1a) $1+x+x^2+\ldots+x^n=\dfrac{x^{n+1}-1}{x-1}$

$x\cdot(1+x+x^2+\ldots+x^n)'=x+2x^2+\ldots+nx^n\to$

$x+2x^2+\ldots+nx^n=x\cdot\left(\dfrac{x^{n+1}-1}{x-1}\right)'=\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$

$x\cdot(x+2x^2+\ldots+nx^n)'=x+2^2x^2+3^2x^3+\ldots+n^2x^n=\sum_{k=1}^n k^2x^k$ (искомая сумма, где $x=\dfrac{1}{4}$) $\to$

$\sum_{k=1}^n k^2x^k=x\cdot(\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2})'=\dfrac{n^2x^{n+3}+(-2n^2-2n+1)x^{n+2}+(n^2+2n+1)x^{n+1}-x^2-x}{(x-1)^3}$

1c) $\sum_{k=0}^\infty x^k=\dfrac{1}{1-x}(|x|<1)\to$

$\dfrac{1}{x}\int\sum_{k=0}^\infty x^k=\dfrac{1}{x}\sum_{k=0}^\infty\dfrac{x^{k+1}}{k+1}=\sum_{k=0}^n\dfrac{x^k}{k+1}$

$\dfrac{1}{x}\int\sum_{k=0}^\infty x^k=\dfrac{1}{x}\int\dfrac{1}{1-x}=-\ln(1-x)\dfrac{1}{x}=-\dfrac{\ln(1-x)}{x}$

1d) $(-\cos(kx))'=k\cdot\sin(kx)$

$\sum_{k=0}^n\cos(kx)=\dfrac{\cos(nx+\frac{x}{2})}{2\cos\frac{x}{2}}-\dfrac{1}{2}$ (см. 2d) $\to$

$\sum_{k=0}^n k\cdot\sin(kx)=-(\dfrac{\cos(nx+\frac{x}{2})}{2\cos\frac{x}{2}}-\dfrac{1}{2})'=\dfrac{\cos(nx+\frac{x}{2})\cdot\sin\frac{x}{2}-\sin(nx+\frac{x}{2})\cdot(n+\frac{1}{2})\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$

1f) $\sum_{k=0}^n\dfrac{5^{2k}}{3^{4n-3k}}C_n^k=\dfrac{1}{81^n}\sum_{k=0}^n 25^k\cdot 27^k\cdot C_n^k=\dfrac{1}{81^n}\sum_{k=0}^n 675^kC_n^k=\dfrac{(1+675)^n}{81^n}=\left(\dfrac{676}{81}\right)^n$

2a) $k\equiv 1 \mod 2\to$

$a_k=\dfrac{-k}{4k^2-1}=-\dfrac{1}{4(2k-1)}-\dfrac{1}{4(2k+1)}$

$a_{k+1}=\dfrac{1}{4(2k+1)}+\dfrac{1}{4(2k+3)}$

$a_{k+2}=-\dfrac{1}{4(2k+3)}-\dfrac{1}{4(2k+5)}$

$a_{k+3}=\dfrac{1}{4(2k+5)}+\dfrac{1}{4(2k+7)}\to$

$\sum_{k=1}^n\dfrac{(-1)^k\cdot k}{4k^2-1}=\dfrac{(-1)^n\cdot n}{4(2n+1)}-\dfrac{1}{12}$

2b) $[x]_k=x(x-2)(x-4)\ldots(x-2(k-1))\to$

$[x+2]_{k+1}-[x]_{k+1}=(x+2)x(x-2)\ldots(x-2(k-1))-x(x-2)\ldots(x-2(k-1))(x-2k)=(x+2-x+2k)x(x-2)\ldots(x-2(k-1))=(2k+2)[x]_k\to$

$[x]_k=\dfrac{[x+2]_{k+1}-[x]_{k+1}}{2n+2}\to$

$\sum_{x=1}^n[x]_k=\dfrac{1}{2k+2}\sum_{x=1}^n[x+2]_{k+1}-[x]_{k+1}=\dfrac{1}{2n+2}([n+2]_{k+1}+[n+1]_{k+1}-[2]_{k+1}-[1]_{k+1})$

2d) $\cos(kx)=\dfrac{2\cdot\cos(kx)\cdot\cos\frac{x}{2}}{2\cdot\cos\frac{x}{2}}=\dfrac{\cos(kx+\frac{x}{2})-\cos(kx-\frac{x}{2})}{2\cdot\cos\frac{x}{2}}\to$

$\sum_{k=0}^n\cos(kx)=\dfrac{1}{2\cos\frac{x}{2}}\sum_{k=0}^n(\cos(kx+\frac{x}{2})-\cos(kx-\frac{x}{2}))=\dfrac{1}{2\cos\frac{x}{2}}(\cos(nx+\frac{x}{2})-\cos\frac{x}{2})=\dfrac{\cos(nx+\frac{x}{2})}{2\cos\frac{x}{2}}-\dfrac{1}{2}$

3e) $(1+i)^{4m}=\sum_{k=0}^{4m}C_{4m}^ki^k$

$k\equiv 0 \mod 4\to C_{4m}^ki^k=C_{4m}^k$

$k\equiv 1 \mod 4\to C_{4m}^ki^k=C_{4m}^ki$

$k\equiv 2 \mod 4\to C_{4m}^ki^k=-C_{4m}^k$

$k\equiv 3 \mod 4\to C_{4m}^ki^k=-C_{4m}^ki$

$\to$ Если $k\equiv 1(3) \mod 4$, то $(4m-k)\equiv 3(1) \mod 4\to C_{4m}^ki^k+C_{4m-k}^ki^{4m-k}=0\to$

$(1+i)^{4m}=C_{4m}^0-C_{4m}^2+\ldots+C_{4m}^{4m}$

3f) $i(1+i)^{4m} = \sum_{k=0}^{4m} C_{4m}^k i^{k+1}$
$k \equiv 0 \mod 4 \rightarrow C_{4m}^k i^{k+1} = C_{4m}^k i$
$k \equiv 1 \mod 4 \rightarrow C_{4m}^k i^{k+1} = -C_{4m}^k$
$k \equiv 2 \mod 4 \rightarrow C_{4m}^k i^{k+1} = -C_{4m}^k i$
$k \equiv 3 \mod 4 \rightarrow C_{4m}^k i^{k+1} = C_{4m}^k$
$\rightarrow k \equiv 0(2) \mod 4 \rightarrow (4m-k) \equiv 2(0) \mod 4 \rightarrow C_{4m}^k i^{k+1} + C_{4m}^{4m-k} i^{4m-k+1} = 0$
$i(1+i)^{4m} = -C_{4m}^1 + C_{4m}^3 + \ldots + C_{4m}^{4m-1} \rightarrow$
$C_{4m}^1 - C_{4m}^3 + \ldots - C_{4m}^{4m-1} = -i(1+i)^{4m}$