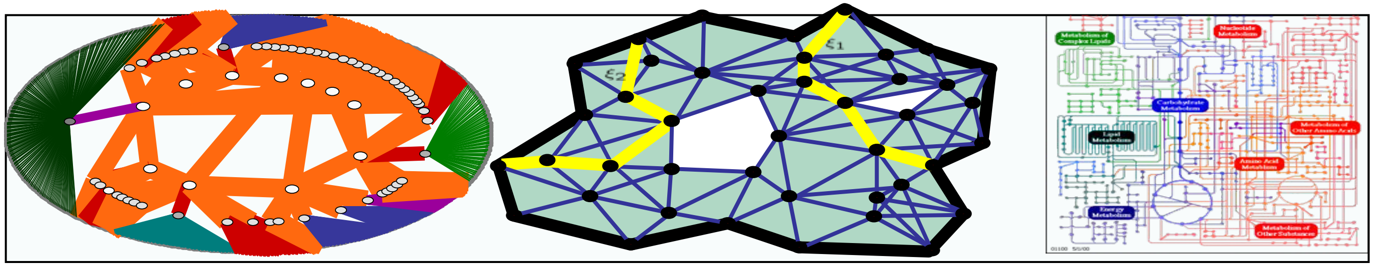

Distributed Coordination, Consensus, and Coverage in Networked Dynamic Systems

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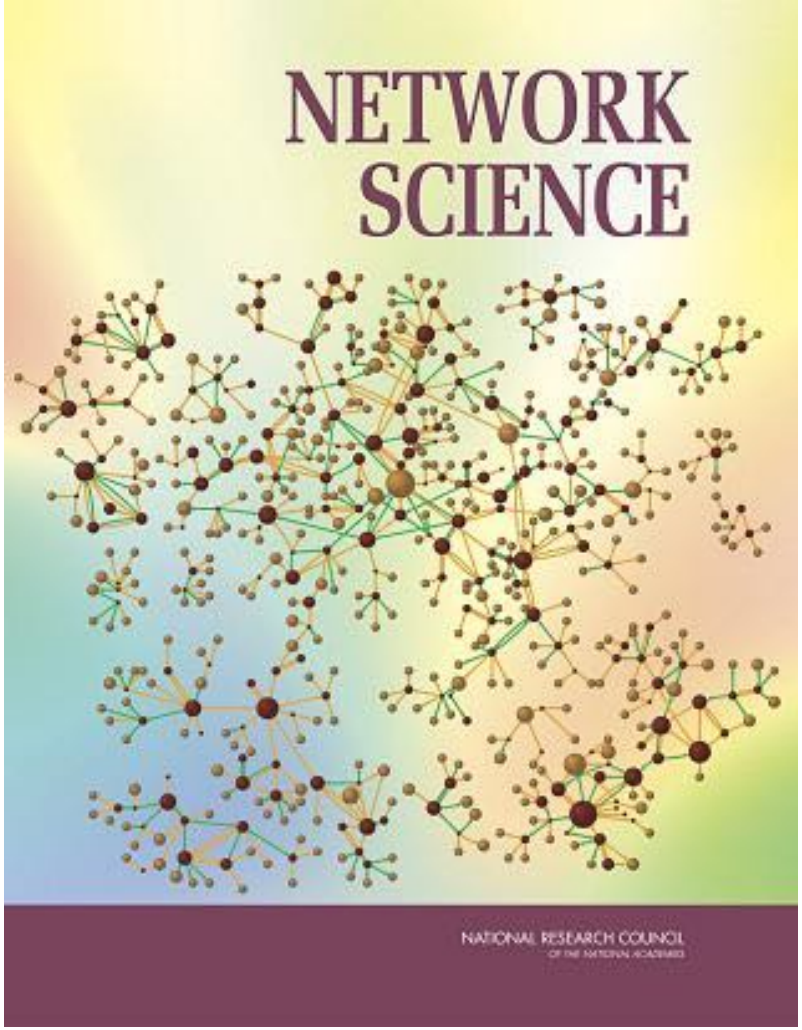
Electrical & Systems Engineering

and GRASP Laboratory

University of Pennsylvania

Kick-off Meeting, July 28, 2008



Good news:
Spectacular progress

Challenges in the NS report:

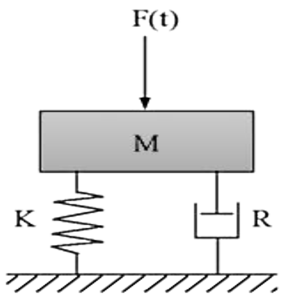
1. ***Dynamics, spatial location, and information propagation in networks.***
2. ***Modeling and analysis of very large networks.***
3. ***Design and synthesis of networks.***
4. ***Increasing the level of rigor and mathematical structure.***
5. *Abstracting common concepts across fields.*
6. *Better experiments and measurements of network structure.*
7. *Robustness and security of networks.*

