IOE 511 Homework 10

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Problem 1

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(a).

$\hat{x} = \text{argmin} \frac{1}{2}\|x-d\|_2^2 = \text{argmin} \frac{1}{2}\|x-c-\frac{1-e^Tc}{n}e\|_2^2$
$= \text{argmin} \frac{1}{2}(\|x-c\|_2^2 + \|\frac{1-e^Tc}{n}e\|_2^2 - 2\frac{1-e^Tc}{n}e^T(x-c))$
$= \text{argmin} \frac{1}{2}(\|x-c\|_2^2 - 2\frac{1-e^Tc}{n}e^T(x-c))$

Since $e^Tx = 1$ $\frac{1-e^Tc}{n}e^T(x-c) = \frac{1-e^Tc}{n}(1-e^Tc)$ without containing x.

So $\hat{x} = \text{argmin} \frac{1}{2}\|x-d\|_2^2$
$= \text{argmin} \frac{1}{2}(\|x-c\|_2^2 - 2\frac{1-e^Tc}{n}e^T(x-c))$
$= \text{argmin} \frac{1}{2}\|x-c\|_2^2$.

Thus, the optimal solution for (P_d) and (P_c) is the same.

(b).

Note that $e^Td = e^Tc + \frac{1-e^Tc}{n}e^Te = e^Tc + (1-e^Tc) = 1$.

The constraints for (P_d) can be rewritten as $e^Tx - e^Td = e^T(x-d) = 0$.

If $d \geq 0$, $x^* = d$, which satisies all the constaints $\begin{cases} e^Tx^* = e^Td = 1 \\ x^* \geq 0 \end{cases}$.

Also, $0 \leq \frac{1}{2}\|x^*-d\|_2^2 = \frac{1}{2}\|d-d\|_2^2 = 0$.

Then 0 is the optimal value for (P_d), which is achieved by $x^* = d$.

Therefore, $x^* = d$ is the optimal solution of (P_d)

(c).

Since (P_d) satisfies the Slater condition, with a Slater point $x = \frac{1}{n}e$, KKT condition is necessary for the optimal solution.

KKT can be expressed as follows.

$$\begin{cases} (x-d) - u + ve = 0 \\ e^T x = 1, x \geq 0 \\ u_i(-x_i) = 0, u \geq 0 \\ e^T d = 1 \end{cases}$$

From the first equation,

$(x-d) - u + ve = 0$
$\Rightarrow e^T(x-d) - e^T u + ve^T e = 0$
$\Rightarrow 1 - 1 - e^T u + nv = 0$
$\Rightarrow v = \frac{e^T u}{n} \geq 0.$

Still from the first equation,

$(x_j - d_j) - u_j + v = 0$, when $d_j < 0$.

$x_j - u_j + v = d_j < 0$
$\Rightarrow x_j + v < u_j$
$\Rightarrow (x_j + v)x_j \leq u_j x_j = 0$, since $x_j \geq 0$.

Because $x_j^2 \geq 0$, $v \geq 0$ and $x_j \geq 0$, $0 \leq (x_j + v)x_j \leq 0$

Thus, $x_j = 0$.

(d).

Step 1: Let $S = \{1, 2, 3, \cdots, n\}$, $k = 0$, $n^0 = n$

Step 2: $d_i = \begin{cases} c_i + \frac{1 - \sum\limits_{i \in S} c_i}{n^k}, i \in S \\ 0, i \notin S \end{cases}$, where n^k is the number of elements in S.

If $d \geq 0$, then $x^* = d$ and stop.

If $d_j < 0$, $S \leftarrow S \backslash \{j\}$, meaning substracting j from Set S. Goto Step 2.

Use this method, we need at most n steps to solve (P_c)

Problem 2

Proof:

(1). If $C = D$,

For any $y \in \mathbb{R}^n$, $S_C = \sup\{y^T x : x \in C\} = sup\{y^T x : x \in D\} = S_D(y)$

(2). If for any $y \in \mathbb{R}^n$, $S_C(y) = S_D(y)$

Proof by contradiction.

