January 2009

2) Given a sequence of finitely generated abelian groups {H_k} with H_0 free, show there is a compact simplicial complex X such that H_k(X) = H_k.

Note: This construction can be found in Hatcher pg. 143-44.

H_0 free $\Rightarrow H_0(X) \cong \mathbb{Z}^n$ so CW structure includes collection of points (0-cells.) We need this for our construction.

Consider the case when $H_k = \mathbb{Z}$. Take S^k ; $H_k(S^k) = \mathbb{Z}$ and $H_i(S^k) = 0$ for $i \neq k$.

Now if $H_k = \mathbb{Z}_m$, simply glue e^{n+1}, an (n+1)-dimensional cell, to S^k to obtain the LES:

$$\cdots \to \underbrace{\mathbb{Z}}_{\text{one (k+1) cell}} \xrightarrow{d_{k+1}} \underbrace{\mathbb{Z}}_{\text{one n-cell}} \xrightarrow{d_k = 0} \underbrace{0 \to \cdots \to}_{\text{no cells in between.}} \underbrace{\mathbb{Z}}_{\text{one 0-cell}} \xrightarrow{d_0} 0$$

(Here, we use the CW structure $S^k = e^k \cup e^0$.)

Define the degree map $S^k \to S^k$, $\alpha \mapsto m\alpha$ to be the gluing map, φ_α

Since $X^k = S^k$, $S^k \xrightarrow{\varphi_\alpha} X^k \xrightarrow{q} S^k$ is simply the gluing map itself. $d_{k+1}(e^{k+1})$

(Here, $X^k \setminus e^k = e^0$, so q is trivial.)

Then $H_k(X) = \ker d_k / \operatorname{Im} d_{k+1} \cong \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_m$.

Note that if $0 < i < k$, $H_i(X) = \ker d_i / \operatorname{Im} d_{i+1} = 0/0 \cong 0$

If $H_k = \bigoplus_{i=1}^{N} G_i$, where $G_i = \mathbb{Z}$ or $\mathbb{Z}_m$ for some $m > 1$, we can take the wedge sum of S^k's or $S^k \sqcup_{\substack{\varphi: S^k \to S^k \\ \text{st } \deg \varphi = m}} e^{k+1}$ as needed

Then $H_k(X) = H_k$ and $H_i(X) = 0$ for $i \neq 0, k$.

Since every finitely generated abelian group takes the form ① $\mathbb{Z}$, ② $\mathbb{Z}_m$, ③ $\mathbb{Z}^m$, or $\bigoplus_{i=1}^{N}$ of ①, ②, or ③, we have constructed a CW complex st $H_k(X) = H_k$ for any finitely generated abelian group, for a fixed $k > 0$.

To construct X st $H_k(X) = H_k$ for all k, simply take $\bigvee_i X_i$ where $H_k(X_i) = H_k$ for $i = k$. Then for any arbitrary k, $H_k(X) = \bigoplus_i H_k(X_i) \cong H_k(X_k) \oplus 0 \cong H_k$.

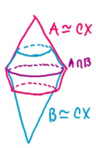
We can add 0-cells to the CW structure as needed to construct X st $H_0(X) \cong \mathbb{Z}^n$.

ie: The CW complex on the left fits the sequence $\{\underset{H_0}{\mathbb{Z}^2}, \underset{H_1}{\mathbb{Z}_2 \oplus \mathbb{Z}}, \mathbb{Z}^3, \ldots\}$.

3) Prove the reduced homology groups of X are equivalent to the reduced homology groups of X*x_0, where * is the join operation.

(This is a rephrasing of Hatcher Ex. 2.32.)

Note that $X * \underbrace{S^0}_{\{0,1\}}$ is just SX (suspension of X.) We can use Mayer-Vietoris to prove the claim.



The LES is:

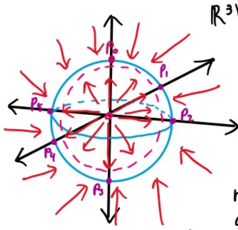
$$\cdots \to \underbrace{\tilde{H}_{k+1}(CX) \oplus \tilde{H}_{k+1}(CX)}_{0} \to \tilde{H}_{k+1}(SX) \to \tilde{H}_k(\underset{A \cap B}{X}) \to \underbrace{\tilde{H}_k(CX) \oplus \tilde{H}_k(CX)}_{0} \to \cdots$$

CX is contractible, so $\tilde{H}_i(CX) \oplus \tilde{H}_i(CX) \cong 0 \oplus 0 \cong 0$, for all i. Then by exactness, that $\tilde{H}_{k+1}(SX) \cong \tilde{H}_k(X)$, so $\tilde{H}_{k+1}(X * x_0) \cong \tilde{H}_k(X)$ for $k \geq 0$. ∎

January 2010

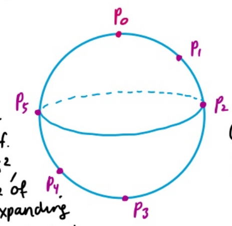
1) Let X be a space obtained from R^3 by removing the three coordinate axes. Calculate pi_1(X) and H_*(X).

$\mathbb{R}^3 \setminus \{x,y,z\text{-axes}\}$



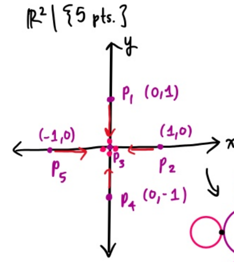
h.e. $\simeq$

points outside $S^2 \subset \mathbb{R}^3$ are def. retracted onto S^2, as is the inside of the sphere (imagine expanding the puncture obtained from removing the origin.)



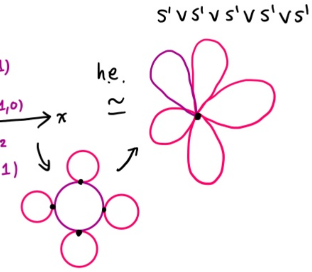
h.e. $\simeq$ (stereographic projection.)

$\mathbb{R}^2 \setminus \{5 \text{ pts.}\}$



h.e. $\simeq$

$S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$



We know $\pi_1\left(\bigvee_{i=1}^{5} S_i\right) = \mathbb{F}_5$, the free group on 5 generators.

And $H_k\left(\bigvee_{i=1}^{5} S_i\right) = \begin{cases} \mathbb{Z}^5 & k=1 \\ 0 & \text{otherwise.} \end{cases}$ So $\pi_1(\mathbb{R}^3 \setminus \{x,y,z\text{-axes}\}) = \mathbb{F}_5$ and $H_k(\mathbb{R}^3 \setminus \{3 \text{ axes}\}) = \begin{cases} \mathbb{Z}^5 & \text{for } k=0,1 \\ 0 & \text{otherwise.} \end{cases}$

4) Is S^2 x S^4 homotopy equivalent to CP^3? (Same cell structure, same cohomology groups, different ring structures.)

No. Let's examine their cohomology rings.

By Künneth, $H^*(S^2 \times S^4; \mathbb{Z}) \cong H^*(S^2;\mathbb{Z}) \otimes H^*(S^4;\mathbb{Z})$.

Recall by Cor. 3.3 (Consequence of Univ. Coeff. Thm.), $H^i(S^n;\mathbb{Z}) \cong H_i(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,n \\ 0 & \text{otherwise.} \end{cases}$

The only nontrivial, nonzero cohomology group is $H^n(S^n;\mathbb{Z})$. Let α be a generator of $H^n(S^n;\mathbb{Z})$ (dual to generator of $H_n(S^n;\mathbb{Z})$)

Note $\alpha \smile \alpha \in H^{2n}(S^n;\mathbb{Z}) \Rightarrow 0 = \alpha \smile \alpha$. Then elements of $H^*(S^n;\mathbb{Z})$ take form $a_0 + a_1\alpha$; $a_0, a_1 \in \mathbb{Z}$. So the cohomology ring $H^*(S^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$, and $|\alpha| = n$.

Determine Cohom. ring stxr of S^n, $\mathbb{Z}$ coeff.

Then $H^*(S^2 \times S^4;\mathbb{Z}) \cong H^*(S^4;\mathbb{Z}) \otimes H^*(S^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2)$ and $|\alpha|=2$ and $|\beta|=4$.

By Thm 3.19, $H^*(\mathbb{C}P^3;\mathbb{Z}) \cong \mathbb{Z}[\gamma]/(\gamma^4)$ where $|\gamma|=2$.

Suppose, for contradiction, $H^*(\mathbb{C}P^3;\mathbb{Z}) \cong H^*(S^2 \times S^4;\mathbb{Z}) \Rightarrow \mathbb{Z}[\gamma]/(\gamma^4) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2)$.

Then the isomorphism $\mathbb{Z}[\gamma]/(\gamma^4) \longrightarrow \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2)$ must map $\gamma \in H^2(\mathbb{C}P^3;\mathbb{Z})$, a nontrivial generator of the LHS, to a generator on the RHS. (In particular, $\gamma \mapsto \alpha$ since $|\gamma|, |\alpha| = 2$ so both are deg 2 generators.)

Consider $\gamma^2 \in H^4(\mathbb{C}P^3;\mathbb{Z})$, which is nonzero in $\mathbb{Z}[\gamma]/(\gamma^4)$.

Now consider α, a gen. of $\mathbb{Z}[\alpha]/(\alpha^2)$. Clearly, $\alpha^2 = 0$, so we have a degree 2 generator that, cupped w/ itself, is nonzero but a degree 2 generator on the RHS that, cupped with itself, is zero. This cannot happen for two isomorphic rings. ∎

5) Show that H*(RP^3; Z) = H*(RP^2 V S^3; Z) is a ring isomorphism. Is RP^3 homotopy equivalent to RP^2 V S^3? Explain.

There are a few ways to compute $H^*(\mathbb{R}P^3;\mathbb{Z})$. Recall the cellular chain complex from Hatcher pg. 144 for $\mathbb{R}P^n$: $\ldots 0 \to \underset{e^n}{\mathbb{Z}} \to \underset{e^{n-1}}{\mathbb{Z}} \to \ldots \xrightarrow{d_2=2} \underset{e^1}{\mathbb{Z}} \xrightarrow{d_1=0} \underset{e^0}{\mathbb{Z}} \xrightarrow{d_0=0} 0$.

We can dualize for $\mathbb{R}P^3$: $\ldots 0 \xleftarrow{\delta_3=0} \underset{e^3}{\mathbb{Z}} \xleftarrow{\delta_2=0} \underset{e^2}{\mathbb{Z}} \xleftarrow{\delta_1=2} \underset{e^1}{\mathbb{Z}} \xleftarrow{\delta_0=0} \underset{e^0}{\mathbb{Z}} \xleftarrow{\delta_{-1}=0} 0$. (think of d as linear maps represented by matrices and dualize by taking the transpose.)

Then $H^0(\mathbb{R}P^3;\mathbb{Z}) \cong \ker\delta_0/\operatorname{Im}\delta_{-1} \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$
$H^1(\mathbb{R}P^3;\mathbb{Z}) \cong \ker\delta_1/\operatorname{Im}\delta_0 \cong 0$
$H^2(\mathbb{R}P^3;\mathbb{Z}) \cong \ker\delta_2/\operatorname{Im}\delta_1 \cong \mathbb{Z}/2\mathbb{Z}$
$H^3(\mathbb{R}P^3;\mathbb{Z}) \cong \ker\delta_3/\operatorname{Im}\delta_2 \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$.

$\boxed{\text{Alternatively, can use the homology groups } H_k(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0, n \text{ for } n \text{ odd} \\ \mathbb{Z}_2 & 1 \le k < n \text{ for } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \text{ and the Univ. Coeff. Thm. corollary that says } H^k(\mathbb{R}P^n;\mathbb{Z}) \cong H_k/T_k \oplus T_{k-1}. \text{ Then } H^0(\mathbb{R}P^3;\mathbb{Z}) \cong \mathbb{Z}/\{0\} \cong \mathbb{Z} \text{ and } H_1(\mathbb{R}P^3;\mathbb{Z}) \cong \mathbb{Z}_2/\mathbb{Z}_2 \oplus 0 \cong 0, \text{ etc.}}$

The nonzero elements are $\alpha \in H^2$ and $\beta \in H^3$, generators. Note $\alpha \in \mathbb{Z}_2$, so $2\alpha = 0$. We find other relations: $H^2 \smile H^2 = H^4 = 0$; $H^2 \smile H^3 = H^5 = 0$; $H^3 \smile H^3 = H^6 = 0$.

Then $\alpha^2 = \alpha\beta = \beta^2 = 0$, so $H^*(\mathbb{R}P^3;\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^2, \alpha\beta, \beta^2)$, $|\alpha|=2$ and $|\beta|=3$. (see Hatcher pg. 222.)

$H^*(\mathbb{R}P^2;\mathbb{Z})$ is computed similarly. The chain complex is: $\ldots 0 \xleftarrow{\delta_2=0} \underset{e^2}{\mathbb{Z}} \xleftarrow{\delta_1=2} \underset{e^1}{\mathbb{Z}} \xrightarrow{\delta_0=0} \underset{e^0}{\mathbb{Z}} \xleftarrow{\delta_{-1}=0} 0$, so $H^0(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$, $H^1(\mathbb{R}P^2;\mathbb{Z}) \cong 0$, and $H^2(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}_2$.

Then we have (and nontrivial) nonzero generator $\alpha \in H^2(\mathbb{R}P^3;\mathbb{Z})$, $\alpha \in \mathbb{Z}_2 \Rightarrow 2\alpha = 0$. $H^2 \smile H^2 = H^4 = 0 \Rightarrow \alpha^2 = 0$. So $H^*(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2, 2\alpha)$, $|\alpha| = 2$.

WTS: $\underbrace{H^*(\mathbb{R}P^3,\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha,\alpha^2,\alpha\beta,\beta^2)}_{|\alpha|=2,\ |\beta|=3} \cong \underbrace{\mathbb{Z}[\gamma]/(\gamma^2,2\gamma)}_{|\gamma|=2,\ H^*(\mathbb{R}P^2;\mathbb{Z})} \oplus \underbrace{\mathbb{Z}[\rho]/(\rho^2)}_{|\rho|=3,\ H^*(S^3;\mathbb{Z})} \cong H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z})$. Note $\mathbb{Z}[\gamma]/(\gamma^2, 2\gamma) \oplus \mathbb{Z}[\rho]/(\rho^2) \cong \mathbb{Z}[\gamma,\rho]/(\gamma^2, 2\gamma, \rho^2, \gamma\rho)$

(Last fact follows from $\mathbb{Z}[\gamma,\rho]/(\gamma\rho) \cong \mathbb{Z}[\gamma] \oplus \mathbb{Z}[\rho]$.)

No, $\mathbb{R}P^3 \overset{h.e.}{\not\simeq} \mathbb{R}P^2 \vee S^3$. This is b/c $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \not\cong H^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) \cong H^*(\mathbb{R}P^2;\mathbb{Z}_2) \oplus H^*(S^3;\mathbb{Z}_2)$.

From Thm 3.19, we know $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^4)$ and $H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^3)$. Here, $|\alpha| = |\beta| = 1$.

By induction on reduced homology / relative homology (using LES $\cdots \to \tilde{H}_k(S^{n-1};\mathbb{Z}_2) \to \tilde{H}_k(D^k;\mathbb{Z}_2) \to \tilde{H}_k(S^n;\mathbb{Z}_2) \to \dots$) deduce $H_k(S^n;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, n \\ 0 & \text{otherwise} \end{cases}$.

By Poincaré Duality, $H_k(S^n;\mathbb{Z}_2) \cong H^{n-k}(S^n;\mathbb{Z}_2)$, so in particular, $H^k(S^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k = 0, n \\ 0 & \text{otherwise} \end{cases}$.

Let $\gamma \in H^n(S^n;\mathbb{Z}_2)$ be a generator and note $H^n \smile H^n = H^{2n} = 0$, so $\gamma^2 = 0$.

Then elements of $H^*(S^n;\mathbb{Z}_2)$ take the form $a_0 + a_1\gamma$; $a_0, a_1 \in \mathbb{Z}_2$, so $H^*(S^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\gamma]/(\gamma^2)$, $|\gamma| = n$.

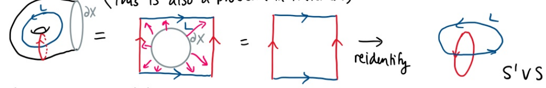
again, as before, if there were a ring isomorphism, it would map degree 1 generators to degree 1 generators. But $\alpha^3 \neq 0$ in $\mathbb{Z}_2[\alpha]/(\alpha^4) \cong H^*(\mathbb{R}P^3;\mathbb{Z}_2)$ while $\beta^3 = 0 \in \mathbb{Z}_2[\beta]/(\beta^3)$, $\lightning$.

Poincaré (Thm 3.30)

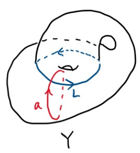
If M is a closed R-orientable (usually, only $\mathbb{Z}$ or $\mathbb{Z}_2$ relevant but note orientable surfaces are R-orientable for any R.), then $H^k(M;R) \cong H_{n-k}(M;R)$.

3a) Show that X deformation retracts onto a wedge W of two circles, one of which is L.

(This is also a problem in Hatcher.)



3b) Compute H_1(Y).