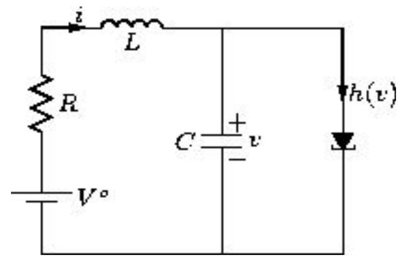
Complexity: dynamics vs. size

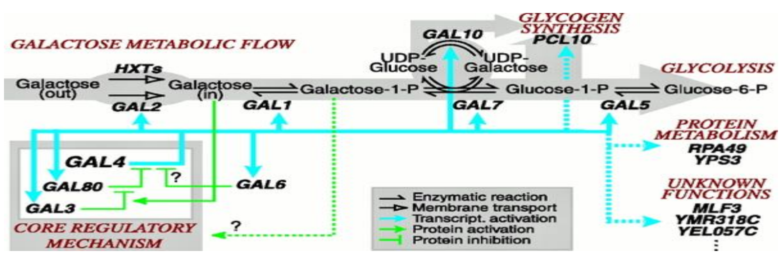
System size

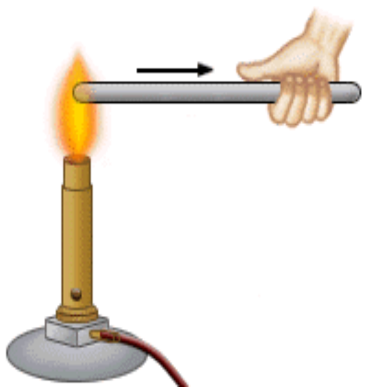
Single agent — Multi-agent

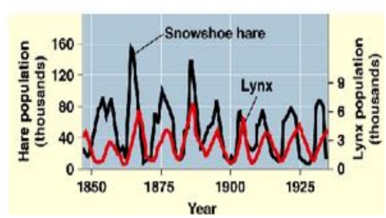
State dimensionality: low — infinite

