1

Tensors

1.1 Introduction

As seen previously in the introductory chapter, the goal of continuum mechanics is to establish a set of equations that governs a physical problem from a macroscopic perspective. The physical variables featuring in a problem are represented by tensor fields, in other words, physical phenomena can be shown mathematically by means of tensors whereas tensor fields indicate how tensor values vary in space and time. In these equations one main condition for these physical quantities is they must be independent of the reference system, *i.e.* they must be the same for different observers. However, for matters of convenience, when solving problems, we need to express the tensor in a given coordinate system, hence we have the concept of tensor components, but while tensors are independent of the coordinate system, their components are not and change as the system change.

In this chapter we will learn the language of TENSORS to help us interpret physical phenomena. These tensors can be classified according to the following order:

Zeroth-Order Tensors (Scalars): Among some of the quantities that have magnitude but not direction are *e.g.:* mass density, temperature, and pressure.

First-Order Tensors (Vectors): Quantities that have both magnitude and direction, *e.g.*: velocity, force. The first-order tensor is symbolized with a boldface letter and by an arrow at the top part of the vector, *i.e.:* $\vec{\bullet}$.

Second-Order Tensors: Quantities that have magnitude and two directions, *e.g.* stress and strain. The second-order and higher-order tensors are symbolized with a boldface letter.

In the first part of this chapter we will study several tools to manage tensors (scalars, vectors, second-order tensors, and higher-order tensors) without heeding their dependence on space and time. At the end of the chapter we will introduce tensor fields and some field operators which can be used to interpret these fields.

In this textbook we will work indiscriminately with the following notations: tensorial, indicial, and matricial. Additionally, when the tensors are symmetrical, it is also possible to represent their components using the Voigt notation.

1.2 Algebraic Operations with Vectors

There now follows a brief review of vectors, in the Euclidean vector space $(\mathcal{E})$, so that we may become acquainted with the nomenclature used in this textbook.

Addition: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors, we can show the sum of adding them, (see Figure 1.1 (a)), with a new vector ($\vec{\mathbf{c}}$) thus defined as:

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \tag{1.1}$$

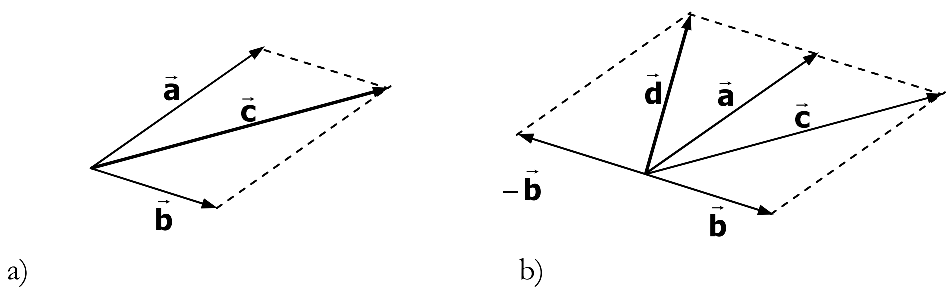


Figure 1.1: Addition and subtraction of vectors.

Subtraction: The subtraction between two arbitrary vectors ($\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$), (see Figure 1.1 (b)), is given as follows:

$$\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}} \tag{1.2}$$

Considering three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ the following properties are satisfied:

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} \tag{1.3}$$

Scalar multiplication: Let $\vec{\mathbf{a}}$ be a vector, we can define the scalar multiplication with $\lambda\vec{\mathbf{a}}$. The product of this operation is another vector with the same direction of $\vec{\mathbf{a}}$, and whose length and orientation is defined with the scalar λ as shown in Figure 1.2.

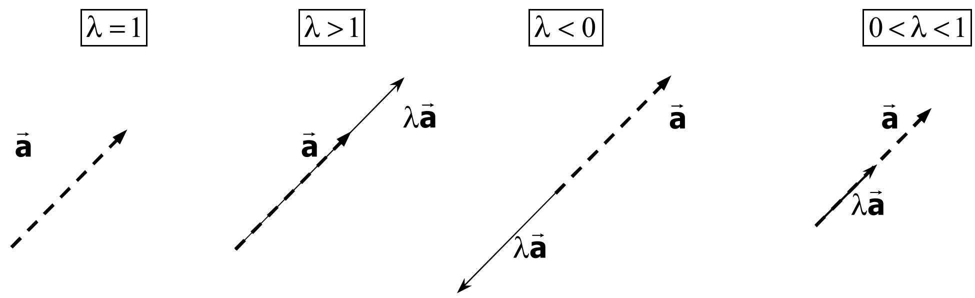


Figure 1.2: Scalar multiplication.

Scalar Product: The *Scalar Product* (also known as the *dot product* or *inner product*) of two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, denoted by $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, is defined as follows:

$$\gamma = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \cos\theta \tag{1.4}$$

where θ is the angle between the two vectors, (see Figure 1.3(a)), and $\|\bullet\|$ represents the Euclidean norm (or magnitude) of $\bullet$. The result of the operation (1.4) is a scalar. Moreover, we can conclude that $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \vec{\mathbf{b}}\cdot\vec{\mathbf{a}}$. The expression (1.4) is also true when $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, therefore:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{a}}\| \cos\theta \xrightarrow{\theta=0^\circ} \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{a}}\| \;\Rightarrow\; \|\vec{\mathbf{a}}\|^2 = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} \tag{1.5}$$

Hence, the norm of a vector is $\|\vec{\mathbf{a}}\| = \sqrt{\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}}$.

Unit Vector: A unit vector, associated with the $\vec{\mathbf{a}}$-direction, is shown with a $\hat{\mathbf{a}}$, which has the same direction and orientation of $\vec{\mathbf{a}}$. In this textbook, the hat symbol ($\hat{\bullet}$) denotes a unit vector. Thus, the unit vector, $\hat{\mathbf{a}}$, codirectional with $\vec{\mathbf{a}}$, is defined as:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \tag{1.6}$$

where $\|\vec{\mathbf{a}}\|$ represents the norm (magnitude) of $\vec{\mathbf{a}}$. If $\hat{\mathbf{a}}$ is the unit vector, then the following must be true:

$$\|\hat{\mathbf{a}}\| = 1 \tag{1.7}$$

Zero Vector (or Null Vector): The zero vector is represented by a:

$$\vec{\mathbf{0}} \tag{1.8}$$

Projection Vector: The projection vector of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$, (see Figure 1.3(b)), is defined as:

$$\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}} = \|\mathrm{proj}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\|\hat{\mathbf{b}} \quad \textit{Projection vector of } \vec{\mathbf{a}} \textit{ onto } \vec{\mathbf{b}} \tag{1.9}$$

where $\|\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}}\|$ is the projection of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$, and $\hat{\mathbf{b}}$ is the unit vector associated with the $\vec{\mathbf{b}}$-direction. The magnitude of $\|\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}}\|$ is obtained by means of the scalar product:

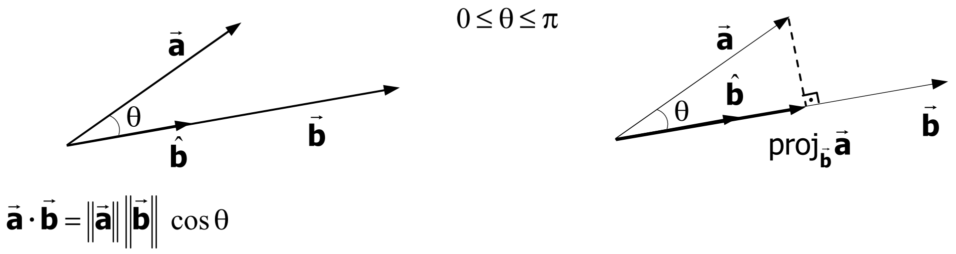
$$\|\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}}\| = \vec{\mathbf{a}}\cdot\hat{\mathbf{b}} \quad \textit{Projection of } \vec{\mathbf{a}} \textit{ onto } \vec{\mathbf{b}} \tag{1.10}$$

So, taking into account the definition of the unit vector, we obtain:

$$\|\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}}\| = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \tag{1.11}$$

Then, the projection vector, $\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}}$, can be calculated by:

$$\overrightarrow{\mathbf{proj}_{\vec{\mathbf{b}}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\hat{\mathbf{b}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} = \underbrace{\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|^2}}_{scalar}\vec{\mathbf{b}} \tag{1.12}$$



a) Scalar product b) Projection vector

Figure 1.3: Scalar product and projection vector.

Orthogonality between vectors: Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if the *scalar product* between them is zero, *i.e.*:

$$\vec{a} \cdot \vec{b} = 0 \tag{1.13}$$

Vector Product (or Cross Product): The *vector product* of two vectors, $\vec{a}$, $\vec{b}$, results in another vector $\vec{c}$, which is perpendicular to the plane defined by the two input vectors, (see Figure 1.4). The vector product has the following characteristics:

- Representation:

$$\vec{c} = \vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a} \tag{1.14}$$

- The vector $\vec{c}$ is orthogonal to the vectors $\vec{a}$ and $\vec{b}$, thus:

$$\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} = 0 \tag{1.15}$$

- The magnitude of $\vec{c}$ is defined by the formula:

$$\|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta \tag{1.16}$$

where θ measures the smallest angle between $\vec{a}$ and $\vec{b}$, (see Figure 1.4).

The magnitude of the vector product $\vec{a} \wedge \vec{b}$ is geometrically expressed as the area of the parallelogram defined by the two vectors, (see Figure 1.4):

$$A = \|\vec{a} \wedge \vec{b}\| \tag{1.17}$$

Therefore, the triangle area defined by the points OCD, (see Figure 1.4 (a)), is:

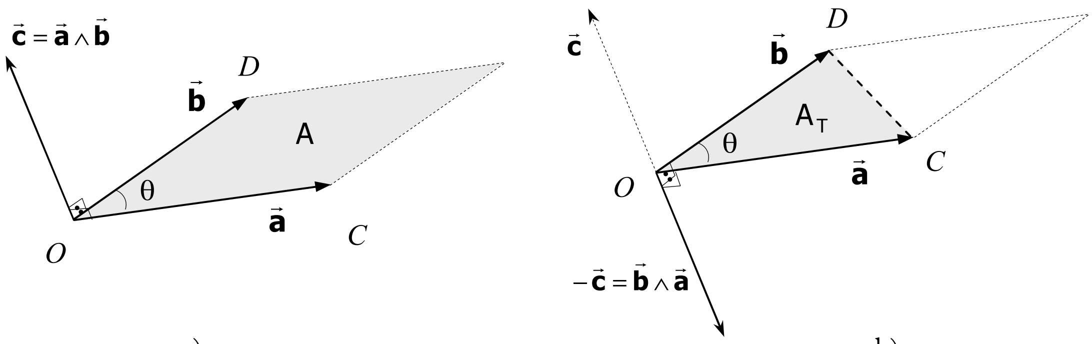
$$A_T = \frac{1}{2}\|\vec{a} \wedge \vec{b}\| \tag{1.18}$$

If $\vec{a}$ and $\vec{b}$ are linearly dependent, *i.e.* $\vec{a} = \alpha\vec{b}$ with α denoting a scalar, the vector product of two linearly dependent vectors becomes a zero vector, $\vec{a} \wedge \vec{b} = \alpha\vec{b} \wedge \vec{b} = \vec{0}$.

Scalar Triple Product (or Mixed Product): Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be arbitrary vectors, we can define the *scalar triple product* as:

$$\begin{aligned} &\vec{a} \cdot (\vec{b} \wedge \vec{c}) = \vec{b} \cdot (\vec{c} \wedge \vec{a}) = \vec{c} \cdot (\vec{a} \wedge \vec{b}) = V \\ &V = -\vec{a} \cdot (\vec{c} \wedge \vec{b}) = -\vec{b} \cdot (\vec{a} \wedge \vec{c}) = -\vec{c} \cdot (\vec{b} \wedge \vec{a}) \end{aligned} \tag{1.19}$$

where the scalar V represents the volume of the parallelepiped defined by $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$, (see Figure 1.5).



a) b)

Figure 1.4: Vector product.

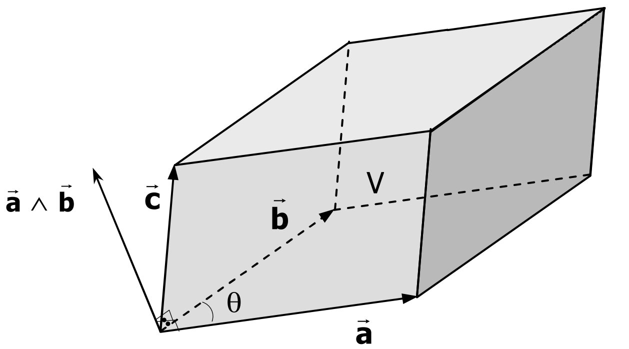
If two vectors are linearly dependent then, the scalar triple product is zero, *i.e.*:

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}\right)=\vec{\mathbf{0}} \tag{1.20}$$

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$, be vectors and α, β be scalars, the following property is satisfied:

$$(\alpha\,\vec{\mathbf{a}}+\beta\,\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})=\alpha\,\vec{\mathbf{a}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})+\beta\,\vec{\mathbf{b}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) \tag{1.21}$$

NOTE: Some authors represent the scalar triple product as, $[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]\equiv\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)$, $[\vec{\mathbf{b}},\vec{\mathbf{c}},\vec{\mathbf{a}}]\equiv\vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right)$, $[\vec{\mathbf{c}},\vec{\mathbf{a}},\vec{\mathbf{b}}]\equiv\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)$. ■



$\mathsf{V}\equiv$ Scalar triple product

Figure 1.5: Scalar triple product.

Vector Triple Product: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be vectors, we can define the *vector triple product* as $\vec{\mathbf{w}}=\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)$. Then, we can demonstrate that the following relationships to be true:

$$\begin{aligned}\vec{\mathbf{w}}=\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)&=-\vec{\mathbf{c}}\wedge\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)=\vec{\mathbf{c}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}\right)\\&=\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}}-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}}\end{aligned} \tag{1.22}$$

whereby it is clear that the result of the vector triple product is another vector $\vec{\mathbf{w}}$, belonging to the plane Π_1 formed by the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, (see Figure 1.6).

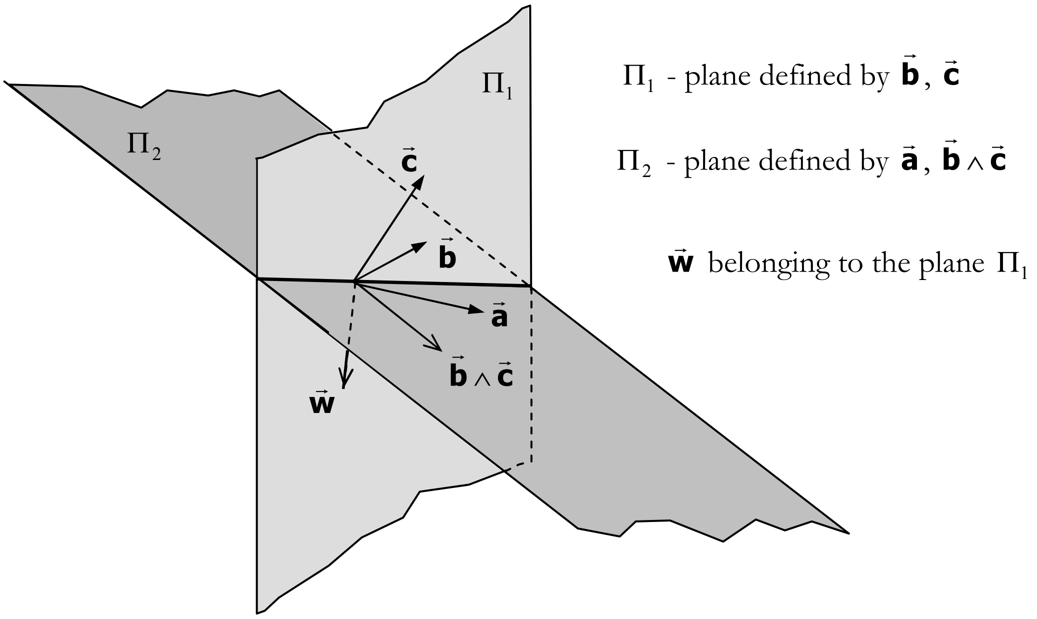


Figure 1.6: Vector triple product.

Problem 1.1: Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors. Prove that the following relationship is true:

$$(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$$

Solution:

$$\begin{aligned}
(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) &= \left\|\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right\|^2 \\
&= \left(\|\vec{\mathbf{a}}\| \|\vec{\mathbf{b}}\| \sin\theta\right)^2 \\
&= \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 \sin^2\theta \\
&= \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 (1 - \cos^2\theta) \\
&= \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 - \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 \cos^2\theta \\
&= \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 - \left(\|\vec{\mathbf{a}}\| \|\vec{\mathbf{b}}\| \cos\theta\right)^2 \\
&= \|\vec{\mathbf{a}}\|^2 \|\vec{\mathbf{b}}\|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2 \\
&= (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2
\end{aligned}$$

Linear Transformation

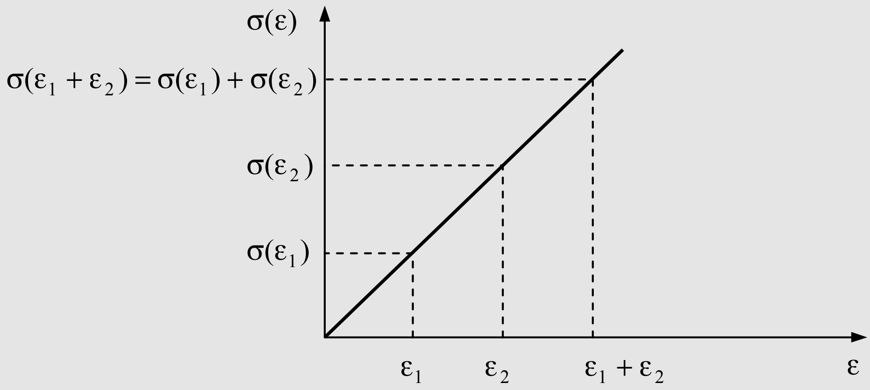
Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be arbitrary vectors, and α be a scalar, we can state F is a linear transformation if the following is true:

- $F(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = F(\vec{\mathbf{u}}) + F(\vec{\mathbf{v}})$
- $F(\alpha\vec{\mathbf{u}}) = \alpha F(\vec{\mathbf{u}})$

Problem 1.2: Given the following functions $\sigma(\varepsilon) = E\varepsilon$ and $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$, demonstrate whether these functions show a linear transformation or not.

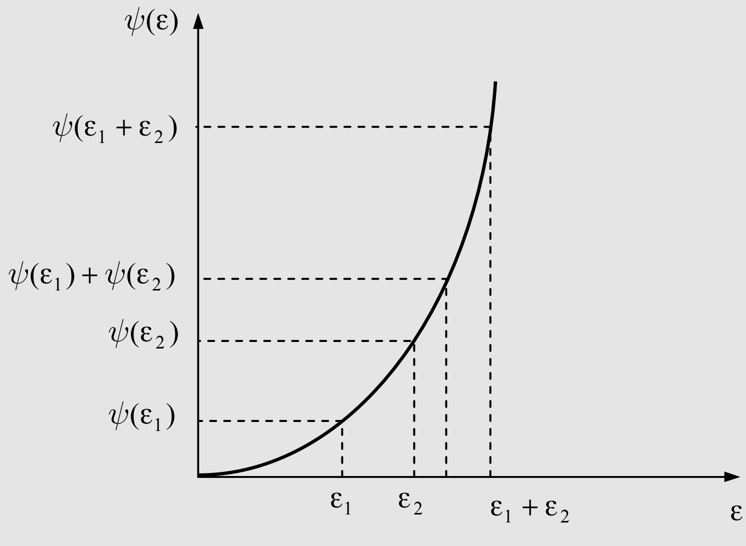
Solution:

$$\sigma(\varepsilon_1 + \varepsilon_2) = E[\varepsilon_1 + \varepsilon_2] = E\varepsilon_1 + E\varepsilon_2 = \sigma(\varepsilon_1) + \sigma(\varepsilon_2) \quad \text{(linear transformation)}$$



The function $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$ does not show a linear transformation because the condition $\psi(\varepsilon_1 + \varepsilon_2) = \psi(\varepsilon_1) + \psi(\varepsilon_2)$ has not been satisfied:

$$\psi(\varepsilon_1 + \varepsilon_2) = \frac{1}{2}E[\varepsilon_1 + \varepsilon_2]^2 = \frac{1}{2}E\left[\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2\right] = \frac{1}{2}E\varepsilon_1^2 + \frac{1}{2}E\varepsilon_2^2 + \frac{1}{2}E2\varepsilon_1\varepsilon_2$$
$$= \psi(\varepsilon_1) + \psi(\varepsilon_2) + E\varepsilon_1\varepsilon_2 \neq \psi(\varepsilon_1) + \psi(\varepsilon_2)$$



1.3 Coordinate Systems

A tensor, which has physical meanings, must be independent of the adopted coordinate system. Sometimes for reasons of convenience, we need to represent a tensor in a specific coordinate system, hence, we have the concept of tensor components, (see Figure 1.7).

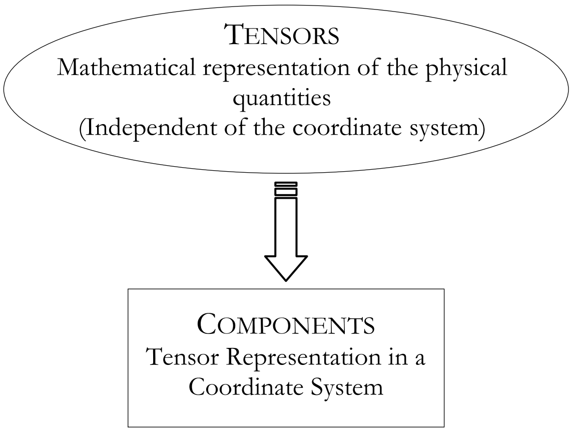


Figure 1.7: Tensor components.

Let $\vec{\mathbf{a}}$ be a first-order tensor (vector) as shown in Figure 1.8 (a), the tensor representation in a general coordinate system, defined as ξ_1, ξ_2, ξ_3, is made up of its components (a_1, a_2, a_3), (see Figure 1.8 (b)). Some examples of coordinate system are: the Cartesian coordinate system; the cylindrical coordinate system; and the spherical coordinate system.

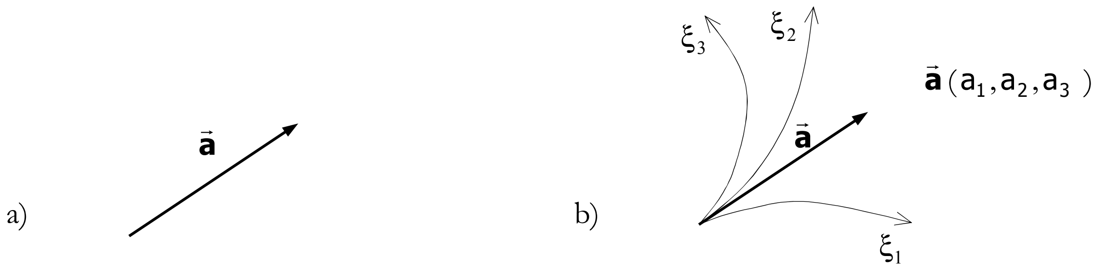


Figure 1.8: Vector representation in a general coordinate system.

1.3.1 Cartesian Coordinate System

The Cartesian coordinate system is defined by three unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, denoted by the Cartesian basis, which make up an *orthonormal basis*. The orthonormal basis has the following properties:

1. The vectors that make up this basis are unit vectors:

$$\|\hat{\mathbf{i}}\| = \|\hat{\mathbf{j}}\| = \|\hat{\mathbf{k}}\| = 1 \tag{1.23}$$

or:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \tag{1.24}$$

2. The unit vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) are mutually orthogonal, *i.e.:*

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0 \tag{1.25}$$

3. The vector product between the vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) is the following:

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}} \quad ; \quad \hat{\mathbf{j}} \wedge \hat{\mathbf{k}} = \hat{\mathbf{i}} \quad ; \quad \hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.26}$$

The direction and orientation of the orthonormal basis can be obtained using the right-hand rule as shown in Figure 1.9.

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \wedge \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.27}$$

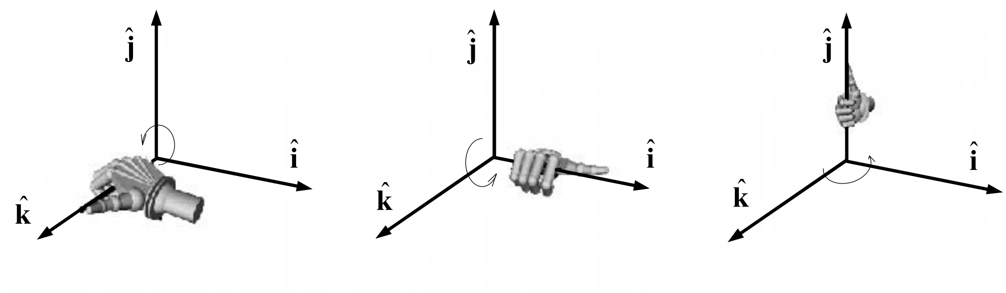


Figure 1.9: The right-hand rule.

1.3.2 Vector Representation in the Cartesian Coordinate System

The vector $\vec{\mathbf{a}}$, (see Figure 1.10), in the Cartesian coordinate system, is represented by its different components (a_x, a_y, a_z) and by the Cartesian bases ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) as:

$$\vec{\mathbf{a}} = \mathrm{a}_x \hat{\mathbf{i}} + \mathrm{a}_y \hat{\mathbf{j}} + \mathrm{a}_z \hat{\mathbf{k}} \tag{1.28}$$

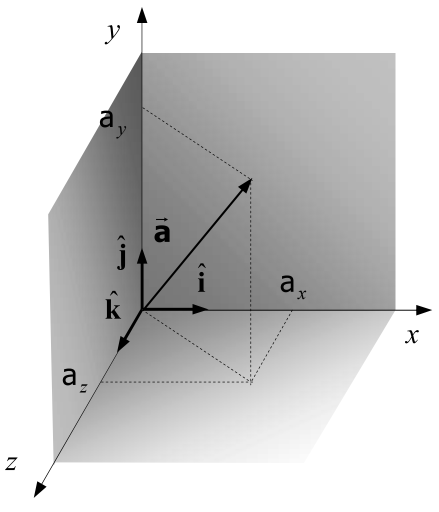


Figure 1.10: Cartesian coordinate system.

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be arbitrary vectors, we can describe some vector operations in the Cartesian coordinate system, as follows:

- The **scalar product** $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ becomes a scalar, which is defined in the Cartesian system as:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})\cdot(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_x b_x+a_y b_y+a_z b_z) \tag{1.29}$$

Thus, it is true that $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}=a_x a_x+a_y a_y+a_z a_z=a_x^2+a_y^2+a_z^2=\|\vec{\mathbf{a}}\|^2$.

NOTE: The projection of a vector onto a given direction was established in the equation (1.10), thus defining the component concept. For example, if we want to know the vector component along the y-direction, all we need to do is calculate: $\vec{\mathbf{a}}\cdot\hat{\mathbf{j}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})\cdot(\hat{\mathbf{j}})=a_y$. ■

- The **norm** of $\vec{\mathbf{a}}$ is:

$$\|\vec{\mathbf{a}}\|=\sqrt{a_x^2+a_y^2+a_z^2} \tag{1.30}$$

- Then, the **unit vector** codirectional with $\vec{\mathbf{a}}$ is:

$$\hat{\mathbf{a}}=\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}=\frac{a_x}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{i}}+\frac{a_y}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{j}}+\frac{a_z}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{k}} \tag{1.31}$$

- The **zero vector** is:

$$\vec{\mathbf{0}}=0\,\hat{\mathbf{i}}+0\,\hat{\mathbf{j}}+0\,\hat{\mathbf{k}} \tag{1.32}$$

- **Addition:** The vector sum of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is represented by:

$$\vec{\mathbf{a}}+\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})+(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_x+b_x)\hat{\mathbf{i}}+(a_y+b_y)\hat{\mathbf{j}}+(a_z+b_z)\hat{\mathbf{k}} \tag{1.33}$$

- **Subtraction:** The difference between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is:

$$\vec{\mathbf{a}}-\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})-(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_x-b_x)\hat{\mathbf{i}}+(a_y-b_y)\hat{\mathbf{j}}+(a_z-b_z)\hat{\mathbf{k}} \tag{1.34}$$

- **Scalar multiplication:** The resulting vector defined by $\lambda\vec{\mathbf{a}}$ is:

$$\lambda\vec{\mathbf{a}}=\lambda a_x\hat{\mathbf{i}}+\lambda a_y\hat{\mathbf{j}}+\lambda a_z\hat{\mathbf{k}} \tag{1.35}$$

- The **vector product** ($\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}$) is evaluated as:

$$\begin{aligned}\vec{\mathbf{c}}=\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}&=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\hat{\mathbf{i}}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\hat{\mathbf{j}}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\hat{\mathbf{k}}\\ &=(a_y b_z-a_z b_y)\hat{\mathbf{i}}-(a_x b_z-a_z b_x)\hat{\mathbf{j}}+(a_x b_y-a_y b_x)\hat{\mathbf{k}}\end{aligned} \tag{1.36}$$

where the symbol $|\bullet|\equiv\det(\bullet)$ denotes the matrix determinant.

- The **scalar triple product** $[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$ is the determinant of the 3 by 3 matrix, defined as:

$$
\begin{aligned}
\mathsf{V}(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}) &= \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right)=\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \\
&= a_x\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \\
&= a_x\left(b_y c_z - b_z c_y\right) - a_y\left(b_x c_z - b_z c_x\right) + a_z\left(b_x c_y - b_y c_x\right)
\end{aligned}
\tag{1.37}
$$

- The **vector triple product** made up of the vectors ($\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$) is obtained, in the Cartesian coordinate system, as:

$$
\begin{aligned}
\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) &= \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}} - \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}} \\
&= \left(\lambda_1 b_x - \lambda_2 c_x\right)\hat{\mathbf{i}} + \left(\lambda_1 b_y - \lambda_2 c_y\right)\hat{\mathbf{j}} + \left(\lambda_1 b_z - \lambda_2 c_z\right)\hat{\mathbf{k}}
\end{aligned}
\tag{1.38}
$$

where $\lambda_1 = \vec{\mathbf{a}}\cdot\vec{\mathbf{c}} = a_x c_x + a_y c_y + a_z c_z$, and $\lambda_2 = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$.

Problem 1.3: Consider the points: $A(1,3,1)$, $B(2,-1,1)$, $C(0,1,3)$ and $D(1,2,4)$, defined in the Cartesian coordinate system.

1) Find the parallelogram area defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$; 2) Find the volume of the parallelepiped defined by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$; 3) Find the projection vector of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$.

Solution:

1) Firstly we calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$
\vec{\mathbf{a}} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = 1\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}}
$$

$$
\vec{\mathbf{b}} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = -1\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}
$$

With reference to the equation (1.36) we can evaluate the vector product as follows:

$$
\vec{\mathbf{a}}\wedge\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -4 & 0 \\ -1 & -2 & 2 \end{vmatrix} = (-8)\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + (-6)\hat{\mathbf{k}}
$$

Then, the parallelogram area can be obtained using definition (1.19), thus:

$$
\mathsf{A} = \left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| = \sqrt{(-8)^2 + (-2)^2 + (-6)^2} = \sqrt{104}
$$

2) Next, we can evaluate the vector $\overrightarrow{AD}$ as:

$$
\vec{\mathbf{c}} = \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = 0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}
$$

and using the equation (1.37) we can obtain the volume of the parallelepiped:

$$
\begin{aligned}
\mathsf{V}(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}) = \left\|\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\right\| &= \left\|\left(0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right)\cdot\left(-8\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}\right)\right\| \\
&= \left\|0 + 2 - 18\right\| = 16
\end{aligned}
$$

3) The $\overrightarrow{BC}$ vector can be calculated as:

$$
\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right) - \left(2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}
$$

Hence, it is possible to evaluate the projection vector of $\vec{AB}$ onto $\vec{BC}$, (see equation (1.12)), as:

$$\overrightarrow{\mathbf{proj}}_{\vec{BC}}\, \vec{AB} = \frac{\vec{BC}\cdot \vec{AB}}{\underbrace{\vec{BC}\cdot \vec{BC}}_{\|\vec{BC}\|^2}}\vec{BC} = \frac{\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)\cdot\left(1\hat{\mathbf{i}}-4\hat{\mathbf{j}}+0\hat{\mathbf{k}}\right)}{\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)\cdot\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)}\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)$$

$$= \frac{(-2-8+0)}{(4+4+4)}\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right) = \frac{5}{3}\hat{\mathbf{i}} - \frac{5}{3}\hat{\mathbf{j}} - \frac{5}{3}\hat{\mathbf{k}}$$

1.3.3 Einstein Summation Convention (Einstein Notation)

As we saw in equation (1.28) $\vec{\mathbf{a}}$ in the Cartesian coordinate system was defined as:

$$\vec{\mathbf{a}} = \mathrm{a}_x\hat{\mathbf{i}} + \mathrm{a}_y\hat{\mathbf{j}} + \mathrm{a}_z\hat{\mathbf{k}} \tag{1.39}$$

Said expression can be rewritten as:

$$\vec{\mathbf{a}} = \mathrm{a}_1\hat{\mathbf{e}}_1 + \mathrm{a}_2\hat{\mathbf{e}}_2 + \mathrm{a}_3\hat{\mathbf{e}}_3 \tag{1.40}$$

where we have considered that: $\mathrm{a}_1 \equiv \mathrm{a}_x$, $\mathrm{a}_2 \equiv \mathrm{a}_y$, $\mathrm{a}_3 \equiv \mathrm{a}_z$, $\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_2 \equiv \hat{\mathbf{j}}$, $\hat{\mathbf{e}}_3 \equiv \hat{\mathbf{k}}$, (see Figure 1.11). In this way we can express equation (1.40) by means of the summation symbol as:

$$\vec{\mathbf{a}} = \mathrm{a}_1\hat{\mathbf{e}}_1 + \mathrm{a}_2\hat{\mathbf{e}}_2 + \mathrm{a}_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3}\mathrm{a}_i\hat{\mathbf{e}}_i \tag{1.41}$$

Then, we introduce the *summation convention*, according to which the "repeated indices" indicate summation. So, equation (1.41) can be represented as follows:

$$\vec{\mathbf{a}} = \mathrm{a}_1\hat{\mathbf{e}}_1 + \mathrm{a}_2\hat{\mathbf{e}}_2 + \mathrm{a}_3\hat{\mathbf{e}}_3 = \mathrm{a}_i\hat{\mathbf{e}}_i \qquad (i=1,2,3)$$
$$\boxed{\vec{\mathbf{a}} = \mathrm{a}_i\hat{\mathbf{e}}_i \qquad (i=1,2,3)} \tag{1.42}$$

NOTE: The summation notation was introduced by Albert Einstein in 1916, which led to the indicial notation. ■

1.4 Indicial Notation

Using indicial notation, the three axes of the coordinate system are designated by the letter x with a subscript. So, x_i is not a single value but i values, *i.e.* x_1, x_2, x_3 (if $i=1,2,3$) where these values x_1, x_2, x_3 correspond to the axes x, y, z, respectively.

Let $\vec{\mathbf{a}}$ be a vector represented in the Cartesian coordinate system as:

$$\vec{\mathbf{a}} = \mathrm{a}_1\hat{\mathbf{e}}_1 + \mathrm{a}_2\hat{\mathbf{e}}_2 + \mathrm{a}_3\hat{\mathbf{e}}_3 \tag{1.43}$$

where the orthonormal basis is represented by $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, (see Figure 1.11), and a_1, a_2, a_3 are the vector components. In indicial notation the vector components are represented by a_i. If the range of the subscript is not indicated, we assume that 1,2,3 show these values. Therefore, the vector components are represented as:

$$(\vec{\mathbf{a}})_i = \mathsf{a}_i = \begin{bmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \\ \mathsf{a}_3 \end{bmatrix} \tag{1.44}$$

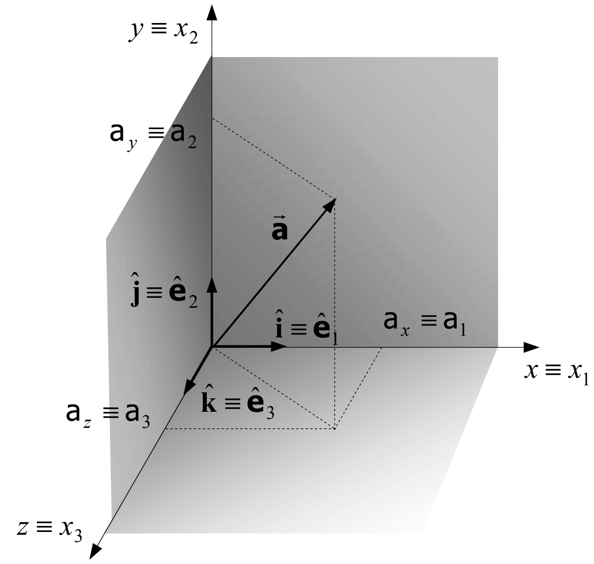


Figure 1.11: Vector representation in the Cartesian coordinate system.

Unit vector components: Let $\vec{\mathbf{a}}$ be a vector, the normalized vector $\hat{\mathbf{a}}$ is defined as:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \quad \text{with} \quad \|\hat{\mathbf{a}}\| = 1 \tag{1.45}$$

whose components are:

$$\hat{\mathsf{a}}_i = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_1^2 + \mathsf{a}_2^2 + \mathsf{a}_3^2}} = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_j \mathsf{a}_j}} = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_k \mathsf{a}_k}} \qquad (i, j, k = 1,2,3) \tag{1.46}$$

In light of the previous equation we can emphasize two types of indices:

The *free index (live index)* is that which only appears once in a term of the expression. In the above equation the free index is the (i). The number of the free index indicates the tensor order.

The *dummy index (summation index)* is that which is repeated only twice in a term of the expression, and indicates summation. In the above equation (1.46) the dummy index is the (j), or the (k) index.

OBS.: An index in a term of an expression can only appear once or twice. If it appears more times, then a large error has occurred.

Scalar product: Using definitions (1.4) and (1.29), we can express the scalar product ($\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$) as follows:

$$\gamma = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\cos\theta = \mathrm{a}_1\mathrm{b}_1 + \mathrm{a}_2\mathrm{b}_2 + \mathrm{a}_3\mathrm{b}_3 = \mathrm{a}_i\mathrm{b}_i = \mathrm{a}_j\mathrm{b}_j \qquad (i,j=1,2,3) \tag{1.47}$$

Problem 1.4: Rewrite the following equations using indicial notation:

1) $a_1x_1x_3 + a_2x_2x_3 + a_3x_3x_3$
Solution: $a_ix_ix_3 \qquad (i=1,2,3)$
2) $x_1x_1 + x_2x_2$
Solution: $x_ix_i \qquad (i=1,2)$
3) $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_x \\ a_{21}x + a_{22}y + a_{23}z = b_y \\ a_{31}x + a_{32}y + a_{33}z = b_z \end{cases}$
Solution:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \xrightarrow[\text{index } j]{\text{dummy}} \begin{cases} a_{1j}x_j = b_1 \\ a_{2j}x_j = b_2 \\ a_{3j}x_j = b_3 \end{cases} \xrightarrow[\text{index } i]{\text{free}} \boxed{a_{ij}x_j = b_i}$$

As we can appreciate in this problem, the use of the indicial notation means that the equation becomes very concise. In many cases, if algebraic operation do not use indicial or tensorial notation they become almost impossible to deal with due to the large number of terms involved.

Problem 1.5: Expand the equation: $A_{ij}x_ix_j \qquad (i,j=1,2,3)$

Solution: The indices i,j are dummy indices, and indicate index summation and there is no free index in the expression $A_{ij}x_ix_j$, therefore the result is a scalar. So, we expand first the dummy index i and later the index j to obtain:

$$A_{ij}x_ix_j \xrightarrow{\text{expanding } i} \underbrace{A_{1j}x_1x_j} + \underbrace{A_{2j}x_2x_j} + \underbrace{A_{3j}x_3x_j}$$

$$\text{expanding } j \downarrow \quad \begin{array}{ccc} A_{11}x_1x_1 & A_{21}x_2x_1 & A_{31}x_3x_1 \\ + & + & + \\ A_{12}x_1x_2 & A_{22}x_2x_2 & A_{32}x_3x_2 \\ + & + & + \\ A_{13}x_1x_3 & A_{23}x_2x_3 & A_{33}x_3x_3 \end{array}$$

Rearranging the terms we obtain:

$$\begin{aligned} A_{ij}x_ix_j = {} & A_{11}x_1x_1 + A_{12}x_1x_2 + A_{13}x_1x_3 + A_{21}x_2x_1 + A_{22}x_2x_2 + \\ & A_{23}x_2x_3 + A_{31}x_3x_1 + A_{32}x_3x_2 + A_{33}x_3x_3 \end{aligned}$$

1.4.1 Some Operators

1.4.1.1 Kronecker Delta

The *Kronecker delta* δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 1 & iff \quad i = j \\ \\ 0 & iff \quad i \neq j \end{cases} \tag{1.48}$$

Also note that the scalar product of the orthonormal basis $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ is equal to 1 if $i = j$ and equal to 0 if $i \neq j$. Hence, $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ can be expressed in matrix form as:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ij} \tag{1.49}$$

An interesting property of the Kronecker delta is shown in the following example. Let V_i be the components of the vector $\vec{V}$, therefore:

$$\delta_{ij} V_i = \delta_{1j} V_1 + \delta_{2j} V_2 + \delta_{3j} V_3 \tag{1.50}$$

As $(j = 1,2,3)$ is a free index, we have three values to be calculated, namely:

$$\left. \begin{array}{l} j = 1 \Rightarrow \delta_{ij} V_i = \delta_{11} V_1 + \delta_{21} V_2 + \delta_{31} V_3 = V_1 \\ j = 2 \Rightarrow \delta_{ij} V_i = \delta_{12} V_1 + \delta_{22} V_2 + \delta_{32} V_3 = V_2 \\ j = 3 \Rightarrow \delta_{ij} V_i = \delta_{13} V_1 + \delta_{23} V_2 + \delta_{33} V_3 = V_3 \end{array} \right\} \Rightarrow \delta_{ij} V_i = V_j \tag{1.51}$$

That is, in the presence of the Kronecker delta symbol we replace the repeated index as follows:

$$\delta_{ij} V_i = V_j \tag{1.52}$$

For this reason, the Kronecker delta is often called the *substitution operator*.

Other examples using the Kronecker delta are presented below:

$$\delta_{ij} A_{ik} = A_{jk} \,, \; \delta_{ij} \delta_{ji} = \delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \,, \; \delta_{ji} a_{ji} = a_{ii} = a_{11} + a_{22} + a_{33} \tag{1.53}$$

To obtain the components of the vector $\vec{\mathbf{a}}$ in the coordinate system represented by $\hat{\mathbf{e}}_i$, it is sufficient to obtain the scalar product with $\vec{\mathbf{a}}$ and $\hat{\mathbf{e}}_i$, *i.e.* $\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_i = \mathrm{a}_p \hat{\mathbf{e}}_p \cdot \hat{\mathbf{e}}_i = \mathrm{a}_p \delta_{pi} = \mathrm{a}_i$. With that, it is also possible to represent the vector as:

$$\vec{\mathbf{a}} = \mathrm{a}_i \hat{\mathbf{e}}_i = (\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_i) \hat{\mathbf{e}}_i \tag{1.54}$$

Problem 1.6: Solve the following equations:

1) $\delta_{ii} \delta_{jj}$

Solution: $\delta_{ii} \delta_{jj} = (\delta_{11} + \delta_{22} + \delta_{33})(\delta_{11} + \delta_{22} + \delta_{33}) = 3 \times 3 = 9$

2) $\delta_{\alpha 1} \delta_{\alpha\gamma} \delta_{\gamma 1}$

Solution: $\delta_{\alpha 1} \delta_{\alpha\gamma} \delta_{\gamma 1} = \delta_{\gamma 1} \delta_{\gamma 1} = \delta_{11} = 1$

NOTE: Note that the following algebraic operation is incorrect $\delta_{\gamma 1} \delta_{\gamma 1} \neq \delta_{\gamma\gamma} = 3 \neq \delta_{11} = 1$, since what must be replaced is the repeated index, not the number ■

1.4.1.2 Permutation Symbol

The *permutation symbol* ϵ_{ijk} (also known as *Levi-Civita symbol* or *alternating symbol)* is defined as:

$$\epsilon_{ijk} = \begin{cases} +1 & if \quad (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & if \quad (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\} \\ 0 & \text{for the remaining cases i.e.: if } (i=j) \text{ or } (j=k) \text{ or } (i=k) \end{cases} \tag{1.55}$$

NOTE: ϵ_{ijk} are the components of the *Levi-Civita pseudo-tensor*, which will be introduced later on. ■

The values of ϵ_{ijk} can be easily memorized using the mnemonic device shown in Figure 1.12(a), in which if the index values are arranged in a clockwise direction, the value of ϵ_{ijk} is equal to 1, if not it has the value of -1. In the same way we can use this mnemonic device to switch indices, (see Figure 1.12(b)).

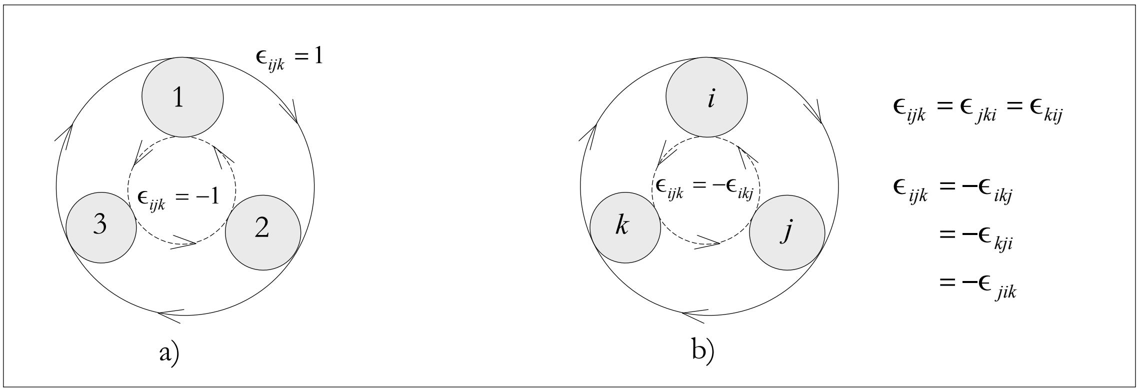


Figure 1.12: Mnemonic device for the permutation symbol.

Another way to express the permutation symbol is by means of its indices:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \tag{1.56}$$

Using both the definition seen in (1.55) and Figure 1.12 (b), it is possible to verify that the following relations are valid:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{jki} = \epsilon_{kij} \\ \epsilon_{ijk} &= -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \end{aligned} \tag{1.57}$$

Using the Kronecker delta property, we can state that:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{lmn}\delta_{li}\delta_{mj}\delta_{nk} \\ &= \delta_{1i}\delta_{2j}\delta_{3k} - \delta_{1i}\delta_{3j}\delta_{2k} - \delta_{2i}\delta_{1j}\delta_{3k} + \delta_{3i}\delta_{1j}\delta_{2k} + \delta_{2i}\delta_{3j}\delta_{1k} - \delta_{3i}\delta_{2j}\delta_{1k} \\ &= \delta_{1i}\left(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}\right) - \delta_{1j}\left(\delta_{2i}\delta_{3k} - \delta_{3i}\delta_{2k}\right) + \delta_{1k}\left(\delta_{2i}\delta_{3j} - \delta_{3i}\delta_{2j}\right) \end{aligned} \tag{1.58}$$

The above equation can be represented by means of the following determinant:

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \tag{1.59}$$

After which, the term $\epsilon_{ijk}\epsilon_{pqr}$ can be evaluated as follows:

$$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \begin{vmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{vmatrix} \tag{1.60}$$

Taking into account that $\det(\mathcal{AB}) = \det(\mathcal{A})\det(\mathcal{B})$, where $\det(\bullet) \equiv |\bullet|$ is the determinant of the matrix $\bullet$, the equation (1.60) can be rewritten as:

$$\epsilon_{ijk}\epsilon_{pqr} = \left| \begin{bmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{bmatrix} \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} \right| \Rightarrow \quad \epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.61}$$

The term δ_{ip} was obtained by means of the operation $\delta_{1i}\delta_{1p} + \delta_{2i}\delta_{2p} + \delta_{3i}\delta_{3p} = \delta_{mi}\delta_{mp}$ and $\delta_{mi}\delta_{mp} = \delta_{ip}$, the other terms were obtained in a similar fashion.

For the special exception when $r = k$, the equation (1.61) is reduced to:

$$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{vmatrix} \Rightarrow \boxed{\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \qquad i,j,k,p,q = 1,2,3} \tag{1.62}$$

Problem 1.7: a) Prove the following is true $\epsilon_{ijk}\epsilon_{pjk} = 2\delta_{ip}$ and $\epsilon_{ijk}\epsilon_{ijk} = 6$. b) Obtain the numerical value of $\epsilon_{ijk}\delta_{2j}\delta_{3k}\delta_{1i}$.

Solution: a) Using the equation in (1.62), *i.e.* $\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$, and by substituting q for j, we obtain:

$$\epsilon_{ijk}\epsilon_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = \delta_{ip}3 - \delta_{ip} = 2\delta_{ip}$$

Based on the above result, it is straight forward to check that:

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6$$

b) $\epsilon_{ijk}\delta_{2j}\delta_{3k}\delta_{1i} = \epsilon_{123} = 1$

- The **vector product** of two vectors ($\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}$) leads to a new vector $\vec{\mathbf{c}}$, defined in (1.36), and the components of $\vec{\mathbf{c}}$, in Cartesian system, are given by:

$$\begin{aligned} \vec{\mathbf{c}} = \vec{\mathbf{a}} \wedge \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \underbrace{(a_2 b_3 - a_3 b_2)}_{c_1}\hat{\mathbf{e}}_1 + \underbrace{(a_3 b_1 - a_1 b_3)}_{c_2}\hat{\mathbf{e}}_2 + \underbrace{(a_1 b_2 - a_2 b_1)}_{c_3}\hat{\mathbf{e}}_3 \end{aligned} \tag{1.63}$$

Using the definition of the permutation symbol ϵ_{ijk}, defined in (1.55), we can express the components of $\vec{\mathbf{c}}$ as follows:

$$\left.\begin{aligned} c_1 &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = \epsilon_{1jk} a_j b_k \\ c_2 &= \epsilon_{231} a_3 b_1 + \epsilon_{213} a_1 b_3 = \epsilon_{2jk} a_j b_k \\ c_3 &= \epsilon_{312} a_1 b_2 + \epsilon_{321} a_2 b_1 = \epsilon_{3jk} a_j b_k \end{aligned}\right\} \Rightarrow c_i = \epsilon_{ijk} a_j b_k \tag{1.64}$$

Then, the vector product ($\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}$) can be represented by means of the permutation symbol as:

$$\begin{aligned} \vec{\mathbf{a}} \wedge \vec{\mathbf{b}} &= \epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i \\ a_j \hat{\mathbf{e}}_j \wedge b_k \hat{\mathbf{e}}_k &= a_j b_k \epsilon_{ijk} \hat{\mathbf{e}}_i \\ a_j b_k (\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k) &= a_j b_k \epsilon_{ijk} \hat{\mathbf{e}}_i = a_j b_k \epsilon_{jki} \hat{\mathbf{e}}_i \end{aligned} \tag{1.65}$$

Therefore, we can also conclude that the following relationship is valid:

$$\boxed{(\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k) = \epsilon_{ijk}\hat{\mathbf{e}}_i} \tag{1.66}$$

The permutation symbol and the orthonormal basis can be interrelated using the triple scalar product as follows:

$$\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j = \epsilon_{ijm}\hat{\mathbf{e}}_m \quad \Rightarrow \quad \left(\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j\right)\cdot \hat{\mathbf{e}}_k = \epsilon_{ijm}\hat{\mathbf{e}}_m \cdot \hat{\mathbf{e}}_k = \epsilon_{ijm}\delta_{mk} \Rightarrow \left(\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j\right)\cdot \hat{\mathbf{e}}_k = \epsilon_{ijk} \tag{1.67}$$

- The **triple scalar product** made up of the vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ is expressed by:

$$\lambda = \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) = a_i\hat{\mathbf{e}}_i \cdot \left(b_j\hat{\mathbf{e}}_j \wedge c_k\hat{\mathbf{e}}_k\right) = a_i b_j c_k \hat{\mathbf{e}}_i \cdot \left(\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k\right) = \epsilon_{ijk} a_i b_j c_k \tag{1.68}$$

$$\boxed{\lambda = \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) = \epsilon_{ijk} a_i b_j c_k \qquad (i,j,k=1,2,3)} \tag{1.69}$$

or

$$\lambda = \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right) = \vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.70}$$

Starting from the equation (1.69) we can prove the following are true:

$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right) = \vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)$:

$$\begin{aligned}
[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] \equiv \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) &= \epsilon_{ijk} a_i b_j c_k \\
&= \epsilon_{jki} a_i b_j c_k = \vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right) \equiv [\vec{\mathbf{b}},\vec{\mathbf{c}},\vec{\mathbf{a}}] \\
&= \epsilon_{kij} a_i b_j c_k = \vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right) \equiv [\vec{\mathbf{c}},\vec{\mathbf{a}},\vec{\mathbf{b}}] \\
&= -\epsilon_{ikj} a_i b_j c_k = -\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{b}}\right) \equiv -[\vec{\mathbf{a}},\vec{\mathbf{c}},\vec{\mathbf{b}}] \\
&= -\epsilon_{jik} a_i b_j c_k = -\vec{\mathbf{b}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}}\right) \equiv -[\vec{\mathbf{b}},\vec{\mathbf{a}},\vec{\mathbf{c}}] \\
&= -\epsilon_{kji} a_i b_j c_k = -\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}\right) \equiv -[\vec{\mathbf{c}},\vec{\mathbf{b}},\vec{\mathbf{a}}]
\end{aligned} \tag{1.71}$$

where we take into account the property of the permutation symbol as given in (1.57).

Problem 1.8: Rewrite the expression $\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}\right)$ without using the vector product symbol.

Solution: The vector product $\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)$ can be expressed as

$\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right) = a_j\hat{\mathbf{e}}_j \wedge b_k\hat{\mathbf{e}}_k = \epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i$. Likewise, it is possible to express $\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}\right)$ as

$\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}\right) = \epsilon_{nlm} c_l d_m \hat{\mathbf{e}}_n$, thus:

$$\begin{aligned}
\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}\right) &= (\epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i)\cdot(\epsilon_{nlm} c_l d_m \hat{\mathbf{e}}_n) = \epsilon_{ijk}\epsilon_{nlm} a_j b_k c_l d_m \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_n \\
&= \epsilon_{ijk}\epsilon_{nlm} a_j b_k c_l d_m \delta_{in} = \epsilon_{ijk}\epsilon_{ilm} a_j b_k c_l d_m
\end{aligned}$$

Taking into account that $\epsilon_{ijk}\epsilon_{ilm} = \epsilon_{jki}\epsilon_{lmi}$ (see equation (1.57)) and by applying the equation (1.62), *i.e.:* $\epsilon_{jki}\epsilon_{lmi} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = \epsilon_{jki}\epsilon_{ilm}$, we obtain:

$$\epsilon_{ijk}\epsilon_{ilm} a_j b_k c_l d_m = \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) a_j b_k c_l d_m = a_l b_m c_l d_m - a_m b_l c_l d_m$$

Since $a_l c_l = \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)$ and $b_m d_m = \left(\vec{\mathbf{b}}\cdot\vec{\mathbf{d}}\right)$ holds true, we can conclude that:

$$\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}\right) = \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{d}}\right) - \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{d}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}\right)$$

Therefore, it is also valid when $\vec{\mathbf{a}} = \vec{\mathbf{c}}$ and $\vec{\mathbf{b}} = \vec{\mathbf{d}}$, thus:

$$(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \left\|\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right\|^2 = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{a}}) = \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$$

which is the same equation obtained in **Problem 1.1**.

