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IOE: Homework 10

Problem 1

9.5

(a) Expand the objective function of (P_c) and (P_d) we have:

$$(\mathrm{P}_c) \quad \frac{1}{2}\|x-c\|_2^2 = \frac{1}{2}(x^Tx - 2x^Tc + \underbrace{c^Tc}_{\textcircled{1}})$$

$$(\mathrm{P}_d) \quad \frac{1}{2}\|x-d\|_2^2 = \frac{1}{2}(x^Tx - 2x^Tc - 2\underbrace{\frac{1-e^Tc}{n} + \frac{2c^Te(1-e^Tc)}{n} + 1 - e^Tc}_{\textcircled{2}})$$

In the above two equations, ① and ② are all constants(because c, n are all fixed) so they do not affect the final optimization result. Plus, (P_c) and (P_d) have the same constraint. Thus, the two problems have the same optimal solutions.

(b) Apparently, $\frac{1}{2}\|x-d\|_2^2 \geq 0$. If $x^* = d$, the objective function reach the lower bound. Plus we already know $e^Td = 1$, so $e^Tx^* = 1$. If $x^* = d \geq 0$, then it also meets all constraints. So it is the an optimal solution of P_d when $d \geq 0$.

(c)

It is obvious to see (P_d) is a convex optimization problem and all constraints are linear. Use KKT condition:

$$\begin{cases} (x-d) + ve - u = 0 & (1) \\ e^Tx = 1 & (2) \\ x \geq 0 & (3) \\ u_ix_i = 0, i = 1, 2, \ldots, n & (4) \\ u \geq 0 & (5) \\ u \in \mathbb{R}^n, v \in \mathbb{R} & (6) \end{cases}$$

Proof by contradiction:

Assume $d_j < 0, x_j^* \neq 0$, by Equ. 1 and Equ. 4, We have $x_j - d_j + v - u_j = 0, u_j = 0 \Rightarrow v < 0$③. For the rest $d_j \geq 0, x_j - u_j = d_j - v > 0 \Rightarrow x_j > 0, u_j = 0$. Therefore, $\forall j, u_j = 0$, $e^T((x-d) + ve - u) = n^2v = 0$ contradicts with ③. Therefore, $d_j < 0, x_j^* = 0$ in all optimal solutions. **Q.E.D.** = nv - e^Tu

(d)

1. Calculate d by c, if $d \geq 0$, set $x = d$, terminate the algorithm; if $\exists d_j < 0$, set $x_j = 0$.

2. If all x_j are determined, terminate the algorithm, otherwise put rest entry of d to be c and rest entry of x to be new x, go to 1.

In the above algorithm, at every run of 1., at least one entry of x can be determined(which means one negative entry of d pops out.), thus at most n steps we need in total.

Problem 2

If $C = D$, it is obvious that $S_C(y) = S_D(y)$.

If $S_C(y) = S_D(y)$, use contradiction to prove $C = D$.

Assume, when $S_C(y) = S_D(y)$①, $\exists x \in C, x \notin D$. $\because C, D$ are closed, $\therefore x \notin \partial D$. $\because C, D$ are convex. By separating hyperplane theorem, $\exists y \in \mathbb{R}^n, \alpha \in \mathbb{R}, y^Tx > \alpha$ and $\forall x_d \in D, y^Tx_d < \alpha$②. However by ①, $\forall y \in \mathbb{R}^n, x \in C$, we can always find $x_d \in D$, $y^Tx_d = y^Tx$, which contradicts with ②. Similarly, we can easily prove, when $S_C(y) = S_D(y)$, $x \in D, x \in C$. Therefore, $S_C(y) = S_D(y), C = D$.

In all, $C = D$ if and only if $S_C(y) = S_D(y)$. **Q.E.D.**

Problem 3

$$
\begin{aligned}
&L^*(u) = \inf_{x\in X} f(x) + u^T g(x), \quad v^* = \sup_{u\geq 0} L^*(u) \\
&\bar{L}^*(\bar{u}) = \inf_{x\in \bar{X}} f(x) + \bar{u}^T \bar{g}(x), \quad \bar{v}^* = \sup_{\bar{u}\geq 0} \bar{L}^*(\bar{u}) \\
&\bar{u} \in \mathbb{R}^r, \quad \bar{g}(\cdot) = \left[g_1(\cdot), g_2(\cdot), \ldots, g_r(\cdot)\right]^T
\end{aligned}
$$

We can easily prove $f^* \geq v^*, f^* \geq \bar{v}^*$: $\because u \geq 0, g(x) \leq 0$, $f(x^*) \geq f(x^*) + u^T g(x^*) \geq \inf_{x\in X} f(x) + u^T g(x) = L^*(u)$①, so for every feasible u, ① stands, so does $f(x^*) \geq \sup_{u\geq 0} L(u) = v^*$. If $L^*(u)$ is minus infinity, dual problem is effectively infeasible, the result is trivial. Similarly, we can prove $f^* \geq \bar{v}^*$.

$\because$ the primal optimal is feasible, $\therefore \bar{X} \neq \emptyset$.

Set $u = \begin{bmatrix} \bar{u} \\ u' \end{bmatrix}$, $u' \in \mathbb{R}^{m-r}, u' \geq 0$, $\because g_i(x) \leq 0, u' \geq 0$ and apparently $\bar{X} \subseteq X$, $\therefore \bar{L}^*(\bar{u}) = \inf_{x\in\bar{X}} f(x) + \bar{u}^T\bar{g}(x) \geq \inf_{x\in X} f(x) + \bar{u}^T\bar{g}(x) \geq \inf_{x\in X} f(x) + u^T g(x)$. Then $\sup_{u\geq 0} \bar{L}^*(\bar{u}) \geq \bar{L}^*(\bar{u}) \geq L^*(u), \forall \bar{u} \geq 0, u' \geq 0 \Rightarrow \sup_{\bar{u}\geq 0} \bar{L}^*(\bar{u}) \geq \sup_{u\geq 0} L^*(u) \Rightarrow \bar{v}^* \geq v^*$

If $\bar{L}^*$ is negative infinity, from above derivation, L^* must also be negative infinity. Then both dual problems are effectively infeasible.

