8.5
Problem 1

(a)

First let's expend the objective function of (P_d). Let $a = \frac{1-e^Tc}{n}e$

$$
\begin{aligned}
&\frac{1}{2}||x-d||_2^2 \\
=&\frac{1}{2}\sum_i (x_i - d_i)^2 \\
=&\frac{1}{2}\sum_i (x_i - c_i - a_i)^2 \\
=&\frac{1}{2}\sum_i (x_i^2 - 2c_ix_i + c_i^2 + 2a_ix_i - 2c_ia_i + a_i^2) \\
=&\frac{1}{2}\sum_i (x_i^2 - 2c_ix_i + c_i^2 + 2(\frac{1-e^Tc}{n}e)_ix_i - 2c_ia_i + a_i^2) \\
=&\frac{1}{2}(||x-c|| + 2\frac{1-e^Tc}{n} - 2c^Ta + ||a||)
\end{aligned}
$$

Thus th objective function of (P_d) is $\frac{1}{2}(||x-c||+2\frac{1-e^Tc}{n} - 2c^Ta + ||a||)$. Because $2\frac{1-e^Tc}{n} - 2c^Ta + ||a||$ is constant, the objective function is mainly related to $||x-c||$. It is similar to the objective function of (P_c), which is also mainly related to $||x-c||$. So the optimal solution of P_c, P_d are the same.

(b)

First, let's check if $x^* = d$ is feasible for this solution.

$$e^Tx^* = e^Td = 1$$

$$x^* = d \geq 0$$

So $x^* = d$ is feasible for P_d. And because $\frac{1}{2}||x-d||_2^2 \geq 0$, the minimum value of P_d is 0 and

$$\frac{1}{2}||x^*-d||_2^2 = \frac{1}{2}||d-d||_2^2 = 0$$

We get the minimum value. Thus $x^* = d$ is an optimal solution of (P_d).

(c)

$$\frac{1}{2}||x-d||_2^2 = \frac{1}{2}\sum_{i=1}^n (x_i-d_i)^2 \geq 0$$

$$L(x,u) = \frac{1}{2}\sum_{i=1}^n (x_i-d_i)^2 + u_E(e^Tx - 1)$$

$$\frac{\partial L(x,u)}{\partial x} = x - d + u_Ee = 0$$

dimension mismatch.
no e, right?

$$x_i = d_i - u_E e$$

$$\frac{\partial^2 L(x,u)}{\partial^2 x} = e > 0$$

Left side is in R^nxn, e \in R^n, “>” can only be use in R

Because it is a quadratic function, on the right hand side of $d_i - u_E e$, it is monotone increase. So the optimal solution of this problem is $x^* = max\{d - u_E, 0\}$. Then the Lagrangian dual is

$$L^*(u) = inf \quad L(x, u_E)$$

$$= \frac{1}{2}||u_E e|| + u_E(e^T(d - u_E e) - 1)$$

$$= -\frac{nu_E^2}{2}$$

The dual problem is

$$sup \quad -\frac{nu_E^2}{2}$$

When $u_E = 0$, we get the optimal value. If $d_j < 0$, then $d - u_E \leq 0$, and $x^* = 0$. Thus, if $d_j < 0$ for some index j, then $x_j^* = 0$ in any optimal solution of (P_d).

(d)
Step 1: Let $d = c + \frac{1-e^T c}{n} e$, $k = 1$
Step 2: If $d^k > 0$, we get optimal solution x_j^*, Stop. Otherwise, if $d_j^k < 0$ for some index j, then $x_j^* = 0$.
Step 3: $k = k + 1$, delete the c_j^k and e_j^k, get new vector c_j^k and e_j^k.
Step 4: $d^k = c^k + \frac{1-(e^k)^T c}{n} e^k$, go to step 2.

Problem 2

(i) If $C = D$, then $S_C(y) = S_D(y)$.
It is clear that if $C = D$, then the following two systems

$$S_C(y) = sup\{y^T x : x \in C\}$$

$$S_D(y) = sup\{y^T x : x \in D\}$$

are exactly the same. Then they will give the same result.