Give CW structure:

Chain complex:

$$\ldots 0 \to \underset{1\ \text{2-cell}}{\mathbb{Z}} \xrightarrow{d_2} \underset{3\ \text{1-cells}}{\mathbb{Z}^3} \xrightarrow{d_1=0} \underset{1\ \text{0-cell}}{\mathbb{Z}} \longrightarrow 0$$

gluing map $d_2 : \cancel{c}+L-\cancel{c}+\cancel{L}+\cancel{a}-\cancel{L}-\cancel{a} = L$

So $H_1(Y) = \ker d_1 / \operatorname{Im} d_2 = \langle a, L, \overset{\in \mathbb{Z}}{c} \mid L \rangle \cong \mathbb{Z}^2$

3c) Let A be the image of H_1(W) in H_1(Y). Draw the covering space $\tilde{Y}$ of Y corresponding to the kernel of the map pi_1(Y) —> H_1(Y)/A.

$H_1(W) \cong \mathbb{Z}^2$, generated by a and L. Note that $L=0$ in $H_1(Y)$ and $a=a$.

Then $H_1(Y)/A = \langle a, c \mid a \rangle = \langle c \rangle$ ($\cong \mathbb{Z}$.) So $\pi_1(Y) \xrightarrow{\phi} H_1(Y)/A$ sends

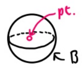
$\pi_1(Y) = \langle a, L, c \mid cLc^{-1}LaL^{-1}a^{-1} \rangle$. So $a \in \ker\phi$ (since $a \in A$), $c \mapsto c$ (so $c \notin \ker\phi$), and $L \mapsto 0$ since $L=0$ in $H_1(Y)$. ($L \in \ker\phi$.) So $\ker\phi = \langle\langle a, L \rangle\rangle$. (w/ relation.)

($H_1(Y) = \text{ab}(\pi_1(Y)) \Rightarrow$ relation $cLc^{-1}LaL^{-1}a^{-1}$ becomes $L\cancel{cc^{-1}}\cancel{LL^{-1}}\cancel{aa^{-1}} = L$.) ← sidenote.

August 2014

1a) Let X be a compact connected subset of S^3 homeomorphic to a 1-dimensional cell complex. Prove that H_1(S^3 - X) is free abelian of the same rank as H_1(X) and that H_n(S^3 - X) = 0 for n > 1.

Page 520: every compact subset of a CW complex (ie X) is contained in a finite subcomplex.
I think this means X has finitely many cells, so it's homeomorphic to a finite graph. Every finite connected graph is h.e. to a finite wedge of circles, $\bigvee_m S^1$; simply contract the edges that are not loops.
If $X \simeq$ graph w/o loops, then $S^3 - X \simeq S^3 - \{pt\} \simeq \mathbb{R}^3 \simeq pt$, so $H_1(S^3-X) \cong H_1(X) \cong 0$, and $H_n(S^n - X) \cong H_n(pt) = 0$ for $n>1$.
Otherwise, consider $S^3 \setminus X$ as a union of subsets, $A = (S^3 \setminus \{pt\}) - X$ and $B = B^3$, where B^3 is a nbhd of our pt., chosen to be disjoint from X.



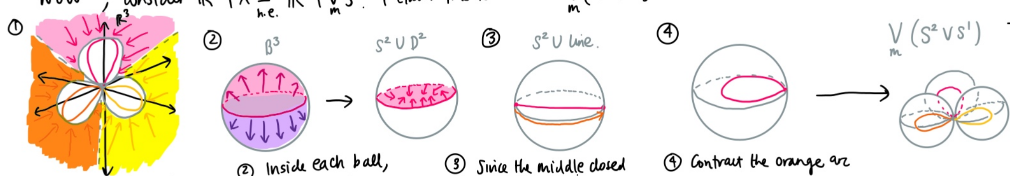
B, nbhd of pt.

Note $A \cap B = B^3 \setminus \{pt\} \simeq S^2$. Then we obtain the LES:

$$\dots \to \tilde H_1(S^2) \to \underbrace{\tilde H_1((S^3\setminus\{pt\})-X)}_{\simeq \mathbb{R}^3 \setminus X} \oplus \underbrace{\tilde H_1(B^3)}_{\text{contractible}} \to \tilde H_1(S^3 - X) \to \tilde H_0(S^2) \to \dots$$

This simplifies to: $\dots \to \underbrace{\tilde H_1(S^2)}_{0} \to \underbrace{\tilde H_1(\mathbb{R}^3 \setminus X)}_{\text{TBD}} \xrightarrow{\cong} \underbrace{\tilde H_1(S^3 - X)}_{\text{goal}} \to \underbrace{\tilde H_0(S^2)}_{0} \to \dots$ So it suffices to calculate $H_1(S^3 \setminus X)$.

Now, consider $\mathbb{R}^3 \setminus X \underset{h.e.}{\simeq} \mathbb{R}^3 \setminus \bigvee_m S^1$. I claim this is h.e. to $\bigvee_m (S^2 \vee S^1)$.



① Deformation retract the space $\mathbb{R}^3 \setminus \bigvee_m B^2$ onto $\bigvee_m B^2$. ($B^2 = \{x \in \mathbb{R}^3 \mid |x| \le 1\}$)

② Inside each ball, push the area above and below the middle disk w/ boundary S^1 onto the S^2.

③ Since the middle closed disk is contractible, d.r. it onto a line segment connecting two antipodal points.

④ Contract the orange arc to obtain $S^1 \vee S^2$.

Inspired by Hatcher's construction on pg. 46.

Then it follows that $H_1(\mathbb{R}^3 \setminus X) = H_1(\bigvee_m (S^2 \vee S^1)) = \bigoplus_m (H_1(S^2) \oplus H_1(S^1)) \cong \mathbb{Z}^m$. Since $H_1(X) = H_1(\bigvee_m S^1) = \mathbb{Z}^m$, $H_1(S^3 \setminus X) \cong H_1(\mathbb{R}^3 \setminus X)$ is indeed free abelian with the same rank as $H_1(X)$.

Note for $i > 3$: $\dots \to \underbrace{H_i(S^2)}_{=0} \to H_i(\mathbb{R}^3 \setminus X) \xrightarrow{\cong} H_i(S^3 \setminus X) \to \underbrace{H_{i-1}(S^2)}_{0} \to \dots$ So $H_i(S^3 \setminus X) \cong H_i(\mathbb{R}^3 \setminus X) \cong H_i(\bigvee_m (S^2 \vee S^1)) \cong 0$.

For $i = 3$: $\dots \to \underbrace{H_3(S^2)}_{0} \xrightarrow{i_*} \underbrace{H_3(\mathbb{R}^3 \setminus X)}_{\substack{0 \\ \bigoplus_m H_3(S^2 \vee S^1) \cong 0}} \xrightarrow{j_*} H_3(S^3 \setminus X) \xrightarrow{k_*} \underbrace{H_2(S^2)}_{\mathbb{Z}} \to \dots$ By exactness, $\operatorname{Im} j_* = \ker k_* = 0$, and $\ker j_* = 0$, so $H_3(S^3 \setminus X) \cong 0$.

For $i = 2$: $\dots \to \underbrace{H_3(S^2)}_{0} \to \underbrace{H_3(\mathbb{R}^3 \setminus X)}_{0} \xrightarrow{j_*} \underbrace{H_3(S^3 \setminus X)}_{0 \text{ from above}} \xrightarrow[\text{injective}]{k_*} H_2(S^2) \xrightarrow[\text{injective}]{i_*} H_2(\mathbb{R}^3 \setminus X) \xrightarrow[\cong]{\varphi} H_2(S^3 \setminus X) \to \underbrace{H_1(S^1)}_{0} \to \dots$

By exactness, $\ker k_* = \operatorname{Im} j_* = 0$ and $\operatorname{Im} k_* = \ker i_* = 0$, so $k_* = 0$. Then $H_2(\mathbb{R}^3 \setminus X) \cong H_2(S^3 \setminus X)$.
Furthermore, $\operatorname{Im} i_* = \ker \varphi = 0 \Rightarrow H_2(\mathbb{R}^3 \setminus X) = 0 \Rightarrow H_2(S^3 \setminus X) = 0$. ∎

1b) Let X in S^3 be homeomorphic to the disjoint union of two circles, and let Y be the disjoint union of two disks. Build a space Z by attaching Y to S^3 and identifying the boundary of Y and X via a homeomorphism. Compute the homology groups of Z.

Let $A = S^3$, $B = Y \cup X$, $A \cap B = X$. Use Mayer-Vietoris:

$$\dots \to \underbrace{H_k(X)}_{0} \to H_k(S^3) \oplus \underbrace{H_k(Y \cup X)}_{\substack{0 \\ \text{(contractible.)}}} \xrightarrow{\cong} H_k(Z) \to \underbrace{H_{k-1}(X)}_{0} \to \dots \quad (k > 2)$$

Then for $k > 3$, $H_k(Z) \cong 0$ and for $k = 3$, $H_k = \mathbb{Z}$.

For $k=2$: $\dots \to \underbrace{H_2(X)}_{0} \to \underbrace{H_2(S^3)}_{0} \oplus \underbrace{H_2(Y \cup X)}_{0} \to H_2(Z) \xrightarrow{\cong} \underbrace{H_1(X)}_{\mathbb{Z}^2} \to \underbrace{H_1(S^3)}_{0} \oplus \underbrace{H_1(Y \cup X)}_{0} \to \dots$ So $H_2(Z) \cong \mathbb{Z}^2$.

For $k=1$: $\dots \to \underbrace{\tilde H_1(S^3)}_{0} \oplus \underbrace{\tilde H_1(Y \cup X)}_{0} \to \tilde H_1(Z) \xrightarrow{\cong} \underbrace{\tilde H_0(X)}_{\mathbb{Z}} \xrightarrow{\text{(Prop. 2.6)}} \underbrace{\tilde H_0(S^3)}_{0} \oplus \underbrace{\tilde H_0(Y \cup X)}_{0} \to \dots$ so $H_1(Z) \cong \tilde H_1(Z) \cong \mathbb{Z}$.

So
$$\boxed{H_k(Z) = \begin{cases} \mathbb{Z} & \text{for } k = 0,1,3 \\ \mathbb{Z}^2 & \text{for } k = 2 \\ 0 & \text{otherwise.} \end{cases}}$$

(Perhaps L ought to be orientable, have connected ∂.)

2) Consider a simplicial complex K homeomorphic to the 3-sphere. Let L be a subcomplex of K homeomorphic to a 3-manifold with nonempty boundary. Show $H_1(L;\mathbb{Z})$ has no torsion.

Consider $A = L \cup N(\partial L) \simeq L$, nbhd around ∂L (here, we use the existence of collar nbhd of ∂L.)

$B = K \setminus L$ so $A \cap B = \partial L$, $A \cup B = S^3$.

Examine Mayer Vietoris sequence:

$$\cdots \to \underbrace{\tilde{H}_2(S^3)}_{0} \to \tilde{H}_1(\partial L) \xrightarrow[\cong]{} \tilde{H}_1(L) \oplus \tilde{H}_1(K\setminus L) \to \underbrace{\tilde{H}_1(S^3)}_{0} \to \tilde{H}_0(\partial L) \to \cdots$$

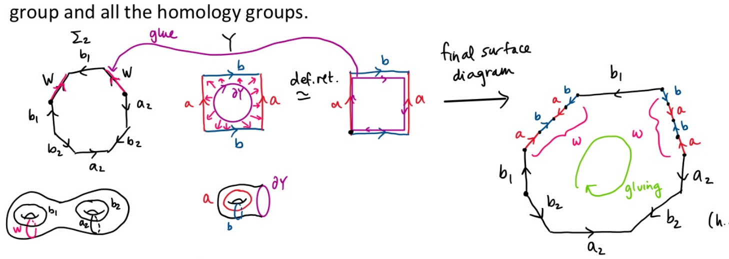
Conclude $\tilde{H}_1(\partial L) \cong \tilde{H}_1(L) \oplus \tilde{H}_1(K \setminus L)$.

We know ∂L is dim 2 $\underset{\text{Corollary 3.28}}{\Rightarrow}$ $H_1(\partial L)$ is torsion-free. (Note: this was not known in 751.)

Since $\tilde{H}_1(L)$ is summand of torsion-free group, it itself must also be torsion-free.

Corollary 3.28 (Proven using UCT for homology.)
If M is a closed connected n-manifold, then the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable, $\mathbb{Z}_2$ if M is nonorientable.

3) Let X be a surface of genus 2 and Y a torus with one boundary component. Let W be a nonseparating circle in X. Let Z be the space obtained by attaching Y to X by identifying W and the boundary of Y via a homeomorphism. Compute the fundamental group and all the homology groups.



(h.e. to our original space.)

(Van Kampen Corollary.)

$\pi_1(Z) = \langle a, b, a_2, b_1, b_2 \mid b_1[a,b]b_1^{-1}[a,b]^{-1}[a_2,b_2] \rangle$
$= \langle a, b, a_2, b_1, b_2 \mid [b_1,[a,b]][a_2,b_2] \rangle.$

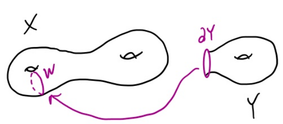
Cellular Chain Complex:

$$\cdots \to 0 \to \underset{1\ \text{2-cell}}{\mathbb{Z}} \xrightarrow{d_2=0} \underset{5\ \text{1-cells}}{\mathbb{Z}^5} \xrightarrow{d_1=0} \underset{1\ \text{0-cell}}{\mathbb{Z}} \xrightarrow{d_0=0} 0$$

2-cell attached by word $b_1 a b a^{-1} b^{-1} b_1^{-1} b a b^{-1} a^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$, which all cancel.
So $d_2 = 0$. So $H_1(Z) = \mathbb{Z}^5$, $H_2(Z) = \ker d_2 / \operatorname{Im} d_3 = \mathbb{Z}$.

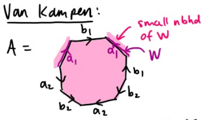
Then $\boxed{H_k(Z) = \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^5 & k=1 \\ 0 & \text{otherwise} \end{cases}}$

Other approaches?

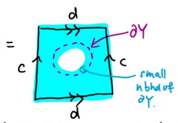


Van Kampen:

A =



B =



$i_{AB}(\gamma) = a_1$
$i_{BA}(\gamma) = [c,d]$
so $N = \langle\langle i_{AB}(\gamma)(i_{BA}(\gamma))^{-1} \rangle\rangle = \langle\langle a[c,d]^{-1} \rangle\rangle$

$A \cap B = \gamma$, ∂Y glued homeomorphically to W.

$\pi_1(Z) = \pi_1(A) * \pi_1(B) / \langle\langle N \rangle\rangle = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1],[a_2,b_2] \rangle * \langle c,d \rangle / \langle\langle a_1[c,d]^{-1} \rangle\rangle$ (Note $\pi_1(B) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$.)
$= \langle a_1, b_1, a_2, b_2, c, d \mid [a_1,b_1],[a_2,b_2], a_1[c,d]^{-1} \rangle$ (note $a_1 = [c,d]$.)
$= \langle b_1, a_2, b_2, c, d \mid [[c,d],b_1],[a_2,b_2] \rangle$

Mayer-Vietoris: There are no 3-cells, so $H_n(Z) = 0$ for $n \geq 3$. And clearly, $H_0(Z) = \mathbb{Z}$ since Z is path-connected.

n=1: $\cdots \to \underbrace{\tilde{H}_1(S^1)}_{\mathbb{Z}} \xrightarrow{\phi} \underbrace{\tilde{H}_1(\Sigma_2)}_{\mathbb{Z}^4} \oplus \underbrace{\tilde{H}_1(S^1 \vee S^1)}_{\mathbb{Z}^2} \to \tilde{H}_1(Z) \to \tilde{H}_0(S^1) \to \tilde{H}_0(\Sigma_2) \oplus \tilde{H}_0(S^1 \vee S^1) \to \tilde{H}_0(X) \to 0$

$H_1(Z) \cong (H_1(\Sigma_2) \oplus H_1(S^1 \vee S^1)) / \operatorname{Im}\phi \cong \mathbb{Z}^6 / \operatorname{Im}\phi \cong \langle a_1, b_1, a_2, b_2, c, d \mid a_1 \rangle \cong \mathbb{Z}^5.$

(Map ϕ: $1 \mapsto ((1,0,0,0),(\underset{c\text{-}c}{0},\underset{d\text{-}d}{0}))$)

For n=2: $\cdots \to \underbrace{H_3(Z)}_{0} \to \underbrace{H_2(S^1)}_{0} \xrightarrow{i_* = 0} \underbrace{H_2(\Sigma_2)}_{\mathbb{Z}} \oplus \underbrace{H_2(S^1 \vee S^1)}_{0} \xrightarrow{j_*} H_2(Z) \xrightarrow{k_*} \underbrace{H_1(S^1)}_{\mathbb{Z}} \xrightarrow{\phi} H_1(\Sigma_2) \oplus H_1(S^1 \vee S^1) \to \cdots$

ϕ ← injective from above calculation

(so $0 = \ker\phi = \operatorname{Im} k_*$.)