In all, we have $f^* \geq \bar{v}^* \geq v^*$. **Q.E.D.**

Problem 4

(a)

$$
\begin{aligned}
(\mathrm{P}_{A1}) \quad & \min \quad f_1(x) \\
& \text{s.t.} \quad f_j(x) \leq b_j, j = 2, \ldots, s \\
& \qquad\;\; g_i(x) \leq 0, i = 1, \ldots, m \\
(\mathrm{P}_{A2}) \quad & \min \quad \sum_{j=1}^{s} w_j f_j(x) \\
& \text{s.t.} \quad g_i(x) \leq 0, i = 1, \ldots, m \\
& \qquad\;\; w_j \geq 0, j = 1, \ldots, s; \quad w_1 > 0
\end{aligned}
$$

(b) For some vector $\bar{x}$ that is the optimal in Approach 2, $\forall x$ is feasible in (P_{A2}), we have:

$$
\begin{aligned}
& \sum_{j=1}^{s} w_j f_j(x) \geq \sum_{j=1}^{s} w_j f_j(\bar{x}) \\
\Rightarrow \quad & f_1(x) + \sum_{j=2}^{s} w_j f_j(x) \geq f_1(\bar{x}) + \sum_{j=2}^{s} w_j f_j(\bar{x}) \\
\Rightarrow \quad & f_1(x) - f_1(\bar{x}) \geq \sum_{j=2} \frac{w_j}{w_1}\Big(f_j(\bar{x}) - f_j(x)\Big) \qquad (7)
\end{aligned}
$$

Apparently, the feasible set of (P_{A1}) belongs to (P_{A2}), so Equation 7 stands for (P_{A1}) too. If $\bar{x}$ is also an optimal for (P_{A1}), (7) should be larger or equal than 0 for all feasible points. So just set $b_j = f_j(\bar{x})$ for $w_j > 0$, $b_j \geq f_j(\bar{x})$ for $w_j = 0$, because $w_j \geq 0$, then (7)≥ 0 and $\bar{x}$ is feasible in (P_{A1}). So in this setting of b_j, $\bar{x}$ is also an optimal of (P_{A1}). **Q.E.D.**

(c)

$$\mathrm{P}_{A1} \quad \begin{cases} \nabla f_1(x) + \sum_{j=2}^{s} u_j \nabla f_j(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0 \\ u_j(f_j(x) - b_j) = 0, j = 2, \ldots, s \\ v_i g_i(x) = 0, i = 1, \ldots, m \\ u_j, v_i \geq 0 \\ f_j(x) \leq b_j, g_i(x) \leq 0 \end{cases}$$

$$\mathrm{P}_{A2} \quad \begin{cases} \sum_{j=1}^{s} w_j \nabla f_j(x) + \sum_{i=1}^{m} v_i \nabla g_i(x) = 0 \\ v_i g_i(x) = 0, i = 1, \ldots, m \\ v_i \geq 0 \\ g_i(x) \leq 0 \end{cases}$$

$w_1 > 0, w_j \geq 0, j = 2, \ldots, s$ is the given value, so not included in the conditions above.

(d) From (c), for P_{A1} we know that if all functions are convex and meet one constraint qualification, the KKT conditions can solve dual problem and primal problem at the same time, strong duality meets, and solution of dual problem attained. So $u_j, j = 2, \ldots, s$ must be solved for some value.

Now, assume $\bar{x}$ is an optimal solution of P_{A1}, let $w_1 = 1$, $w_j = 0$ when $f_j(\bar{x}) < b_j$, $w_j = u_j$ when $f_j(\bar{x}) = b_j$ and v_i has the same value as P_{A1} in P_{A2}. $\therefore$ then we can easily see the first order necessary optimization condition between the two problem is the same. Because, all functions are convex and meet slater condition, they are also sufficient conditions. $\therefore \bar{x}$ is optimal in P_{A2} when $w_1 = 1, w_j = u_j, j = 2, \ldots, s$

Problem 5

Abbreviate Kirchhoff's current law to law (K) and abbreviate Ohm's law to law (O). If current $x_{ij}, (i,j) \in A$ that satisfies law (K) and law (O). Then it is at least a feasible solution of the nonlinear programming problem. Since we know there is a set of current value that meets the equality constraint and the objective function is apparently convex, so the problem has a unique optimal with unique solution and can attain its dual solution.

Write the first order derivative optimization condition:

$$R_{ij} x_{ij} - t_{ij} = w_i - w_j \tag{8}$$

The original Lagrangian function is:

$$\sum_{(i,j)\in A} \left(\frac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij}\right) + \sum_{i=1}^{N} w_i \Big(\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ij} \Big), \quad w_i \in \mathbb{R}$$

Since $x_{ij}, (i,j) \in A$ satisfies law (O): $v_i - v_j = R_{ij} x_{ij} - t_{ij}$, then compared with Equation 8, set $w_i = v_i, w_j = v_j$, now the $x_{ij}, (i,j) \in A$ which satisfies (O), (K) satisfies all KKT conditions of a convex optimization problem with a constraint qualification, thus it is the unique solution of the optimization problem.