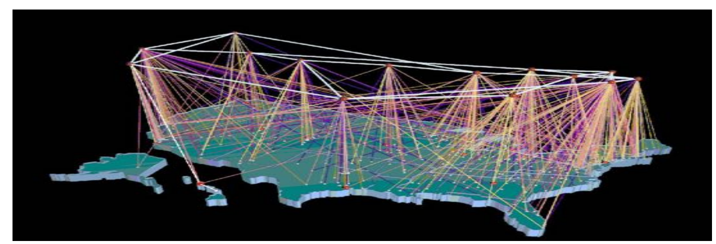


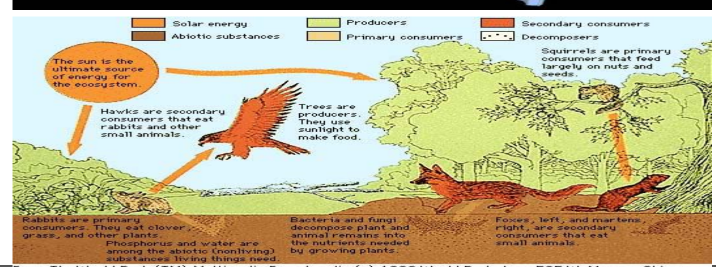












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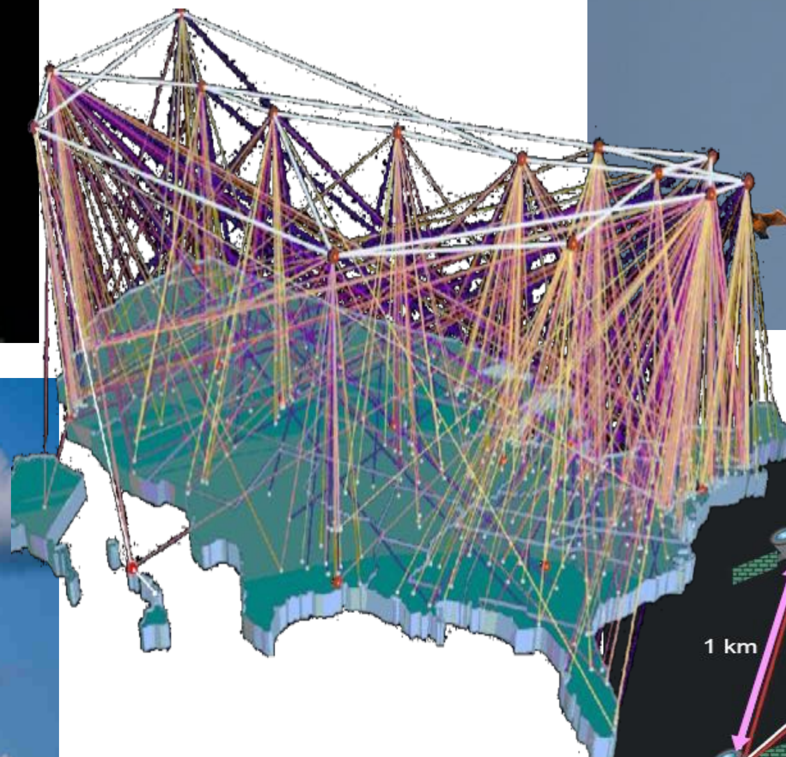
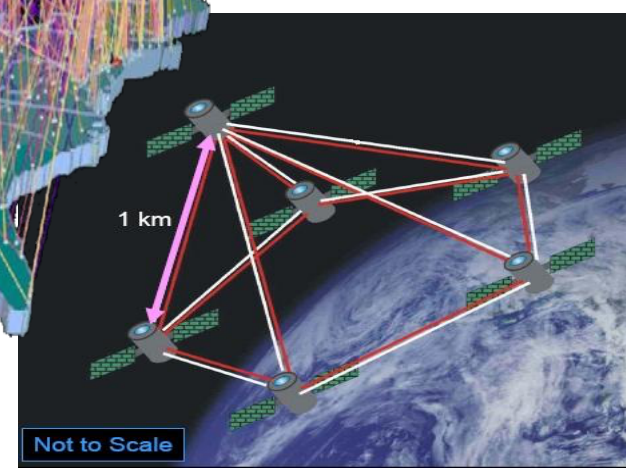
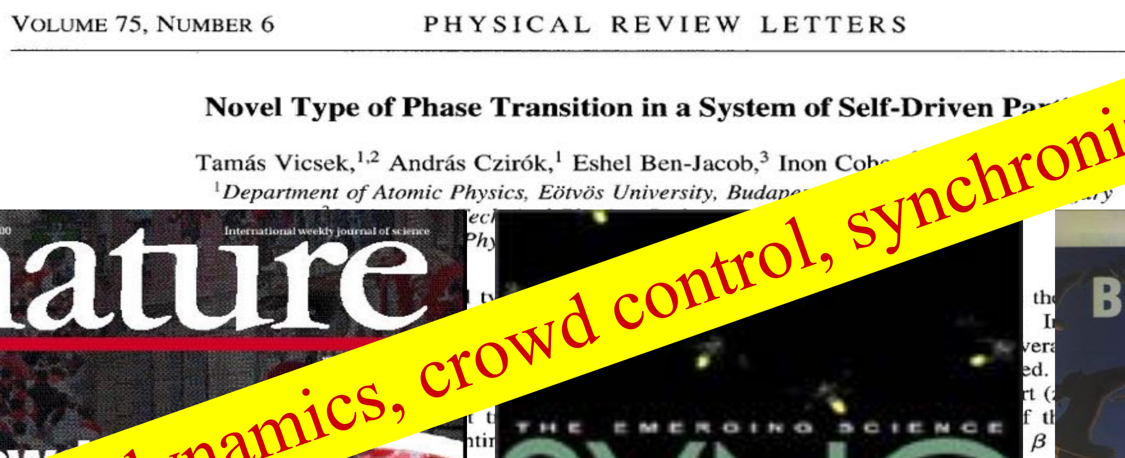
Substantial Recent Progress