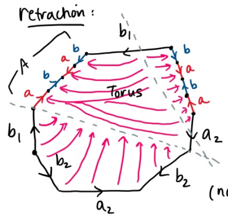
$\operatorname{Im}(j_*) = \ker(k_*) = H_2(Z) = $ j_* surjective.
j_* is also injective. So $H_2(Z) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$.

Hatcher Ex. 2.36
$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^{2g} & k=1 \\ 0 & \text{otherwise} \end{cases}$

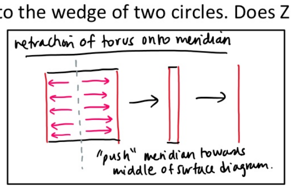
Conclude: $\boxed{H_k(Z) = \begin{cases} \mathbb{Z} & \text{for } k=0,2 \\ \mathbb{Z}^5 & \text{for } k=1 \\ 0 & \text{otherwise.} \end{cases}}$

3b) Show that Z retracts onto the wedge of two circles. Does Z deformation retract onto the wedge of two circles? (similar to Hatcher 1.2.9)

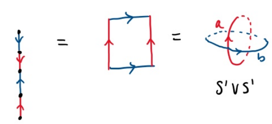
retraction:



(note A is fixed.)



Result:



Deformation retract Z (as in part a) onto the space represented by the surface diagram. Then compose the deformation retraction with the retraction seen to the left to obtain a wedge of two circles.

No, there is no deformation retraction; $\pi_1(Z) = \langle a, b, a_2, b_1, b_2 \mid [b_1,[a,b]] \cdot [a_2,b_2] \rangle$, as calculated before, but $\pi_1(S^1 \vee S^1) = \langle a, b \rangle$. Clearly, these two groups are not isomorphic (a map $\pi_1(Z) \to \pi_1(S^1 \vee S^1)$ would not map generators bijectively.) so $Z \underset{h.e.}{\not\simeq} S^1 \vee S^1$. Since deformation retractions are homotopy equivalences, we reach our conclusion.

Lens space: orbit space $S^{2n-1}/\mathbb{Z}_m$.

6) Prove the following theorem:

If L is a lens space that embeds nicely into S^4, then L is homeomorphic to S^3 or S^2 x S^1.

Defn: $f: X \to Y$ is an embedding if it is a homeomorphism onto its image.

"nice" embedding: $Y - f(X)$ has 2 components A, B w/ nbhds $N(A)$, $N(B)$ h.e. to A, B, $N(A) \cap N(B)$ he to X.

Assume L is not S^3 or $S^2 \times S^1$, suppose L admits nice embedding $f: L \to S^4$, A and B are two components of $S^4 - f(L)$.

a) Show $H_1(L) \cong H_1(A) \oplus H_1(B)$.

$N(A) \cup N(B) = S^4 - f(L)$, $N(A) \simeq A$ and $N(B) \simeq B$, $N(A) \cap N(B) \simeq L$.

$$\cdots \to \tilde{H}_2(L) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(S^4 - f(L)) \to \tilde{H}_1(L) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(S^4 - f(L)) \to \underbrace{\tilde{H}_0(L)}_{0} \to \cdots$$

4) Prove that the ring Q[x, y]/ (x^3, y^3, x^2y^2) is not isomorphic to the rational cohomology ring H*(M; Q) of any closed connected orientable 195-manifold M.

6) Let X be a simplicial complex.

a) Prove that H^1(X; Z) has no torsion. X is a simplicial complex, so we obtain a chain complex of free abelian groups.

Recall by a corollary of the Univ. Coefficient Theorem (see Corollary 3.3), $H^1(X;\mathbb{Z}) \cong H_1(X;\mathbb{Z})/T_1 \oplus T_0$, where $T_1 \subset H_1(X;\mathbb{Z})$ is the torsion subgroup of H_1 and T_0 is the torsion subgroup of H_0. Note $H_1(X;\mathbb{Z})/T_1$ is torsion-free, and $H_0(X;\mathbb{Z}) \cong \mathbb{Z}^n$, where n is the number of path components of X, so $T_0 = 0$. Then it follows $H^1(X;\mathbb{Z})$ is torsion-free.

[Alternatively, the Univ. Coeff. Thm says $0 \to \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),\mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X);\mathbb{Z}) \to 0$. Again, $H_0(X;\mathbb{Z}) = \mathbb{Z}^n$, which is free, so $\mathrm{Ext}(H_0(X);\mathbb{Z}) = 0 \Rightarrow H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X),\mathbb{Z})$. If we examine elements $\phi \in \mathrm{Hom}(H_1(X);\mathbb{Z})$, note $\phi^n(1) = n\phi(1) = n$, so ϕ is finite order iff it's the identity (this is a consequence of $\mathbb{Z}$ being torsion-free), so $\mathrm{Hom}(H_1(X);\mathbb{Z})$ is torsion-free. Then $H^1(X;\mathbb{Z})$ is also torsion-free.]

$\boxed{\text{Torsion-free abelian Group: • group operation commutative • only identity element is finite-order}}$

b) Let a be an element of H^1(X; Z). Show a⌣ a = 0.

January 2014

5a) Let M be a compact manifold with boundary. Is there a retraction of M onto its boundary?

This is Hatcher Ex. 32 (pg. 260).

No. Suppose there were. Then the map $H_{n-1}(\partial M;\mathbb{Z}_2) \xrightarrow{i_*} H_{n-1}(M;\mathbb{Z}_2)$ must be injective. (since $\partial M \hookrightarrow M \to \partial M$, $r \circ i = \mathbb{1} \Rightarrow (r \circ i)_* = \mathbb{1}_* \Rightarrow i_*$ injective.)

Recall the relative LES: $\dots \to \underbrace{H_n(M;\mathbb{Z}_2)}_{0} \to \underbrace{H_n(M,\partial M;\mathbb{Z}_2)}_{\mathbb{Z}_2} \xrightarrow{\phi} \underbrace{H_{n-1}(\partial M;\mathbb{Z}_2)}_{\mathbb{Z}_2} \xrightarrow{i_*} H_{n-1}(M;\mathbb{Z}_2) \to H_{n-1}(M,\partial M;\mathbb{Z}_2) \to \dots$

Note that by Lefschetz, $H_n(M;\mathbb{Z}_2) \cong H^0(M,\partial M;\mathbb{Z}_2) \cong \tilde{H}^0(M/\partial M;\mathbb{Z}_2) \cong 0$. So ϕ is injective.

Furthermore, $H_n(M,\partial M;\mathbb{Z}_2) \cong H^0(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ since M is path-connected. (again by Lefschetz)

Since ∂M is an (n-1)-manifold, $H_{n-1}(\partial M;\mathbb{Z}_2) \cong H^0(\partial M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, by Poincaré Duality.

Since $\ker\phi = 0$, it follows that $\operatorname{Im}\phi = \mathbb{Z}_2 \Rightarrow \ker i_* = \mathbb{Z}_2 \neq 0$. ↯.

> Lefschetz (pg. 254) Duality
> Special case of Thm 3.43: If M is compact and R-orientable n-manifold, ∂M is composed of 2 (n-1)-dim manifolds A and B (can be empty) and $\partial A = \partial B = A \cap B$, (here, $A=\emptyset$ and $B = \partial M$, $\partial A = \partial B = 0$ since $\partial\partial M = 0$.) then $H^k(M,A;R) \cong H_{n-k}(M,B;R)$.

5b) Is there a retraction of RP^n onto RP^k for k < n? Rigorously justify your answer.

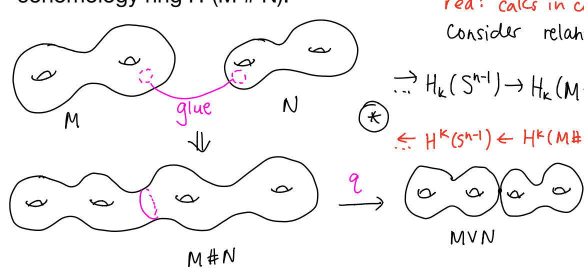
No. Suppose there were. Since $r \circ i: \mathbb{RP}^k \xrightarrow{i} \mathbb{RP}^n \xrightarrow{r} \mathbb{RP}^k$ is the id, i must be injective. So it induces an injective homomorphism $i^*: H^*(\mathbb{RP}^k;\mathbb{Z}_2) \to H^*(\mathbb{RP}^n;\mathbb{Z}_2)$, $\mathbb{Z}_2[\alpha]/(\alpha^{k+1}) \to \mathbb{Z}_2[\beta]/(\beta^{n+1})$, $|\alpha| = |\beta| = 1$. Since $|\alpha| = |\beta|$, $i^*(\alpha) = \beta$ necessarily.

Then we must have $0 = i^*(0) = i^*(\alpha^{k+1}) = (i^*(\alpha))^{k+1} = \beta^{k+1}$; however $\beta^{k+1} \neq 0$ since $k \lneq n$. ↯

6a) Let M, N be closed, connected $\boxed{\text{oriented}}$ n-manifolds, and consider their connected sum M#N. Describe the cohomology ring H*(M # N).

red: calcs in cohomology

$\operatorname{Ext}(X) = 0\ \forall$ spaces here, so $H^k(X) \cong \operatorname{Hom}(H_k(X);\mathbb{Z})$.



Consider relative LES:

(*) $\dots \to H_k(S^{n-1}) \to H_k(M\#N) \xrightarrow{q_*} \underbrace{H_k(M\#N, S^{n-1})}_{\cong \tilde{H}_k(M \vee N)} \to H_{k-1}(S^{n-1}) \to \dots$

$\leftarrow H^k(S^{n-1}) \leftarrow H^k(M\#N) \leftarrow H^k(M \vee N) \leftarrow H_{k-1}(S^{n-1}) \leftarrow \dots$ (dualize)

$k>0$: $\dots \leftarrow 0 \leftarrow \tilde{H}^k(0) \leftarrow \tilde{H}^k(M) \oplus \tilde{H}^k(N) \leftarrow \tilde{H}^k(M \vee N) \leftarrow 0 \dots$

By Mayer Vietoris, $\dots \to \underbrace{\tilde{H}_k(pt)}_{0} \to \tilde{H}_k(M) \oplus \tilde{H}_k(N) \xrightarrow{\cong} \tilde{H}_k(M \vee N) \to \underbrace{\tilde{H}_{k-1}(pt)}_{0} \to \dots$

so $\forall k \geq 0$ $H_k(M) \oplus H_k(N) \cong H_k(M \vee N)$

Also, M#N is orientable dim n, $\Rightarrow H_n(M\#N;\mathbb{Z}) \cong \mathbb{Z}$.

Then (*) becomes: $\dots \to \underbrace{H_k(S^{n-1})}_{0 \text{ if } k \neq n-1} \to H_k(M\#N) \xrightarrow[\text{if } k \neq n, n-1]{\cong} H_k(M) \oplus H_k(N) \to \underbrace{H_{k-1}(S^{n-1})}_{0 \text{ if } k \neq n} \to \dots$

$\leftarrow \underbrace{\tilde{H}^k(S^{n-1})}_{0 \text{ if } k \neq 1} \leftarrow H^k(M\#N) \leftarrow H^k(M) \oplus H^k(N) \leftarrow \underbrace{H^{k-1}(S^{n-1})}_{0 \text{ if } k \neq 2} \leftarrow \dots$

So the only interesting case is when $k = n-1$. The only interesting part of LES:

$$\dots \to \underbrace{H_n(S^{n-1})}_{0} \to \underbrace{H_n(M\#N)}_{\mathbb{Z}} \to \underbrace{\underbrace{H_n(M)}_{\mathbb{Z}} \oplus \underbrace{H_n(N)}_{\mathbb{Z}}}_{H_n(M \vee N)} \xrightarrow{\varphi} \underbrace{H_{n-1}(S^{n-1})}_{\mathbb{Z}} \xrightarrow{\alpha} H_{n-1}(M\#N) \xrightarrow[\text{surjective}]{\psi} H_{n-1}(M) \oplus H_{n-1}(N) \to 0 \to \dots$$

> Thm 3.26 (and consequences)
> ① If M is R-orientable, then $H_n(M;R) \cong R$.
> ② $H_i(M;R) \cong 0$ for $i > n$. ($n = \dim(M)$.)
> ③ $H_n(M;\mathbb{Z}) \cong 0$ if M is nonorientable.

Examine the map $\varphi: H_n(M \vee N) \to H_{n-1}(S^{n-1})$. Note that S^{n-1} is a pt. in $M \vee N$, so $\operatorname{Im}(\varphi) = 0$. So $\ker(\psi) = 0$ as well, and ψ is an isomorphism.

Then $H_{n-1}(M\#N) \cong H_{n-1}(M) \oplus H_{n-1}(N)$.

In general, $H_k(M\#N) \cong H_k(M) \oplus H_k(N)$ (ofc for $k>0$.)

By same logic, can conclude $H^k(M\#N) \cong H^k(M) \oplus H^k(N)$. So $\alpha \oplus \beta$ generates $H^k(M\#N)$ ($\alpha \in H^k(M)$, $\beta \in H^k(N)$) iff $\exists$ $\gamma \oplus \delta$, ($\gamma \in H^{n-k}(M)$, $\delta \in H^{n-k}(M)$) gen. of $H^{n-k}(M\#N)$ st $\alpha \smile \beta$ generates $H^n(M\#N) \cong H^n(M) \oplus H^n(M)$.

So $H^*(M\#N) \cong H^*(M) \oplus H^*(N)$. (mod identif. of $H^0(M\#N)$, $H^n(M\#N)$ elements.)

b) Compute the Euler characteristic of M # N.

careful w/ terms where $H_k(M\#N)\not\cong H_k(M)\oplus H_k(N)$.

$$\chi(M\#N)=\sum_{k=0}^{n}(-1)^k\operatorname{rank}(H_k(M\#N)) = \sum_{k=1}^{n-1}(-1)^k\operatorname{rank}(\underbrace{H_k(M\#N)}_{H_k(M)\oplus H_k(N)}) + \underbrace{(-1)^0\operatorname{rank}(\underbrace{H_0(M\#N)}_{\mathbb{Z}})+(-1)^n\operatorname{rank}(\underbrace{H_n(M\#N)}_{\mathbb{Z}})}_{\chi(S^n)}$$

$$=\sum_{k=1}^{n-1}(-1)^k\operatorname{rank}(H_k(M))+\sum_{k=1}^{n-1}(-1)^k\operatorname{rank}(H_k(N))+\chi(S^n)$$

$$=\boxed{\chi(M)+\chi(N)-\chi(S^{n-1}).}$$

c) Are S^2 x S^2 and CP^2 # CP^2 homotopy equivalent? No.

Using Euler characteristic: Recall from b) that $\chi(M\#N)=\chi(M)+\chi(N)-\chi(S^4)$, where $n=\dim(\mathbb{C}P^2)=4$. (CW structure of $\mathbb{C}P^2=e^0\cup e^2\cup e^4$.)

Then $\underbrace{\chi(\mathbb{C}P^2)}_{e^0\cup e^2\cup e^4}+\underbrace{\chi(\mathbb{C}P^2)}_{e^0\cup e^2\cup e^4}-\underbrace{\chi(S^4)}_{e^0\cup e^4}=3+3-2=4.$

Using the CW structure $S^2=e^0\cup e^2$, deduce, using the construction of pdts of CW structures (Hatcher pg. 8) we have the CW structure:

$\left.\begin{array}{l} S^2=e^0_1\cup e^2_1 \\ S^2=e^0_2\cup e^2_2\end{array}\right\}$ $S^2\times S^2$: 0-cell $e^0_1\times e^0_2$, 2-cell $e^0_1\times e^2_2$, 2-cell $e^2_1\times e^0_2$, 4-cell $e^2_1\times e^2_2$. So it follows that $\chi(S^2\times S^2)=1+2+1=4$. (This didn't work.)

They have the same Euler characteristic, so this doesn't work.

$$H^*(S^2\times S^2)\underset{\text{künneth}}{\cong} H^*(S^2)\otimes_{\mathbb{Z}}H^*(S^2)\cong \mathbb{Z}[\alpha]/(\alpha^2)\otimes_{\mathbb{Z}}\mathbb{Z}[\beta]/(\beta^2),\quad |\alpha|,|\beta|=2.$$

$$H^*(\mathbb{C}P^2\#\mathbb{C}P^2)\cong \underset{(\text{mod identif.})}{H^*(\mathbb{C}P^2)\oplus H^*(\mathbb{C}P^2)}\cong \mathbb{Z}[\gamma]/(\gamma^3)\oplus\mathbb{Z}(\delta)/(\delta^3),\quad |\gamma|=|\delta|=2.$$

Note $\alpha\cup\beta\in H^4(S^2\times S^2)\neq 0$ but $(\gamma,0)\cup(0,\delta)\in H^4(\mathbb{C}P^2\#\mathbb{C}P^2)=0$, so their cohomology rings are not isomorphic.

Note: for part a), suppose M and N are not necessarily orientable. Then need to do some casework.

