Introduction to Tensors

PFI Seminar

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Self Introduction

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- PSS(Professional and Support Service) Division
- Jubatus Project
  - msgpack-idl, generation of clients
- Survey and Prototyping Team
- 2010 PFI Summer Intern → 2011 Engineer

Agenda

- What is Tensor?
- Decomposition of Matrices and Tensors
- Symmetry Parametrized by Young Diagram

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Vector as a Special Case of Tensor

Consider a n dimensional vector (Below is the case $n = 3$)

$$v = \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} \tag{1}$$

When we write this vector as $v = (v_i)$, we have $v_1 = 7$, $v_2 = 4$, and $v_3 = 9$. As v has **One** index, this vector is regarded as **Rank 1 Tensor**.

Note

- We use 1-index notation, not 0-index.
- We somtimes use rank in a different meaning, (also in the context of tensors).

Matrix as a Special Case of Tensor

Consider a $n \times n$ Matrix (Below is the case $n = 3$).

$$A = \begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix} \tag{2}$$

We write this matrix as $A = (a_{ij})_{1 \leq i,j \leq 3}$.

- e.g. $a_{11} = 7$, $a_{23} = -2$, $a_{32} = 4$.

A has **Two** indices (namely i and j) and these indeices run through 1 to 3. A can be regarded as **Rank 2 Tensor**.

$\rightarrow$ What happens if the number of indices are not just 1 or 2 ?

An Example of Rank 3 Tensor

Consider the set of real numbers $X = (x_{ijk})_{1 \leq i,j,k \leq 3}$. As i, j, k run throuth 1 to 3 independently, this set contains $3 \times 3 \times 3 = 27$ elements.
If we fix the value of i, we can write this set in a matrix form.

$$x_{1**} = \begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix}, x_{2**} = \begin{bmatrix} 10 & 2 & 1 \\ 3 & -1 & 0 \\ 8 & 1 & 4 \end{bmatrix}, x_{3**} = \begin{bmatrix} 5 & -10 & 18 \\ -4 & 1 & -2 \\ 9 & -4 & -10 \end{bmatrix} \tag{3}$$

Note

- In some field, we concatenate these matrix and name it $X_{(1)}$
  - This is called **Flattening** or **Matricing** of X.

Is This a Multidimensional Array?

We can realize a tensor as a multiple dimensional array in most programming languages.

PseudoCode(C like):

```
int[3][3][3] x;
x[1][1][2] = -1;
```

Inner Product of Tensors

Let $X = (x_{ijk})$, $Y = (y_{ijk})$ be two rank 3 tensors and $G = (g^{ij})$ be a symmetric (i.e. $g^{ij} = g^{ji}$) and positive definite matrix. We can introduce an inner product of X and Y by:

$$< X, Y >= \sum_{a,b,c,i,j,k=1}^{n} g^{ai} g^{bj} g^{ck} x_{abc} \, y_{ijk} \tag{4}$$

Note:

- We can similarly define an inner product of two arbitrary rank tensor
    - X and Y must have same rank.

An Example of Inner Product (Vector)

Let's consider the case of rank 1 tensor ($=$ vector). We can write $X = (x_i)$, $Y = (y_i)$. And we set $G = (g^{ij})$ an identity matrix:

$$G = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \tag{5}$$

In this case, this definition of inner product reduce to ordinal inner product of two vectors, namely,

$$< X, Y > = \sum_{a,i} g^{ai} x_a y_i = \sum_i x_i y_i \tag{6}$$

An Example of Tensor

Suppose we have a (smooth) function $f : \mathbb{R}^3 \to \mathbb{R}$. We can derive tensors of arbitrary rank from this function.