Problem 1.9: Prove that $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}}) = \vec{\mathbf{c}}[\vec{\mathbf{d}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})] - \vec{\mathbf{d}}[\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]$

Solution: Expressing the correct equality term in indicial notation we obtain:

$$\left\{\vec{\mathbf{c}}[\vec{\mathbf{d}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})] - \vec{\mathbf{d}}[\vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})]\right\}_p = \mathrm{c}_p [\mathrm{d}_i (\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k)] - \mathrm{d}_p [\mathrm{c}_i (\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k)]$$

$$\Rightarrow \epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k \mathrm{c}_p \mathrm{d}_i - \epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k \mathrm{c}_i \mathrm{d}_p \quad \Rightarrow \quad \epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k (\mathrm{c}_p \mathrm{d}_i - \mathrm{c}_i \mathrm{d}_p)$$

Using the Kronecker delta the above equation becomes:

$$\Rightarrow \epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k (\delta_{pm} \mathrm{c}_m \mathrm{d}_n \delta_{ni} - \delta_{im} \mathrm{c}_m \mathrm{d}_n \delta_{np}) \quad \Rightarrow \quad (\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k) \mathrm{c}_m \mathrm{d}_n (\delta_{pm} \delta_{ni} - \delta_{im} \delta_{np})$$

and by applying the equation $\delta_{pm} \delta_{ni} - \delta_{im} \delta_{np} = \epsilon_{pil} \epsilon_{mnl}$, (see eq. (1.62)), the above equation can be rewritten as follows:

$$\Rightarrow (\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k) \mathrm{c}_m \mathrm{d}_n (\epsilon_{pil} \epsilon_{mnl}) \quad \Rightarrow \quad \epsilon_{pil} [(\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k)(\epsilon_{mnl} \mathrm{c}_m \mathrm{d}_n)]$$

Since $\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k$ and $\epsilon_{mnl} \mathrm{c}_m \mathrm{d}_n$ represent the components of $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ and $(\vec{\mathbf{c}} \wedge \vec{\mathbf{d}})$, respectively, we can conclude that:

$$\epsilon_{pil} [(\epsilon_{ijk} \mathrm{a}_j \mathrm{b}_k)(\epsilon_{mnl} \mathrm{c}_m \mathrm{d}_n)] = [(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \wedge (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}})]_p$$

Problem 1.10: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be linearly independent vectors, and $\vec{\mathbf{v}}$ be a vector, demonstrate that:

$$\vec{\mathbf{v}} = \alpha \vec{\mathbf{a}} + \beta \vec{\mathbf{b}} + \gamma \vec{\mathbf{c}} \neq \vec{\mathbf{0}}$$

where the scalars α, β, γ are given by:

$$\alpha = \frac{\epsilon_{ijk} \mathrm{v}_i \mathrm{b}_j \mathrm{c}_k}{\epsilon_{pqr} \mathrm{a}_p \mathrm{b}_q \mathrm{c}_r} \quad ; \quad \beta = \frac{\epsilon_{ijk} \mathrm{a}_i \mathrm{v}_j \mathrm{c}_k}{\epsilon_{pqr} \mathrm{a}_p \mathrm{b}_q \mathrm{c}_r} \quad ; \quad \gamma = \frac{\epsilon_{ijk} \mathrm{a}_i \mathrm{b}_j \mathrm{v}_k}{\epsilon_{pqr} \mathrm{a}_p \mathrm{b}_q \mathrm{c}_r}$$

Solution: The scalar product made up of $\vec{\mathbf{v}}$ and $(\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})$ becomes:

$$\vec{\mathbf{v}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = \alpha \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) + \beta \underbrace{\vec{\mathbf{b}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}_{=0} + \gamma \underbrace{\vec{\mathbf{c}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}_{=0} \quad \Rightarrow \quad \alpha = \frac{\vec{\mathbf{v}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}{\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})}$$

which is the same as:

$$\alpha = \frac{\begin{vmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}}{\begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix}} = \frac{\begin{vmatrix} \mathrm{v}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{v}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{v}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}}{\begin{vmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}} = \frac{\epsilon_{ijk} \mathrm{v}_i \mathrm{b}_j \mathrm{c}_k}{\epsilon_{pqr} \mathrm{a}_p \mathrm{b}_q \mathrm{c}_r}$$

One can obtain the parameters β and γ in a similar fashion.

Problem 1.11: Prove the relationship given in (1.38) is valid, *i.e.*:

$$\vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \vec{\mathbf{c}}.$$

Solution: Taking into account that $(\vec{\mathbf{d}})_i = (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})_i = \epsilon_{ijk} \mathrm{b}_j \mathrm{c}_k$ and that $(\vec{\mathbf{a}} \wedge \vec{\mathbf{d}})_q = \epsilon_{qjk} \mathrm{b}_j \mathrm{c}_k$, we obtain:

$$
\begin{aligned}
\left[\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)\right]_q &= \epsilon_{qsi}\mathsf{a}_s(\epsilon_{ijk}\mathsf{b}_j\mathsf{c}_k)=\epsilon_{qsi}\epsilon_{ijk}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k=\epsilon_{qsi}\epsilon_{jki}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k\\
&=\left(\delta_{qj}\delta_{sk}-\delta_{qk}\delta_{sj}\right)\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k=\delta_{qj}\delta_{sk}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k-\delta_{qk}\delta_{sj}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k\\
&=\mathsf{a}_k\mathsf{b}_q\mathsf{c}_k-\mathsf{a}_j\mathsf{b}_j\mathsf{c}_q=\mathsf{b}_q\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)-\mathsf{c}_q\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\\
&\Rightarrow\ \left[\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)\right]_q=\left[\vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)-\vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\right]_q
\end{aligned}
$$

1.5 Algebraic Operations with Tensors

1.5.1 Dyadic

The *tensor product,* made up of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$, becomes a dyad, which is a particular case of a second-order tensor. The dyad is represented by:

$$\vec{\mathbf{u}}\vec{\mathbf{v}}\equiv\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}=\mathbf{A} \tag{1.72}$$

where the operator $\otimes$ denotes the *tensor product*. Then, we define a *dyadic* as a linear combination of dyads. Furthermore, as we will see later, any tensor can be represented by means of a linear combination of dyads, (see Holzapfel (2000)).

The tensor product has the following properties:

1. $$(\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})\cdot\vec{\mathbf{x}}=\vec{\mathbf{u}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{x}})\equiv\vec{\mathbf{u}}\otimes(\vec{\mathbf{v}}\cdot\vec{\mathbf{x}}) \tag{1.73}$$

2. $$\vec{\mathbf{u}}\otimes(\alpha\vec{\mathbf{v}}+\beta\vec{\mathbf{w}})=\alpha\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}+\beta\vec{\mathbf{u}}\otimes\vec{\mathbf{w}} \tag{1.74}$$

3. $$\begin{aligned}(\alpha\vec{\mathbf{v}}\otimes\vec{\mathbf{u}}+\beta\vec{\mathbf{w}}\otimes\vec{\mathbf{r}})\cdot\vec{\mathbf{x}}&=\alpha(\vec{\mathbf{v}}\otimes\vec{\mathbf{u}})\cdot\vec{\mathbf{x}}+\beta(\vec{\mathbf{w}}\otimes\vec{\mathbf{r}})\cdot\vec{\mathbf{x}}\\&=\alpha\left[\vec{\mathbf{v}}\otimes(\vec{\mathbf{u}}\cdot\vec{\mathbf{x}})\right]+\beta\left[\vec{\mathbf{w}}\otimes(\vec{\mathbf{r}}\cdot\vec{\mathbf{x}})\right]\end{aligned} \tag{1.75}$$

where α and β are scalars. By definition, the dyad does not contain the commutative property, *i.e.*, $\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}\neq\vec{\mathbf{v}}\otimes\vec{\mathbf{u}}$.

The equation (1.72) can also be expressed in the Cartesian system as:

$$\begin{aligned}\mathbf{A}=\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}&=(\mathsf{u}_i\hat{\mathbf{e}}_i)\otimes(\mathsf{v}_j\hat{\mathbf{e}}_j)\\&=\mathsf{u}_i\mathsf{v}_j(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\qquad (i,j=1,2,3)\\&=\mathsf{A}_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\end{aligned} \tag{1.76}$$

$$\boxed{\underbrace{\mathbf{A}}_{Tensor}=\underbrace{\mathsf{A}_{ij}}_{components}\ \underbrace{\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j}_{basis}}\qquad (i,j=1,2,3) \tag{1.77}$$

In this textbook, the components of a second-order tensor can be represented in different ways, namely:

$$\begin{gathered}\underbrace{\mathbf{A}=\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}}\\ \downarrow\\ \textit{components}\\ \downarrow\\ (\mathbf{A})_{ij}=(\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})_{ij}=\mathsf{u}_i\mathsf{v}_j=\mathsf{A}_{ij}\end{gathered} \tag{1.78}$$

These components are explicitly expressed in matrix form as:

$$(\mathbf{A})_{ij} = \mathsf{A}_{ij} = \mathcal{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33} \end{bmatrix} \tag{1.79}$$

It is easy to identify the tensor order by the number of free indices in the tensor components, *i.e.*:

$$\begin{aligned} &\textit{Second-order tensor} && \mathbf{U} = \mathsf{U}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\ &\textit{Third-order tensor} && \mathbf{T} = \mathsf{T}_{ijk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \qquad (i,j,k,l = 1,2,3) \\ &\textit{Fourth-order tensor} && \mathbb{I} = \mathbb{I}_{ijkl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \end{aligned} \tag{1.80}$$

OBS.: The tensor order is given by the number of free indices in its components.

OBS.: The number of tensor components is given by a^n, where the base a is the maximum value in the index range, and the exponent n is the number of the free index.

Problem 1.12: Define the order of the tensors represented by their Cartesian components: v_i, Φ_{ijk}, F_{ijj}, ε_{ij}, $\mathbb{C}_{ijkl}$, σ_{ij}. Determine the number of components in tensor $\mathbb{C}$.

Solution: The order of the tensor is given by the number of free indices, so it follows that:
First-order tensor (vector): $\vec{v}$, $\vec{\mathbf{F}}$; Second-order tensor: $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$; Third-order tensor: Φ; Fourth-order tensor: $\mathbb{C}$

The number of tensor components is given by the maximum index range value, *i.e.* $i,j,k,l = 1,2,3$, to the power of the number of free indices which is equal to 4 in the case of $\mathbb{C}_{ijkl}$. Thus, the number of independent components in $\mathbb{C}$ is given by:

$$3^4 = (i=3)\times(j=3)\times(k=3)\times(l=3) = 81$$

The fourth-order tensor $\mathbb{C}_{ijkl}$ has 81 components.

Let $\mathbf{A}$ and $\mathbf{B}$ be second-order tensors, we can then define some algebraic operations including:

- **Addition:** The sum of two tensors of same order is a new tensor defined as follows:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.81}$$

The components of $\mathbf{C}$ are represented by:

$$(\mathbf{C})_{ij} = (\mathbf{A}+\mathbf{B})_{ij} \qquad \text{or} \qquad \mathsf{C}_{ij} = \mathsf{A}_{ij} + \mathsf{B}_{ij} \tag{1.82}$$

or, in matrix notation as:

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \tag{1.83}$$

- **Multiplication** of a tensor by a scalar: The multiplication of a second-order tensor ($\mathbf{A}$) by a scalar (λ) is defined by a new tensor $\mathbf{D}$, so that:

$$\mathbf{D} = \lambda\mathbf{A} \xrightarrow{\textit{in components}} (\mathbf{D})_{ij} = \lambda(\mathbf{A})_{ij} \tag{1.84}$$

or, in matrix form:

$$\mathcal{A}=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \longrightarrow \lambda\mathcal{A}=\begin{bmatrix} \lambda A_{11} & \lambda A_{12} & \lambda A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33} \end{bmatrix} \tag{1.85}$$

It is also true that:

$$(\lambda\mathbf{A})\cdot\vec{\mathbf{v}}=\lambda(\mathbf{A}\cdot\vec{\mathbf{v}}) \tag{1.86}$$

for any vector $\vec{\mathbf{v}}$.

- **Scalar Product (or Dot Product)**: The scalar product (also known as single contraction) between a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{x}}$ is another vector (first-order tensor) $\vec{\mathbf{y}}$, defined as:

$$\begin{aligned} \vec{\mathbf{y}} &= \mathbf{A}\cdot\vec{\mathbf{x}} \\ &= (A_{jk}\hat{\mathbf{e}}_j \otimes \overbrace{\hat{\mathbf{e}}_k)\cdot(x_l\hat{\mathbf{e}}_l}^{\delta_{kl}}) \\ &= A_{jk}x_l\delta_{kl}\hat{\mathbf{e}}_j \\ &= \underbrace{A_{jk}x_k}_{y_j}\hat{\mathbf{e}}_j \\ &= y_j\hat{\mathbf{e}}_j \end{aligned} \tag{1.87}$$

The scalar product between two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is another second-order tensor, that verifies: $\mathbf{A}\cdot\mathbf{B}\neq\mathbf{B}\cdot\mathbf{A}$:

$$\begin{aligned} \mathbf{C}=\mathbf{A}\cdot\mathbf{B} &= (A_{ij}\hat{\mathbf{e}}_i\otimes\overbrace{\hat{\mathbf{e}}_j)\cdot(B_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}}\otimes\hat{\mathbf{e}}_l) \\ &= A_{ij}B_{kl}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \\ &= \underbrace{A_{ik}B_{kl}}_{\mathcal{AB}}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \\ &= C_{il}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \end{aligned} \qquad \begin{aligned} \mathbf{D}=\mathbf{B}\cdot\mathbf{A} &= (B_{ij}\hat{\mathbf{e}}_i\otimes\overbrace{\hat{\mathbf{e}}_j)\cdot(A_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}}\otimes\hat{\mathbf{e}}_l) \\ &= B_{ij}A_{kl}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \\ &= \underbrace{B_{ik}A_{kl}}_{\mathcal{BA}}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \\ &= D_{il}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \end{aligned} \tag{1.88}$$

It also satisfies the following properties:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \quad ; \quad \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})=(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} \tag{1.89}$$

- **The powers of second-order tensors**

The scalar product allows us to define the power of second-order tensors, as seen below:

$$\mathbf{A}^0=\mathbf{1} \quad ; \quad \mathbf{A}^1=\mathbf{A} \quad ; \quad \mathbf{A}^2=\mathbf{A}\cdot\mathbf{A} \quad ; \quad \mathbf{A}^3=\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\text{, and so on,} \tag{1.90}$$

where $\mathbf{1}$ is the *second-order unit tensor* (also called the *identity tensor*).

- **Double Scalar Product (or Double contraction)**

Consider two dyads, $\mathbf{A}=\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}$ and $\mathbf{B}=\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}$. The double contraction between them is defined in different ways, namely: $\mathbf{A}:\mathbf{B}$ and $\mathbf{A}\cdot\cdot\mathbf{B}$.

Double contraction $(\cdot\cdot)$:

$$(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})\cdot\cdot(\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})=(\vec{\mathbf{c}}\cdot\vec{\mathbf{v}})(\vec{\mathbf{d}}\cdot\vec{\mathbf{u}}) \tag{1.91}$$

In components

$$\delta_{il} \qquad \delta_{jk}$$

$$\begin{aligned}\mathbf{A}\cdot\cdot\mathbf{B} &= (\mathrm{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\cdot\cdot(\mathrm{B}_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l)\\ &= \mathrm{A}_{ij}\mathrm{B}_{kl}\delta_{jk}\delta_{il}\\ &= \mathrm{A}_{ij}\mathrm{B}_{ji}\\ &= \gamma \quad (\textit{scalar})\end{aligned} \tag{1.92}$$

The double contraction $(\cdot\cdot)$ is commutative, *i.e.* $\mathbf{A}\cdot\cdot\mathbf{B} = \mathbf{B}\cdot\cdot\mathbf{A}$.

Double contraction $(:)$:

$$\mathbf{A}:\mathbf{B} = (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}}):(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{v}}) \tag{1.93}$$

The double contraction $(:)$ is commutative, so:

$$\mathbf{B}:\mathbf{A} = (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}):(\vec{\mathbf{c}} \otimes \vec{\mathbf{d}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{c}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{d}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{v}}) = \mathbf{A}:\mathbf{B} \tag{1.94}$$

The breakdown into its components appears like this:

$$\delta_{ik}$$

$$\begin{aligned}\mathbf{A}:\mathbf{B} &= (\mathrm{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j):(\mathrm{B}_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l)\\ &= \mathrm{A}_{ij}\mathrm{B}_{kl}\delta_{ik}\delta_{jl} \qquad \delta_{jl}\\ &= \mathrm{A}_{ij}\mathrm{B}_{ij}\\ &= \lambda \quad (\textit{scalar})\end{aligned} \tag{1.95}$$

In general, $\mathbf{A}:\mathbf{B} \neq \mathbf{A}\cdot\cdot\mathbf{B}$, however, they are equal if at least one of them is symmetric, *i.e.* $\mathbf{A}^{sym}:\mathbf{B} = \mathbf{A}^{sym}\cdot\cdot\mathbf{B}$ or $\mathbf{A}:\mathbf{B}^{sym} = \mathbf{A}\cdot\cdot\mathbf{B}^{sym}$, so $\mathbf{A}^{sym}:\mathbf{B}^{sym} = \mathbf{A}^{sym}\cdot\cdot\mathbf{B}^{sym}$.

The double contraction with a third-order tensor ($\mathbf{S}$) and a second-order tensor ($\mathbf{B}$) becomes:

$$\begin{aligned}\mathbf{S}:\mathbf{B} &= (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}} \otimes \vec{\mathbf{a}}):(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{v}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{u}})\vec{\mathbf{c}}\\ \mathbf{B}:\mathbf{S} &= (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}):(\vec{\mathbf{c}} \otimes \vec{\mathbf{d}} \otimes \vec{\mathbf{a}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{c}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{d}})\vec{\mathbf{a}}\end{aligned} \tag{1.96}$$

As we can verify the result is a vector. In symbolic notation, the double contraction ($\mathbf{B}:\mathbf{S}$) is represented by:

$$\mathrm{S}_{ijk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k : \mathrm{B}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q = \mathrm{S}_{ijk}\mathrm{B}_{pq}\delta_{jp}\delta_{kq}\hat{\mathbf{e}}_i = \mathrm{S}_{ijk}\mathrm{B}_{jk}\hat{\mathbf{e}}_i \tag{1.97}$$

The double contraction of a fourth-order tensor ($\mathbb{C}$) with a second-order tensor ($\boldsymbol{\varepsilon}$) is defined as:

$$\begin{aligned}\mathbb{C}_{ijkl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l : \varepsilon_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q = \mathbb{C}_{ijkl}\varepsilon_{pq}\delta_{kp}\delta_{lq}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j &= \mathbb{C}_{ijkl}\varepsilon_{kl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\\ &= \sigma_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\end{aligned} \tag{1.98}$$

where σ_{ij} are the components of $\boldsymbol{\sigma} = \mathbb{C}:\boldsymbol{\varepsilon}$.

Next, we express some properties of the double contraction $(:)$:

$$
\begin{aligned}
&a)\ \mathbf{A}:\mathbf{B}=\mathbf{B}:\mathbf{A}\\
&b)\ \mathbf{A}:(\mathbf{B}+\mathbf{C})=\mathbf{A}:\mathbf{B}+\mathbf{A}:\mathbf{C}\\
&c)\ \lambda(\mathbf{A}:\mathbf{B})=(\lambda\mathbf{A}):\mathbf{B}=\mathbf{A}:(\lambda\mathbf{B})
\end{aligned}
\tag{1.99}
$$

where $\mathbf{A},\mathbf{B},\mathbf{C}$ are second-order tensors, and λ is a scalar.

Via the definition of the double scalar product, it is possible to obtain the components of the second-order tensor $\mathbf{A}$ in the Cartesian system, *i.e.*:

$$
(\mathbf{A})_{ij}=(\mathrm{A}_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l):(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)=\hat{\mathbf{e}}_i\cdot(\mathrm{A}_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l)\cdot\hat{\mathbf{e}}_j=\mathrm{A}_{kl}\delta_{ki}\delta_{lj}=\mathrm{A}_{ij}
\tag{1.100}
$$

If we consider any two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and an arbitrary second-order tensor, $\mathbf{A}$, we can demonstrate that:

$$
\begin{aligned}
\vec{\mathbf{a}}\cdot\mathbf{A}\cdot\vec{\mathbf{b}}&=\mathrm{a}_p\hat{\mathbf{e}}_p\cdot\mathrm{A}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\cdot\mathrm{b}_r\hat{\mathbf{e}}_r=\mathrm{a}_p\mathrm{A}_{ij}\mathrm{b}_r\delta_{pi}\delta_{jr}=\mathrm{a}_i\mathrm{A}_{ij}\mathrm{b}_j=\mathrm{A}_{ij}(\mathrm{a}_i\mathrm{b}_j)\\
&=\mathbf{A}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})
\end{aligned}
\tag{1.101}
$$

- **Vector product**

The vector product between a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{x}}$ is a second-order tensor given by:

$$
\mathbf{A}\wedge\vec{\mathbf{x}}=(\mathrm{A}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\wedge(\mathrm{x}_k\hat{\mathbf{e}}_k)=\epsilon_{ljk}\mathrm{A}_{ij}\mathrm{x}_k\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l
\tag{1.102}
$$

where we have used the definition (1.67), *i.e.* $\hat{\mathbf{e}}_j\wedge\hat{\mathbf{e}}_k=\epsilon_{ljk}\hat{\mathbf{e}}_l$. In **Problem 1.11**, we have shown that the relation $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}}-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}$ holds, which is also represented by means of dyads as:

$$
\left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]_j=(\mathrm{a}_k\mathrm{c}_k)\mathrm{b}_j-(\mathrm{a}_k\mathrm{b}_k)\mathrm{c}_j=(\mathrm{b}_j\mathrm{c}_k-\mathrm{c}_j\mathrm{b}_k)\mathrm{a}_k=\left[(\vec{\mathbf{b}}\otimes\vec{\mathbf{c}}-\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}}\right]_j
\tag{1.103}
$$

In the particular case when $\vec{\mathbf{a}}=\vec{\mathbf{c}}$ we obtain:

$$
\begin{aligned}
\left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}})\right]_j&=(\mathrm{a}_k\mathrm{a}_k)\mathrm{b}_j-(\mathrm{a}_k\mathrm{b}_k)\mathrm{a}_j=(\mathrm{a}_k\mathrm{a}_k)\mathrm{b}_p\delta_{jp}-(\mathrm{a}_k\mathrm{b}_p\delta_{kp})\mathrm{a}_j\\
&=\left[(\mathrm{a}_k\mathrm{a}_k)\delta_{jp}-(\mathrm{a}_k\delta_{kp})\mathrm{a}_j\right]\mathrm{b}_p=\left[(\mathrm{a}_k\mathrm{a}_k)\delta_{jp}-\mathrm{a}_p\mathrm{a}_j\right]\mathrm{b}_p\\
&=\left\{\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1}-\vec{\mathbf{a}}\otimes\vec{\mathbf{a}}\right]\cdot\vec{\mathbf{b}}\right\}_j
\end{aligned}
\tag{1.104}
$$

Thus, the following relationships are valid:

$$
\boxed{
\begin{aligned}
&\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}}-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}=(\vec{\mathbf{b}}\otimes\vec{\mathbf{c}}-\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}}\\
&\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}})=\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1}-\vec{\mathbf{a}}\otimes\vec{\mathbf{a}}\right]\cdot\vec{\mathbf{b}}
\end{aligned}}
\tag{1.105}
$$

1.5.1.1 Component Representation of a Second-Order Tensor in the Cartesian Basis

As seen before, a vector which has 3 independent components is represented in a Cartesian space as shown in Figure 1.11. An arbitrary second-order tensor has 9 independent components, so we would need a hyperspace to represent all its components. Afterwards, a device is introduced to represent the second-order tensor components in the Cartesian basis.

An arbitrary second-order tensor $\mathbf{T}$ is represented in the Cartesian basis by:

$$\begin{aligned}\mathbf{T} &= \mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \mathsf{T}_{i1}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{i2}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{i3}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_3 \\ &= \mathsf{T}_{11}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{12}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{13}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 + \\ &+ \mathsf{T}_{21}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 + \\ &+ \mathsf{T}_{31}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3\end{aligned} \tag{1.106}$$

Next, we calculate the projection of $\mathbf{T}$ onto $\hat{\mathbf{e}}_k$:

$$\mathbf{T}\cdot\hat{\mathbf{e}}_k = \mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \mathsf{T}_{ij}\hat{\mathbf{e}}_i\delta_{jk} = \mathsf{T}_{ik}\hat{\mathbf{e}}_i = \mathsf{T}_{1k}\hat{\mathbf{e}}_1 + \mathsf{T}_{2k}\hat{\mathbf{e}}_2 + \mathsf{T}_{3k}\hat{\mathbf{e}}_3 \tag{1.107}$$

thereby defining three vectors, namely:

$$\mathbf{T}\cdot\hat{\mathbf{e}}_k = \mathsf{T}_{ik}\hat{\mathbf{e}}_i \quad \Rightarrow \quad \begin{cases} k=1 \;\Rightarrow\; \mathsf{T}_{i1}\hat{\mathbf{e}}_i = \mathsf{T}_{11}\hat{\mathbf{e}}_1 + \mathsf{T}_{21}\hat{\mathbf{e}}_2 + \mathsf{T}_{31}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)} \\ k=2 \;\Rightarrow\; \mathsf{T}_{i2}\hat{\mathbf{e}}_i = \mathsf{T}_{12}\hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)} \\ k=3 \;\Rightarrow\; \mathsf{T}_{i3}\hat{\mathbf{e}}_i = \mathsf{T}_{13}\hat{\mathbf{e}}_1 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)} \end{cases} \tag{1.108}$$

Graphical representation of these three vectors $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$, $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)}$, $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)}$, in the Cartesian basis, is shown in Figure 1.13. Note also that $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$ is the projection of $\mathbf{T}$ onto $\hat{\mathbf{e}}_1$, $\hat{\mathsf{n}}_i^{(1)} = [1,0,0]$, which can be verified by:

$$(\mathbf{T}\cdot\hat{\mathbf{n}})_i = \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{21} \\ \mathsf{T}_{31} \end{bmatrix} = \mathsf{t}_i^{(\hat{\mathbf{e}}_1)} \tag{1.109}$$

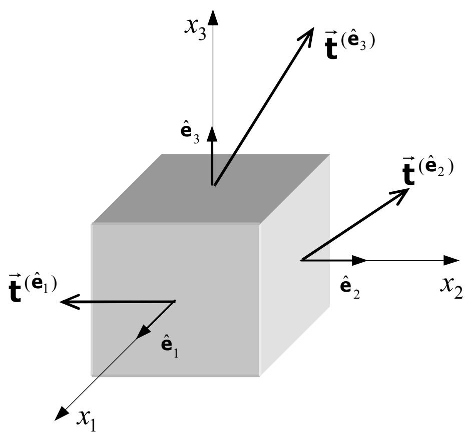


Figure 1.13: The projection of $\mathbf{T}$ in the Cartesian basis.

The same result obtained in (1.109) could have been evaluated by the scalar product of $\mathbf{T}$, given in (1.106), with the basis $\hat{\mathbf{e}}_1$, *i.e.*:

$$\begin{aligned}\mathbf{T}\cdot\hat{\mathbf{e}}_1 = [\; & \mathsf{T}_{11}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{12}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{13}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 + \\ & + \mathsf{T}_{21}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 + \\ & + \mathsf{T}_{31}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \;]\cdot\hat{\mathbf{e}}_1 \\ = \; & \mathsf{T}_{11}\hat{\mathbf{e}}_1 + \mathsf{T}_{21}\hat{\mathbf{e}}_2 + \mathsf{T}_{31}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}\end{aligned} \tag{1.110}$$

where we have used the orthogonality property of the basis, *i.e.* $\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1$, $\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 = 0$, $\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 = 0$. Taking into account the components are represented in matrix form, (see Figure 1.14), we can establish that, the diagonal terms (T_{11}, T_{22}, T_{33}) are normal to the plane defined by the unit vectors ($\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$), hence they will be referred to as *normal* *components*. The components displayed tangentially to the plane are called *tangential components*, and correspond to the off-diagonal terms of T_{ij}.

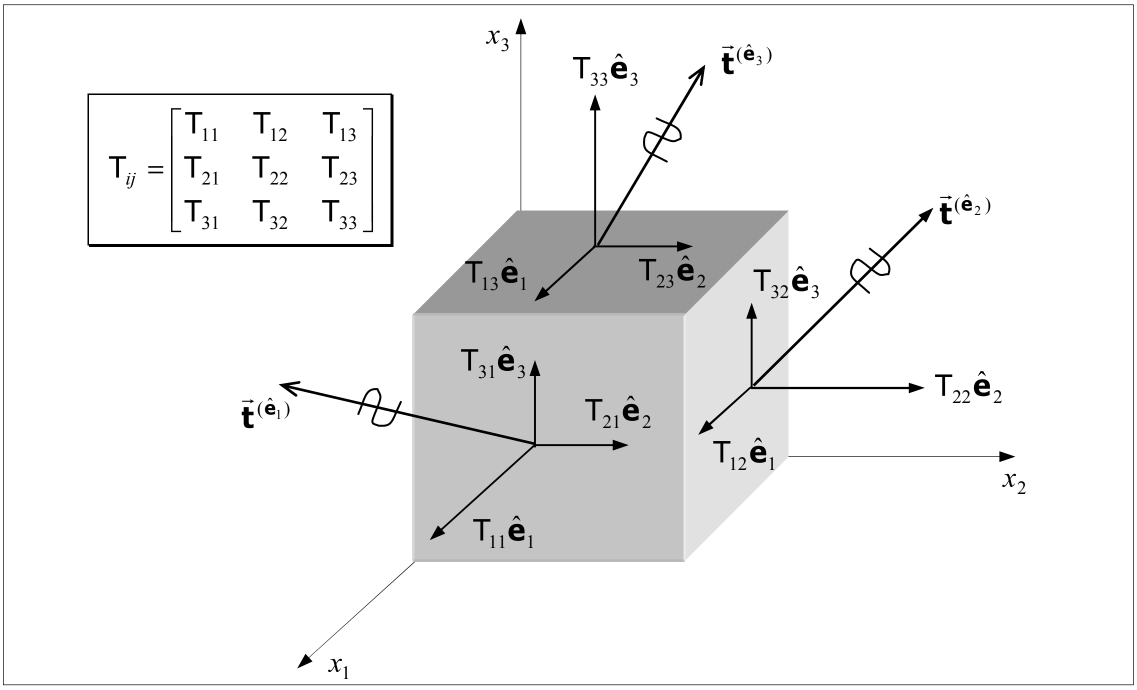


Figure 1.14: Representation of the second-order tensor components in the Cartesian coordinate system.

NOTE: Throughout the textbook, we will use the following notations:

Tensorial notation

Symbolic notation

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} &= (\mathsf{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\cdot(\mathsf{B}_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) \\ &= \mathsf{A}_{ij}\mathsf{B}_{kl}\delta_{jk}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l) \\ &= \mathsf{A}_{ij}\mathsf{B}_{jl}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l) \end{aligned} \tag{1.111}$$

Cartesian basis

Indicial notation

Note that the index is not repeated more than twice either in symbolic notation or in indicial notation. Also note that the indicial notation is equivalent to the tensor notation only when dealing with scalars, *e.g.* $\mathbf{A}:\mathbf{B} = \mathsf{A}_{ij}\mathsf{B}_{ij} = \lambda$, or $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \mathsf{a}_i\mathsf{b}_i$. ■

1.5.2 Properties of Tensors

1.5.2.1 Tensor Transpose

Let $\mathbf{A}$ be a second-order tensor, the *transpose* of $\mathbf{A}$ is defined as:

$$\mathbf{A}^T = \mathsf{A}_{ji}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) \tag{1.112}$$

If A_{ij} are the components of $\mathbf{A}$, it follows that the components of the transpose are:

$$(\mathbf{A}^T)_{ij} = \mathrm{A}_{ji} \tag{1.113}$$

If $\mathbf{A} = \vec{\mathbf{u}} \otimes \vec{\mathbf{v}}$, the transpose of the dyad $\mathbf{A}$ is given by $\mathbf{A}^T = \vec{\mathbf{v}} \otimes \vec{\mathbf{u}}$:

$$\begin{aligned}
\mathbf{A}^T \quad &= (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})^T && = \vec{\mathbf{v}} \otimes \vec{\mathbf{u}} \\
&= (\mathrm{u}_i \hat{\mathbf{e}}_i \otimes \mathrm{v}_j \hat{\mathbf{e}}_j)^T && = \mathrm{v}_j \hat{\mathbf{e}}_j \otimes \mathrm{u}_i \hat{\mathbf{e}}_i \quad = \mathrm{v}_i \hat{\mathbf{e}}_i \otimes \mathrm{u}_j \hat{\mathbf{e}}_j \\
&= (\mathrm{u}_i \mathrm{v}_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)^T && = \mathrm{u}_i \mathrm{v}_j \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \quad = \mathrm{u}_j \mathrm{v}_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\
&= (\mathrm{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)^T && = \mathrm{A}_{ij} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \quad = \mathrm{A}_{ji} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j
\end{aligned} \tag{1.114}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be second-order tensors and α, β be scalars, and the following relationships are valid:

$$(\mathbf{A}^T)^T = \mathbf{A} \quad ; \quad (\alpha \mathbf{B} + \beta \mathbf{A})^T = \alpha \mathbf{B}^T + \beta \mathbf{A}^T \quad ; \quad (\mathbf{B} \cdot \mathbf{A})^T = \mathbf{A}^T \cdot \mathbf{B}^T \tag{1.115}$$

$$\begin{aligned}
\mathbf{A} : \mathbf{B}^T &= (\mathrm{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) : (\mathrm{B}_{kl} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k) = \mathrm{A}_{ij} \mathrm{B}_{kl} \delta_{il} \delta_{jk} = \mathrm{A}_{ij} \mathrm{B}_{ji} = \mathbf{A} \cdot\cdot \mathbf{B} \\
\mathbf{A}^T : \mathbf{B} &= (\mathrm{A}_{ij} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) : (\mathrm{B}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \mathrm{A}_{ij} \mathrm{B}_{kl} \delta_{jk} \delta_{il} = \mathrm{A}_{ij} \mathrm{B}_{ji} = \mathbf{A} \cdot\cdot \mathbf{B}
\end{aligned} \tag{1.116}$$

The transpose of the matrix $\mathcal{A}$ is formed by changing rows for columns and vice versa, *i.e.:*

$$\mathcal{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \xrightarrow{transpose} \mathcal{A}^T = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix}^T = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{21} & \mathrm{A}_{31} \\ \mathrm{A}_{12} & \mathrm{A}_{22} & \mathrm{A}_{32} \\ \mathrm{A}_{13} & \mathrm{A}_{23} & \mathrm{A}_{33} \end{bmatrix} \tag{1.117}$$

Problem 1.13: Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be arbitrary second-order tensors. Demonstrate that:

$$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}^T) : \mathbf{B}$$

Solution: Expressing the term $\mathbf{A} : (\mathbf{B} \cdot \mathbf{C})$ in indicial notation we obtain:

$$\begin{aligned}
\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) &= \mathrm{A}_{ij} \hat{\mathbf{e}}_i \otimes \mathbf{e}_j : (\mathrm{B}_{lk} \hat{\mathbf{e}}_l \otimes \mathbf{e}_k \cdot \mathrm{C}_{pq} \hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) \\
&= \mathrm{A}_{ij} \mathrm{B}_{lk} \mathrm{C}_{pq} \hat{\mathbf{e}}_i \otimes \mathbf{e}_j : (\delta_{kp} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_q) \\
&= \mathrm{A}_{ij} \mathrm{B}_{lk} \mathrm{C}_{pq} \delta_{kp} \delta_{il} \delta_{jq} = \mathrm{A}_{ij} \mathrm{B}_{ik} \mathrm{C}_{kj}
\end{aligned}$$

Note that, when we are dealing with indicial notation the position of the terms does not matter, *i.e.:*

$$\mathrm{A}_{ij} \mathrm{B}_{ik} \mathrm{C}_{kj} = \mathrm{B}_{ik} \mathrm{A}_{ij} \mathrm{C}_{kj} = \mathrm{A}_{ij} \mathrm{C}_{kj} \mathrm{B}_{ik}$$

We can now observe that the algebraic operation $\mathrm{B}_{ik} \mathrm{A}_{ij}$ is equivalent to the components of the second-order tensor $(\mathbf{B}^T \cdot \mathbf{A})_{kj}$, thus,

$$\mathrm{B}_{ik} \mathrm{A}_{ij} \mathrm{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A})_{kj} \mathrm{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C}.$$

Likewise, we can state that $\mathrm{A}_{ij} \mathrm{C}_{kj} \mathrm{B}_{ik} = (\mathbf{A} \cdot \mathbf{C}^T) : \mathbf{B}$.

Problem 1.14: Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ be vectors and $\mathbf{A}$ be a second-order tensor. Show that the following relationship holds:

$$\vec{\mathbf{u}} \cdot \mathbf{A}^T \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \mathbf{A} \cdot \vec{\mathbf{u}}$$

Solution:

$$\begin{aligned}
&\vec{\mathbf{u}} \cdot \mathbf{A}^T \cdot \vec{\mathbf{v}} && = \vec{\mathbf{v}} \cdot \mathbf{A} \cdot \vec{\mathbf{u}} \\
&\mathrm{u}_i \hat{\mathbf{e}}_i \cdot \mathrm{A}_{jl} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_j \cdot \mathrm{v}_k \hat{\mathbf{e}}_k && = \mathrm{v}_k \hat{\mathbf{e}}_k \cdot \mathrm{A}_{jl} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_l \cdot \mathrm{u}_i \hat{\mathbf{e}}_i \\
&\mathrm{u}_i \mathrm{A}_{jl} \delta_{il} \mathrm{v}_k \delta_{jk} && = \mathrm{v}_k \delta_{kj} \mathrm{A}_{jl} \mathrm{u}_i \delta_{il} \\
&\mathrm{u}_l \mathrm{A}_{jl} \mathrm{v}_j && = \mathrm{v}_j \mathrm{A}_{jl} \mathrm{u}_l
\end{aligned}$$

1.5.2.2 Symmetry and Antisymmetry

1.5.2.2.1 Symmetric tensor

A second-order tensor $\mathbf{A}$ is symmetric, *i.e.:* $\mathbf{A} \equiv \mathbf{A}^{sym}$, if the tensor is equal to its transpose:

$$\text{if} \quad \mathbf{A} = \mathbf{A}^T \xrightarrow{\text{in components}} \mathsf{A}_{ij} = \mathsf{A}_{ji} \quad \Leftrightarrow \quad \mathbf{A} \text{ is symmetric} \tag{1.118}$$

in matrix form:

$$\mathcal{A} = \mathcal{A}^T \longrightarrow \mathcal{A} \equiv \mathcal{A}^{sym} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{12} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{13} & \mathsf{A}_{23} & \mathsf{A}_{33} \end{bmatrix} \tag{1.119}$$

From the above it is clear that a symmetric second-order tensor has 6 independent components, namely: A_{11}, A_{22}, A_{33}, A_{12}, A_{23}, A_{13}.

According to equation (1.118), a symmetric tensor can be represented by:

$$\begin{aligned} &\mathsf{A}_{ij} = \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} + \mathsf{A}_{ij} = \mathsf{A}_{ij} + \mathsf{A}_{ji} \\ &2\mathsf{A}_{ij} = \mathsf{A}_{ij} + \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} = \frac{1}{2}(\mathsf{A}_{ij} + \mathsf{A}_{ji}) \quad \Rightarrow \quad \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \end{aligned} \tag{1.120}$$

A fourth-order tensor $\mathbb{C}$, whose components are $\mathbb{C}_{ijkl}$, may have the following types of symmetries:

Minor symmetry:

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{jilk} \tag{1.121}$$

Major symmetry:

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} \tag{1.122}$$

A fourth-order tensor that does not exhibit any kind of symmetry has 81 independent components. If the tensor $\mathbb{C}$ has only minor symmetry, *i.e.* symmetry in $ij = ji(6)$, and symmetry in $kl = lk(6)$, the tensor features 36 independent components. If besides presenting minor symmetry it also provides major symmetry, the tensor features 21 independent components.

1.5.2.2.2 Antisymmetric tensor

A tensor $\mathbf{A}$ is antisymmetric (also called *skew-symmetric tensor* or *skew tensor*), *i.e.:* $\mathbf{A} \equiv \mathbf{A}^{skew}$:

$$\text{if} \quad \mathbf{A} = -\mathbf{A}^T \xrightarrow{\text{in components}} \mathsf{A}_{ij} = -\mathsf{A}_{ji} \quad \Leftrightarrow \quad \mathbf{A} \text{ is antisymmetric} \tag{1.123}$$

which broken down into its components is the same as:

$$\mathcal{A} = -\mathcal{A}^T \longrightarrow \mathcal{A}^{skew} = \begin{bmatrix} 0 & \mathsf{A}_{12} & \mathsf{A}_{13} \\ -\mathsf{A}_{12} & 0 & \mathsf{A}_{23} \\ -\mathsf{A}_{13} & -\mathsf{A}_{23} & 0 \end{bmatrix} \tag{1.124}$$

Therefore, an antisymmetric second-order tensor has 3 independent components, namely: A_{12}, A_{23}, A_{13}.

Under the conditions expressed in (1.123), an antisymmetric tensor can be represented by:

$$\begin{aligned} &\mathsf{A}_{ij} + \mathsf{A}_{ij} = \mathsf{A}_{ij} - \mathsf{A}_{ji} \\ &2\mathsf{A}_{ij} = \mathsf{A}_{ij} - \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} = \frac{1}{2}(\mathsf{A}_{ij} - \mathsf{A}_{ji}) \quad \Rightarrow \quad \mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \end{aligned} \tag{1.125}$$

Let us consider an antisymmetric second-order tensor denoted by $\mathbf{W}$, then satisfy the above relationship (1.125):

$$\mathsf{W}_{ij} = \frac{1}{2}(\mathsf{W}_{ij} - \mathsf{W}_{ji}) = \frac{1}{2}(\mathsf{W}_{kl}\delta_{ik}\delta_{jl} - \mathsf{W}_{kl}\delta_{jk}\delta_{il}) = \frac{1}{2}\mathsf{W}_{kl}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \tag{1.126}$$

Using the relation between the Kronecker delta and the permutation symbol given by (1.62), *i.e.* $\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = -\epsilon_{ijr}\epsilon_{lkr}$, the equation (1.126) is rewritten as:

$$\mathsf{W}_{ij} = -\frac{1}{2}\mathsf{W}_{kl}\epsilon_{ijr}\epsilon_{lkr} \tag{1.127}$$

Expanding the term $\mathsf{W}_{kl}\epsilon_{lkr}$, for the dummy indices (k, l), we can obtain the following nonzero terms:

$$\mathsf{W}_{kl}\epsilon_{lkr} = \mathsf{W}_{12}\epsilon_{21r} + \mathsf{W}_{13}\epsilon_{31r} + \mathsf{W}_{21}\epsilon_{12r} + \mathsf{W}_{23}\epsilon_{32r} + \mathsf{W}_{31}\epsilon_{13r} + \mathsf{W}_{32}\epsilon_{23r} \tag{1.128}$$

thus,

$$\left.\begin{aligned} r=1 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = -\mathsf{W}_{23} + \mathsf{W}_{32} = -2\mathsf{W}_{23} = 2w_1 \\ r=2 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = \mathsf{W}_{13} - \mathsf{W}_{31} = 2\mathsf{W}_{13} = 2w_2 \\ r=3 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = -\mathsf{W}_{12} + \mathsf{W}_{21} = -2\mathsf{W}_{12} = 2w_3 \end{aligned}\right\} \Rightarrow \mathsf{W}_{kl}\epsilon_{lkr} = 2w_r \tag{1.129}$$

In which we assume the following variables have changed:

$$\mathsf{W}_{ij} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ \mathsf{W}_{21} & 0 & \mathsf{W}_{23} \\ \mathsf{W}_{31} & \mathsf{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{1.130}$$

Hence, we introduce the *axial vector* $\vec{w}$ associated with the antisymmetric tensor, $\mathbf{W}$, as:

$$\vec{w} = w_1\hat{\mathbf{e}}_1 + w_2\hat{\mathbf{e}}_2 + w_3\hat{\mathbf{e}}_3 \tag{1.131}$$

The magnitude of the axial vector $\vec{w}$ is given by:

$$\omega^2 = \|\vec{w}\|^2 = \vec{w}\cdot\vec{w} = w_1^2 + w_2^2 + w_3^2 = \mathsf{W}_{23}^2 + \mathsf{W}_{13}^2 + \mathsf{W}_{12}^2 \tag{1.132}$$

Substituting (1.129) into (1.127) and by considering that $\epsilon_{ijr} = \epsilon_{rij}$ we obtain:

$$\boxed{\mathsf{W}_{ij} = -w_r\epsilon_{rij}} \tag{1.133}$$

Multiplying both sides of the equation (1.133) by ϵ_{kij} we can obtain:

$$\epsilon_{kij}\mathsf{W}_{ij} = -w_r\epsilon_{rij}\epsilon_{kij} = -2w_r\delta_{rk} = -2w_k \tag{1.134}$$

where we have applied the relation $\epsilon_{rij}\epsilon_{kij} = 2\delta_{rk}$, which was evaluated in **Problem 1.7**, thus we can conclude that:

$$\boxed{w_k = -\frac{1}{2}\epsilon_{kij}\mathsf{W}_{ij}} \tag{1.135}$$

Graphical representation of the antisymmetric tensor components and its corresponding axial vector, in the Cartesian system, is shown in Figure 1.15.

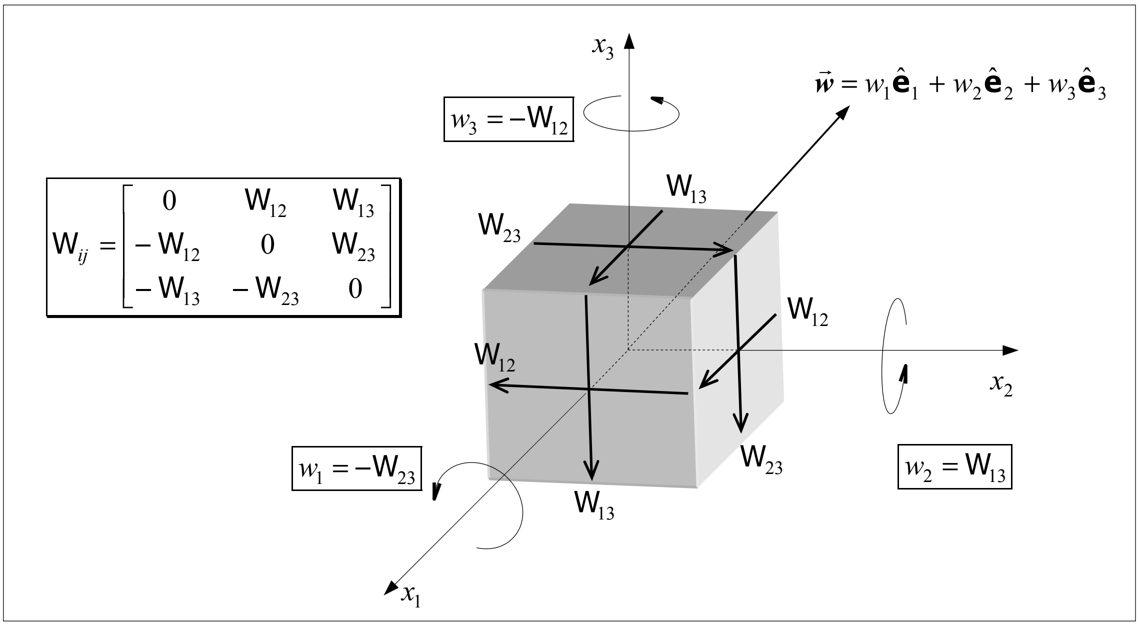


Figure 1.15: Antisymmetric tensor components and the axial vector.

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors and $\mathbf{W}$ be an antisymmetric tensor, it follows that:

$$\vec{\mathbf{b}}\cdot\mathbf{W}\cdot\vec{\mathbf{a}} = \vec{\mathbf{a}}\cdot\mathbf{W}^T\cdot\vec{\mathbf{b}} = -\vec{\mathbf{a}}\cdot\mathbf{W}\cdot\vec{\mathbf{b}} \tag{1.136}$$

when $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, it holds that:

$$\vec{\mathbf{a}}\cdot\mathbf{W}\cdot\vec{\mathbf{a}} = \mathbf{W}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{a}}) = 0 \tag{1.137}$$

NOTE: Note that $(\vec{\mathbf{a}}\otimes\vec{\mathbf{a}})$ is a symmetric second-order tensor. Later on we will show that the result of the double contraction between a symmetric tensor and an antisymmetric tensor equals zero. ■

Let us consider an antisymmetric tensor $\mathbf{W}$ and an arbitrary vector $\vec{\mathbf{a}}$. The components of the scalar product $\mathbf{W}\cdot\vec{\mathbf{a}}$ are given by:

$$\begin{aligned} &\mathsf{W}_{ij}\mathsf{a}_j = \mathsf{W}_{i1}\mathsf{a}_1 + \mathsf{W}_{i2}\mathsf{a}_2 + \mathsf{W}_{i3}\mathsf{a}_3 \\ &i=1 \Rightarrow \quad \mathsf{W}_{11}\mathsf{a}_1 + \mathsf{W}_{12}\mathsf{a}_2 + \mathsf{W}_{13}\mathsf{a}_3 \\ &i=2 \Rightarrow \quad \mathsf{W}_{21}\mathsf{a}_1 + \mathsf{W}_{22}\mathsf{a}_2 + \mathsf{W}_{23}\mathsf{a}_3 \\ &i=3 \Rightarrow \quad \mathsf{W}_{31}\mathsf{a}_1 + \mathsf{W}_{32}\mathsf{a}_2 + \mathsf{W}_{33}\mathsf{a}_3 \end{aligned} \tag{1.138}$$

Bearing in mind that the normal components are equal to zero for an antisymmetric tensor, *i.e.*, $\mathsf{W}_{11} = 0$, $\mathsf{W}_{22} = 0$, $\mathsf{W}_{33} = 0$, the scalar product (1.138) becomes:

$$(\mathbf{W}\cdot\vec{\mathbf{a}})_i \Rightarrow \begin{cases} i=1 \Rightarrow \quad \mathsf{W}_{12}\mathsf{a}_2 + \mathsf{W}_{13}\mathsf{a}_3 \\ i=2 \Rightarrow \quad \mathsf{W}_{21}\mathsf{a}_1 + \mathsf{W}_{23}\mathsf{a}_3 \\ i=3 \Rightarrow \quad \mathsf{W}_{31}\mathsf{a}_1 + \mathsf{W}_{32}\mathsf{a}_2 \end{cases} \tag{1.139}$$

The above components are the same as the result of the algebraic operation $\vec{w}\wedge\vec{\mathbf{a}}$:

$$\vec{w} \wedge \vec{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ w_1 & w_2 & w_3 \\ \mathsf{a}_1 & \mathsf{a}_2 & \mathsf{a}_3 \end{vmatrix}$$

$$\begin{aligned} &= (-w_3\mathsf{a}_2 + w_2\mathsf{a}_3)\hat{\mathbf{e}}_1 + (w_3\mathsf{a}_1 - w_1\mathsf{a}_3)\hat{\mathbf{e}}_2 + (-w_2\mathsf{a}_1 + w_1\mathsf{a}_2)\hat{\mathbf{e}}_3 \\ &= (\mathsf{W}_{12}\mathsf{a}_2 + \mathsf{W}_{13}\mathsf{a}_3)\hat{\mathbf{e}}_1 + (\mathsf{W}_{21}\mathsf{a}_1 + \mathsf{W}_{23}\mathsf{a}_3)\hat{\mathbf{e}}_2 + (\mathsf{W}_{31}\mathsf{a}_1 + \mathsf{W}_{32}\mathsf{a}_2)\hat{\mathbf{e}}_3 \end{aligned} \tag{1.140}$$

where $w_1 = -\mathsf{W}_{23} = \mathsf{W}_{32}$, $w_2 = \mathsf{W}_{13} = -\mathsf{W}_{31}$, $w_3 = -\mathsf{W}_{12} = \mathsf{W}_{21}$. Then, given an antisymmetric tensor $\mathbf{W}$ and the axial vector $\vec{w}$ associated with $\mathbf{W}$, it holds that:

$$\boxed{\mathbf{W} \cdot \vec{\mathbf{a}} = \vec{w} \wedge \vec{\mathbf{a}}} \tag{1.141}$$

for any vector $\vec{\mathbf{a}}$. The property (1.141) could easily have been obtained by taking into account the components of $\mathbf{W}$ given by (1.133), *i.e.*:

$$(\mathbf{W} \cdot \vec{\mathbf{a}})_i = \mathsf{W}_{ik}\mathsf{a}_k = -w_j \epsilon_{jik}\mathsf{a}_k = \epsilon_{ijk} w_j \mathsf{a}_k = (\vec{w} \wedge \vec{\mathbf{a}})_i \tag{1.142}$$

The vector $\vec{w}$ can be represented by its magnitude, $\|\vec{w}\| = \omega$, and by the unit vector codirectional with $\vec{w}$, *i.e.* $\vec{w} = \omega\hat{\mathbf{e}}_1^*$. Then, the equation (1.141) can still be expressed as:

$$\mathbf{W} \cdot \vec{\mathbf{a}} = \vec{w} \wedge \vec{\mathbf{a}} = \omega\hat{\mathbf{e}}_1^* \wedge \vec{\mathbf{a}} \tag{1.143}$$

Additionally, we can choose two unit vectors $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$, which make up an orthonormal basis with the unit vector $\hat{\mathbf{e}}_1^*$, (see Figure 1.16), so that:

$$\hat{\mathbf{e}}_1^* = \hat{\mathbf{e}}_2^* \wedge \hat{\mathbf{e}}_3^* \quad ; \quad \hat{\mathbf{e}}_2^* = \hat{\mathbf{e}}_3^* \wedge \hat{\mathbf{e}}_1^* \quad ; \quad \hat{\mathbf{e}}_3^* = \hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_2^* \tag{1.144}$$

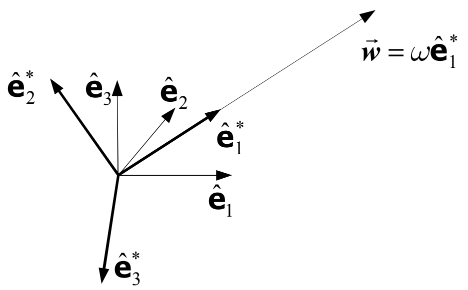


Figure 1.16: Orthonormal basis defined by the axial vector.

