IOE511 HW10

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1.
(a)
We have $\frac{1}{2}||x-d||_2^2 = \frac{1}{2}(x-d)^T(x-d) = \frac{1}{2}[x-c-(\frac{1-e^Tc}{n}e)]^T[x-c-(\frac{1-e^Tc}{n}e)] \Rightarrow$
$= \frac{1}{2}(x-c)^T(x-c) - (\frac{1-e^Tc}{n})e^T(x-c) + \frac{1}{2}(\frac{1-e^Tc}{n})^2 e^Te$, as $e^Tx = 1$, we can get

$$\min \frac{1}{2}||x-d||_2^2 = \min \frac{1}{2}(x-c)^T(x-c) = \min \frac{1}{2}||x-c||_2^2$$

So that (P_c) and (P_d) have the same optimal solutions.
(b)
As $e^Td = 1 \Rightarrow e^Tx^* = 1, d \geq 0 \Rightarrow x^* \geq 0$, so that x^* is a feasible solution.
On the other hand, $\frac{1}{2}||x-d||_2^2 \geq 0, \frac{1}{2}||x^*-d||_2^2 = 0 \Rightarrow x^*$ is an optimal solution.
(c)
We have $\nabla g_i(x) = e_i$, which are linearly independent. So that x^* must satisfy KKT conditions: $\nabla f(x) + \sum u_i \nabla g_i(x) + v\nabla h(x) = 0, u_i \geq 0, i = 1, \ldots, n, u_i g_i(x) = 0. \Rightarrow$ nv = sum u_i
$x_i - d_i - u_i + v = 0, i = 1, \ldots, n$. By combining all n equations we can get $v = \sum_{i=1}^n u_i \geq 0$.
If $x_j^* > 0$, then $u_j^* = 0$, we have $x_j - d_j - u_j + v > 0$, yielding the contradiction needed. So that $x_j^* = 0$.
(d)
Calculate d, if $d \geq 0$, then $x^* = d$. Otherwise, set $x_j^* = 0$ for $d_j < 0$. Eliminate x_j, d_j in the objective function to form a new function $\frac{1}{2}||x-c_2||_2^2$, where c_2 is d eliminating negative entries. It's trivial to see x_i^*, where $i \neq j$ in the original problem is the same as the new optimal x_i^* in the new problem. Then calculate d_2 and repeat the step. At each step, at least one x_i^* can be found so that it takes at most n steps to find all x_i^*.

2.
If $C = D$, it's trivial to see $S_C(y) = S_D(y)$.
For $S_C(y) = S_D(y)$, if $C \neq D$, without loss of generality, assume there exists $x^* \in C, x^* \notin D$, as D is a closed convex set, then there exists $p \neq 0$ and α, such that for all x in $D, p^Tx < \alpha$, and $p^Tx^* > \alpha$. As $x^* \in C$, we have $S_C(p) > S_D(p)$, yielding the contradiction needed.

3.
We have

$$L^*(u) = \inf_{x \in X} f(x) + \sum_{i=1}^{m} g_i(x) \le \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{m} g_i(x) \le \inf_{x \in \bar{X}} f(x) + \sum_{i=1}^{r} g_i(x) = \bar{L}^*(u),$$

so that $L^*(u) \le \bar{L}^*(u)$ for all u. So that $v^* = \sup_u L^*(u) \le \sup_u L^*(u) = \bar{v}^*$.
Combining weak duality, we can get $v^* \le \bar{v}^* \le f^*$.

4.
(a)
Approach 1:

$$\begin{aligned} &\min f_1(x) \\ &s.t. \quad -f_j(x) - b_j \le 0, j = 2, \dots, s \\ &\qquad\quad g_i(x) \le 0, i = 1, \dots, m \end{aligned}$$

Approach 2:

$$\begin{aligned} &\min \textstyle\sum_{j=1}^{s} w_j f_j(x) \\ &s.t. \quad g_i(x) \le 0, i = 1, \dots, m \end{aligned}$$

(b)
Suppose the optimal solution for Approach 2 is $\bar{x}$, then we can find corresponding target level for approach 1 which is $b_j = f_j(\bar{x})$.
It's trivial to see that $\bar{x}$ is a feasible solution in Approach 1. If $\bar{x}$ is not an optimal solution, then we can find $\hat{x}$ such that $f(\hat{x}) < f(\bar{x})$ and satisfies all the constraints in Approach 1. So that $\sum_{j=1}^{s} w_j f_j(\hat{x}) < \sum_{j=1}^{s} w_j f_j(\bar{x})$, and $\hat{x}$ satisfy all the constraints in Approach 2, which means $\bar{x}$ is not an optimal solution for Approach 2, yielding the contradiction needed.
(c)
Approach 1:

$$\nabla f_1(\bar{x}) + \sum_{i=2}^{s} \nabla f_i(\bar{x}) u_i + \sum_{j=1}^{m} \nabla g_j(\bar{x}) v_j = 0,$$

$$[u \ v]^T \ge 0,$$

$$u_i f_i(\bar{x}) = 0, i = 2, \dots, s, v_j g_j(\bar{x}) = 0, j = 1, \dots, m$$

Approach 2:

$$\sum_{i=1}^{s} w_i f_i(x) + \sum_{j=1}^{m} \nabla g_j(\bar{x}) v_j = 0,$$

$$v \geq 0,$$

$$v_j g_j(x) = 0, j = 1, \dots, m$$

(d)
Suppose the optimal solution for approach 1 is $\bar{x}$, then it satisfies the first order necessary conditions. Let $\frac{w_i}{w_1} = u_i, i = 2, \dots, s$, as Approach 2 is now a convex problem, we can find that $\bar{x}$ satisfies KKT sufficient conditions. So that $\bar{x}$ is an optimal solution in Approach 2.

5.
As $f(x), h(x)$ are all convex, x_{ij} evidently belongs to a convex set, the problem is a convex problem. From KKT sufficient condition, we have $\nabla f(x) + u\nabla h(x) = 0$, that is for any x_{ij}, $\frac{\partial f(x)}{\partial x_{ij}} + u_i(-1) + u_j(1) = 0 \Rightarrow v_i - v_j - u_i + u_j = 0$. So that for the solution $\bar{x}$ we can find $u_i = v_i$ for all i, such that $\bar{x}$ satisfies KKT sufficient condition.$\Rightarrow$ $\bar{x}$ is an optimal solution. If there is another optimal solution, as all constraints are linear, it must satisfy KKT necessary condition, which are the same conditions that will lead to $\bar{x}$. So that $\bar{x}$ is the unique solution.