(ii) If $S_C(y) = S_D(y)$, then $C = D$.(Proof by contradiction)
Assume that $C \not\subset D$, then there exist $\bar{x}_c \in C$, $\bar{x}_c \notin D$.
By theorem $B.3.1$, we know that there exists a nonzero vector p and scalar α such that $p^T \bar{x}_C > \alpha$ and $p^T x_D < \alpha, \forall x_D \in D$. Then let $y = p$, then

$$S_C(y) > \alpha, S_D(y) < \alpha$$

, which is conflict with,

$$S_C(y) = S_D(y)$$

Thus our assumption is incorrect, then we know $C \subset D$.
Then let's prove in other direction. Assume that $D \not\subset C$, then there exist $\bar{x}_D \in D$, $\bar{x}_D \notin C$.
By theorem $B.3.1$, we know that there exists a nonzero vector p and scalar α such that $p^T\bar{x}_D > \alpha$ and $p^T x_C < \alpha, \forall x_C \in C$. Then let $y = p$, then

$$S_D(y) > \alpha, S_C(y) < \alpha$$

, which is conflict with,

$$S_C(y) = S_D(y)$$

Thus our assumption is incorrect, then we know $D \subset C$. Now we can conclude that because of $D \subset C, C \subset D$, $C = D$.

Problem 3

By Lagrangian dual theorem,

$$L(x, u) = f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

$$L^*(u) = inf_x \quad f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

Assume the optimal solution is $x^* \in X$,

$$v^* = sup_u \quad f(x^*) + \sum_{i=1}^{m} u_i g_i(x^*)$$

$$u_i \geq 0$$

Because $u_i \geq 0, g_i(x^*) \leq 0$, $u_i g_i(x^*) \leq 0$, by weak duality

$$v^* = sup_u \quad f(x^*) + \sum_{i=1}^{m} u_i g_i(x^*) \leq f(x^*) = f^*$$

Then let's consider another case. By Lagrangian dual theorem,

$$\bar{L}(x, u) = f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

$$\bar{L}^*(u) = inf_x \quad f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

Assume the optimal solution is $\bar{x}^* \in X$,

$$\bar{v}^* = sup_u \quad f(\bar{x}^*) + \sum_{i=1}^{r} u_i g_i(\bar{x}^*)$$

$$u_i \geq 0$$

Because $u_i \geq 0, g_i(x^*) \leq 0$, $u_i g_i(x^*) \leq 0$, by weak duality

$$\bar{v}^* = sup_u \quad f(\bar{x}^*) + \sum_{i=1}^{r} u_i g_i(\bar{x}^*) \leq f(\bar{x}^*) = f^*$$

Now let's consider

$$L^*(u) = inf_{x \in X} \quad f(x) + \sum_{i=1}^{r} u_i g_i(x) + \sum_{i=r+1}^{m} u_i g_i(x)$$

Since $\bar{X} \subset X$, $u_i \geq 0, g_i(x^*) \leq 0$, $u_i g_i(x^*) \leq 0$, for any u, we have

$$\begin{aligned} L^*(u) &= inf_{x \in X} \quad f(x) + \sum_{i=1}^{r} u_i g_i(x) + \sum_{i=r+1}^{m} u_i g_i(x) \\ &\leq inf_{x \in \bar{X}} \quad f(x) + \sum_{i=1}^{r} u_i g_i(x) + \sum_{i=r+1}^{m} u_i g_i(x) \\ &\leq \bar{L}^*(u) \\ &\leq \bar{v}^* \end{aligned}$$

then $(sup_u \quad L^*(u)) = v^* \leq \bar{v}^*$. Due to former proof, we have $v^* \leq \bar{v}^* \leq f^*$.

Problem 4

(a)

(i)

$$\begin{aligned} min \quad & f_1(x) \\ f_j(x) \leq b_j \quad & j = 2, ..., s \\ g_i(x) \leq 0 \quad & i = 1, ..., m \end{aligned}$$

(ii)

$$\begin{aligned} min \quad & f(x) = \sum_{j=1}^{s} w_j f_j(x) \\ g_i(x) \leq 0 \quad & i = 1, ..., m \end{aligned}$$

(b)

We would like to prove that every solution $\bar{x}^2$ obtained by Approach 2 for some selection of weights w_j with $w_1 > 0$, we can find target levels b_j such that the same solution $\bar{x}^1$ is optimal in Approach 1. Here $\bar{x}^1 = \bar{x}^2$.

Proof by contradiction, assume $\bar{x}^1 \neq \bar{x}^2$, $\bar{x}^1, \bar{x}^2$ are optimal solution for approach 1 and approach 2. Let $b_j = f(\bar{x}^2)$, then we have

$$\sum_{j=1}^{s} w_j f_j(\bar{x}^1) \leq w_1 f(\bar{x}^1) + \sum_{j=2}^{s} w_j b_j \quad \leq \sum_{j=1}^{s} w_j f(\bar{x}^2)$$

Thus, $\bar{x}^2$ is not an optimal solution of approach 2, which is contradict to assumption. Then, $\bar{x}^1 = \bar{x}^2$.