(case I) one of M, N is orientable (WLOG, assume it's M.) Then the LES is:

$$\cdots\leftarrow\underbrace{\tilde H^2(S^{n-1})}_{H_{n-2}(S^{n-1})=0}\leftarrow\underbrace{\tilde H^2(M\#N)}_{H_{n-2}(M\#N)}\leftarrow\underbrace{\tilde H^2(M)\oplus\tilde H^2(N)}_{H_{n-2}(M)\oplus H_{n-2}(N)\ (\text{known.})}\leftarrow\underbrace{\tilde H^1(S^{n-1})}_{H_{n-1}(S^{n-1})=\mathbb{Z}}\leftarrow\underbrace{\tilde H^1(M\#N)}_{H_{n-1}(M\#N)}\leftarrow\underbrace{\tilde H^1(M)\oplus\tilde H^1(N)}_{H_{n-1}(M)\oplus H_{n-1}(N)\ (\text{need to find this.})}\leftarrow\underbrace{\tilde H^0(S^{n-1})}_{0}\leftarrow\cdots$$

Finding $H_{n-1}(M\#N)$:

$$\cdots\to\underbrace{H_n(S^{n-1})}_{0}\to\underbrace{H_n(M\#N)}_{0\ (\text{nonorientable.})}\to\underbrace{\underbrace{H_n(M)}_{\mathbb{Z}}\oplus\underbrace{H_n(N)}_{0}}_{H_n(M\vee N)}\xrightarrow{\varphi=0}\underbrace{H_{n-1}(S^{n-1})}_{\mathbb{Z}}\to H_{n-1}(M\#N)\xrightarrow[\text{surjective}]{\psi}H_{n-1}(M)\oplus H_{n-1}(N)\to 0\to\cdots$$

Since $\operatorname{Im}(\varphi)=0$ still, ψ is also injective, so $H_{n-1}(M\#N)\cong H_{n-1}(M)\oplus H_{n-1}(N)$.

(Case II) Both M and N are nonorientable.

Get
$$\cdots\to\underbrace{H_n(M\#N)}_{0\ (M\#N\text{ nonorientable.})}\to\underbrace{H_n(M)\oplus H_n(N)}_{0}\to\underbrace{H_{n-1}(S^{n-1})}_{\mathbb{Z}}\xrightarrow{\text{injective.}}H_{n-1}(M\#N)\to H_{n-1}(M)\oplus H_{n-1}(N)\to 0\to\cdots$$

4a) Show that a closed manifold of dimension 4n + 2 with odd Euler characteristic is non-orientable.

Suppose M is closed and orientable, and has dimension $4n+2$.

Recall $\chi(M) = \sum_k (-1)^k \operatorname{rank}(H_k(M))$ (Thm 2.44), and by Poincaré, $H_k(M) \cong H^{4n+2-k}(M)$.

By UCT, $H_{4n+2-k}(M) \cong H^k(M) \cong H_k(M)/\underline{T_k} \oplus \underline{T_{k-1}}$, so $\operatorname{rank}(H_{4n+2-k}(M)) = \operatorname{rank}(H_k(M))$.

do not affect rank.

Then $\chi(M) = \sum_{k=0}^{4n+2} (-1)^k \operatorname{rank}(H_k(M)) = \sum_{k=0}^{2n+1} (-1)^k \operatorname{rank}(H_k(M)) + \underbrace{\sum_{k=2n+1}^{4n+2} (-1)^k \operatorname{rank}(H_k(M))}_{} - \underbrace{(-1)^{2n+1}\operatorname{rank}(H_{2n+1}(M))}_{\text{double counted}} = 2\left(\sum_{k=0}^{2n+1}(-1)^k \operatorname{rank}(H_k(M))\right) + \operatorname{rank}(H_{2n+1}(M))$.

$\llcorner = \sum_{k=0}^{2n+1} (-1)^{4n+2-k} \operatorname{rank}(H_{4n+2-k}(M))$

$= \sum_{k=0}^{2n+1} (-1)^k \operatorname{rank}(H_k(M))$ since $4n+2-k$, k have same parity.

Now show $\operatorname{rank}(H_{2n+1}(M))$ is even. (This is hard!)

Thm 3.11 $\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$ for $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$, given R is commutative.

Let $\alpha, \beta \in H^{2n+1}(M;\mathbb{Z})$.

By Thm 3.11, $\alpha \smile \alpha = (-1)^{(2n+1)(2n+1)} \alpha \smile \alpha = -\alpha \smile \alpha \Rightarrow \alpha \smile \alpha = 0$.

Furthermore, $\alpha \smile \beta = (-1)^{(2n+1)(2n+1)} \beta \smile \alpha = -\beta \smile \alpha$.

(The cup pdt pairing is nonsingular for closed R-orientable manifolds when R is a field, or $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.)

Nonsingular: Bilinear pairing $\varphi: A \times B \to R$ is nonsingular if maps $A \to \operatorname{Hom}_R(B,R)$, $B \to \operatorname{Hom}_R(A,R)$ are isomorphisms. Boils down to: φ can be represented by $GL(n,R)$ matrix.

Consider the cup-pdt pairing $H^k(M;\mathbb{R}) \times H^{n-k}(M;\mathbb{R}) \to \mathbb{R}$, $(\alpha,\beta) \mapsto (\alpha,\beta)[M]$. By corollary 3.38, this pairing is nonsingular.

By Corollary 3.39, $\alpha \in H^{2n+1}(M)$ is a generator for an infin. cyclic summand in $H^{2n+1}(M;\mathbb{R})$ iff $\exists\, \beta \in H^{2n+1}(M)$ st $\alpha \smile \beta$ generates $H^{4n+2}(M)$.

Corollary 3.39 An element $0 \neq \alpha \in H^k(M;\mathbb{F})$ generates an infinite cyclic summand of $H^k(M;\mathbb{F})$ iff $\exists$ element $\beta \in H^{n-k}(M)$ st $\alpha \smile \beta$ is a generator of $H^n(M;\mathbb{F})$.

As a result, if we order generators of $H^{2n+1}(M;\mathbb{R})$ st generators that are paired together to generate elements in H^{4n+2} are adjacent to ea. other, we get the block diagonal form:

$$\begin{pmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & & & 0 \\ & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & & \\ & & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & \\ & & & \ddots & \\ 0 & & & & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \end{pmatrix}$$

To explain ea. block:

paired together st $\alpha \smile \beta$ generates $H^{4n+2}(M)$

$$\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & 0 & 1 \\ \beta & -1 & 0 \end{array}$$

$\alpha \smile \alpha = 0$ (1st entry)
$\alpha \smile \beta = 1$ (generator)
$\beta \smile \alpha = -1$

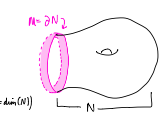
Column space of ea. block: $\binom{0}{-1}, \binom{1}{0}$

We conclude that $\operatorname{rank}(A) = 2n$, $A \in GL(\operatorname{rank}(H^{2n+1}(M)), \mathbb{R})$, so $\operatorname{rank}(H^{2n+1}(M)) = 2n$.

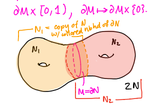
Then it follows that $\chi(M)$ is even. ∎

b) Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even.
Is RP^2n an oriented boundary?

Suppose M is even-dimensional (for if it were odd-dimensional, $\chi(M) = 0$ by Corollary 3.37.)



$\partial N = M$ has a collared nbhd (open nbhd homeomorphic to $\partial M \times [0,1)$, $\partial M \mapsto \partial M \times \{0\}$.)



Generalized Rank-Nullity

Given an exact sequence $0 \to V_1 \to V_2 \to \dots \to V_r \to 0$, $\sum_{i=1}^r (-1)^i \operatorname{rank}(V_i) = 0$. (This is proven by induction using the usual Rank-Nullity Thm, $T: V \to W$, $\dim V = \operatorname{Rank}(T) + \operatorname{Null}(T)$.)

By Prop. 3.42, M has a collar nbhd since $M = \partial N$, N compact and w/ boundary.

Let N_1, N_2 be two copies of N, along with a collared nbhd around $M = \partial N$ (thus, they are two open sets where $N_1 \cup N_2 = 2N$ and $N_1 \cap N_2 = \partial N = M$.) We can write a Mayer-Vietoris LES: ($n = \dim(N)$)

$$\dots 0 \to \underbrace{H_n(M)}_{V_1} \to \underbrace{H_n(N_1) \oplus H_n(N_2)}_{V_2} \to \underbrace{H_n(2N)}_{V_3} \to \underbrace{H_{n-1}(M)}_{V_4} \to \dots \to H_0(M) \to H_0(N_1) \oplus H_0(N_2) \to H_0(2M) \to 0.$$

($M = \partial N$ written above M; "conclude" written below.)

Then by the generalized Rank-Nullity Thm (stated above),
$$0 = \underbrace{\sum_{i=1}^n (-1)^i \operatorname{rank}(H_i(M))}_{\chi(M)} + \underbrace{\sum_{i=1}^n (-1)^i \operatorname{rank}(H_i(2N))}_{\chi(2N)} + \underbrace{\sum_{i=1}^n (-1)^{i+1} \operatorname{rank}(H_i(N_1) \oplus H_i(N_2))}_{-2\chi(N)}$$
$\operatorname{rank}(H_i(N_1) \oplus H_i(N_2)) = \operatorname{rank}(H_i(N_1)) + \operatorname{rank}(H_i(N_2)) = 2\operatorname{rank}(H_i(N))$

$\Rightarrow \chi(M) + \chi(2N) - 2\chi(N) = 0$. If M is even-dimensional, Note N is odd-dimensional, so its double, $2N$ is a closed odd-dimensional manifold $\Rightarrow \chi(2N) = 0$.

Then $\chi(M) = 2\chi(N) \Rightarrow \chi(M)$ is even. ∎

We use the fact to show $\mathbb{R}P^{2n}$ is not an oriented boundary.

Recall we can compute $H_k(\mathbb{R}P^{2n})$:
$$H_k(\mathbb{R}P^{2n}) = \ker d_k / \operatorname{im} d_{k+1} = \begin{cases} \mathbb{Z} & \text{if } k = 0 \quad (\text{rank} = 1) \\ \mathbb{Z}_2 & \text{if } k \text{ is odd, b/w } 0 \text{ and } 2n \quad (\text{rank} = 0) \\ 0 & \text{otherwise} \quad (\text{rank} = 0) \end{cases}$$

$$\dots \to 0 \to \underset{e^{2n}}{\mathbb{Z}_2} \xrightarrow{d_{2n}=2} \mathbb{Z}_2 \xrightarrow{d_{2n-1}=0} \dots \xrightarrow{d_2=2} \mathbb{Z}_2 \xrightarrow{d_1=0} \mathbb{Z}_2 \xrightarrow{d_0=0} 0$$

So $\chi(\mathbb{R}P^{2n}) = \sum_{k=0}^{2n} (-1)^k \operatorname{rank}(H_k(\mathbb{R}P^2)) = 1 \neq 0$

c) Is the suspension of RP^2 homotopy equivalent to a manifold? Explain. No. We show this using Euler characteristic.

Recall that $H_k(X) \cong H_k(SX)$.

$\Rightarrow \chi(SX) = \sum_{k=0}^n (-1)^k \operatorname{rank}(SX) = \sum_{k=0}^n (-1)^k \operatorname{rank}(X)$, so here, $\chi(S(\mathbb{R}P^2)) = 1$.

$$H_k(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & k = 0 \quad \text{rank } 1 \\ \mathbb{Z}_2 & k = 1 \quad \text{rank } 0 \\ 0 & \text{otherwise} \quad \text{rank } 0 \end{cases}$$

$$\dots \to 0 \xrightarrow{d_3=0} \underset{\text{2-cell}}{\mathbb{Z}} \xrightarrow{d_2=2} \underset{\text{1-cell}}{\mathbb{Z}} \xrightarrow{d_1=0} \underset{\text{0-cell}}{\mathbb{Z}} \xrightarrow{d_0=0} 0$$

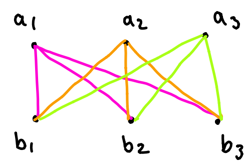
However, examining the CW structure of $S(\mathbb{R}P^2)$, we conclude $S(\mathbb{R}P^2)$ has dimension 3. (CW structure of $\mathbb{R}P^2$: $e^0 \cup e^1 \cup e^2 \Rightarrow$ highest dimensional cells of
I: $e_1^0 \cup e_2^0 \cup e^1$

$S\mathbb{R}P^2 = \mathbb{R}P^2 \times I \big/ \substack{\mathbb{R}P^2 \times \{0\} \sim x_0 \\ \mathbb{R}P^2 \times \{1\} \sim x_1}$ are 3-cells. So $\dim(S(\mathbb{R}P^2)) = 3$.)

Furthermore, $S\mathbb{R}P^2$ is closed. So by Thm 3.37, $\chi(S(\mathbb{R}P^2)) = 0$, ↯

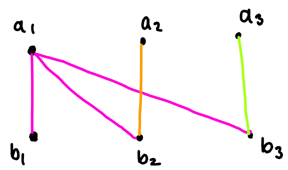
January 2016

1) the graph K has six vertices a_1, a_2, a_3, b_1, b_2, b_3 and nine edges a_ib_j (i, j = 1, 2, 3). The space X is obtained from K by attaching a 2-cell along each loop formed by a cycle of four edges in K. Find the fundamental group of X.



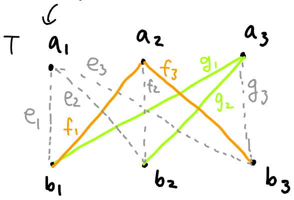
K

maximal spanning tree T



X − T

choose as base vertex.



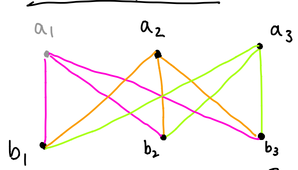
Generators of $\pi_1(K)$:

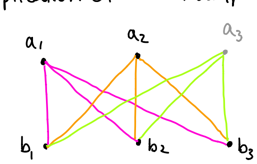
① $e_1 f_1 f_2 e_2$ (crosses f_1)
② $e_1 g_1 g_3 e_3$ (crosses g_1)
③ $e_2 g_2 g_3 e_3$ (crosses g_2)
④ $e_3 f_3 f_2 e_2$ (crosses f_3)

Conclude that $\pi_1(K)$ is generated by four cycles w/ four edges.

Recall by corollary of Van Kampen that $\pi_1(X) = \pi_1(K)/N$, N generated by the gluing maps. The gluing instr. are cycles with four edges, and are hence a subset of $\pi_1(K)$. On the other hand, $\pi_1(K)$ is generated by a cycle w/ 4 edges, a subset of N. Then $N \cong \pi_1(K) \Rightarrow \pi_1(X) \cong 0$.

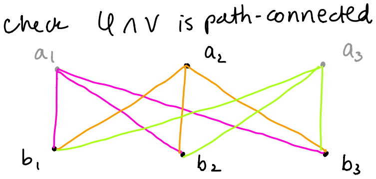
Another approach(?) Direct application of Van Kampen. (unfinished.)





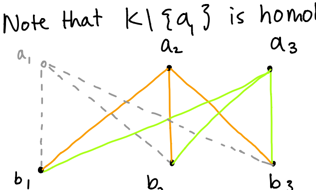
consider $U = X \setminus \{a_1\}$ and $V = X \setminus \{a_3\}$

check $U \cap V$ is path-connected:



$U \cap V = X \setminus \{a_1, a_3\}$.

Note that $K \setminus \{a_1\}$ is homotopy equivalent to K':

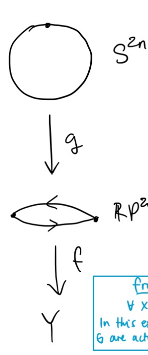


K'

gray: contractible portions of U.

2) Show that if f: RP^2n —> X is a covering map of a CW complex X, then f is a homeomorphism.

Sketch of proof:



Note $f: \mathbb{R}P^{2n} \to Y$ is a finite-sheeted covering map and $g: S^{2n} \to \mathbb{R}P^{2n}$ is a two-sheeted covering, so $gf: S^{2n} \to Y$ is also a finite sheeted covering. In particular, since S^{2n} is simply-connected, S^{2n} is the universal cover of Y. Then by Prop. 1.39, the group of deck transformations G of $S^{2n} \to S^{2n}$ ($G \circlearrowright S^{2n}$) is isomorphic to $\pi_1(Y)$. Since group actions are free, by Prop. 2.29, $\mathbb{Z}_2 \cong G$ necessarily. Then $\mathbb{Z}_2 \cong \pi_1(Y) \overset{\text{Prop 1.32}}{\Rightarrow} S^{2n}$ is a two-sheeted cover of Y. Since S^{2n} is a two-sheeted cover of both $\mathbb{R}P^{2n}$ and Y, f must be a one-sheeted covering, and thus a homeomorphism.

Prop. 1.39 $p: \tilde{X} \to X$; $H = p_*(\pi_1(\tilde{X}))$ (path-connected; pc, lpc)
$S^{2n} \to Y$; $H = p_*(\pi_1(S^{2n})) = \{1\}$
b) $G(\tilde{X}) \cong N(H)/H$; here, $G(S^{2n}) \cong \pi_1(Y)/\{1\} \cong \pi_1(Y)$.

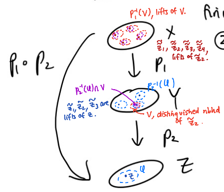
Prop. 2.29 $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even. (proof involves degree maps.)

free action: $G \times X \to X$ is free if $\forall x \in X$, $gx = x \Rightarrow g = Id$ for all g, x. In this example, $X = S^{2n}$ and deck transformations G are actions on S^{2n}, which by defn. are free.

