IOE 511 / Math 562– Homework 10

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1 Problem 1

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(a).

It is clear that the feasible region of the (P_c) and (P_d) are the same. We call it F. Then we only need to show that $\min_{x\in F}\|x-c\|_2^2$ is equivalent with $\min_{x\in F}\|x-d\|_2^2$.

$$
\begin{aligned}
\|x-d\|_2^2 &= (x-d)^{\mathrm{T}}(x-d) = x^{\mathrm{T}}x - 2x^{\mathrm{T}}d + d^{\mathrm{T}}d \\
&= x^{\mathrm{T}}x - 2x^{\mathrm{T}}(c + \frac{1-e^{\mathrm{T}}c}{n}e) + (c + \frac{1-e^{\mathrm{T}}c}{n}e)^{\mathrm{T}}(c + \frac{1-e^{\mathrm{T}}c}{n}e) \\
&= x^{\mathrm{T}}x + 2x^{\mathrm{T}}c - 2\frac{1-e^{\mathrm{T}}c}{n}x^{\mathrm{T}}e + c^{\mathrm{T}}c + 2c^{\mathrm{T}}e(\frac{1-e^{\mathrm{T}}c}{n}e + n\cdot(\frac{1-e^{\mathrm{T}}c}{n}e)^2)
\end{aligned}
$$

Since for any $x \in F$, $x^{\mathrm{T}}e = 1$, $\min_{x\in F}\|x-d\|_2^2$ is equivalent with $\min_{x\in F} x^{\mathrm{T}}x + 2x^{\mathrm{T}}c$, which is equivalent with $\min_{x\in F}\|x-c\|_2^2$.

So we have shown that (P_c) and (P_d) have the same optimal solutions.

(b).

If $d \geq 0$, then $x^* = d$ is a feasible solution. Also $\|x^*-d\|_2^2 = \inf_x \|x-d\|_2^2$. So $x^* = d$ is an optimal solution of (P_d).

(c). It is easy to check that problem (P_d) satisfied slater condition. So if x^* is an optimal solution, then it must satisfy KKT conditions. There exist nonzero vector $(u, v) \in \mathbb{R}^{n+1}$ such that

$$
\begin{aligned}
& u_i x_i^* = 0, i = 1, 2, \cdots, n \\
& e^{\mathrm{T}}x = 1; -x \leq 0 \\
& x^* - d - u + ve = 0 \Rightarrow u = x^* - d + ve
\end{aligned}
$$

If $d_j < 0$ and $x_j^* > 0$, then $u_j = 0$. So we have $v = -(x_j^* - d_j) < 0$.

On the other hand, since $d^{\mathrm{T}}e = 1$, and $u^{\mathrm{T}}e = \sum_{i=1}^n u_i \geq 0$, we have

$$
x^{\mathrm{T}}e - d^{\mathrm{T}}e + ve^{\mathrm{T}}e = nv = u^{\mathrm{T}}e \geq 0 \Rightarrow v \geq 0
$$

This is a contradiction. So $x_j^* = 0$ in any optimal solutions of (P_d).

(d).

To solve (P_c), first calculate d and then solve the problem (P_d). For any j such that $d_j < 0$, set $x_j = 0$. Let I be the set of index such that $\forall i \in I, d_i \geq 0$.

If $|I| = n$, i.e. $d \geq 0$, then set $x^* = d$. The optimal solution is found.

Otherwise $|I| < n$ and we need to solve the problem below:

$$
\begin{aligned}
\min \ & \sum_{i \in I} (x_i - d_i)^2 \\
s.t. \ & \sum_{i \in I} x_i = 1 \\
& x_i \geq 0, \ \forall i \in I
\end{aligned}
$$

This is a similar problem as (P_c), but with lower dimension. So we can repeat the process above. At each step, either we have $d \geq 0$ and thus the algorithm stops, or the dimension of the problem is reduced by at least 1. So within n steps, the algorithm will stop.

2 Problem 2

It is obvious that if $C = D$, then their support functions are equal. Now suppose the support functions of two closed convex set C and D are equal, then we want to show that $C = D$.

If $C \neq D$, then there exists a point $x_0 \in C$ but $x_0 \notin D$. Since D is a closed convex set, there must be a vector y such that for any $w \in D$:

$$
y^{\mathrm{T}} w < y^{\mathrm{T}} x_0
$$

Then we have

$$
S_D(y) \leq \sup_{w \in D} y^{\mathrm{T}} w < y^{\mathrm{T}} x_0 \leq S_C(y)
$$

This contradicts the assumption that $S_C = S_D$. So we have proved that if the support functions of two closed convex set C and D are equal, then $C = D$.