Case Study: Emergence of Consensus, synchronization, flocking

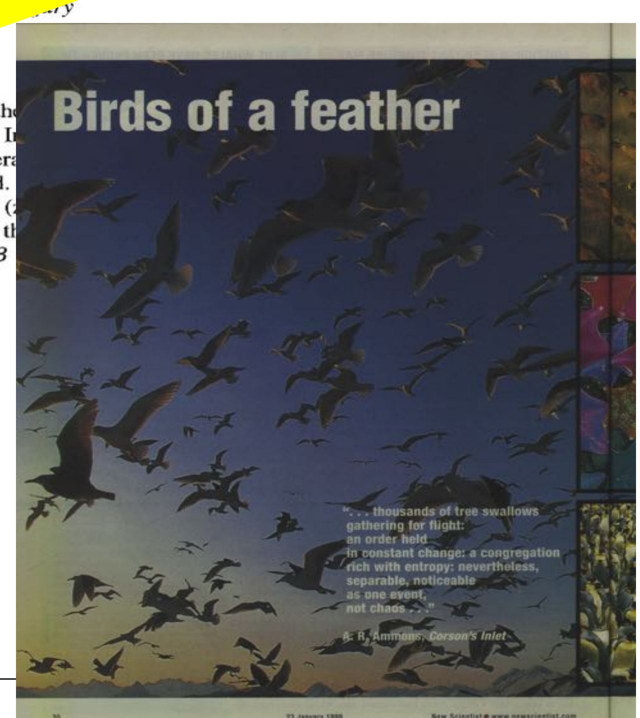

VOLUME 75, NUMBER 6 PHYSICAL REVIEW LETTERS 7 AUGUST 19

Novel Type of Phase Transition in a System of Self-Driven Par

Tamás Vicsek,[1,2] András Czirók,[1] Eshel Ben-Jacob,[3] Inon Coh

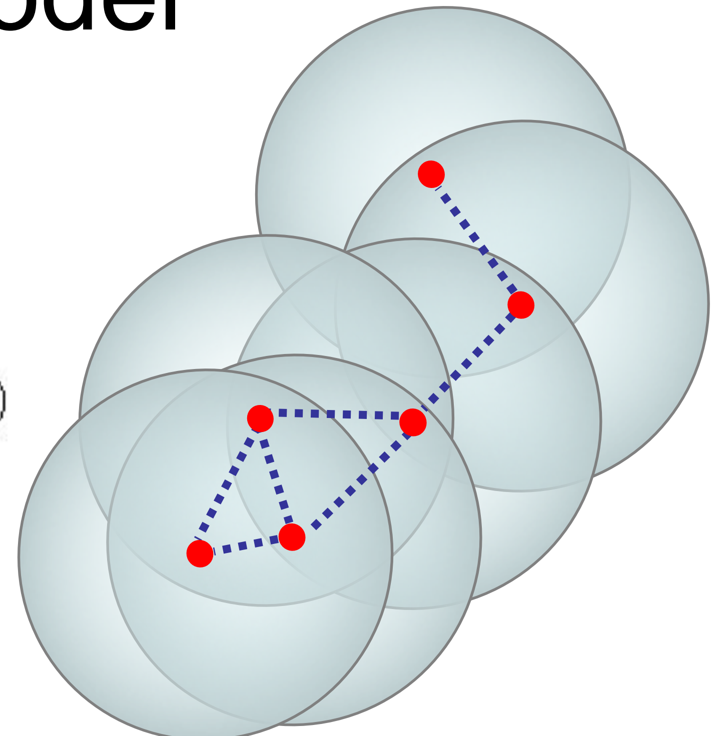
[1]Department of Atomic Physics, Eötvös University, Budape

Opinion dynamics, crowd control, synchronization and flocking



UCSD

An intuitive model



The value of each agent is updated (in discrete time) as a weighted **average** of the value of its neighbors:

$$\theta_i(k+1) = \frac{1}{d_i(k)+1}\Big(\sum_{j \in \mathcal{N}_i(k)} w_{ij}\theta_j(k) + w_{ii}\theta_i(k)\Big)$$

Neighborhood relation might depend on actual value, resulting in change in topology

The neighboring relationship between the agents was represented by a **graph**. $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$

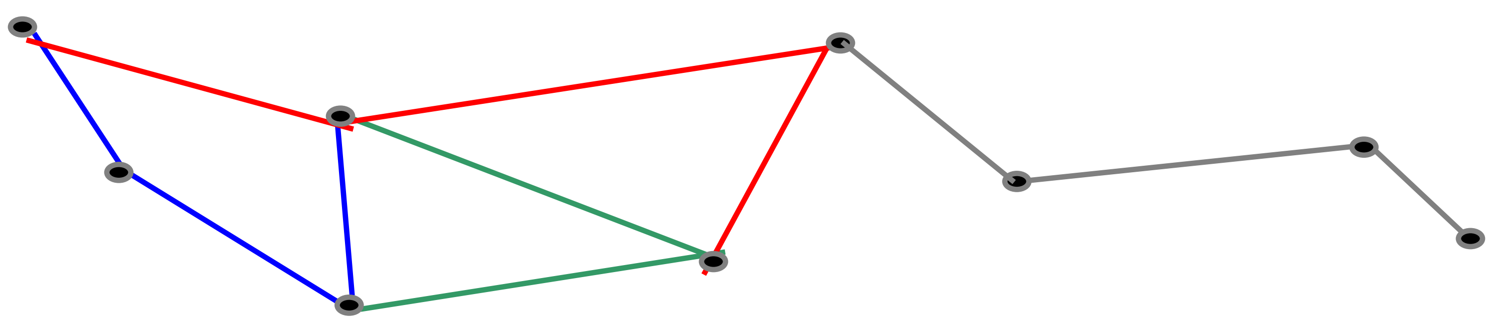
The connectivity graph could be **fixed** or **dynamic**.
When do values converge?