By representing the vector $\vec{\mathbf{a}}$ in this new basis, $\vec{\mathbf{a}} = \mathsf{a}_1^*\hat{\mathbf{e}}_1^* + \mathsf{a}_2^*\hat{\mathbf{e}}_2^* + \mathsf{a}_3^*\hat{\mathbf{e}}_3^*$, the relationship shown in (1.143) obtains the form below:

$$\begin{aligned} \mathbf{W} \cdot \vec{\mathbf{a}} &= \omega\hat{\mathbf{e}}_1^* \wedge \vec{\mathbf{a}} = \omega\hat{\mathbf{e}}_1^* \wedge (\mathsf{a}_1^*\hat{\mathbf{e}}_1^* + \mathsf{a}_2^*\hat{\mathbf{e}}_2^* + \mathsf{a}_3^*\hat{\mathbf{e}}_3^*) \\ &= \omega(\mathsf{a}_1^* \underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_1^*}_{=\vec{\mathbf{0}}} + \mathsf{a}_2^* \underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_2^*}_{=\hat{\mathbf{e}}_3^*} + \mathsf{a}_3^* \underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_3^*}_{=-\hat{\mathbf{e}}_2^*}) = \omega(\mathsf{a}_2^*\hat{\mathbf{e}}_3^* - \mathsf{a}_3^*\hat{\mathbf{e}}_2^*) \\ &= \left[\omega(\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^*)\right] \cdot \vec{\mathbf{a}} \end{aligned} \tag{1.145}$$

Thus, an antisymmetric tensor can be represented, in the space defined by the axial vector, as follows:

$$\mathbf{W} = \omega(\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^*) \tag{1.146}$$

Note that by using the antisymmetric tensor representation shown in (1.146), the projections of the tensor $\mathbf{W}$ according to directions $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$ and $\hat{\mathbf{e}}_3^*$ are respectively:

$$\mathbf{W}\cdot\hat{\mathbf{e}}_1^* = \vec{\mathbf{0}} \quad ; \quad \mathbf{W}\cdot\hat{\mathbf{e}}_2^* = \omega\hat{\mathbf{e}}_3^* \quad ; \quad \mathbf{W}\cdot\hat{\mathbf{e}}_3^* = -\omega\hat{\mathbf{e}}_2^* \tag{1.147}$$

We can also verify that:

$$\begin{aligned}\hat{\mathbf{e}}_3^*\cdot\mathbf{W}\cdot\hat{\mathbf{e}}_2^* &= \hat{\mathbf{e}}_3^*\cdot\left[\omega(\hat{\mathbf{e}}_3^*\otimes\hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^*\otimes\hat{\mathbf{e}}_3^*)\right]\cdot\hat{\mathbf{e}}_2^* = \omega \\ \hat{\mathbf{e}}_2^*\cdot\mathbf{W}\cdot\hat{\mathbf{e}}_3^* &= \hat{\mathbf{e}}_2^*\cdot\left[\omega(\hat{\mathbf{e}}_3^*\otimes\hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^*\otimes\hat{\mathbf{e}}_3^*)\right]\cdot\hat{\mathbf{e}}_3^* = -\omega\end{aligned} \tag{1.148}$$

Then, the tensor components of $\mathbf{W}$ in the basis formed by the orthonormal basis $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$, are given by:

$$\mathsf{W}_{ij}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \tag{1.149}$$

In Figure 1.17 we can see these components and the axial vector representation. Note that if we take any basis that is formed just by rotation along the $\hat{\mathbf{e}}_1^*$-axis, the components of $\mathbf{W}$ in this new basis will be the same as those provided in (1.149).

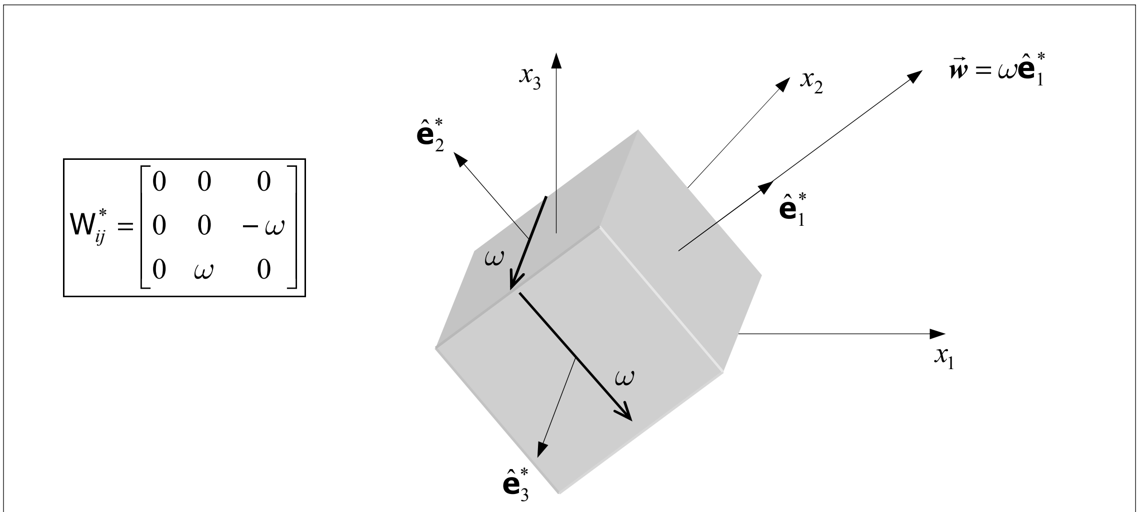


Figure 1.17: Antisymmetric tensor components in the space defined by the axial vector.

1.5.2.2.3 Additive decomposition. Symmetric and antisymmetric part

Any arbitrary second-order tensor $\mathbf{A}$ can be split additively into a symmetric and an antisymmetric part:

$$\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)}_{\mathbf{A}^{sym}} + \underbrace{\frac{1}{2}(\mathbf{A}-\mathbf{A}^T)}_{\mathbf{A}^{skew}} = \mathbf{A}^{sym} + \mathbf{A}^{skew} \tag{1.150}$$

or, into its components:

$$\mathsf{A}_{ij}^{sym} = \frac{1}{2}(\mathsf{A}_{ij} + \mathsf{A}_{ji}) \quad and \quad \mathsf{A}_{ij}^{skew} = \frac{1}{2}(\mathsf{A}_{ij} - \mathsf{A}_{ji}) \tag{1.151}$$

If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-order tensors, it holds that:

$$\begin{aligned}\left(\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A}\right)^{sym} &= \frac{1}{2}\left[\left(\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A}\right)+\left(\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A}\right)^T\right] = \frac{1}{2}\left[\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A}+\mathbf{A}^T\cdot\mathbf{B}^T\cdot\mathbf{A}\right] \\ &= \mathbf{A}^T\cdot\frac{1}{2}\left[\mathbf{B}+\mathbf{B}^T\right]\cdot\mathbf{A} = \mathbf{A}^T\cdot\mathbf{B}^{sym}\cdot\mathbf{A}\end{aligned} \tag{1.152}$$

Problem 1.15: Show that $\boldsymbol{\sigma}:\mathbf{W}=0$ is always true when $\boldsymbol{\sigma}$ is a symmetric second-order tensor and $\mathbf{W}$ is an antisymmetric second-order tensor.

Solution:

$$\boldsymbol{\sigma}:\mathbf{W}=\sigma_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j):\mathsf{W}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)=\sigma_{ij}\mathsf{W}_{lk}\delta_{il}\delta_{jk}=\sigma_{ij}\mathsf{W}_{ij}\quad\text{(scalar)}$$

Thus,

$$\sigma_{ij}\mathsf{W}_{ij}=\underbrace{\sigma_{1j}\mathsf{W}_{1j}}_{\substack{\sigma_{11}\mathsf{W}_{11}\\+\\\sigma_{12}\mathsf{W}_{12}\\+\\\sigma_{13}\mathsf{W}_{13}}}+\underbrace{\sigma_{2j}\mathsf{W}_{2j}}_{\substack{\sigma_{21}\mathsf{W}_{21}\\+\\\sigma_{22}\mathsf{W}_{22}\\+\\\sigma_{23}\mathsf{W}_{23}}}+\underbrace{\sigma_{3j}\mathsf{W}_{3j}}_{\substack{\sigma_{31}\mathsf{W}_{31}\\+\\\sigma_{32}\mathsf{W}_{32}\\+\\\sigma_{33}\mathsf{W}_{33}}}$$

Taking into account the characteristics of a symmetric and an antisymmetric tensor, *i.e.* $\sigma_{12}=\sigma_{21}$, $\sigma_{31}=\sigma_{13}$, $\sigma_{32}=\sigma_{23}$, and $\mathsf{W}_{11}=\mathsf{W}_{22}=\mathsf{W}_{33}=0$, $\mathsf{W}_{21}=-\mathsf{W}_{12}$, $\mathsf{W}_{31}=-\mathsf{W}_{13}$, $\mathsf{W}_{32}=-\mathsf{W}_{23}$, the equation above becomes:

$$\boldsymbol{\sigma}:\mathbf{W}=0$$

Problem 1.16: Show that a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}$; b) $\mathbf{A}:\mathbf{B}=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\mathbf{A}^{skew}:\mathbf{B}^{skew}$

where $\vec{\mathbf{M}}$ is a vector, and $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{B}$ are arbitrary second-order tensors.

Solution:

a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\left(\mathbf{Q}^{sym}+\mathbf{Q}^{skew}\right)\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}+\vec{\mathbf{M}}\cdot\mathbf{Q}^{skew}\cdot\vec{\mathbf{M}}$

Since the relation $\vec{\mathbf{M}}\cdot\mathbf{Q}^{skew}\cdot\vec{\mathbf{M}}=\mathbf{Q}^{skew}:\underbrace{\left(\vec{\mathbf{M}}\otimes\vec{\mathbf{M}}\right)}_{symmetric\ tensor}=0$ holds, it follows that:

$$\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}$$

b)

$$\begin{aligned}\mathbf{A}:\mathbf{B}&=(\mathbf{A}^{sym}+\mathbf{A}^{skew}):(\mathbf{B}^{sym}+\mathbf{B}^{skew})\\&=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\underbrace{\mathbf{A}^{sym}:\mathbf{B}^{skew}}_{=0}+\underbrace{\mathbf{A}^{skew}:\mathbf{B}^{sym}}_{=0}+\mathbf{A}^{skew}:\mathbf{B}^{skew}\\&=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\mathbf{A}^{skew}:\mathbf{B}^{skew}\end{aligned}$$

Then, it is also valid that:

$$\mathbf{A}:\mathbf{B}^{sym}=\mathbf{A}^{sym}:\mathbf{B}^{sym}\quad;\quad\mathbf{A}:\mathbf{B}^{skew}=\mathbf{A}^{skew}:\mathbf{B}^{skew}$$

Problem 1.17: Let $\mathbf{T}$ be an arbitrary second-order tensor, and $\vec{\mathbf{n}}$ be a vector. Check if the relationship $\vec{\mathbf{n}}\cdot\mathbf{T}=\mathbf{T}\cdot\vec{\mathbf{n}}$ is valid.

Solution:

$$\begin{aligned}\vec{\mathbf{n}}\cdot\mathbf{T}&=\mathsf{n}_i\hat{\mathbf{e}}_i\cdot\mathsf{T}_{kl}(\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l)\\&=\mathsf{n}_i\mathsf{T}_{kl}\delta_{ik}\hat{\mathbf{e}}_l\\&=\mathsf{n}_k\mathsf{T}_{kl}\hat{\mathbf{e}}_l\\&=(\mathsf{n}_1\mathsf{T}_{1l}+\mathsf{n}_2\mathsf{T}_{2l}+\mathsf{n}_3\mathsf{T}_{3l})\hat{\mathbf{e}}_l\end{aligned}\qquad\text{and}\qquad\begin{aligned}\mathbf{T}\cdot\vec{\mathbf{n}}&=\mathsf{T}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)\cdot\mathsf{n}_i\hat{\mathbf{e}}_i\\&=\mathsf{n}_i\mathsf{T}_{lk}\delta_{ki}\hat{\mathbf{e}}_l\\&=\mathsf{n}_k\mathsf{T}_{lk}\hat{\mathbf{e}}_l\\&=(\mathsf{n}_1\mathsf{T}_{l1}+\mathsf{n}_2\mathsf{T}_{l2}+\mathsf{n}_3\mathsf{T}_{l3})\hat{\mathbf{e}}_l\end{aligned}$$

With the above we can prove that $\mathsf{n}_k\mathsf{T}_{kl}\neq\mathsf{n}_k\mathsf{T}_{lk}$, then:

$$\vec{\mathbf{n}}\cdot\mathbf{T}\neq\mathbf{T}\cdot\vec{\mathbf{n}}$$

If $\mathbf{T}$ is a symmetric tensor, it follows that the relationship $\vec{\mathbf{n}} \cdot \mathbf{T}^{sym} = \mathbf{T}^{sym} \cdot \vec{\mathbf{n}}$ holds.

Problem 1.18: Obtain the axial vector $\vec{w}$ associated with the antisymmetric tensor $(\vec{x} \otimes \vec{a})^{skew}$.

Solution: Let $\vec{z}$ be an arbitrary vector, it then holds that:

$$(\vec{x} \otimes \vec{a})^{skew} \cdot \vec{z} = \vec{w} \wedge \vec{z}$$

where $\vec{w}$ is the axial vector associated with $(\vec{x} \otimes \vec{a})^{skew}$. Using the definition of an antisymmetric tensor:

$$(\vec{x} \otimes \vec{a})^{skew} = \frac{1}{2}\left[(\vec{x} \otimes \vec{a}) - (\vec{x} \otimes \vec{a})^{T}\right] = \frac{1}{2}\left[\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}\right]$$

and by replacing it with $(\vec{x} \otimes \vec{a})^{skew} \cdot \vec{z} = \vec{w} \wedge \vec{z}$, we obtain:

$$\frac{1}{2}\left[\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}\right] \cdot \vec{z} = \vec{w} \wedge \vec{z} \quad \Rightarrow \quad \left[\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}\right] \cdot \vec{z} = 2\vec{w} \wedge \vec{z}$$

By using the equation $\left[\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}\right] \cdot \vec{z} = \vec{z} \wedge (\vec{x} \wedge \vec{a})$, (see Eq. (1.105)), the above equation becomes:

$$\left[\vec{x} \otimes \vec{a} - \vec{a} \otimes \vec{x}\right] \cdot \vec{z} = \vec{z} \wedge (\vec{x} \wedge \vec{a}) = (\vec{a} \wedge \vec{x}) \wedge \vec{z} = 2\vec{w} \wedge \vec{z}$$

with the above we can conclude that:

$$\vec{w} = \frac{1}{2}(\vec{a} \wedge \vec{x}) \quad \textit{is the axial vector associated with } (\vec{x} \otimes \vec{a})^{skew}$$

1.5.2.3 Cofactor Tensor. Adjugate of a Tensor

Let $\mathbf{A}$ be a second-order tensor and $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors then there is then a unique tensor $\mathrm{cof}(\mathbf{A})$, known as the *cofactor* of $\mathbf{A}$, as we can see below:

$$\mathrm{cof}(\mathbf{A}) \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\mathbf{A} \cdot \vec{\mathbf{a}}) \wedge (\mathbf{A} \cdot \vec{\mathbf{b}}) \tag{1.153}$$

We can also define the adjugate of $\mathbf{A}$ as:

$$\mathrm{adj}(\mathbf{A}) = \left[\mathrm{cof}(\mathbf{A})\right]^{T} \tag{1.154}$$

which satisfies the following condition:

$$\left[\mathrm{adj}(\mathbf{A})\right]^{T} = \mathrm{adj}(\mathbf{A}^{T}) \tag{1.155}$$

The components of $\mathrm{cof}(\mathbf{A})$ are obtained by expressing the equation (1.153) in terms of its components, *i.e.*:

$$\left[\mathrm{cof}(\mathbf{A})\right]_{it} \epsilon_{tpr} \mathrm{a}_p \mathrm{b}_r = \epsilon_{ijk} \mathrm{A}_{jp} \mathrm{a}_p \mathrm{A}_{kr} \mathrm{b}_r \quad \Rightarrow \quad \left[\mathrm{cof}(\mathbf{A})\right]_{it} \epsilon_{tpr} = \epsilon_{ijk} \mathrm{A}_{jp} \mathrm{A}_{kr} \tag{1.156}$$

By multiplying both sides of the equation by ϵ_{qpr} and by also considering that $\epsilon_{tpr}\epsilon_{qpr} = 2\delta_{tq}$, we can conclude that:

$$\begin{gathered}
\left[\mathrm{cof}(\mathbf{A})\right]_{it} \epsilon_{tpr} = \epsilon_{ijk} \mathrm{A}_{jp} \mathrm{A}_{kr} \quad \Rightarrow \quad \left[\mathrm{cof}(\mathbf{A})\right]_{it} \underbrace{\epsilon_{tpr}\epsilon_{qpr}}_{=2\delta_{tq}} = \epsilon_{ijk}\epsilon_{qpr} \mathrm{A}_{jp} \mathrm{A}_{kr} \\
\Rightarrow \left[\mathrm{cof}(\mathbf{A})\right]_{iq} = \frac{1}{2}\epsilon_{ijk}\epsilon_{qpr} \mathrm{A}_{jp} \mathrm{A}_{kr}
\end{gathered} \tag{1.157}$$

1.5.2.4 Tensor Trace

Let's start by defining the trace of the basis $(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$:

$$\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \tag{1.158}$$

Thus, we can define the trace of a second-order tensor $\mathbf{A}$ as follows:

$$\begin{aligned} \mathrm{Tr}(\mathbf{A}) &= \mathrm{Tr}(\mathsf{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}\,\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}\delta_{ij} = \mathsf{A}_{ii} \\ &= \mathsf{A}_{11} + \mathsf{A}_{22} + \mathsf{A}_{33} \end{aligned} \tag{1.159}$$

And, the trace of the dyad $(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})$ can be evaluated as:

$$\begin{aligned} \mathrm{Tr}(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) &= \mathrm{Tr}(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = \mathsf{u}_i \mathsf{v}_j\,\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{u}_i \mathsf{v}_j(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \mathsf{u}_i \mathsf{v}_j \delta_{ij} = \mathsf{u}_i \mathsf{v}_i \\ &= \mathsf{u}_1\mathsf{v}_1 + \mathsf{u}_2\mathsf{v}_2 + \mathsf{u}_3\mathsf{v}_3 = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \end{aligned} \tag{1.160}$$

NOTE: As we will show later, the tensor trace is an *invariant*, *i.e.* it is independent of the coordinate system. ■

Let $\mathbf{A}$, $\mathbf{B}$ be arbitrary tensors, then:

- The transposed tensor trace is equal to the tensor trace:

$$\mathrm{Tr}\left(\mathbf{A}^T\right) = \mathrm{Tr}(\mathbf{A}) \tag{1.161}$$

- The trace of $(\mathbf{A} + \mathbf{B})$ is the sum of traces:

$$\mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B}) \tag{1.162}$$

Proving this is very simple:

$$\begin{aligned} &\mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B}) \\ &[(\mathsf{A}_{11} + \mathsf{B}_{11}) + (\mathsf{A}_{22} + \mathsf{B}_{22}) + (\mathsf{A}_{33} + \mathsf{B}_{33})] = (\mathsf{A}_{11} + \mathsf{A}_{22} + \mathsf{A}_{33}) + (\mathsf{B}_{11} + \mathsf{B}_{22} + \mathsf{B}_{33}) \end{aligned} \tag{1.163}$$

- The scalar product trace $(\mathbf{A} \cdot \mathbf{B})$ becomes:

$$\begin{aligned} \mathrm{Tr}(\mathbf{A} \cdot \mathbf{B}) &= \mathrm{Tr}\left[(\mathsf{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \cdot (\mathsf{B}_{lm}\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_m)\right] \\ &= \mathsf{A}_{ij}\mathsf{B}_{lm}\delta_{jl}\underbrace{\mathrm{Tr}\left[(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_m)\right]}_{\delta_{im}} \\ &= \mathsf{A}_{il}\mathsf{B}_{li} = \mathbf{A} \cdot\cdot\, \mathbf{B} = \mathrm{Tr}(\mathbf{B} \cdot \mathbf{A}) \end{aligned} \tag{1.164}$$

and, the double scalar product ($:$) can be expressed in trace terms as:

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= \mathsf{A}_{ij}\mathsf{B}_{ij} \\ &= \mathsf{A}_{kj}\mathsf{B}_{lj}\delta_{ik}\delta_{il} && = \mathsf{A}_{ik}\mathsf{B}_{il}\delta_{jk}\delta_{jl} \\ &= \underbrace{\mathsf{A}_{kj}\mathsf{B}_{lj}}_{(\mathbf{A}\cdot\mathbf{B}^T)_{kl}}\delta_{kl} && = \underbrace{\mathsf{A}_{ik}\mathsf{B}_{il}}_{(\mathbf{A}^T\cdot\mathbf{B})_{kl}}\delta_{kl} \\ &= \left(\mathbf{A} \cdot \mathbf{B}^T\right)_{kk} && = \left(\mathbf{A}^T \cdot \mathbf{B}\right)_{kk} \\ &= \mathrm{Tr}(\mathbf{A} \cdot \mathbf{B}^T) && = \mathrm{Tr}(\mathbf{A}^T \cdot \mathbf{B}) \end{aligned} \tag{1.165}$$

Similarly, it is possible to show that:

$$\mathrm{Tr}(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}) = \mathrm{Tr}(\mathbf{B} \cdot \mathbf{C} \cdot \mathbf{A}) = \mathrm{Tr}(\mathbf{C} \cdot \mathbf{A} \cdot \mathbf{B}) = \mathsf{A}_{ij}\mathsf{B}_{jk}\mathsf{C}_{ki} \tag{1.166}$$

$$\begin{aligned} &\mathrm{Tr}(\mathbf{A}) = \mathsf{A}_{ii} \\ &[\mathrm{Tr}(\mathbf{A})]^2 = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{A}) = \mathsf{A}_{ii}\mathsf{A}_{jj} \\ &\mathrm{Tr}(\mathbf{A} \cdot \mathbf{A}) = \mathrm{Tr}\left(\mathbf{A}^2\right) = \mathsf{A}_{il}\mathsf{A}_{li}\,; \quad \mathrm{Tr}(\mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}) = \mathrm{Tr}\left(\mathbf{A}^3\right) = \mathsf{A}_{ij}\mathsf{A}_{jk}\mathsf{A}_{ki} \end{aligned} \tag{1.167}$$

Problem 1.19: Let $\mathbf{T}$ be a second-order tensor. Show that:

$$(\mathbf{T}^m)^T = (\mathbf{T}^T)^m \quad and \quad \mathrm{Tr}(\mathbf{T}^T)^m = \mathrm{Tr}(\mathbf{T}^m).$$

Solution:

$$(\mathbf{T}^m)^T = (\mathbf{T}\cdot\mathbf{T}\cdots\mathbf{T})^T = \mathbf{T}^T\cdot\mathbf{T}^T\cdots\mathbf{T}^T = (\mathbf{T}^T)^m$$

For the second demonstration we can use the trace property $\mathrm{Tr}(\mathbf{T}^T) = \mathrm{Tr}(\mathbf{T})$, thus:

$$\mathrm{Tr}(\mathbf{T}^T)^m = \mathrm{Tr}(\mathbf{T}^m)^T = \mathrm{Tr}(\mathbf{T}^m)$$

1.5.2.5 Particular Tensors

1.5.2.5.1 Unit Tensors

- The second-order unit tensor, also called the *identity tensor*, is defined as:

$$\mathbf{1} = \delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i = \mathit{1}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.168}$$

where $\mathit{1}$ is the matrix with the components of tensor $\mathbf{1}$. δ_{ij} is the Kronecker delta symbol defined in (1.48).

- Fourth-order unit tensors can be defined as follows:

$$\mathbb{I} = \mathbf{1}\overline{\otimes}\mathbf{1} = \delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \mathbb{I}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.169}$$

$$\overline{\mathbb{I}} = \mathbf{1}\underline{\otimes}\mathbf{1} = \delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \overline{\mathbb{I}}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.170}$$

$$\overline{\overline{\mathbb{I}}} = \mathbf{1}\otimes\mathbf{1} = \delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \overline{\overline{\mathbb{I}}}_{ijk\ell}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \tag{1.171}$$

Taking into account the fourth-order unit tensors defined above, it holds that:

$$\begin{aligned}\mathbb{I}:\mathbf{A} &= (\delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell):(\mathsf{A}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) = \delta_{ik}\delta_{j\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \delta_{ik}\delta_{j\ell}\mathsf{A}_{k\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \mathbf{A}\end{aligned} \tag{1.172}$$

and

$$\begin{aligned}\overline{\mathbb{I}}:\mathbf{A} &= (\delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell):(\mathsf{A}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) = \delta_{i\ell}\delta_{jk}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \delta_{i\ell}\delta_{jk}\mathsf{A}_{k\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ji}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \mathbf{A}^T\end{aligned} \tag{1.173}$$

and

$$\begin{aligned}\overline{\overline{\mathbb{I}}}:\mathbf{A} &= (\delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell):(\mathsf{A}_{pq}\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) = \delta_{ij}\delta_{k\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \delta_{ij}\delta_{k\ell}\mathsf{A}_{k\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{kk}\delta_{ij}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\\ &= \mathrm{Tr}(\mathbf{A})\mathbf{1}\end{aligned} \tag{1.174}$$

The symmetric part of the fourth-order unit tensor $\mathbb{I}$ is defined as:

$$\mathbb{I}^{sym} \equiv \mathbf{I} = \frac{1}{2}(\mathbf{1}\overline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}\mathbf{1}) \xrightarrow{\text{in components}} \mathrm{I}_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{1.175}$$

The property of the tensor product $\overline{\otimes}$ is presented below. Consider a second-order unit tensor, $\mathbf{1} = \delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. Then, the tensor product $\overline{\otimes}$ can be defined as:

$$\mathbf{1}\overline{\otimes}\mathbf{1}=(\delta_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\overline{\otimes}(\delta_{k\ell}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell)=\delta_{ij}\delta_{k\ell}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_\ell) \tag{1.176}$$

which is the same as:

$$\mathbb{I}=\mathbf{1}\overline{\otimes}\mathbf{1}=\delta_{ik}\delta_{j\ell}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell) \tag{1.177}$$

and the tensor product $\underline{\otimes}$ is defined as:

$$\mathbf{1}\underline{\otimes}\mathbf{1}=(\delta_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\underline{\otimes}(\delta_{k\ell}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell)=\delta_{ij}\delta_{k\ell}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_\ell\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_j) \tag{1.178}$$

or

$$\bar{\mathbb{I}}=\mathbf{1}\underline{\otimes}\mathbf{1}=\delta_{i\ell}\delta_{jk}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell) \tag{1.179}$$

The antisymmetric part of $\mathbb{I}$ is defined as:

$$\mathbb{I}^{skew}=\frac{1}{2}(\mathbf{1}\overline{\otimes}\mathbf{1}-\mathbf{1}\underline{\otimes}\mathbf{1}) \xrightarrow{\text{in components}} \mathbb{I}^{skew}_{ijk\ell}=\frac{1}{2}(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}) \tag{1.180}$$

With a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{b}}$, the following relationships are valid:

$$\begin{aligned}
&\vec{\mathbf{b}}\cdot\mathbf{1}=\vec{\mathbf{b}}\\
&\mathbb{I}:\mathbf{A}=\mathbf{A} \quad ; \quad \mathbb{I}^{sym}:\mathbf{A}=\mathbf{A}^{sym}\\
&\mathbf{A}:\mathbf{1}=\mathrm{Tr}(\mathbf{A})=\mathsf{A}_{ii}\\
&\mathbf{A}^2:\mathbf{1}=\mathrm{Tr}(\mathbf{A}^2)=\mathrm{Tr}(\mathbf{A}\cdot\mathbf{A})=\mathsf{A}_{il}\mathsf{A}_{li}\\
&\mathbf{A}^3:\mathbf{1}=\mathrm{Tr}(\mathbf{A}^3)=\mathrm{Tr}(\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A})=\mathsf{A}_{ij}\mathsf{A}_{jk}\mathsf{A}_{ki}
\end{aligned} \tag{1.181}$$

Problem 1.20: Show that $\mathbf{T}:\mathbf{1}=\mathrm{Tr}(\mathbf{T})$, where $\mathbf{T}$ is an arbitrary second-order tensor.
Solution:

$$\begin{aligned}
\mathbf{T}:\mathbf{1} &= \mathsf{T}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j:\delta_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l\\
&= \mathsf{T}_{ij}\delta_{kl}\delta_{ik}\delta_{jl}\\
&= \mathsf{T}_{ij}\delta_{ij}=\mathsf{T}_{ii}=\mathsf{T}_{jj}\\
&= \mathrm{Tr}(\mathbf{T})
\end{aligned}$$

1.5.2.5.2 Levi-Civita Pseudo-Tensor

The *Levi-Civita Pseudo-Tensor*, also known as the *Permutation Tensor*, is a third-order pseudo-tensor and is denoted by:

$$\boldsymbol{\epsilon}=\epsilon_{ijk}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k \tag{1.182}$$

which is not a "true" tensor in the strict meaning of the word, and whose components ϵ_{ijk} were defined in (1.55), the permutation symbol.

1.5.2.6 Determinant of a Tensor

The determinant of a tensor is a scalar and is expressed as:

$$\det(\mathbf{A})\equiv|\mathbf{A}|=\epsilon_{ijk}\mathsf{A}_{1i}\mathsf{A}_{2j}\mathsf{A}_{3k}=\underbrace{\epsilon_{ijk}\mathsf{A}_{i1}\mathsf{A}_{j2}\mathsf{A}_{k3}}_{|\mathbf{A}^T|} \tag{1.183}$$

It is also an invariant (independent of the adopted system). Demonstrating the equation above (1.183) can be done starting directly from the determinant:

$$\begin{aligned}\det(\mathbf{A}) &= |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\ &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{21}(A_{12}A_{33} - A_{13}A_{32}) + A_{31}(A_{12}A_{23} - A_{13}A_{22}) \\ &= A_{11}(\epsilon_{1jk}A_{j2}A_{k3}) - A_{21}(-\epsilon_{2jk}A_{j2}A_{k3}) + A_{31}(\epsilon_{3jk}A_{j2}A_{k3}) \\ &= \epsilon_{1jk}A_{11}A_{j2}A_{k3} + \epsilon_{2jk}A_{21}A_{j2}A_{k3} + \epsilon_{3jk}A_{31}A_{j2}A_{k3} \\ &= \epsilon_{ijk}A_{i1}A_{j2}A_{k3}\end{aligned} \tag{1.184}$$

Some determinant properties with second-order tensors are described below:

$$\det(\mathbf{1}) = 1 \tag{1.185}$$

- We can conclude from (1.183) that:

$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \tag{1.186}$$

- We can also show that:

$$\det(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B}), \quad \det(\alpha\mathbf{A}) = \alpha^3\det(\mathbf{A}) \quad \text{where } \alpha \text{ is a scalar} \tag{1.187}$$

- A tensor $(\mathbf{A})$ is said to be *singular* if $\det(\mathbf{A}) = 0$.
- If you exchange two rows or columns, the determinant sign changes.
- If all elements of a row or column equal zero, the determinant is also zero.
- If you multiply all the elements of a row or column by a constant c (scalar), the determinant is $c|\mathbf{A}|$.
- If two or more rows (or column) are linearly dependent, the determinant is zero.

Problem 1.21: Show that $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}A_{rt}A_{jp}A_{kq}$.

Solution:

We start with the following definition:

$$|\mathbf{A}| = \epsilon_{rjk}A_{r1}A_{j2}A_{k3} \quad \Rightarrow \quad |\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\epsilon_{tpq}A_{r1}A_{j2}A_{k3} \tag{1.188}$$

and also taking into account that the term $\epsilon_{rjk}\epsilon_{tpq}$ can be replaced by (1.61):

$$\begin{aligned}\epsilon_{rjk}\epsilon_{tpq} &= \begin{vmatrix} \delta_{rt} & \delta_{rp} & \delta_{rq} \\ \delta_{jt} & \delta_{jp} & \delta_{jq} \\ \delta_{kt} & \delta_{kp} & \delta_{kq} \end{vmatrix} \\ &= \delta_{rt}\delta_{jp}\delta_{kq} + \delta_{rp}\delta_{jq}\delta_{kt} + \delta_{rq}\delta_{jt}\delta_{kp} - \delta_{rq}\delta_{jp}\delta_{kt} - \delta_{jq}\delta_{kp}\delta_{rt} - \delta_{kq}\delta_{jt}\delta_{rp}\end{aligned} \tag{1.189}$$

Then, by substituting (1.189) into (1.188) we can obtain:

$$\begin{aligned}|\mathbf{A}|\epsilon_{tpq} &= A_{t1}A_{p2}A_{q3} + A_{p1}A_{q2}A_{t3} + A_{q1}A_{t2}A_{p3} - A_{q1}A_{p2}A_{t3} - A_{t1}A_{q2}A_{p3} - A_{p1}A_{t2}A_{q3} \\ &= A_{t1}\left(\epsilon_{1jk}A_{pj}A_{qk}\right) + A_{t2}\left(\epsilon_{2jk}A_{pj}A_{qk}\right) + A_{t3}\left(\epsilon_{3jk}A_{pj}A_{qk}\right) \\ &= \epsilon_{rjk}A_{rt}A_{jp}A_{kq} = \epsilon_{rjk}A_{tr}A_{pj}A_{qk}\end{aligned}$$

Problem 1.22: Show that $|\mathbf{A}| = \dfrac{1}{6}\epsilon_{rjk}\epsilon_{tpq}A_{rt}A_{jp}A_{kq}$.

Solution:

Starting with the definition $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$, (see **Problem 1.21**), and by multiplying both sides of the equation by ϵ_{tpq}, we obtain:

$$|\mathbf{A}|\epsilon_{tpq}\epsilon_{tpq} = \epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq} \tag{1.190}$$

Using the property defined in expression (1.62), we obtain $\epsilon_{tpq}\epsilon_{tpq} = \delta_{tt}\delta_{pp} - \delta_{tp}\delta_{tp} = \delta_{tt}\delta_{pp} - \delta_{tt} = 6$. Then, the relationship (1.190) becomes:

$$|\mathbf{A}| = \frac{1}{6}\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$$

Problem 1.23: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors and α be a scalar. Show that:

$$\boxed{\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mu^3 + \mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}} \tag{1.191}$$

Solution: The determinant of $\mathbf{A}$ is given by $|\mathbf{A}| = \epsilon_{ijk}\mathsf{A}_{i1}\mathsf{A}_{j2}\mathsf{A}_{k3}$. If we denote by $\mathsf{A}_{ij} = \mu\delta_{ij} + \alpha\mathsf{a}_i\mathsf{b}_j$, thus, $\mathsf{A}_{i1} = \mu\delta_{i1} + \alpha\mathsf{a}_i\mathsf{b}_1$, $\mathsf{A}_{k3} = \mu\delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3$, $\mathsf{A}_{k3} = \delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3$, then the equation in (1.191) can be rewritten as:

$$\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \epsilon_{ijk}(\mu\delta_{i1} + \alpha\mathsf{a}_i\mathsf{b}_1)(\mu\delta_{j2} + \alpha\mathsf{a}_j\mathsf{b}_2)(\mu\delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3) \tag{1.192}$$

By developing the equation (1.192), we obtain:

$$\begin{aligned}\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \epsilon_{ijk}\big[&\mu^3\delta_{i1}\delta_{j2}\delta_{k3} + \mu^2\alpha\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \mu^2\alpha\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \mu^2\alpha\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3} + \\ &+ \mu\alpha^2\mathsf{a}_j\mathsf{b}_2\mathsf{a}_k\mathsf{b}_3\delta_{i1} + \mu\alpha^2\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} + \mu\alpha^2\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3} + \alpha^3\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3\big]\end{aligned}$$

Note that: $\mu^3\epsilon_{ijk}\delta_{i1}\delta_{j2}\delta_{k3} = \mu^3\epsilon_{123} = \mu^3$,

$$\mu^2\alpha(\epsilon_{ijk}\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \epsilon_{ijk}\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \epsilon_{ijk}\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3}) =$$

$$\mu^2\alpha(\epsilon_{12k}\mathsf{a}_k\mathsf{b}_3 + \epsilon_{1j3}\mathsf{a}_j\mathsf{b}_2 + \epsilon_{i23}\mathsf{a}_i\mathsf{b}_1) = \mu^2\alpha(\mathsf{a}_3\mathsf{b}_3 + \mathsf{a}_2\mathsf{b}_2 + \mathsf{a}_1\mathsf{b}_1) = \mu^2\alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} = \epsilon_{i2k}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3 = \mathsf{a}_1\mathsf{a}_3\mathsf{b}_1\mathsf{b}_3 - \mathsf{a}_3\mathsf{a}_1\mathsf{b}_1\mathsf{b}_3 = 0$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3} = \epsilon_{ij3}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2 = \epsilon_{123}\mathsf{a}_1\mathsf{a}_2\mathsf{b}_1\mathsf{b}_2 - \epsilon_{213}\mathsf{a}_2\mathsf{a}_1\mathsf{b}_1\mathsf{b}_2 = 0$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3 = 0$$

Notice that, there was no need to expand the terms $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}$, $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3}$, and $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3$ to realize that these terms equal zero, since $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} = (\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})_j\mathsf{b}_1\mathsf{b}_3\delta_{j2} = 0$, similarly for other terms.

Taking into account the above considerations we can prove that:

$$\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mu^3 + \mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$$

For the particular case when $\mu = 1$ the above equation becomes:

$$\boxed{\det(\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 1 + \alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}$$

Then, it is simple to prove that $\det(\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 0$, since

$$\det(\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \alpha^3\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3 = \alpha^3\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3[\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})] = 0$$

The following relations are satisfied:

$$\det\left[\mathbf{1} + \alpha\,(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) + \beta\,(\vec{\mathbf{b}}\otimes\vec{\mathbf{a}})\right] = 1 + \alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) + \beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) + \alpha\beta\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2 - (\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}})\right] \tag{1.193}$$

where α, β are scalars. If $\beta = 0$ we can regain the equation $\det(\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 1 + \alpha\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, (see **Problem 1.23**). If $\alpha = \beta$ we obtain:

$$\det\left(\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\alpha\,\vec{\mathbf{b}}\otimes\vec{\mathbf{a}}\right)=1+\alpha\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)+\alpha\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)+\alpha^2\left[\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)^2-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}\right)\right]$$
$$=1+\alpha\left[2\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)-\alpha\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)^2\right] \tag{1.194}$$

where we have used the property $\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)^2-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}\right)=-\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)^2$, (see **Problem 1.1**).

It is also true that:

$$\det(\alpha\,\mathbf{A}+\beta\,\mathbf{B})=\alpha^3\det(\mathbf{A})+\alpha^2\beta\,\mathrm{Tr}[\mathbf{B}\cdot\mathrm{adj}(\mathbf{A})]+\alpha\beta^2\,\mathrm{Tr}[\mathbf{A}\cdot\mathrm{adj}(\mathbf{B})]+\beta^3\det(\mathbf{B}) \tag{1.195}$$

Moreover, in the particular case when $\alpha=1$, $\mathbf{A}=\mathbf{1}$, $\mathbf{B}=\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}$, and bearing in mind that $\det\left(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=0$, and $\mathrm{cof}\left(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=\mathbf{0}$, we can conclude that:

$$\det\left(\mathbf{1}+\beta\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=\det(\mathbf{1})+\beta\mathrm{Tr}\left[\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\cdot\mathbf{1}\right]=1+\beta\mathrm{Tr}\left[\mathrm{a}_i\mathrm{b}_j\right]=1+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} \tag{1.196}$$

which has already been demonstrated in **Problem 1.23**.

We next show that the following property is valid:

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]=\det(\mathbf{A})\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]} \tag{1.197}$$

To achieve this goal we start with the definition of the scalar triple product given in (1.69), *i.e.* $\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)=\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k$, and by multiplying both sides of this equation by the determinant of $\mathbf{A}$ we obtain:

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)|\mathbf{A}|=\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k|\mathbf{A}| \tag{1.198}$$

It was proven in **Problem 1.21** that $|\mathbf{A}|\epsilon_{ijk}=\epsilon_{pqr}\mathrm{A}_{pi}\mathrm{A}_{qj}\mathrm{A}_{rk}$, thus:

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)|\mathbf{A}|=\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k|\mathbf{A}|=\epsilon_{pqr}\mathrm{A}_{pi}\mathrm{A}_{qj}\mathrm{A}_{rk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k=\epsilon_{pqr}(\mathrm{A}_{pi}\mathrm{a}_i)(\mathrm{A}_{qj}\mathrm{b}_j)(\mathrm{A}_{rk}\mathrm{c}_k)$$
$$=(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]=\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]\cdot(\mathbf{A}\cdot\vec{\mathbf{a}}) \tag{1.199}$$

1.5.2.7 Inverse of a Tensor

We use the notation $\mathbf{A}^{-1}$ to denote the inverse of $\mathbf{A}$, which is defined as follows:

$$\text{if } |\mathbf{A}|\neq 0 \quad\Leftrightarrow\quad \exists\mathbf{A}^{-1}\;\left|\mathbf{A}\cdot\mathbf{A}^{-1}=\mathbf{A}^{-1}\cdot\mathbf{A}=\mathbf{1}\right. \tag{1.200}$$

Or in indicial notation:

$$\text{if } |\mathbf{A}|\neq 0 \quad\Leftrightarrow\quad \exists\mathrm{A}^{-1}_{ij}\;\left|\mathrm{A}_{ik}\mathrm{A}^{-1}_{kj}=\mathrm{A}^{-1}_{ik}\mathrm{A}_{kj}=\delta_{ij}\right. \tag{1.201}$$

To obtain the inverse of a tensor we start from the definition of the adjugate tensor given in (1.153), *i.e.* $\mathrm{adj}(\mathbf{A}^T)\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})$. Then by applying the dot product between an arbitrary vector $\vec{\mathbf{d}}$ and this equation we obtain:

$$\left\{[\mathrm{adj}(\mathbf{A})]^T\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\right\}\cdot\vec{\mathbf{d}}=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\vec{\mathbf{d}}=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\mathbf{1}\cdot\vec{\mathbf{d}}$$
$$=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\mathbf{A}\cdot\underbrace{\mathbf{A}^{-1}\cdot\vec{\mathbf{d}}}_{=\vec{\mathbf{c}}} \tag{1.202}$$

In equation (1.199) we demonstrated that $\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)|\mathbf{A}|=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot(\mathbf{A}\cdot\vec{\mathbf{c}})$ thus,

$$\left\{[\mathrm{adj}(\mathbf{A})]^T \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})\right\} \cdot \vec{\mathbf{d}} = \left[(\mathbf{A} \cdot \vec{\mathbf{a}}) \wedge (\mathbf{A} \cdot \vec{\mathbf{b}})\right] \cdot \mathbf{A} \cdot \mathbf{A}^{-1} \cdot \vec{\mathbf{d}} = |\mathbf{A}| \left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right) \cdot \mathbf{A}^{-1} \cdot \vec{\mathbf{d}} \tag{1.203}$$

Denoted by $\vec{\mathbf{p}} = (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$, the above equation (1.203) can be rearranged as follows:

$$\begin{aligned} &\{[\mathrm{adj}(\mathbf{A})]_{ki} \mathrm{p}_k\} \mathrm{d}_i = |\mathbf{A}| \mathrm{p}_k \mathrm{A}^{-1}_{ki} \mathrm{d}_i \\ &\Rightarrow [\mathrm{adj}(\mathbf{A})]_{ki} \mathrm{p}_k \mathrm{d}_i = |\mathbf{A}| \mathrm{A}^{-1}_{ki} \mathrm{p}_k \mathrm{d}_i \\ &\Rightarrow [\mathrm{adj}(\mathbf{A})] : \left[(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \otimes \vec{\mathbf{d}}\right] = |\mathbf{A}| \mathbf{A}^{-1} : \left[(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) \otimes \vec{\mathbf{d}}\right] \end{aligned} \tag{1.204}$$

Thus, we can conclude that:

$$[\mathrm{adj}(\mathbf{A})] = |\mathbf{A}| \mathbf{A}^{-1} \quad \Rightarrow \quad \boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}[\mathrm{adj}(\mathbf{A})] = \frac{1}{|\mathbf{A}|}[\mathrm{cof}(\mathbf{A})]^T} \tag{1.205}$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be invertible tensors, the following properties hold:

$$\begin{aligned} &(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \\ &\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A} \\ &(\beta \mathbf{A})^{-1} = \frac{1}{\beta} \mathbf{A}^{-1} \\ &\det(\mathbf{A}^{-1}) = [\det(\mathbf{A})]^{-1} \end{aligned} \tag{1.206}$$

- The following notation will be used to represent the inverse transpose:

$$\mathbf{A}^{-T} \equiv (\mathbf{A}^{-1})^T \equiv (\mathbf{A}^T)^{-1} \tag{1.207}$$

Next, we prove the relation $\mathrm{adj}(\mathbf{A} \cdot \mathbf{B}) = \mathrm{adj}(\mathbf{B}) \cdot \mathrm{adj}(\mathbf{A})$ holds. To do this, we take the definition of the inverse of a tensor given in (1.205) as a starting point:

$$\begin{aligned} &\mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = \frac{[\mathrm{adj}(\mathbf{B})]}{|\mathbf{B}|} \cdot \frac{[\mathrm{adj}(\mathbf{A})]}{|\mathbf{A}|} \Rightarrow |\mathbf{A}||\mathbf{B}| \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = [\mathrm{adj}(\mathbf{B})] \cdot [\mathrm{adj}(\mathbf{A})] \\ &\Rightarrow |\mathbf{A}||\mathbf{B}| (\mathbf{A} \cdot \mathbf{B})^{-1} = [\mathrm{adj}(\mathbf{B})] \cdot [\mathrm{adj}(\mathbf{A})] \Rightarrow |\mathbf{A}||\mathbf{B}| \frac{[\mathrm{adj}(\mathbf{A} \cdot \mathbf{B})]}{|\mathbf{A} \cdot \mathbf{B}|} = [\mathrm{adj}(\mathbf{B})] \cdot [\mathrm{adj}(\mathbf{A})] \\ &\Rightarrow \mathrm{adj}(\mathbf{A} \cdot \mathbf{B}) = [\mathrm{adj}(\mathbf{B})] \cdot [\mathrm{adj}(\mathbf{A})] \end{aligned} \tag{1.208}$$

where we have used the property $|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$. Similarly, it is possible to show that $\mathrm{cof}(\mathbf{A} \cdot \mathbf{B}) = [\mathrm{cof}(\mathbf{A})] \cdot [\mathrm{cof}(\mathbf{B})]$.

Procedure for obtaining the inverse of the matrix $\mathcal{A}$

1) Obtain the cofactor matrix: $\mathrm{cof}(\mathcal{A})$ as follows:

Consider the matrix $\mathcal{A}$ as:

$$\mathcal{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} \tag{1.209}$$

We define the matrix $\mathcal{M}$, where the component $\mathcal{M}_{ij}$ is the determinant of the resulting matrix by removing the i^{th} row and the j^{th} column, *i.e.*:

$$\mathcal{M}=\begin{bmatrix} \begin{vmatrix} \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{32} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{21} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{21} & \mathsf{A}_{22} \\ \mathsf{A}_{31} & \mathsf{A}_{32} \end{vmatrix} \\ \begin{vmatrix} \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{32} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{13} \\ \mathsf{A}_{31} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{31} & \mathsf{A}_{32} \end{vmatrix} \\ \begin{vmatrix} \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{22} & \mathsf{A}_{23} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{23} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{vmatrix} \end{bmatrix} \tag{1.210}$$

Thus, we define the cofactor matrix as:

$$\mathsf{cof}(\mathcal{A})=(-1)^{i+j}\mathcal{M}_{ij} \tag{1.211}$$

2) Obtain the adjugate matrix, $\mathsf{adj}(\mathcal{A})$, as follows:

$$\mathsf{adj}(\mathcal{A})=[\mathsf{cof}(\mathcal{A})]^T \tag{1.212}$$

3) Obtain the inverse matrix:

$$\mathcal{A}^{-1}=\frac{\mathsf{adj}(\mathcal{A})}{|\mathcal{A}|} \tag{1.213}$$

So, the relation $\mathcal{A}[\mathsf{adj}(\mathcal{A})]=|\mathcal{A}|\boldsymbol{1}$ holds, where $\boldsymbol{1}$ is the identity matrix.

Taking into account (1.64), we can express the components of the first, second, and third row of the cofactor matrix, (1.211), as: $\mathsf{M}_{1i}=\epsilon_{ijk}\mathsf{A}_{2j}\mathsf{A}_{3k}$, $\mathsf{M}_{2i}=\epsilon_{ijk}\mathsf{A}_{1j}\mathsf{A}_{3k}$, $\mathsf{M}_{3i}=\epsilon_{ijk}\mathsf{A}_{1j}\mathsf{A}_{2k}$, respectively.

Problem 1.24: Let $\mathbf{A}$ be an arbitrary second-order tensor. Show that there is a nonzero vector $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$ so that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$ if and only if $\det(\mathbf{A})=0$, Chadwick (1976).

Solution: Firstly, we show that, if $\det(\mathbf{A})\equiv|\mathbf{A}|=0\Rightarrow\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$. Secondly, we show that, if $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}\Rightarrow\det(\mathbf{A})\equiv|\mathbf{A}|=0$.

We assume that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, and we choose an arbitrary basis $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ (linearly independent). We apply these vectors in the definition seen in (1.197):

$$\vec{\mathbf{f}}\cdot\left(\vec{\mathbf{g}}\wedge\vec{\mathbf{h}}\right)|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})\right]$$

Due to the fact that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, the implication is that:

$$(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})\right]=0$$

Thus, we can conclude that the vectors $(\mathbf{A}\cdot\vec{\mathbf{f}})$, $(\mathbf{A}\cdot\vec{\mathbf{g}})$, $(\mathbf{A}\cdot\vec{\mathbf{h}})$, are linearly dependent. This implies that there are nonzero scalars α, β, γ so that:

$$\alpha(\mathbf{A}\cdot\vec{\mathbf{f}})+\beta(\mathbf{A}\cdot\vec{\mathbf{g}})+\gamma(\mathbf{A}\cdot\vec{\mathbf{h}})=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot\left(\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}}\right)=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$$

where $\vec{\mathbf{n}}=\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}}\neq\vec{\mathbf{0}}$ since $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ is linearly independent, (see **Problem 1.10**).

Now we choose two vectors $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, which are linearly independent to $\vec{\mathbf{n}}$. We apply definition (1.199) once more:

$$\vec{\mathbf{k}}\cdot\left(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}}\right)|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{k}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{m}})\wedge(\mathbf{A}\cdot\vec{\mathbf{n}})\right]$$

Considering that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$, and $\vec{\mathbf{k}}\cdot\left(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}}\right)\neq 0$ owing to the fact that $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, $\vec{\mathbf{n}}$ are linearly independent, we can conclude that:

$$\underbrace{\vec{\mathbf{k}} \cdot (\vec{\mathbf{m}} \wedge \vec{\mathbf{n}})}_{\neq 0} |\mathbf{A}| = 0 \quad \Rightarrow \quad |\mathbf{A}| = 0$$

1.5.2.8 Orthogonal Tensors

Orthogonal tensors play an important role in continuum mechanics. A second-order tensor ($\mathbf{Q}$) is said to be orthogonal when the transpose ($\mathbf{Q}^T$) is equal to the inverse ($\mathbf{Q}^{-1}$), *i.e.* $\mathbf{Q}^T = \mathbf{Q}^{-1}$. Then, it follows that:

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{1} \quad ; \quad \mathrm{Q}_{ik}\mathrm{Q}_{jk} = \mathrm{Q}_{ki}\mathrm{Q}_{kj} = \delta_{ij} \tag{1.214}$$

A *proper orthogonal tensor* has the following properties:

- The inverse of $\mathbf{Q}$ is equal to the transpose (orthogonality):

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \tag{1.215}$$

- The tensor $\mathbf{Q}$ is a proper, *rotation tensor*, if:

$$\det(\mathbf{Q}) \equiv |\mathbf{Q}| = +1 \tag{1.216}$$

If $|\mathbf{Q}| = -1$, the orthogonal tensor is said to be *improper (a reflection tensor).*

If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal tensors, the resulting tensor $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is also an orthogonal tensor, *i.e.*:

$$\mathbf{C}^{-1} = (\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = \mathbf{B}^T \cdot \mathbf{A}^T = (\mathbf{A} \cdot \mathbf{B})^T = \mathbf{C}^T \tag{1.217}$$

Consider two arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. An orthogonal transformation applied to these vectors becomes:

$$\tilde{\vec{\mathbf{a}}} = \mathbf{Q} \cdot \vec{\mathbf{a}} \quad ; \quad \tilde{\vec{\mathbf{b}}} = \mathbf{Q} \cdot \vec{\mathbf{b}} \tag{1.218}$$

And the dot product between these new vectors ($\tilde{\vec{\mathbf{a}}}$) and ($\tilde{\vec{\mathbf{b}}}$) is given by:

$$\begin{aligned} \tilde{\vec{\mathbf{a}}} \cdot \tilde{\vec{\mathbf{b}}} &= (\mathbf{Q} \cdot \vec{\mathbf{a}}) \cdot (\mathbf{Q} \cdot \vec{\mathbf{b}}) = \vec{\mathbf{a}} \cdot \underbrace{\mathbf{Q}^T \cdot \mathbf{Q}}_{=\mathbf{1}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \\ \tilde{\mathrm{a}}_i \tilde{\mathrm{b}}_i &= (\mathrm{Q}_{ik}\mathrm{a}_k)(\mathrm{Q}_{ij}\mathrm{b}_j) = \mathrm{a}_k \underbrace{\mathrm{Q}_{ik}\mathrm{Q}_{ij}}_{\delta_{kj}} \mathrm{b}_j = \mathrm{a}_k \mathrm{b}_k \end{aligned} \tag{1.219}$$

So, it is also true when $\tilde{\vec{\mathbf{a}}} = \tilde{\vec{\mathbf{b}}}$, thus $\tilde{\vec{\mathbf{a}}} \cdot \tilde{\vec{\mathbf{a}}} = \left\|\tilde{\vec{\mathbf{a}}}\right\|^2 = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = \left\|\vec{\mathbf{a}}\right\|^2$. Therefore, we can conclude that in an orthogonal transformation, the magnitude vectors and the angle between them are maintained, (see Figure 1.18).

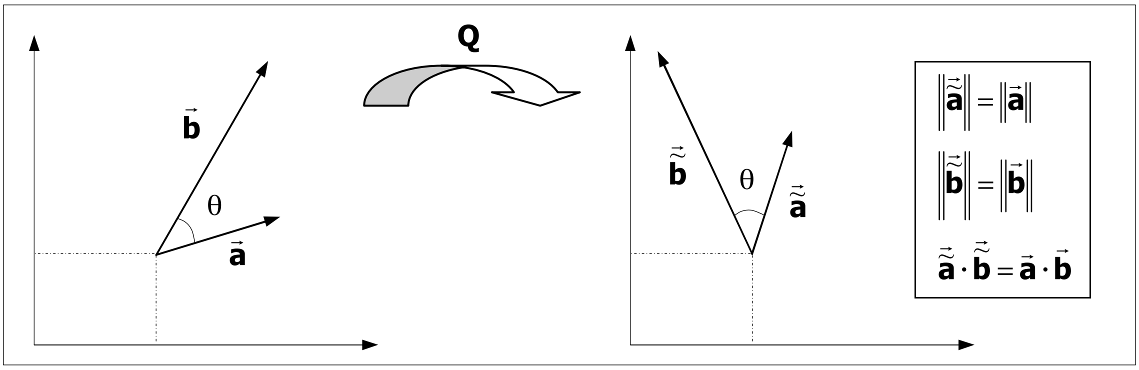


Figure 1.18: Orthogonal transformation applied to vectors.

1.5.2.9 Positive Definite Tensor, Negative Definite Tensor and Semi-Definite Tensors

A tensor is said to be *positive definite* when the following notations hold:

Tensorial notation	*Indicial notation*	*Matrix notation*	
$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} > 0$	$\mathrm{x}_i \mathrm{T}_{ij} \mathrm{x}_j > 0$	$x^T \mathcal{T}\, x > 0$	(1.220)

for all vectors $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Conversely, a tensor is said to be *negative definite* when these notations hold:

Tensorial notation	*Indicial notation*	*Matrix notation*	
$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} < 0$	$\mathrm{x}_i \mathrm{T}_{ij} \mathrm{x}_j < 0$	$x^T \mathcal{T}\, x < 0$	(1.221)

for all vectors $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$.

A tensor is said to be *semi-positive definite* if $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0$ for all vectors $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Similarly, we define a *semi-negative definite tensor* when the following holds: $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \leq 0$.