Assume $\exists \bar{x} \in D$, but $\bar{x} \notin C$ with no loss of generality.

By Theorem B.3.1, there exists a hyperplane $H = \{x : p^T x = \alpha\}$ strongly separating the closed convex set C and the point $\bar{x}$.

i.e. $\forall x \in C, p^T x < \alpha$ and $p^T \bar{x} > \alpha$.

Since $S_C(p) = \{p^T x, x \in C\}$ and $S_D(p) = \{p^T x, x \in D\}$, $S_C(p) \leq \alpha$ and $S_D(p) > \alpha$. $S_C(p) \neq S_D(p)$, where contradicts with the condtions.

Thus, $\forall x \in D, x \in C$. Similarly, we can get $\forall x \in C, x \in D$. $C = D$.

Problem 3

Denote $\bar{u} = [u_1, u_2, \cdots, u_r]^T$, $u = [u_1, u_2, \cdots, u_m]^T$

$L(x, u) = f(x) + u^T g(x)$

$\bar{L}(x, \bar{u}) = f(x) + \bar{u}^T \bar{g}(x)$.

For $u = [\bar{u}^T, u_{r+1}, \cdots u_m]^T \geq 0$, $L(x, u) = f(x) + u^T g(x) = f(x) + \bar{u}^T \bar{g}(x) + u_{r+1} g_{r+1}(x) + \cdots + u_m g_m(x) \leq f(x) + \bar{u}^T \bar{g}(x) = \bar{L}(x, \bar{u})$

Because $\bar{X} \in X$,

$\bar{L}^*(\bar{u}) = \inf_{x \in \bar{X}} \bar{L}(x, \bar{u}) \geq \inf_{x \in X} \bar{L}(x, \bar{u}) \geq \inf_{x \in X} L(x, u) = L^*(u)$, where $u = [\bar{u}^T, u_{r+1}, \cdots u_m]^T \geq 0$.

$\bar{v}^* = \sup_{\bar{u} \geq 0} \bar{L}^*(\bar{u}) = \sup_{u \geq 0} \bar{L}^*(\bar{u}) \geq \sup_{u \geq 0} L^*(u) = v^*$.

Since the $(\bar{D})$ is a Lagragian dual problem on $(\bar{P})$ and $(\bar{P})$ has the same optimal solution with (P), $\bar{v}^* \leq f^*$.

Therefore, $v^* \leq \bar{v}^* \leq f^*$

Problem 4

(a).

Approach 1:

$$
\begin{aligned}
(P_1) \min \quad & f_1(x) \\
& f_j(x) \leq b_j, j = 2, 3, \cdots n \\
& g_i(x) \leq 0, i = 1, 2, \cdots, m
\end{aligned}
$$

Approach 2: $f(x) = \sum\limits_{j=1}^{s} w_j f_j(x)$, $w_j \geq 0$ and $w_1 > 0$

$$
\begin{aligned}
(P_2) \min \quad & f(x) \\
& g_i(x) \leq 0, i = 1, 2, \cdots, m
\end{aligned}
$$

(b).

The set $X = \{x \in \mathbb{R}^n, g_i(x) \leq 0, i = 1, 2, \cdots, m\}$

Let z_1^* and z_2^* are the optimal values of (P_1) and (P_2), respectively. And let x_2^* is the optimal solution for (P_2).

$z_2^* = f(x_2^*) \le f(x) = \sum\limits_{j=1}^{s} w_j f_j(x), x \in X$

$\Rightarrow z_2^* = f(x_2^*) \le f(x) = \sum\limits_{j=1}^{s} w_j f_j(x),$ s.t. $x \in X, f_j(x) \le f_j(x_2^*), j = 2, 3, \cdots, s.$

