1 (a).

$$\|x-c\|^2 = (x-c)^T(x-c) = x^Tx - 2c^Tx + c^Tc$$

$$\|x-d\|^2 = (x-d)^T(x-d) = (x^T-d^T)(x-d) = x^Tx - 2d^Tx + d^Td$$

$$d^Td = \left(c^T + \left(\frac{1-e^Tc}{n}\right)e^T\right)d = c^Td + \frac{1-e^Tc}{n}e^Td$$

Since $e^Td = 1$

$$\Rightarrow \quad d^Td = c^Td + \frac{1-e^Tc}{n}$$

$$= c^T\left(c + \left(\frac{1-e^Tc}{n}\right)e\right) + \frac{1-e^Tc}{n}$$

$$= c^Tc + \left(\frac{1-e^Tc}{n}\right)(1+e^Tc)$$

$$-2d^Tx = -2\left(c^T + \left(\frac{1-e^Tc}{n}\right)e^T\right)x$$

$$= -2c^Tx - 2\left(\frac{1-e^Tc}{n}\right)e^Tx$$

Since $e^Tx = 1$

$$\Rightarrow \quad -2d^Tx = -2c^Tx - 2\left(\frac{1-e^Tc}{n}\right)$$

$$\Rightarrow \quad \|x-d\|^2 = x^Tx - 2c^Tx + c^Tc + \left(\frac{1-e^Tc}{n}\right)(e^Tc-1)$$

$$= \|x-c\|^2 - \frac{1}{n}(1-e^Tc)^2$$

$\Rightarrow$ (P_c) and (P_d) have the same optimal solutions.

(b). Rewrite (P_d):

$$(P_d) \quad \min \; f(x)$$
$$\text{s.t.} \quad h_1(x) = 0$$
$$x \in X$$

where, $f(x) := \frac{1}{2}\|x-d\|_2^2$

$h_1(x) = e^T x - 1$

$X = \{x : x \geq 0\}$

Want to show: (P_d) is a convex problem.

Since for any $x^1 \in X$, $x^2 \in X$ and $\forall \lambda \in [0,1]$

$x^1 \geq 0$, $x^2 \geq 0$

$\Rightarrow \lambda x^1 + (1-\lambda) x^2 \geq 0 \Rightarrow \lambda x^1 + (1-\lambda) x^2 \in X$

$\Rightarrow X$ is a convex set.

Also, $h_1(x)$ is a linear function.

$\Rightarrow (P_d)$ is a convex problem.

$f(x) = \frac{1}{2}\|x-d\|_2^2 = \frac{1}{2}(x^T - d^T)(x-d) = \frac{1}{2}(x^T x - 2d^T x + d^T d)$

$\nabla f(x) = \frac{1}{2}(2x - 2d) = x - d$, $\nabla h_1(x) = x$.

For $x = d \geq 0$, try to solve v_1 in the following system:

$$\nabla f(x)\Big|_{x=d} + \nabla h_1(x)\Big|_{x=d} \cdot v_1 = 0$$

$$(x-d)\Big|_{x=d} + x\Big|_{x=d} \; v_1 = 0 \iff (d-d) + v_1 d = 0$$

$\Rightarrow v_1 = 0$.

By KKT sufficient conditions for convex problems,

$x^* = d \geq 0$ is a global optimal solution of (P_d)

(c). If x^* is an optimal solution of (P_d), it must satisfy the KKT Necessary Conditions.

$\Rightarrow \exists v_1$, s.t. $\nabla f(x^*) + \nabla h_1(x^*) v_1 = 0$

$\Rightarrow \exists v_1$, s.t. $(x^* - d) + v_1 x^* = 0$ (1)

$\Rightarrow \exists v_1$. s.t. $x_j^* - d_j + x_j^* \cdot v_1 = 0$, $\forall j = 1, \ldots, n$.

From equation (1), we have

$$e^T \left((x^* - d) + v_1 x^* \right) = 0$$

$$\Rightarrow e^T x^* - e^T d + v_1 e^T x^* = 0$$

Since $e^T x^* = 1$. $e^T d = 1$

$\Rightarrow 1 - 1 + v_1 = 0 \Rightarrow v_1 = 0$

Suppose, for $d_j < 0$, we have $x_j^* \neq 0 \Rightarrow x_j^* > 0$

then $x_j^* - d_j + x_j^* \cdot v_1 = 0 \Rightarrow v_1 = \dfrac{d_j - x_j^*}{x_j^*} < 0$,

which contradicts the fact that $v_1 = 0$!

$\Rightarrow$ Thus, if $d_j < 0$ for some index j, then $x_j^* = 0$ in any optimal solution of (P_d).

(d). Want to solve (P_c).

Step 1. Define $d = c + \left(\frac{1 - e^T c}{n}\right) e$, and define a new problem (P_d).

Then (P_c) and (P_d) have the same optimal solutions.

Step 2. Check all the coordinates of d,

find out all the indexes j with $d_j < 0$, and set

$x_j^* = 0$.

Take all the other coordinates

2. $(\Rightarrow)$.

If $C=D \Rightarrow \forall y \in \mathbb{R}^n$, $S_C(y) = S_D(y)$

$(\Leftarrow)$. if $\forall y \in \mathbb{R}^n$, $S_C(y) = S_D(y)$.

$\Rightarrow$ (P_c) $\sup\limits_{x \in C} y^T x$

(P_d) $\sup\limits_{x \in D} y^T x$

(P_c) and (P_d) have the same optimal solution.

C and D are both convex and closed sets.

Suppose $C \neq D$.

3. (P) $\min f(x)$

$s.t.\ g_i(x) \le 0. \quad i=1\ldots m$

$x \in X$

($\bar{P}$) $\min f(x)$

$s.t.\ g_i(x) \le 0 \ . \ i=1\cdots r$

$x \in \bar{X}$.

where $\bar{X} = \{x \in X : g_{r+1}(x) \le 0, \ldots, g_m(x) \le 0\}$.

Assume this problem is feasible and its optimal value is $f^* > -\infty$.

$L^*(u) := \min_x f(x) + \sum_{i=1}^{m} u_i g_i(x)$

$s.t. \quad x \in X$

$D: \ v^* = \max_u L^*(u)$

$s.t. \quad u_i \geqslant 0 \quad i=1,\cdots,m.$

$\bar{L}^*(\bar{u}) := \min_x f(x) + \sum_{i=1}^{r} \bar{u}_i g_i(x)$

$s.t. \quad x \in \bar{X}$

$\bar{D}: \ \bar{v}^* = \max_{\bar{u}} L^*(\bar{u})$

$s.t. \quad \bar{u}_i \geqslant 0. \quad i=1\cdots r.$

①. WTS: $f^* \geqslant v^*$, $f^* \geqslant \bar{v}^*$.

This is obvious by Weak Duality Theorem.

②. WTS: $\bar{v}^* \geqslant v^*$.