Prop. 1.32 The # of sheets of $p: \tilde{X} \to X$ w/ $\tilde{X}, X$ path-connected equals the index of $p_*(\pi_1(\tilde{X}))$ in $\pi_1(X)$. Here, $p_*(\pi_1(S^{2n})) = \{1\}$ and $\pi_1(Y) = \mathbb{Z}_2$, so $|\mathbb{Z}_2 : \{1\}| = |\mathbb{Z}_2| = 2$.

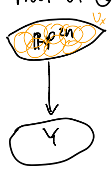
Taken for granted: ① Given finite sheeted covers $p_1: X \to Y$ and $p_2: Y \to Z$, $p_1 \circ p_2 : X \to Z$ is also a finite-sheeted cover.

② $f: \mathbb{R}P^{2n} \to Y$ is a finite-sheeted cover.



① Choose an arbitrary $z \in Z$. $\exists$ $\mathcal{U}$, nbhd of z, st $p_2^{-1}(\mathcal{U})$ maps homeomorphically onto $\mathcal{U}$. Since p_2 is finite-sheeted, $\exists$ finitely many lifts of z, $\tilde{z}_i$. For each of these, choose nbhd W_i st $p_1^{-1}(W_i)$ maps homeomorphically onto W_i. (again, there are finitely many disjoint nbhds in the preimage.) Consider nbhds of $\tilde{\tilde{z}}_i$, $p^{-1}(p_2^{-1}(\mathcal{U}) \cap W_i)$. Restricted to these nbhds, $p_1 \circ p_2$ is a homeomorphism, and since there are finitely many preimages of W_i, there are finitely many such nbhds.

Proof of ②: $\forall$ $y \in Y$, choose a nbhd $\mathcal{U}_y$ st $f^{-1}(\mathcal{U}_y)$ maps homeomorphically onto $\mathcal{U}_y$ (exists b/c f is covering map). Construct an open cover of $\mathbb{R}P^{2n}$ containing $\bigcup_y f^{-1}(\mathcal{U}_y)$, which has a finite subcover. Then $f^{-1}(\mathcal{U}_y)$ can be covered finitely, so $|f^{-1}(\mathcal{U}_y)| < \infty$, so f is finite-cheeted.



3) Show that the reduced homology groups of the join X * Y are given by the relative homology groups of the pair (X x Y, X V Y).

Tullia was supposed to email me the answer to this one, but she never did. She promised that if she writes the qual, this will not be one of the questions!

January 2017

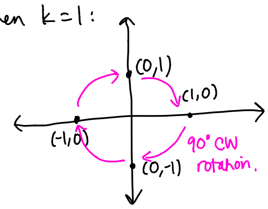
1a) Is there a continuous map f: RP^(2k - 1) to RP^(2k - 1) with no fixed points? See Hatcher pg. 180.

Yes; consider the rotation $f: \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ defined by $f(x_1, x_2, x_3, \dots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \dots, x_{2k}, -x_{2k-1})$.

Note $\mathbb{R}P^{n-1} \cong (\mathbb{R}^n \setminus \{0\})/v \sim \lambda v$, and since f takes lines through the origin to lines through the origin, it induces a well-defined map on $\mathbb{R}P^{2k-1}$, ie $\bar{f}: \mathbb{R}P^{2k-1} \to \mathbb{R}P^{2k-1}$, $[(x_1, x_2, \dots, x_{2k})] \mapsto [(x_2, -x_1, \dots, x_{2k}, -x_{2k-1})]$.

This map clearly has no fixed pts.

ie when $k=1$: rotation of xy-plane by $90°$ CW.



We know fixed pts. of $\bar{f}$ correspond to eigenvectors of f. (since if v is eigenvector of f, then $\bar{f}(v) = \lambda v \sim v$.)

Note map $f: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is described by matrix:

$$\begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & -1 & 0 & & 0 & 0 \\ & & \vdots & \vdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & & -1 & 0 \end{bmatrix} \\ \begin{array}{ccccccc} f(e_1) & f(e_2) & f(e_3) & f(e_4) & \cdots & f(e_{2k-1}) & f(e_{2k}) \end{array} \end{array}$$

This matrix has characteristic polynomial w/ even degree and has no real roots (see calculation below.)

$$\underbrace{\begin{bmatrix} \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} \lambda & & & & 0 \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ 0 & & & & \lambda \end{bmatrix}}_{\lambda I} = \begin{bmatrix} \boxed{\begin{smallmatrix} \lambda & 1 \\ -1 & \lambda \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} \lambda & 1 \\ -1 & \lambda \end{smallmatrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{smallmatrix} \lambda & 1 \\ -1 & \lambda \end{smallmatrix}} \end{bmatrix}$$

block diagonal matrix. det of ea. of blocks: λ^2+1

Characteristic polynomial: $(\lambda^2+1)^n = 0$ iff $\lambda = \pm i$. $\Rightarrow$ No real eigenvectors so no fixed points for $\bar{f}$.

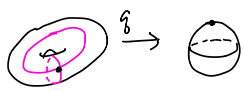
Note if we change to $f: \mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$, induced map $\bar{f}: \mathbb{R}P^{2k} \to \mathbb{R}P^{2k}$ will have fixed pts.

Modulo torsion, $H_k(\mathbb{R}P^{2k}) \cong H_k(\text{pt.})$, so by Lefschetz fixed pt. thm, $\bar{f}$ will have fixed pts.

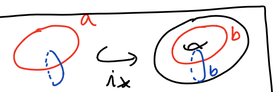
1b) Construct a homotopically essential map S^1 x S^1 —> S^2 and justify your construction.

(Def.: "essential" $\Rightarrow$ not $\overbrace{\text{homotopic to constant}}^{\text{(nullhomotopic.)}}$.) Inspiration from Hatcher Chp. 2.2 Q 12.

$q: S^1 \times S^1 \to S^2$, $S^1 \vee S^1 \mapsto pt$. is not nullhomotopic. Consider the relative LES:



$$\cdots \to \underbrace{H_3(S^2)}_{0} \to \underbrace{H_2(S^1 \vee S^1)}_{0} \xrightarrow{\phi} \underbrace{H_2(S^1 \times S^1)}_{\mathbb{Z}} \xrightarrow{q_*} \underbrace{H_2(S^2)}_{\mathbb{Z}} \xrightarrow{\Psi} \underbrace{H_1(S^1 \vee S^1)}_{\mathbb{Z}^2} \underset{\cong}{\xrightarrow{i_*}} \underbrace{H_1(S^1 \times S^1)}_{\mathbb{Z}^2} \to \cdots$$



$i_*(a) = a$, $i_*(b) = b$

i_* is an isomorphism.

① We know i_* is an isomorphism, so $\ker(i_*) = \operatorname{Im}(\Psi) \Rightarrow \operatorname{Im}(\Psi) = 0$.

② Then $\ker(\Psi) = \mathbb{Z} \Rightarrow \operatorname{Im}(q_*) \cong \mathbb{Z} \Rightarrow q_*$ is surjection.

③ q_* is also injective by exactness.

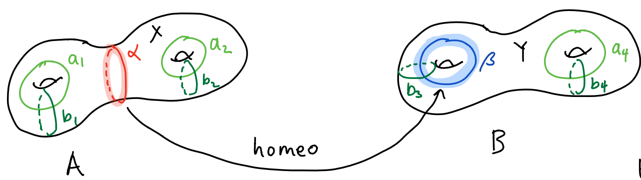
So q_* is an isomorphism $\Rightarrow H_2(S^1 \times S^1) \cong H_2(S^2)$.

I claim this shows q is not nullhomotopic.

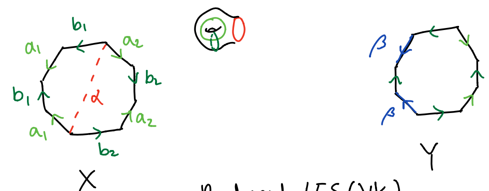
A nullhomotopic map $f: S^1 \times S^1 \to S^2$ induces a zero map on homology since $f \simeq c$, where c is constant map means $H_n(f(S^1 \times S^1)) = H_n(c(S^1 \times S^1)) = H_n(s_0^{\in S^2}) \cong 0$ for $n=2$. But if q_* is an isomorphism and since $H_2(S^1 \times S^1) \not\cong 0$, we know q_* is not the zero map. So q cannot be nullhomotopic, as desired.

[Alternatively, show f is degree 1 map by showing $\underbrace{S^1 \times S^1 / (S^1 \vee S^1)}_{S^1 * S^1} \underset{\text{homeo}}{=} S^2$, so $f_*: H_2(S^1 \times S^1 / S^1 \vee S^1) \to H_2(S^2)$ is deg 1 map b/c homeos have degree 1.]

2a) Compute the integral homology groups of Z.



final space : Z.



Clearly, Z path-connected, so $H_0(Z) \cong \mathbb{Z}$.
Also Z is CW complex of dim 2, so $H_k(Z) \cong 0$ for $k \geq 3$.
It suffices to find $H_k(Z)$ for $k = 1, 2$.
For this, we use Mayer Vietoris on:
$A = X \cup \{\text{collar nbhd of } \alpha\}$ ($\overset{h.e.}{\simeq} X$.)
$B = Y \cup \{\text{collar nbhd of } \beta\}$ ($\overset{h.e.}{\simeq} Y$.)

Reduced LES (VK)

(obviously injective)
$1 \mapsto ((0,0,0,0),(1,0,0,0))$

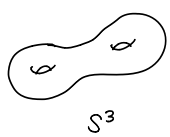
$$\cdots \to \underbrace{\tilde{H}_3(Z)}\to \underbrace{\tilde{H}_2(S^1)}_{0} \to \underbrace{\tilde{H}_2(X)}_{\mathbb{Z}} \oplus \underbrace{\tilde{H}_2(Y)}_{\mathbb{Z}} \xrightarrow{\varphi} \tilde{H}_2(Z) \xrightarrow{\alpha} \underbrace{\tilde{H}_1(S^1)}_{\mathbb{Z}} \xrightarrow{\psi} \underbrace{\tilde{H}_1(X)}_{\mathbb{Z}^4} \oplus \underbrace{\tilde{H}_1(Y)}_{\mathbb{Z}^4} \to \tilde{H}_1(Z) \to \underbrace{\tilde{H}_0(S^1)}_{0} \to \cdots$$

① $\tilde{H}_1(Z) \cong \tilde{H}_1(X) \oplus \tilde{H}_1(Y) / \operatorname{Im}(\psi)$. $\cong \mathbb{Z}^8 / \operatorname{Im}\psi = \langle a_1, a_2, \beta, a_4, b_1, b_2, b_3, b_4 \mid \beta \rangle \cong \mathbb{Z}_7$.

② know $0 = \ker\psi = \operatorname{Im}\alpha$. So $\operatorname{Im}(\varphi) = \ker\alpha = \tilde{H}_2(Z) \Rightarrow \varphi$ is surjective. But also, φ is injective by exactness, so $\tilde{H}_2(X) \oplus \tilde{H}_2(Y) \cong \tilde{H}_2(Z) \Rightarrow H_2(Z) \cong \mathbb{Z}^2$.

So
$$\boxed{H_k(Z;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=0 \\ \mathbb{Z}^7 & \text{for } k=1 \\ \mathbb{Z}^2 & \text{for } k=2 \\ 0 & \text{otherwise.} \end{cases}}$$

2b) Compute the integral homology groups of W.



F, closed surface embedded in S^3.

Again, use van Kampen. Since W is CW complex w/ no dim. ≥ 4 cells, $H_k(W) \cong 0$, $k \geq 4$.
And W is path-connected, so $H_0(W) \cong \mathbb{Z}$.
Suffices to find $H_k(W)$ for $k = 1, 2, 3$.

$A, B = S^3 \cup \{\text{nbd of } F \text{ homeo to } F \times \mathbb{R}\}$. $\overset{he}{\simeq} S^3$. We get LES:

$$\cdots \to \underbrace{\tilde{H}_4(Z)}_{0} \to \underbrace{\tilde{H}_3(F)}_{0} \to \underbrace{\tilde{H}_3(S^3)}_{\mathbb{Z}} \oplus \underbrace{\tilde{H}_3(S^3)}_{\mathbb{Z}} \xrightarrow{\cong} \tilde{H}_3(W) \xrightarrow{\alpha} \underbrace{\tilde{H}_2(S^1)}_{0} \to \underbrace{\tilde{H}_2(S^3)}_{0} \oplus \underbrace{\tilde{H}_2(S^3)}_{0} \xrightarrow{\cong} \tilde{H}_2(W) \to \underbrace{\tilde{H}_1(S^3)}_{0} \to \cdots$$

$$\cdots \to \underbrace{\tilde{H}_1(S^3)}_{0} \oplus \underbrace{\tilde{H}_1(S^3)}_{0} \xrightarrow{\cong} \tilde{H}_1(W) \to \underbrace{\tilde{H}_1(S^3)}_{0} \oplus \underbrace{\tilde{H}_1(S^3)}_{0} \to \tilde{H}_1(W) \xrightarrow{\cong} \underbrace{\tilde{H}_0(S^1)}_{0} \to \underbrace{\tilde{H}_0(S^3) \oplus \tilde{H}_0(S^3)}_{0} \to \underbrace{\tilde{H}_0(W)}_{0} \to 0.$$

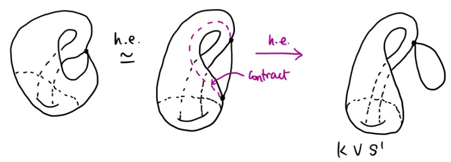
From exactness, we immediately conclude $H_2(W) \cong 0$ and $H_1(W) \cong 0$.

So $H_3(S^3) \oplus H_3(S^3) \cong H_3(W) \Rightarrow H_3(W) \cong \mathbb{Z}^2$.

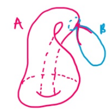
Deduce
$$\boxed{H_k(W) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^2 & k=3 \\ 0 & \text{otherwise.} \end{cases}}$$

January 2018

1) Let X be a quotient space of the Klein bottle obtained by identifying two distinct points. Compute the fundamental group and all the homology groups of X.



Apply Van Kampen: A deformation retracts to K; B deformation retracts to S^1.



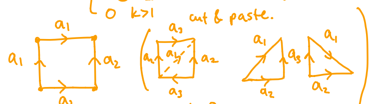
Then by VK, $\pi_1(X) = \pi_1(K) * \pi_1(S_1) / 0$ since their intersection is contractible. So $\boxed{\pi_1(X) \cong \langle a, b, c \mid aba^{-1}b \rangle.}$

To compute homology groups, use Mayer–Vietoris.

A and B are as described before. Recall $B \simeq S^1$ and $A \simeq K$. Then the (nontrivial parts) MV sequence is: $A \simeq K$, $B \simeq S^1$, $A \cap B \simeq pt.$

$H_k(S^1) = \begin{cases} 0 & \text{for } k \neq 1 \\ \mathbb{Z} & \text{for } k = 1 \end{cases}$

$H_k(K) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } k = 1 \\ 0 & k > 1 \end{cases}$ (in gen., $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.)



$\ldots \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1 = 0} \mathbb{Z} \xrightarrow{d_0 = 0} 0$

$H_1(K) = \langle \underbrace{a_1, a_2}_{\mathbb{Z}^2} \mid \underbrace{2a_1 + 2a_2}_{Img(d_2)} \rangle = \langle a_1, a_1 + a_2 \mid 2a_1 + 2a_2 \rangle$

① $\ldots 0 \to \underbrace{H_2(pt)}_{0} \to \underbrace{H_2(K)}_{0} \oplus \underbrace{H_2(S^1)}_{0} \xrightarrow{\cong} H_2(X) \to \underbrace{H_1(pt.)}_{0} \to \ldots$

So $H_2(X) \cong 0$.

② $\ldots \to \underbrace{\tilde{H}_1(pt)}_{0} \to \underbrace{\tilde{H}_1(K)}_{\mathbb{Z} \oplus \mathbb{Z}_2} \oplus \underbrace{\tilde{H}_1(S^1)}_{\mathbb{Z}} \xrightarrow{\cong} \tilde{H}_1(X) \to \underbrace{\tilde{H}_0(pt)}_{0} \to \ldots$

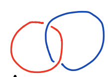
So $\tilde{H}_1(X) \cong H_1(X) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$.

Conclude $\boxed{H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \text{ (path-connected.)} \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}}$

2) Determine which of the following spaces are homotopy equivalent to each other.

a) $S^1 \vee S^1$.

b) The complement in S^3 of Hopf Link.



(Hopf Link)

c) The complement in S^3 of unlink of 2 circles.

d) The complement in $\mathbb{R}^3$ of two parallel lines.

e) The complement in $\mathbb{R}^3$ of two intersecting lines.

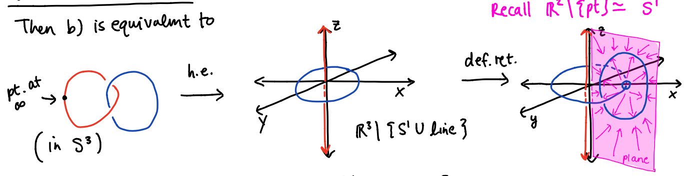
Claim: $S^1 \vee S^1 \overset{h.e.}{\simeq} \mathbb{R}^3 \setminus \{2 \text{ parallel lines}\}$

The others are not homotopic to ea. other.