If $\alpha = \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \mathbf{T} : (\hat{\mathbf{x}} \otimes \hat{\mathbf{x}}) = \mathrm{T}_{ij} \mathrm{x}_i \mathrm{x}_j$, then the derivative of α with respect to $\vec{\mathbf{x}}$ is given by:

$$\frac{\partial \alpha}{\partial \mathrm{x}_k} = \mathrm{T}_{ij} \frac{\partial \mathrm{x}_i}{\partial \mathrm{x}_k} \mathrm{x}_j + \mathrm{T}_{ij} \mathrm{x}_i \frac{\partial \mathrm{x}_j}{\partial \mathrm{x}_k} = \mathrm{T}_{ij} \delta_{ik} \mathrm{x}_j + \mathrm{T}_{ij} \mathrm{x}_i \delta_{jk} = \mathrm{T}_{kj} \mathrm{x}_j + \mathrm{T}_{ik} \mathrm{x}_i = \left(\mathrm{T}_{ki} + \mathrm{T}_{ik} \right) \mathrm{x}_i \tag{1.222}$$

Thus, we can conclude that:

$$\frac{\partial \alpha}{\partial \hat{\mathbf{x}}} = 2\mathbf{T}^{sym} \cdot \hat{\mathbf{x}} \quad \Rightarrow \quad \frac{\partial^2 \alpha}{\partial \hat{\mathbf{x}} \otimes \partial \hat{\mathbf{x}}} = 2\mathbf{T}^{sym} \tag{1.223}$$

Remember that it is also true that $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{x}}$, therefore if the symmetric part of a tensor is positive definite the tensor is too.

NOTE: As we will demonstrate later, the eigenvalues must be positive for $\mathbf{T}$ to be positive definite. The proof is in the subsection “Spectral Representation of Tensors”. ■

Problem 1.25: Let $\boldsymbol{F}$ be an arbitrary second-order tensor. Show that the resulting tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are *symmetric tensors* and *semi-positive definite tensors*. Also check in what condition are $\boldsymbol{C}$ and $\boldsymbol{b}$ *positive definite tensors*.
Solution: Symmetry:

$$\boldsymbol{C}^T = (\boldsymbol{F}^T \cdot \boldsymbol{F})^T = \boldsymbol{F}^T \cdot (\boldsymbol{F}^T)^T = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C}$$
$$\boldsymbol{b}^T = (\boldsymbol{F} \cdot \boldsymbol{F}^T)^T = (\boldsymbol{F}^T)^T \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{b}$$

Thus, we have shown that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are symmetric tensors.

To prove that the tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are semi-positive definite tensors, we start with the definition of a semi-positive definite tensor, *i.e.*, a tensor $\mathbf{A}$ is semi-positive definite if $\hat{\mathbf{x}} \cdot \mathbf{A} \cdot \hat{\mathbf{x}} \geq 0$ holds, for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Thus:

$$\begin{aligned} \hat{\mathbf{x}} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot \hat{\mathbf{x}} &= \boldsymbol{F} \cdot \hat{\mathbf{x}} \cdot \boldsymbol{F} \cdot \hat{\mathbf{x}} \\ &= (\boldsymbol{F} \cdot \hat{\mathbf{x}}) \cdot (\boldsymbol{F} \cdot \hat{\mathbf{x}}) \\ &= \left\| \boldsymbol{F} \cdot \hat{\mathbf{x}} \right\|^2 \geq 0 \end{aligned} \qquad \begin{aligned} \hat{\mathbf{x}} \cdot (\boldsymbol{F} \cdot \boldsymbol{F}^T) \cdot \hat{\mathbf{x}} &= \hat{\mathbf{x}} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \hat{\mathbf{x}} \\ &= (\boldsymbol{F}^T \cdot \hat{\mathbf{x}}) \cdot (\boldsymbol{F}^T \cdot \hat{\mathbf{x}}) \\ &= \left\| \boldsymbol{F}^T \cdot \hat{\mathbf{x}} \right\|^2 \geq 0 \end{aligned}$$

Or in indicial notation:

$$\begin{aligned} \mathrm{x}_i C_{ij} \mathrm{x}_j &= \mathrm{x}_i (F_{ki} F_{kj}) \mathrm{x}_j \\ &= (F_{ki} \mathrm{x}_i)(F_{kj} \mathrm{x}_j) \\ &= \left\| F_{ki} \mathrm{x}_i \right\|^2 \geq 0 \end{aligned} \qquad \begin{aligned} \mathrm{x}_i b_{ij} \mathrm{x}_j &= \mathrm{x}_i (F_{ik} F_{jk}) \mathrm{x}_j \\ &= (F_{ik} \mathrm{x}_i)(F_{jk} \mathrm{x}_j) \\ &= \left\| F_{ik} \mathrm{x}_i \right\|^2 \geq 0 \end{aligned}$$

Thus, we proved that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are semi-positive definite tensors. Note that $\hat{\mathbf{x}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{x}} = \left\| \boldsymbol{F} \cdot \hat{\mathbf{x}} \right\|^2$ equals zero, when $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$, if $\boldsymbol{F} \cdot \hat{\mathbf{x}} = \vec{\mathbf{0}}$. Furthermore, by definition $\boldsymbol{F} \cdot \hat{\mathbf{x}} = \vec{\mathbf{0}}$ with $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$ if and only if $\det(\boldsymbol{F}) = 0$, (see **Problem 1.24**). Then, the tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are positive definite if and only if $\det(\boldsymbol{F}) \neq 0$.

1.5.2.10 Additive Decomposition of Tensors

Given two arbitrary tensors $\mathbf{S}$, $\mathbf{T} \neq \mathbf{0}$, and a scalar α, we can represent $\mathbf{S}$ by means of the following additive decomposition of tensors:

$$\mathbf{S} = \alpha \mathbf{T} + \mathbf{U} \qquad \textit{where} \qquad \mathbf{U} = \mathbf{S} - \alpha \mathbf{T} \tag{1.224}$$

Note that, depending on the value of α, we have an infinite number of possibilities for representing $\mathbf{S}$. But, if $\mathrm{Tr}(\mathbf{T} \cdot \mathbf{U}^T) = \mathrm{Tr}(\mathbf{U} \cdot \mathbf{T}^T) = 0$, the additive decomposition is unique. From the relationship in (1.224), we can evaluate the value of α as follows:

$$\mathbf{S} \cdot \mathbf{T}^T = \alpha \mathbf{T} \cdot \mathbf{T}^T + \mathbf{U} \cdot \mathbf{T}^T \Rightarrow \mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T) = \alpha \mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T) + \underbrace{\mathrm{Tr}(\mathbf{U} \cdot \mathbf{T}^T)}_{=0} = \alpha \mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T) \tag{1.225}$$

$$\Rightarrow \qquad \alpha = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T)}{\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T)} \tag{1.226}$$

For example, let us suppose that $\mathbf{T} = \mathbf{1}$. In this case α is evaluated as follows:

$$\alpha = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T)}{\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T)} = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{1})}{\mathrm{Tr}(\mathbf{1} \cdot \mathbf{1})} = \frac{\mathrm{Tr}(\mathbf{S})}{\mathrm{Tr}(\mathbf{1})} = \frac{\mathrm{Tr}(\mathbf{S})}{3} \tag{1.227}$$

We can then define $\mathbf{U}$ as:

$$\mathbf{U} = \mathbf{S} - \alpha \mathbf{T} = \mathbf{S} - \frac{\mathrm{Tr}(\mathbf{S})}{3} \mathbf{1} \equiv \mathbf{S}^{dev} \tag{1.228}$$

Thus:

$$\mathbf{S} = \frac{\mathrm{Tr}(\mathbf{S})}{3} \mathbf{1} + \mathbf{S}^{dev} = \mathbf{S}^{sph} + \mathbf{S}^{dev} \tag{1.229}$$

NOTE: $\boldsymbol{\mathsf{S}}^{sph} = \dfrac{\mathrm{Tr}(\boldsymbol{\mathsf{S}})}{3}\boldsymbol{1}$ is the *spherical* part of the tensor $\boldsymbol{\mathsf{S}}$, and $\boldsymbol{\mathsf{S}}^{dev} = \boldsymbol{\mathsf{S}} - \dfrac{\mathrm{Tr}(\boldsymbol{\mathsf{S}})}{3}\boldsymbol{1}$ is known as a *deviatoric tensor*. ■

Now suppose that $\boldsymbol{\mathsf{T}}$ is given by $\boldsymbol{\mathsf{T}} = \frac{1}{2}(\boldsymbol{\mathsf{S}} + \boldsymbol{\mathsf{S}}^T)$ then α can be evaluated as follows:

$$\alpha = \frac{\mathrm{Tr}(\boldsymbol{\mathsf{S}}\cdot\boldsymbol{\mathsf{T}}^T)}{\mathrm{Tr}(\boldsymbol{\mathsf{T}}\cdot\boldsymbol{\mathsf{T}}^T)} = \frac{\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\mathsf{S}}\cdot(\boldsymbol{\mathsf{S}}+\boldsymbol{\mathsf{S}}^T)^T\right]}{\frac{1}{4}\mathrm{Tr}\left[(\boldsymbol{\mathsf{S}}+\boldsymbol{\mathsf{S}}^T)\cdot(\boldsymbol{\mathsf{S}}+\boldsymbol{\mathsf{S}}^T)^T\right]} = 1 \tag{1.230}$$

We can then define $\boldsymbol{\mathsf{U}}$ as $\boldsymbol{\mathsf{U}} = \boldsymbol{\mathsf{S}} - \alpha\boldsymbol{\mathsf{T}} = \boldsymbol{\mathsf{S}} - \frac{1}{2}(\boldsymbol{\mathsf{S}} + \boldsymbol{\mathsf{S}}^T) = \frac{1}{2}(\boldsymbol{\mathsf{S}} - \boldsymbol{\mathsf{S}}^T)$. Then we obtain $\boldsymbol{\mathsf{S}}$ represented by the additive decomposition as follows:

$$\boldsymbol{\mathsf{S}} = \frac{1}{2}(\boldsymbol{\mathsf{S}} + \boldsymbol{\mathsf{S}}^T) + \frac{1}{2}(\boldsymbol{\mathsf{S}} - \boldsymbol{\mathsf{S}}^T) = \boldsymbol{\mathsf{S}}^{sym} + \boldsymbol{\mathsf{S}}^{skew} \tag{1.231}$$

which is the same as the equation obtained in (1.150) in which we split the tensor into symmetric and antisymmetric parts.

Problem 1.26: Find a fourth-order tensor $\mathbb{P}$ so that $\mathbb{P} : \boldsymbol{\mathsf{A}} = \boldsymbol{\mathsf{A}}^{dev}$, where $\boldsymbol{\mathsf{A}}$ is a second-order tensor.
Solution: Taking into account the additive decomposition into spherical and deviatoric parts, we obtain:

$$\boldsymbol{\mathsf{A}} = \boldsymbol{\mathsf{A}}^{sph} + \boldsymbol{\mathsf{A}}^{dev} = \frac{\mathrm{Tr}(\boldsymbol{\mathsf{A}})}{3}\boldsymbol{1} + \boldsymbol{\mathsf{A}}^{dev} \quad \Rightarrow \quad \boldsymbol{\mathsf{A}}^{dev} = \boldsymbol{\mathsf{A}} - \frac{\mathrm{Tr}(\boldsymbol{\mathsf{A}})}{3}\boldsymbol{1}$$

Referring to the definition of fourth-order unit tensors seen in (1.172), and (1.174), where the relations $\bar{\bar{\mathbb{I}}} : \boldsymbol{\mathsf{A}} = \mathrm{Tr}(\boldsymbol{\mathsf{A}})\boldsymbol{1}$ and $\mathbb{I} : \boldsymbol{\mathsf{A}} = \boldsymbol{\mathsf{A}}$ hold, we can now state:

$$\boldsymbol{\mathsf{A}}^{dev} = \boldsymbol{\mathsf{A}} - \frac{\mathrm{Tr}(\boldsymbol{\mathsf{A}})}{3}\boldsymbol{1} = \mathbb{I} : \boldsymbol{\mathsf{A}} - \frac{1}{3}\bar{\bar{\mathbb{I}}} : \boldsymbol{\mathsf{A}} = \left(\mathbb{I} - \frac{1}{3}\bar{\bar{\mathbb{I}}}\right) : \boldsymbol{\mathsf{A}} = \left(\mathbb{I} - \frac{1}{3}\boldsymbol{1}\otimes\boldsymbol{1}\right) : \boldsymbol{\mathsf{A}}$$

Therefore, we can conclude that:

$$\boxed{\mathbb{P} = \mathbb{I} - \frac{1}{3}\boldsymbol{1}\otimes\boldsymbol{1}}$$

The tensor $\mathbb{P}$ is known as a *fourth-order projection tensor*, Holzapfel(2000).

1.5.3 Transformation Law of the Tensor Components

The tensor components depend on the coordinate system, so, if the coordinate system is changed due to a rotation so do the tensor components. The tensor components between these coordinate systems are interrelated to each other by the component transformation law, which is defined below, (see Figure 1.19).

Consider a Cartesian coordinate system (x_1, x_2, x_3) formed by the orthogonal basis $(\hat{\boldsymbol{\mathsf{e}}}_1, \hat{\boldsymbol{\mathsf{e}}}_2, \hat{\boldsymbol{\mathsf{e}}}_3)$, (see Figure 1.20). In this system, an arbitrary vector $\vec{\boldsymbol{\mathsf{v}}}$ is represented by its components as follows:

$$\vec{\boldsymbol{\mathsf{v}}} = \mathsf{v}_i\hat{\boldsymbol{\mathsf{e}}}_i = \mathsf{v}_1\hat{\boldsymbol{\mathsf{e}}}_1 + \mathsf{v}_2\hat{\boldsymbol{\mathsf{e}}}_2 + \mathsf{v}_3\hat{\boldsymbol{\mathsf{e}}}_3 \tag{1.232}$$

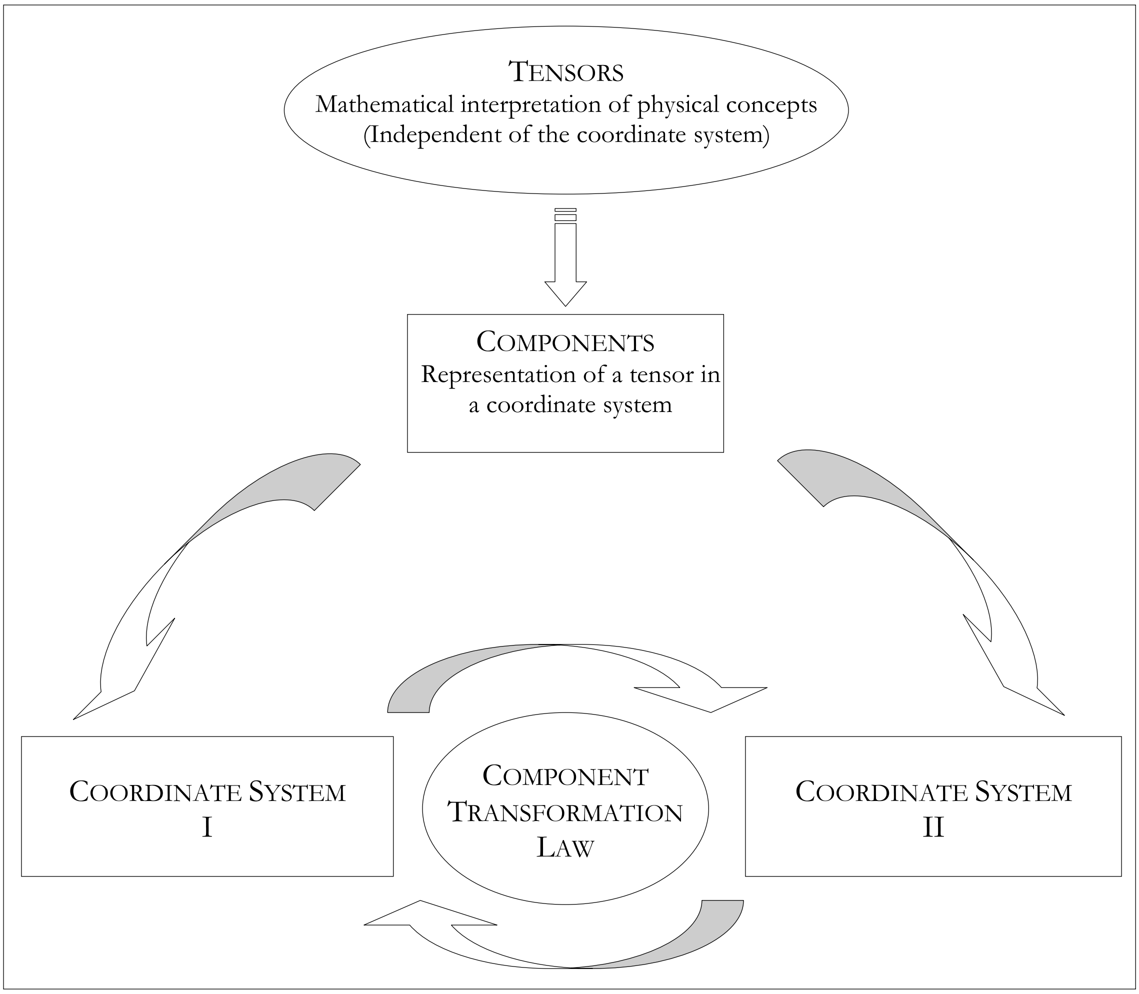


Figure 1.19: Transformation law of the tensor components.

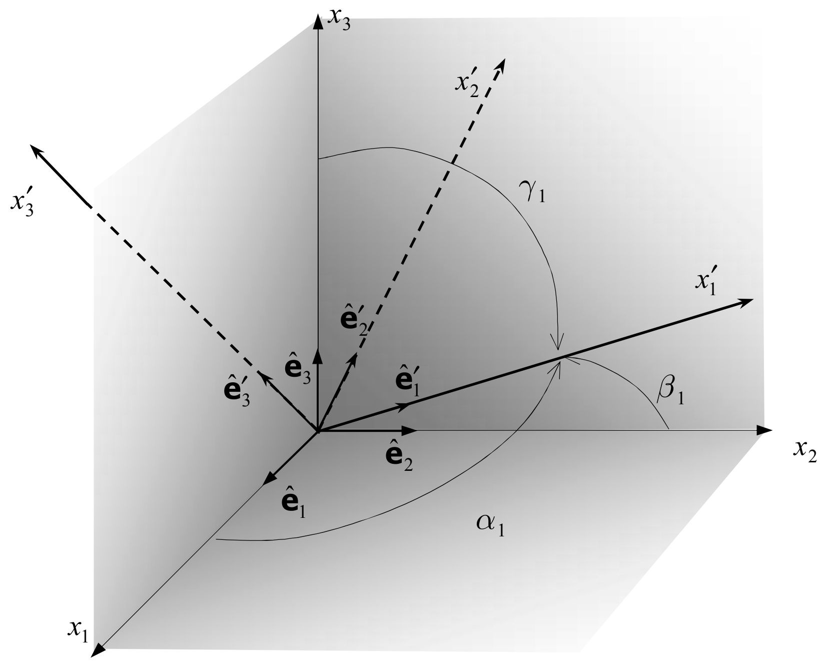


Figure 1.20: Rotation of the Cartesian system.

The components, v_i, are represented in matrix form as:

$$(\vec{\mathbf{v}})_i = \mathrm{v}_i = \boldsymbol{v} = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix} \tag{1.233}$$

Now consider a new coordinate system (x_1', x_2', x_3') represented by the orthogonal basis $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3')$, (see Figure 1.20). In this new system, the vector $\vec{\mathbf{v}}$ is represented by $\mathrm{v}_j'\hat{\mathbf{e}}_j'$. As mentioned before, a tensor is independent of the adopted system, so:

$$\vec{\mathbf{v}} = \mathrm{v}_k'\hat{\mathbf{e}}_k' = \mathrm{v}_j\hat{\mathbf{e}}_j \tag{1.234}$$

To obtain the components of a tensor in a given system one need only make the dot product between the tensor and the system basis:

$$\begin{aligned} \mathrm{v}_k'\hat{\mathbf{e}}_k' \cdot \hat{\mathbf{e}}_i' &= (\mathrm{v}_j\hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_i' \\ \mathrm{v}_k'\delta_{ki} &= (\mathrm{v}_j\hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_i' \\ \mathrm{v}_i' &= (\mathrm{v}_1\hat{\mathbf{e}}_1 + \mathrm{v}_2\hat{\mathbf{e}}_2 + \mathrm{v}_3\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}_i' \end{aligned} \tag{1.235}$$

Or in matrix form:

$$\begin{bmatrix} \mathrm{v}_1' \\ \mathrm{v}_2' \\ \mathrm{v}_3' \end{bmatrix} = \begin{bmatrix} (\mathrm{v}_1\hat{\mathbf{e}}_1 + \mathrm{v}_2\hat{\mathbf{e}}_2 + \mathrm{v}_3\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}_1' \\ (\mathrm{v}_1\hat{\mathbf{e}}_1 + \mathrm{v}_2\hat{\mathbf{e}}_2 + \mathrm{v}_3\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}_2' \\ (\mathrm{v}_1\hat{\mathbf{e}}_1 + \mathrm{v}_2\hat{\mathbf{e}}_2 + \mathrm{v}_3\hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}_3' \end{bmatrix} \tag{1.236}$$

After restructuring, the previous equation looks like:

$$\begin{bmatrix} \mathrm{v}_1' \\ \mathrm{v}_2' \\ \mathrm{v}_3' \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1' \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2' \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3' \end{bmatrix} \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix} \quad \therefore \quad a_{ij} = \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i' = \hat{\mathbf{e}}_i' \cdot \hat{\mathbf{e}}_j \tag{1.237}$$

Or in indicial notation:

$$\mathrm{v}_i' = a_{ij}\mathrm{v}_j \tag{1.238}$$

where we have introduced the transformation matrix $\mathcal{A} \equiv a_{ij}$ as:

$$\begin{gathered} \mathcal{A} \equiv a_{ij} = \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1' \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2' \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3' & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3' & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3' \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}_1' \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1' \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_1' \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_2' \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2' \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2' \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_3' \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3' \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3' \cdot \hat{\mathbf{e}}_3 \end{bmatrix} \\ a_{ij} \equiv \mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{gathered} \tag{1.239}$$

The matrix ($\mathcal{A}$) is not symmetric, *i.e.* $\mathcal{A} \neq \mathcal{A}^T$. With reference to the scalar product $\hat{\mathbf{e}}_i' \cdot \hat{\mathbf{e}}_j = \|\hat{\mathbf{e}}_i'\| \|\hat{\mathbf{e}}_j\| \cos(x_i', x_j) = \cos(x_i', x_j)$, (see equation (1.4)), the relationship in (1.237) is expressed by means of the direction cosines as:

$$\underbrace{\begin{bmatrix} \mathsf{v}'_1 \\ \mathsf{v}'_2 \\ \mathsf{v}'_3 \end{bmatrix}}_{\boldsymbol{v}'} = \underbrace{\begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) & \cos(x'_1, x_3) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) & \cos(x'_2, x_3) \\ \cos(x'_3, x_1) & \cos(x'_3, x_2) & \cos(x'_3, x_3) \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \mathsf{v}_3 \end{bmatrix}}_{\boldsymbol{v}} \qquad (1.240)$$

$$\boxed{\boldsymbol{v}' = \mathcal{A}\boldsymbol{v}}$$

The direction cosines of a vector are those of the angles between the vector and the three coordinate axes. According to Figure 1.20 we can verify that $\cos\alpha_1 = \cos(x'_1, x_1)$, $\cos\beta_1 = \cos(x'_1, x_2)$ and $\cos\gamma_1 = \cos(x'_1, x_3)$.

In the equation (1.235) we have projected the vector onto $\hat{\mathbf{e}}'_i$. Now, we can project the vector onto $\hat{\mathbf{e}}_i$:

$$\begin{aligned} \mathsf{v}_k \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i &= \mathsf{v}'_j \hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i \\ \mathsf{v}_k \delta_{ki} &= \mathsf{v}'_j a_{ji} \\ \mathsf{v}_i &= \mathsf{v}'_j a_{ji} \\ \boldsymbol{v} &= \mathcal{A}^T \boldsymbol{v}' \end{aligned} \qquad (1.241)$$

Therefore, it is also true that:

$$\boxed{\hat{\mathbf{e}}_i = a_{ji} \hat{\mathbf{e}}'_j} \qquad (1.242)$$

The inverse relationship of equation (1.240) is obtained as follows:

$$\mathcal{A}^{-1}\boldsymbol{v}' = \mathcal{A}^{-1}\mathcal{A}\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{v} = \mathcal{A}^{-1}\boldsymbol{v}' \qquad (1.243)$$

and by comparing the equations (1.243) with (1.241) we can conclude that the matrix $\mathcal{A}$ is an orthogonal matrix, *i.e.*:

$$\mathcal{A}^{-1} = \mathcal{A}^T \quad \Rightarrow \quad \mathcal{A}^T\mathcal{A} = \boldsymbol{1} \xrightarrow[]{\text{Indicial notation}} a_{ki}a_{kj} = \delta_{ij} \qquad (1.244)$$

<u>Second-order tensor</u>

Consider a coordinate system formed by the orthogonal basis $\hat{\mathbf{e}}_i$ then, how the basis changes from the $\hat{\mathbf{e}}_i$ system to a new one represented by the orthogonal basis $\hat{\mathbf{e}}'_i$. This is illustrated in transformation law as $\hat{\mathbf{e}}_k = a_{ik}\hat{\mathbf{e}}'_i$, which allow us to represent a second-order tensor $\mathbf{T}$ as follows:

$$\begin{aligned} \mathbf{T} &= \mathsf{T}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \\ &= \mathsf{T}_{kl} a_{ik} \hat{\mathbf{e}}'_i \otimes a_{jl} \hat{\mathbf{e}}'_j \\ &= \mathsf{T}_{kl} a_{ik} a_{jl} \, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \\ &= \mathsf{T}'_{ij} \, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \end{aligned} \qquad (1.245)$$

Then, the transformation law of the components between systems for a second-order tensor is given by:

$$\mathsf{T}'_{ij} = \mathsf{T}_{kl} a_{ik} a_{jl} = a_{ik} \mathsf{T}_{kl} a_{jl} \xrightarrow{\text{Matrix form}} \mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \qquad (1.246)$$

<u>Third-order tensor</u>

A third-order tensor ($\mathbf{S}$) can be shown in two systems represented by orthogonal bases $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ as follows:

$$\begin{aligned}\mathbf{S} &= \mathsf{S}_{lmn}\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_m \otimes \hat{\mathbf{e}}_n \\ &= \mathsf{S}_{lmn} a_{il}\hat{\mathbf{e}}'_i \otimes a_{jm}\hat{\mathbf{e}}'_j \otimes a_{kn}\hat{\mathbf{e}}'_k \\ &= \mathsf{S}_{lmn} a_{il} a_{jm} a_{kn}\; \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \otimes \hat{\mathbf{e}}'_k \\ &= \mathsf{S}'_{ijk}\; \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \otimes \hat{\mathbf{e}}'_k\end{aligned} \tag{1.247}$$

In conclusion the components of the third-order tensor in the new basis ($\hat{\mathbf{e}}'_i$) are:

$$\mathsf{S}'_{ijk} = \mathsf{S}_{lmn} a_{il} a_{jm} a_{kn} \tag{1.248}$$

The following table summarizes the transformation law of the components according to the tensor rank:

rank	from $(x_1, x_2, x_3) \xrightarrow{to} (x'_1, x'_2, x'_3)$	from $(x'_1, x'_2, x'_3) \xrightarrow{to} (x_1, x_2, x_3)$
0 (scalar)	$\lambda' = \lambda$	$\lambda = \lambda'$
1 (vector)	$S'_i = a_{ij} S_j$	$S_i = a_{ji} S'_j$
2	$S'_{ij} = a_{ik} a_{jl} S_{kl}$	$S_{ij} = a_{ki} a_{lj} S'_{kl}$
3	$S'_{ijk} = a_{il} a_{jm} a_{kn} S_{lmn}$	$S_{ijk} = a_{li} a_{mj} a_{nk} S'_{lmn}$
4	$S'_{ijkl} = a_{im} a_{jn} a_{kp} a_{lq} S_{mnpq}$	$S_{ijkl} = a_{mi} a_{nj} a_{pk} a_{ql} S'_{mnpq}$

(1.249)

Problem 1.27: Obtain the components of $\mathbf{T}'$, given by the transformation:

$$\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T$$

where the components of $\mathbf{T}$ and $\mathbf{A}$ are shown, respectively, as T_{ij} and a_{ij}. Afterwards, given that a_{ij} are the components of the transformation matrix, represent graphically the components of the tensors $\mathbf{T}$ and $\mathbf{T}'$ on both systems.

Solution: The expression $\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T$ in symbolic notation is given by:

$$\begin{aligned}\mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) &= a_{rs}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_s) \cdot \mathsf{T}_{pq}(\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) \cdot a_{kl}(\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k) \\ &= a_{rs}\mathsf{T}_{pq} a_{kl}\delta_{sp}\delta_{ql}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) \\ &= a_{rp}\mathsf{T}_{pq} a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k)\end{aligned}$$

To obtain the components of $\mathbf{T}'$ one only need make the double scalar product with the basis $(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$, the result of which is:

$$\begin{aligned}&\mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = a_{rp}\mathsf{T}_{pq} a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &\mathsf{T}'_{ab}\delta_{ai}\delta_{bj} = a_{rp}\mathsf{T}_{pq} a_{kq}\delta_{ri}\delta_{kj} \\ &\mathsf{T}'_{ij} = a_{ip}\mathsf{T}_{pq} a_{jq}\end{aligned}$$

The above eqaution is shown in matrix notation as:

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \xrightarrow{inverse} \mathcal{T} = \mathcal{A}^{-1}\,\mathcal{T}'\,\mathcal{A}^{-T}$$

Since $\mathcal{A}$ is an orthogonal matrix, it holds that $\mathcal{A}^T = \mathcal{A}^{-1}$. Thus, $\mathcal{T} = \mathcal{A}^T\,\mathcal{T}'\,\mathcal{A}$. The graphical representation of the tensor components in both systems can be seen in Figure 1.21.

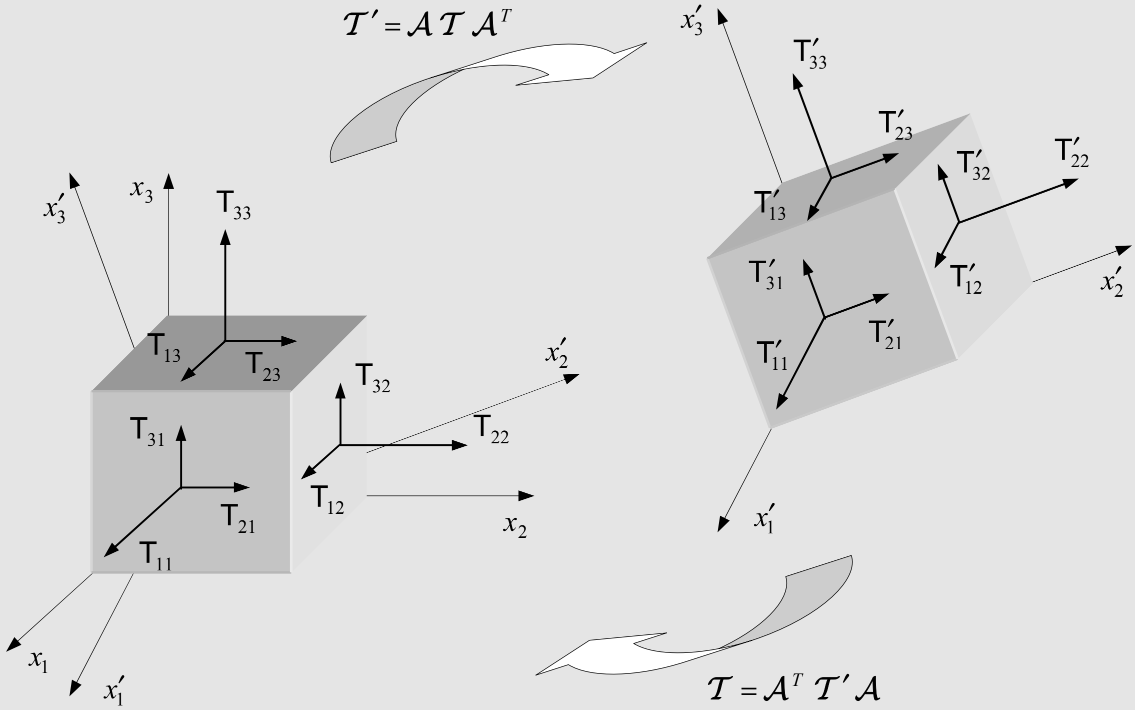


Figure 1.21: Transformation law of the second-order tensor components.

Problem 1.28: Let $\mathbf{T}$ be a symmetric second-order tensor and $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ be scalars, where:

$$\boxed{I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = \mathsf{T}_{ii} \quad ; \quad II_{\mathbf{T}} = \frac{1}{2}\left\{ I_{\mathbf{T}}^{\,2} - \mathrm{Tr}(\mathbf{T}^2) \right\} \quad ; \quad III_{\mathbf{T}} = \det(\mathbf{T})}$$

Show that $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ are invariant with a change of basis.

Solution:

a) Taking into account the transformation law for the second-order tensor components given in (1.249), *i.e.* $\mathsf{T}'_{ij} = a_{ik} a_{jl} \mathsf{T}_{kl}$ or in matrix form $\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T$. Then, T'_{ii} is:

$$\mathsf{T}'_{ii} = a_{ik} a_{il} \mathsf{T}_{kl} = \delta_{kl} \mathsf{T}_{kl} = \mathsf{T}_{kk} = I_{\mathbf{T}}$$

Hence we have proved that $I_{\mathbf{T}}$ is independent of the adopted system.

b) To prove that $II_{\mathbf{T}}$ is an invariant, one only need show that $\mathrm{Tr}(\mathbf{T}^2)$ is one also, since $I_{\mathbf{T}}^{\,2}$ is already an invariant.

$$\begin{aligned}
\mathrm{Tr}(\mathbf{T}'^2) = \mathrm{Tr}(\mathbf{T}' \cdot \mathbf{T}') = \mathbf{T}' : \mathbf{T}' = \mathsf{T}'_{ij}\mathsf{T}'_{ij} &= (a_{ik} a_{jl} \mathsf{T}_{kl})(a_{ip} a_{jq} \mathsf{T}_{pq}) \\
&= \underbrace{a_{ik} a_{ip}}_{\delta_{kp}} \underbrace{a_{jl} a_{jq}}_{\delta_{lq}} \mathsf{T}_{kl} \mathsf{T}_{pq} \\
&= \mathsf{T}_{pl} \mathsf{T}_{pl} \\
&= \mathbf{T} : \mathbf{T} = \mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}) = \mathrm{Tr}(\mathbf{T}^2)
\end{aligned}$$

c)

$$\det(\mathbf{T}') = \det(\mathcal{T}') = \det(\mathcal{A}\,\mathcal{T}\,\mathcal{A}^T) = \underbrace{\det(\mathcal{A})}_{=1} \det(\mathcal{T}) \underbrace{\det(\mathcal{A}^T)}_{=1} = \det(\mathcal{T})$$

Consider now four sets of coordinate systems, represented by (x_1, x_2, x_3), (x'_1, x'_2, x'_3), (x''_1, x''_2, x''_3) and (x'''_1, x'''_2, x'''_3), (see Figure 1.22), and consider also the following transformation matrices:

$\mathcal{A}$: Transformation matrix from (x_1, x_2, x_3) to (x'_1, x'_2, x'_3);

$\mathcal{B}$: Transformation matrix from (x_1', x_2', x_3') to (x_1'', x_2'', x_3'');

$\mathcal{C}$: Transformation matrix from (x_1'', x_2'', x_3'') to (x_1''', x_2''', x_3''').

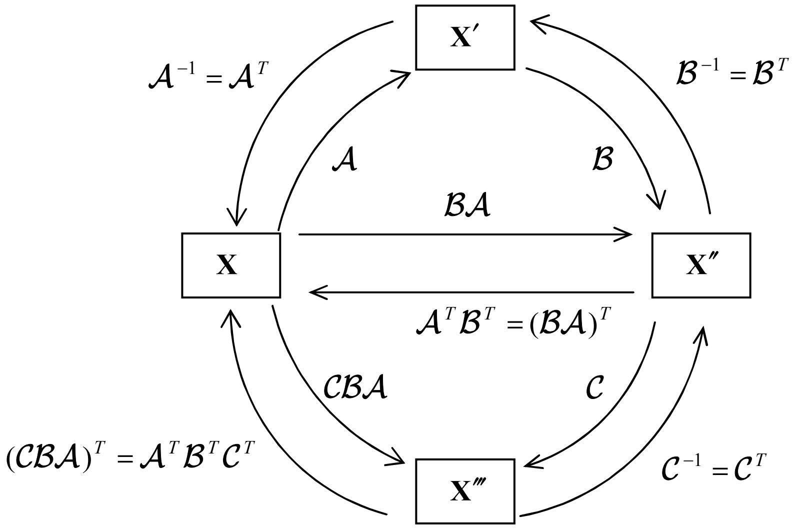


Figure 1.22: Transformations matrices between several systems.

If we consider a $\boldsymbol{v}$ column matrix made up of components of $\vec{\mathbf{v}}$ in the coordinate system (x_1, x_2, x_3), the components of this vector in the system (x_1', x_2', x_3') are given by:

$$\boldsymbol{v}' = \mathcal{A}\boldsymbol{v} \tag{1.250}$$

and the inverse transformation of relation (1.250) is:

$$\boldsymbol{v} = \mathcal{A}^T\boldsymbol{v}' \tag{1.251}$$

Now, starting with the system (x_1', x_2', x_3'), the components of the vector in the system (x_1'', x_2'', x_3'') are given by:

$$\boldsymbol{v}'' = \mathcal{B}\boldsymbol{v}' \tag{1.252}$$

and the inverse transformation is:

$$\boldsymbol{v}' = \mathcal{B}^T\boldsymbol{v}'' \tag{1.253}$$

By substituting the equation (1.250) into (1.252) we obtain:

$$\boldsymbol{v}'' = \mathcal{BA}\boldsymbol{v} \tag{1.254}$$

The resulting matrix $\mathcal{BA}$ is also an orthogonal matrix, and shows the transformation matrix from (x_1, x_2, x_3) to (x_1'', x_2'', x_3''), (see Figure 1.22). The inverse form of (1.254) is evaluated by substituting (1.253) into (1.251), the result of which is:

$$\boldsymbol{v} = \mathcal{A}^T\mathcal{B}^T\boldsymbol{v}'' \tag{1.255}$$

This equation could have been obtained by using equation (1.254), *i.e.*:

$$(\mathcal{BA})^{-1}\boldsymbol{v}'' = (\mathcal{BA})^{-1}(\mathcal{BA})\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{v} = (\mathcal{BA})^{-1}\boldsymbol{v}'' = \mathcal{A}^{-1}\mathcal{B}^{-1}\boldsymbol{v}'' = \mathcal{A}^T\mathcal{B}^T\boldsymbol{v}'' \tag{1.256}$$

Then, it is easy to find the components of the vector in the coordinate system (x_1''', x_2''', x_3'''), (see Figure 1.22):

$$\boldsymbol{v}''' = \mathcal{CBA}\boldsymbol{v} \quad \xrightarrow{\text{inverse form}} \quad \boldsymbol{v} = \mathcal{A}^T\mathcal{B}^T\mathcal{C}^T\boldsymbol{v}''' \tag{1.257}$$

1.5.3.1 Component Transformation Law in Two Dimensions (2D)

Now, consider two sets of coordinate systems, shown in Figure 1.23.

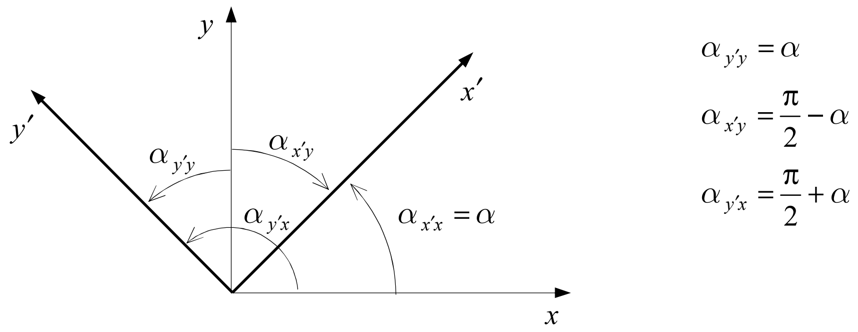


Figure 1.23: Transformation of a coordinate system in 2D.

The transformation matrix from $(x - y)$ to $(x' - y')$ is given by direction cosines, (see Figure 1.23), as:

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha_{x'x}) & \cos(\alpha_{x'y}) & 0 \\ \cos(\alpha_{y'x}) & \cos(\alpha_{y'y}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.258}$$

By using trigonometric identities we can deduce that:

$$\begin{aligned} &\alpha_{x'x} = \alpha_{y'y} \Rightarrow \cos(\alpha_{x'x}) = \cos(\alpha_{y'y}) = \cos(\alpha),\ \cos(\alpha_{x'y}) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha), \\ &\cos(\alpha_{y'x}) = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) \end{aligned} \tag{1.259}$$

Thus, the transformation matrix in 2D is dependent on a single parameter, α, *i.e.*:

$$\mathcal{A} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{1.260}$$

Another way to prove (1.260) is by considering the vector position of the point P in both systems, (see Figure 1.24).

Moreover, in view of Figure 1.24, said coordinates are interrelated as shown below:

$$\begin{gathered} \begin{cases} x'_P = x_P \cos(\alpha) + y_P \cos(\beta) \\ y'_P = -x_P \cos(\beta) + y_P \cos\left(\frac{\pi}{2} - \beta\right) \end{cases} \Rightarrow \begin{cases} x'_P = x_P \cos(\alpha) + y_P \cos\left(\frac{\pi}{2} - \alpha\right) \\ y'_P = -x_P \cos\left(\frac{\pi}{2} - \alpha\right) + y_P \cos(\alpha) \end{cases} \\ \Rightarrow \begin{cases} x'_P = x_P \cos(\alpha) + y_P \sin(\alpha) \\ y'_P = -x_P \sin(\alpha) + y_P \cos(\alpha) \end{cases} \end{gathered} \tag{1.261}$$

Or in matrix form:

$$\begin{bmatrix} x'_P \\ y'_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x_P \\ y_P \end{bmatrix} \xrightarrow[\textit{transformation}]{\textit{Inverse}} \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}^{-1} \begin{bmatrix} x'_P \\ y'_P \end{bmatrix} \tag{1.262}$$

Since $\mathcal{A}^{-1} = \mathcal{A}^T$, it is true that:

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x'_P \\ y'_P \end{bmatrix} \tag{1.263}$$

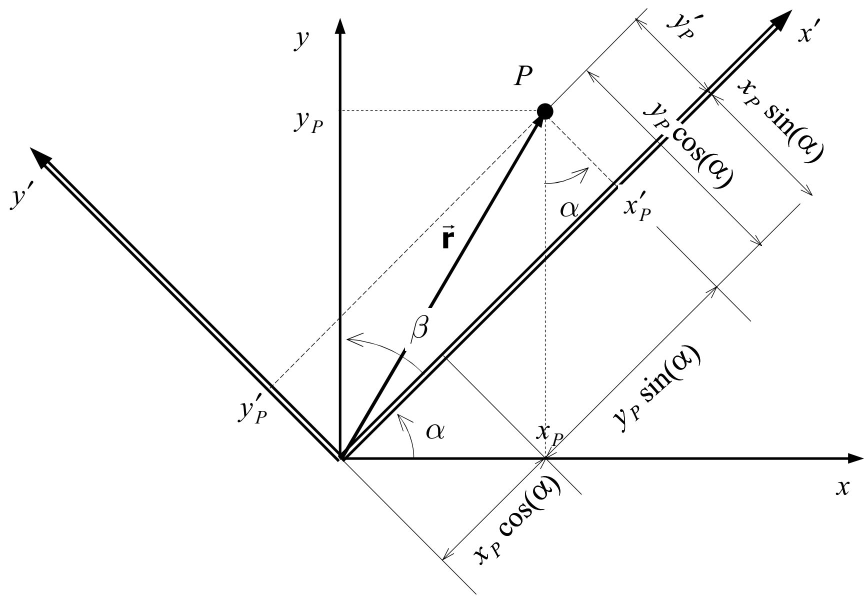


Figure 1.24: Transformation of a coordinate system in 2D.

Problem 1.29: Find the transformation matrix between the systems: x, y, z and x''', y''', z'''. These systems are represented in Figure 1.25.

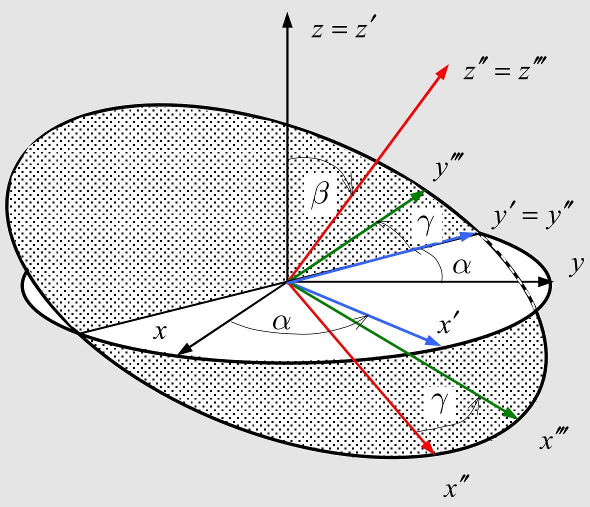
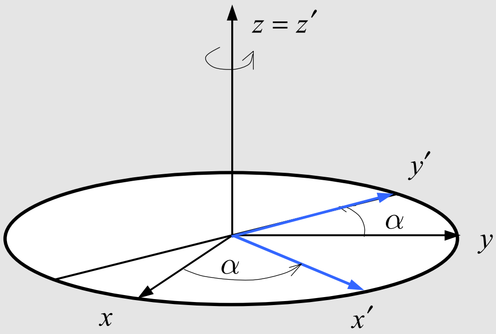


Figure 1.25: Rotation.

Solution: The coordinate system x''', y''', z''' can be obtained by different combinations of rotations as follows:

- Rotation along the z-axis

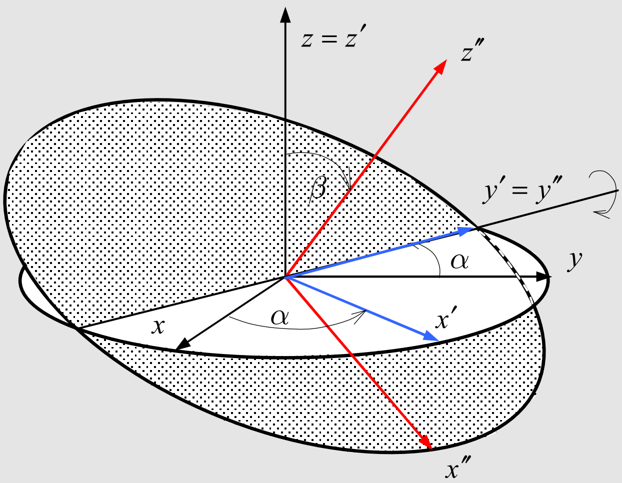


from x, y, z to x', y', z'

$$\mathcal{A} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $0 \le \alpha \le 360^{\circ}$

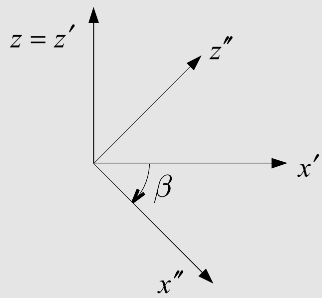
- Rotation along the y'-axis



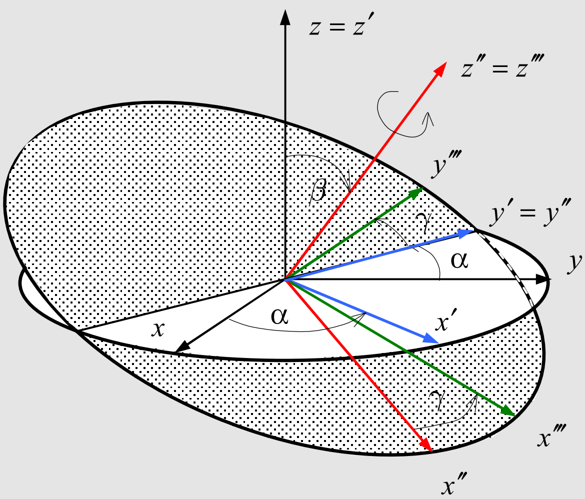
from x', y', z' to x'', y'', z''

$$\mathcal{B} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

with $0 \le \beta \le 180^{\circ}$



- Rotation along the z''-axis



from x'', y'', z'' to x''', y''', z'''

$$\mathcal{C} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $0 \le \gamma \le 360^{\circ}$

The transformation matrix from (x, y, z) to (x''', y''', z'''), (see Figure 1.22), is given by:

$$\mathcal{D} = \mathcal{C}\mathcal{B}\mathcal{A}$$

After multiplying the matrices, we obtain:

$$\mathcal{D} = \begin{bmatrix} (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma) & (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma) & -\sin\beta\cos\gamma \\ (-\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma) & (-\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma) & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix}$$

The angles α, β, γ are known as *Euler angles* and were introduced by Leonhard Euler to describe the orientation of a rigid body motion.

Problem 1.30: Let $\mathbf{T}$ be a second-order tensor whose components in the Cartesian system (x_1, x_2, x_3) are given by:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given that the transformation matrix between two systems, (x_1, x_2, x_3)- (x_1', x_2', x_3'), is:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Obtain the tensor components T_{ij} in the new coordinate system (x_1', x_2', x_3').

Solution: As defined in equation (1.249), the transformation law for second-order tensor components is:

$$\mathsf{T}'_{ij} = a_{ik} a_{jl} \mathsf{T}_{kl}$$

To enable the previous calculation to be carried out in matrix form we use:

$$\mathsf{T}'_{ij} = [a_{i\underbrace{k}}] [\mathsf{T}_{\underbrace{k\,l}}] [a_{\underbrace{l}\,j}]^T$$

Thus

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T$$

$$\mathcal{T}' = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

On carrying out the operation of the previous matrices we now have:

$$\mathcal{T}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

NOTE: As we can verify in the above example, the components of the tensor $\mathbf{T}$, in the new basis, have one particular feature, *i.e.* the off-diagonal terms are equal to zero. The question now is: Given an arbitrary tensor $\mathbf{T}$, is there a transformation which results in the off-diagonal terms being zero? This type of problem is called the *eigenvalue and eigenvector* problem. ■

1.5.4 Eigenvalue and Eigenvector Problem

As we have seen, the scalar product between a second-order tensor $\mathbf{T}$ and a vector (or unit vector $\hat{\mathbf{n}}'$) leads to a vector. In other words, projecting a second-order tensor onto a certain direction results in a vector that does not necessarily have the same direction as $\hat{\mathbf{n}}'$, (see Figure 1.26(a)).

The aim of the eigenvalue and eigenvector problem is to find a direction $\hat{\mathbf{n}}$, in such a way that the resulting vector, $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \mathbf{T} \cdot \hat{\mathbf{n}}$, coincides with it, (see Figure 1.26 (b)).

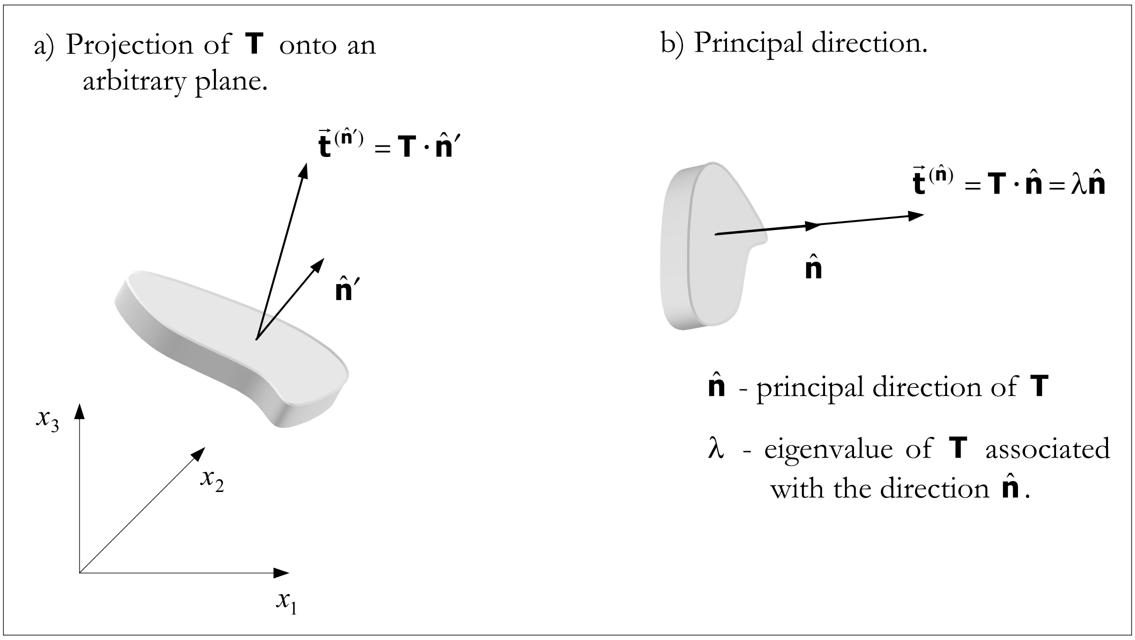


Figure 1.26: Projecting a tensor onto a direction.