What regimes of topology change are good and which ones are bad?

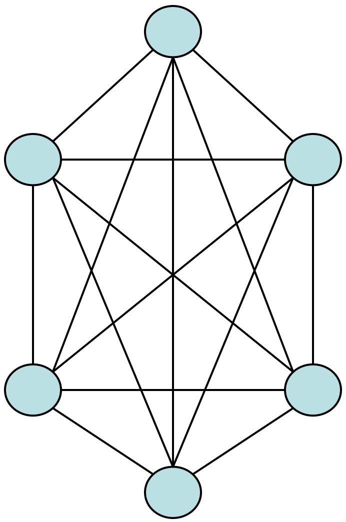
Conditions for reaching consensus

Theorem (Jadbabaie et al. 2003): *If there is a sequence of bounded, non-overlapping time intervals T_k, such that over any interval of length T_k, the network of agents is "jointly connected ", then all agents will reach consensus on their velocity vectors.*

Theorem (Tahbaz Salehi and Jadbabaie '08): when graph process is random, almost sure consensus iff the "average" network reaches deterministic agreement.



Synchronization



$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

N : Number of oscillators

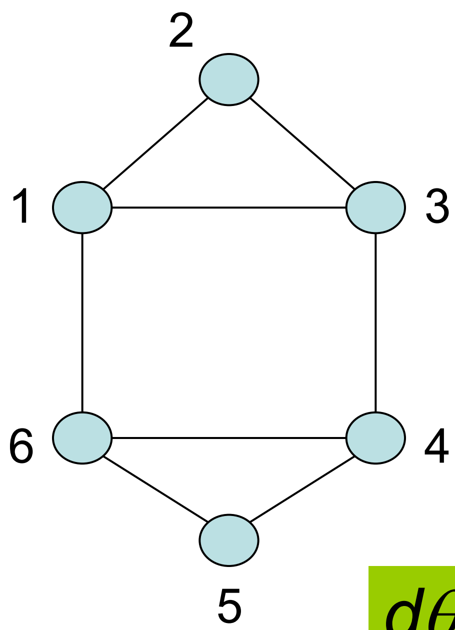
ω_i: Natural frequency of oscillator i, $i = 1,\ldots,N$.

θ_i : Phase of oscillator i, $i = 1,\ldots,N$.

K : Coupling strength

- Model for *pacemaker cells in the heart and nervous system, collective synchronization of pancreatic beta cells, synchronously flashing fire flies, rhythmic applause, gait generation for bipedal robots, …*
- Benchmark problem in physics
- Not very well understood over arbitrary networks

Kuramoto model & graph topology



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i)$$

- B is the incidence matrix of the graph

$$\dot{\theta} = \omega - \frac{K}{N} B \sin(B^T \theta)$$

Kuramoto model, dual decomposition and nonlinear utility minimization

Minimize the misalignment

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\left(1-\cos(\theta_i-\theta_j)\right), \\ \text{s.t.} \quad & \sum_{j=1}^{N} A_{ij}\sin(\theta_i-\theta_j)=\frac{N\omega_i}{K} \end{aligned}$$

$$L=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\left(1-\cos(\theta_i-\theta_j)\right)-\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\nu_i\sin(\theta_i-\theta_j)+\sum_{i=1}^{N}\frac{N\nu_i\omega_i}{K}$$

$$\frac{\partial L}{\partial\left(\theta_i-\theta_j\right)}=\sin(\theta_i-\theta_j)-(\nu_i-\nu_j)\cos(\theta_i-\theta_j)$$

$$\frac{\partial L}{\partial \nu_i}=\sum_{j=1}^{N} A_{ij}\sin(\theta_i-\theta_j)-\frac{N\omega_i}{K}$$

$$\dot{\nu}_i=-\frac{K}{N}\frac{\partial L}{\partial \nu_i}=\omega_i+\frac{K}{N}\sum_{j=1}^{N} A_{ij}\frac{(\nu_i-\nu_j)}{\sqrt{1+(\nu_i-\nu_j)^2}}$$

Kuramoto model is the just a gradient algorithm for minimization of a global utility which measures misalignment between phasors (exactly like TCP!)