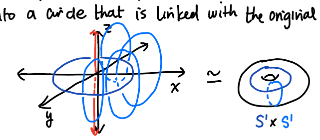
b) is h.e. to a torus: consider $S^3 \setminus \{pt\}$ as the one-pt. compactification (Alexandroff extension) of $\mathbb{R}^3$. (analogous to $\underset{S^2 \cup \{\infty\}}{\overset{\infty}{\circ}} \simeq \underset{\mathbb{R}^2}{\square}$.)

Then b) is equivalent to

Recall $\mathbb{R}^2 \setminus \{pt\} \overset{h.e.}{\simeq} S^1$



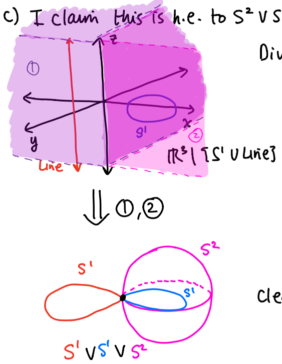
Notice S^1 intersects an open half-plane that borders the line (in pink) at one point. This punctured open half-plane deformation retracts onto a circle that is linked with the original circle.



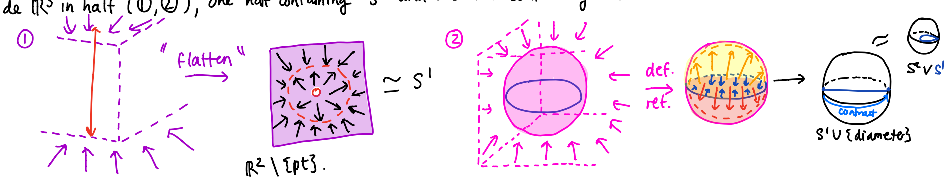
If we consider all open half planes that intersect S^1 at a point, and all deformation retractions, we obtain a torus.

Obviously, $S^1 \times S^1 \simeq T^2 \setminus \{pt\} \not\simeq T^2 \simeq S^3 \setminus \{\text{Hopf Link}\}$.

c) I claim this is h.e. to $S^2 \vee S^1 \vee S^1$. Again, take a pt. on one of the circles as $\{\infty\}$ and recall $S^3 \cup \{\infty\} \cong \mathbb{R}^3$. So our space becomes:



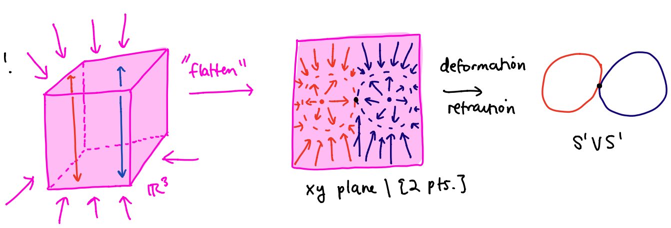
Divide $\mathbb{R}^3$ in half (①,②), one half containing S^1 and the other containing the line. One half (①) deformation retracts to S^1.



And the other (via ex. 1.23 in Hatcher) deformation retracts to $S^2 \vee S^1$.

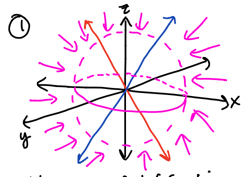
Clearly, since $H_2(S^2 \vee S^1 \vee S^1) \cong \mathbb{Z} \neq 0 = H_2(S^1 \vee S^1)$, $\pi_1(S^2 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \not\cong \mathbb{Z}^2 \cong \pi_1(T^2)$, so c) is not h.e. to a) or b).

d) I claim this is h.e. to $S^1 \vee S^1$.

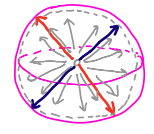


Flatten $\mathbb{R}^3 \setminus \{L_1, L_2\}$ onto the x-y-plane, which L_1, L_2 intersect at 2 pts. Then $\mathbb{R}^2 \setminus \{p_1, p_2\}$ deformation retracts onto $S^1 \vee S^1$.

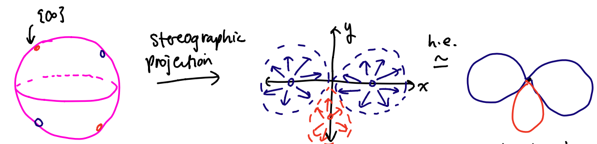
e) I claim this def. retracts onto $S^1 \vee S^1 \vee S^1$.

① 

Points outside S^2 deformation retract onto S^2.

② 

We can "enlarge" the point of intersection of L_1 and L_2, deformation retracting the space onto $S^2 \setminus \{4 \text{ pts.}\}$.

③ 

Again, consider $S^2 \setminus \{pt\}$ as the one-point compactification of $\mathbb{R}^2$, so our space is h.e. to $\mathbb{R}^2 \setminus \{3 \text{ pts.}\}$.

$S^1 \vee S^1 \vee S^1$.

Note $\pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}^2 \cong \pi_1(T^2)$
$\neq \mathbb{Z} * \mathbb{Z} \cong \pi_1(S^1 \vee S^1), \pi_1(S^2 \vee S^1 \vee S^1)$ so it follows e) is not h.e. to any of the spaces above.

1a) Show that S^2 x RP^4 and S^4 x RP^2 are not homotopy equivalent.

Künneth Formula The cross product
$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module $\forall k$.

UCT for Homology (Corollary) (3A.6)
b) If $H_n(X;\mathbb{Z})$, $H_{n-1}(X;\mathbb{Z})$ are finitely generated, then for p prime, $H_n(X;\mathbb{Z}_p)$ consists of (i) a $\mathbb{Z}_p$ summand for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$
(ii) a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand in $H_n(X;\mathbb{Z})$, $k \geq 1$.
(iii) a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand in $H_{n-1}(X;\mathbb{Z})$, $k \geq 1$.

$$H^*(S^2 \times \mathbb{RP}^4;\mathbb{Z}_2) \underset{\text{Künneth}}{\cong} H^*(S^2;\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*(\mathbb{RP}^4;\mathbb{Z}_2)$$

$$H^k(S^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k=0,2 \\ 0 & \text{otherwise.} \end{cases} \underset{\substack{\text{UCT} \\ \text{for Homology}}}{\Rightarrow} H^k(S^2;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } k=0,2 \\ 0 & \text{otherwise} \end{cases}$$

Let $\alpha \in H^2(S^2;\mathbb{Z}_2)$ generate $H^2(S^2;\mathbb{Z}_2)$. The only poss. nontrivial cup products are $\alpha \smile 1$ and $\alpha \smile \alpha$. $\alpha \smile \alpha \in H^4(S^2;\mathbb{Z}_2) \cong 0$, so $\alpha^2 = 0$. Then elements of $H^*(S^2;\mathbb{Z}_2)$ take the form $\mathbb{Z}_2[\alpha]/(\alpha^2)$, $|\alpha| = 2$.

We know from Thm in Hatcher $H^*(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^5)$, $|\beta| = 1$.

But also $H^k(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2$ for $k \leq n$. See Ex. 3.40. So $H^*(S^2;\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^2) \underset{\mathbb{Z}_2}{\otimes} \mathbb{Z}_2[\beta]/(\beta^5)$, $|\alpha| = 2$ and $|\beta| = 1$.

$$\left(\cdots \leftarrow 0 \leftarrow \mathbb{Z}_2 \xleftarrow{d_n^* = 0} \mathbb{Z}_2 \xleftarrow{d_{n-1}^* = 0} \cdots \xleftarrow{d_2^* = 0} \mathbb{Z}_2 \xleftarrow{d_1^* = 0} \mathbb{Z}_2 \xleftarrow{d_0^* = 0} 0 \right)$$
Cochain complex

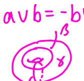
By similar calculations, $H^*(S^4 \times \mathbb{RP}^2;\mathbb{Z}_2) \cong H^*(S^4;\mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\gamma]/(\gamma^2) \underset{\mathbb{Z}_2}{\otimes} \mathbb{Z}_2[\delta]/(\delta^3)$, $|\gamma| = 4$ and $|\delta| = 1$.

If there were an isomorphism $H^*(S^2 \times \mathbb{RP}^4;\mathbb{Z}_2) \to H^*(S^4 \times \mathbb{RP}^2;\mathbb{Z}_2)$, it would send $\beta \mapsto \pm\delta$ (since $|\beta| = |\delta| = 1$), but $1 \times \beta^3 \in H^*(S^2 \times \mathbb{RP}^4;\mathbb{Z}_2)$ is nontrivial while $1 \times \delta^3 \in H^*(S^4 \times \mathbb{RP}^2;\mathbb{Z}_2)$ is trivial, contradiction.

Since the cohomology rings are different, $S^2 \times \mathbb{RP}^4 \not\simeq S^4 \times \mathbb{RP}^2$.

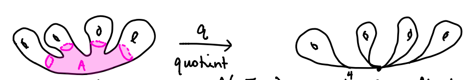
1b) Assuming the cup product structure on S^1 x S^1, compute the cup product structure in H^*(M_4), the orientable surface of genus

Given: (see below for explicit calculations.)
$H^*(S^1 \times S^1) = \Lambda_{\mathbb{Z}}(a,b)$
i.e.: a, b generate $H^1(S^1 \times S^1)$, and
$a \smile a = b \smile b = 0$ and
$a \smile b = -b \smile a = c \in H^2(S^1 \times S^1)$



a, b, c dual to α, β, γ.
(notation changed from calcs, oops.)

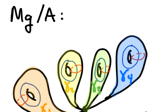
Consider the quotient map: q:



$\bigvee_{i=1}^{4} T_i^2 = M_4/A$

Consider the homology groups of M_4/A:

$$H_k(M_4/A) = \begin{cases} 0 & \text{for } k \geq 3 \\ \mathbb{Z}^4 & \text{for } k=2 \\ \mathbb{Z}^8 & \text{for } k=1 \\ \mathbb{Z} & \text{for } k=0 \end{cases}$$

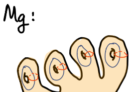


$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^1$
$\beta_1, \beta_2, \beta_3, \beta_4 \in H^1$
$\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in H^2$.

(Recall $H_k(\bigvee_j X_j) \cong \bigoplus_j H_k(X_j)$.)

And the homology groups of M_4:

$$H_k(M_4) = \begin{cases} 0 & \text{for } k \geq 3 \\ \mathbb{Z} & \text{for } k=2 \\ \mathbb{Z}^8 & \text{for } k=1 \\ \mathbb{Z} & \text{for } k=0 \end{cases}$$



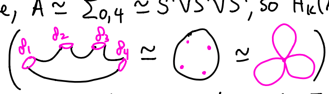
generators. $\begin{bmatrix} \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^1(M_g) \\ \beta_1, \beta_2, \beta_3, \beta_4 \in H^1(M_g) \\ \gamma \in H^2(M_g) \end{bmatrix}$

Since $H_k(M_g, A;\mathbb{Z})$ is free, $\mathrm{Ext}(H_k(M_g, A), \mathbb{Z}) = 0$. So by UCT, $H^k(M_g, A) \cong \mathrm{Hom}(H_k(M_g, A), \mathbb{Z})$. The same is true for $H^k(M_g)$.

Then $\exists\, a_i, b_i \in H^1(M_g, A;\mathbb{Z})$ and $c_i \in H^2(M_g, A;\mathbb{Z})$ $(i = 1,2,3,4)$ and $a_i, b_i \in H^1(M_g;\mathbb{Z})$ and $c \in H^2(M_g;\mathbb{Z})$, the appropriate duals to generators of H_k.
└ dual to $\alpha_i, \beta_i \in H_1(M_g, A)$ └ dual to $\gamma_i \in H_2(M_g, A)$ └ dual to $\alpha_i, \beta_i \in H_1(M_g)$ └ dual to $\gamma \in H_2(M_g)$

Note that q induces a map $q_*: H_k(M_g;\mathbb{Z}) \to H_k(M_g, A;\mathbb{Z})$, $\alpha_i \mapsto \alpha_i$, $\beta_i \mapsto \beta_i$, $\sum_{i=1}^{4} \gamma_i \mapsto \gamma$; if we dualize, we obtain:
$q^*: H^k(M_g, A;\mathbb{Z}) \to H^k(M_g, \mathbb{Z})$, $a_i \mapsto a_i$, $b_i \mapsto b_i$, $c \mapsto c_i$.] claim

Proof of claim:
Furthermore, $A \overset{\text{h.e.}}{\simeq} \Sigma_{0,4} \simeq S^1 \vee S^1 \vee S^1$, so $\tilde{H}_k(A) = \begin{cases} 0 & \text{for } k=0, k \geq 2 \\ \mathbb{Z}^3 & \text{for } k=1 \end{cases}$, so $i_*: H_k(A) \hookrightarrow H_k(M_g)$ is zero map for $k \geq 2$. Furthermore, $i_*: H_1(A) \hookrightarrow H_1(M_g)$ is zero as well since $\delta_1, \delta_2, \delta_3, \delta_4 \mapsto 0$.



Then have relative homology chain: (in $\mathbb{Z}$ coeff.) $(\mathrm{Im}\, i_* = \ker q_* = \ker q_* = 0)$

$$\cdots \to \underbrace{\tilde{H}_2(A)}_{0} \xhookrightarrow{i_*} \tilde{H}_2(M_g) \xrightarrow[\cong]{q_*} \tilde{H}_2(M_g, A) \xrightarrow{j_*} \tilde{H}_1(A) \xhookrightarrow[i_*]{0} \tilde{H}_1(M_g) \xrightarrow[\cong]{q_*} \tilde{H}_1(M_g, A) \to \underbrace{\tilde{H}_0(A)}_{0} \to \cdots$$

$\tilde{H}_1(A) = \ker i_* = \mathrm{Im}\, j_* \Rightarrow j_*$ surjective

So $\tilde{H}_1(M_g) \cong \tilde{H}_1(M_g, A)$ and $\tilde{H}_2(M_g) \cong \tilde{H}_2(M_g, A)$.

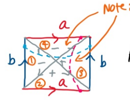
(see bottom of next page.)

(From 2009 August.) Computing $H^*(S^1\times S^1)$. $H^*(S^1\times S^1;\mathbb{Z})$

4) Compute the cup product structure of the cohomology ring of the torus. Use this to compute the cup product structure of the closed genus g surface.

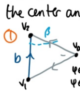
Note: it's important these 2 simplices are labeled with 2 different signs.

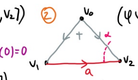
This is example 3.7 in Hatcher.

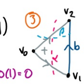
Use the following Δ-complex structure for $S^1\times S^1$:  A basis for $H^1(S^1\times S^1)$ is formed by edges a and b.

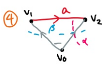
By the Univ. Coefficient Thm for Cohomology, we have the Exact sequence $0\to \mathrm{Ext}(H_0(T);\mathbb{Z})\to H^1(T;\mathbb{Z})\to \mathrm{Hom}(H_1(T);\mathbb{Z})\to 0$. Since T is path-connected, here, $H_0(T)\cong\mathbb{Z}$, which has trivial torsion subgroup. Then by Corollary 3.3, $\mathrm{Ext}(H_0(T);\mathbb{Z})=0$, so $H^1(T;\mathbb{Z})\cong \mathrm{Hom}(H_1(T),\mathbb{Z})$. A basis for $H_1(T)$ thus represents a dual basis for $\mathrm{Hom}(H_0(T),\mathbb{Z})$, so α is dual to a and β is dual to b, where α,β are cohomology classes st $\alpha(a)=1$, $\alpha(b)=0$ and $\beta(a)=0$, $\beta(b)=1$.

Let φ count the intersections of the arc α (shown in diagram) with ea. edge radiating from the center and ψ count the number of intersections of the arc β. We then compute the cup product:

①  $(\varphi\cup\psi)(\sigma_1)=\varphi([v_0,v_1])\psi([v_1,v_2])$
$= 0(1)=0$.
$\varphi\in\mathcal{C}^1(T;\mathbb{Z})$, $\psi\in\mathcal{C}^1(T;\mathbb{Z})$; σ_i: ①→①
$(\beta\cup\alpha)(\sigma_1)=0(0)=0$

②  $(\varphi\cup\psi)(\sigma_2)=\varphi([v_0,v_1])\psi([v_1,v_2])$
$=0(0)=0$
$(\beta\cup\alpha)(\sigma_2)=\beta([v_0,v_1])\alpha([v_1,v_2])=0(1)=0$

③  $(\varphi\cup\psi)(\sigma_3)=\varphi([v_0,v_1])\psi([v_1,v_2])$
$=(1)(1)=1$
$(\beta\cup\alpha)(\sigma_3)=\beta([v_0,v_1])\alpha([v_1,v_2])$
$=0(0)=0$

④  $(\varphi\cup\psi)(\sigma_4)=\varphi([v_0,v_1])\psi([v_1,v_2])$
$=(0)(0)=0$.
$(\beta\cup\alpha)(\sigma_4)=\beta([v_0,v_1])\alpha([v_1,v_2])$
$=(1)(1)=0$

Conclude that Δ^2 labeled 3 is the only one with nonzero cup product.
Then on the $\overbrace{\text{two-chain}}^{c}$ formed by the sum of all 2-simplices w/ signs indicated at the center of the figure, $(\varphi\cup\psi)(c) = -0-0+1+0=1$.

$$\cdots\to 0\xrightarrow{\partial_3}\underset{\substack{\psi\\ c}}{\mathbb{Z}^4}\xrightarrow{\partial_2}\underset{\substack{\psi\\ a,b}}{\mathbb{Z}^2}\to\cdots$$

$\partial c=0$, so $c\in\ker\partial_2$.
Also, $c\notin \mathrm{Im}\,\partial_3 \Rightarrow c$ is nontrivial element of $H_2(T)=\ker\partial_2/\mathrm{Im}\,\partial_3$.