Let $\mathbf{T}$ be a second-order tensor. A vector $\hat{\mathbf{n}}$ is said to be *eigenvector* of $\mathbf{T}$ if there is a scalar λ, called the *eigenvalue*, so that:

$$\mathbf{T} \cdot \hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \tag{1.264}$$

The equation (1.264) can be rearranged in indicial notation as:

$$\begin{aligned} & \mathrm{T}_{ij}\hat{\mathrm{n}}_j = \lambda\hat{\mathrm{n}}_i \\ & \Rightarrow \mathrm{T}_{ij}\hat{\mathrm{n}}_j - \lambda\hat{\mathrm{n}}_i = 0_i \\ & \Rightarrow (\mathrm{T}_{ij} - \lambda\delta_{ij})\hat{\mathrm{n}}_j = 0_i \quad \xrightarrow[\textit{notation}]{\textit{Tensorial}} \quad (\mathbf{T} - \lambda\mathbf{1}) \cdot \hat{\mathbf{n}} = \vec{\mathbf{0}} \end{aligned} \tag{1.265}$$

The previous set of homogeneous equations only have nontrivial solution, *i.e.* $\hat{\mathbf{n}} \neq \vec{\mathbf{0}}$, if and only if:

$$\det(\mathbf{T} - \lambda\mathbf{1}) = 0 \quad ; \quad \left|\mathrm{T}_{ij} - \lambda\delta_{ij}\right| = 0 \tag{1.266}$$

The determinant (1.266) is called the *characteristic determinant* of the tensor $\mathbf{T}$, explicitly given by:

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0 \tag{1.267}$$

Developing this determinant, we obtain the *characteristic polynomial*, which is shown by a cubic equation in λ:

$$\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0 \tag{1.268}$$

where $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ are the *principal invariants* of $\mathbf{T}$, and are defined in components terms as:

$$\begin{aligned}
I_{\mathbf{T}} &= \mathrm{Tr}(\mathbf{T}) = T_{ii} \\
II_{\mathbf{T}} &= \frac{1}{2}\left[(\mathrm{Tr}\mathbf{T})^2 - \mathrm{Tr}(\mathbf{T}^2)\right] \\
&= \frac{1}{2}\left\{ \mathrm{Tr}(T_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\mathrm{Tr}(T_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) - \mathrm{Tr}\left[(T_{ij}\hat{\mathbf{e}}_i \otimes \overbrace{\hat{\mathbf{e}}_j)\cdot(T_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}} \otimes \hat{\mathbf{e}}_l)\right] \right\} \\
&= \frac{1}{2}\left\{ T_{ij}\delta_{ij}T_{kl}\delta_{kl} - T_{ij}T_{kl}\delta_{jk}\mathrm{Tr}\left[(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l)\right] \right\} \\
&= \frac{1}{2}\left\{ T_{ii}T_{kk} - T_{ij}T_{kl}\delta_{jk}\delta_{il} \right\} \\
&= \frac{1}{2}\left\{ T_{ii}T_{kk} - T_{ij}T_{ji} \right\} = M_{ii} = \mathrm{Tr}[\mathrm{cof}(\mathbf{T})] \\
III_{\mathbf{T}} &= \det(\mathbf{T}) = \left|T_{ij}\right| = \epsilon_{ijk}T_{i1}T_{j2}T_{k3}
\end{aligned} \tag{1.269}$$

where M_{ii} is the matrix trace defined in equation (1.210), $M_{ii} = M_{11} + M_{22} + M_{33}$. More explicitly the invariants are given by:

$$\begin{aligned}
I_{\mathbf{T}} &= T_{11} + T_{22} + T_{33} \\
II_{\mathbf{T}} &= \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} \\
&= T_{22}T_{33} - T_{23}T_{32} + T_{11}T_{33} - T_{13}T_{31} + T_{11}T_{22} - T_{12}T_{21} \\
III_{\mathbf{T}} &= T_{11}(T_{22}T_{33} - T_{32}T_{23}) - T_{12}(T_{21}T_{33} - T_{31}T_{23}) + T_{13}(T_{21}T_{32} - T_{31}T_{22})
\end{aligned} \tag{1.270}$$

If $\mathbf{T}$ is a symmetric tensor, the principal invariants are summarized as follows:

$$\begin{aligned}
I_{\mathbf{T}} &= T_{11} + T_{22} + T_{33} \\
II_{\mathbf{T}} &= T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33} - T_{12}^2 - T_{13}^2 - T_{23}^2 \\
III_{\mathbf{T}} &= T_{11}T_{22}T_{33} + T_{12}T_{13}T_{23} + T_{13}T_{12}T_{23} - T_{12}^2T_{33} - T_{23}^2T_{11} - T_{13}^2T_{22}
\end{aligned} \tag{1.271}$$

The eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, are found by solving the characteristic polynomial (1.268). Once the eigenvalues are evaluated, the eigenvectors are found by applying equation (1.265), *i.e.* $(T_{ij} - \lambda_1\delta_{ij})\hat{n}_j^{(1)} = 0_i$, $(T_{ij} - \lambda_2\delta_{ij})\hat{n}_j^{(2)} = 0_i$, $(T_{ij} - \lambda_3\delta_{ij})\hat{n}_j^{(3)} = 0_i$. These eigenvectors constitute a new space denoted as the *principal space*.

If $\mathbf{T}$ is a symmetric tensor, the principal space is defined by an orthonormal basis and all eigenvalues are real numbers. If the three eigenvalues are different, $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the three principal directions are unique. If two of them are equal, *e.g.* $\lambda_1 = \lambda_2 \neq \lambda_3$, we can state that the principal direction, $\hat{\mathbf{n}}^{(3)}$, associated with the eigenvalue λ_3, is unique, and, any direction defined in the plane normal to $\hat{\mathbf{n}}^{(3)}$ is a principal direction, and othorgonality is the only constraint to determining $\hat{\mathbf{n}}^{(1)}$ and $\hat{\mathbf{n}}^{(2)}$. If $\lambda_1 = \lambda_2 = \lambda_3$, any direction is principal. A tensor that has three equal eigenvalues is called a *Spherical Tensor*, (see Appendix A-The Tensor ellipsoid).

The $\mathbf{T}$-components in the principal space are only made up of normal components, *i.e.*:

$$\mathsf{T}'_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \tag{1.272}$$

Therefore, the principal invariants can also be evaluated by:

$$I_{\mathbf{T}} = \mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3\,,\; I\!I_{\mathbf{T}} = \mathsf{T}_1\mathsf{T}_2 + \mathsf{T}_2\mathsf{T}_3 + \mathsf{T}_1\mathsf{T}_3\,,\; I\!I\!I_{\mathbf{T}} = \mathsf{T}_1\mathsf{T}_2\mathsf{T}_3 \tag{1.273}$$

whose values must match the values obtained in (1.270), since they are invariant with a change of basis.

If $\mathbf{T}$ is a spherical tensor, *i.e.* $\mathsf{T}_1 = \mathsf{T}_2 = \mathsf{T}_3 = \mathsf{T}$, it holds that $I_{\mathbf{T}}^2 = 3 I\!I_{\mathbf{T}}$, $I\!I\!I_{\mathbf{T}} = \mathsf{T}^3$.

Let $\mathbf{W}$ be an antisymmetric tensor. The principal invariants of $\mathbf{W}$ are given by:

$$\begin{aligned}
I_{\mathbf{W}} &= \mathrm{Tr}(\mathbf{W}) = 0 \\
I\!I_{\mathbf{W}} &= \frac{1}{2}\left[(\mathrm{Tr}\mathbf{W})^2 - \mathrm{Tr}(\mathbf{W}^2)\right] = \frac{-\mathrm{Tr}(\mathbf{W}^2)}{2} \\
&= \begin{vmatrix} 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{23} & 0 \end{vmatrix} + \begin{vmatrix} 0 & \mathsf{W}_{13} \\ -\mathsf{W}_{13} & 0 \end{vmatrix} + \begin{vmatrix} 0 & \mathsf{W}_{12} \\ -\mathsf{W}_{12} & 0 \end{vmatrix} \\
&= \mathsf{W}_{23}\mathsf{W}_{23} + \mathsf{W}_{13}\mathsf{W}_{13} + \mathsf{W}_{12}\mathsf{W}_{12} \\
&= \omega^2 \\
I\!I\!I_{\mathbf{W}} &= 0
\end{aligned} \tag{1.274}$$

where $\omega^2 = \left\|\vec{w}\right\|^2 = \vec{w}\cdot\vec{w} = \mathsf{W}_{23}^2 + \mathsf{W}_{13}^2 + \mathsf{W}_{12}^2$ as defined in (1.132). Then, the characteristic equation for an antisymmetric tensor is reduced to:

$$\lambda^3 - \lambda^2 I_{\mathbf{W}} + \lambda I\!I_{\mathbf{W}} - I\!I\!I_{\mathbf{W}} = 0 \quad\Rightarrow\quad \lambda^3 + \omega^2\lambda = 0 \quad\Rightarrow\quad \lambda\left(\lambda^2 + \omega^2\right) = 0 \tag{1.275}$$

In this case, one eigenvalue is real and equal to zero and the others are imaginary roots:

$$\lambda^2 + \omega^2 = 0 \quad\Rightarrow\quad \lambda^2 = -\omega^2 = 0 \quad\Rightarrow\quad \lambda_{(1,2)} = \pm\omega\sqrt{-1} = \pm\omega\, i \tag{1.276}$$

1.5.4.1 The Orthogonality of the Eigenvectors

Consider a symmetric second-order tensor $\mathbf{T}$. By the definition of eigenvalues, given in (1.264), if λ_1, λ_2, λ_3 are the eigenvalues of $\mathbf{T}$, then it follows that:

$$\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_1\hat{\mathbf{n}}^{(1)} \quad;\quad \mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)} \qquad \mathbf{T}\cdot\hat{\mathbf{n}}^{(3)} = \lambda_3\hat{\mathbf{n}}^{(3)} \tag{1.277}$$

Applying the dot product between $\hat{\mathbf{n}}^{(2)}$ and $\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_1\hat{\mathbf{n}}^{(1)}$, and the dot product between $\hat{\mathbf{n}}^{(1)}$ and $\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)}$ we obtain:

$$\begin{aligned}
\hat{\mathbf{n}}^{(2)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} &= \lambda_1\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} \\
\hat{\mathbf{n}}^{(1)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)} &= \lambda_2\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)}
\end{aligned} \tag{1.278}$$

Since $\mathbf{T}$ is symmetric, it holds that $\hat{\mathbf{n}}^{(2)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(1)}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}^{(2)}$, so:

$$\lambda_1\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} = \lambda_2\hat{\mathbf{n}}^{(1)}\cdot\hat{\mathbf{n}}^{(2)} = \lambda_2\hat{\mathbf{n}}^{(2)}\cdot\hat{\mathbf{n}}^{(1)} \tag{1.279}$$

$$\Rightarrow (\lambda_1 - \lambda_2)\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)} = 0 \tag{1.280}$$

To satisfy the equation (1.280), with $\lambda_1 \neq \lambda_2 \neq 0$, the following must be true:

$$\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)} = 0 \tag{1.281}$$

Similarly, it is possible to show that $\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(3)} = 0$ and $\hat{\mathbf{n}}^{(2)} \cdot \hat{\mathbf{n}}^{(3)} = 0$ and then we can conclude that the eigenvectors are mutually orthogonal, and constitute an orthogonal basis, (see Figure 1.27), where the transformation matrix between systems is:

$$\mathcal{A} = \begin{bmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{n}_1^{(1)} & \hat{n}_2^{(1)} & \hat{n}_3^{(1)} \\ \hat{n}_1^{(2)} & \hat{n}_2^{(2)} & \hat{n}_3^{(2)} \\ \hat{n}_1^{(3)} & \hat{n}_2^{(3)} & \hat{n}_3^{(3)} \end{bmatrix} \tag{1.282}$$

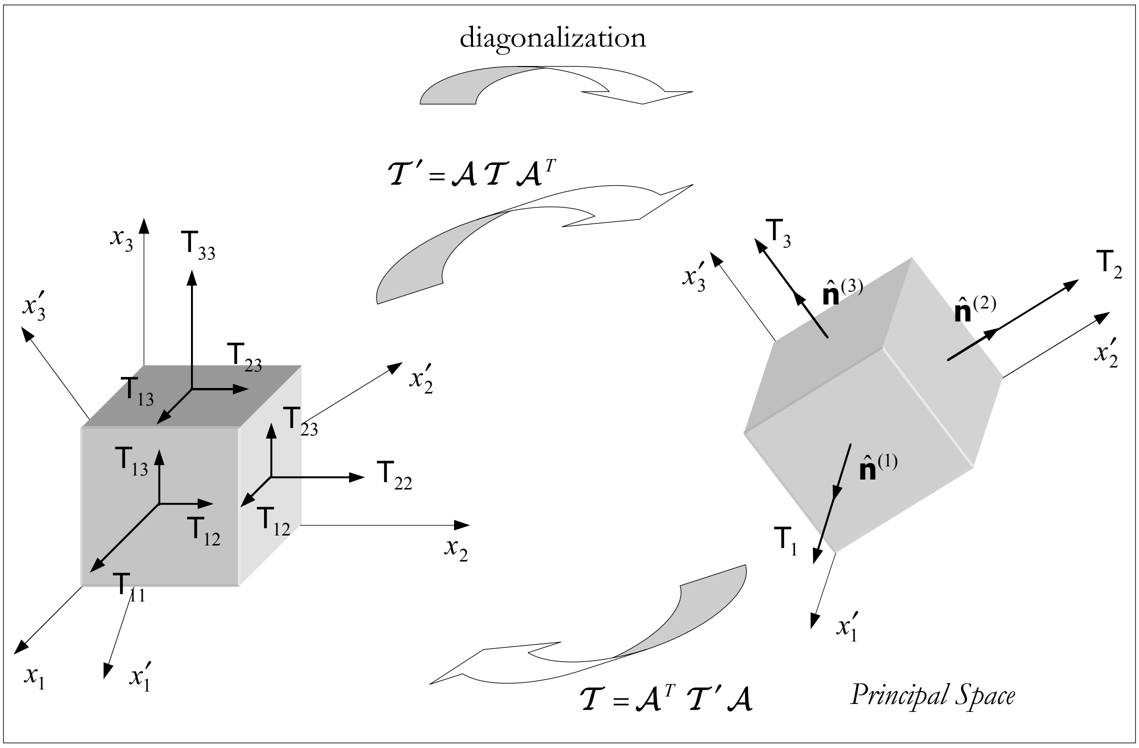


Figure 1.27: Diagonalization.

Problem 1.31: Show that the following relations are invariants:

$$C_1^2 + C_2^2 + C_3^2 \quad ; \quad C_1^3 + C_2^3 + C_3^3 \quad ; \quad C_1^4 + C_2^4 + C_3^4$$

where C_1, C_2, C_3 are the eigenvalues of the second-order tensor $\boldsymbol{C}$.

Solution: Any combination of invariants is also an invariant, so, on this basis, we can try to express the above expressions in terms of their principal invariants.

$$I_{\boldsymbol{C}}^2 = (C_1 + C_2 + C_3)^2 = C_1^2 + C_2^2 + C_3^2 + 2\underbrace{(C_1 C_2 + C_1 C_3 + C_2 C_3)}_{II_{\boldsymbol{C}}} \Rightarrow C_1^2 + C_2^2 + C_3^2 = I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}}$$

So, we have proved that $C_1^2 + C_2^2 + C_3^2$ is an invariant. Similarly, we can obtain:

$$C_1^3 + C_2^3 + C_3^3 = I_{\boldsymbol{C}}^3 - 3\,II_{\boldsymbol{C}} I_{\boldsymbol{C}} + 3\,III_{\boldsymbol{C}}$$

$$C_1^4 + C_2^4 + C_3^4 = I_{\boldsymbol{C}}^4 - 4\,II_{\boldsymbol{C}} I_{\boldsymbol{C}}^2 + 4\,III_{\boldsymbol{C}} I_{\boldsymbol{C}} + 2\,II_{\boldsymbol{C}}^2$$

Problem 1.32: Let $\mathbf{Q}$ be a proper orthogonal tensor, and $\mathbf{E}$ be an arbitrary second-order tensor. Show that the eigenvalues of $\mathbf{E}$ do not change with the following orthogonal transformation:

$$\mathbf{E}^* = \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T$$

Solution: We can prove this as follows:

$$\begin{aligned}
0 &= \det(\mathbf{E}^* - \lambda \mathbf{1}) \\
&= \det(\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T - \lambda \mathbf{1}) \\
&= \det(\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T - \mathbf{Q} \cdot \lambda \mathbf{1} \cdot \mathbf{Q}^T) \\
&= \det[\mathbf{Q} \cdot (\mathbf{E} - \lambda \mathbf{1}) \cdot \mathbf{Q}^T] \\
&= \underbrace{\det(\mathbf{Q})}_{1} \det(\mathbf{E} - \lambda \mathbf{1}) \underbrace{\det(\mathbf{Q}^T)}_{1} \\
&= \det(\mathbf{E} - \lambda \mathbf{1})
\end{aligned}
\qquad
\begin{aligned}
0 &= \det(\mathsf{E}^*_{ij} - \lambda \delta_{ij}) \\
&= \det(\mathsf{Q}_{ik} \mathsf{E}_{kp} \mathsf{Q}_{jp} - \lambda \delta_{ij}) \\
&= \det(\mathsf{Q}_{ik} \mathsf{E}_{kp} \mathsf{Q}_{jp} - \lambda \mathsf{Q}_{ik} \mathsf{Q}_{jp} \delta_{kp}) \\
&= \det[\mathsf{Q}_{ik} (\mathsf{E}_{kp} - \lambda \delta_{kp}) \mathsf{Q}_{jp}] \\
&= \det(\mathsf{Q}_{ik}) \det(\mathsf{E}_{kp} - \lambda \delta_{kp}) \det(\mathsf{Q}_{jp}) \\
&= \det(\mathsf{E}_{kp} - \lambda \delta_{kp})
\end{aligned}$$

Thus, we have proved that $\mathbf{E}$ and $\mathbf{E}^*$ have the same eigenvalues.

1.5.4.2 Solution of the Cubic Equation

Let $\mathbf{T}$ be a symmetric second-order tensor. The roots of the characteristic equation ($\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0$) are all real numbers, and are expressed as:

$$\begin{aligned}
\lambda_1 &= 2S\left[\cos\left(\frac{\alpha}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3} \\
\lambda_2 &= 2S\left[\cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3} \\
\lambda_3 &= 2S\left[\cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3}
\end{aligned} \tag{1.283}$$

where

$$R = \frac{I_{\mathbf{T}}^2 - 3 II_{\mathbf{T}}}{3}; \quad S = \sqrt{\frac{R}{3}}; \quad Q = \frac{I_{\mathbf{T}} II_{\mathbf{T}}}{3} - III_{\mathbf{T}} - \frac{2 I_{\mathbf{T}}^3}{27}; \quad T = \sqrt{\frac{R^3}{27}}; \quad \alpha = \arccos\left(-\frac{Q}{2T}\right)$$

where α is in radians. (1.284)

By restructuring the solution (1.283) in matrix form, we obtain:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \underbrace{\frac{I_{\mathbf{T}}}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Spherical part}} + \underbrace{2S\begin{bmatrix} \cos\left(\frac{\alpha}{3}\right) & 0 & 0 \\ 0 & \cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right) & 0 \\ 0 & 0 & \cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right) \end{bmatrix}}_{\text{Deviatoric part}} \tag{1.285}$$

where we clearly distinguish the spherical and the deviatoric part of the tensor in the principal space. Note that, if $\mathbf{T}$ is a spherical tensor the following relationship holds $I_{\mathbf{T}}^2 = 3 II_{\mathbf{T}}$, then $S = 0$.

Problem 1.33: Find the principal values and directions of the second-order tensor $\mathbf{T}$, where the Cartesian components of $\mathbf{T}$ are:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: We need to find nontrivial solutions for $(\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = 0_i$, which are constrained by $\hat{\mathsf{n}}_j\hat{\mathsf{n}}_j = 1$ (unit vector). As we have seen, the nontrivial solution requires that:

$$\left|\mathsf{T}_{ij} - \lambda\delta_{ij}\right| = 0$$

Explicitly, the above equation is:

$$\begin{vmatrix} \mathsf{T}_{11}-\lambda & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22}-\lambda & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33}-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Developing the above determinant, we can obtain the cubic equation:

$$(1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0$$

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

We could have obtained the characteristic equation directly in terms of invariants:

$$I_{\mathbf{T}} = \mathrm{Tr}(\mathsf{T}_{ij}) = \mathsf{T}_{ii} = \mathsf{T}_{11} + \mathsf{T}_{22} + \mathsf{T}_{33} = 7$$

$$II_{\mathbf{T}} = \frac{1}{2}\left(\mathsf{T}_{ii}\mathsf{T}_{jj} - \mathsf{T}_{ij}\mathsf{T}_{ij}\right) = \begin{vmatrix} \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{32} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{13} \\ \mathsf{T}_{31} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{vmatrix} = 14$$

$$III_{\mathbf{T}} = \left|\mathsf{T}_{ij}\right| = \epsilon_{ijk}\mathsf{T}_{i1}\mathsf{T}_{j2}\mathsf{T}_{k3} = 8$$

Then, using the equation in (1.268), the characteristic equation is:

$$\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0 \quad \rightarrow \quad \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

On solving the cubic equation we obtain three real roots, namely:

$$\lambda_1 = 1; \quad \lambda_2 = 2; \quad \lambda_3 = 4$$

We can also verify that:

$$I_{\mathbf{T}} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 4 = 7 \checkmark$$

$$II_{\mathbf{T}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\times 2 + 2\times 4 + 4\times 1 = 14 \checkmark$$

$$III_{\mathbf{T}} = \lambda_1\lambda_2\lambda_3 = 8 \checkmark$$

Thus, we can see that the invariants are the same as those evaluated previously.

<u>Principal directions:</u>

Each eigenvalue, λ_i, is associated with a corresponding eigenvector, $\hat{\mathbf{n}}^{(i)}$. We can use the equation in (1.265), *i.e.* $(\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = \mathbf{0}_i$, to obtain the principal directions.

- $\lambda_1 = 1$

$$\begin{bmatrix} 3-\lambda_1 & -1 & 0 \\ -1 & 3-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 3-1 & -1 & 0 \\ -1 & 3-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These become the following system of equations:

$$\begin{cases} 2n_1 - n_2 = 0 \\ -n_1 + 2n_2 = 0 \\ 0n_3 = 0 \end{cases} \Rightarrow n_1 = n_2 = 0$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1$$

Then we can conclude that: $\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$.

NOTE: This solution could have been directly determined by the specific features of the $\mathcal{T}$ matrix. As the terms $T_{13} = T_{23} = T_{31} = T_{32} = 0$ imply that $T_{33} = 1$ is already a principal value, then, consequently, the original direction is a principal direction. ■

$$\lambda_2 = 2$$

$$\begin{bmatrix} 3-\lambda_2 & -1 & 0 \\ -1 & 3-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \\ 0 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} n_1 - n_2 = 0 \Rightarrow n_1 = n_2 \\ -n_1 + n_2 = 0 \\ -n_3 = 0 \end{cases}$$

The first two equations are linearly dependent, after which we need an additional equation:

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_1^2 = 1 \Rightarrow n_1 = \pm\sqrt{\frac{1}{2}}$$

Thus:

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \left[\pm\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

$$\lambda_3 = 4$$

$$\begin{bmatrix} 3-\lambda_3 & -1 & 0 \\ -1 & 3-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \\ 0 & 0 & 1-4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -n_1 - n_2 = 0 \\ -n_1 - n_2 = 0 \\ -3n_3 = 0 \end{cases} \Rightarrow n_1 = -n_2$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_2^2 = 1 \Rightarrow n_2 = \pm\sqrt{\frac{1}{2}}$$

Then:

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \left[\mp\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

Afterwards, we summarize the eigenvalues and eigenvectors of $\mathbf{T}$:

$$\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$$

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \left[\pm\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \left[\mp\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

NOTE: The tensor components of this problem are the same as those used in **Problem 1.30**. Additionally, we can verify that the eigenvectors make up the transformation matrix, $\mathcal{A}$, between the original system, (x_1, x_2, x_3), and the principal space, (x'_1, x'_2, x'_3), (see **Problem 1.30**). ■

1.5.5 Spectral Representation of Tensors

Based on the solution of the equation in (1.268), if $\mathbf{T}$ is a symmetric second-order tensor there are three real eigenvalues: T_1, T_2, T_3 each of which is associated with an eigenvector, *i.e.*:

$$\begin{aligned} \mathsf{T}_1 &\Rightarrow \hat{\mathsf{n}}_i^{(1)} = \left[\hat{\mathsf{n}}_1^{(1)} \quad \hat{\mathsf{n}}_2^{(1)} \quad \hat{\mathsf{n}}_3^{(1)}\right] \\ \mathsf{T}_2 &\Rightarrow \hat{\mathsf{n}}_i^{(2)} = \left[\hat{\mathsf{n}}_1^{(2)} \quad \hat{\mathsf{n}}_2^{(2)} \quad \hat{\mathsf{n}}_3^{(2)}\right] \\ \mathsf{T}_3 &\Rightarrow \hat{\mathsf{n}}_i^{(3)} = \left[\hat{\mathsf{n}}_1^{(3)} \quad \hat{\mathsf{n}}_2^{(3)} \quad \hat{\mathsf{n}}_3^{(3)}\right] \end{aligned} \tag{1.286}$$

The principal space is formed by the orthogonal basis $\hat{\mathbf{n}}^{(1)}, \hat{\mathbf{n}}^{(2)}, \hat{\mathbf{n}}^{(3)}$, and the tensor components are represented by their eigenvalues as:

$$\mathsf{T}'_{ij} = \mathcal{T}' = \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \tag{1.287}$$

With reference to the fact that eigenvectors form a transformation matrix, $\mathcal{A}$, so that:

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \tag{1.288}$$

Since $\mathcal{A}^{-1} = \mathcal{A}^T$, the inverse form is:

$$\mathcal{T} = \mathcal{A}^T\,\mathcal{T}'\,\mathcal{A} \tag{1.289}$$

where

$$\mathcal{A} = \begin{bmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_3^{(1)} \\ \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_3^{(2)} \\ \hat{\mathsf{n}}_1^{(3)} & \hat{\mathsf{n}}_2^{(3)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \tag{1.290}$$

Explicitly, the relation in (1.289) is given by:

$$\begin{aligned} \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} &= \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_1^{(3)} \\ \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_2^{(3)} \\ \hat{\mathsf{n}}_3^{(1)} & \hat{\mathsf{n}}_3^{(2)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_3^{(1)} \\ \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_3^{(2)} \\ \hat{\mathsf{n}}_1^{(3)} & \hat{\mathsf{n}}_2^{(3)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \\ &= \mathcal{A}^T \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \mathcal{A} \\ &= \mathsf{T}_1 \mathcal{A}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathsf{T}_2 \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathsf{T}_3 \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{A} \end{aligned} \tag{1.291}$$

Whereas:

$$
\begin{aligned}
&\mathcal{A}^T\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\mathcal{A}=\begin{bmatrix}\hat{n}_1^{(1)}\hat{n}_1^{(1)} & \hat{n}_1^{(1)}\hat{n}_2^{(1)} & \hat{n}_1^{(1)}\hat{n}_3^{(1)}\\ \hat{n}_2^{(1)}\hat{n}_1^{(1)} & \hat{n}_2^{(1)}\hat{n}_2^{(1)} & \hat{n}_2^{(1)}\hat{n}_3^{(1)}\\ \hat{n}_3^{(1)}\hat{n}_1^{(1)} & \hat{n}_3^{(1)}\hat{n}_2^{(1)} & \hat{n}_3^{(1)}\hat{n}_3^{(1)}\end{bmatrix}=\hat{n}_i^{(1)}\hat{n}_j^{(1)}\\
&\mathcal{A}^T\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\mathcal{A}=\begin{bmatrix}\hat{n}_1^{(2)}\hat{n}_1^{(2)} & \hat{n}_1^{(2)}\hat{n}_2^{(2)} & \hat{n}_1^{(2)}\hat{n}_3^{(2)}\\ \hat{n}_2^{(2)}\hat{n}_1^{(2)} & \hat{n}_2^{(2)}\hat{n}_2^{(2)} & \hat{n}_2^{(2)}\hat{n}_3^{(2)}\\ \hat{n}_3^{(2)}\hat{n}_1^{(2)} & \hat{n}_3^{(2)}\hat{n}_2^{(2)} & \hat{n}_3^{(2)}\hat{n}_3^{(2)}\end{bmatrix}=\hat{n}_i^{(2)}\hat{n}_j^{(2)}\\
&\mathcal{A}^T\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\mathcal{A}=\begin{bmatrix}\hat{n}_1^{(3)}\hat{n}_1^{(3)} & \hat{n}_1^{(3)}\hat{n}_2^{(3)} & \hat{n}_1^{(3)}\hat{n}_3^{(3)}\\ \hat{n}_2^{(3)}\hat{n}_1^{(3)} & \hat{n}_2^{(3)}\hat{n}_2^{(3)} & \hat{n}_2^{(3)}\hat{n}_3^{(3)}\\ \hat{n}_3^{(3)}\hat{n}_1^{(3)} & \hat{n}_3^{(3)}\hat{n}_2^{(3)} & \hat{n}_3^{(3)}\hat{n}_3^{(3)}\end{bmatrix}=\hat{n}_i^{(3)}\hat{n}_j^{(3)}
\end{aligned}
\tag{1.292}
$$

Then, it is possible to represent the components of a second-order tensor in function of their eigenvalues and eigenvectors (spectral representation) as:

$$
\mathsf{T}_{ij}=\mathsf{T}_1\,\hat{n}_i^{(1)}\hat{n}_j^{(1)}+\mathsf{T}_2\hat{n}_i^{(2)}\hat{n}_j^{(2)}+\mathsf{T}_3\,\hat{n}_i^{(3)}\hat{n}_j^{(3)} \tag{1.293}
$$

As we can see, the tensor is represented as a linear combination of dyads and the above representation in tensorial notation becomes:

$$
\mathbf{T}=\mathsf{T}_1\,\hat{\mathbf{n}}^{(1)}\otimes\hat{\mathbf{n}}^{(1)}+\mathsf{T}_2\,\hat{\mathbf{n}}^{(2)}\otimes\hat{\mathbf{n}}^{(2)}+\mathsf{T}_3\,\hat{\mathbf{n}}^{(3)}\otimes\hat{\mathbf{n}}^{(3)} \tag{1.294}
$$

or:

$$
\boxed{\mathbf{T}=\sum_{a=1}^{3}\mathsf{T}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}} \quad \textit{Spectral representation of a second-order tensor} \tag{1.295}
$$

which is the *spectral representation* of the tensor. Note that, in the above equation we have to resort to the summation symbol, because the dummy index appears thrice in the expression.

NOTE: The spectral representation in (1.295) could easily have been obtained from the definition of the second-order unit tensor, given in (1.168), *i.e.* $\mathbf{1}=\hat{\mathbf{n}}_i\otimes\hat{\mathbf{n}}_i$, which can also be represented by means of the summation symbol as $\mathbf{1}=\sum_{a=1}^{3}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}$. Then, it follows that:

$$
\mathbf{T}=\mathbf{T}\cdot\mathbf{1}=\mathbf{T}\cdot\left(\sum_{a=1}^{3}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)=\sum_{a=1}^{3}\mathbf{T}\cdot\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}=\sum_{a=1}^{3}\mathsf{T}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)} \tag{1.296}
$$

where we have used the definition of eigenvalue and eigenvector $\mathbf{T}\cdot\hat{\mathbf{n}}^{(a)}=\mathsf{T}_a\,\hat{\mathbf{n}}^{(a)}$. ■

We now consider the orthogonal tensor $\mathbf{R}$. The orthogonal transformation applied to the unit vector $\hat{\mathbf{N}}$ leads to the unit vector $\hat{\mathbf{n}}$, *i.e.* $\hat{\mathbf{n}}=\mathbf{R}\cdot\hat{\mathbf{N}}$. Therefore, it is also possible to represent the orthogonal tensor $\mathbf{R}$ as follows:

$$
\mathbf{R}=\mathbf{R}\cdot\mathbf{1}=\mathbf{R}\cdot\left(\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)=\sum_{a=1}^{3}\mathbf{R}\cdot\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=\sum_{a=1}^{3}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} \tag{1.297}
$$

The spectral representation is very useful for making algebraic operations with tensors. For example, tensor power in the principal space can be expressed as:

$$
\left(\mathbf{T}^{n}\right)_{ij} = \begin{bmatrix} \mathsf{T}_1^n & 0 & 0 \\ 0 & \mathsf{T}_2^n & 0 \\ 0 & 0 & \mathsf{T}_3^n \end{bmatrix} \tag{1.298}
$$

So, the spectral representation of $\mathbf{T}^n$ is given by:

$$
\mathbf{T}^n = \sum_{a=1}^{3} \mathsf{T}_a^n \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.299}
$$

Now, if we need the square root of the tensor, $\sqrt{\mathbf{T}}$, this can easily be obtained from the spectral representation as:

$$
\sqrt{\mathbf{T}} = \sum_{a=1}^{3} \sqrt{\mathsf{T}_a} \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.300}
$$

Next, we can show that a positive definite tensor has positive eigenvalues. For this purpose, we can consider a semi-positive definite tensor, $\mathbf{T}$, by which the condition $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0$ holds for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Replacing the tensor by its spectral representation, we obtain:

$$
\begin{aligned}
&\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0 \\
&\Rightarrow \hat{\mathbf{x}} \cdot \left(\sum_{a=1}^{3} \mathsf{T}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \hat{\mathbf{x}} \geq 0 \\
&\Rightarrow \sum_{a=1}^{3} \mathsf{T}_a \, \hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{x}} \geq 0
\end{aligned} \tag{1.301}
$$

Note that the result of the operation $(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)})$ is a scalar, thus:

$$
\begin{aligned}
&\sum_{a=1}^{3} \mathsf{T}_a \, \hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{x}} \geq 0 \quad \Rightarrow \quad \sum_{a=1}^{3} \mathsf{T}_a \underbrace{\left[(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)})^2 \right]}_{>0} \geq 0 \\
&\Rightarrow \mathsf{T}_1 \, (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(1)})^2 + \mathsf{T}_2 \, (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(2)})^2 + \mathsf{T}_3 \, (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(3)})^2 \geq 0
\end{aligned} \tag{1.302}
$$

The above expression must hold for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. If we take $\hat{\mathbf{x}} = \hat{\mathbf{n}}^{(1)}$, the above equation is reduced to $\mathsf{T}_1 \, (\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(1)})^2 = \mathsf{T}_1 \geq 0$. The same is true for T_2 and T_3. Thus, we have demonstrated that if a tensor is semi-positive definite, its eigenvalues are greater than or equal to zero, *i.e.* $\mathsf{T}_1 \geq 0$, $\mathsf{T}_2 \geq 0$, $\mathsf{T}_3 \geq 0$. Therefore we can conclude that a tensor is positive definite, *i.e.* $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} > 0$, if and only if its eigenvalues are positive and nonzero, *i.e.* $\mathsf{T}_1 > 0$, $\mathsf{T}_2 > 0$, $\mathsf{T}_3 > 0$. Consequently, the positive definite tensor trace is greater than zero. If the positive definite tensor trace is zero, this implies that the tensor is the zero tensor.

The spectral representation of the fourth-order unit tensor, $\mathbb{I}$, can be obtained starting from the definition in (1.169), *i.e.*:

$$
\mathbb{I} = \delta_{ik}\delta_{j\ell} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \sum_{a=1}^{3}\sum_{b=1}^{3} \hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b \otimes \hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b \tag{1.303}
$$

As $\mathbb{I}$ is an isotropic tensor, (see 1.5.8 Isotropic and Anisotropic tensors), then the representation in (1.303) is also valid in any orthonormal basis, $\hat{\mathbf{n}}^{(a)}$, so:

$$
\mathbb{I} = \sum_{a=1}^{3}\sum_{b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \tag{1.304}
$$

Similarly, we obtain the spectral representation for $\overline{\mathbb{I}}$ and $\overline{\overline{\mathbb{I}}}$ as:

$$\overline{\mathbb{I}} = \delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \tag{1.305}$$

$$\overline{\mathbb{I}} = \sum_{a,b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.306}$$

and

$$\overline{\overline{\mathbb{I}}} = \delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k \tag{1.307}$$

$$\overline{\overline{\mathbb{I}}} = \sum_{a,b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} \tag{1.308}$$

Problem 1.34: Let $\boldsymbol{\omega}$ be an antisymmetric second-order tensor and $\mathbf{V}$ be a positive definite symmetric tensor whose spectral representation is given by:

$$\mathbf{V} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$$

Show that the antisymmetric tensor $\boldsymbol{\omega}$ can be represented by:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab} \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Demonstrate also that:

$$\boldsymbol{\omega} \cdot \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab} (\lambda_b - \lambda_a) \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Solution:

It is true that

$$\boldsymbol{\omega} \cdot \mathbf{1} = \boldsymbol{\omega} \cdot \left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) = \sum_{a=1}^{3} \boldsymbol{\omega} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \left(\vec{w} \wedge \hat{\mathbf{n}}^{(a)} \right) \otimes \hat{\mathbf{n}}^{(a)}$$
$$= \sum_{a,b=1}^{3} w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(a)} \right) \otimes \hat{\mathbf{n}}^{(a)}$$

where we have applied an antisymmetric tensor property $\boldsymbol{\omega} \cdot \hat{\mathbf{n}} = \vec{w} \wedge \hat{\mathbf{n}}$, where $\vec{w}$ is the axial vector associated with $\boldsymbol{\omega}$. Expanding the above equation, we obtain:

$$\begin{aligned}
\boldsymbol{\omega} &= w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(1)} + w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(2)} + w_b \left(\hat{\mathbf{n}}^{(b)} \wedge \hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(3)} = \\
&= w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(1)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(1)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(1)} + \\
&\quad + w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(2)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(2)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(2)} + \\
&\quad + w_1 \left(\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(3)} + w_2 \left(\hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(3)} + w_3 \left(\hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(3)}
\end{aligned}$$

On simplifying the above expression we obtain:

$$\begin{aligned}
\boldsymbol{\omega} = &- w_2 \left(\hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(1)} + w_3 \left(\hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(1)} + \\
&+ w_1 \left(\hat{\mathbf{n}}^{(3)} \right) \otimes \hat{\mathbf{n}}^{(2)} - w_3 \left(\hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(2)} + \\
&- w_1 \left(\hat{\mathbf{n}}^{(2)} \right) \otimes \hat{\mathbf{n}}^{(3)} + w_2 \left(\hat{\mathbf{n}}^{(1)} \right) \otimes \hat{\mathbf{n}}^{(3)}
\end{aligned}$$

Taking into account that $w_1 = -\omega_{23} = \omega_{32}$, $w_2 = \omega_{13} = -\omega_{31}$, $w_3 = -\omega_{12} = \omega_{21}$, the above equation becomes:

$$\boldsymbol{\omega} = \omega_{31}\,\hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_{21}\,\hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(1)} + \\ + \omega_{32}\,\hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(2)} + \omega_{12}\,\hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(2)} + \\ + \omega_{23}\,\hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(3)} + \omega_{13}\,\hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(3)}$$

which is the same as:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

The terms $\boldsymbol{\omega}\cdot\mathbf{V}$ and $\mathbf{V}\cdot\boldsymbol{\omega}$ can be expressed as follows:

$$\boldsymbol{\omega}\cdot\mathbf{V} = \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right)\cdot\left(\sum_{b=1}^{3} \lambda_b\,\hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)}\right)$$

$$= \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\cdot\hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

and

$$\mathbf{V}\cdot\boldsymbol{\omega} = \left(\sum_{a=1}^{3} \lambda_a\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right) = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Then,

$$\boldsymbol{\omega}\cdot\mathbf{V} - \mathbf{V}\cdot\boldsymbol{\omega} = \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right) - \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right)$$

$$= \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b - \lambda_a)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Similarly, it is possible to show that:

$$\boldsymbol{\omega}\cdot\mathbf{V}^2 - \mathbf{V}^2\cdot\boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b^2 - \lambda_a^2)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

1.5.6 Cayley-Hamilton Theorem

The Cayley-Hamilton theorem states that any tensor, $\mathbf{T}$, satisfies its own characteristic equation, *i.e.* if the eigenvalues of $\mathbf{T}$ satisfy the equation $\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0$, so does the tensor $\mathbf{T}$:

$$\mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0} \tag{1.309}$$

One of the applications of the Cayley-Hamilton theorem is to express the power of tensor, $\mathbf{T}^n$, as a combination of $\mathbf{T}^{n-1}$, $\mathbf{T}^{n-2}$, $\mathbf{T}^{n-3}$. For example, $\mathbf{T}^4$ is obtained as:

$$\mathbf{T}^3\cdot\mathbf{T} - \mathbf{T}^2\cdot\mathbf{T}I_{\mathbf{T}} + \mathbf{T}\cdot\mathbf{T}II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1}\cdot\mathbf{T} = \mathbf{0} \quad \Rightarrow \quad \mathbf{T}^4 = \mathbf{T}^3 I_{\mathbf{T}} - \mathbf{T}^2 II_{\mathbf{T}} + III_{\mathbf{T}}\mathbf{T} \tag{1.310}$$

Using the Cayley-Hamilton theorem, it is possible to express the third invariant as a function of traces. According to the Cayley-Hamilton theorem, the expression

$\mathbf{T}^3 - I_{\mathbf{T}}\mathbf{T}^2 + II_{\mathbf{T}}\mathbf{T} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}$ remains valid. Additionally, by applying the double scalar product with the second-order unit tensor, $\mathbf{1}$, we obtain:

$$\mathbf{T}^3 : \mathbf{1} - I_{\mathbf{T}}\mathbf{T}^2 : \mathbf{1} + II_{\mathbf{T}}\mathbf{T} : \mathbf{1} - III_{\mathbf{T}}\mathbf{1} : \mathbf{1} = \mathbf{0} : \mathbf{1} \tag{1.311}$$

Taking into consideration $\mathbf{T}^3 : \mathbf{1} = \mathrm{Tr}(\mathbf{T}^3)$, $\mathbf{T}^2 : \mathbf{1} = \mathrm{Tr}(\mathbf{T}^2)$, $\mathbf{T} : \mathbf{1} = \mathrm{Tr}(\mathbf{T})$, $\mathbf{1} : \mathbf{1} = \mathrm{Tr}(\mathbf{1}) = 3$, $\mathbf{0} : \mathbf{1} = \mathrm{Tr}(\mathbf{0}) = 0$ in the equation (1.311) we obtain:

$$\begin{aligned} &\mathrm{Tr}(\mathbf{T}^3) - I_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^2) + II_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}) - III_{\mathbf{T}}\underbrace{\mathrm{Tr}(\mathbf{1})}_{=3} = 0 \\ &\Rightarrow III_{\mathbf{T}} = \frac{1}{3}\left[\mathrm{Tr}(\mathbf{T}^3) - I_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^2) + II_{\mathbf{T}}\mathrm{Tr}(\mathbf{T})\right] \end{aligned} \tag{1.312}$$

Replacing the values of the invariants, $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, given by equation (1.269), we obtain:

$$III_{\mathbf{T}} = \frac{1}{3}\left\{\mathrm{Tr}(\mathbf{T}^3) - \frac{3}{2}\mathrm{Tr}(\mathbf{T}^2)\mathrm{Tr}(\mathbf{T}) + \frac{1}{2}\left[\mathrm{Tr}(\mathbf{T})\right]^3\right\} \tag{1.313}$$

or in indicial notation

$$III_{\mathbf{T}} = \frac{1}{3}\left\{\mathrm{T}_{ij}\mathrm{T}_{jk}\mathrm{T}_{ki} - \frac{3}{2}\mathrm{T}_{ij}\mathrm{T}_{ji}\mathrm{T}_{kk} + \frac{1}{2}\mathrm{T}_{ii}\mathrm{T}_{jj}\mathrm{T}_{kk}\right\} \tag{1.314}$$

Problem 1.35: Based on the Cayley-Hamilton theorem, find the inverse of a tensor $\mathbf{T}$ in terms of tensor power.

Solution: The Cayley-Hamilton theorem states that:

$$\mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}$$

Carrying out the dot product between the previous equation and the tensor $\mathbf{T}^{-1}$, we obtain:

$$\begin{aligned} &\mathbf{T}^3 \cdot \mathbf{T}^{-1} - \mathbf{T}^2 \cdot \mathbf{T}^{-1} I_{\mathbf{T}} + \mathbf{T} \cdot \mathbf{T}^{-1} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} \cdot \mathbf{T}^{-1} = \mathbf{0} \cdot \mathbf{T}^{-1} \\ &\mathbf{T}^2 - \mathbf{T} I_{\mathbf{T}} + \mathbf{1} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{0} \\ &\Rightarrow \mathbf{T}^{-1} = \frac{1}{III_{\mathbf{T}}}\left(\mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1}\right) \end{aligned}$$

The Cayley-Hamilton theorem also applies to square matrices of order n. Let $\mathcal{A}_{n\times n}$ be a square n by n matrix. The characteristic determinant is given by:

$$\left|\lambda \mathbf{1}_{n\times n} - \mathcal{A}\right| = 0 \tag{1.315}$$

where $\mathbf{1}_{n\times n}$ is the identity n by n matrix. Developing the determinant (1.315) we ontain:

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} - \cdots (-1)^n I_n = 0 \tag{1.316}$$

where $I_1, I_2, \cdots, I_n$ are the invariants of $\mathcal{A}$. In the particular case when $n = 3$, the invariants are the same obtained for a second-order tensor, *i.e.*: $I_1 = I_{\mathbf{A}}$, $I_2 = II_{\mathbf{A}}$, $I_3 = III_{\mathbf{A}}$. Applying the Cayley-Hamilton theorem it is true that:

$$\mathcal{A}^n - I_1\mathcal{A}^{n-1} + I_2\mathcal{A}^{n-2} - \cdots + (-1)^n I_n \mathbf{1} = \mathbf{0} \tag{1.317}$$

By means of the relationship (1.317), we can obtain the inverse of the matrix $\mathcal{A}_{n\times n}$ by multiplying all the terms by the inverse, $\mathcal{A}^{-1}$, *i.e.*:

$$\begin{aligned}&\mathcal{A}^n\mathcal{A}^{-1}-I_1\mathcal{A}^{n-1}\mathcal{A}^{-1}+I_2\mathcal{A}^{n-2}\mathcal{A}^{-1}-\cdots+(-1)^n I_n\boldsymbol{1}\mathcal{A}^{-1}=\boldsymbol{0}\\&\Rightarrow\mathcal{A}^{n-1}-I_1\mathcal{A}^{n-2}+I_2\mathcal{A}^{n-3}-\cdots(-1)^{n-1}I_{n-1}\boldsymbol{1}+(-1)^n I_n\mathcal{A}^{-1}=\boldsymbol{0}\end{aligned} \tag{1.318}$$

then

$$\mathcal{A}^{-1}=\frac{(-1)^{n-1}}{I_n}\left(\mathcal{A}^{n-1}-I_1\mathcal{A}^{n-2}+I_2\mathcal{A}^{n-3}-\cdots(-1)^{n-1}I_{n-1}\boldsymbol{1}\right) \tag{1.319}$$

I_n is the determinant of $\mathcal{A}_{n\times n}$. Then, the inverse exists if $I_n=\det(\mathcal{A})\neq 0$.

Problem 1.36: Check the Cayley-Hamilton theorem by using a second-order tensor whose Cartesian components are given by:

$$\mathcal{T}=\begin{bmatrix}5&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

Solution:
The Cayley-Hamilton theorem states that:

$$\mathcal{T}^3-\mathcal{T}^2 I_{\mathbf{T}}+\mathcal{T}I\!I_{\mathbf{T}}-I\!I\!I_{\mathbf{T}}\boldsymbol{1}=\boldsymbol{0}$$

where $I_{\mathbf{T}}=5+2+1=8$, $I\!I_{\mathbf{T}}=10+2+5=17$, $I\!I\!I_{\mathbf{T}}=10$, and

$$\mathcal{T}^3=\begin{bmatrix}5^3&0&0\\0&2^3&0\\0&0&1\end{bmatrix}=\begin{bmatrix}125&0&0\\0&8&0\\0&0&1\end{bmatrix}\quad;\quad\mathcal{T}^2=\begin{bmatrix}5^2&0&0\\0&2^2&0\\0&0&1\end{bmatrix}=\begin{bmatrix}25&0&0\\0&4&0\\0&0&1\end{bmatrix}$$

By applying the Cayley-Hamilton theorem, we can verify that it is true:

$$\begin{bmatrix}125&0&0\\0&8&0\\0&0&1\end{bmatrix}-8\begin{bmatrix}25&0&0\\0&4&0\\0&0&1\end{bmatrix}+17\begin{bmatrix}5&0&0\\0&2&0\\0&0&1\end{bmatrix}-10\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

1.5.7 Norms of Tensors

The magnitude (module) of a tensor, also known as the *Frobenius norm*, is given below:

$$\|\vec{v}\|=\sqrt{\vec{v}\cdot\vec{v}}=\sqrt{v_i v_i}\quad\text{(vector)} \tag{1.320}$$

$$\|\mathbf{T}\|=\sqrt{\mathbf{T}:\mathbf{T}}=\sqrt{\mathsf{T}_{ij}\mathsf{T}_{ij}}\quad\text{(second-order tensor)} \tag{1.321}$$

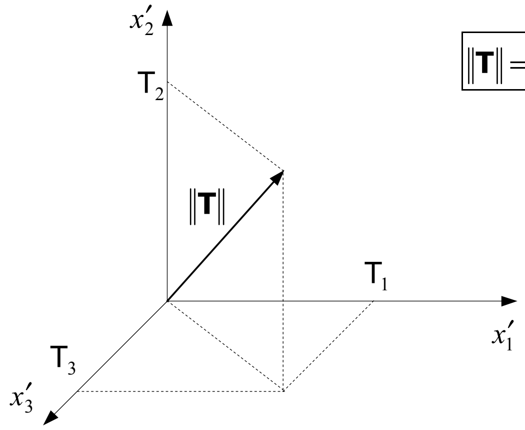
$$\|\mathbf{A}\|=\sqrt{\mathbf{A}:\cdot\mathbf{A}}=\sqrt{\mathsf{A}_{ijk}\mathsf{A}_{ijk}}\quad\text{(third-order tensor)} \tag{1.322}$$

$$\|\mathbb{C}\|=\sqrt{\mathbb{C}::\mathbb{C}}=\sqrt{\mathbb{C}_{ijkl}\mathbb{C}_{ijkl}}\quad\text{(fourth-order tensor)} \tag{1.323}$$

Interpreting the Frobenius norm of $\mathbf{T}$ is done by considering the principal space of $\mathbf{T}$ where $\mathsf{T}_1,\mathsf{T}_2,\mathsf{T}_3$ are the eigenvalues of $\mathbf{T}$. In this space, it follows that:

$$\|\mathbf{T}\|=\sqrt{\mathbf{T}:\mathbf{T}}=\sqrt{\mathsf{T}_{ij}\mathsf{T}_{ij}}=\sqrt{\mathsf{T}_1^2+\mathsf{T}_2^2+\mathsf{T}_3^2}=\sqrt{I_{\mathbf{T}}^2-2\,I\!I_{\mathbf{T}}} \tag{1.324}$$

As we can verify $\|\mathbf{T}\|$ is an invariant, and $\|\mathbf{T}\|$ represents a measurement of distance as shown in Figure 1.28.



$$\boxed{\|\mathbf{T}\| = \sqrt{\mathbf{T}:\mathbf{T}} = \sqrt{I_{\mathbf{T}}^2 - 2\,II_{\mathbf{T}}}}$$

Figure 1.28: Norm of a second-order tensor.

1.5.8 Isotropic and Anisotropic Tensor

A tensor is called *isotropic* when its components are the same in any coordinate system, otherwise the tensor is said to be *anisotropic*.

Let $\mathcal{T}$ and $\mathcal{T}'$ represent the tensor components $\mathbf{T}$ in the systems $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_i'$, respectively, so, the tensor is isotropic if $\mathcal{T} = \mathcal{T}'$ on any arbitrary basis.

Isotropic first-order tensor

Let $\vec{\mathbf{v}}$ be a vector that is represented by its components, $\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3$, in the coordinate system x_1, x_2, x_3. The representation of these components in a new coordinate system, x_1', x_2', x_3', are given by $\mathsf{v}_1', \mathsf{v}_2', \mathsf{v}_3'$, so the transformation law for these components is:

$$\vec{\mathbf{v}} = \mathsf{v}_i \hat{\mathbf{e}}_i = \mathsf{v}_j' \hat{\mathbf{e}}_j' \quad \Rightarrow \quad \mathsf{v}_i' = a_{ij} \mathsf{v}_j \tag{1.325}$$

By definition, $\vec{\mathbf{v}}$ is an isotropic tensor if it holds that $\mathsf{v}_i = \mathsf{v}_i'$, and this is only possible if $\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_j'$, *i.e.* there is no change of system, or if the tensor is the zero vector, *i.e.* $\mathsf{v}_i = \mathsf{v}_i' = 0_i$. Then, the unique isotropic first-order tensor is the zero vector $\vec{\mathbf{0}}$.

Isotropic second-order tensor

An example of a second-order isotropic tensor is the unit tensor, $\mathbf{1}$, whose components are represented by δ_{kl} (Kronecker delta). In the demonstration, we use the transformation law for a second-order tensor components, obtained in (1.248), thus:

$$\delta_{ij}' = a_{ik} a_{jl} \delta_{kl} = \underbrace{a_{ik} a_{jk}}_{\mathcal{A}\mathcal{A}^T = \mathbf{1}} = \delta_{ij} \tag{1.326}$$

An immediate observation of the isotropy of unit tensor $\mathbf{1}$ is that any spherical tensor ($\alpha\mathbf{1}$) is also an isotropic tensor. So, if a second-order tensor is isotropic it is spherical and vice versa.

Isotropic third-order tensor

An example of a third-order isotropic tensor is the Levi-Civita pseudo-tensor, defined in (1.182), which is not a "real" tensor in the strict meaning of the word. With reference to the transformation law for the third-order tensor components, (see equation (1.248)), we can conclude that:

$$\epsilon'_{ijk} = a_{il} a_{jm} a_{kn} \epsilon_{lmn} = \underbrace{|\mathcal{A}|}_{1} \epsilon_{ijk} = \epsilon_{ijk} \quad \text{(see **Problem 1.21**)} \tag{1.327}$$

Isotropic fourth-order tensor

With reference to the transformation law for fourth-order tensor components, (see equation (1.249)), it is possible to demonstrate that the following tensors are isotropic:

$$\bar{\bar{\mathbb{I}}}_{ijkl} = \delta_{ij}\delta_{kl} \quad ; \quad \mathbb{I}_{ijkl} = \delta_{ik}\delta_{jl} \quad ; \quad \bar{\mathbb{I}}_{ijkl} = \delta_{il}\delta_{jk} \tag{1.328}$$

Therefore, any fourth-order isotropic tensor can be represented by a linear combination of the three tensors given in (1.328), *e.g.*:

$$\begin{gathered} \mathbb{D} = a_0 \bar{\bar{\mathbb{I}}} + a_1 \mathbb{I} + a_2 \bar{\mathbb{I}} \\ \mathbb{D} = a_0 \mathbf{1} \otimes \mathbf{1} + a_1 \mathbf{1} \overline{\otimes} \mathbf{1} + a_2 \mathbf{1} \underline{\otimes} \mathbf{1} \\ \mathbb{D}_{ijkl} = a_0 \delta_{ij}\delta_{kl} + a_1 \delta_{ik}\delta_{jl} + a_2 \delta_{il}\delta_{jk} \end{gathered} \tag{1.329}$$

Problem 1.37: Let $\mathbb{C}$ be a fourth-order tensor, whose components are given by:

$$\mathbb{C}_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where λ, μ are constant real numbers. Show that $\mathbb{C}$ is an isotropic tensor.

Solution:

Applying the transformation law for fourth-order tensor components:

$$\mathbb{C}'_{ijkl} = a_{im} a_{jn} a_{kp} a_{lq} \mathbb{C}_{mnpq}$$

and by replacing the relation $\mathbb{C}_{mnpq} = \lambda \delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})$ in the above equation, we obtain:

$$\begin{aligned} \mathbb{C}'_{ijkl} &= a_{im} a_{jn} a_{kp} a_{lq} \left[\lambda \delta_{mn}\delta_{pq} + \mu(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})\right] \\ &= \lambda a_{im} a_{jn} a_{kp} a_{lq} \delta_{mn}\delta_{pq} + \mu(a_{im} a_{jn} a_{kp} a_{lq} \delta_{mp}\delta_{nq} + a_{im} a_{jn} a_{kp} a_{lq} \delta_{mq}\delta_{np}) \\ &= \lambda a_{in} a_{jn} a_{kq} a_{lq} + \mu(a_{ip} a_{jq} a_{kp} a_{lq} + a_{iq} a_{jn} a_{kn} a_{lq}) \\ &= \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &= \mathbb{C}_{ijkl} \end{aligned}$$

which is proof that $\mathbb{C}$ is an isotropic tensor.

1.5.9 Coaxial Tensors

Two arbitrary second-order tensors, $\mathbf{T}$ and $\mathbf{S}$, are coaxial tensors if they have the same eigenvectors. It is easy to show that if two tensors are coaxial, this means the dot product between them is commutative, and vice versa, *i.e.*:

$$\text{if} \quad \mathbf{T} \cdot \mathbf{S} = \mathbf{S} \cdot \mathbf{T} \quad \Leftrightarrow \quad \mathbf{S}, \mathbf{T} \text{ are coaxial} \tag{1.330}$$

If $\mathbf{T}$ and $\mathbf{S}$ are coaxial as well as symmetric tensors, the spectral representations of these tensors are given by:

$$\mathbf{T} = \sum_{a=1}^{3} \mathrm{T}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \quad ; \quad \mathbf{S} = \sum_{a=1}^{3} \mathrm{S}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.331}$$

An immediate result of (1.330) is that the tensor $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ are coaxial tensors:

$$\mathbf{S}^{-1}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{S}^{-1}=\mathbf{1}$$
$$\mathbf{S}=\sum_{a=1}^{3}\mathsf{S}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\qquad;\qquad\mathbf{S}^{-1}=\sum_{a=1}^{3}\frac{1}{\mathsf{S}_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\tag{1.332}$$

where S_a, $\dfrac{1}{\mathsf{S}_a}$, are the eigenvalues of $\mathbf{S}$ and $\mathbf{S}^{-1}$, respectively.