Taking **Gradient**, we obtain rank 1 tensor. (Please replace (1, 2, 3) with (x, y, z) and vice versa.

$$\nabla f = (\nabla_i f) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \tag{7}$$

Similarly, **Hessian** is an example of rank 2 tensor.

$$H(f) = (H_{ij}(f)) = \begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z \partial z} \end{bmatrix} \tag{8}$$

Supersymmetry Property

By taking derivative, we can create an arbitrary rank of tensors. These tensors are **(Super)Symmetric** in the sense that interchange of any two indices remains the tensor identical. For example,

$$f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zxy} = f_{zyx}. \tag{9}$$

Note:

- Afterward, we write this symmetry as $\square\square\square$.

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Decompose!

It is fundamental in mathematics to decompose some object into simpler, easier-to-handle objects.

- Fourier Expansion : $f = \sum_{n \in \mathbb{Z}} a_n \exp(\sqrt{-1}nx)$
- Legendre Polynomial : $P = \sum_k a_k P_k$
    - where $P_k(x) = \frac{1}{k!2^k} \frac{d^k}{dx^k} (x^2 - 1)^k$
- Jordan-Hölder : $e = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$

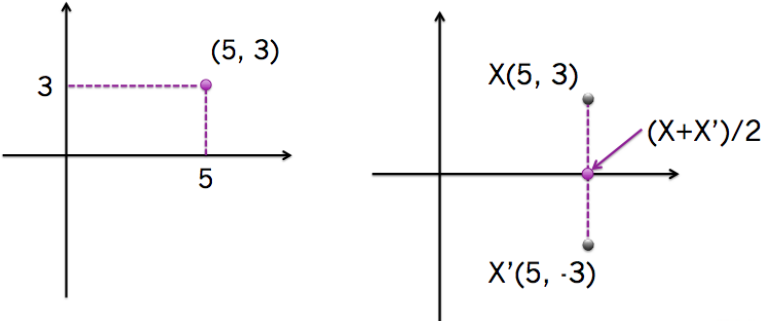
Decomposition of Matrix

There are many theories regarding the decomposition (or, factorization) of matrices.

- SVD
- LU
- QR
- Cholesky
- Jordan
- Diagonalization ...

Although we have theories of factorization of Tensors (e.g. Tucker Decomposition, higher-order SVD etc.) We do **NOT** go this direction. Instead, we consider decomposition of matrix into **Summation** of matrix.

Example: Projection to Axis



Example of Decomposition of Matrix

We can decompose matrix into **Symmetric** part and **Antisymmetric** part.

Example:

$$\underbrace{\begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 7 & 7 & 6 \\ 7 & -1 & 1 \\ 6 & 1 & 5 \end{bmatrix}}_{\substack{\text{Symmetric Part} \\ A^{sym}}} + \underbrace{\begin{bmatrix} 0 & 3 & -3 \\ -3 & 0 & -3 \\ 3 & 3 & 0 \end{bmatrix}}_{\substack{\text{Antisymmetric Part} \\ A^{anti}}} \tag{10}$$

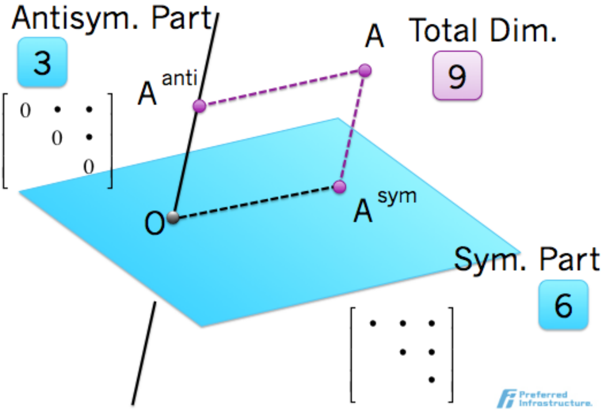
How to Calculate Sym. and Antisym. Part

We can calculate the symmetic and antisymmetric part by simple calculation (Exercise!).