Internet congestion control

S sources (Users):

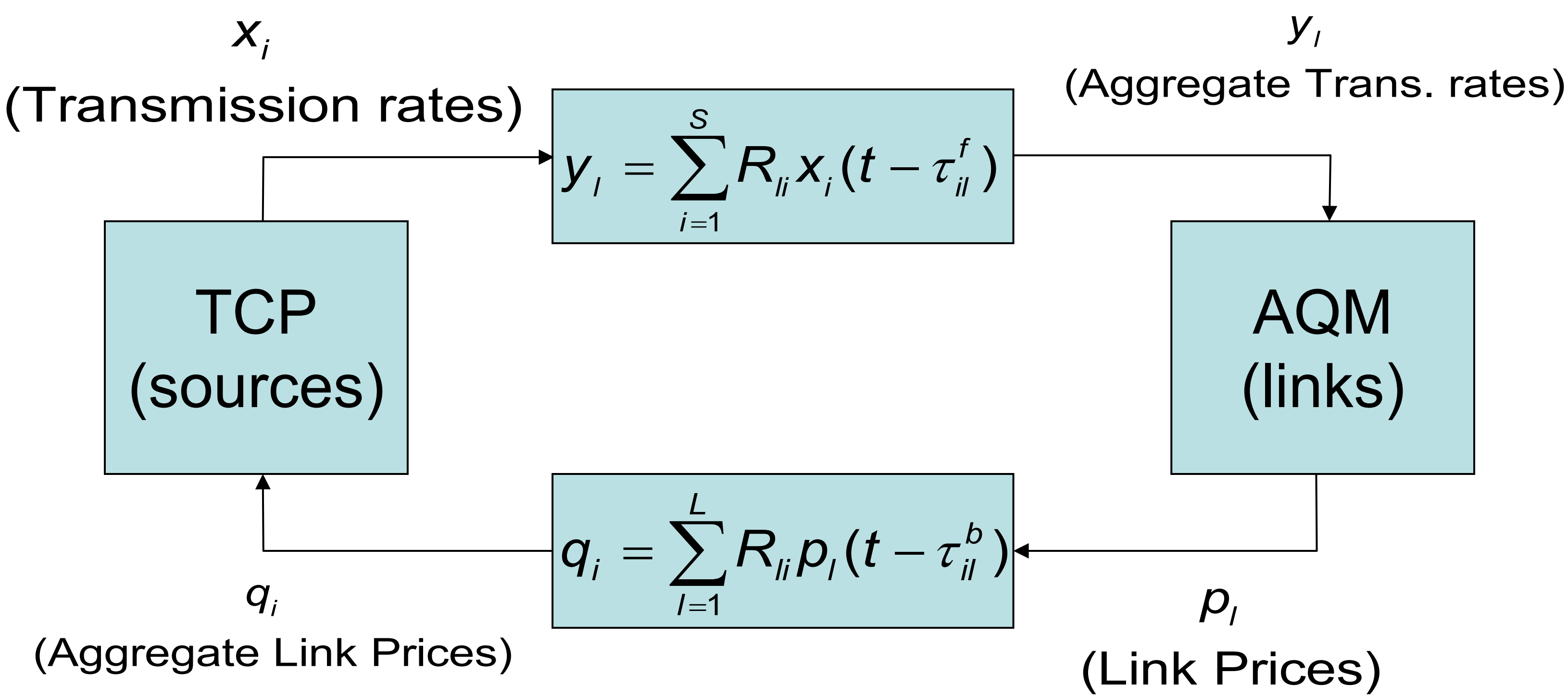
Round Trip time $\tau_i,\ i = 1,\dots,S$

L links (Routers):

Capacity $c_l,\ l = 1,\dots,L$

R Routing Matrix:

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases}$$



Consensus, synchronization, and congestion control

- All 3 involve dynamics over networks
- All can be posed as nonlinear utility maximization /network flow problems
- Common algorithms in all 3 are implementation of a distributed gradient algorithm for solving an implicit optimization
- Proofs scale to arbitrary dimensions, arbitrary topologies, with arbitrary delays
- Success in analysis due to interplay of dynamics with the combinatorial interconnection structure (eg. graphs)

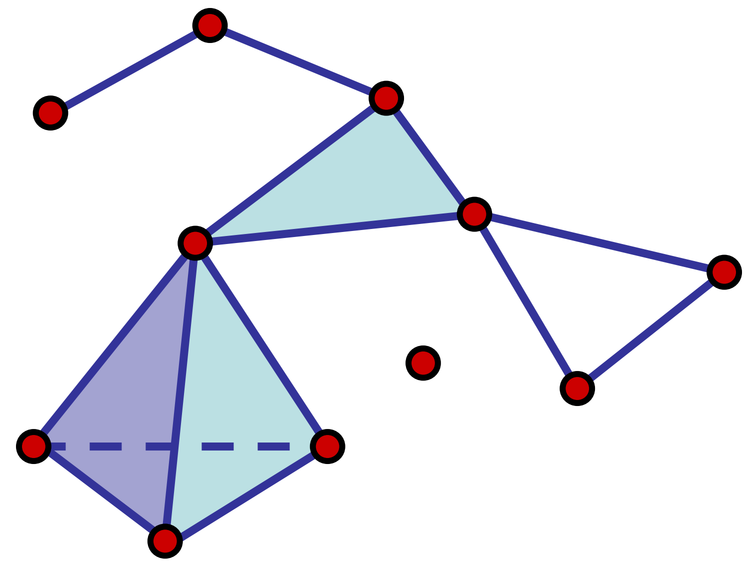
Beyond graphs: Higher order combinatorial specs