3 Problem 3

We write the Lagrangian functions and Lagrangian dual functions of (P) and $(\bar{P})$ as the following:

$$
\begin{aligned}
L(x,u) &= f(x) + \sum_{i=1}^{m} u_i g_i(x), x \in X, u \geq 0 \\
\bar{L}(\bar{x},\bar{u}) &= f(\bar{x}) + \sum_{i=1}^{r} \bar{u}_i g_i(\bar{x}), \bar{x} \in \bar{X}, \bar{u} \geq 0 \\
L^*(u) &= \inf_{x\in X} L(x,u) \\
\bar{L}^*(\bar{u}) &= \inf_{\bar{x}\in \bar{X}} \bar{L}(\bar{x},\bar{u})
\end{aligned}
$$

To show that $v^* \leq \bar{v}^* \leq f^*$, we only need to show that for any feasible point x for (P)(or equivalently for $(\bar{P})$), and any $u \geq 0$ we have $L^*(u) \leq L^*(\bar{u}) \leq f(x)$, where $u \in \mathbb{R}^m$ and $\bar{u}$ is the first r component of u.

For any feasible point x_0 for (P), $g_i(x_0) \leq 0$ for $i = 1, 2, \cdots, m$, and for any $u \geq 0$, we have

$$
\sum_{i=1}^{m} u_i g_i(x_0) \leq 0, \quad \sum_{i=1}^{r} u_i g_i(x_0) \leq 0, \quad \sum_{i=1}^{m} u_i g_i(x_0) \leq \sum_{i=1}^{r} u_i g_i(x_0)
$$

which follows

$$
L(x_0,u) \leq \bar{L}(x_0,\bar{u}) \leq f(x_0)
$$

So $L^*(u) = \inf_{x\in X} L(x,u) \leq \bar{L}(x_0,u) \leq f(x_0)$ and similarly $\bar{L}^*(\bar{u}) \leq f(x_0)$.

For any $x_1 \in \bar{X}$, we have $g_i(x_0) \leq 0$ for $i = r+1, r+2, \cdots, m$. Then for any $u \geq 0$,

$$
L^*(u) \leq L(x_1,u) \leq \bar{L}(x_1,\bar{u})
$$

Now take the infimum over all $x_i \in \bar{X}$, we have

$$
L^*(u) \leq \bar{L}^*(\bar{u})
$$

So we have shown that $L^*(u) \leq \bar{L}^*(\bar{u}) \leq f(x_0)$ for any $u \geq 0$ and feasible point x_0 for (P). So $\sup_{u\geq 0} L^*(u) \leq \sup_{\bar{u}\geq 0} \bar{L}^*(\bar{u}) \leq \inf_{x \text{ feasible}} f(x)$, i.e. $v^* \leq \bar{v}^* \leq f^*$.

4 Problem 4

(a).
The nonlinear optimization problem of Approach 1is

$$\begin{aligned} &\min\ f_1(x) \\ &s.t.\ f_j(x) \leq b_j\ j = 2, 3, \cdots, s. \\ &\quad\ \ g_i(x) \leq 0\ i = 1, 2, \cdots, m. \end{aligned}$$

The nonlinear optimization problem of Approach 2 is

$$\begin{aligned} &\min\ \sum_{i=1}^{s} w_i f_i(x) \\ &s.t.\ g_i(x) \leq 0\ i = 1, 2, \cdots, m. \end{aligned}$$

(b). For some weight with $w_1 > 0$ and the corresponding optimal solution of Approach 2 x^*, let $b_j = f_j(x^*)$ for $j = 2, \cdots, s$. Now prove that x^* is also an optimal solution for Approach 1 given these b_j.

Suppose x^* is not the optimal solution, then there exists x' such that

$$g_i(x') \leq 0,\ i = 1, 2, \cdots, m$$
$$f_j(x') \leq b_j = f_j(x^*),\ j = 2, \cdots, s.$$
$$f_1(x') < f_1(x^*)$$

So this x' is also feasible for (P2), and

$$\sum_{i=1}^{s} w_i f_i(x') < \sum_{i=1}^{s} w_i f_i(x^*)$$

which contradicts that x^* is optimal for (P2). So x^* is optimal in Approach 1.

(c).

If a constraint qualification is satisfied for both problem. Then we can write down the first order necessary conditions for optimality for both problems as below.

For (P1), there exists a vector $x \in \mathbb{R}^n$ and $u = (u^1, u^2) \in \mathbb{R}^{s-1+m}$ such that

$$
\begin{aligned}
&g_i(x) \le 0,\ i = 1, 2, \cdots, m \\
&f_j(x) \le b_j,\ j = 2, \cdots, s. \\
&\nabla f_1(x) + \sum_{i=1}^{s-1} u_i^1 \nabla f_{i+1}(x) + \sum_{i=1}^{m} u_i^2 \nabla g_i(x) = 0 \\
&u_j^1(f_{j+1}(x) - b_{j+1}) = 0,\ j = 2, \cdots, s. \\
&u_j^2 g_j(x) = 0,\ j = 1, \cdots, m. \\
&(u^1, u^2) \ge 0
\end{aligned}
$$

For (P2), there exists a vector $x \in \mathbb{R}^n$ and $u = \in \mathbb{R}^m$ such that

$$
\begin{aligned}
&g_i(x) \le 0,\ i = 1, 2, \cdots, m \\
&\sum_{i=1}^{s} w_i \nabla f_i(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) = 0 \\
&u_j^2 g_j(x) = 0,\ j = 1, \cdots, m. \\
&u \ge 0
\end{aligned}
$$

(d).

