Virtual Network Mapping in Cloud Computing: A Graph Pattern Matching Approach

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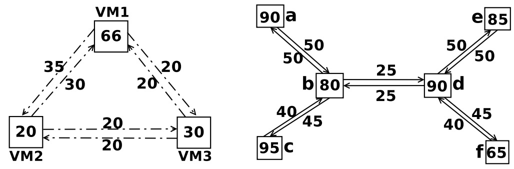
ABSTRACT

Virtual network mapping is to build a network on demand by deploying virtual machines in a substrate network, *e.g.,* data center networks in a cloud platform, subject to constraints on machine capacity and on connection bandwidth or latency. This paper presents a systematic study of the problem. (1) We propose to model virtual network mapping as graph pattern matching: we specify a virtual network request as a graph pattern carrying various constraints, and treat a substrate network as a graph in which nodes and edges bear attributes specifying their capacity. We show that a variety of mapping requirements can be expressed in this model, such as virtual machine placement, network embedding and priority mapping, with node sharing or not. (2) In this model, we formulate the virtual network mapping problem and its optimization problem with respect to a mapping cost function. We establish complexity bounds of these problems for various mapping constraints, ranging from PTIME to NP-complete. For intractable optimization problems, we show that they are approximation-hard, *i.e.,* NPO-complete in general and APX-hard even for special cases. (3) We also develop heuristic algorithms for priority mapping, with node sharing or not. (4) We experimentally verify that our algorithms are efficient and are able to find high-quality mappings, using real-life and synthetic data.

1. INTRODUCTION

Cloud computing has found prevalent use in database applications [6, 7, 9, 38, 39, 41]. These include data center transformations and deployment of database appliances in *Infrastructure as a Service* (IaaS) clouds such as Amazon's Elastic Computing Cloud (EC2) [3] and HP Blade System [1]. In these applications, users are typically allowed to access *virtual machines* (VMs) in a *data center network*, on which they can install and run software [7, 27, 36, 39]. To support these, network virtualization is being commonly employed to allocate physical machine resources to virtual machines [7].

This highlights the need for studying *virtual network mapping*, referred to as VNM and also known as virtual network embedding and assignment. In a nutshell, a *virtual network* (VN) is specified in terms of a set of virtual nodes (machines or routers, denoted as VMs) and their virtual links, along with constraints imposed on the capacities of the nodes (*e.g.,* CPU and storage) and on the links (*e.g.,* bandwidth and latency). VNM is to deploy the VN in a *substrate network* (SN), *e.g.,* data center networks in a cloud platform [36], such that virtual nodes are hosted on substrate nodes, virtual links are instantiated with physical paths in the SN, and moreover, the constraints on the virtual nodes and links are satisfied. The need for VNM is evident in, *e.g.,* data center transformations, cloud infrastructure provision and computing resource deployment on demand.



(a) VN Request (b) SN

Figure 1: VN requests found in practice

Several models have been proposed to specify VNM in various settings (notations are summarized in Table 1):

(1) *Virtual machine placement* (VMP): it is to find a mapping f from virtual machines in a VN to substrate nodes in an SN such that for each VM v, its capacity is no greater than that of $f(v)$, *i.e.,* $f(v)$ is able to conduct the computation of the VM v that it hosts [12, 29].

(2) *Single-path* VN *embedding* ($\mathsf{VNE_{SP}}$): it is to find

- (a) an injective mapping f_v that maps nodes in VN to nodes in SN, subject to node capacity constraints;
- (b) a function that maps a virtual link (v, v') in VN to a path from $f_v(v)$ to $f_v(v')$ in SN that satisfies a bandwidth constraint, *i.e.,* the bandwidth of each link in the SN is no smaller than the sum of the bandwidth requirements of all those virtual links that are mapped to a path containing it [28, 30, 31].

(3) *Multi-path* VN *embedding* ($\mathsf{VNE_{MP}}$): it is to find a node mapping f_v as in $\mathsf{VNE_{SP}}$ and a function that maps each virtual link (v, v') to *a set* of paths from $f_v(v)$ to $f_v(v')$ in SN, subject to bandwidth constraints [17, 40].

However, there are a number of VN requests that are commonly found in practice, but cannot be expressed in any of these models, as illustrated by the following.

Example 1: Consider a VN request and an SN, depicted in Figures 1(a) and 1(b), respectively. The VN has three virtual nodes VM_1, VM_2 and VM_3, each specifying a capacity constraint, along with a constraint on each virtual link. In the SN, each substrate node bears a resource capacity and each connection (edge) has an attribute, indicating either bandwidth or latency. Consider the following cases.

(1) Mapping with latency constraints ($\mathsf{VNM_L}$). Assume that the numbers attached to the virtual nodes and links in Fig. 1(a) denote requirements on CPUs and latencies for SN, respectively. The VNM problem here aims to map each virtual node to a substrate node with sufficient computational power, and to map each virtual link (v, v') in the VN to a path in the SN such that the latency of the path, *i.e., the sum* of the latencies of the edges on the path, does not exceed the latency specified for (v, v'). The need for studying VNM_L arises from latency sensitive applications such as multimedia transmitting networks [34], where constraints on virtual links concern latency rather than bandwidth.