Given a set of points: $V = \{v_1, \cdots, v_n\}$

k-simplex: An unordered subset of elements of V of size $k+1$.

Face of a k-simplex: Any size k subset of a simplex

Simplicial Complex:
A collection of simplices closed under the inclusion of faces



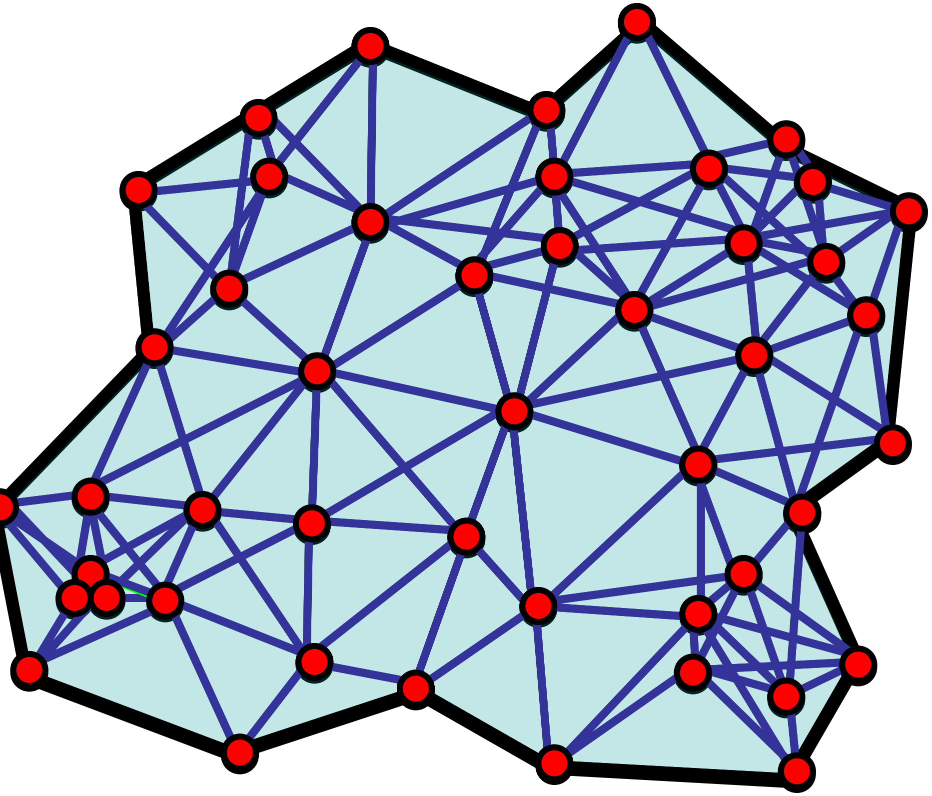
Higher order simplices are like higher order terms in a combinatorial "Taylor Expansion"

Graphs are just the "linear term"

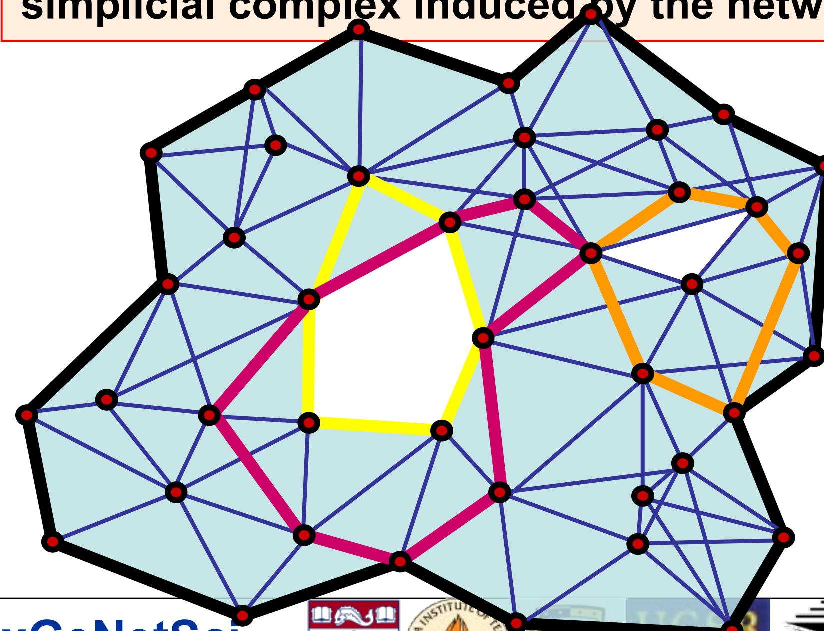
Lots of tools form graph theory extend to Simplicial complexes