Since $(\varphi\cup\psi)(c)=1$, a generator of $\mathbb{Z}$, c is also a generator for $H_2(T)\cong\mathbb{Z}$.
Its dual, γ, must also be a generator for $H^2(T)\cong\mathrm{Hom}(H_2(T),\mathbb{Z})\cong\mathbb{Z}$. γ is represented by $\varphi\cup\psi$.
(By Univ. Coeff. Thm, $0\to \underbrace{\mathrm{Ext}(\underbrace{H_1(T)}_{\mathbb{Z}};\mathbb{Z})}_{=0}\to H^2(T)\xrightarrow{\cong}\mathrm{Hom}(H_2(T),\mathbb{Z})\to 0$.)

Thus, since α is represented by φ, a simplicial cocycle and ψ represents β, $\alpha\cup\beta=\gamma$. By calculations in purple, we have $(\beta\cup\alpha)(c)=-1$. Since neither α nor β intersects $[v_0,v_1]$ and $[v_1,v_2]$ simultaneously in ea. simplex, $(\alpha\cup\alpha)=0$ and $\beta\cup\beta=0$. The conclude: $\alpha\cup\beta=\gamma=-\beta\cup\alpha$, $\alpha\cup\alpha=\beta\cup\beta=0$. It then follows that $H^*(T;\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha,\beta]$.

Perhaps this is too complicated.

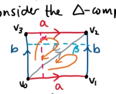
<u>Simpler approach</u>: observe $H_0(T;\mathbb{Z})\cong\mathbb{Z}$, $H_1(T)\cong\mathbb{Z}^2$, $H_2(T)\cong\mathbb{Z}$, $H_k(T)\cong 0$ for $k\geq 3$. In particular, $T_0,T_1,T_2=0$ (trivial torsion subgroups.)

Chain complex (cellular Homology): $\cdots\to \underset{\mathcal{C}_3}{0}\to\underset{\mathcal{C}_2}{\mathbb{Z}^2}\to\underset{\mathcal{C}_1}{\mathbb{Z}^2}\xrightarrow{\partial=0}\underset{\mathcal{C}_0}{\mathbb{Z}}\xrightarrow{\partial=0}0$ (all fin. gen. free abelian groups.)

Cochain complex (Cohomology): $\cdots\leftarrow\underset{\mathcal{C}_3^*}{0}\leftarrow\underset{\mathcal{C}_2^*}{\mathbb{Z}^2}\leftarrow\underset{\mathcal{C}_1^*}{\mathbb{Z}^2}\xleftarrow{\delta=0}\underset{\mathcal{C}_0^*}{\mathbb{Z}}\xleftarrow{\delta=0}0$ $\underset{\text{Corollary 3.3}}{\Longrightarrow}$
$H^1(T;\mathbb{Z})\cong H_1(T)/T_1\oplus T_0\cong\mathbb{Z}^2/\{0\}\oplus 0\cong\mathbb{Z}^2\cong H_1(T)$ and
$H^2(T;\mathbb{Z})\cong H_2(T)/T_2\oplus T_1\cong\mathbb{Z}/\{0\}\oplus 0\cong\mathbb{Z}\cong H_2(T)$.
$H^0(T;\mathbb{Z})\cong\ker\delta/\mathrm{Im}\,\delta\cong\mathbb{Z}/\{0\}\cong\mathbb{Z}\cong H_0(T;\mathbb{Z})$.

Note we only care about the cup pdt $H^1(T;\mathbb{Z})\times H^1(T;\mathbb{Z})\xrightarrow{\cup}H^2(T;\mathbb{Z})$.
<u>Why</u>: 1 gen. for $\mathbb{Z}\Rightarrow 1\in H^0(T;\mathbb{Z})$ is id element for cup pdt, so $H^0\times H^0\xrightarrow{\cup}H^0$ is just the identity map.
$H^1\times H^2\xrightarrow{\cup}H^3\cong 0$; $H^k\times H^l\xrightarrow{\cup}H^{k+l}\cong 0$ for $k,l\geq 2$.

Consider the Δ-complex structure: (chosen so that $\varphi\in H^1(T;\mathbb{Z})$ acts on $[v_0,v_1]$ and $\psi\in H^1(T;\mathbb{Z})$ acts on $[v_1,v_2]$ and there are no vertices left over.)



α is dual to $a\in H_1(T)$, a generator for $H_1(T;\mathbb{Z})$ and β is dual to $b\in H_1(T;\mathbb{Z})$.
In particular, $\alpha(a)=1$ (since a intersects α once), $\alpha(b)=0$ (since b does not intersect α) and $\beta(a)=0$, $\beta(b)=1$. (Here, $\alpha,\beta\in H^1(T;\mathbb{Z})$ generate $H^1(T;\mathbb{Z})$.)
We now calculate $\alpha\cup\beta$, $\alpha\cup\alpha$, $\beta\cup\beta$, $\beta\cup\alpha$, which will give us cup pdt. structure. (since cup pdt is distributive.)

we need to find a 2-chain, $\sigma\in\mathcal{C}_2(T;\mathbb{Z})$ st σ generates $H_2(T;\mathbb{Z})$.
Let $\sigma=[v_0,v_1,v_2]-[v_0,v_3,v_2]$. Note $\partial(\sigma)=(b+a-c)-(a+b-c)=0$, so $\sigma\in\ker\partial_2$. And $\mathrm{Im}\,\partial_3=0$, so $\sigma\notin\mathrm{Im}\,\partial_3\Rightarrow\sigma$ is a nontrivial element of $H_2(T)\cong\mathbb{Z}$.
Furthermore, $(\alpha\cup\beta)(\sigma)=1$ (see below) so $(\alpha\cup\beta)(\sigma)$ is a generator of $\mathbb{Z}\cong H^2(T), H_2(T)$. Then it follows that σ generates $H_2(T)$ and its dual, $\alpha\cup\beta$, generates $H^2(T)$.

$(\alpha\cup\beta)([v_0,v_1,v_2])=\alpha(\underbrace{[v_0,v_1]}_{a})\beta(\underbrace{[v_1,v_2]}_{b})=1(1)=1$
$(\alpha\cup\beta)([v_0,v_3,v_2])=\alpha(\underbrace{[v_0,v_3]}_{b})\beta(\underbrace{[v_3,v_2]}_{a})=0(0)=0$
$\Big]\ (\alpha\cup\beta)(\sigma)=1$

$(\alpha\cup\alpha)([v_0,v_1,v_2])=\alpha([v_0,v_1])\alpha([v_1,v_2])=1(0)=0$
$(\alpha\cup\alpha)([v_0,v_3,v_2])=\alpha([v_0,v_3])\alpha([v_3,v_2])=0(1)=0$
$\Big]\ (\alpha\cup\alpha)(\sigma)=0-0=0$.

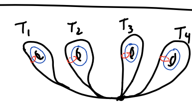
$(\beta\cup\alpha)([v_0,v_1,v_2])=\beta([v_0,v_1])\alpha([v_1,v_2])=0(0)=0$
$(\beta\cup\alpha)([v_0,v_3,v_2])=\beta([v_0,v_3])\alpha([v_3,v_2])=1(1)=1$
$\Big]\ (\beta\cup\alpha)(\sigma)=-1$

$(\beta\cup\beta)([v_0,v_1,v_2])=\beta([v_0,v_1])\beta([v_1,v_2])=0(1)=0$
$(\beta\cup\beta)([v_0,v_3,v_2])=\beta([v_0,v_3])\beta([v_3,v_2])=1(0)=0$
$\Big]\ (\beta\cup\beta)(\sigma)=0-0=0$.

So $\alpha\cup\beta=-\beta\cup\alpha$, $\alpha\cup\alpha=\beta\cup\beta=0\Rightarrow H^*(T;\mathbb{Z})\cong\Lambda_{\mathbb{Z}}[\alpha,\beta]$. (exterior algebra generated by α,β.)

Determine cup pdt structure of $H^*(T;\mathbb{Z})$.

(b) (Cont.) In each of the wedge components, the tori T_1, T_2, T_3, and T_4, from our assumption, we know $\Lambda_{\mathbb{Z}}(a_i,b_i)$ ($a_i,b_i\in H^1(T_i)$) give the cohomology ring structures of $H^*(T_i;\mathbb{Z})$, so $a_i\cup a_i=b_i\cup b_i=0$, $a_i\cup b_i=c_i$ for $i=1,2,3,4$.



Since $H^1(M_g,A)\cong H^1(M_g)$ and $H^2(M_g,A)\cong H^2(M_g)$, we have the relations:

$$\boxed{\begin{cases} a_i\cup a_i=b_i\cup b_i=0 & \text{for } i=1,2,3,4\\ a_i\cup b_i=-b_i\cup a_i=c\end{cases}\quad\text{cohomology ring structure of } H^*(M_g).}$$

$a_i,b_i\in H^1(M_g)$
$c\in H^2(M_g)$

2) Let Z_6 act on S^3 via (z, w) —> (ez, ew), where e is the sixth root of unity. Denote L by the quotient S^3/Z_6.

a) What is the fundamental group of L?

$\mathbb{Z}_6 \curvearrowright S^3$ since $(1,0) \to (j\varepsilon, 0) \neq (1,0)$ for $1 \leq j \leq 5$. So only the identity element fixes every pt. in S^3.

Then $S^3 \to S^3/\mathbb{Z}^6$ is a covering map and $\mathbb{Z}_6$ is a covering space action.

Apply Prop. 1.40: $\mathbb{Z}_6 \cong \pi_1(S^3/\mathbb{Z}_6) / \underbrace{p_*(\pi_1(S^3))}_{0} \cong \pi_1(S^3/\mathbb{Z}_6)$. Conclude $\pi_1(S^3/\mathbb{Z}_6) \cong \mathbb{Z}_6$.

since S^3 is simply-connected

Prop. 1.40 If an action of a group G on a space Y acts freely on Y (only identity element fixes any pt. of Y) then c) $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$. (This cover is also normal.) and G is group of deck transf.

b) Describe all coverings of L.

By the Galois correspondence, $\forall$ subgroup of $\pi_1(S^3/\mathbb{Z}_6) \cong \mathbb{Z}_6$, $\exists$ covering space $p_H : X_H \to X$ st $p_*(\pi_1(X_H)) = H$.

We can find all subgroups of $\mathbb{Z}_6$; since $\mathbb{Z}_6$ is cyclic, all its subgroups are also cyclic. Elements of $\mathbb{Z}_6$ are of order 6 (ie 1,5), 3 (ie 2,4), 2 (ie 3), or 1 (ie 0),

so its subgroups are $\langle 0 \rangle \cong 0$, $\langle 1 \rangle \cong \mathbb{Z}_6$, $\langle 2 \rangle \cong \mathbb{Z}_3$, and $\langle 3 \rangle \cong \mathbb{Z}_2$.

S^3 is simply-connected, so the universal cover is S^3.

$S^3/\mathbb{Z}^6$ corresponds to the $\mathbb{Z}_6$ subgroup.

$S^3/\mathbb{Z}_3$, $S^3/\mathbb{Z}_2$.

c) Show that any continuous map L —> S^1 is nullhomotopic.

Recall the homotopy lifting property.

Homotopy Lifting Property

Suppose given a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Let $f : L \to S^1$ be an arbitrary continuous map.

Recall $p : S^3 \to L$ is a covering map, and $\pi_1(S^3) = 0 \Rightarrow p_*(\pi_1(S^3)) = 0$. (so want $f_*(\pi_1(L)) = 0$.)

f induces a homomorphism $f_* : \pi_1(L) \to \pi_1(S^1)$, $\mathbb{Z}_6 \to \mathbb{Z}$. I claim $f_* = 0$ necessarily.

f_* is determined by where it sends generators, namely $f_*(1)$.

Note $f_*(1+1+1+1+1+1) = f_*(0) = 0$ and OTOH, $6f_*(1) = f_*(1+1+1+1+1+1) \Rightarrow 6f_*(1) = 0$.

So $|f_*(1)|$ must divide 6. But 0 is the only element in $\mathbb{Z}$ of finite order, so $f_*(1) = 0$.

Then $f_*(n) = n f_*(1) = 0$ as well $\forall n \in \mathbb{Z}_6$, so $f_* = 0$, and $f_*(\pi_1(L)) \subseteq p_*(\pi_1(S^1))$. We can then apply HLP: $f = p \circ \tilde{f}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ L \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Note that $\tilde{f} : L \to \mathbb{R}$ is nullhomotopic since $\mathbb{R}$ is contractible. So $\tilde{f} \underset{h.e.}{\simeq} c$

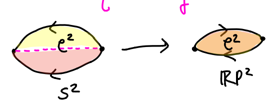
Then $\tilde{f} \circ p \simeq c'$, $c' : \mathbb{R} \to S^1$ as well since $\tilde{f} \circ p \underset{h.e.}{\simeq} c \circ p$, which is constant.

Then f is nullhomotopic. ∎

3) A map f: S^n —> S^n satisfying f(x) = f(-x) for all x is called an even map. Show that an even map has even degree, and this degree is 0 when n is even. When n is odd, Show there exist even maps of given even degree.

Note that every even map $f : S^n \to S^n$ factors through $\mathbb{RP}^n$, ie: $f : S^n \xrightarrow{q} \mathbb{RP}^n \xrightarrow{g} S^n$, where $q : S^n \to \mathbb{RP}^n$ is the quotient map $S^n \to \mathbb{RP}^n$, $x \sim -x$ ($[x] = [-x]$) and $g([x]) = f(\pm x)$.

In particular, under q, an n-cell $\underset{S^n}{e^n} \mapsto \underset{\mathbb{RP}^n}{2e^n}$, 2 copies of $e^n \in \mathbb{RP}^n$.



Suppose n is odd.

Recall by the cellular chain complexes of S^n and $\mathbb{RP}^n$: S^n (1 n-cell) $\cdots 0 \xrightarrow{d_{n+1}=0} \mathbb{Z} \xrightarrow{d_n = 0} 0 \to \cdots$ and $\mathbb{RP}^n$ (1 n-cell) $\cdots 0 \xrightarrow{d_{n+1}=0} \mathbb{Z} \xrightarrow{d_n=0} \mathbb{Z} \to \cdots$

$H_n(S^n) \cong \ker d_n / \operatorname{Im} d_{n+1} \cong \mathbb{Z}/\{0\} \cong \mathbb{Z}$, generated by an element of $e_n(S^n)$ (ie, an n-cell),

and $H_n(\mathbb{RP}^n) \cong \ker d_n / \operatorname{Im} d_{n+1} \cong \mathbb{Z}/\{0\} = \mathbb{Z}$, gen. by an n-cell in $\mathcal{C}_n(\mathbb{RP}^n)$.

Note that under q, $e^n_{S^n} \in \mathcal{C}_n(S^n)$ maps to two copies of $e^n_{\mathbb{RP}^n} \in \mathcal{C}_n(\mathbb{RP}^n)$, so $q_* : \underset{\mathbb{Z}}{H_n(S^n)} \to \underset{\mathbb{Z}}{H_n(\mathbb{RP}^n)}$, $\alpha \mapsto 2\alpha$.

Then if $g_* : H^n(\mathbb{RP}^n) \to H^n(S^n)$ maps $\alpha \mapsto d\alpha$, we have $f_* : H^n(S^n) \to H^n(S^n)$ maps $\alpha \mapsto d(2\alpha) = 2d\alpha \Rightarrow \deg(f_*) = 2d$.

Suppose n is even.

Then $q_* : H^n(S^n) \to H^n(\mathbb{RP}^n)$ is a map $\mathbb{Z} \to 0 \Rightarrow q_* = 0$. So $f_* : H^n(S^n) \to H^n(S^n)$ maps $\alpha \mapsto 0 \mapsto 0 \Rightarrow \deg(f_*) = 0$.

When n is odd, take any arbitrary degree $2k$.

Construct map $g : H^n(\mathbb{RP}^n) \to H^n(S^n)$, $\alpha \mapsto k\alpha$. Note $\mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n$. (since $\mathbb{RP}^n = e^n \cup \underbrace{e^{n-1} \cup e^{n-2} \cup \dots \cup e^1 \cup e^0}_{\mathbb{RP}^{n-1}}$, $S^n = e^n \cup e^0$.)