(2) Priority mapping ($\mathsf{VNM_P}$). Assume that the constraints on the nodes in Fig. 1(a) are CPU capacities, and constraints imposed on edges are bandwidth capacities. The VNM problem is to map each virtual node to a node in SN with sufficient CPU capacity, and each virtual link (v, v') in the VN to a path in SN such that the *minimum* bandwidth of all edges on the path is no less than the bandwidth specified for (v, v'). The need for this is evident in emerging applications such as Internet-based virtualized infrastructure computing platform (iVIC [4]), which gives different priorities at run time to virtual links that share some physical links, and requires the mapping only to provide bandwidth guarantee for the connection with the highest priority.

(3) Mapping with node sharing ($\mathsf{VNE_{SP(NS)}}$). Assume that the numbers attached to the virtual nodes and links in Fig. 1(a) denote requirements on CPUs and bandwidths for SN, respectively. The VNM problem here is an extension of the single-path VN embedding ($\mathsf{VNE_{SP}}$) by supporting node sharing, *i.e.,* by allowing multiple virtual nodes to be mapped to the same substrate node. This is needed by, *e.g.,* X-Bone [5], among others.

Similarly, there is also practical need to extend other mappings with node sharing, such as virtual machine placement (VMP), latency mapping ($\mathsf{VNM_L}$), priority mapping $\mathsf{VNM_P}$ and multi-path VN embedding ($\mathsf{VNE_{MP}}$). We denote such an extension by adding a subscript NS (see Table 1).

Observe the following. (a) The VNM problem varies depending on different practical requirements, *e.g.,* when latency, high-priority connections and node sharing are concerned. (b) Existing models are not capable of expressing such requirements; indeed, none of them is able to specify $\mathsf{VNM_L}$, $\mathsf{VNM_P}$ or $\mathsf{VNE_{SP(NS)}}$. (c) It would be an overkill to develop a model for each of the large variety of requirements, and to study it individually. □

Table 1: Notations and various VNM cases

Notation	Description
VNM	virtual network mapping
VN	virtual network
SN	substrate network
VMs	virtual nodes (machines or routers)
VMP ($\mathsf{VMP_{(NS)}}$)	VM Placement (node sharing (NS))
$\mathsf{VNM_P}$ ($\mathsf{VNM_{P(NS)}}$)	priority mapping (with NS)
$\mathsf{VNE_{SP}}$ ($\mathsf{VNE_{SP(NS)}}$)	single-path embedding (with NS)
$\mathsf{VNE_{MP}}$ ($\mathsf{VNE_{MP(NS)}}$)	multi-path embedding (with NS)
$\mathsf{VNM_L}$ ($\mathsf{VNM_{L(NS)}}$)	latency constrained mapping (NS)

As suggested by the example, we need a generic model to express virtual network mappings in various practical settings, including both those already studied (*e.g.,*VMP, $\mathsf{VNE_{SP}}$ and $\mathsf{VNE_{MP}}$) and those that have been overlooked (*e.g.,* $\mathsf{VNM_L}$, $\mathsf{VNM_P}$ and $\mathsf{VNE_{SP(NS)}}$). The uniform model allows us to characterize and compare VNMs in different settings, and better still, to study generic properties that pertain to all the variants. Among these are the complexity and approximation analyses of VNMs, which are obviously important but have not yet been studied by and large.

Contributions & Roadmap. This work takes a step toward providing a uniform model to characterize VNMs. It is also among the first effort to establish the complexity and approximation bounds for VNMs. For intractable and VNM cases, we develop effective heuristic methods to find high-quality mappings.

(1) We propose a generic model to express VNMs in terms of graph pattern matching [24] (Section 2). In this model a VN request is specified as a graph pattern, bearing various constraints on nodes and links defined with aggregation functions, and an SN is simply treated as a graph with attributes associated with its nodes and edges. The decision and optimization problems for VNMs are then simply graph pattern matching problems. We show that the model is able to express VNMs commonly found in practice, including all the mappings we have seen so far (all the cases in Table 1).

(2) We establish complexity and approximation bounds for VNMs (Section 3). We give a uniform upper bound for the VNM problems expressed in this model, by showing that all these problems are in NP. We also show that VNM is polynomial time (PTIME) solvable if only node constraints are present (VMP), but it becomes NP-complete when either node sharing is allowed or constraints on edges are imposed (all the other cases in Table 1). Moreover, we propose a VNM cost function and study optimization problems for VNM based on the metric. We show that the optimization problems are NP-complete in most cases and worse still, are NPO-complete in general and APX-hard [10] for special cases. To the best of our knowledge, these are among the first complexity and approximation results on VNMs.