$$A^{sym} = \frac{1}{2}\left(\begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 4 & 9 \\ 10 & -1 & 4 \\ 3 & -2 & 5 \end{bmatrix}\right) \tag{11}$$
$$a_{ij}^{sym} = \frac{1}{2}(a_{ij} + a_{ji})$$

$$A^{anti} = \frac{1}{2}\left(\begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 9 \\ 10 & -1 & 4 \\ 3 & -2 & 5 \end{bmatrix}\right) \tag{12}$$
$$a_{ij}^{anti} = \frac{1}{2}(a_{ij} - a_{ji})$$

Picture of Projection to Sym. and Antisym. Part



Notation Using Young Diagram

We can symbolize symmetry and antisymmetry with **Young Diagram**

$$\underset{A}{\begin{bmatrix} 7 & 10 & 3 \\ 4 & -1 & -2 \\ 9 & 4 & 5 \end{bmatrix}} = \underset{A^{sym}\ \begin{array}{|c|c|}\hline & \\ \hline\end{array}}{\begin{bmatrix} 7 & 7 & 6 \\ 7 & -1 & 1 \\ 6 & 1 & 5 \end{bmatrix}} + \underset{A^{anti}\ \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}}{\begin{bmatrix} 0 & 3 & -3 \\ -3 & 0 & -3 \\ 3 & 3 & 0 \end{bmatrix}} \tag{13}$$

The number of boxes indicates rank of the tensor (one box for vector and two boxes for matrix).

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Note:

- From now on, we concentrate on **Rank 3 Tensors** (i.e. $k = 3$).
- And we assume that $n = 3$, that is, indices run from 1 to 3.

Symmetrizing Operator

We consider the transformation of tensor:

$$T_{\square\square\square} : X = (x_{ijk}) \mapsto X' = (x'_{ijk}), \tag{14}$$

$$\begin{aligned} \text{where } x'_{ijk} &= \frac{1}{6}\left(x_{ijk} + x_{ikj} + x_{jik} + x_{jki} + x_{kij} + x_{kji}\right) \\ &= x_{(ijk)}. \end{aligned} \tag{15}$$

Young Diagram and Symmetry of Tensor (Sym. Case)

Let $X = (x_{123})$ be a tensor of rank 3, we call X **Has a Symmetry of** $\square\square\square$, if interchange of any of two indices doesn't change each entry of X.

Example:

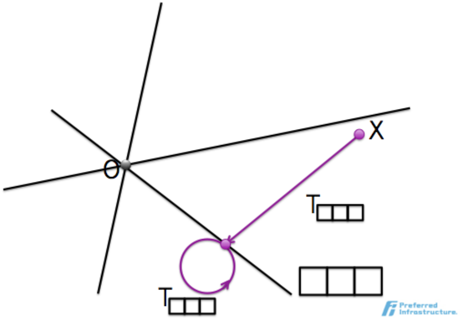
- If X has a symmetry of $\square\square\square$, then

$$\begin{gathered} x_{112} = x_{121} = x_{211} \\ x_{123} = x_{132} = x_{213} = x_{231} = x_{312} = x_{321} \quad etc... \end{gathered} \tag{16}$$

- Symmetric matrices have a symmetry of $\square\square$

Property of $T_{\square\square\square}$

- For all X, $T_{\square\square\square}(X)$ has symmetry of $\square\square\square$.
- If X has symmetry of $\square\square\square$, then $T_{\square\square\square}(X) = X$.
- For all X, $T_{\square\square\square}(T_{\square\square\square}(X)) = T_{\square\square\square}(X)$.

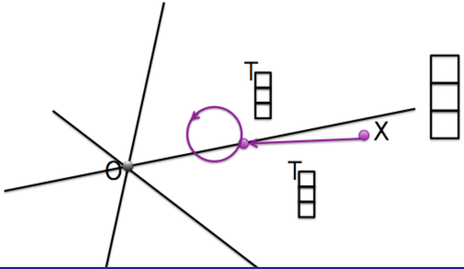


Antisymmetrizing Operator

Next, we consider **Anti**symmetrization of tensors:

$$T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} : X = (x_{ijk}) \mapsto X' = (x'_{ijk}), \tag{17}$$

$$\text{where } x'_{ijk} \begin{aligned}[t] &= \frac{1}{6}\left(x_{ijk} - x_{ikj} - x_{jik} + x_{jki} + x_{kij} - x_{kji}\right) \\ &= x_{[ijk]}. \end{aligned} \tag{18}$$



Young Diagram and Symmetry of Tensor (Antisym. Case)

Let $X = (x_{123})$ be a tensor of rank k, we call X has a symmetry of $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ if interchange of any of two indices change **only the sign** of each entry of X.