If $\mathbf{S}$ and $\mathbf{T}$ are coaxial symmetric tensors, the resulting tensor ($\mathbf{S}\cdot\mathbf{T}$) becomes another symmetric tensor. To prove this we start from the definition of coaxial tensors:

$$\mathbf{T}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{T}\quad\Rightarrow\quad\mathbf{T}\cdot\mathbf{S}-\mathbf{S}\cdot\mathbf{T}=\mathbf{0}\quad\Rightarrow\quad\mathbf{T}\cdot\mathbf{S}-(\mathbf{T}\cdot\mathbf{S})^T=\mathbf{0}\Rightarrow 2(\mathbf{T}\cdot\mathbf{S})^{skew}=\mathbf{0}\tag{1.333}$$

Then, if the antisymmetric part of a tensor is a zero tensor, it follows that this tensor is symmetric:

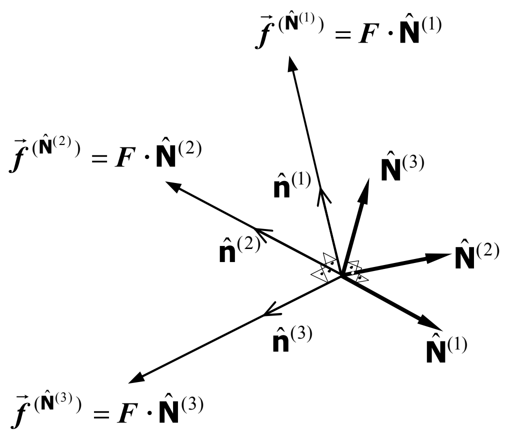
$$(\mathbf{T}\cdot\mathbf{S})^{skew}=\mathbf{0}\quad\Rightarrow\quad(\mathbf{T}\cdot\mathbf{S})\equiv(\mathbf{T}\cdot\mathbf{S})^{sym}\tag{1.334}$$

1.5.10 Polar Decomposition

Let $\boldsymbol{F}$ be an arbitrary nonsingular second-order tensor, *i.e.* ($\det(\boldsymbol{F})\neq 0\Rightarrow\exists\boldsymbol{F}^{-1}$). Additionally, as previously seen, it satisfies the condition $\boldsymbol{F}\cdot\hat{\mathbf{N}}=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}})}\right\|\hat{\mathbf{n}}=\lambda_{(\hat{\mathbf{n}})}\hat{\mathbf{n}}\neq\vec{\mathbf{0}}$, since $\det(\boldsymbol{F})\neq 0$. After that, given an orthonormal basis $\hat{\mathbf{N}}^{(a)}$, we can obtain:

$$\boldsymbol{F}^{-1}\cdot\boldsymbol{F}=\mathbf{1}=\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}$$
$$\Rightarrow\boldsymbol{F}=\boldsymbol{F}\cdot\mathbf{1}=\boldsymbol{F}\cdot\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=\sum_{a=1}^{3}\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\tag{1.335}$$
$$\Rightarrow\boldsymbol{F}=\sum_{a=1}^{3}\lambda_a\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}$$

NOTE: The representation of $\boldsymbol{F}$, given in (1.335), is not the spectral representation of $\boldsymbol{F}$ in the strict sense of the word, *i.e.*, λ_a are not eigenvalues of $\boldsymbol{F}$, and neither $\hat{\mathbf{n}}^{(a)}$ nor $\hat{\mathbf{N}}^{(a)}$ are eigenvectors of $\boldsymbol{F}$. ■



$$\boxed{\begin{aligned}\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(1)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(1)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(1)})}\right\|\hat{\mathbf{n}}^{(1)}=\lambda_1\hat{\mathbf{n}}^{(1)}\\\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(2)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(2)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(2)})}\right\|\hat{\mathbf{n}}^{(2)}=\lambda_2\hat{\mathbf{n}}^{(2)}\\\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(3)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(3)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(3)})}\right\|\hat{\mathbf{n}}^{(3)}=\lambda_3\hat{\mathbf{n}}^{(3)}\end{aligned}}$$

Figure 1.29: Projecting $\boldsymbol{F}$ onto $\hat{\mathbf{N}}^{(a)}$.

Note that for the arbitrary orthonormal basis $\hat{\mathbf{N}}^{(a)}$, the new basis $\hat{\mathbf{n}}^{(a)}$ will not necessarily be orthonormal. We seek to find a basis $\hat{\mathbf{N}}^{(a)}$ so that the new basis $\hat{\mathbf{n}}^{(a)}$ is orthonormal, (see Figure 1.29), *i.e.* $\vec{f}^{(\hat{\mathbf{N}}^{(1)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(2)})} = 0$, $\vec{f}^{(\hat{\mathbf{N}}^{(2)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(3)})} = 0$, $\vec{f}^{(\hat{\mathbf{N}}^{(3)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(1)})} = 0$. Then we look for a space in accordance with the following orthogonal transformation $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$, which ensures $\hat{\mathbf{n}}^{(a)}$ orthonormality since an orthogonal transformation changes neither angles between vectors nor their magnitudes.

Now, consider that there is a transformation from $\hat{\mathbf{N}}^{(a)}$ to $\hat{\mathbf{n}}^{(a)}$, which is given by the following orthogonal transformation $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$, then we can state that:

$$
\begin{gathered}
\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{R} \cdot \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{R} \cdot \mathbf{U} \\
\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} \quad \Rightarrow \quad \mathbf{U} = \mathbf{R}^T \cdot \boldsymbol{F}
\end{gathered}
\tag{1.336}
$$

where we have defined the tensor $\mathbf{U} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$. Note that $\mathbf{U}$ is a symmetric tensor, *i.e.* $\mathbf{U} = \mathbf{U}^T$. This condition is easily verified by the fact that $\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ is also symmetric. Now considering that $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \Rightarrow \hat{\mathbf{N}}^{(a)} = \mathbf{R}^T \cdot \hat{\mathbf{n}}^{(a)} = \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R}$, we obtain:

$$
\begin{gathered}
\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} = \mathbf{V} \cdot \mathbf{R} \\
\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R} \quad \Rightarrow \quad \mathbf{V} = \boldsymbol{F} \cdot \mathbf{R}^T
\end{gathered}
\tag{1.337}
$$

where we have defined the symmetric second-order tensor $\mathbf{V} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$. By comparing the spectral representation of $\mathbf{U}$ with $\mathbf{V}$, we can conclude that they have the same eigenvalues but different eigenvectors, and they are related by $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$.

With reference to the above considerations, we can define the polar decomposition:

$$
\boxed{\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}} \quad \textit{Polar Decomposition} \tag{1.338}
$$

Carrying out the dot product between $\boldsymbol{F}^T$ and $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$, we obtain:

$$
\underbrace{\boldsymbol{F}^T \cdot \boldsymbol{F}}_{\boldsymbol{C}} = \boldsymbol{F}^T \cdot \mathbf{R} \cdot \mathbf{U} = (\mathbf{R}^T \cdot \boldsymbol{F})^T \cdot \mathbf{U} = \mathbf{U}^T \cdot \mathbf{U} = \mathbf{U}^2 \quad \Rightarrow \quad \mathbf{U} = \pm\sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}} = \pm\sqrt{\boldsymbol{C}} \tag{1.339}
$$

Moreover, by carrying out the dot product between $\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R}$ and $\boldsymbol{F}^T$, we obtain:

$$
\underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^T}_{\boldsymbol{b}} = \mathbf{V} \cdot \mathbf{R} \cdot \boldsymbol{F}^T = \mathbf{V} \cdot (\boldsymbol{F} \cdot \mathbf{R}^T)^T = \mathbf{V} \cdot \mathbf{V}^T = \mathbf{V}^2 \quad \Rightarrow \quad \mathbf{V} = \pm\sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T} = \pm\sqrt{\boldsymbol{b}} \tag{1.340}
$$

Since $\det(\boldsymbol{F}) \neq 0$, the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are positive definite symmetric tensors, (see **Problem 1.25**), which implies that the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{b}$ are all real and positive. However, up to now, $\det(\boldsymbol{F}) \neq 0$ is the only restriction imposed on the tensor $\boldsymbol{F}$. Therefore, we have the following possibilities:

- If $\det(\boldsymbol{F}) > 0$

In this scenario, we have $\det(\boldsymbol{F}) = \det(\mathbf{R})\det(\mathbf{U}) = \det(\mathbf{V})\det(\mathbf{R}) > 0$, which results in the following cases:

$$\begin{cases}\mathbf{R} - \text{Proper ortogonal tensor} \\ \mathbf{U},\mathbf{V} - \text{Positive definite tensors}\end{cases} \text{or} \begin{cases}\mathbf{R} - \text{Improper orthogonal tensor} \\ \mathbf{U},\mathbf{V} - \text{Negative definite tensors}\end{cases}$$

- If $\det(\boldsymbol{F}) < 0$

In this situation, we have $\det(\boldsymbol{F}) = \det(\mathbf{R})\det(\mathbf{U}) = \det(\mathbf{V})\det(\mathbf{R}) < 0$, which give us the following cases:

$$\begin{cases}\mathbf{R} - \text{Proper orthogonal tensor} \\ \mathbf{U},\mathbf{V} - \text{Negative definite tensors}\end{cases} \text{or} \begin{cases}\mathbf{R} - \text{Improper orthogonal tensor} \\ \mathbf{U},\mathbf{V} - \text{Positive definite tensors}\end{cases}$$

NOTE: In Chapter 2 we will work with some special tensors where $\boldsymbol{F}$ is a nonsingular tensor, $\det(\boldsymbol{F}) \neq 0$, and $\det(\boldsymbol{F}) > 0$. $\mathbf{U}$ and $\mathbf{V}$ are positive definite tensors and $\mathbf{R}$ is a rotation tensor, *i.e.* a proper orthogonal tensor. ■

1.5.11 Partial Derivative with Tensors

The first derivative of a tensor with respect to itself is defined as:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}} \equiv \mathbf{A}_{,\mathbf{A}} = \frac{\partial \mathrm{A}_{ij}}{\partial \mathrm{A}_{kl}} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \delta_{ik}\delta_{jl} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \mathbb{I} \tag{1.341}$$

The derivative of a tensor trace with respect to a tensor:

$$\frac{\partial [\mathrm{Tr}(\mathbf{A})]}{\partial \mathbf{A}} \equiv [\mathrm{Tr}(\mathbf{A})]_{,\mathbf{A}} = \frac{\partial \mathrm{A}_{kk}}{\partial \mathrm{A}_{ij}} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \delta_{ki}\delta_{kj} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \delta_{ij} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathbf{1} \tag{1.342}$$

The derivative of the tensor trace squared with respect to the tensor is given by:

$$\frac{\partial [\mathrm{Tr}(\mathbf{A})]^2}{\partial \mathbf{A}} = 2\mathrm{Tr}(\mathbf{A}) \frac{\partial [\mathrm{Tr}(\mathbf{A})]}{\partial \mathbf{A}} = 2\mathrm{Tr}(\mathbf{A})\mathbf{1} \tag{1.343}$$

And, the derivative of the trace of the tensor squared with respect to tensor is given by:

$$\begin{aligned} \frac{\partial [\mathrm{Tr}(\mathbf{A}^2)]}{\partial \mathbf{A}} &= \frac{\partial (\mathrm{A}_{sr}\mathrm{A}_{rs})}{\partial \mathrm{A}_{ij}} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \left[\mathrm{A}_{rs} \frac{\partial (\mathrm{A}_{sr})}{\partial \mathrm{A}_{ij}} + \mathrm{A}_{sr} \frac{\partial (\mathrm{A}_{rs})}{\partial \mathrm{A}_{ij}} \right] (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &= [\mathrm{A}_{rs}\delta_{si}\delta_{rj} + \mathrm{A}_{sr}\delta_{ri}\delta_{sj}] (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = [\mathrm{A}_{ji} + \mathrm{A}_{ji}] (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &= 2\mathrm{A}_{ji} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = 2\mathbf{A}^T \end{aligned} \tag{1.344}$$

We leave the reader with the following demonstration:

$$\frac{\partial [\mathrm{Tr}(\mathbf{A}^3)]}{\partial \mathbf{A}} = 3(\mathbf{A}^2)^T \tag{1.345}$$

Then, if we are considering a symmetric second-order tensor, $\mathbf{C}$, it is true that $\frac{\partial [\mathrm{Tr}(\mathbf{C})]}{\partial \mathbf{C}} = \mathbf{1}$, $\frac{\partial [\mathrm{Tr}(\mathbf{C})]^2}{\partial \mathbf{C}} = 2\mathrm{Tr}(\mathbf{C})\mathbf{1}$, $\frac{\partial [\mathrm{Tr}(\mathbf{C}^2)]}{\partial \mathbf{C}} = 2\mathbf{C}^T = 2\mathbf{C}$, $\frac{\partial [\mathrm{Tr}(\mathbf{C}^3)]}{\partial \mathbf{C}} = 3(\mathbf{C}^2)^T = 3\mathbf{C}^2$.

Moreover, we can say that the derivative of the Frobenius norm of $\mathbf{C}$ is given by:

$$\frac{\partial\|\mathbf{C}\|}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathbf{C}:\mathbf{C}}\right)}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathrm{Tr}(\mathbf{C}\cdot\mathbf{C}^T)}\right)}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathrm{Tr}(\mathbf{C}^2)}\right)}{\partial\mathbf{C}}=\frac{1}{2}\left[\mathrm{Tr}(\mathbf{C}^2)\right]^{\frac{-1}{2}}\left[\mathrm{Tr}(\mathbf{C}^2)\right]_{,\mathbf{C}}$$
$$=\frac{1}{2}\left[\mathrm{Tr}(\mathbf{C}^2)\right]^{\frac{-1}{2}}2\mathbf{C} \tag{1.346}$$

or:

$$\frac{\partial\|\mathbf{C}\|}{\partial\mathbf{C}}=\frac{\mathbf{C}}{\sqrt{\mathrm{Tr}(\mathbf{C}^2)}}=\frac{\mathbf{C}}{\|\mathbf{C}\|} \tag{1.347}$$

Another interesting derivative is presented below:

$$\frac{\partial\left(\mathrm{n}_i\mathrm{C}_{ij}\mathrm{n}_j\right)}{\partial\mathrm{n}_k}=\frac{\partial\mathrm{n}_i}{\partial\mathrm{n}_k}\mathrm{C}_{ij}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ij}\frac{\partial\mathrm{n}_j}{\partial\mathrm{n}_k}=\delta_{ik}\mathrm{C}_{ij}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ij}\delta_{jk}=\mathrm{C}_{kj}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ik}$$
$$=\mathrm{C}_{kj}\mathrm{n}_j+\mathrm{C}_{jk}\mathrm{n}_j=(\mathrm{C}_{kj}+\mathrm{C}_{jk})\mathrm{n}_j=2\mathrm{C}_{kj}^{sym}\mathrm{n}_j=2\mathrm{C}_{kj}\mathrm{n}_j \tag{1.348}$$

where we have assumed that $\mathbf{C}$ is symmetric, *i.e.*, $\mathrm{C}_{kj}=\mathrm{C}_{jk}$.

Let $\mathbf{C}$ be a symmetric second-order tensor. The partial derivative of $\mathbf{C}^{-1}$ with respect to the tensor $\mathbf{C}$ is obtained by using the following relationship:

$$\frac{\partial\mathbf{1}}{\partial\mathbf{C}}=\frac{\partial\left(\mathbf{C}^{-1}\cdot\mathbf{C}\right)}{\partial\mathbf{C}}=\mathbb{O} \tag{1.349}$$

where $\mathbb{O}$ is the fourth-order zero tensor and the above equation in indicial notation becomes:

$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}=\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}+\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}=\mathbb{O}_{ikjl}$$
$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\quad\Rightarrow\quad\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}\mathrm{C}_{jr}^{-1}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{jr}^{-1} \tag{1.350}$$
$$\Rightarrow\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\delta_{qr}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{jr}^{-1}$$

whereas $\mathrm{C}_{qj}=\frac{1}{2}\left(\mathrm{C}_{qj}+\mathrm{C}_{jq}\right)$, so we can conclude that:

$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\delta_{qr}=-\mathrm{C}_{iq}^{-1}\frac{1}{2}\left[\frac{\partial\left(\mathrm{C}_{qj}+\mathrm{C}_{jq}\right)}{\partial\mathrm{C}_{kl}}\right]\mathrm{C}_{jr}^{-1}$$
$$\frac{\partial\left(\mathrm{C}_{ir}^{-1}\right)}{\partial\mathrm{C}_{kl}}=-\mathrm{C}_{iq}^{-1}\frac{1}{2}\left[\delta_{qk}\delta_{jl}+\delta_{jk}\delta_{ql}\right]\mathrm{C}_{jr}^{-1}=-\frac{1}{2}\left[\mathrm{C}_{iq}^{-1}\delta_{qk}\delta_{jl}\mathrm{C}_{jr}^{-1}+\mathrm{C}_{iq}^{-1}\delta_{jk}\delta_{ql}\mathrm{C}_{jr}^{-1}\right] \tag{1.351}$$
$$\frac{\partial\left(\mathrm{C}_{ir}^{-1}\right)}{\partial\mathrm{C}_{kl}}=-\frac{1}{2}\left[\mathrm{C}_{ik}^{-1}\mathrm{C}_{lr}^{-1}+\mathrm{C}_{il}^{-1}\mathrm{C}_{kr}^{-1}\right]$$

Or in tensorial notation:

$$\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}}=-\frac{1}{2}\left[\mathbf{C}^{-1}\overline{\otimes}\mathbf{C}^{-1}+\mathbf{C}^{-1}\underline{\otimes}\mathbf{C}^{-1}\right] \tag{1.352}$$

NOTE: Note that, if we had not replaced the symmetric part of C_{qj} in (1.351), we would have found that $\dfrac{\partial(\mathsf{C}_{iq}^{-1})}{\partial \mathsf{C}_{kl}}\delta_{qr} = -\mathsf{C}_{iq}^{-1}\dfrac{\partial(\mathsf{C}_{qj})}{\partial \mathsf{C}_{kl}}\mathsf{C}_{jr}^{-1} = -\mathsf{C}_{iq}^{-1}\delta_{qk}\delta_{jl}\mathsf{C}_{jr}^{-1} = -\mathsf{C}_{ik}^{-1}\mathsf{C}_{lr}^{-1}$, which is a non-symmetric tensor. ■

1.5.11.1 Partial Derivative of Invariants

Let $\mathbf{T}$ be a second-order tensor. The partial derivative of $I_{\mathbf{T}}$ with respect to $\mathbf{T}$, (see equation (1.342)), is:

$$\frac{\partial[I_{\mathbf{T}}]}{\partial \mathbf{T}} = \frac{\partial[\mathrm{Tr}(\mathbf{T})]}{\partial \mathbf{T}} = [\mathrm{Tr}(\mathbf{T})]_{,\mathbf{T}} = \mathbf{1} \tag{1.353}$$

The partial derivative of $II_{\mathbf{T}}$ with respect to $\mathbf{T}$, (see equation (1.342)), is:

$$\begin{aligned}
\frac{\partial[II_{\mathbf{T}}]}{\partial \mathbf{T}} = \frac{\partial}{\partial \mathbf{T}}\left\{\frac{1}{2}\left[[\mathrm{Tr}(\mathbf{T})]^2 - \mathrm{Tr}(\mathbf{T}^2)\right]\right\} &= \frac{1}{2}\left[\frac{\partial[\mathrm{Tr}(\mathbf{T})]^2}{\partial \mathbf{T}} - \frac{\partial[\mathrm{Tr}(\mathbf{T}^2)]}{\partial \mathbf{T}}\right] \\
&= \frac{1}{2}\left[2(\mathrm{Tr}\mathbf{T})\mathbf{1} - 2\mathbf{T}^T\right] \\
&= \mathrm{Tr}(\mathbf{T})\mathbf{1} - \mathbf{T}^T
\end{aligned} \tag{1.354}$$

Next, we apply the Cayley-Hamilton theorem so as to represent $\mathbf{T}$ as:

$$\begin{aligned}
&\mathbf{T}^3 : \mathbf{T}^{-2} - I_{\mathbf{T}}\mathbf{T}^2 : \mathbf{T}^{-2} + II_{\mathbf{T}}\mathbf{T} : \mathbf{T}^{-2} - III_{\mathbf{T}}\mathbf{1} : \mathbf{T}^{-2} = \mathbf{0} \\
&\mathbf{T} - I_{\mathbf{T}}\mathbf{1} + II_{\mathbf{T}}\mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{T}^{-2} = \mathbf{0} \\
&\Rightarrow \mathbf{T} = I_{\mathbf{T}}\mathbf{1} - II_{\mathbf{T}}\mathbf{T}^{-1} + III_{\mathbf{T}}\mathbf{T}^{-2}
\end{aligned} \tag{1.355}$$

By substituting (1.355) into the equation in (1.354), we obtain:

$$\frac{\partial[II_{\mathbf{T}}]}{\partial \mathbf{T}} = \mathrm{Tr}(\mathbf{T})\mathbf{1} - \mathbf{T}^T = \mathrm{Tr}(\mathbf{T})\mathbf{1} - \left(I_{\mathbf{T}}\mathbf{1} - II_{\mathbf{T}}\mathbf{T}^{-1} + III_{\mathbf{T}}\mathbf{T}^{-2}\right)^T = \left(II_{\mathbf{T}}\mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{T}^{-2}\right)^T \tag{1.356}$$

To find the partial derivative of the third invariant, we can start with the definition given in (1.313), so:

$$\begin{aligned}
\frac{\partial[III_{\mathbf{T}}]}{\partial \mathbf{T}} &= \frac{\partial}{\partial \mathbf{T}}\left\{\frac{1}{3}\mathrm{Tr}(\mathbf{T}^3) - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\mathrm{Tr}(\mathbf{T}) + \frac{1}{6}[\mathrm{Tr}(\mathbf{T})]^3\right\} \\
&= \frac{1}{3}3(\mathbf{T}^2)^T - \frac{1}{2}\frac{\partial[\mathrm{Tr}(\mathbf{T}^2)]}{\partial \mathbf{T}}\mathrm{Tr}(\mathbf{T}) - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\frac{\partial[\mathrm{Tr}(\mathbf{T})]}{\partial \mathbf{T}} + \frac{3}{6}[\mathrm{Tr}(\mathbf{T})]^2\mathbf{1} \\
&= (\mathbf{T}^2)^T - \mathrm{Tr}(\mathbf{T})\mathbf{T}^T - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\mathbf{1} + \frac{1}{2}[\mathrm{Tr}(\mathbf{T})]^2\mathbf{1} \\
&= (\mathbf{T}^2)^T - \mathrm{Tr}(\mathbf{T})\mathbf{T}^T + \frac{1}{2}\left[[\mathrm{Tr}(\mathbf{T})]^2 - \mathrm{Tr}(\mathbf{T}^2)\right]\mathbf{1} \\
&= (\mathbf{T}^2)^T - I_{\mathbf{T}}\mathbf{T}^T + II_{\mathbf{T}}\mathbf{1}
\end{aligned} \tag{1.357}$$

Once again using the Cayley-Hamilton theorem we obtain:

$$\begin{aligned}
&\mathbf{T}^3 \cdot \mathbf{T}^{-1} - I_{\mathbf{T}}\mathbf{T}^2 \cdot \mathbf{T}^{-1} + II_{\mathbf{T}}\mathbf{T} \cdot \mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{1} \cdot \mathbf{T}^{-1} = \mathbf{0} \\
&\mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1} - III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{0} \\
&\Rightarrow III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1}
\end{aligned} \tag{1.358}$$

and the transpose:

$$\left(III_{\mathbf{T}} \mathbf{T}^{-1} \right)^T = \left(\mathbf{T}^2 - I_{\mathbf{T}} \mathbf{T} + II_{\mathbf{T}} \mathbf{1} \right)^T = \left(\mathbf{T}^2 \right)^T - I_{\mathbf{T}} \mathbf{T}^T + II_{\mathbf{T}} \mathbf{1} \tag{1.359}$$

By comparing (1.357) with (1.359) we find another way to express the derivative of $III_{\mathbf{T}}$ with respect to $\mathbf{T}$, *i.e.*:

$$\frac{\partial [III_{\mathbf{T}}]}{\partial \mathbf{T}} = \left(III_{\mathbf{T}} \mathbf{T}^{-1} \right)^T = III_{\mathbf{T}} \mathbf{T}^{-T} \tag{1.360}$$

1.5.11.2 Time Derivative of Tensors

Let us assume a second-order tensor depends on the time, t, *i.e.* $\mathbf{T} = \mathbf{T}(t)$. Then, we define the first time derivative and the second time derivative of the tensor $\mathbf{T}$, respectively, as:

$$\frac{D}{Dt}\mathbf{T} = \dot{\mathbf{T}} \quad ; \quad \frac{D^2}{Dt^2}\mathbf{T} = \ddot{\mathbf{T}} \tag{1.361}$$

The time derivative of a tensor determinant is defined as:

$$\frac{D}{Dt}[\det(\mathbf{T})] = \frac{D\mathsf{T}_{ij}}{Dt} \operatorname{cof}(\mathsf{T}_{ij}) \tag{1.362}$$

where $\operatorname{cof}(\mathsf{T}_{ij})$ is the cofactor of T_{ij} and defined as $[\operatorname{cof}(\mathsf{T}_{ij})]^T = \det(\mathbf{T})(\mathbf{T}^{-1})_{ij}$.

Problem 1.38: Consider that $J = [\det(\boldsymbol{b})]^{\frac{1}{2}} = (III_{\boldsymbol{b}})^{\frac{1}{2}}$, where $\boldsymbol{b}$ is a symmetric second-order tensor, *i.e.* $\boldsymbol{b} = \boldsymbol{b}^T$. Obtain the partial derivatives of J and $\ln(J)$ with respect to $\boldsymbol{b}$.

Solution:

$$\Rightarrow \quad \frac{\partial J}{\partial \boldsymbol{b}} = \frac{\partial \left[(III_{\boldsymbol{b}})^{\frac{1}{2}} \right]}{\partial \boldsymbol{b}}$$

$$= \frac{1}{2} (III_{\boldsymbol{b}})^{\frac{-1}{2}} \frac{\partial III_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2} (III_{\boldsymbol{b}})^{\frac{-1}{2}} III_{\boldsymbol{b}} \boldsymbol{b}^{-T}$$

$$= \frac{1}{2} (III_{\boldsymbol{b}})^{\frac{1}{2}} \boldsymbol{b}^{-1} = \frac{1}{2} J \boldsymbol{b}^{-1}$$

$$\Rightarrow \quad \frac{\partial [\ln(J)]}{\partial \boldsymbol{b}} = \frac{\partial \left[\ln\left(III_{\boldsymbol{b}}^{\frac{1}{2}} \right) \right]}{\partial \boldsymbol{b}} = \frac{1}{2\, III_{\boldsymbol{b}}} \frac{\partial III_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2} \boldsymbol{b}^{-1}$$

1.5.12 Spherical and Deviatoric Tensors

Any tensor can be decomposed into a spherical and a deviatoric part, so, for a given second-order tensor $\mathbf{T}$, this decomposition is represented by:

$$\mathbf{T} = \mathbf{T}^{sph} + \mathbf{T}^{dev} = \frac{\operatorname{Tr}(\mathbf{T})}{3}\mathbf{1} + \mathbf{T}^{dev} = \frac{I_{\mathbf{T}}}{3}\mathbf{1} + \mathbf{T}^{dev} = \mathsf{T}_m \mathbf{1} + \mathbf{T}^{dev} \tag{1.363}$$

The deviatoric part of the tensor $\mathbf{T}$ is defined as:

$$\mathbf{T}^{dev} = \mathbf{T} - \frac{\mathrm{Tr}(\mathbf{T})}{3}\mathbf{1} = \mathbf{T} - \mathrm{T}_m \mathbf{1} \tag{1.364}$$

For the following operations, we consider that $\mathbf{T}$ is a symmetric tensor, $\mathbf{T} = \mathbf{T}^T$, then under this condition the deviatoric tensor components, T_{ij}^{dev}, become:

$$\begin{aligned}\mathrm{T}_{ij}^{dev} &= \begin{bmatrix} \mathrm{T}_{11}^{dev} & \mathrm{T}_{12}^{dev} & \mathrm{T}_{13}^{dev} \\ \mathrm{T}_{12}^{dev} & \mathrm{T}_{22}^{dev} & \mathrm{T}_{23}^{dev} \\ \mathrm{T}_{13}^{dev} & \mathrm{T}_{23}^{dev} & \mathrm{T}_{33}^{dev} \end{bmatrix} = \begin{bmatrix} \mathrm{T}_{11} - \mathrm{T}_m & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{12} & \mathrm{T}_{22} - \mathrm{T}_m & \mathrm{T}_{23} \\ \mathrm{T}_{13} & \mathrm{T}_{23} & \mathrm{T}_{33} - \mathrm{T}_m \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}(2\mathrm{T}_{11} - \mathrm{T}_{22} - \mathrm{T}_{33}) & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{12} & \frac{1}{3}(2\mathrm{T}_{22} - \mathrm{T}_{11} - \mathrm{T}_{33}) & \mathrm{T}_{23} \\ \mathrm{T}_{13} & \mathrm{T}_{23} & \frac{1}{3}(2\mathrm{T}_{33} - \mathrm{T}_{11} - \mathrm{T}_{22}) \end{bmatrix}\end{aligned} \tag{1.365}$$

Graphical representations of the Cartesian components of the spherical and deviatoric parts are shown in Figure 1.30.

In the following subsections we obtain the deviatoric tensor invariants in terms of the principal invariants of $\mathbf{T}$.

1.5.12.1 First Invariant of the Deviatoric Tensor

$$I_{\mathbf{T}^{dev}} = \mathrm{Tr}(\mathbf{T}^{dev}) = \mathrm{Tr}\left[\mathbf{T} - \frac{\mathrm{Tr}(\mathbf{T})}{3}\mathbf{1}\right] = \mathrm{Tr}(\mathbf{T}) - \frac{\mathrm{Tr}(\mathbf{T})}{3}\underbrace{\mathrm{Tr}(\mathbf{1})}_{\delta_{ii}=3} = 0 \tag{1.366}$$

Thus, we can conclude that the trace of any deviatoric tensor is equal to zero.

1.5.12.2 Second Invariant of the Deviatoric Tensor

For simplicity we can use the principal space to obtain the second and third invariant of the deviatoric tensor. In the principal space the components of $\mathbf{T}$ are given by:

$$\mathrm{T}_{ij} = \begin{bmatrix} \mathrm{T}_1 & 0 & 0 \\ 0 & \mathrm{T}_2 & 0 \\ 0 & 0 & \mathrm{T}_3 \end{bmatrix} \tag{1.367}$$

The principal invariants of $\mathbf{T}$: $I_{\mathbf{T}} = \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3$, $II_{\mathbf{T}} = \mathrm{T}_1\mathrm{T}_2 + \mathrm{T}_2\mathrm{T}_3 + \mathrm{T}_3\mathrm{T}_1$, $III_{\mathbf{T}} = \mathrm{T}_1\mathrm{T}_2\mathrm{T}_3$.

The deviatoric components, $\mathbf{T}^{dev} = \mathbf{T} - \mathrm{T}_m\mathbf{1}$, in the principal space are:

$$\mathrm{T}_{ij}^{dev} = \begin{bmatrix} \mathrm{T}_1 - \mathrm{T}_m & 0 & 0 \\ 0 & \mathrm{T}_2 - \mathrm{T}_m & 0 \\ 0 & 0 & \mathrm{T}_3 - \mathrm{T}_m \end{bmatrix} \tag{1.368}$$

So, the second invariant of deviatoric tensor $\mathbf{T}^{dev}$ is evaluated as follows:

$$\begin{aligned} II_{\mathbf{T}^{dev}} &= (\mathrm{T}_1 - \mathrm{T}_m)(\mathrm{T}_2 - \mathrm{T}_m) + (\mathrm{T}_1 - \mathrm{T}_m)(\mathrm{T}_3 - \mathrm{T}_m) + (\mathrm{T}_2 - \mathrm{T}_m)(\mathrm{T}_3 - \mathrm{T}_m) \\ &= (\mathrm{T}_1\mathrm{T}_2 + \mathrm{T}_1\mathrm{T}_3 + \mathrm{T}_2\mathrm{T}_3) - 2\mathrm{T}_m(\mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3) + 3\mathrm{T}_m^2 \\ &= II_{\mathbf{T}} - \frac{2I_{\mathbf{T}}}{3}(I_{\mathbf{T}}) + \frac{I_{\mathbf{T}}^2}{3} \\ &= \frac{1}{3}\left(3\,II_{\mathbf{T}} - I_{\mathbf{T}}^2\right) \end{aligned} \tag{1.369}$$

We could also have obtained the above result, by directly starting from the definition of the second invariant of a tensor given in (1.269), *i.e.*:

$$
\begin{aligned}
I\!I_{\mathbf{T}^{dev}} &= \frac{1}{2}\left\{(I_{\mathbf{T}^{dev}})^2 - \mathrm{Tr}\left[(\mathbf{T}^{dev})^2\right]\right\} = -\frac{1}{2}\left\{\mathrm{Tr}\left[(\mathbf{T}^{dev})^2\right]\right\} \\
&= \frac{1}{2}\left\{-\mathrm{Tr}\left[(\mathbf{T} - \mathrm{T}_m \mathbf{1})^2\right]\right\} \\
&= \frac{1}{2}\left\{-\mathrm{Tr}\left[(\mathbf{T}^2 - 2\mathrm{T}_m \mathbf{T}\cdot\mathbf{1} + \mathrm{T}_m^2 \mathbf{1})\right]\right\} \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + 2\mathrm{T}_m \mathrm{Tr}(\mathbf{T}) - \mathrm{T}_m^2 \mathrm{Tr}(\mathbf{1})\right] \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + 2\frac{I_{\mathbf{T}}}{3} I_{\mathbf{T}} - \frac{I_{\mathbf{T}}^2}{9} 3\right] \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + \frac{I_{\mathbf{T}}^2}{3}\right]
\end{aligned}
\tag{1.370}
$$

Observing that $\mathrm{Tr}(\mathbf{T}^2) = \mathrm{T}_1^2 + \mathrm{T}_2^2 + \mathrm{T}_3^2 = I_{\mathbf{T}}^2 - 2 I\!I_{\mathbf{T}}$, (see **Problem 1.31**), the equation (1.370) becomes:

$$
I\!I_{\mathbf{T}^{dev}} = \frac{1}{2}\left[-I_{\mathbf{T}}^2 + 2I\!I_{\mathbf{T}} + \frac{I_{\mathbf{T}}^2}{3}\right] = \frac{1}{2}\left[2I\!I_{\mathbf{T}} - \frac{2I_{\mathbf{T}}^2}{3}\right] = \frac{1}{3}\left(3I\!I_{\mathbf{T}} - I_{\mathbf{T}}^2\right)
\tag{1.371}
$$

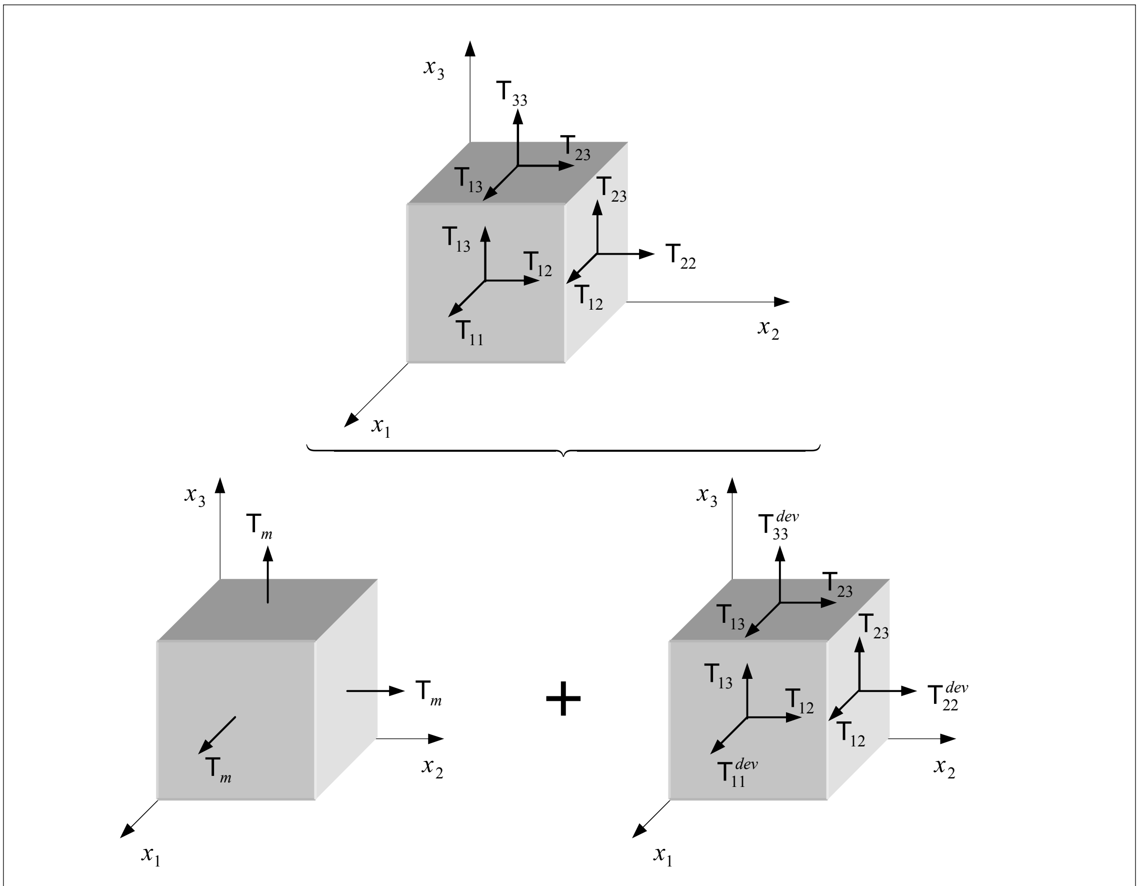


Figure 1.30: Spherical and deviatoric part.

Another equation for $II_{\mathbf{T}^{dev}}$ is presented in terms of deviatoric tensor components. To calculate this, we can apply the equation (1.370):

$$II_{\mathbf{T}^{dev}} = -\frac{1}{2}\mathrm{Tr}\left[(\mathbf{T}^{dev})^2\right] = -\frac{1}{2}\mathrm{Tr}\left[(\mathbf{T}^{dev}\cdot\mathbf{T}^{dev})\right] = -\frac{1}{2}\mathbf{T}^{dev}\cdot\cdot\mathbf{T}^{dev} = -\frac{1}{2}\mathsf{T}^{dev}_{ij}\mathsf{T}^{dev}_{ji} \tag{1.372}$$

Expanding the previous equation we obtain:

$$II_{\mathbf{T}^{dev}} = -\frac{1}{2}\left[(\mathsf{T}^{dev}_{11})^2 + (\mathsf{T}^{dev}_{22})^2 + (\mathsf{T}^{dev}_{33})^2 + 2(\mathsf{T}^{dev}_{12})^2 + 2(\mathsf{T}^{dev}_{13})^2 + 2(\mathsf{T}^{dev}_{23})^2\right] \tag{1.373}$$

Additionally, in the space of the principal directions we obtain:

$$II_{\mathbf{T}^{dev}} = -\frac{1}{2}\mathsf{T}^{dev}_{ij}\mathsf{T}^{dev}_{ji} = -\frac{1}{2}\left[(\mathsf{T}^{dev}_{1})^2 + (\mathsf{T}^{dev}_{2})^2 + (\mathsf{T}^{dev}_{3})^2\right] \tag{1.374}$$

Another way to express the second invariant is shown below:

$$\begin{aligned} II_{\mathbf{T}^{dev}} &= \begin{vmatrix} \mathsf{T}^{dev}_{22} & \mathsf{T}^{dev}_{23} \\ \mathsf{T}^{dev}_{23} & \mathsf{T}^{dev}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}^{dev}_{11} & \mathsf{T}^{dev}_{13} \\ \mathsf{T}^{dev}_{13} & \mathsf{T}^{dev}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}^{dev}_{11} & \mathsf{T}^{dev}_{12} \\ \mathsf{T}^{dev}_{12} & \mathsf{T}^{dev}_{22} \end{vmatrix} \\ &= -\frac{1}{2}\left[-2\mathsf{T}^{dev}_{22}\mathsf{T}^{dev}_{33} - 2\mathsf{T}^{dev}_{11}\mathsf{T}^{dev}_{33} - 2\mathsf{T}^{dev}_{11}\mathsf{T}^{dev}_{22}\right] - (\mathsf{T}^{dev}_{12})^2 - (\mathsf{T}^{dev}_{23})^2 - (\mathsf{T}^{dev}_{13})^2 \end{aligned} \tag{1.375}$$

or

$$\begin{aligned} II_{\mathbf{T}^{dev}} = -\frac{1}{2}\Big[&\left(\mathsf{T}^{dev}_{22}\right)^2 - 2\mathsf{T}^{dev}_{22}\mathsf{T}^{dev}_{33} + \left(\mathsf{T}^{dev}_{33}\right)^2 + \left(\mathsf{T}^{dev}_{11}\right)^2 - 2\mathsf{T}^{dev}_{11}\mathsf{T}^{dev}_{33} + \left(\mathsf{T}^{dev}_{33}\right)^2 + \\ &\left(\mathsf{T}^{dev}_{11}\right)^2 - 2\mathsf{T}^{dev}_{11}\mathsf{T}^{dev}_{22} + \left(\mathsf{T}^{dev}_{22}\right)^2\Big] + \left(\mathsf{T}^{dev}_{11}\right)^2 + \left(\mathsf{T}^{dev}_{22}\right)^2 + \left(\mathsf{T}^{dev}_{33}\right)^2 \\ &- (\mathsf{T}^{dev}_{12})^2 - (\mathsf{T}^{dev}_{23})^2 - (\mathsf{T}^{dev}_{13})^2 \end{aligned} \tag{1.376}$$

Note that, from equation (1.373), we can state that:

$$(\mathsf{T}^{dev}_{11})^2 + (\mathsf{T}^{dev}_{22})^2 + (\mathsf{T}^{dev}_{33})^2 = -2\,II_{\mathbf{T}^{dev}} - 2(\mathsf{T}^{dev}_{12})^2 - 2(\mathsf{T}^{dev}_{13})^2 - 2(\mathsf{T}^{dev}_{23})^2 \tag{1.377}$$

Substituting (1.377) into (1.376), we find:

$$II_{\mathbf{T}^{dev}} = -\frac{1}{6}\left[(\mathsf{T}^{dev}_{22} - \mathsf{T}^{dev}_{33})^2 + (\mathsf{T}^{dev}_{11} - \mathsf{T}^{dev}_{33})^2 + (\mathsf{T}^{dev}_{11} - \mathsf{T}^{dev}_{22})^2\right] - (\mathsf{T}^{dev}_{12})^2 - (\mathsf{T}^{dev}_{23})^2 - (\mathsf{T}^{dev}_{13})^2 \tag{1.378}$$

Moreover, if we consider the principal space we obtain:

$$II_{\mathbf{T}^{dev}} = -\frac{1}{6}\left[(\mathsf{T}^{dev}_{2} - \mathsf{T}^{dev}_{3})^2 + (\mathsf{T}^{dev}_{1} - \mathsf{T}^{dev}_{3})^2 + (\mathsf{T}^{dev}_{1} - \mathsf{T}^{dev}_{2})^2\right] \tag{1.379}$$

1.5.12.3 Third Invariant of Deviatoric Tensor

The third invariant of the deviatoric tensor is given by:

$$\begin{aligned} III_{\mathbf{T}^{dev}} &= (\mathsf{T}_1 - \mathsf{T}_m)(\mathsf{T}_2 - \mathsf{T}_m)(\mathsf{T}_3 - \mathsf{T}_m) \\ &= \mathsf{T}_1\mathsf{T}_2\mathsf{T}_3 - \mathsf{T}_m(\mathsf{T}_1\mathsf{T}_2 + \mathsf{T}_1\mathsf{T}_3 + \mathsf{T}_2\mathsf{T}_3) + \mathsf{T}_m^2(\mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3) - \mathsf{T}_m^3 \\ &= III_{\mathbf{T}} - \frac{I_{\mathbf{T}}}{3}II_{\mathbf{T}} + \frac{I_{\mathbf{T}}^2}{9}I_{\mathbf{T}} - \frac{I_{\mathbf{T}}^3}{27} \\ &= III_{\mathbf{T}} - \frac{I_{\mathbf{T}}II_{\mathbf{T}}}{3} + \frac{2I_{\mathbf{T}}^3}{27} \\ &= \frac{1}{27}\left(2I_{\mathbf{T}}^3 - 9I_{\mathbf{T}}II_{\mathbf{T}} + 27\,III_{\mathbf{T}}\right) \end{aligned} \tag{1.380}$$

Another way of expressing the third invariant is:

$$III_{\mathbf{T}^{dev}} = \mathsf{T}_1^{dev}\mathsf{T}_2^{dev}\mathsf{T}_3^{dev} = \frac{1}{3}\mathsf{T}_{ij}^{dev}\mathsf{T}_{jk}^{dev}\mathsf{T}_{ki}^{dev} \tag{1.381}$$

Problem 1.39: Let $\boldsymbol{\sigma}$ be a symmetric second-order tensor, and $\mathbf{s} \equiv \boldsymbol{\sigma}^{dev}$ be a deviatoric tensor. Prove that $\mathbf{s} : \frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$. Also show that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors.

Solution: First, we make use of the definition of a deviatoric tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma}^{sph} + \mathbf{s} = \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} + \mathbf{s} \quad \Rightarrow \quad \mathbf{s} = \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}.$$

Afterwards we calculate:

$$\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \frac{\partial\left[\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right]}{\partial \boldsymbol{\sigma}} = \frac{\partial[\boldsymbol{\sigma}]}{\partial \boldsymbol{\sigma}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \boldsymbol{\sigma}}\mathbf{1}$$

which in indicial notation is:

$$\frac{\partial \mathsf{s}_{ij}}{\partial \sigma_{kl}} = \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \sigma_{kl}}\delta_{ij} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}$$

Therefore

$$\begin{aligned}\mathsf{s}_{ij}\frac{\partial \mathsf{s}_{ij}}{\partial \sigma_{kl}} &= \mathsf{s}_{ij}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right) = \mathsf{s}_{ij}\delta_{ik}\delta_{jl} - \frac{1}{3}\mathsf{s}_{ij}\delta_{kl}\delta_{ij} = \mathsf{s}_{kl} - \frac{1}{3}\delta_{kl}\underbrace{\mathsf{s}_{ii}}_{=0} \\ &= \mathsf{s}_{kl}\end{aligned}$$

$$\mathbf{s} : \frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$$

To show that two tensors are coaxial, we must prove that $\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev}$:

$$\begin{aligned}\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev} &= \boldsymbol{\sigma} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{sph}) = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{sph} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \\ &= \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \cdot \boldsymbol{\sigma} \\ &= \left(\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right) \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma}\end{aligned}$$

Therefore, we have shown that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors. In other words, they have the same principal directions.

1.6 The Tensor-Valued Tensor Function

Tensor-valued tensor function can be of the types: scalar, vector, or higher-order tensors. As examples of scalar-valued tensor functions we can list:

$$\begin{aligned}\Psi &= \Psi(\mathbf{T}) = \det(\mathbf{T})\\ \Psi &= \Psi(\mathbf{T},\mathbf{S}) = \mathbf{T}:\mathbf{S}\end{aligned} \tag{1.382}$$

where $\mathbf{T}$ and $\mathbf{S}$ are second-order tensors. Additionally, as an example of a second-order-valued tensor function we have:

$$\Pi = \Pi(\mathbf{T}) = \alpha\mathbf{1} + \beta\mathbf{T} \tag{1.383}$$

where α and β are scalars.

1.6.1 The Tensor Series

The function $f(x)$ can be approximated by the Taylor series as $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n f(a)}{\partial x^n}(x-a)^n$, where $n!$ denotes the factorial of n, and $f(a)$ is the value of the function at the application point $x=a$. We can extrapolate that definition for use on tensors. For example, let us suppose we have a scalar-valued tensor function ψ in terms of a second-order tensor, $\boldsymbol{E}$, then we can approximate $\psi(\boldsymbol{E})$ as:

$$\begin{aligned}\psi(\boldsymbol{E}) &\approx \frac{1}{0!}\psi(\boldsymbol{E}_0) + \frac{1}{1!}\frac{\partial\psi(\boldsymbol{E}_0)}{\partial E_{ij}}(E_{ij} - E_{0ij}) + \frac{1}{2!}\frac{\partial^2\psi(\boldsymbol{E}_0)}{\partial E_{ij}\partial E_{kl}}(E_{ij} - E_{ij0})(E_{kl} - E_{kl0}) + \cdots\\ &\approx \psi_0 + \frac{\partial\psi(\boldsymbol{E}_0)}{\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \frac{1}{2}(\boldsymbol{E}-\boldsymbol{E}_0):\frac{\partial^2\psi(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \cdots\end{aligned} \tag{1.384}$$

A second-order-valued tensor function, $\mathbf{S}(\boldsymbol{E})$, can be approximated as:

$$\begin{aligned}\mathbf{S}(\boldsymbol{E}) &\approx \frac{1}{0!}\mathbf{S}(\boldsymbol{E}_0) + \frac{1}{1!}\frac{\partial\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \frac{1}{2!}(\boldsymbol{E}-\boldsymbol{E}_0):\frac{\partial^2\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \cdots\\ &\approx \mathbf{S}_0 + \frac{\partial\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \frac{1}{2}(\boldsymbol{E}-\boldsymbol{E}_0):\frac{\partial^2\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_0) + \cdots\end{aligned} \tag{1.385}$$

Other tensor algebraic expressions can be represented by series, *e.g.*:

$$\begin{aligned}\exp^{\mathbf{S}} &= \mathbf{1} + \mathbf{S} + \frac{1}{2!}\mathbf{S}^2 + \frac{1}{3!}\mathbf{S}^3 + \cdots\\ \ln(\mathbf{1}+\mathbf{S}) &= \mathbf{S} - \frac{1}{2}\mathbf{S}^2 + \frac{1}{3}\mathbf{S}^3 - \cdots\\ \sin(\mathbf{S}) &= \mathbf{S} - \frac{1}{3!}\mathbf{S}^3 + \frac{1}{5!}\mathbf{S}^5 - \cdots\end{aligned} \tag{1.386}$$

With reference to the spectral representation of a symmetric second-order tensor, $\mathbf{S}$, it is also true that:

$$
\begin{aligned}
\exp^{\mathbf{S}} &= \sum_{a=1}^{3}\left(1+\lambda_a+\frac{\lambda_a^2}{2!}+\frac{\lambda_a^3}{3!}+\cdots\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3}\exp^{\lambda_a}\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)} \\
\ln(\mathbf{1}+\mathbf{S}) &= \sum_{a=1}^{3}\left(\lambda_a-\frac{1}{2}\lambda_a^2+\frac{1}{3}\lambda_a^3+\cdots\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3}\ln(1+\lambda_a)\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}
\end{aligned}
\tag{1.387}
$$

where λ_a and $\hat{\mathbf{n}}^{(a)}$ are the eigenvalues and eigenvectors, respectively, of the tensor $\mathbf{S}$.

1.6.2 The Tensor-Valued Isotropic Tensor Function

A second-order-valued tensor function, $\Pi=\Pi(\mathbf{T})$, is isotropic if after an orthogonal transformation the following condition is satisfied:

$$
\Pi^*(\mathbf{T})=\mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T=\underbrace{\Pi\left(\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T\right)}_{\Pi(\mathbf{T}^*)}
\tag{1.388}
$$

We can show that $\Pi(\mathbf{T})$ has the same principal directions of $\mathbf{T}$, *i.e.* $\Pi(\mathbf{T})$ and $\mathbf{T}$ are coaxial tensors. To demonstrate this we can regard the components of $\mathbf{T}$ in the principal space as:

$$
(\mathbf{T})_{ij}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}
\tag{1.389}
$$

Then the tensor function is given in terms of the principal values of $\mathbf{T}$: $\Pi=\Pi(\lambda_1,\lambda_2,\lambda_3)$ and, the transformation of $\mathbf{T}$ is given by:

$$
\mathbf{T}^*=\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T
\tag{1.390}
$$

Likewise, for the tensor function Π:

$$
\Pi^*(\mathbf{T})=\mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T
\tag{1.391}
$$

If we take as the orthogonal tensor components:

$$
(\mathbf{Q})_{ij}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}
\tag{1.392}
$$

After having done the calculation for the matrices (1.391), we obtain:

$$
\begin{aligned}
\Pi^* &= \begin{bmatrix}\Pi_{11} & -\Pi_{12} & -\Pi_{13}\\ -\Pi_{12} & \Pi_{22} & -\Pi_{23}\\ -\Pi_{13} & -\Pi_{23} & \Pi_{33}\end{bmatrix} = \begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ \Pi_{12} & \Pi_{22} & \Pi_{23}\\ \Pi_{13} & \Pi_{23} & \Pi_{33}\end{bmatrix} = \Pi \\
\Rightarrow \Pi^* &= \begin{bmatrix}\Pi_{11} & 0 & 0\\ 0 & \Pi_{22} & 0\\ 0 & 0 & \Pi_{33}\end{bmatrix}
\end{aligned}
\tag{1.393}
$$

To satisfy that $\Pi^*=\Pi$ (isotropy), we conclude that $\Pi_{12}=\Pi_{13}=\Pi_{23}=0$. Therefore, $\Pi(\mathbf{T})$ and $\mathbf{T}$ have the same principal directions.

Once again we observe, a tensor function $\Pi(\mathbf{T})$. This tensor function is isotropic if and only if it can be represented by the following linear transformation, Truesdell & Noll (1965):

$$\Pi = \Pi(\mathbf{T}) = \Phi_0 \mathbf{1} + \Phi_1 \mathbf{T} + \Phi_2 \mathbf{T}^2 \tag{1.394}$$

where Φ_0, Φ_1, Φ_2 are functions of the tensor $\mathbf{T}$ invariants or functions of the $\mathbf{T}$ eigenvalues.