So, $b_j = f_j(x_2^*), j = 2, 3, \cdots, s.$

For x satisfies $x \in X$ and $f_j(x) \le b_j, j = 2, 3, \cdots, s$

$w_1 f_1(x) + \sum\limits_{j=2}^{s} w_j f_j(x) \ge z_2^*$

$\Rightarrow w_1 f_1(x) \ge z_2^* - \sum\limits_{i=2}^{s} w_j f_j(x) \ge z_2^* - \sum\limits_{i=2}^{s} w_j b_j$, because $w_j \ge 0, j = 2, 3, \cdots, s$

$f_1(x) \ge \frac{1}{w_1}(z_2^* - \sum\limits_{i=2}^{s} w_j b_j) \Rightarrow z_1^* \ge \frac{1}{w_1}(z_2^* - \sum\limits_{i=2}^{s} w_j b_j)$. It achieves the equality, when $x = x_2^*$. Thus x_2^* is the optimal solution for Approach 1 when $b_j = f_j(x_2^*), j = 2, 3, \cdots, s$

(c).

Since a constraint qualification is satisfied for both problems, the optimal solution should satisfy the KKT condition.

For (P_1)

The KKT condition is $\exists(x, c, u)$, such that

$$\begin{cases} f_j(x) \le b_j,\ g_i(x) \le 0,\ j = 2, 3, \cdots s,\ i = 1, 2, \cdots, m \\ \nabla f_1(x) + \sum\limits_{j=2}^{s} c_j \nabla f_j(x) + \sum\limits_{i=1}^{m} u_i \nabla g_i(x) = 0 \\ c \ge 0, u \ge 0 \\ c_j(f_j(x) - b_j) = 0,\ u_i g_i(x) = 0,\ j = 2, 3, \cdots, s,\ i = 1, 2, \cdots, m \end{cases}$$

For (P_2)

The KKT condition is $\exists(x, u)$, such that

$$\begin{cases} g_i(x) \le 0,\ i = 1, 2, \cdots m \\ \sum\limits_{j=1}^{s} w_j \nabla f_j(x) + \sum\limits_{i=1}^{m} u_i \nabla g_i(x) = 0 \\ u \ge 0 \\ u_i g_i(x) = 0,\ i = 1, 2, \cdots, m \end{cases},$$

where $w_j \ge 0, j = 1, 2, \cdots, s$ and $w_1 > 0$

(d).

Let z_1^* be the optimal value of (P_1)

There are two possible duals of (P_1).

The first one:

$L_1^1(x, c, u) = f_1(x) + \sum_{j=2}^{s} c_j(f_j(x) - b_j(x)) + \sum_{i=1}^{n} u_i g_i(x).$

$L_1^{1*}(c, u) = \inf_x L_1^1(x, c, u)$

(D_1^1) $v_1^{1*} = \sup L_1^{1^*}(c, u), \text{ s.t. } , c \geq 0, u \geq 0$

The second one:

$L_1^2(x, c) = f_1(x) + \sum_{j=2}^{s} c_j(f_j(x) - b_j(x)).$

$L_1^{2*}(c) = \inf_{x \in X} L_1^1(x, c)$

(D_1^2) $v_1^{2*} = \sup L_1^{2^*}(c), \text{ s.t. } , c \geq 0$

For Approach 1, (P_1) has a constraint qualification and convexity, so the KKT condition is N&S for the global optimal of (P_1), i.e. $\exists(\bar{x}, \bar{c}, \bar{u})$ satisfies the KKT conditions of (P_1) mentioned above.

By the Theorem 7.7.2, $v_1^{1*} = z_1^*$, and $(\bar{x})$ and $(\bar{c}, \bar{u})$ are the optimal solutions for (P_1) and (D_1^1), respectively.

From Problem 3, we know that $v_1^{1*} \leq v_1^{2*} \leq z_1^*$. So $v_1^{2*} = z_1^*$, which means (P_1) and (D_1^2) are strong duals too. All inequalities in the problem 3 achieve equality. So $\bar{x}, \bar{c}$ should be the optimal solutions for (P_1) and D_1^2.