If all functions are convex and a constraint qualification is satisfied, the KKT conditions above are necessary and sufficient for an optimal solution.

So x^* is an optimal solution for (P1) if and only if there exist $(u^1, u^2) \in \mathbb{R}^{s-1+m}$ such that the above KKT conditions for (P1) are satisfied.

Now let $w_1 = 1$ and $w_{i+1} = u_i^1$ for $i = 1, 2, \cdots s - 1$. Then it is easy to check that x^* and $u^2 \in \mathbb{R}^m$ satisfy the KKT conditions for (P2), so x^* is also an optimal solution for (P2).

5 Problem 5

Denote $f : \mathbb{R}^{|A|} \to \mathbb{R}, f(x) = \sum_{(i,j)\in A}(\frac{1}{2}R_{ij}x_{ij}^2 - t_{ij}x_{ij})$ and $g_i : \mathbb{R}^{|A|} \to \mathbb{R}, g_i(x) = \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ij}$ for $i \in N$. Then the problem is

$$
\begin{aligned}
&\min\ f(x) \\
&\ s.t.\ g_i(x) = 0,\ i \in N
\end{aligned}
$$

Since $R_{ij} > 0$, $f(x)$ is a convex function. We also notice that the equality constraints are all linear. So the KKT conditions of this problem is necessary and sufficient for an optimal solution. So $x_{ij}, (i,j) \in A$ is an optimal solution of this problem if and only if there exists a nonzero vector $v \in \mathbb{R}^{|N|}$ such that

$$g_i(x) = \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0, i \in N$$

$$\nabla f(x) + \sum_{k=1}^{|N|} v_k \nabla g_k(x) = 0$$

The condition $\nabla f(x) + \sum_{i=1}^{|N|} v_i \nabla g_i(x) = 0$ means for any $(i,j) \in A$,

$$\frac{\partial f}{\partial x_{ij}} + \sum_{k=1}^{|N|} v_k \frac{\partial g_k}{\partial x_{ij}} = 0$$

Since

$$\frac{\partial f}{\partial x_{ij}} = R_{ij} x_{ij} - t_{ij}, \quad \frac{\partial g_k}{\partial x_{ij}} = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{otherwise} \end{cases}$$

So we get for any $(i,j) \in A$

$$R_{ij} x_{ij} - t_{ij} = v_i - v_j$$

So we have shown that $x_{ij}, (i,j) \in A$ is an optimal solution if and only if $x_{ij}, (i,j) \in A$ satisfies laws (1) and (2).

Then we need to show such solution is unique. Suppose x^* is an optimal solution, i.e. x^* satisfies the above KKT condition. Then we can show that for any other feasible point x, $f(x^*) < f(x)$. This means the optimal solution is unique.

Since f is a strictly convex quadratic function, we have

$$f(x) = f(x^*) + \nabla f(x^*)^{\mathrm{T}}(x - x^*) + (x - x^*)^{\mathrm{T}} \nabla^2 f(x^*)(x - x^*)$$

Since $\nabla^2 f(x^*) = diag(R_i j)$ where $(i,j) \in A$ and $x - x^* \neq 0$, the last term $(x - x^*)^{\mathrm{T}} \nabla^2 f(x^*)(x - x^*)$ is strictly greater than 0.

Since x, x^* are feasible, $x - x^*$ also satisfies

$$\sum_{j:(i,j)\in A} x_{ij} - x^*_{ij} = \sum_{j:(j,i)\in A} x_{ji} - x^*_{ji}$$

And we also know that $\frac{\partial f}{\partial x_{ij}} = R_{ij}x_{ij} - t_{ij} = v_i - v_j$. So the middle term $\nabla f(x^*)^{\mathrm{T}}(x - x^*)$ can be written as

$$\begin{aligned}\sum_{(i,j)\in A}(v_i - v_j)(x_{ij} - x^*_{ij}) &= \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) - \sum_{(i,j)\in A} v_j(x_{ij} - x^*_{ij}) \\ &= \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) - \sum_{(j,i)\in A} v_j(x_{ji} - x^*_{ji}) \\ &= \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) - \sum_{(i,j)\in A} v_i(x_{ij} - x^*_{ij}) = 0\end{aligned}$$

So we have

$$\begin{aligned}f(x) &= f(x^*) + \nabla f(x^*)^{\mathrm{T}}(x - x^*) + (x - x^*)^{\mathrm{T}}\nabla^2 f(x^*)(x - x^*) \\ &= f(x^*) + 0 + (x - x^*)^{\mathrm{T}}\nabla^2 f(x^*)(x - x^*) > f(x^*)\end{aligned}$$

So the optimal solution is uinque.