Case study: Blind Coverage

- A bounded domain of interest $\mathcal{D}$ filled with a finite number of sensors
- Each sensor can communicate with other sensors within a distance r
- Any group of sensors in pairwise communication cover their entire convex hull.
- Can we find the minimal set of sensors to cover the entire domain?

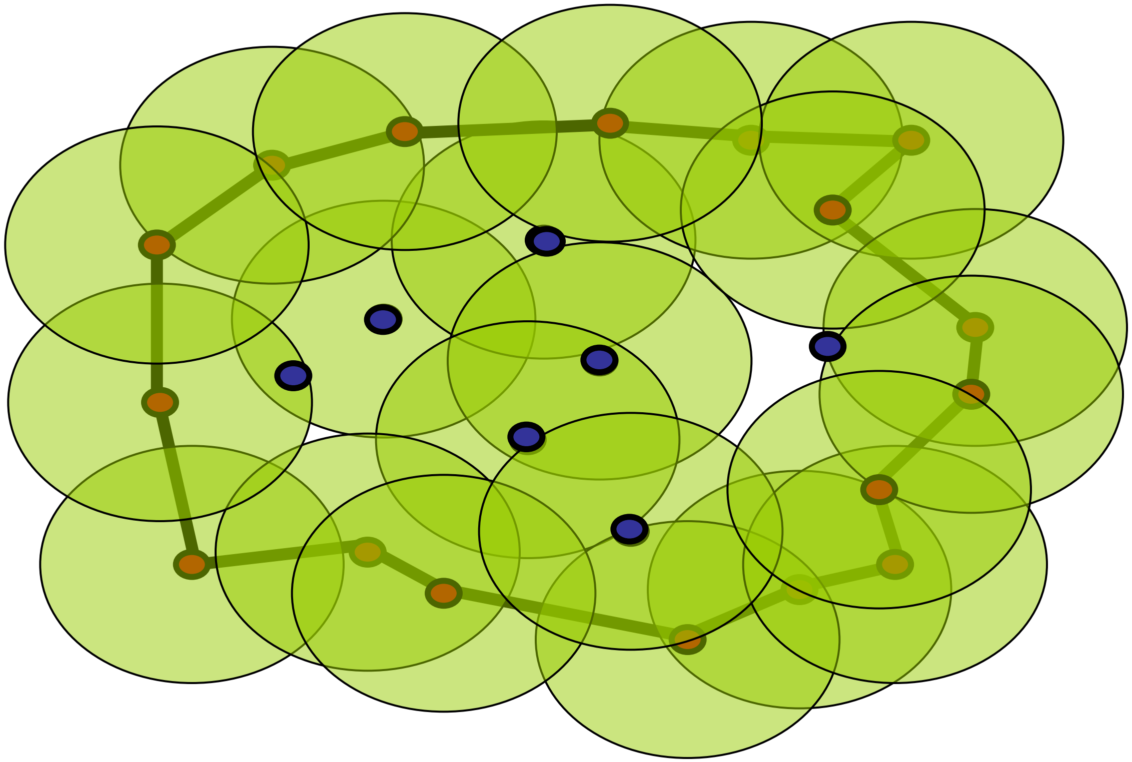


Coverage verification amounts to finding shortest representative cycles in homology classes (e.g non-contractible cycles) of a simplicial complex induced by the network



Example II: Intruder-free coverage

- Given a set of moving sensors, is there an escape strategy for an intruder in $\mathcal{D}$?



- Algebraic topology (Hodge theory) + "consensus-like" algorithms give us a decentralized test for intruder detection

Final thoughts

- Need to analyze network dynamics when nodes are dynamical systems, and change in node values changes topology
- “Network Topology” has a precise topological meaning, more than the interconnection structure
- Interplay of algebraic topology, spectral and algebraic graph theory, distributed systems and dynamics and control lead to better understanding of networks
- Need to develop distributed algorithms for computing network topological invariants
- Work presented here partially supported by ARO MURI SWARMS and DARPA DSO StoMP projects