(c)

(i) The first order KKT condition of approach 1.

$$\nabla f_1(\bar{x}) + \sum_{j=2}^{s} u_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0$$

$$u_i \geq 0$$
$$u_j \geq 0$$
$$u_j(f_j(x) - b_j) = 0$$
$$f_j(x) \leq b_j$$
$$g_i(x) \leq 0$$
$$u_i g_i(x) = 0$$
$$i = 1, ..., m$$
$$j = 2, ..., s$$

(ii) The first order KKT condition of approach 2.

$$\sum_{j=1}^{s} w_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0$$

$$u_i \geq 0$$
$$g_i(x) \leq 0$$
$$u_i g_i(x) = 0$$
$$i = 1, ..., m$$

(d)

If this is a convex optimization problem, then by theorem 3.2.1, the KKT point is an optimal solution. We would like to prove that a KKT point $\bar{x}^1$ of approach 1 with b_j is the optimal solution, then there exist w_j, such that $\bar{x}^1 = \bar{x}^2$, $\bar{x}^2$ satisfy the KKT condition of approach 2.
Then for $\bar{x}^1$ and u_j, u_i, that satisfied the KKT condition for approach 1, we have

$$\nabla f_1(\bar{x}^1) + \sum_{j=2}^{s} u_j \nabla f_j(\bar{x}^1) = -\sum_{i=1}^{m} u_i \nabla g_i(\bar{x}^1)$$

$$u_i \geq 0$$
$$g_i(\bar{x}^1) \leq 0$$
$$u_i g_i(\bar{x}^1) = 0$$
$$i = 1, ..., m$$

take it into the following equation, and because $\bar{x}^1 = \bar{x}^2$

$$\begin{aligned}
&\sum_{j=1}^{s} w_j \nabla f_j(\bar{x}^2) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}^2) \\
=&\sum_{j=1}^{s} w_j \nabla f_j(\bar{x}^2) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}^1) \\
=&- f_1(\bar{x}^1) - \sum_{j=2}^{s} u_j \nabla f_j(\bar{x}^1) + \sum_{j=1}^{s} w_j \nabla f_j(\bar{x}^2) \\
=&- f_1(\bar{x}^2) - \sum_{j=2}^{s} u_j \nabla f_j(\bar{x}^2) + \sum_{j=1}^{s} w_j \nabla f_j(\bar{x}^2) \\
=&(w_1 - 1)\nabla f_1(\bar{x}^2) + \sum_{j=1}^{s} (w_j - u_j) \nabla f_j(\bar{x}^2)
\end{aligned}$$

If $w_1 = 1 > 0$, $w_j = u_j$, it is clear that

$$\begin{aligned}
&\sum_{j=1}^{s} w_j \nabla f_j(\bar{x}^2) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}^2) \\
=&(w_1 - 1)\nabla f_1(\bar{x}^2) + \sum_{j=1}^{s} (w_j - u_j) \nabla f_j(\bar{x}^2) \\
=&0
\end{aligned}$$

Also because $\bar{x}^1 = \bar{x}^2$

$$\begin{gathered}
u_i \geq 0 \\
g_i(\bar{x}^1) = g_i(\bar{x}^2) \leq 0 \\
u_i g_i(\bar{x}^1) = u_i g_i(\bar{x}^2) = 0 \\
i = 1, ..., m
\end{gathered}$$

are also satisfied. In this condition, $\bar{x}^2$ and w_j satisfied the KKT condition is an optimal solution, and $\bar{x}^1 = \bar{x}^2$.

Problem 5

Because $\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij}$ is an affine constraint, this problem satisfy the constraint quality. Also the objective function $f(x) = \sum_{(i,j)\in A}(\frac{1}{2}R_{ij}x_{ij}^2 - t_{ij}x_{ij})$ is a convex function, because

$$\frac{\partial^2 f(x)}{\partial x^2} = R_{ij} > 0$$

Thus, KKT condition is sufficient to optimal condition. The KKT condition is following

$$(R_{ij}x_{ij} - t_{ij}) + u_i - u_j = 0$$

$$\sum_{j:(i,j)\in A} x_{ij} = \sum_{j:(j,i)\in A} x_{ij}$$

Then we get an unique optimal solution $x_{ij} = \frac{t_{ij}-u_i+u_j}{R_{ij}}$. Because $(R_{ij}x_{ij} - t_{ij}) = v_i - v_j$. Let $u_j - u_i = v_i - v_j$, then the current $x_{ij}, (i,j) \in A$ that satisfies both laws (1) and (2) is the unique solution of the following nonlinear programming problem, $x_{ij} = \frac{t_{ij}-v_j+v_i}{R_{ij}}$.