, \quad i = 1, ..., m \end{aligned}$$

(b)(d) For the first approach, KKT condition looks like

$$\begin{aligned} & \nabla f_1(x) + \sum_{j=2}^{s} u_j \nabla f_j(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0 \\ & u \ge 0, v \ge 0 \\ & f_j(x) \le b_j, \quad j = 2, ..., s \\ & g_i(x) \le 0, \quad i = 1, ..., m \\ & u_j(f_j(x) - b_j) = 0, \quad j = 2, ..., s \\ & v_i g_i(x) = 0, \quad i = 1, ..., m \end{aligned}$$

This is the first order necessary conditions for optimality for both problems.

For the second approach, KKT condition looks like

$$\sum_{j=1}^{s} w_j \nabla f_j(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0$$

Since $w_j \ge 0$ and $w_1 > 0$, from the correspondence of other constraints, these two problems are equivalent.

5 The corresponding KKT conditions are

$$\begin{aligned} R_{ij} x_{ij} - t_{ij} + \sum_{k=1}^{N} v_k (\mathbb{1}_{(k=i)} - \mathbb{1}_{(k=j)}) = 0, \text{ for } (i,j) \in A \\ \sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ji}, \text{ for } i \in N \end{aligned}$$

which is exactly Kirchhoff's law and Ohm's law. The multipliers of equality constraints just stand for the corresponding voltages.

Since this problem is convex, both laws is the solution. Since the objective function is quadratic with respect to every x_{ij}, the solution is unique.