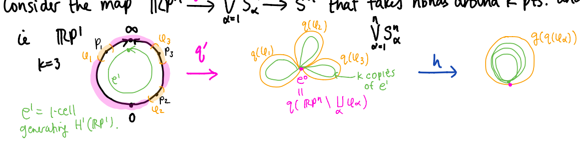
Consider the relative chain $\cdots \to \underbrace{\tilde{H}_n(\mathbb{RP}^{n-1})}_{0} \to \tilde{H}_n(\mathbb{RP}^n) \xrightarrow{\cong} \tilde{H}_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \to \underbrace{\tilde{H}_{n-1}(\mathbb{RP}^{n-1})}_{0} \to \cdots$ where n is odd

no n-cells in $\mathbb{RP}^{n-1}$ n-1 even

So $\tilde{H}_n(\mathbb{RP}^n) \cong \tilde{H}_n(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \cong \tilde{H}_n(S^n)$. In particular, $q_* : \tilde{H}_n(\mathbb{RP}^n) \to \tilde{H}_n(S^n)$ maps a generator $e^n \in \tilde{H}_n(\mathbb{RP}^n)$ to a generator of $\tilde{H}_n(S^n)$.

Consider the map $\mathbb{RP}^n \xrightarrow{q'} \bigvee_{\alpha=1}^{k} S^n_\alpha \xrightarrow{h} S^n$ that takes nbhds U_α around k pts. and quotients $\mathbb{RP}^n \setminus \coprod_{\alpha=1}^{k} U_\alpha$ to a point and then identifies the k copies of S^n.

ie



Then $g = h \circ q' : \mathbb{RP}^n \to S^n$ maps $e^n \mapsto ke^n$, so f maps $e^n \mapsto 2e^n \mapsto k(2e^n)$.

So $\deg f = \deg(g \circ q') = 2k$, as desired. ∎

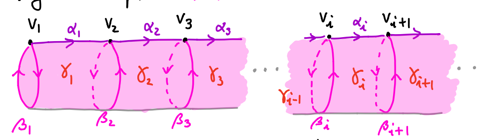
1) **Mapping Telescope.** (Using only 751 techniques.)

(n ≥ 1 fixed.)

Construction: $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \dots$, $f_i : S^n \to S^n$, degree i map. So $(f_i)_* : \underset{\substack{\cup \\ H_n(X_i)}}{\beta_i} \mapsto \underset{\substack{\cup \\ H_n(X_{i+1})}}{i\beta_i}$

$$M := \left(\coprod_{i \geq 1} X_i \times [0,1]\right) \Big/ \left(\underbrace{(x_i,1)}_{X_i \times \{1\}} \sim \underbrace{(f(x_i),0)}_{X_{i+1} \times \{0\}}\right)$$

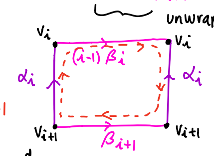
$\boxed{\text{ie for } n=1}$



Generators of $H_n(M)$: n-cells β_i, $i \geq 1$ (for $i=1$, also α_i.)

Generators of $H_1(M)$: 1-cells α_i, $i \geq 1$ (for $i=1$, also β_i.)

(Here, $v_i \cup \beta_i = X_i$.)



Generators of $H_{n+1}(M)$: $n+1$-cells, γ_i, $i \geq 1$

Gluing map:

φ: $\gamma_i \mapsto \cancel{\alpha_i} + (i-1)\beta_i - \cancel{\alpha_i} - \beta_{i+1}$

$\gamma_i \mapsto (i-1)\beta_i - \beta_{i+1}$

d_{n+1}: $\gamma_i \mapsto (i-1)\beta_i - \beta_{i+1}$ for $i>1$, $\beta_1 - \beta_2$ for $i=1$.

d_1: $\alpha_i \mapsto v_{i+1} - v_i$, so the cellular chain complex is:

$$\dots 0 \to \underbrace{\mathbb{Z}[\gamma_1, \gamma_2, \dots, \gamma_i, \dots]}_{n+1\text{-cells}} \xrightarrow{d_{n+1}} \underbrace{\mathbb{Z}[\beta_1, \beta_2, \dots, \beta_i, \dots]}_{n\text{-cells}} \xrightarrow{d_n = 0} 0 \to \dots \to 0 \to \underbrace{\mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_i, \dots]}_{1\text{-cells}} \xrightarrow{d_1} \underbrace{\mathbb{Z}[v_1, v_2, \dots, v_i, \dots]}_{0\text{-cells}} \xrightarrow{d_0 = 0} 0$$

So $H_{n+1}(M) = \ker d_{n+1} / \underbrace{\operatorname{Im} d_{n+2}}_{0} = 0$ since d_{n+1} is injective.

$H_n(M) = \ker d_n / \operatorname{Im} d_{n+1} = \langle \beta_1, \beta_2, \dots, \beta_i, \dots \mid \beta_1 - \beta_2, (i-1)\beta_i - \beta_{i+1} \text{ for } i>1 \rangle$

$H_1(M) = \ker d_1 / \underbrace{\operatorname{Im} d_2}_{0} = 0$ (since d_1 is injective.) unless $n=1$, in which case, get $\langle \beta_1, \beta_2, \dots, \beta_i, \dots \mid \beta_1 - \beta_2, (i-1)\beta_i - \beta_{i+1} \text{ for } i>1 \rangle$.

$H_0(M) \cong \mathbb{Z}$ since M is path-connected. Otherwise, since $C_k(M) = 0$ for $k \neq 0, 1, n, n+1$, $H_k(M) = 0$.

1) Show that if M is a compact contractible n-manifold with boundary, then the boundary of M is a homology n-1 sphere. (assume M is connected.)

(ie $H_k(\partial M; \mathbb{Z}) \cong H_k(S^{n-1}; \mathbb{Z})$.) Since M is not necessarily orientable, consider hom w/ $\mathbb{Z}_2$ coefficients. Then M is certainly $\mathbb{Z}_2$-orientable.

Since M is contractible, $H_k(M) = \begin{cases} \mathbb{Z}_2 & k=0 \\ 0 & \text{otherwise.} \end{cases}$ Consider the relative LES: Suppose here $k>1$

$$\dots \to \underbrace{H_k(M)}_{0} \to H_k(M, \partial M) \xrightarrow{\cong} H_{k-1}(\partial M) \to \underbrace{H_{k-1}(M)}_{0} \to \dots$$

Then conclude for $k>1$, $H_{k-1}(\partial M; \mathbb{Z}_2) \cong H_k(M, \partial M; \mathbb{Z}_2)$. By Lefschetz, $H_k(M, \partial M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2)$, for $k \leq n$.

Note Cannot use Poincaré to conclude $H_k(M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2)$ since M is not closed.

By consequence of UCT, since M is contractible, $H^i(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k=0 \\ 0 & \text{otherwise.} \end{cases}$

By consequence of UCT for homology, $H^i(M; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } i=0 \\ 0 & \text{otherwise.} \end{cases}$

In particular, for $1<k<n$

$H_k(M, \partial M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2) \cong 0$,

so $H_{k-1}(\partial M; \mathbb{Z}_2) \cong 0$ for $1<k<n$.

So $H_i(\partial M; \mathbb{Z}_2) \cong 0$ for $1 \leq i \leq n-2$.

Corollary 3.3 If H_n, H_{n-1} are finitely-gen., then $H^n(C; \mathbb{Z}) \cong H_n/T_n \oplus T_{n-1}$

Here, $H_k(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{otherwise,} \end{cases}$ so

$H^k(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{otherwise.} \end{cases}$

UCT Cohomology (Corollary 3A.6)

If $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$ are finitely generated, p prime, then:

$H_n(X; \mathbb{Z}_p)$ has: 1 $\mathbb{Z}_p$ summand for ea. $\mathbb{Z}$ summand of H_n;

2 $\mathbb{Z}_p$ summand for ea. $\mathbb{Z}_{p^k}$ summand of H_n ($k \geq 1$);

3 $\mathbb{Z}_p$ summand for ea. $\mathbb{Z}_{p^k}$ summand of H_{n-1} ($k \geq 1$).

So by UCT, we can conclude $H_i(\partial M; \mathbb{Z}) \cong 0$ for $1 \leq i \leq n-2$, $H_i(\partial M; \mathbb{Z}) \cong \mathbb{Z}$ (Since ∂M is path-connected.) Since ∂M is $n-1$-dimensional, $H_i(\partial M; \mathbb{Z}) = 0$ for $i \geq n$. Then the only nontrivial case left is $H_{n-1}(\partial M; \mathbb{Z}) \cong \mathbb{Z}$.

$H_{n-1}(\partial M; \mathbb{Z}_2) \cong H_n(M, \partial M; \mathbb{Z}_2) \cong H_0(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, so by UCT for Homology,

(i) $H_{n-1}(\partial M; \mathbb{Z}) \cong \mathbb{Z}$ (in which case $H_{n-1}(\partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ by 1 above.)

or (ii) $H_{n-1}(\partial M; \mathbb{Z}) \cong \mathbb{Z}_2$ (in which case $H_{n-1}(\partial M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ by 2 above.)

or (iii) $H_{n-1}(\partial M; \mathbb{Z}) \cong 0 \Rightarrow H_{n-2}(\partial M; \mathbb{Z}) \cong \mathbb{Z}_2$ (by 3 above.)

Note clearly (iii) is impossible since we already determined $H_{n-2}(\partial M; \mathbb{Z}) \cong 0$.

If ∂M is nonorientable, $H_{n-1}(\partial M; \mathbb{Z}) \cong 0$, but this is impossible by the logic above. So ∂M must be orientable.

On the other hand, if ∂M is orientable, $H_{n-1}(\partial M; \mathbb{Z}) = \mathbb{Z}$, so (ii) is impossible. Then it must follow that $H_{n-1}(\partial M; \mathbb{Z}) \cong \mathbb{Z}$.

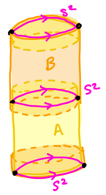
Thm 3.26 for M closed, connected n-dim manifold:
$H_n(M; \mathbb{Z}) = \mathbb{Z}$ if M is orientable,
$H_n(M; \mathbb{Z}_2) = 0$ if M is nonorientable.

2) Show that $S^2 \times S^1$ and $S^1 \vee S^2 \vee S^3$ have isomorphic homology but nonisomorphic cohomology ring structures.

$H_k(S^n) = \begin{cases} \mathbb{Z} & k=0,n \\ 0 & \text{otherwise} \end{cases} \Rightarrow H_k(S^1 \vee S^2 \vee S^3) = \begin{cases} \mathbb{Z} & k=0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$.

We can calculate homology of $S^1 \times S^2$ using Mayer-Vietoris.
Clearly, $H_0(S^1 \times S^2) \cong \mathbb{Z}$.

$$\cdots \to \underbrace{\tilde{H}_1(S^2)}_{0} \oplus \underbrace{\tilde{H}_1(S^2)}_{0} \to \tilde{H}_1(S^2 \times S^1) \xrightarrow{\cong} \underbrace{\tilde{H}_0(S^2 \sqcup S^2)}_{\mathbb{Z}} \to \underbrace{\tilde{H}_0(S^2)}_{0} \oplus \underbrace{\tilde{H}_0(S^2)}_{0} \to \cdots$$ so $H_1(S^2 \times S^1) \cong \mathbb{Z}$.



$S^2 \sqcup S^2 = A \cap B$.

$$\cdots \to H_2(S^2 \sqcup S^2) \xrightarrow{i_* : (1,1) \mapsto (1,1)} H_2(S^2) \oplus H_2(S^2) \xrightarrow[\alpha]{} H_2(S^1 \times S^2) \xrightarrow[\phi]{0} \underbrace{H_1(S^2 \sqcup S^2)}_{0} \xrightarrow{0} \underbrace{H_1(S^2) \oplus H_1(S^2)}_{0} \to \cdots$$

$\mathbb{Z} \cong \mathrm{Im}(i_*) \cong \ker(\alpha) \Rightarrow \ker \alpha = \mathbb{Z}$.
also $\mathrm{Im}(\alpha) \cong H_2(S^1 \times S^2)$.

$\mathrm{Im}(\alpha) = \ker(\phi) \cong H_2(S^1 \times S^2)$

$$\cdots \to \underbrace{H_3(S^1) \oplus H_3(S^2)}_{0} \to H_3(S^1 \times S^2) \xrightarrow{j_*} H_2(S^2 \sqcup S^2) \xrightarrow{i_*} \underbrace{H_2(S^2) \oplus H_2(S^2)}_{\mathbb{Z} \oplus \mathbb{Z}} \to H_2(S^2 \times S^1) \to \cdots$$ So $H_3(S^1 \times S^2) \cong \mathbb{Z}$.

$i_* : (1,1) \mapsto (1,1)$ so $\mathrm{Im}(i_*) \cong \mathbb{Z}$. So $\ker(i_*) \cong \mathbb{Z}$.

So $\mathrm{Im}(j_*) = \ker(i_*) = \mathbb{Z}$, and $\ker(j_*) \cong 0 \Rightarrow H_3(S^1 \times S^2) \cong \mathbb{Z}$.

Künneth Formula — The cross product
$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y; R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module $\forall k$.

Cohomology rings not isomorphic:

$H^*(S^1 \times S^2) \underset{\text{Künneth}}{\cong} H^*(S^1) \otimes_{\mathbb{Z}} H^*(S^2) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^2)$, $|\alpha| = 1$ and $|\beta| = 2$. Then $\alpha \smile \beta \in H^3(S^1 \times S^2)$ is a nonzero generator.

OTOH,
$H^*(S^1 \vee S^2 \vee S^3) \cong H^*(S^1) \oplus H^*(S^2) \oplus H^*(S^3) \cong \mathbb{Z}[\alpha]/(\alpha^2) \oplus \mathbb{Z}[\beta]/(\beta^2) \oplus \mathbb{Z}[\gamma]/(\gamma^2)$ $\quad |\alpha| = 1, |\beta| = 2, |\gamma| = 3$.

$\alpha \smile \beta \in H^3(S^1 \vee S^2) = 0 \Rightarrow \alpha \smile \beta = 0$ ($\iota : S^1 \vee S^2 \hookrightarrow S^1 \vee S^2 \vee S^3$ induces map on cohomology.)

So there is no isomorphism b/w $H^*(S^1 \times S^2)$ and $H^*(S^1 \vee S^2 \vee S^3)$.

Another approach to finding homologies:

General Künneth Formula (Corollary 3B.7)
If F is a field and X and Y are CW complexes, then $\bigoplus_i (H_i(X;F)) \otimes_F H_{n-i}(Y;F)) \cong H_n(X \times Y; F)$ for all n.

So $H_1(S^1 \times S^2) \cong \bigoplus_{i=0}^{1} H_i(S^1) \otimes_{\mathbb{Z}} H_{1-i}(S^2)$

$\cong (\underbrace{H_0(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_1(S^2)}_{0}) \oplus (\underbrace{H_1(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_0(S^2)}_{\mathbb{Z}})$

$\cong 0 \oplus \mathbb{Z} \cong \mathbb{Z}$.

Ring Tensor Basics
- $R \otimes_R M \cong M$ for R commutative, M an R-module.
- $R \otimes_R 0 = 0 \otimes_R R = 0$.

and $H_2(S^1 \times S^2) \cong \bigoplus_{i=0}^{2} H_i(S^1) \otimes H_{2-i}(S^2)$

$\cong (\underbrace{H_0(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_2(S^2)}_{\mathbb{Z}}) \oplus (\underbrace{H_1(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_1(S^2)}_{0}) \oplus (\underbrace{H_2(S^1)}_{0} \otimes \underbrace{H_0(S^2)}_{\mathbb{Z}})$

$\cong \mathbb{Z}$.

$H_3(S^1 \times S^2) \cong \bigoplus_{i=0}^{3} H_i(S^1) \otimes H_{3-i}(S^2) \cong (\underbrace{H_0(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_3(S^2)}_{0}) \oplus (\underbrace{H_1(S^1)}_{\mathbb{Z}} \otimes \underbrace{H_2(S^2)}_{\mathbb{Z}}) \oplus (\underbrace{H_2(S^1)}_{0} \otimes \underbrace{H_1(S^2)}_{0}) \oplus (\underbrace{H_3(S^1)}_{0} \otimes \underbrace{H_0(S^2)}_{\mathbb{Z}})$

$\cong 0 \oplus \mathbb{Z} \oplus 0 \oplus 0 \cong \mathbb{Z}$.

2b) Let k < n and $\mathbb{R}P^k \subset \mathbb{R}P^n$ the standard inclusion. Is RP^k a retract of RP^n? Justify.

No. Suppose there were. Since $r \circ i : \mathbb{R}P^k \xrightarrow{i} \mathbb{R}P^n \xrightarrow[r]{} \mathbb{R}P^k$ is the id, i must be injective. So it induces an injective homomorphism $i^* : H^*(\mathbb{R}P^k; \mathbb{Z}_2) \to H^*(\mathbb{R}P^n; \mathbb{Z}_2)$, $\mathbb{Z}_2[\alpha]/(\alpha^{k+1}) \to \mathbb{Z}_2[\beta]/(\beta^{n+1})$, $|\alpha| = |\beta| = 1$. Since $|\alpha| = |\beta|$, $i^*(\alpha) = \beta$ necessarily. Then we must have $0 = i^*(0) = i^*(\alpha^{k+1}) = (i^*(\alpha))^{k+1} = \beta^{k+1}$; however $\beta^{k+1} \neq 0$ since $k \lneq n$. ↯