A brief demonstration follows. We can now consider the spectral representations of $\mathbf{T}$ and Π, respectively:

$$\mathbf{T} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \lambda_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \tag{1.395}$$

$$\Pi = \sum_{a=1}^{3} \omega_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \omega_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \omega_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \tag{1.396}$$

Note that $\mathbf{T}$ and Π have the same principal directions $\hat{\mathbf{n}}^{(i)}$. Then, we can put the following set of equations together:

$$\begin{cases} \mathbf{1} = \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \\ \mathbf{T} = \lambda_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \\ \mathbf{T}^2 = \lambda_1^2 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_3^2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3^2 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \end{cases} \tag{1.397}$$

Solving the set above, we obtain $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ as a function of the tensor $\mathbf{T}$, and we obtain:

$$\begin{aligned} \mathbf{M}^{(1)} &= \frac{\lambda_2 \lambda_3}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} \mathbf{1} - \frac{(\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} \mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} \\ \mathbf{M}^{(2)} &= \frac{\lambda_1 \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \mathbf{1} - \frac{(\lambda_1 + \lambda_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\ \mathbf{M}^{(3)} &= \frac{\lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \mathbf{1} - \frac{(\lambda_1 + \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{aligned} \tag{1.398}$$

It is evident that, if we substitute the values of $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ in equation (1.395) we obtain: $\mathbf{T} = \mathbf{T}$. Now, if we substitute the values of $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ in equation (1.396), we obtain:

$$\Pi = \Pi(\mathbf{T}) = \Phi_0 \mathbf{1} + \Phi_1 \mathbf{T} + \Phi_2 \mathbf{T}^2 \tag{1.399}$$

where the coefficients Φ_0, Φ_1, and Φ_2 are functions of the eigenvalues of $\mathbf{T}$, $(\lambda_1 \neq \lambda_2 \neq \lambda_3)$, and given by:

$$\begin{aligned} \Phi_0 &= \frac{\omega_1 \lambda_2 \lambda_3}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} + \frac{\omega_2 \lambda_1 \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\omega_3 \lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ \Phi_1 &= -\frac{\omega_1 (\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} - \frac{\omega_2 (\lambda_1 + \lambda_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} - \frac{\omega_3 (\lambda_1 + \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \\ \Phi_2 &= \frac{\omega_1}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)} + \frac{\omega_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\omega_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{aligned} \tag{1.400}$$

We can now show that if a tensor function $\Pi(\mathbf{T})$ is given in (1.399), this tensor function is isotropic:

$$\begin{aligned}\Pi^*(\mathbf{T}) &= \mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T = \mathbf{Q}\cdot\left(\Phi_0\mathbf{1}+\Phi_1\mathbf{T}+\Phi_2\mathbf{T}^2\right)\cdot\mathbf{Q}^T \\ &= \Phi_0\mathbf{Q}\cdot\mathbf{1}\cdot\mathbf{Q}^T+\Phi_1\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T+\Phi_2\mathbf{Q}\cdot\mathbf{T}^2\cdot\mathbf{Q}^T = \Phi_0\mathbf{1}+\Phi_1\mathbf{T}^*+\Phi_2\mathbf{T}^{*2} \\ &= \Pi(\mathbf{T}^*)\end{aligned} \tag{1.401}$$

1.6.3 The Derivative of the Tensor-Valued Tensor Function

Firstly, we refer to a scalar-valued tensor function:

$$\Pi = \Pi(\mathbf{A}) \tag{1.402}$$

The partial derivative of $\Pi(\mathbf{A})$ with respect to $\mathbf{A}$ is defined as:

$$\frac{\partial\Pi}{\partial\mathbf{A}} \equiv \Pi_{,\mathbf{A}} = \frac{\partial\Pi}{\partial\mathrm{A}_{ij}}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j) \tag{1.403}$$

where the comma denotes a partial derivative.

Then the second derivative of $\Pi(\mathbf{A})$ becomes a fourth-order tensor:

$$\frac{\partial^2\Pi}{\partial\mathbf{A}\otimes\partial\mathbf{A}} = \Pi_{,\mathbf{AA}} = \frac{\partial^2\Pi}{\partial\mathrm{A}_{ij}\partial\mathrm{A}_{kl}}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l) = \mathbb{D}_{ijkl}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l) \tag{1.404}$$

Let $\boldsymbol{C}$ and $\boldsymbol{b}$ be positive definite symmetric second-order tensors defined as:

$$\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F} \quad ; \quad \boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T \tag{1.405}$$

where $\boldsymbol{F}$ is an arbitrary second-order tensor with the restriction $\det(\boldsymbol{F})>0$ imposed on it. We must also bear in mind that there is a scalar-valued isotropic tensor function, $\Psi = \Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})$, expressed in terms of the principal invariants of $\boldsymbol{C}$, where $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$, $II_{\boldsymbol{C}} = II_{\boldsymbol{b}}$, $III_{\boldsymbol{C}} = III_{\boldsymbol{b}}$. Next, we can find the partial derivative of Ψ with respect to $\boldsymbol{C}$, and with respect to $\boldsymbol{b}$. We must also verify that the following relation holds:

$$\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T = \Psi_{,\boldsymbol{b}}\cdot\boldsymbol{b} \tag{1.406}$$

By applying the chain rule for derivative we obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})}{\partial\boldsymbol{C}} = \frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} + \frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} + \frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} \tag{1.407}$$

Considering the partial derivatives of the principal invariants, we can state that:

$$\begin{aligned}&\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = \mathbf{1} \\ &\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1}-\boldsymbol{C}^T = I_{\boldsymbol{C}}\mathbf{1}-\boldsymbol{C} = II_{\boldsymbol{C}}\boldsymbol{C}^{-1} - III_{\boldsymbol{C}}\boldsymbol{C}^{-2} \\ &\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = III_{\boldsymbol{C}}\boldsymbol{C}^{-T} = III_{\boldsymbol{C}}\boldsymbol{C}^{-1} = \boldsymbol{C}^2 - I_{\boldsymbol{C}}\boldsymbol{C} + II_{\boldsymbol{C}}\mathbf{1}\end{aligned} \tag{1.408}$$

Now, by substituting the following values $\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = \mathbf{1}$, $\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1}-\boldsymbol{C}$ and $\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = III_{\boldsymbol{C}}\boldsymbol{C}^{-1}$ into the equation in (1.407), we obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\mathbf{1} + \frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}(I_{\boldsymbol{C}}\mathbf{1}-\boldsymbol{C}) + \frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}\, III_{\boldsymbol{C}}\boldsymbol{C}^{-1} \tag{1.409}$$

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}} \tag{1.410}$$

Another way to express the relation (1.410) is by substituting $\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}$, $\frac{\partial II_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}$ and $\frac{\partial III_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \boldsymbol{C}^2 - I_{\boldsymbol{C}}\boldsymbol{C} + II_{\boldsymbol{C}}\mathbf{1}$ into the equation in (1.407), thus:

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} II_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial III_{\boldsymbol{C}}}\right)\boldsymbol{C}^2} \tag{1.411}$$

If we now consider the values $\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}$, $\frac{\partial II_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = II_{\boldsymbol{C}}\boldsymbol{C}^{-1} - III_{\boldsymbol{C}}\boldsymbol{C}^{-2}$, $\frac{\partial III_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = III_{\boldsymbol{C}}\boldsymbol{C}^{-1}$, in the equation in (1.407), we obtain:

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1} + \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} II_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}} \tag{1.412}$$

If we now observe both $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$, $II_{\boldsymbol{C}} = II_{\boldsymbol{b}}$, $III_{\boldsymbol{C}} = III_{\boldsymbol{b}}$, and the equation in (1.410), we can draw the conclusion that:

$$\boxed{\Psi_{,\boldsymbol{b}} = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} I_{\boldsymbol{b}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}}\boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}}\boldsymbol{b}^{-1}} \tag{1.413}$$

Using the equation in (1.410), the equation $\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T$ becomes:

$$\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T = \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{F}\cdot\mathbf{1}\cdot\boldsymbol{F}^T - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^T + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{F}\cdot\boldsymbol{C}^{-1}\cdot\boldsymbol{F}^T \tag{1.414}$$

Then, if we observe that:

$$\begin{aligned} &\Rightarrow \boldsymbol{F}\cdot\mathbf{1}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b} \\ &\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F} \\ &\Rightarrow \boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}\cdot\boldsymbol{b} = \boldsymbol{b}^2 \end{aligned} \tag{1.415}$$

$$\begin{aligned} &\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{F} \\ &\Rightarrow \boldsymbol{F}\cdot\boldsymbol{C}^{-1}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}^{-1}\cdot\boldsymbol{b} \end{aligned} \tag{1.416}$$

The equation (1.414) can be rewritten as:

$$\begin{aligned} \boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T &= \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{b} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{b}^2 + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{b}^{-1}\cdot\boldsymbol{b} \\ &= \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{b}^{-1}\right]\cdot\boldsymbol{b} \end{aligned} \tag{1.417}$$

In light of the equation in (1.413) and (1.417), we can draw the conclusion that:

$$\begin{aligned} \boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T &= \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} I_{\boldsymbol{b}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}}\boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}}\boldsymbol{b}^{-1}\right]\cdot\boldsymbol{b} \\ &= \Psi_{,\boldsymbol{b}}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\Psi_{,\boldsymbol{b}} \end{aligned} \tag{1.418}$$

which indicates that $\Psi_{,\boldsymbol{b}}$ and $\boldsymbol{b}$ are coaxial tensors.

Once again, we can observe $\boldsymbol{C}$ given by the equation in (1.405). Next, we can evaluate the derivative of the scalar-valued tensor function, $\Psi = \Psi(\boldsymbol{C})$, with respect to the tensor $\boldsymbol{F}$:

$$\Psi_{,\boldsymbol{F}} = \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{F}} = \frac{\partial \Psi}{\partial \boldsymbol{C}} : \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{F}} \xrightarrow{\text{indicial notation}} (\Psi_{,\boldsymbol{F}})_{kl} = \frac{\partial \Psi}{\partial C_{ij}} \frac{\partial C_{ij}}{\partial F_{kl}} \tag{1.419}$$

The derivative of tensor $\boldsymbol{C}$ with respect to $\boldsymbol{F}$ is evaluated as follows:

$$\begin{aligned} \frac{\partial C_{ij}}{\partial F_{kl}} &= \frac{\partial (F_{qi} F_{qj})}{\partial F_{kl}} \\ &= \frac{\partial (F_{qi})}{\partial F_{kl}} F_{qj} + F_{qi} \frac{\partial (F_{qj})}{\partial F_{kl}} \\ &= \delta_{qk} \delta_{il} F_{qj} + \delta_{qk} \delta_{jl} F_{qi} \\ &= \delta_{il} F_{kj} + \delta_{jl} F_{ki} \end{aligned} \tag{1.420}$$

Then, by substituting (1.420) into (1.419), we obtain:

$$\begin{aligned} (\Psi_{,\boldsymbol{F}})_{kl} &= \frac{\partial \Psi}{\partial C_{ij}} (\delta_{il} F_{kj} + \delta_{jl} F_{ki}) \\ &= F_{kj} \frac{\partial \Psi}{\partial C_{lj}} + F_{ki} \frac{\partial \Psi}{\partial C_{il}} \end{aligned} \tag{1.421}$$

Due to the symmetry of $\boldsymbol{C}$, *i.e.* $C_{lj} = C_{jl}$, we can draw the conclusion that:

$$(\Psi_{,\boldsymbol{F}})_{kl} = 2 \frac{\partial \Psi}{\partial C_{lj}} F_{kj} = 2 \frac{\partial \Psi}{\partial C_{jl}} F_{kj} \quad \Rightarrow \quad \boxed{\Psi_{,\boldsymbol{F}} = 2 \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = 2 \boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}}} \tag{1.422}$$

Now, suppose that $\boldsymbol{C}$ is given by the equation $\boldsymbol{C} = \mathbf{U}^T \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U} = \mathbf{U}^2$, where $\mathbf{U}$ is a symmetric second-order tensor. To find $\Psi(\boldsymbol{C})_{,\mathbf{U}}$ we can use the same equation as in (1.422), *i.e.*:

$$\Psi_{,\mathbf{U}} = 2 \Psi_{,\boldsymbol{C}} \cdot \mathbf{U} = 2 \mathbf{U} \cdot \Psi_{,\boldsymbol{C}} \tag{1.423}$$

Therefore, we can draw the conclusion that $\Psi_{,\boldsymbol{C}}$ and $\mathbf{U}$ are coaxial tensors.

Let $\mathbf{A}$ be a symmetric second-order tensor, and $\Psi = \Psi(\mathbf{A})$ be a scalar-valued tensor function. The following relationships hold:

$$\begin{aligned} &\Psi_{,\boldsymbol{b}} = 2 \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} \quad \text{for} \quad \mathbf{A} = \boldsymbol{b}^T \cdot \boldsymbol{b} \\ &\Psi_{,\boldsymbol{b}} = 2 \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \quad \text{for} \quad \mathbf{A} = \boldsymbol{b} \cdot \boldsymbol{b}^T \\ &\Psi_{,\boldsymbol{b}} = 2 \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} = 2 \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \\ &\qquad = \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} + \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \quad \text{for} \quad \mathbf{A} = \boldsymbol{b} \cdot \boldsymbol{b} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{b}^T \end{aligned} \tag{1.424}$$

1.7 The Voigt Notation

When dealing with symmetric tensors, it may be advantageous to just work with the independent components. For example, a symmetric second-order tensor has 6 independent components, so, it is possible to represent these components by a column matrix as follows:

$$\mathsf{T}_{ij} = \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} \xrightarrow{Voigt} \{\mathcal{T}\} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix} \tag{1.425}$$

This representation is called the *Voigt Notation*. It is also possible to represent a second-order tensor as:

$$\mathsf{E}_{ij} = \begin{bmatrix} \mathsf{E}_{11} & \mathsf{E}_{12} & \mathsf{E}_{13} \\ \mathsf{E}_{12} & \mathsf{E}_{22} & \mathsf{E}_{23} \\ \mathsf{E}_{13} & \mathsf{E}_{23} & \mathsf{E}_{33} \end{bmatrix} \xrightarrow{Voigt} \{\mathcal{E}\} = \begin{bmatrix} \mathsf{E}_{11} \\ \mathsf{E}_{22} \\ \mathsf{E}_{33} \\ 2\mathsf{E}_{12} \\ 2\mathsf{E}_{23} \\ 2\mathsf{E}_{13} \end{bmatrix} \tag{1.426}$$

As we have seen before, a fourth-order tensor, $\mathbb{C}$, that presents minor symmetry, *i.e.* $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{jilk}$, has $6 \times 6 = 36$ independent components. Note that, due to the symmetry of (ij) we have 6 independent components, and due to the symmetry of (kl) we have 6 independent components. In Voigt Notation we can represent these components in a 6-by-6 matrix as:

$$[\mathcal{C}] = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & \mathbb{C}_{1123} & \mathbb{C}_{1113} \\ \mathbb{C}_{2211} & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & \mathbb{C}_{2223} & \mathbb{C}_{2213} \\ \mathbb{C}_{3311} & \mathbb{C}_{3322} & \mathbb{C}_{3333} & \mathbb{C}_{3312} & \mathbb{C}_{3323} & \mathbb{C}_{3313} \\ \mathbb{C}_{1211} & \mathbb{C}_{1222} & \mathbb{C}_{1233} & \mathbb{C}_{1212} & \mathbb{C}_{1223} & \mathbb{C}_{1213} \\ \mathbb{C}_{2311} & \mathbb{C}_{2322} & \mathbb{C}_{2333} & \mathbb{C}_{2312} & \mathbb{C}_{2323} & \mathbb{C}_{2313} \\ \mathbb{C}_{1311} & \mathbb{C}_{1322} & \mathbb{C}_{1333} & \mathbb{C}_{1312} & \mathbb{C}_{1323} & \mathbb{C}_{1313} \end{bmatrix} \tag{1.427}$$

In addition to minor symmetry the tensor also has major symmetry, *i.e.* $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$, and the number of independent components have reduced to 21. One can easily memorize the order of the components in the matrix $[\mathcal{C}]$ if we consider the order of the second-order tensor in Voigt Notation, *i.e.*:

$$\begin{bmatrix} (11) \\ (22) \\ (33) \\ (12) \\ (23) \\ (13) \end{bmatrix} \begin{bmatrix} (11) & (22) & (33) & (12) & (23) & (13) \end{bmatrix} \tag{1.428}$$

1.7.1 The Unit Tensors in Voigt Notation

The second-order unit tensor is represented in the Voigt notation as:

$$\delta_{ij} \equiv \boldsymbol{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{Voigt} \{\delta\} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.429}$$

In the subsection *1.5.2.5.1 Unit Tensors* we have defined three fourth-order unit tensors, namely, $\mathbb{I}_{ijk\ell} = \delta_{ik}\delta_{j\ell}$, $\bar{\mathbb{I}}_{ijk\ell} = \delta_{i\ell}\delta_{jk}$ and $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$, among which only $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$ is a symmetric tensor. The representation of $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$ in Voigt notation can be evaluated by observing how a symmetric fourth-order tensor is represented in (1.427), thus:

$$\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell} \xrightarrow{Voigt} [\bar{\bar{\mathcal{I}}}] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.430}$$

where $\bar{\bar{\mathbb{I}}}_{1111} = \delta_{11}\delta_{11} = 1$, $\bar{\bar{\mathbb{I}}}_{1122} = \delta_{11}\delta_{22} = 1$, and so on.

The components of a fourth-order unit tensor, $\mathbb{I}^{sym}$, are represented by $\mathrm{I}_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$, which in Voigt notation becomes:

$$\mathrm{I}_{ijk\ell} \xrightarrow{Voigt} [\mathcal{I}] = \begin{bmatrix} \mathrm{I}_{1111} & \mathrm{I}_{1122} & \mathrm{I}_{1133} & \mathrm{I}_{1112} & \mathrm{I}_{1123} & \mathrm{I}_{1113} \\ \mathrm{I}_{2211} & \mathrm{I}_{2222} & \mathrm{I}_{2233} & \mathrm{I}_{2212} & \mathrm{I}_{2223} & \mathrm{I}_{2213} \\ \mathrm{I}_{3311} & \mathrm{I}_{3322} & \mathrm{I}_{3333} & \mathrm{I}_{3312} & \mathrm{I}_{3323} & \mathrm{I}_{3313} \\ \mathrm{I}_{1211} & \mathrm{I}_{1222} & \mathrm{I}_{1233} & \mathrm{I}_{1212} & \mathrm{I}_{1223} & \mathrm{I}_{1213} \\ \mathrm{I}_{2311} & \mathrm{I}_{2322} & \mathrm{I}_{2333} & \mathrm{I}_{2312} & \mathrm{I}_{2323} & \mathrm{I}_{2313} \\ \mathrm{I}_{1311} & \mathrm{I}_{1322} & \mathrm{I}_{1333} & \mathrm{I}_{1312} & \mathrm{I}_{1323} & \mathrm{I}_{1313} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{1.431}$$

and the inverse of the equation in (1.431) becomes:

$$[\mathcal{I}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{1.432}$$

1.7.2 The Scalar Product in Voigt Notation

The dot product between a symmetric second-order tensor, $\mathbf{T}$, and a vector $\vec{\mathbf{n}}$, is given by $\vec{\mathbf{b}} = \mathbf{T} \cdot \vec{\mathbf{n}}$ where the components of $\vec{\mathbf{b}}$ can be evaluated as follows:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{cases} b_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 \\ b_2 = T_{12}n_1 + T_{22}n_2 + T_{23}n_3 \\ b_3 = T_{13}n_1 + T_{23}n_2 + T_{33}n_3 \end{cases} \tag{1.433}$$

By observing how a second-order tensor is presented in Voigt notation, as in (1.425), the scalar product (1.433) can be represented in the Voigt notation as:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{bmatrix}}_{[\overline{\mathcal{N}}]^T} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix} \quad \Rightarrow \quad \{\boldsymbol{b}\} = [\overline{\mathcal{N}}]^T \{\mathcal{T}\} \tag{1.434}$$

1.7.3 The Component Transformation Law in Voigt Notation

The component transformation law for a second-order tensor is defined as:

$$T'_{ij} = T_{kl} a_{ik} a_{jl} \tag{1.435}$$

or in matrix form:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{12} & T'_{22} & T'_{23} \\ T'_{13} & T'_{23} & T'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \tag{1.436}$$

By multiplying the matrices and by rearranging the result in Voigt notation we obtain:

$$\{\mathcal{T}'\} = [\mathcal{M}]\{\mathcal{T}\} \tag{1.437}$$

where:

$$\{\mathcal{T}'\} = \begin{bmatrix} T'_{11} \\ T'_{22} \\ T'_{33} \\ T'_{12} \\ T'_{23} \\ T'_{13} \end{bmatrix} \quad ; \quad \{\mathcal{T}\} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix} \tag{1.438}$$

and $[\mathcal{M}]$ is the transformation matrix for the second-order tensor components in Voigt Notation. The matrix $[\mathcal{M}]$ is given by:

$$[\mathcal{M}] = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\ a_{21}a_{11} & a_{22}a_{12} & a_{13}a_{23} & (a_{11}a_{22} + a_{12}a_{21}) & (a_{13}a_{22} + a_{12}a_{23}) & (a_{13}a_{21} + a_{11}a_{23}) \\ a_{31}a_{21} & a_{32}a_{22} & a_{33}a_{23} & (a_{31}a_{22} + a_{32}a_{21}) & (a_{33}a_{22} + a_{32}a_{23}) & (a_{33}a_{21} + a_{31}a_{23}) \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12} + a_{32}a_{11}) & (a_{33}a_{12} + a_{32}a_{13}) & (a_{33}a_{11} + a_{31}a_{13}) \end{bmatrix} \tag{1.439}$$

If the representation of tensor components is shown in (1.426), equation (1.436) in Voigt Notation becomes:

$$\{\mathcal{E}'\} = [\mathcal{N}]\{\mathcal{E}\} \tag{1.440}$$

where

$$[\mathcal{N}] = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & a_{13}{}^2 & a_{11}a_{12} & a_{12}a_{13} & a_{11}a_{13} \\ a_{21}{}^2 & a_{22}{}^2 & a_{23}{}^2 & a_{21}a_{22} & a_{22}a_{23} & a_{21}a_{23} \\ a_{31}{}^2 & a_{32}{}^2 & a_{33}{}^2 & a_{31}a_{32} & a_{32}a_{33} & a_{31}a_{33} \\ 2a_{21}a_{11} & 2a_{22}a_{12} & 2a_{13}a_{23} & (a_{11}a_{22} + a_{12}a_{21}) & (a_{13}a_{22} + a_{12}a_{23}) & (a_{13}a_{21} + a_{11}a_{23}) \\ 2a_{31}a_{21} & 2a_{32}a_{22} & 2a_{33}a_{23} & (a_{31}a_{22} + a_{32}a_{21}) & (a_{33}a_{22} + a_{32}a_{23}) & (a_{33}a_{21} + a_{31}a_{23}) \\ 2a_{31}a_{11} & 2a_{32}a_{12} & 2a_{33}a_{13} & (a_{31}a_{12} + a_{32}a_{11}) & (a_{33}a_{12} + a_{32}a_{13}) & (a_{33}a_{11} + a_{31}a_{13}) \end{bmatrix} \tag{1.441}$$

The matrices (1.439) and (1.441) are not orthogonal matrices, *i.e.* $[\mathcal{M}]^{-1} \neq [\mathcal{M}]^T$ and $[\mathcal{N}]^{-1} \neq [\mathcal{N}]^T$. However, it is possible to show that $[\mathcal{M}]^{-1} = [\mathcal{N}]^T$.

1.7.4 Spectral Representation in Voigt Notation

Regarding the spectral representation of a symmetric tensor $\mathbf{T}$:

$$\mathbf{T} = \sum_{a=1}^{3} \mathsf{T}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \quad \xrightarrow{\text{Matricial form}} \quad \mathcal{T} = \mathcal{A}^T \, \mathcal{T}' \, \mathcal{A} \tag{1.442}$$

where $\mathcal{A}$ is the transformation matrix between the original set and the principal space, made up of the eigenvectors $\hat{\mathbf{n}}^{(a)}$. The above equation can be rewritten in terms of components as follows:

$$\begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} = \mathcal{A}^T \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \mathcal{A} \tag{1.443}$$

or

$$\begin{aligned} \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} &= \mathsf{T}_1 \begin{bmatrix} a_{11}^2 & a_{11}a_{12} & a_{11}a_{13} \\ a_{11}a_{12} & a_{12}^2 & a_{12}a_{13} \\ a_{11}a_{13} & a_{12}a_{13} & a_{13}^2 \end{bmatrix} + \mathsf{T}_2 \begin{bmatrix} a_{21}^2 & a_{21}a_{22} & a_{21}a_{23} \\ a_{21}a_{22} & a_{22}^2 & a_{22}a_{23} \\ a_{21}a_{23} & a_{22}a_{23} & a_{23}^2 \end{bmatrix} \\ &+ \mathsf{T}_3 \begin{bmatrix} a_{31}^2 & a_{31}a_{32} & a_{31}a_{33} \\ a_{31}a_{32} & a_{32}^2 & a_{32}a_{33} \\ a_{31}a_{33} & a_{32}a_{33} & a_{33}^2 \end{bmatrix} \end{aligned} \tag{1.444}$$

By regarding how second-order tensors are presented in Voigt Notation as in (1.438), the spectral representation of a second-order tensor in Voigt notation becomes:

$$\{\mathcal{T}\} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix} = \mathsf{T}_1 \begin{bmatrix} a_{11}^2 \\ a_{12}^2 \\ a_{13}^2 \\ a_{11}a_{12} \\ a_{12}a_{13} \\ a_{11}a_{13} \end{bmatrix} + \mathsf{T}_2 \begin{bmatrix} a_{21}^2 \\ a_{22}^2 \\ a_{23}^2 \\ a_{21}a_{22} \\ a_{22}a_{23} \\ a_{21}a_{23} \end{bmatrix} + \mathsf{T}_3 \begin{bmatrix} a_{31}^2 \\ a_{32}^2 \\ a_{33}^2 \\ a_{31}a_{32} \\ a_{32}a_{33} \\ a_{31}a_{33} \end{bmatrix} \tag{1.445}$$

1.7.5 Deviatoric Tensor Components in Voigt Notation

Observing the components of the deviatoric tensor:

$$\mathsf{T}_{ij}^{dev} = \begin{bmatrix} \frac{1}{3}(2\mathsf{T}_{11} - \mathsf{T}_{22} - \mathsf{T}_{33}) & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \frac{1}{3}(2\mathsf{T}_{22} - \mathsf{T}_{11} - \mathsf{T}_{33}) & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \frac{1}{3}(2\mathsf{T}_{33} - \mathsf{T}_{11} - \mathsf{T}_{22}) \end{bmatrix} \tag{1.446}$$

T_{ij}^{dev} in Voigt notation is given by:

$$\begin{bmatrix} \mathsf{T}_{11}^{dev} \\ \mathsf{T}_{22}^{dev} \\ \mathsf{T}_{33}^{dev} \\ \mathsf{T}_{12}^{dev} \\ \mathsf{T}_{23}^{dev} \\ \mathsf{T}_{13}^{dev} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix} \tag{1.447}$$

Problem 1.40: Let $\mathbf{T}(\vec{x},t)$ be a symmetric second-order tensor, which is expressed in terms of the position $(\vec{x})$ and time (t). Also, bear in mind that the tensor components, along direction x_3, are equal to zero, *i.e.* $\mathsf{T}_{13} = \mathsf{T}_{23} = \mathsf{T}_{33} = 0$.

NOTE: In the next section we will define $\mathbf{T}(\vec{x},t)$ as a field tensor, *i.e.* the value of $\mathbf{T}$ depends on position and time. As we will see later, if the tensor is independent of any one direction at all points $(\vec{x})$, *e.g.* if $\mathbf{T}(\vec{x},t)$ is independent of the x_3-direction, (see Figure 1.31), the problem becomes a two-dimensional problem (plane state) so that the problem is greatly simplified. ■

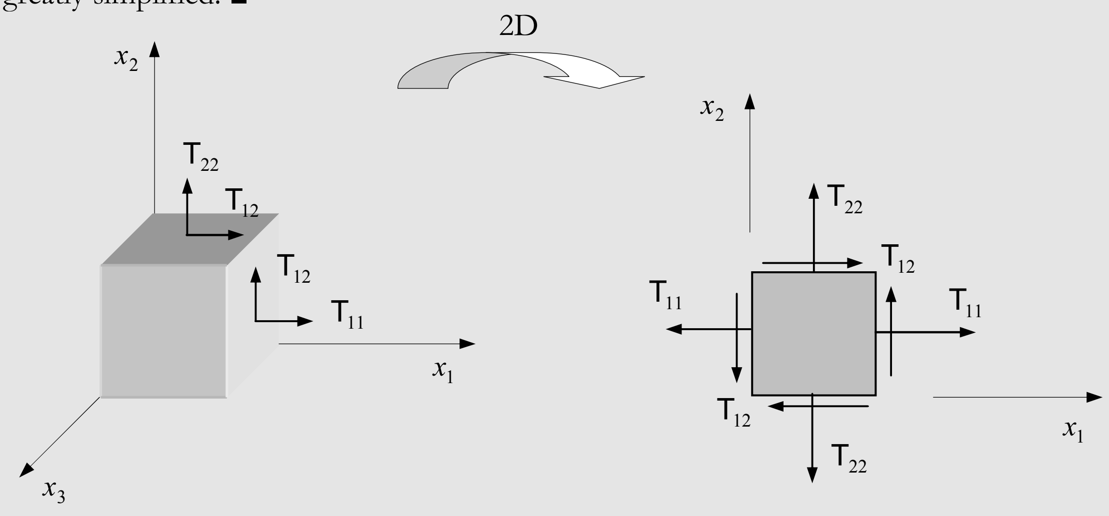


Figure 1.31: A two-dimensional problem (2D).

a) Obtain T'_{11}, T'_{22}, T'_{12} in the new reference system $(x'_1 - x'_2)$ defined in Figure 1.32.
b) Obtain the value of θ so that θ corresponds to the principal direction of $\mathbf{T}$, and also find an equation for the principal values of $\mathbf{T}$.
c) Evaluate the values of T'_{ij}, $(i,j=1,2)$, when $\mathsf{T}_{11}=1$, $\mathsf{T}_{22}=2$, $\mathsf{T}_{12}=-4$ and $\theta=45^{\circ}$. Also, obtain the principal values and principal directions.
d) Draw a graph that shows the relationship between θ and components T'_{11}, T'_{22} and T'_{12}, and in which the angle varies from 0° to 360°.
Hint: Use the Voigt Notation, and express the results in terms of 2θ.

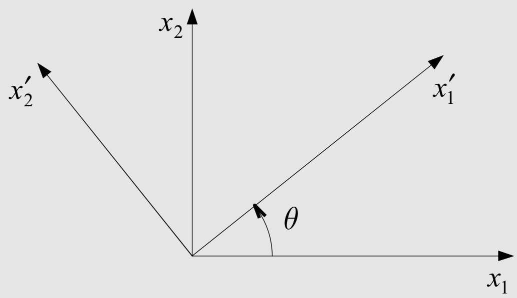


Figure 1.32: A two-dimensional problem (2D).

Solution:

a) Here we can apply the transformation law obtained in (1.437), which after removing rows and columns associated with the x_3-direction becomes:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & 2a_{11}a_{12} \\ a_{21}{}^2 & a_{22}{}^2 & 2a_{21}a_{22} \\ a_{21}a_{11} & a_{22}a_{12} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.448}$$

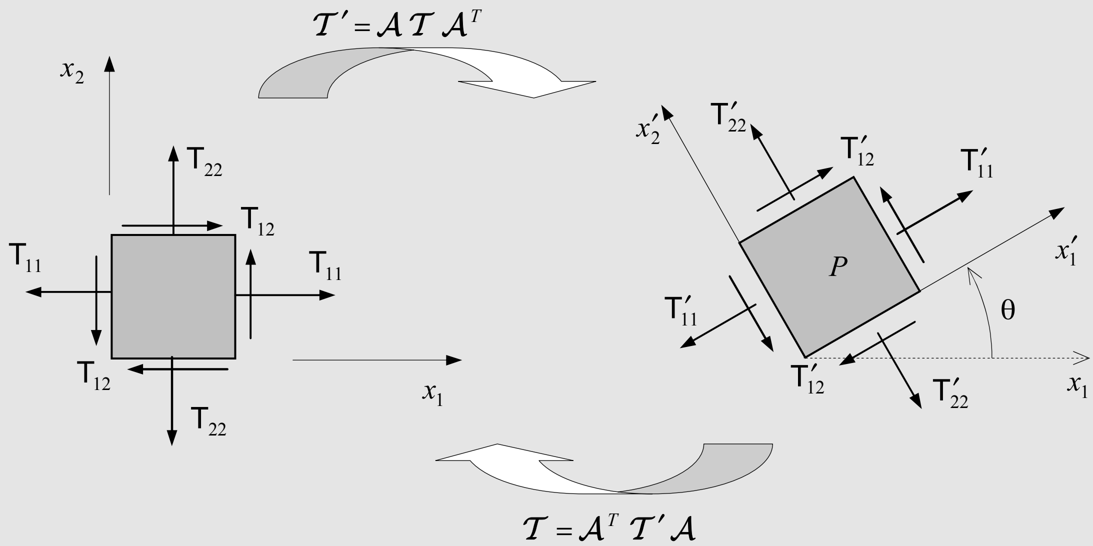


Figure 1.33: Transformation law for (2D) tensor components.

The transformation matrix, a_{ij}, in the plane, can be evaluated in terms of a single parameter, θ:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.449}$$

By substituting the matrix components a_{ij} given in (1.449) into (1.448) we obtain:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.450}$$

Making use of the following trigonometric identities, $2\cos\theta\sin\theta = \sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, (1.450) becomes:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1+\cos 2\theta}{2}\right) & \left(\dfrac{1-\cos 2\theta}{2}\right) & \sin 2\theta \\ \left(\dfrac{1-\cos 2\theta}{2}\right) & \left(\dfrac{1+\cos 2\theta}{2}\right) & -\sin 2\theta \\ \left(-\dfrac{\sin 2\theta}{2}\right) & \left(\dfrac{\sin 2\theta}{2}\right) & \cos 2\theta \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix}$$

Explicitly, the above components are given by:

$$\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{1+\cos 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{1-\cos 2\theta}{2}\right)\mathsf{T}_{22} + \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{22} = \left(\dfrac{1-\cos 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{1+\cos 2\theta}{2}\right)\mathsf{T}_{22} - \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{12} = \left(-\dfrac{\sin 2\theta}{2}\right)\mathsf{T}_{11} + \left(\dfrac{\sin 2\theta}{2}\right)\mathsf{T}_{22} + \mathsf{T}_{12}\cos 2\theta \end{cases}$$

Reordering the previous equation, we obtain:

$$\boxed{\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2}\right) + \left(\dfrac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)\cos 2\theta + \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{22} = \left(\dfrac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2}\right) - \left(\dfrac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)\cos 2\theta - \mathsf{T}_{12}\sin 2\theta \\ \mathsf{T}'_{12} = -\left(\dfrac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)\sin 2\theta + \mathsf{T}_{12}\cos 2\theta \end{cases}} \tag{1.451}$$

b) Recalling that the principal directions are characterized by the lack of any tangential components, *i.e.* $\mathsf{T}_{ij} = 0$ if $i \neq j$, in order to find the principal directions in the plane, we let $\mathsf{T}'_{12} = 0$, hence:

$$\mathsf{T}'_{12} = -\left(\frac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)\sin 2\theta + \mathsf{T}_{12}\cos 2\theta = 0 \Rightarrow \left(\frac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)\sin 2\theta = \mathsf{T}_{12}\cos 2\theta$$

$$\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}} \Rightarrow \mathrm{tg}(2\theta) = \frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}$$

Then, the angle corresponding to the principal direction is:

$$\boxed{\theta = \frac{1}{2}\,\mathrm{arctg}\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}\right)} \tag{1.452}$$

To find the principal values (eigenvalues) we must solve the following characteristic equation:

$$\begin{vmatrix} \mathsf{T}_{11}-\mathsf{T} & \mathsf{T}_{12} \\ \mathsf{T}_{12} & \mathsf{T}_{22}-\mathsf{T} \end{vmatrix} = 0 \quad \Rightarrow \quad \mathsf{T}^2 - \mathsf{T}(\mathsf{T}_{11}+\mathsf{T}_{22}) + (\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2) = 0$$

And by evaluating the quadratic equation we obtain:

$$\begin{aligned} \mathsf{T}_{(1,2)} &= \frac{-[-(\mathsf{T}_{11}+\mathsf{T}_{22})] \pm \sqrt{[-(\mathsf{T}_{11}+\mathsf{T}_{22})]^2 - 4(1)(\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2)}}{2(1)} \\ &= \frac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2} \pm \sqrt{\frac{[(\mathsf{T}_{11}+\mathsf{T}_{22})]^2 - 4(\mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}^2)}{4}} \end{aligned}$$

By rearranging the above equation we obtain the principal values for the two-dimensional case as:

$$\boxed{\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2}} \tag{1.453}$$

c) We directly apply equation (1.451) to evaluate the values of the components T'_{ij}, $(i,j=1,2)$, where $\mathsf{T}_{11}=1$, $\mathsf{T}_{22}=2$, $\mathsf{T}_{12}=-4$ and $\theta=45^\circ$, *i.e.*:

$$\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{1+2}{2}\right) + \left(\dfrac{1-2}{2}\right)\cos 90^\circ - 4\sin 90^\circ = -2.5 \\ \mathsf{T}'_{22} = \left(\dfrac{1+2}{2}\right) - \left(\dfrac{1-2}{2}\right)\cos 90^\circ + 4\sin 90^\circ = 5.5 \\ \mathsf{T}'_{12} = -\left(\dfrac{1-2}{2}\right)\sin 90^\circ - 4\cos 90^\circ = 0.5 \end{cases}$$

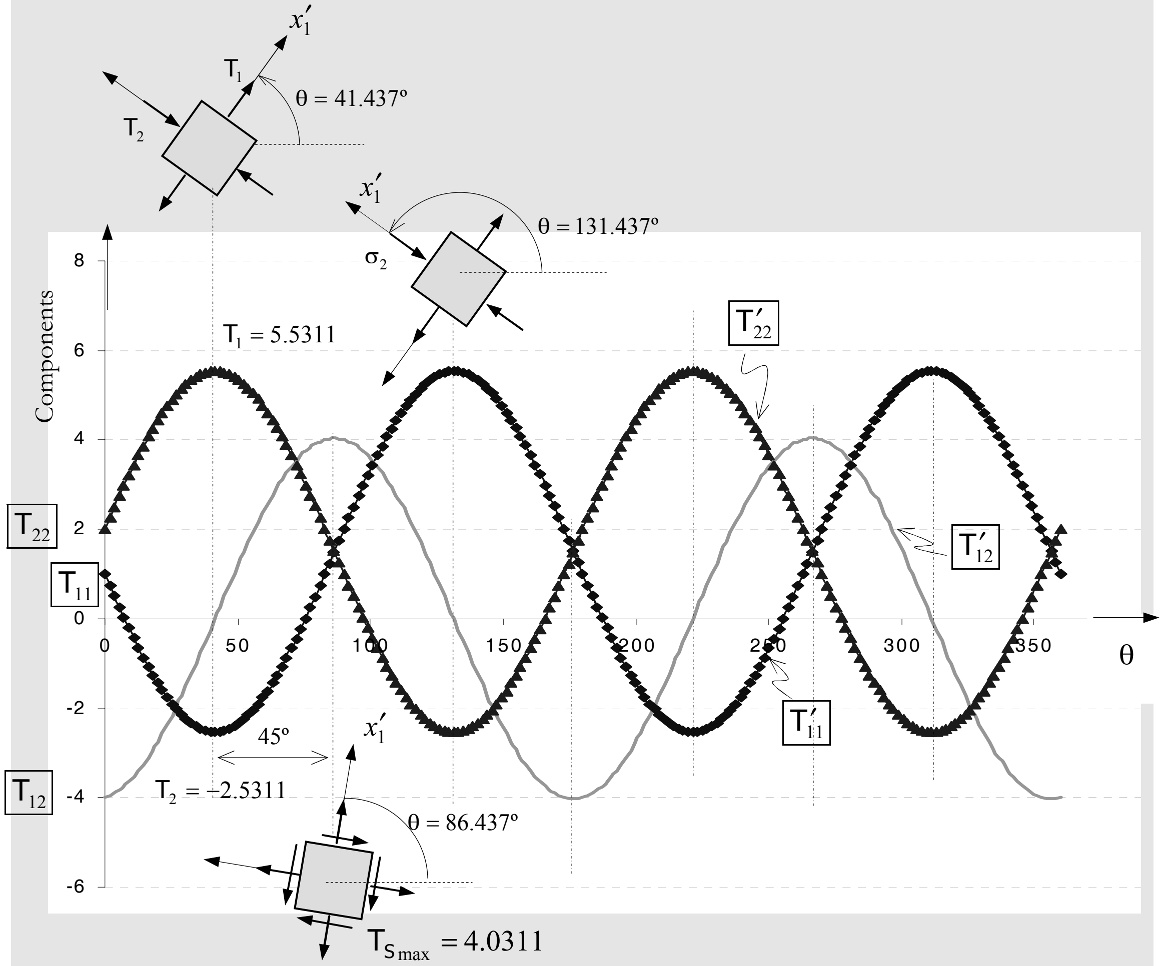
And the angle corresponding to the principal direction is:

$$\theta = \frac{1}{2}\operatorname{arctg}\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11}-\mathsf{T}_{22}}\right) = \frac{2\times(-4)}{1-2} \quad \Rightarrow \quad (\theta = 41.4375^\circ)$$

The principal values of $\mathbf{T}(\vec{x},t)$ can be evaluated as follows:

$$\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11}+\mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11}-\mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2} \quad \Rightarrow \quad \begin{cases} \mathsf{T}_1 = 5.5311 \\ \mathsf{T}_2 = -2.5311 \end{cases}$$

d) By referring to equation in (1.451) and by varying θ from 0° to 360°, we can obtain different values of T'_{11}, T'_{22}, T'_{12}, which are illustrated in the following graph:



1.8 Tensor Fields

A *tensor field* indicates how the tensor, $\mathbf{T}(\vec{\boldsymbol{x}},t)$, varies in space ($\vec{\boldsymbol{x}}$) and time ($t$). In this section, we regard the tensor field as a differentiable function of position and time. For more information about it, we need to define some operators, *e.g. gradient*, *divergence*, *curl*, which we can use as indicators of how these fields vary in space.

A tensor field which is independent of time is called a stationary or steady-state tensor field, *i.e.* $\mathbf{T}=\mathbf{T}(\vec{\boldsymbol{x}})$. However, if the field is only dependent on t then it is said to be *homogeneous* or *uniform*. That is, $\mathbf{T}(t)$ has the same value at every $\vec{\boldsymbol{x}}$ position.

Tensor fields can be classified according to their order as: scalar, vector, second-order tensor fields, etc. As an example of a scalar field we can quote temperature $T(\vec{\boldsymbol{x}},t)$ and in Figure 1.34(a) we can see temperature distribution over time $t=t_1$. Then, as an example of a vector field we can quote velocity $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)$ and Figure 1.34(b) shows velocity distribution, in which each point is associated with a vector $\vec{\boldsymbol{v}}$ over time $t=t_1$.

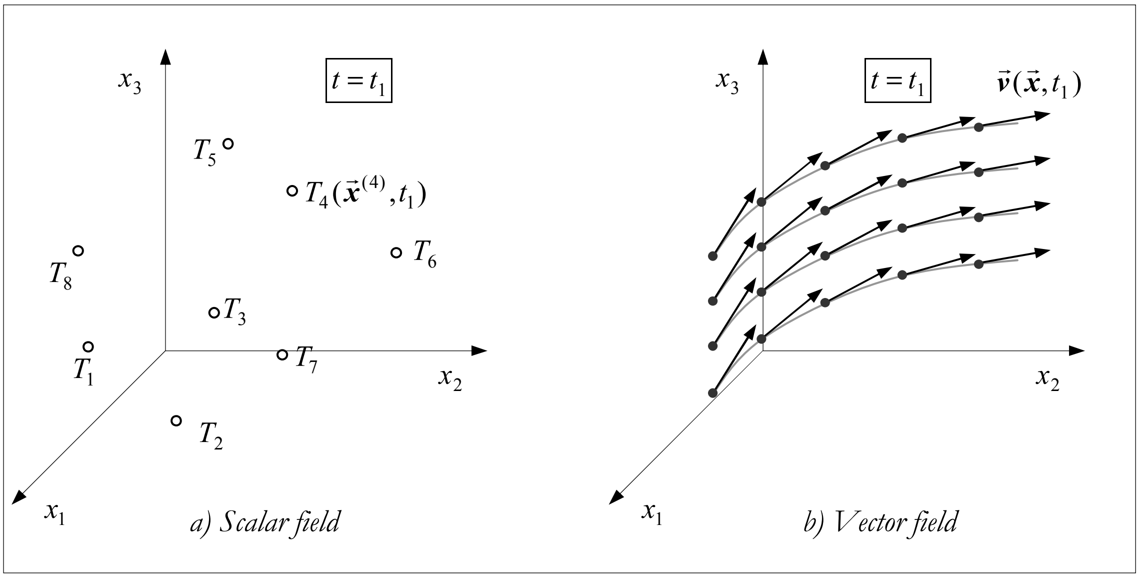


Figure 1.34: Examples of tensor fields.

Scalar Field

$$\phi=\phi(\vec{\boldsymbol{x}},t) \tag{1.454}$$

Vector Field

Tensorial notation $\quad \vec{\mathbf{v}}=\vec{\mathbf{v}}(\vec{\boldsymbol{x}},t)$

Indicial notation $\quad \mathsf{v}_i=\mathsf{v}_i(\vec{\boldsymbol{x}},t)$ (1.455)

Second-Order Tensor Field

Tensorial notation $\quad \mathbf{T}=\mathbf{T}(\vec{\boldsymbol{x}},t)$

Indicial notation $\quad \mathsf{T}_{ij}=\mathsf{T}_{ij}(\vec{\boldsymbol{x}},t)$ (1.456)

1.8.1 Scalar Fields

The next analysis is carry out with reference to a stationary scalar field, *i.e.* $\phi = \phi(\vec{x})$, with continuous values of $\partial\phi / \partial x_1$, $\partial\phi / \partial x_2$ and $\partial\phi / \partial x_3$. Then, observe that the value of the scalar function at point $(\vec{x})$ is $\phi(\vec{x})$, and if we observe a second point located at $(\vec{x} + d\vec{x})$, the *total derivative* (*differential*) of the function ϕ is defined as:

$$
\begin{gathered}
\phi(\vec{x} + d\vec{x}) - \phi(\vec{x}) \equiv d\phi \\
\phi(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3) - \phi(x_1,\ x_2,\ x_3) \equiv d\phi
\end{gathered}
\tag{1.457}
$$

For any continuous function $\phi(x_1, x_2, x_3)$, $d\phi$ is linearly related to dx_1, dx_2, dx_3. This linear relationship can be evaluated by the chain rule of differentiation as:

$$
\begin{gathered}
d\phi = \frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 \\
d\phi = \phi_{,i}\, dx_i
\end{gathered}
\tag{1.458}
$$

The differentiation of the components of a tensor, with respect to coordinates x_i, is expressed by the differential operator:

$$
\frac{\partial \bullet}{\partial x_i} \equiv \bullet_{,i} \tag{1.459}
$$

1.8.2 Gradient

The gradient of a scalar field

The gradient $\nabla_{\vec{x}}\phi$ or $\mathbf{grad}\phi$ is defined as:

$$
\nabla_{\vec{x}}\phi \longrightarrow d\phi = \nabla_{\vec{x}}\phi \cdot d\vec{x} \tag{1.460}
$$

where the operator $\nabla_{\vec{x}}$ is known as the *Nabla symbol*. Expressing the equation (1.460) in the Cartesian basis we obtain:

$$
\begin{aligned}
&\frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 = \\
&\quad = \left[(\nabla_{\vec{x}}\phi)_{x_1} \hat{\mathbf{e}}_1 + (\nabla_{\vec{x}}\phi)_{x_2} \hat{\mathbf{e}}_2 + (\nabla_{\vec{x}}\phi)_{x_3} \hat{\mathbf{e}}_3\right] \cdot \left[(dx_1)\hat{\mathbf{e}}_1 + (dx_2)\hat{\mathbf{e}}_2 + (dx_3)\hat{\mathbf{e}}_3\right]
\end{aligned}
\tag{1.461}
$$

Evaluating the above scalar product we find:

$$
\frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 = (\nabla_{\vec{x}}\phi)_{x_1} dx_1 + (\nabla_{\vec{x}}\phi)_{x_2} dx_2 + (\nabla_{\vec{x}}\phi)_{x_3} dx_3 \tag{1.462}
$$

Therefore, we can draw the conclusion that the $\nabla_{\vec{x}}\phi$ components in the Cartesian basis are:

$$
(\nabla_{\vec{x}}\phi)_1 \equiv \frac{\partial\phi}{\partial x_1} \quad ; \quad (\nabla_{\vec{x}}\phi)_2 \equiv \frac{\partial\phi}{\partial x_2} \quad ; \quad (\nabla_{\vec{x}}\phi)_3 \equiv \frac{\partial\phi}{\partial x_3} \tag{1.463}
$$

Hence, the gradient in terms of components is defined as such:

$$
\nabla_{\vec{x}}\phi = \frac{\partial\phi}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial x_2} \hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial x_3} \hat{\mathbf{e}}_3 \tag{1.464}
$$

The Nabla symbol $\nabla_{\vec{x}}$ is defined as:

$$\boxed{\nabla_{\vec{x}} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \equiv \partial_{,i}\,\hat{\mathbf{e}}_i} \quad \textit{Nabla symbol} \tag{1.465}$$

The geometric meaning of $\nabla_{\vec{x}}\phi$

- The direction of $\nabla_{\vec{x}}\phi$ is normal to the equiscalar surface, *i.e.* it is perpendicular to the isosurface $\phi = const$. The direction of $\nabla_{\vec{x}}\phi$ points to the direction where ϕ is increasing the most, (see Figure 1.35).
- The magnitude of $\nabla_{\vec{x}}\phi$ is the rate of change of ϕ, *i.e.* the gradient of ϕ.

The normal vector to this surface is obtained as follows:

$$\hat{\mathbf{n}} = \frac{\nabla_{\vec{x}}\phi}{\left\|\nabla_{\vec{x}}\phi\right\|} \tag{1.466}$$

The surface $\phi = const$, called the surface level, or isosurface or equiscalar surface, is the surface formed by points which all have the same value of ϕ, so, if we move along the level surface the values of the function do not change.

The gradient of a vector field $\vec{\mathbf{v}}(\vec{x})$:

$$\mathrm{grad}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}}\vec{\mathbf{v}} \tag{1.467}$$

Using the definition of $\nabla_{\vec{x}}$, given in (1.465), the gradient of the vector field becomes:

$$\nabla_{\vec{x}}\vec{\mathbf{v}} = \frac{\partial(\mathrm{v}_i\hat{\mathbf{e}}_i)}{\partial x_j} \otimes \hat{\mathbf{e}}_j = (\mathrm{v}_i\hat{\mathbf{e}}_i)_{,j} \otimes \hat{\mathbf{e}}_j = \mathrm{v}_{i,j}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.468}$$

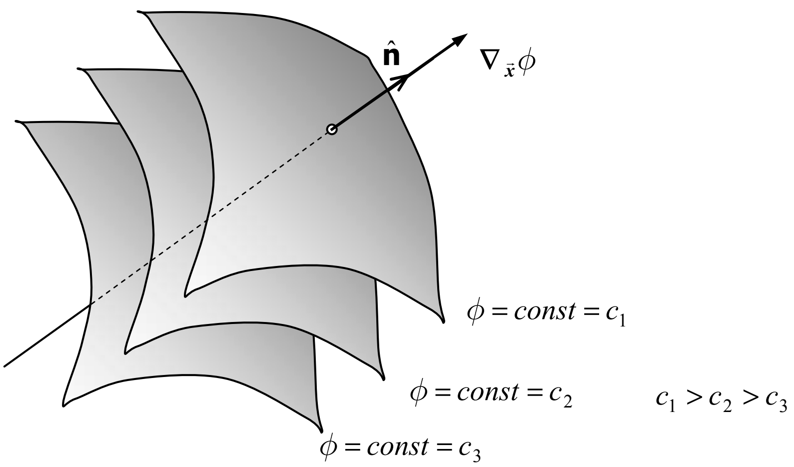


Figure 1.35: Gradient of ϕ.

Therefore, we can define the gradient of a tensor field $(\bullet(\vec{x},t))$ in the Cartesian basis as:

$$\boxed{\nabla_{\vec{x}}(\bullet) = \frac{\partial(\bullet)}{\partial x_j} \otimes \hat{\mathbf{e}}_j} \quad \textit{Gradient of a tensor field in the Cartesian basis} \tag{1.469}$$

As noted, the gradient of a vector field becomes a second-order tensor field, whose components are:

$$\mathsf{v}_{i,j} \equiv \frac{\partial \mathsf{v}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathsf{v}_1}{\partial x_1} & \frac{\partial \mathsf{v}_1}{\partial x_2} & \frac{\partial \mathsf{v}_1}{\partial x_3} \\ \frac{\partial \mathsf{v}_2}{\partial x_1} & \frac{\partial \mathsf{v}_2}{\partial x_2} & \frac{\partial \mathsf{v}_2}{\partial x_3} \\ \frac{\partial \mathsf{v}_3}{\partial x_1} & \frac{\partial \mathsf{v}_3}{\partial x_2} & \frac{\partial \mathsf{v}_3}{\partial x_3} \end{bmatrix} \tag{1.470}$$

The gradient of a second-order tensor field $\mathbf{T}(\vec{x})$:

$$\nabla_{\vec{x}} \mathbf{T} = \frac{\partial (T_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)}{\partial x_k} \otimes \hat{\mathbf{e}}_k = T_{ij,k} \; \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \tag{1.471}$$

and its components are represented by:

$$(\nabla_{\vec{x}} \mathbf{T})_{ijk} \equiv T_{ij,k} \tag{1.472}$$

Problem 1.41: Find the gradient of the function $f(x_1, x_2) = \cos(x_1) + \exp^{x_1 x_2}$ at the point $(x_1 = 0, x_2 = 1)$.

Solution: By definition, the gradient of a scalar function is given by:

$$\nabla_{\vec{x}} f = \frac{\partial f}{\partial x_1} \; \hat{\mathbf{e}}_1 + \frac{\partial f}{\partial x_2} \; \hat{\mathbf{e}}_2$$

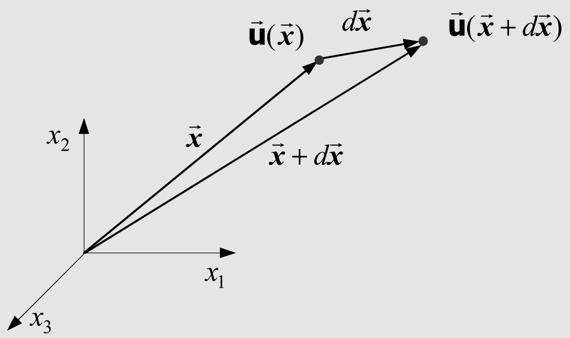
where: $\frac{\partial f}{\partial x_1} = -\sin(x_1) + x_2 \exp^{x_1 x_2}$; $\frac{\partial f}{\partial x_2} = x_1 \exp^{x_1 x_2}$

$$\nabla_{\vec{x}} f(x_1, x_2) = \left[-\sin(x_1) + x_2 \exp^{x_1 x_2} \right] \hat{\mathbf{e}}_1 + \left[x_1 \exp^{x_1 x_2} \right] \hat{\mathbf{e}}_2 \Rightarrow \nabla_{\vec{x}} f(0,1) = [1] \; \hat{\mathbf{e}}_1 + [0] \; \hat{\mathbf{e}}_2 = 2\hat{\mathbf{e}}_1$$

Problem 1.42: Let $\vec{\mathbf{u}}(\vec{x})$ be a stationary vector field. a) Obtain the components of the differential $d\vec{\mathbf{u}}$. b) Now, consider that $\vec{\mathbf{u}}(\vec{x})$ represents a displacement field, and is independent of x_3. With these conditions, graphically illustrate the displacement field in the differential area element $dx_1 dx_2$.