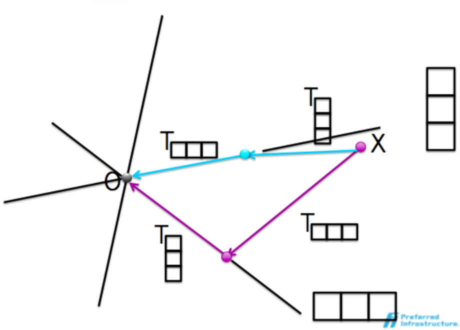
Example.

- If X has a symmetry of $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$, then
  - $x_{123} = -x_{132} = -x_{213} = x_{231} = x_{312} = -x_{321}$
  - $x_{112} = 0$, because x_{112} must be equal to $-x_{112}$.
- Antisymmetric matrix has a symmetry of $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$

Orthogonality of Projection

$T_{\square\square\square}$ and $T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ are orthogonal in the sense

$$T_{\square\square\square} \circ T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(X) = T_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} \circ T_{\square\square\square}(X) = 0. \tag{19}$$



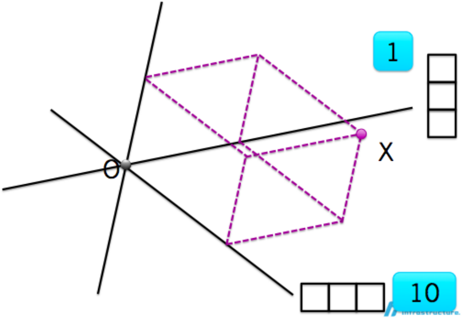
Counting Dimension

How many dimension do the vector space consisting of tensors ...

- with symmetry $\ydiagram{3}$ have ?
- with symmetry $\ydiagram{1,1,1}$ have ?

Dimensions of each subvector spaces.

If $n = 3$ (that is, i, j and k runs through 1 to 3), then each dimension is as follows.



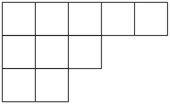
But dimension of total space is $3 \times 3 \times 3 = 27$. This is larger than $1 + 10 = 11$.

Young Diagram

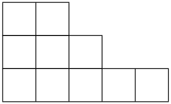
Young Diagram is an arrangent of boxes of the same size so that:

- two adjacent boxes share one of their side.
- the number of boxes in each row is non-increasing.
- the number of boxes in each column is non-increasing.

Here is an example of Young Diagram with 10 boxes.



English Notation



French Notation

(20)

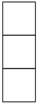
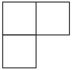
We have three types of Young Diagram which have three boxes, namely,

$$\underset{\textit{Symmetric}}{\ydiagram{3}}, \quad \underset{\textit{Antisymmetric}}{\ydiagram{1,1,1}}, \quad \textit{and} \quad \underset{???}{\ydiagram{2,1}} \tag{21}$$

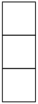
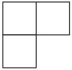
$\to$ What symmetry does $\ydiagram{2,1}$ represent?

Coefficients of Symmetrizers

We list coneffients of each operator in a tabular form.