Thus, $\bar{x} = \operatorname{argmin} L_1^2(x, \bar{c})$
$= \operatorname{argmin} f_1(x) + \sum_{j=2}^{s} \bar{c}_j(f_j(x) - b_j(x))$
$= \operatorname{argmin} f_1(x) + \sum_{j=2}^{s} \bar{c}_j f_j(x)$
$= \operatorname{argmin} w_1 f_1(x) + w_1 \sum_{j=2}^{s} \bar{c}_j f_j(x)$
$= \operatorname{argmin} \sum_{j=1}^{s} w_j f_j(x),$

where $w_1 > 0$ and $w_j = w_1 * \bar{c}_j, j = 2, 3, \cdots s$.

In this case, $\bar{x}$ is the optimal solution for both (P_1) and (P_2)

Problem 5

Proof

1. prove the solution of (1) and (2) is the same as the solution of (3)

$f(x) = \sum\limits_{(i,j)\in A} (\frac{1}{2}R_{ij}x_{ij}^2 - t_{ij}x_{ij}) = \frac{1}{2}x^T Rx - t^T x$. R is a diagonal matrix, whose diagonal elements are R_{ij}s, $(i,j) \in A$. Since $R_{ij} > 0$, the matrix R is positive definite.

$H(x) = R$ is positive definite, so $f(x)$ is a strictly convex function.

(3) has no inequality constraints. All the points in the feasible region $\mathcal{F}$ are slater points, so (3) satisfies the Slater condition. $f(x)$ is convex on $\mathcal{F}$, so the KKT condition is the N&S condition for the optimal solution of (3).

Define $h_i(x) = \sum\limits_{j:(j,i)\in A} x_{ji} - \sum\limits_{j:(i,j)\in A} x_{ij}, i \in N$

The KKT condition of (3) is as follows:

$$\begin{cases} \sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji}, i \in N \\ \nabla f(x) + \sum\limits_{i\in N} v_j \nabla h_j(x) = 0 \end{cases} \Rightarrow \begin{cases} \sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji}, i \in N \\ R_{ij}x_{ij} - t_{ij} + v_j - v_i = 0, \forall (i,j) \in A \end{cases}$$

The first set of equations is Kirchhoff's current law. The second set of equations is Ohm's law.

Thus, the solutions satisfying both (1) and (2) and the optimal solutions of (3) are same.

2. prove (3) has at most solution.

$\mathcal{F} = \{x : \sum\limits_{j:(i,j)\in A} x_{ij} = \sum\limits_{j:(j,i)\in A} x_{ji}, i \in N\}$. Since the constraints are linear, the set $\mathcal{F}$ is a convex set, which can be verified easily by $x^1, x^2 \in \mathcal{F}, \lambda x^1 + (1-\lambda)x^2 \in \mathcal{F}, \lambda \in [0,1]$.

Suppose the solution to (3) isn't unique, i.e. $x^1, x^2 \in \mathcal{F}$ are optimal solutions of (3).

$a = f(x^1) = f(x^2) \leq f(x), \forall x \in X$. Since X is convex, so $\bar{x} = \lambda x^1 + (1-\lambda)x^2 \in X \Rightarrow f(\bar{x}) \geq a$.

Because $f(x)$ is strictly convex, $f(\bar{x}) < \lambda f(x^1) + (1-\lambda)f(x^2) = a$. It contradicts with the statement above.

Thus, (3) must have at most one solution.

3. prove the current $x_{ij}, (i,j) \in A$ satisfying both (1) and (2) has at least one solution.

Suppose we have n nodes and a arcs, i.e. $n = Card(N), a = Card(A)$.

We have n variables to denote the voltage of n nodes, and a variables to denote the current of a arcs. We have $n + a$ variables.

For each node, we have a equation of Kirchhoff's current law. For every arc, we have a equation of Ohm's law. We totally have $n + a$ equations.

A system of $n + a$ equations of $n + a$ variables must have at least one solution.

Thus, (1) must have at least one solution.

Therefore combine 1,2 and 3, we can conclude that the current $x_{ij}, (i,j) \in A$ that satisfies both laws (1) and (2) is the unique solution of (3).