Solution: According to the differential and gradient definitions, it holds that:

$$\boxed{\begin{aligned} d\vec{\mathbf{u}} &\equiv \vec{\mathbf{u}}(\vec{x} + d\vec{x}) - \vec{\mathbf{u}}(\vec{x}) \\ d\vec{\mathbf{u}} &= \nabla_{\vec{x}} \vec{\mathbf{u}} \cdot d\vec{x} \end{aligned}}$$



Thus, the components are defined as:

$$d\mathsf{u}_i = \frac{\partial \mathsf{u}_i}{\partial x_j} dx_j \quad \Rightarrow \quad \begin{bmatrix} d\mathsf{u}_1 \\ d\mathsf{u}_2 \\ d\mathsf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathsf{u}_1}{\partial x_1} & \frac{\partial \mathsf{u}_1}{\partial x_2} & \frac{\partial \mathsf{u}_1}{\partial x_3} \\ \frac{\partial \mathsf{u}_2}{\partial x_1} & \frac{\partial \mathsf{u}_2}{\partial x_2} & \frac{\partial \mathsf{u}_2}{\partial x_3} \\ \frac{\partial \mathsf{u}_3}{\partial x_1} & \frac{\partial \mathsf{u}_3}{\partial x_2} & \frac{\partial \mathsf{u}_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

---

or:

$$\begin{cases} d\mathrm{u}_1 = \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_1}{\partial x_3} dx_3 \\ d\mathrm{u}_2 = \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_2}{\partial x_3} dx_3 \\ d\mathrm{u}_3 = \dfrac{\partial \mathrm{u}_3}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_3}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_3}{\partial x_3} dx_3 \end{cases}$$

with

$$\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_1(x_1, x_2, x_3) \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_2(x_1, x_2, x_3) \\ d\mathrm{u}_3 = \mathrm{u}_3(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_3(x_1, x_2, x_3) \end{cases}$$

As the field is independent of x_3, the displacement field in the differential area element is defined as:

$$\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_1(x_1, x_2) = \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_2(x_1, x_2) = \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}$$

or:

$$\begin{cases} \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_1(x_1, x_2) + \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_2(x_1, x_2) + \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}$$

Note that the above equation is equivalent to the Taylor series expansion taking into account only up to linear terms. The representation of the displacement field in the differential area element is shown in Figure 1.36.

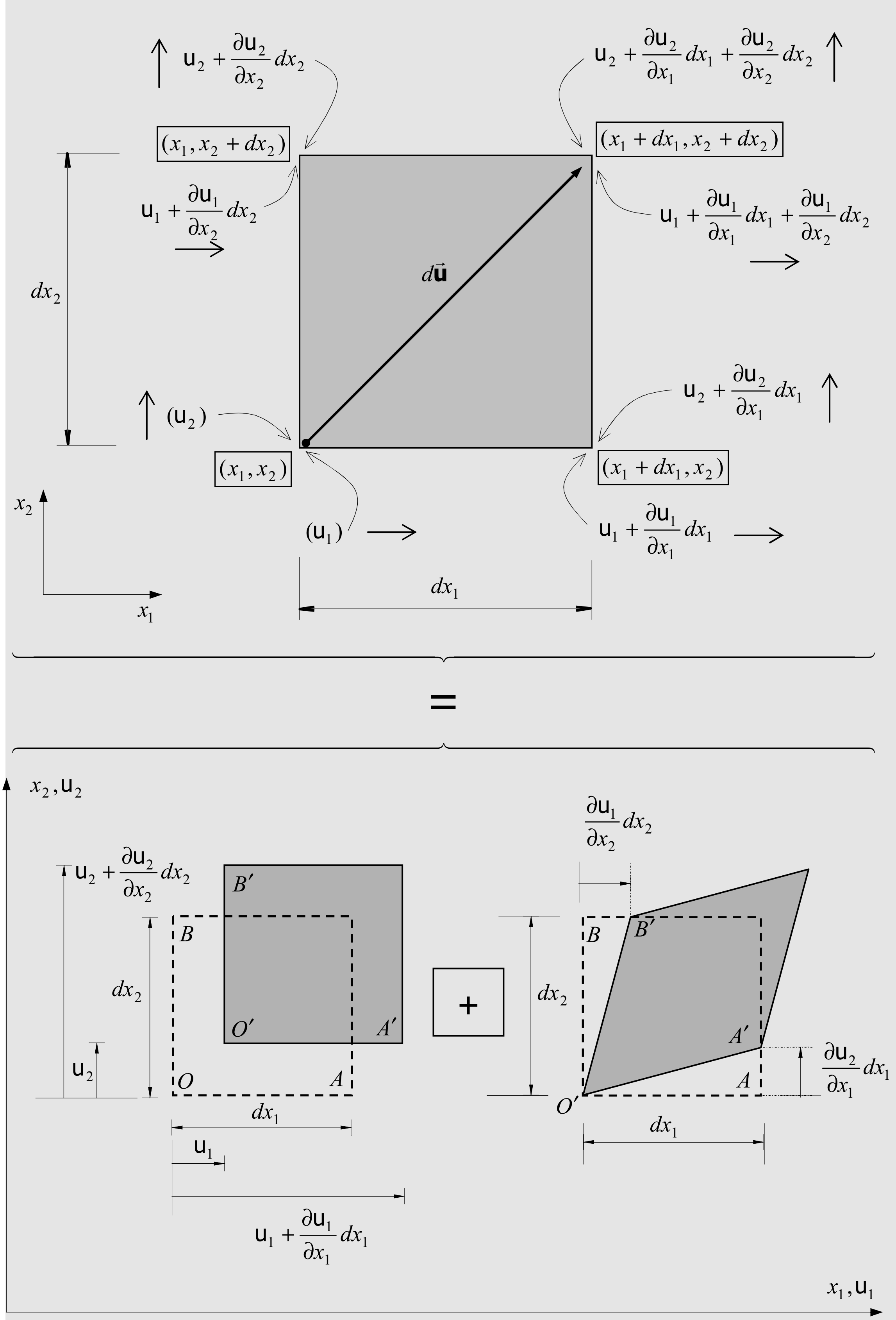


Figure 1.36: Displacement field in the differential area element.

1.8.3 Divergence

The **divergence of a vector field**, $\vec{\mathbf{v}}(\vec{x})$, is denoted as follows:

$$\mathrm{div}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} \tag{1.473}$$

which by definition is:

$$\mathrm{div}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}}\vec{\mathbf{v}} : \mathbf{1} = \mathrm{Tr}(\nabla_{\vec{x}}\vec{\mathbf{v}}) \tag{1.474}$$

Then:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}}\vec{\mathbf{v}} : \mathbf{1} &= \left[\mathrm{v}_{i,j}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right] : \left[\delta_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l\right] = \mathrm{v}_{i,j}\delta_{kl}\delta_{ik}\delta_{jl} = \mathrm{v}_{k,k} \\ &= \frac{\partial \mathrm{v}_1}{\partial x_1} + \frac{\partial \mathrm{v}_2}{\partial x_2} + \frac{\partial \mathrm{v}_3}{\partial x_3} \end{aligned} \tag{1.475}$$

or

$$\begin{aligned} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}}\vec{\mathbf{v}} : \mathbf{1} &= \left[\mathrm{v}_{i,j}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right] : \left[\delta_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l\right] \\ &= \left[\mathrm{v}_{i,j}\delta_{kl}\delta_{lj}\hat{\mathbf{e}}_i\right] \cdot \hat{\mathbf{e}}_k \\ &= \left[\mathrm{v}_{i,k}\hat{\mathbf{e}}_i\right] \cdot \hat{\mathbf{e}}_k \\ &= \frac{\partial\left[\mathrm{v}_i\hat{\mathbf{e}}_i\right]}{\partial x_k} \cdot \hat{\mathbf{e}}_k \end{aligned} \tag{1.476}$$

Which we can use to insert the following operator into the Cartesian basis:

$$\boxed{\nabla_{\vec{x}} \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_k} \cdot \hat{\mathbf{e}}_k} \quad \textit{Divergence of } (\bullet) \textit{ in Cartesian basis} \tag{1.477}$$

We can also verify that, when divergence is applied to a tensor field its rank decreases by one order.

Divergence of a second-order tensor field $\mathbf{T}(\vec{x})$

The divergence of a second-order tensor field $\mathbf{T}$ is denoted by $\nabla_{\vec{x}} \cdot \mathbf{T} = \nabla_{\vec{x}}\mathbf{T} : \mathbf{1}$, which becomes a vector:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \mathbf{T} \equiv \mathrm{div}\mathbf{T} &= \frac{\partial(T_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)}{\partial x_k} \cdot \hat{\mathbf{e}}_k \\ &= \frac{\partial T_{ij}}{\partial x_k}\delta_{jk}\hat{\mathbf{e}}_i \\ &= \mathrm{T}_{ik,k}\hat{\mathbf{e}}_i \end{aligned} \tag{1.478}$$

NOTE: In this text book, when dealing with *gradient* or *divergence* of a tensor field, *e.g.* $\nabla_{\vec{x}}\vec{\mathbf{v}}$ (the gradient of the vector field), $\nabla_{\vec{x}}\mathbf{T}$ (the gradient of a second-order tensor field), $\nabla_{\vec{x}} \cdot \mathbf{T}$ (divergence of a second-order tensor field), this does not indicate that we are making a tensor operation between a vector and a tensor, *i.e.* $\nabla_{\vec{x}}\vec{\mathbf{v}} \neq (\vec{\nabla}_{\vec{x}}) \otimes (\vec{\mathbf{v}})$, $\nabla_{\vec{x}}\mathbf{T} \neq (\vec{\nabla}_{\vec{x}}) \otimes (\mathbf{T})$ and $\nabla_{\vec{x}} \cdot \mathbf{T} \neq (\vec{\nabla}_{\vec{x}}) \cdot (\mathbf{T})$ and so on. In this textbook, $\nabla_{\vec{x}}$ is an operator which must be applied to the entire tensor field, so, the tensor must be inside the operator, (see equations (1.477) and (1.469)). Nevertheless, it is possible to relate $\nabla_{\vec{x}}\vec{\mathbf{v}}$, $\nabla_{\vec{x}}\mathbf{T}$ or $\nabla_{\vec{x}} \cdot \mathbf{T}$ to tensor operations between tensors, and it is easy to show that:

$$\nabla_{\vec{x}}\vec{\mathbf{v}} = (\vec{\mathbf{v}}) \otimes (\vec{\nabla}_{\vec{x}})$$
$$\nabla_{\vec{x}}\mathbf{T} = (\mathbf{T}) \otimes (\vec{\nabla}_{\vec{x}}) \tag{1.479}$$
$$\nabla_{\vec{x}} \cdot \mathbf{T} = (\mathbf{T}) \cdot (\vec{\nabla}_{\vec{x}}) = (\vec{\nabla}_{\vec{x}}) \cdot (\mathbf{T}^T) \quad \blacksquare$$

Once the Nabla symbol is defined we introduce the *Laplacian operator* ∇^2 as:

$$\nabla_{\vec{x}}^2 = \nabla_{\vec{x}} \cdot \nabla_{\vec{x}} = \left(\frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \hat{\mathbf{e}}_j \right) \cdot \hat{\mathbf{e}}_i \right) = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \delta_{ij} = \frac{\partial^2}{\partial x_i \partial x_i}$$
$$\nabla_{\vec{x}}^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \partial,_k \partial,_k = \partial,_{kk} \tag{1.480}$$

Then, the vector Laplacian of a vector field, $\vec{\mathbf{v}}(\vec{x})$, is given by:

$$\nabla_{\vec{x}}^2 \vec{\mathbf{v}} = \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \vec{\mathbf{v}}) \xrightarrow{components} \left[\nabla_{\vec{x}}^2 \vec{\mathbf{v}} \right]_i = \left[\nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \vec{\mathbf{v}}) \right]_i = \mathrm{v}_{i,kk} \tag{1.481}$$

Problem 1.43: Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be vectors. Show that the following identity $\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$ holds.

Solution:

Observing that $\vec{\mathbf{a}} = \mathrm{a}_j \hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathrm{b}_k \hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$, we can express $\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}})$ as:

$$\frac{\partial (\mathrm{a}_j \hat{\mathbf{e}}_j + \mathrm{b}_k \hat{\mathbf{e}}_k)}{\partial x_i} \cdot \hat{\mathbf{e}}_i = \frac{\partial \mathrm{a}_j}{\partial x_i} \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_i + \frac{\partial \mathrm{b}_k}{\partial x_i} \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i = \frac{\partial \mathrm{a}_i}{\partial x_i} + \frac{\partial \mathrm{b}_i}{\partial x_i} = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$$

Working directly with indicial notation we obtain:

$$\nabla_{\vec{x}} \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = (\mathrm{a}_i + \mathrm{b}_i),_i = \mathrm{a}_i,_i + \mathrm{b}_i,_i = \nabla_{\vec{x}} \cdot \vec{\mathbf{a}} + \nabla_{\vec{x}} \cdot \vec{\mathbf{b}}$$

Problem 1.44: Find the components of $(\nabla_{\vec{x}} \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}$.

Solution: Bearing in mind that $\vec{\mathbf{a}} = \mathrm{a}_j \hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathrm{b}_k \hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$ $(i = 1,2,3)$, the following is true:

$$(\nabla_{\vec{x}} \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} = \left(\frac{\partial (\mathrm{a}_j \hat{\mathbf{e}}_j)}{\partial x_i} \otimes \hat{\mathbf{e}}_i \right) \cdot (\mathrm{b}_k \hat{\mathbf{e}}_k) = \left(\frac{\partial \mathrm{a}_j}{\partial x_i} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \right) \cdot (\mathrm{b}_k \hat{\mathbf{e}}_k) = \mathrm{b}_k \delta_{ik} \frac{\partial \mathrm{a}_j}{\partial x_i} \hat{\mathbf{e}}_j = \mathrm{b}_k \frac{\partial \mathrm{a}_j}{\partial x_k} \hat{\mathbf{e}}_j$$

Expanding the dummy index k, we obtain:

$$\mathrm{b}_k \frac{\partial \mathrm{a}_j}{\partial x_k} = \mathrm{b}_1 \frac{\partial \mathrm{a}_j}{\partial x_1} + \mathrm{b}_2 \frac{\partial \mathrm{a}_j}{\partial x_2} + \mathrm{b}_3 \frac{\partial \mathrm{a}_j}{\partial x_3}$$

Thus,

$$j = 1 \quad \Rightarrow \quad \mathrm{b}_1 \frac{\partial \mathrm{a}_1}{\partial x_1} + \mathrm{b}_2 \frac{\partial \mathrm{a}_1}{\partial x_2} + \mathrm{b}_3 \frac{\partial \mathrm{a}_1}{\partial x_3}$$

$$j = 2 \quad \Rightarrow \quad \mathrm{b}_1 \frac{\partial \mathrm{a}_2}{\partial x_1} + \mathrm{b}_2 \frac{\partial \mathrm{a}_2}{\partial x_2} + \mathrm{b}_3 \frac{\partial \mathrm{a}_2}{\partial x_3}$$

$$j = 3 \quad \Rightarrow \quad \mathrm{b}_1 \frac{\partial \mathrm{a}_3}{\partial x_1} + \mathrm{b}_2 \frac{\partial \mathrm{a}_3}{\partial x_2} + \mathrm{b}_3 \frac{\partial \mathrm{a}_3}{\partial x_3}$$

Problem 1.45: Prove that the following relationship is valid:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T} \right) = \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T$$

where $\vec{\mathbf{q}}(\vec{x},t)$ is an arbitrary vector field, and $T(\vec{x},t)$ is a scalar field.

Solution:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T} \right) = \frac{\partial}{\partial x_i} \left(\frac{\mathrm{q}_i}{T} \right) \equiv \left(\frac{\mathrm{q}_i}{T} \right)_{,i} = \frac{1}{T} \mathrm{q}_{i,i} - \frac{1}{T^2} \mathrm{q}_i T_{,i}$$

$$= \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \quad \text{(scalar)}$$

1.8.4 The Curl

The curl of a vector field

The curl (or rotor) of a vector field, $\vec{\mathbf{v}}(\vec{x})$ is denoted by $\mathrm{curl}(\vec{\mathbf{v}}) \equiv \mathrm{rot}(\vec{\mathbf{v}}) \equiv \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$, and is defined in the Cartesian basis as:

$$\boxed{\vec{\nabla}_{\vec{x}} \wedge (\bullet) = \frac{\partial}{\partial x_j} \hat{\mathbf{e}}_j \wedge (\bullet)} \quad \textit{The curl (rotor) of a tensor field in the Cartesian basis} \tag{1.482}$$

Note that the curl is already a tensor operator between two vectors. Using the definition of the vector product we obtain the curl of a vector field as:

$$\mathrm{rot}(\vec{\mathbf{v}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} = \frac{\partial}{\partial x_j} \hat{\mathbf{e}}_j \wedge (\mathrm{v}_k \hat{\mathbf{e}}_k) = \frac{\partial \mathrm{v}_k}{\partial x_j} \hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k = \frac{\partial \mathrm{v}_k}{\partial x_j} \epsilon_{ijk} \hat{\mathbf{e}}_i = \epsilon_{ijk} \mathrm{v}_{k,j} \hat{\mathbf{e}}_i \tag{1.483}$$

where ϵ_{ijk} is the permutation symbol defined in (1.55). Moreover, we have applied the definition $\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k = \epsilon_{ijk} \hat{\mathbf{e}}_i$ and we can also note that:

$$\begin{aligned} \mathrm{rot}(\vec{\mathbf{v}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{vmatrix} = \epsilon_{ijk} \mathrm{v}_{k,j} \hat{\mathbf{e}}_i \\ &= \left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 \end{aligned} \tag{1.484}$$

We can verify that the antisymmetric part of a vector field gradient, which is illustrated by $(\nabla_{\vec{x}} \vec{\mathbf{v}})^{skew} \equiv \mathbf{W}$, has as components:

$$\begin{aligned} \left[(\nabla_{\vec{x}} \vec{\mathbf{v}})^{skew} \right]_{ij} \equiv \mathrm{v}_{i,j}^{skew} &= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial \mathrm{v}_1}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_1} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \\ \frac{1}{2}\left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) & 0 & \frac{1}{2}\left(\frac{\partial \mathrm{v}_2}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_2} \right) \\ \frac{1}{2}\left(\frac{\partial \mathrm{v}_3}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_3} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mathrm{W}_{12} & \mathrm{W}_{13} \\ \mathrm{W}_{21} & 0 & \mathrm{W}_{23} \\ \mathrm{W}_{31} & \mathrm{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{W}_{12} & \mathrm{W}_{13} \\ -\mathrm{W}_{12} & 0 & \mathrm{W}_{23} \\ -\mathrm{W}_{13} & -\mathrm{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \end{aligned} \tag{1.485}$$

where w_1, w_2, w_3 are the components of the axial vector $\vec{w}$ associated with $\mathbf{W}$, (see subsection: 1.5.2.2.2. Antisymmetric Tensor).

With reference to the definition of the curl in (1.484) and the relationship in (1.485), we can conclude that:

$$
\begin{aligned}
\operatorname{rot}(\vec{\mathbf{v}}) \equiv \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} &= \left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 \\
&= 2\mathrm{W}_{32} \hat{\mathbf{e}}_1 + 2\mathrm{W}_{13} \hat{\mathbf{e}}_2 + 2\mathrm{W}_{21} \hat{\mathbf{e}}_3 \\
&= 2\left(w_1 \hat{\mathbf{e}}_1 + w_2 \hat{\mathbf{e}}_2 + w_3 \hat{\mathbf{e}}_3 \right) \\
&= 2\vec{w}
\end{aligned}
\tag{1.486}
$$

And, if we use the identity in (1.141), we obtain:

$$
\mathbf{W} \cdot \vec{\mathbf{v}} = \vec{w} \wedge \vec{\mathbf{v}} = \frac{1}{2} \left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} \right) \wedge \vec{\mathbf{v}} \tag{1.487}
$$

It could be interesting to note that the equation in (1.486) can be obtained by means of **Problem 1.18**, in which we showed that $\frac{1}{2}(\vec{a} \wedge \vec{x})$ is the axial vector associated with the antisymmetric tensor $(\vec{x} \otimes \vec{a})^{skew}$. Therefore, the axial vector associated with the antisymmetric tensor $\mathbf{W} = (\nabla_{\vec{x}} \vec{\mathbf{v}})^{skew} = \left[(\vec{\mathbf{v}}) \otimes (\vec{\nabla}) \right]^{skew}$ is the vector $\frac{1}{2}\left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} \right)$.

As we can see, the curl describes the rotational tendency of the vector field.

Summary

$\bullet$	Divergence $\operatorname{div}(\bullet) \equiv \nabla_{\vec{x}} \cdot \bullet$	Gradient $\operatorname{grad}(\bullet) \equiv \nabla_{\vec{x}} \bullet$	Curl $\operatorname{rot}(\bullet) \equiv \vec{\nabla}_{\vec{x}} \wedge \bullet$
Scalar		vector	
Vector	Scalar	Second-order tensor	Vector
Second-order tensor	Vector	Third-order tensor	Second-order tensor

We can now present some equations:

- $\operatorname{rot}(\lambda \vec{\mathbf{a}}) = \vec{\nabla}_{\vec{x}} \wedge (\lambda \vec{\mathbf{a}}) = \lambda (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}} \lambda \wedge \vec{\mathbf{a}})$

  The result of the algebraic operation $\vec{\nabla}_{\vec{x}} \wedge (\lambda \vec{\mathbf{a}})$ is a vector, whose components are given by:

$$
\begin{aligned}
\left[\vec{\nabla}_{\vec{x}} \wedge (\lambda \vec{\mathbf{a}}) \right]_i &= \epsilon_{ijk} (\lambda \mathrm{a}_k)_{,j} \\
&= \epsilon_{ijk} (\lambda_{,j} \mathrm{a}_k + \lambda \mathrm{a}_{k,j}) \\
&= \epsilon_{ijk} \lambda \mathrm{a}_{k,j} + \epsilon_{ijk} \lambda_{,j} \mathrm{a}_k \\
&= \lambda (\nabla_{\vec{x}} \wedge \vec{\mathbf{a}})_i + \epsilon_{ijk} (\nabla_{\vec{x}} \lambda)_j \mathrm{a}_k \\
&= \lambda (\nabla_{\vec{x}} \wedge \vec{\mathbf{a}})_i + (\nabla_{\vec{x}} \lambda \wedge \vec{\mathbf{a}})_i
\end{aligned}
\tag{1.488}
$$

We can use the above equation to check that the relationship
$\operatorname{rot}(\lambda \vec{\mathbf{a}}) = \vec{\nabla}_{\vec{x}} \wedge (\lambda \vec{\mathbf{a}}) = \lambda (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}} \lambda \wedge \vec{\mathbf{a}})$ holds.

■ $$\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}} - (\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}} + (\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} - (\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}} \tag{1.489}$$

The components of the vector product $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ are given by $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})_k = \epsilon_{kij}\mathrm{a}_i\mathrm{b}_j$, thus:

$$\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})\right]_l = \epsilon_{lpk}(\epsilon_{kij}\mathrm{a}_i\mathrm{b}_j)_{,p} = \epsilon_{kij}\epsilon_{lpk}(\mathrm{a}_{i,p}\mathrm{b}_j + \mathrm{a}_i\mathrm{b}_{j,p}) \tag{1.490}$$

Regarding that $\epsilon_{kij} = \epsilon_{ijk}$ and $\epsilon_{ijk}\epsilon_{lpk} = \delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl}$, the above equation becomes:

$$\begin{aligned}\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})\right]_l &= \epsilon_{kij}\epsilon_{lpk}(\mathrm{a}_{i,p}\mathrm{b}_j + \mathrm{a}_i\mathrm{b}_{j,p}) = (\delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl})(\mathrm{a}_{i,p}\mathrm{b}_j + \mathrm{a}_i\mathrm{b}_{j,p}) \\ &= \delta_{il}\delta_{jp}\mathrm{a}_{i,p}\mathrm{b}_j - \delta_{ip}\delta_{jl}\mathrm{a}_{i,p}\mathrm{b}_j + \delta_{il}\delta_{jp}\mathrm{a}_i\mathrm{b}_{j,p} - \delta_{ip}\delta_{jl}\mathrm{a}_i\mathrm{b}_{j,p} \\ &= \mathrm{a}_{l,p}\mathrm{b}_p - \mathrm{a}_{p,p}\mathrm{b}_l + \mathrm{a}_l\mathrm{b}_{p,p} - \mathrm{a}_p\mathrm{b}_{l,p}\end{aligned} \tag{1.491}$$

We can also verify that $\left[(\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}\right]_l = \mathrm{a}_{l,p}\mathrm{b}_p$, $\left[(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}}\right]_l = \mathrm{a}_{p,p}\mathrm{b}_l$, $\left[(\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}}\right]_l = \mathrm{a}_l\mathrm{b}_{p,p}$, $\left[(\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}\right]_l = \mathrm{a}_p\mathrm{b}_{l,p}$.

■ $$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) = \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}}) - \nabla_{\vec{x}}^2\vec{\mathbf{a}} \tag{1.492}$$

The components of $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})$ are given by $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})_i = \underbrace{\epsilon_{ijk}\mathrm{a}_{k,j}}_{\mathrm{c}_i}$, thus:

$$\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})\right]_q = \epsilon_{qli}\mathrm{c}_{i,l} = \epsilon_{qli}(\epsilon_{ijk}\mathrm{a}_{k,j})_{,l} = \epsilon_{qli}\epsilon_{ijk}\mathrm{a}_{k,jl} \tag{1.493}$$

Once again considering that $\epsilon_{qli}\epsilon_{ijk} = \epsilon_{qli}\epsilon_{jki} = \delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj}$, the above equation becomes:

$$\begin{aligned}\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})\right]_q &= \epsilon_{qli}\epsilon_{ijk}\mathrm{a}_{k,jl} = (\delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj})\mathrm{a}_{k,jl} = \delta_{qj}\delta_{lk}\mathrm{a}_{k,jl} - \delta_{qk}\delta_{lj}\mathrm{a}_{k,jl} \\ &= \mathrm{a}_{k,kq} - \mathrm{a}_{q,ll}\end{aligned} \tag{1.494}$$

where $\left[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\right]_q = \mathrm{a}_{k,kq}$ and $\left[\nabla_{\vec{x}}^2\vec{\mathbf{a}}\right]_q = \mathrm{a}_{q,ll}$.

■ $$\nabla_{\vec{x}} \cdot (\psi\nabla_{\vec{x}}\phi) = \psi\nabla_{\vec{x}}^2\phi + (\nabla_{\vec{x}}\psi) \cdot (\nabla_{\vec{x}}\phi) \tag{1.495}$$

$$\nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) = (\phi\psi_{,i})_{,i} = \phi\psi_{,ii} + \phi_{,i}\psi_{,i} = \phi\nabla_{\vec{x}}^2\psi + (\nabla_{\vec{x}}\phi) \cdot (\nabla_{\vec{x}}\psi) \tag{1.496}$$

where ϕ and ψ are scalar fields. Other interesting equations derived from the above are:

$$\begin{aligned}\nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) &= \phi\nabla_{\vec{x}}^2\psi + (\nabla_{\vec{x}}\phi) \cdot (\nabla_{\vec{x}}\psi) \\ \nabla_{\vec{x}} \cdot (\psi\nabla_{\vec{x}}\phi) &= \psi\nabla_{\vec{x}}^2\phi + (\nabla_{\vec{x}}\psi) \cdot (\nabla_{\vec{x}}\phi)\end{aligned} \tag{1.497}$$

After subtracting the above two identities we obtain:

$$\begin{aligned}&\nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi) - \nabla_{\vec{x}} \cdot (\psi\nabla_{\vec{x}}\phi) = \phi\nabla_{\vec{x}}^2\psi - \psi\nabla_{\vec{x}}^2\phi \\ &\Rightarrow \nabla_{\vec{x}} \cdot (\phi\nabla_{\vec{x}}\psi - \psi\nabla_{\vec{x}}\phi) = \phi\nabla_{\vec{x}}^2\psi - \psi\nabla_{\vec{x}}^2\phi\end{aligned} \tag{1.498}$$

1.8.5 The Conservative Field

A vector field, $\vec{\mathbf{b}}(\vec{x},t)$, is said to be conservative if there exists a differentiable scalar field, $\phi(\vec{x},t)$, so that:

$$\vec{\mathbf{b}} = \nabla_{\vec{x}}\phi \tag{1.499}$$

If the function ϕ satisfies the relation (1.499), then ϕ is a *potential function* of $\vec{\mathbf{b}}(\vec{x},t)$.

A necessary but insufficient condition for $\vec{\mathbf{b}}(\vec{x},t)$ to be conservative is that $\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\mathbf{0}}$. In other words, given a conservative field, the curl $\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}}$ equals zero. However, if the curl of a vector field equals zero, this does not necessarily mean that the field is conservative.

Problem 1.46: Let ϕ be a scalar field, and $\vec{\mathbf{u}}$ be a vector field. a) Show that $\nabla_{\vec{x}} \cdot \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) = 0$ and $\vec{\nabla}_{\vec{x}} \wedge \left(\nabla_{\vec{x}}\phi\right) = \vec{\mathbf{0}}$.
b) Show that $\vec{\nabla}_{\vec{x}} \wedge \left[\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}\ \right] = (\nabla_{\vec{x}} \cdot \vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + \left[\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})$;
c) Referring $\vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \wedge \vec{v}$, show that $\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^{2}\vec{v}) = \nabla_{\vec{x}}^{2}(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \nabla_{\vec{x}}^{2}\vec{\boldsymbol{\omega}}$.

Solution:
Regarding that: $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i$

$$\nabla_{\vec{x}} \cdot \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) = \frac{\partial}{\partial x_l}\left(\epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i\right) \cdot \hat{\mathbf{e}}_l = \epsilon_{ijk} \frac{\partial}{\partial x_l}\left(v_{k,j}\right)\delta_{il} = \epsilon_{ijk} \frac{\partial}{\partial x_i}\left(v_{k,j}\right) = \epsilon_{ijk} v_{k,ji}$$

The second derivative of $\vec{v}$ is symmetrical with ij, *i.e.* $v_{k,ji} = v_{k,ij}$, while ϵ_{ijk} is antisymmetric with ij, *i.e.*, $\epsilon_{ijk} = -\epsilon_{jik}$, thus:

$$\epsilon_{ijk} v_{k,ji} = \epsilon_{ij1} v_{1,ji} + \epsilon_{ij2} v_{2,ji} + \epsilon_{ij3} v_{3,ji} = 0$$

We can observe that $\epsilon_{ij1} v_{1,ji}$ equals the double scalar product by using a symmetric and an antisymmetric tensor, so $\epsilon_{ij1} v_{1,ji} = 0$.
Likewise, we can show that:

$$\vec{\nabla}_{\vec{x}} \wedge \left(\nabla_{\vec{x}}\phi\right) = \epsilon_{ijk}\phi_{,kj}\ \hat{\mathbf{e}}_i = 0_i \hat{\mathbf{e}}_i = \vec{\mathbf{0}}$$

b) Denoting by $\vec{\boldsymbol{\omega}} = \vec{\nabla}_{\vec{x}} \wedge \vec{v}$ we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge \left[\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}\ \right] = \vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v})$$

Observing the equation in (1.489), it holds that:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) = (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} - (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}})\vec{v} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}}$$

Note that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = 0$. Then, we can draw the conclusion that:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) &= (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}} \\ &= (\nabla_{\vec{x}} \cdot \vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + \left[\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

c) Observing the equation in (1.492) we obtain:

$$\begin{aligned}\nabla_{\vec{x}}^{2}\vec{v} &= \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \\ &= \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}\end{aligned}$$

Applying the curl to the above equation we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^{2}\vec{v}) = \underbrace{\vec{\nabla}_{\vec{x}} \wedge \left[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})\right]}_{=\vec{\mathbf{0}}} - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$$

Referring once again to the equation in (1.492) to express the term $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^{2}\vec{v}) &= -\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}) = -\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}}) + \nabla_{\vec{x}}^{2}\vec{\boldsymbol{\omega}} = -\nabla_{\vec{x}}\underbrace{\left[\nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right]}_{=0} + \nabla_{\vec{x}}^{2}\vec{\boldsymbol{\omega}} \\ &= \nabla_{\vec{x}}^{2}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

1.9 Theorems Involving Integrals

1.9.1 Integration by Parts

Integration by parts states that:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx \tag{1.500}$$

where $v'(x) = \dfrac{dv}{dx}$, and the functions $u(x)$, $v(x)$ are differentiable in $a \le x \le b$.

1.9.2 The Divergence Theorem

Given a domain $\mathcal{B}$ with a volume V, and bounded by the surface S, (see Figure 1.37), the divergence theorem, also called the *Gauss' theorem*, applied to the vector field states that:

$$\begin{aligned} &\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{v}}\ dV = \int_S \vec{\mathbf{v}} \cdot \hat{\mathbf{n}}\, dS = \int_S \vec{\mathbf{v}} \cdot d\vec{\boldsymbol{S}} \\ &\int_V \mathrm{v}_{i,i}\ dV = \int_S \mathrm{v}_i \hat{\mathrm{n}}_i\ dS = \int_S \mathrm{v}_i\ dS_i \end{aligned} \tag{1.501}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to surface S.

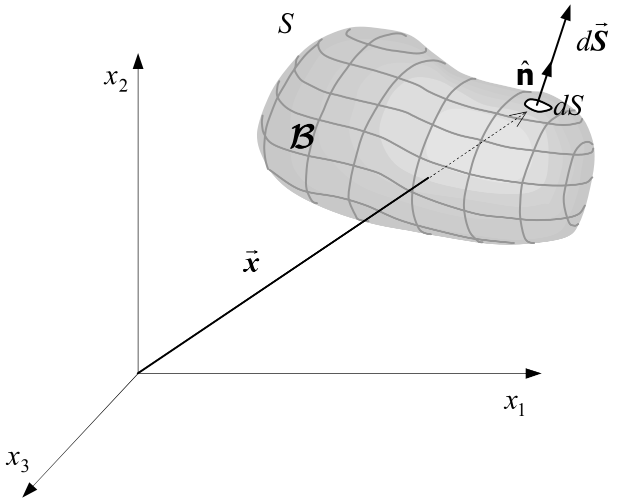


Figure 1.37.

Let $\mathbf{T}$ be a second-order tensor field defined in the domain $\mathcal{B}$. The divergence theorem applied to this field is defined as:

$$\int_V \nabla_{\vec{x}} \cdot \mathbf{T}\ dV = \int_S \mathbf{T} \cdot \hat{\mathbf{n}}\, dS = \int_S \mathbf{T} \cdot d\vec{\boldsymbol{S}} \qquad \Bigg| \qquad \int_V \mathrm{T}_{ij,j}\ dV = \int_S \mathrm{T}_{ij} \hat{\mathrm{n}}_j\ dS = \int_S \mathrm{T}_{ij}\ dS_j \tag{1.502}$$

By using the divergence theorem we can also demonstrate that:

$$
\begin{aligned}
\int_V (x_k)_{,j}\; dV &= \int_V (\delta_{ik} x_i)_{,j}\; dV &&= \int_S \delta_{ik} x_i \hat{\mathrm{n}}_j \; dS \\
&= \int_V \left[\delta_{ik,j} x_i + \delta_{ik} x_{i,j}\right] dV &&= \int_S x_k \hat{\mathrm{n}}_j \; dS \\
&= \int_V x_{k,j}\; dV &&= \int_S x_k \hat{\mathrm{n}}_j \; dS
\end{aligned}
\tag{1.503}
$$

in which we have assumed that $\delta_{ik,j} = 0_{ikj}$. Additionally, by observing that $x_{k,j} = \delta_{kj}$, we can obtain:

$$
\delta_{kj} \int_V dV = \int_S x_k \hat{\mathrm{n}}_j \; dS \quad \Rightarrow \quad
\begin{aligned}
V\delta_{kj} &= \int_S x_k \hat{\mathrm{n}}_j \; dS \\
V\mathbf{1} &= \int_S \vec{x} \otimes \hat{\mathbf{n}} \; dS
\end{aligned}
\tag{1.504}
$$

Given a second-order tensor $\boldsymbol{\sigma}$ defined in the domain $\mathcal{B}$, the following is valid:

$$
\begin{aligned}
\int_V (x_i \sigma_{jk})_{,k}\; dV = \int_V (x_i \sigma_{jk})_{,k}\; dV &= \int_S x_i \sigma_{jk} \hat{\mathrm{n}}_k \; dS \\
= \int_V \left[x_{i,k} \sigma_{jk} + x_i \sigma_{jk,k}\right] dV &= \int_S x_i \sigma_{jk} \hat{\mathrm{n}}_k \; dS \\
= \int_V \left[\delta_{ik} \sigma_{jk} + x_i \sigma_{jk,k}\right] dV &= \int_S x_i \sigma_{jk} \hat{\mathrm{n}}_k \; dS
\end{aligned}
\tag{1.505}
$$

Hence proving that:

$$
\begin{aligned}
\int_V x_i \sigma_{jk,k}\; dV &= \int_S x_i \sigma_{jk} \hat{\mathrm{n}}_k \; dS - \int_V \sigma_{ji}\; dV \\
\int_V \vec{x} \otimes \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}\, dV &= \int_S \vec{x} \otimes (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\; dS - \int_V \boldsymbol{\sigma}^T\; dV
\end{aligned}
\tag{1.506}
$$

or

$$
\int_V \vec{x} \otimes \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}\, dV = \int_S \left(\vec{x} \otimes \boldsymbol{\sigma}\right) \cdot d\vec{S} - \int_V \boldsymbol{\sigma}^T\; dV
\tag{1.507}
$$

Likewise, one can prove that:

$$
\int_V (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{x}\, dV = \int_S (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \otimes \vec{x}\; dS - \int_V \boldsymbol{\sigma}\, dV
\tag{1.508}
$$

Problem 1.47: Let Ω be a domain bounded by Γ as shown in Figure 1.38. Further consider that $\mathbf{m}$ is a second-order tensor field and ω is a scalar field. Show that the following relationship holds:

$$
\int_\Omega \left[\mathbf{m} : \nabla_{\vec{x}} (\nabla_{\vec{x}} \omega)\right] d\Omega = \int_\Gamma \left[(\nabla_{\vec{x}} \omega) \cdot \mathbf{m}\right] \cdot \hat{\mathbf{n}}\, d\Gamma - \int_\Omega \left[(\nabla_{\vec{x}} \cdot \mathbf{m}) \cdot \nabla_{\vec{x}} \omega\right] d\Omega
$$

$$\int_{\Omega} [\mathsf{m}_{ij}\omega_{,ij}] d\Omega = \int_{\Gamma} (\omega_{,i}\, \mathsf{m}_{ij})\hat{\mathsf{n}}_j \; d\Gamma - \int_{\Omega} [\mathsf{m}_{ij,j}\omega_{,i}] d\Omega$$

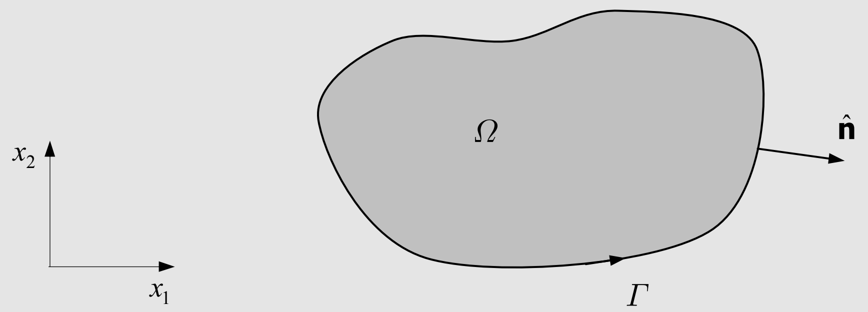


Figure 1.38

Solution: We could directly apply the definition of integration by parts to demonstrate the above relationship. But, here we will start with the definition of the divergence theorem. That is, given a tensor field $\vec{\mathbf{v}}$, it is true that:

$$\int_{\Omega} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} \; d\Omega = \int_{\Gamma} \vec{\mathbf{v}} \cdot \hat{\mathbf{n}} \, d\Gamma \xrightarrow{\text{indicial}} \int_{\Omega} \mathsf{v}_{j,j} \; d\Omega = \int_{\Gamma} \mathsf{v}_j \hat{\mathsf{n}}_j \; d\Gamma$$

Observing that the tensor $\vec{\mathbf{v}}$ is the result of the algebraic operation $\vec{\mathbf{v}} = \nabla_{\vec{x}}\omega \cdot \mathbf{m}$ and the equivalent in indicial notation to $\mathsf{v}_j = \omega_{,i}\, \mathsf{m}_{ij}$, and by substituting it in the above equation we obtain:

$$\begin{aligned}
\int_{\Omega} \mathsf{v}_{j,j} \; d\Omega = \int_{\Gamma} \mathsf{v}_j \hat{\mathsf{n}}_j \; d\Gamma \quad &\Rightarrow \int_{\Omega} [\omega_{,i}\, \mathsf{m}_{ij}]_{,j} \; d\Omega = \int_{\Gamma} \omega_{,i}\, \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma \\
&\Rightarrow \int_{\Omega} [\omega_{,ij}\, \mathsf{m}_{ij} + \omega_{,i}\, \mathsf{m}_{ij,j}] d\Omega = \int_{\Gamma} \omega_{,i}\, \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma \\
&\Rightarrow \int_{\Omega} [\omega_{,ij}\, \mathsf{m}_{ij}] d\Omega = \int_{\Gamma} \omega_{,i}\, \mathsf{m}_{ij} \hat{\mathsf{n}}_j \; d\Gamma - \int_{\Omega} [\omega_{,i}\, \mathsf{m}_{ij,j}] d\Omega
\end{aligned}$$

The above equation in tensorial notation becomes:

$$\int_{\Omega} [\mathbf{m} : \nabla_{\vec{x}}(\nabla_{\vec{x}}\omega)] d\Omega = \int_{\Gamma} [(\nabla_{\vec{x}}\omega) \cdot \mathbf{m}] \cdot \hat{\mathbf{n}} d\Gamma - \int_{\Omega} [\nabla_{\vec{x}}\omega \cdot (\nabla_{\vec{x}} \cdot \mathbf{m})] d\Omega$$

NOTE: Consider now the domain defined by the volume V, which is bounded by the surface S with the outward unit normal to the surface $\hat{\mathbf{n}}$. If $\vec{N}$ is a vector field and T is a scalar field, it is also true that:

$$\int_V N_i T_{,ij} \; dV = \int_S N_i T_{,i}\, \hat{\mathsf{n}}_j dS - \int_V N_{i,j} T_{,i} \; dV$$

$$\Rightarrow \int_V \vec{N} \cdot \nabla_{\vec{x}}(\nabla_{\vec{x}} T) dV = \int_S (\nabla_{\vec{x}} T \cdot \vec{N}) \otimes \hat{\mathbf{n}} dS - \int_V \nabla_{\vec{x}} T \cdot \nabla_{\vec{x}} \vec{N} \; dV$$

where we have directly applied the definition of integration by parts.

1.9.3 Independence of Path

A curve which connects two points A and B is denoted by the path from A to B, (see Figure 1.39). We can then establish the condition by which a line integral is independent of path, (see Figure 1.39).

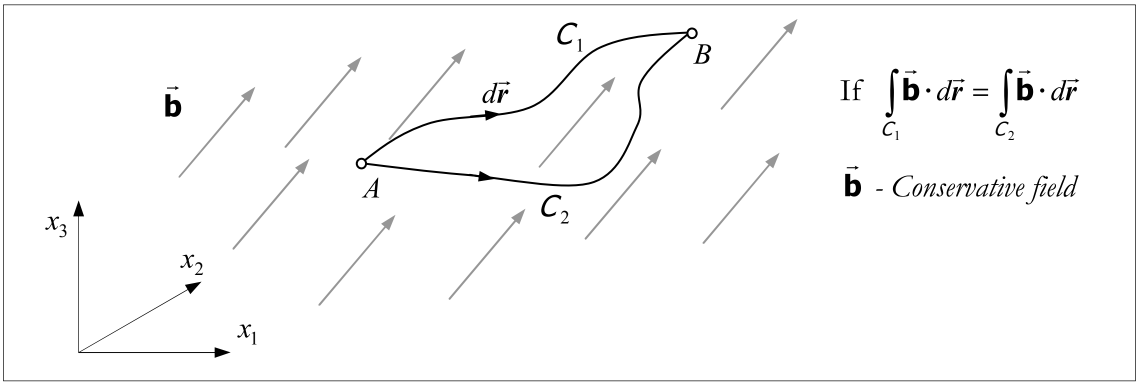


Figure 1.39: Path independence.

Let $\vec{\mathbf{b}}(\vec{x})$ be a continuous vector fields, then the integral $\int_C \vec{\mathbf{b}} \cdot d\vec{r}$ is *independent of the path* if and only if $\vec{\mathbf{b}}$ is a *conservative field*. This means that there is a scalar field ϕ so that $\vec{\mathbf{b}} = \nabla_{\vec{x}}\phi$. Regarding the above, we can draw the conclusion that:

$$\int_A^B \vec{\mathbf{b}} \cdot d\vec{r} = \int_A^B \nabla_{\vec{x}}\phi \cdot d\vec{r}$$
$$\int_A^B (\mathrm{b}_1\hat{\mathbf{e}}_1 + \mathrm{b}_2\hat{\mathbf{e}}_2 + \mathrm{b}_3\hat{\mathbf{e}}_3) \cdot d\vec{r} = \int_A^B \left(\frac{\partial\phi}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{e}}_3 \right) \cdot d\vec{r} \tag{1.509}$$

Thus

$$\mathrm{b}_1 = \frac{\partial\phi}{\partial x_1} \quad ; \quad \mathrm{b}_2 = \frac{\partial\phi}{\partial x_2} \quad ; \quad \mathrm{b}_3 = \frac{\partial\phi}{\partial x_3} \tag{1.510}$$

As the field is conservative, the curl of $\vec{\mathbf{b}}$ is the zero vector:

$$\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \end{vmatrix} = 0_i \tag{1.511}$$

We can therefore conclude that:

$$\begin{cases} \dfrac{\partial \mathrm{b}_3}{\partial x_2} - \dfrac{\partial \mathrm{b}_2}{\partial x_3} = 0 \\ \dfrac{\partial \mathrm{b}_1}{\partial x_3} - \dfrac{\partial \mathrm{b}_3}{\partial x_1} = 0 \\ \dfrac{\partial \mathrm{b}_2}{\partial x_1} - \dfrac{\partial \mathrm{b}_1}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \mathrm{b}_3}{\partial x_2} = \dfrac{\partial \mathrm{b}_2}{\partial x_3} \\ \dfrac{\partial \mathrm{b}_1}{\partial x_3} = \dfrac{\partial \mathrm{b}_3}{\partial x_1} \\ \dfrac{\partial \mathrm{b}_2}{\partial x_1} = \dfrac{\partial \mathrm{b}_1}{\partial x_2} \end{cases} \tag{1.512}$$

Therefore, if the above condition is not satisfied, the field is not conservative.

1.9.4 The Kelvin-Stokes' Theorem

Let S be a regular surface, (see Figure 1.40), and $\vec{\mathbf{F}}(\vec{x},t)$ be a vector field. According to the Kelvin-Stokes' Theorem:

$$\boxed{\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\Gamma}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{\boldsymbol{S}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{n}} dS} \tag{1.513}$$

If $\hat{\mathbf{p}}$ denotes the unit vector tangent to the boundary Γ, the Stokes' theorem becomes:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot \hat{\mathbf{p}} d\Gamma = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{\boldsymbol{S}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{n}} dS \tag{1.514}$$

With reference to the vector representation in the Cartesian basis: $\vec{\mathbf{F}} = \mathsf{F}_1 \hat{\mathbf{e}}_1 + \mathsf{F}_2 \hat{\mathbf{e}}_2 + \mathsf{F}_3 \hat{\mathbf{e}}_3$, $d\vec{\boldsymbol{S}} = dS_1 \hat{\mathbf{e}}_1 + dS_2 \hat{\mathbf{e}}_2 + dS_3 \hat{\mathbf{e}}_3$, $d\vec{\boldsymbol{\Gamma}} = dx_1 \hat{\mathbf{e}}_1 + dx_2 \hat{\mathbf{e}}_2 + dx_3 \hat{\mathbf{e}}_3$, the components of the curl of $\vec{\mathbf{F}}$ are given by:

$$\left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}\right)_i = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathsf{F}_1 & \mathsf{F}_2 & \mathsf{F}_3 \end{vmatrix} = \left(\frac{\partial \mathsf{F}_3}{\partial x_2} - \frac{\partial \mathsf{F}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathsf{F}_1}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 \tag{1.515}$$

Next, the Stokes' theorem expressed in terms of components becomes:

$$\oint_{\Gamma} \mathsf{F}_1 dx_1 + \mathsf{F}_2 dx_2 + \mathsf{F}_3 dx_3 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_3}{\partial x_2} - \frac{\partial \mathsf{F}_2}{\partial x_3} \right) dS_1 + \left(\frac{\partial \mathsf{F}_1}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_1} \right) dS_2 + \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) dS_3 \tag{1.516}$$

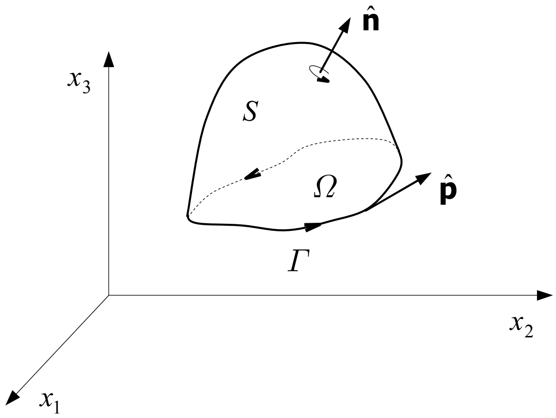


Figure 1.40: Stokes' theorem.

In the special case when the surface S coincides with the plane Ω, (see Figure 1.41), the equation (1.516) remains valid. Then, if the domain Ω coincides with the plane $x_1 - x_2$, the equation (1.513) becomes:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\Gamma}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{e}}_3 dS \tag{1.517}$$

which is known as the Stokes' theorem in the plane or *Green's theorem*, which is expressed in terms of components as:

$$\oint_{\Gamma} \mathsf{F}_1 dx_1 + \mathsf{F}_2 dx_2 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) dS_3 \tag{1.518}$$

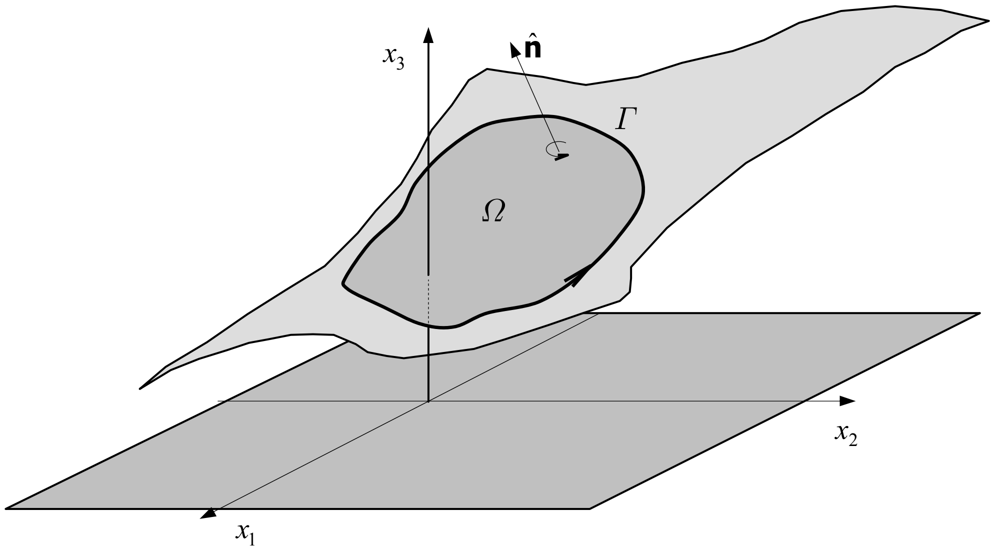


Figure 1.41.

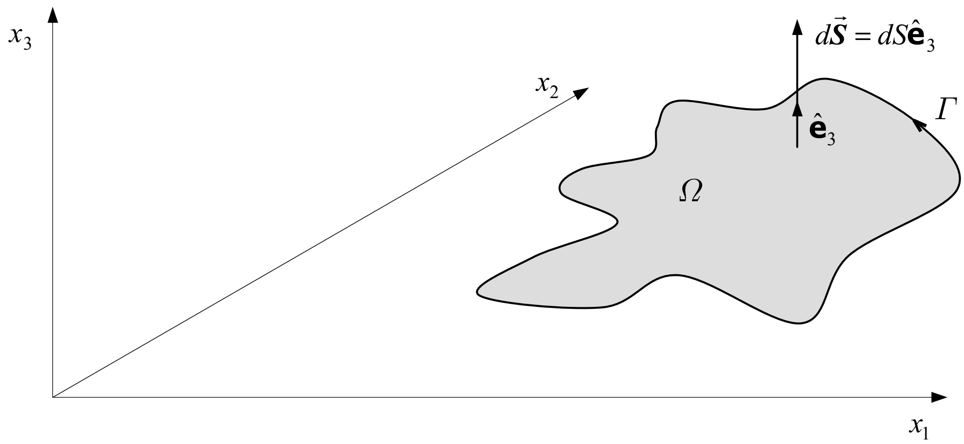


Figure 1.42: Green's theorem.

1.9.5 Green's Identities

Let $\vec{\mathbf{F}}$ be a vector field, and by applying the divergence theorem we obtain:

$$\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{F}} \, dV = \int_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, dS \tag{1.519}$$

With reference to the equations (1.496) and (1.498), *i.e.*:

$$\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi) = \phi \nabla_{\vec{x}}{}^2 \psi + (\nabla_{\vec{x}} \phi) \cdot (\nabla_{\vec{x}} \psi) \tag{1.520}$$

$$\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi - \psi \nabla_{\vec{x}} \phi) = \phi \nabla_{\vec{x}}{}^2 \psi - \psi \nabla_{\vec{x}}{}^2 \phi \tag{1.521}$$

and also regarding that $\vec{\mathbf{F}} = \phi \nabla_{\vec{x}} \psi$, and by substituting (1.520) into (1.519) we obtain:

$$\int_V \phi\nabla_{\vec{x}}^2\psi + (\nabla_{\vec{x}}\phi)\cdot(\nabla_{\vec{x}}\psi)\; dV = \int_S \phi\nabla_{\vec{x}}\psi\cdot\hat{\mathbf{n}}\, dS$$
$$\Rightarrow \int_V (\nabla_{\vec{x}}\phi)\cdot(\nabla_{\vec{x}}\psi)\; dV = \int_S \phi\nabla_{\vec{x}}\psi\cdot\hat{\mathbf{n}}\, dS - \int_V \phi\nabla_{\vec{x}}^2\psi\; dV \tag{1.522}$$

which is known as *Green's first identity*.

Now, if we substituting (1.521) into (1.519) we obtain:

$$\int_V \phi\nabla_{\vec{x}}^2\psi - \psi\nabla_{\vec{x}}^2\phi\; dV = \int_S (\phi\nabla_{\vec{x}}\psi - \psi\nabla_{\vec{x}}\phi)\cdot\hat{\mathbf{n}}\, dS \tag{1.523}$$

which is known as *Green's second identity*.

Problem 1.48: Let $\vec{\mathbf{b}}$ be a vector field, which is defined as $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$. Show that:

$$\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_V \lambda_{,i}\, \mathrm{b}_i\; dV$$

where $\lambda = \lambda(\vec{x})$.

Solution: The Cartesian components of $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$ are $\mathrm{b}_i = \epsilon_{ijk}\mathrm{v}_{k,j}$ and by substituting them in the above surface integral we obtain:

$$\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_S \lambda\epsilon_{ijk}\mathrm{v}_{k,j}\hat{\mathrm{n}}_i\; dS$$

Applying the divergence theorem we obtain:

$$\begin{aligned}\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_S \lambda\epsilon_{ijk}\mathrm{v}_{k,j}\hat{\mathrm{n}}_i\; dS &= \int_V (\epsilon_{ijk}\lambda\mathrm{v}_{k,j})_{,i}\; dV \\ &= \int_V (\epsilon_{ijk}\lambda_{,i}\,\mathrm{v}_{k,j} + \epsilon_{ijk}\lambda\mathrm{v}_{k,ji})\, dV \\ &= \int_V (\lambda_{,i}\underbrace{\epsilon_{ijk}\mathrm{v}_{k,j}}_{\mathrm{b}_i} + \lambda\underbrace{\epsilon_{ijk}\mathrm{v}_{k,ji}}_{0})\, dV = \int_V \lambda_{,i}\,\mathrm{b}_i\; dV\end{aligned}$$