	Coefficient of x_{ijk}					
	ijk	ikj	jik	jki	kij	kji
	1/6	1/6	1/6	1/6	1/6	1/6
	1/6	-1/6	-1/6	1/6	1/6	-1/6
						
Sum	1					

Coefficients of Symmetrizers

	Coefficient of x_{ijk}					
	ijk	ikj	jik	jki	kij	kji
	1/6	1/6	1/6	1/6	1/6	1/6
	1/6	-1/6	-1/6	1/6	1/6	-1/6
	4/6			-2/6	-2/6	
Sum	1					

Symmetrizer $T_{\yng(2,1)}$

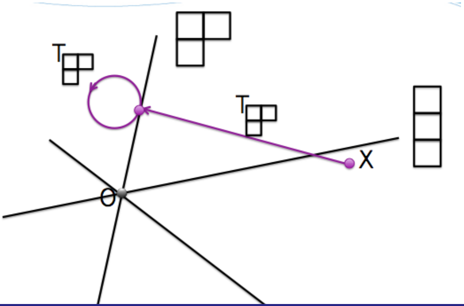
We consider the transformation of tensor:

$$T_{\yng(2,1)} : X = (x_{ijk}) \mapsto X' = (x'_{ijk}), \tag{22}$$

where $x'_{ijk} = \frac{1}{3}\left(2x_{ijk} - x_{jki} - x_{kij}\right)$.

Property of $T_{\ydiagram{2,1}}$

- For all X, $T_{\ydiagram{2,1}}(X)$ has symmetry of $\ydiagram{2,1}$.
- If X has symmetry of $\ydiagram{2,1}$, then $T_{\ydiagram{2,1}}(X) = X$.
- For all X, $T_{\ydiagram{2,1}}\left(T_{\ydiagram{2,1}}(X)\right) = T_{\ydiagram{2,1}}(X)$.



Tensors with Symmetry of $\ydiagram{2,1}$

Let $X = (x_{ijk})$ be a tensor of rank 3. We say X **Has a Symmetry of** $\ydiagram{2,1}$, if

$$x_{ijk} = \frac{1}{3}\left(2x_{ijk} - x_{jki} - x_{kij}\right) \tag{23}$$

for all $1 \leq i, j, k \leq n$.

Where This Equation "Symmetric" ?

Remember that X has a symmetry of , if

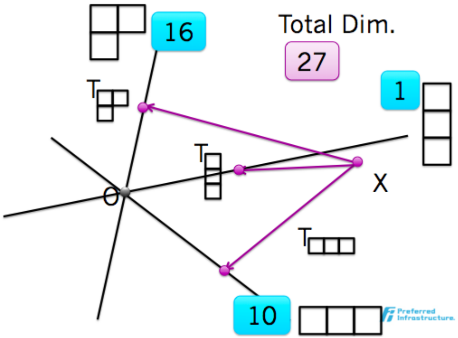
$$x_{ijk} = \frac{1}{3}\left(2x_{ijk} - x_{jki} - x_{kij}\right) \tag{24}$$

for all $1 \leq i, j, k \leq n$.

This equation can be transormed as follows :

$$\begin{aligned} & x_{ijk} = \frac{1}{3}\left(2x_{ijk} - x_{jki} - x_{kij}\right) \\ \Leftrightarrow \quad & 3x_{ijk} = 2x_{ijk} - x_{jki} - x_{kij} \\ \Leftrightarrow \quad & x_{ijk} + x_{jki} + x_{kij} = 0 \end{aligned} \tag{25}$$

Projection to Each Symmetrized Tensors



Decomposition of Higher Rank Tensors

Higher rank tensors are also decompose into symmetric tensors parametrized by Young Diagram.

$$
\begin{aligned}
\mathbb{R}^{n\times n\times n\times n\times n} &= \yng(5) \oplus \yng(4,1) \oplus \yng(3,2) \\
&\oplus \yng(3,1,1) \oplus \yng(2,2,1) \oplus \yng(2,1,1,1) \oplus \yng(1,1,1,1,1)
\end{aligned} \tag{26}
$$

Summary

- Tensors as a Generalization of Vectors and Matrices.
- Decomposition of Matrices and Tensors
- Symmetry of Tensors Parametrized by Young Diagrams.

References

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- And Google "Representation Theory", "Symmetric Group", "Young Diagram" and so on.