(3) These results tell us that it is beyond reach in practice to find PTIME algorithms for VNMs with edge constraints such as $\mathsf{VNM_P}$ and $\mathsf{VNE_{SP}}$, or to find efficient approximation algorithms with decent performance guarantees. In light of these, we develop heuristic algorithms for priority mapping, with node sharing or not (Section 4), which reduce unnecessary computations by minimizing VNs requests and utilizing auxiliary graphs of SNs. While a number of algorithms are available for VN embedding ($\mathsf{VNE_{SP}}$, *e.g.,* [28, 30, 31]), no previous work has studied algorithms for $\mathsf{VNM_P}$.
(4) Finally, we experimentally verify the effectiveness and efficiency of our algorithm by providing a simulation study (Section 5). We evaluate our algorithm for priority mapping and VN embedding (with or without node sharing). We find that our algorithm is able to find high-quality mappings and is efficient on large VNs and SNs. In particular, it outperforms previous algorithms for VN embedding in the cost of mappings, with better or comparable efficiency.

All the proofs of the results can be found in [2].

We contend that these results are useful for developing IaaS clouds and data center transformations, among other things. By modeling VNM as graph pattern matching, we are able to characterize various VN requests with different classes of graph patterns, and study the expressive power and complexity of these graph pattern languages. Furthermore, techniques developed for graph pattern matching can be leveraged to study VNMs. Indeed, the proofs of some of the results in this work capitalize on graph pattern techniques. On the other hand, the results of this work are also of interest to the study of graph pattern matching [24].

Related Work. In light of the rapid development of cloud computing and data centers [27], virtualization techniques have recently been investigated for a number of database applications, such as database appliance deployment and intelligent management of virtualized resources for database systems [6,7,36,38,39]. However, none of these has provided a systematic study of VNM, by modeling VNM as graph pattern matching. While subgraph isomorphism was adopted for VNM [30], it is only a special case of the generic model proposed in this work. Furthermore, complexity and approximation analyses associated with VNM have not been studied for cloud computing in database applications.

Several models have been developed for VNM. (a) The VM placement problem (VMP) was studied in [12, 29], which is similar to the bin packing problem and aims to map a set of VMs onto an SN with constraints on node capacities. (b) Single-path VN embedding ($\mathsf{VNE_{SP}}$) was investigated in [31,35,42], which is to map a VN to an SN by a node-to-node injection and an edge-to-path function, subject to constraints on the CPU capacities of nodes and constraints on the bandwidths of physical connections. (c) Different from $\mathsf{VNE_{SP}}$, multi-path embedding ($\mathsf{VNE_{MP}}$) was studied in [17,40], which allows an edge of a VN to be mapped to multiple parallel paths of an SN such that the sum of the bandwidth capacities of those paths is no smaller than the bandwidth of that edge. (d) Graph layout problems, while similar to VN mapping, do not have bandwidth constraints on edges but instead, impose certain topological constraints (see [20] for a survey). In contrast to our work, these models are studied for specific domains. No previous work has studied generic models to support various VN requests that commonly arise in practice. Moreover, no previous work has considered newly emerging settings such as priority mapping, mappings with only latency constraints on links, and mappings with node sharing, which are tackled in this paper.

Very few complexity results are known for VNM. The only work we are aware of is [8], which claimed that the testbed mapping problem is NP-hard in the presence of node types and some links with infinite capacity. Several complexity and approximation results are established for graph pattern matching (see [24] for a survey). However, those results are for edge-to-edge mappings, whereas VNM typically needs to map virtual links to physical paths. There have been recent extensions to support edge-to-path mappings for graph pattern matching [21–23, 43], with several intractability and approximation bounds established there. Those differ from this work in that either no constraints on links are considered [23], or graph simulation is adopted [21, 22, 43], which does not work for VNM. The complexity and approximation bounds developed in this work are among the first results that have been developed for VNM in cloud computing.

A number of algorithms have been developed for VNM. There are greedy algorithms for the VM placement problem [12, 29]. When considering bandwidth constraints on links, [42] provided a heuristic algorithm to find mappings with load balance with infitite SN resources. A special case of mapping to SNs of a backbone-star shape was studied in [31], allowing constraints on both nodes and links. A path-splitting assumption was proposed in [40], to rectify limitations of mapping an edge to a single path. Based on the assumption, [17] developed an MIP model and algorithms for finding such mappings. Unfortunately none of these works for priority mappings studied in this paper.

2. GRAPH PATTERN MATCHING MODEL

Below we first represent virtual networks (VNs) and substrate networks (SNs) as weighted directed graphs. We then introduce a generic model to express virtual network mapping (VNM) in terms of graph pattern matching [24].

2.1 Substrate and Virtual Networks

An SN consists of a set of substrate nodes connected with physical links, in which the nodes and links are associated with resources of a certain capacity, *e.g.,* CPU and storage capacity for nodes, and bandwidth and latency for links. A VN is specified in terms of a set of virtual nodes and a set of virtual links, along with requirements on the capacities of the nodes and the capacities of the links. Both VNs and SNs can be naturally modeled as weighted directed graphs.

Weighted directed graphs. A *weighted directed graph* is defined as $G = (V, E, f_V, f_E)$, where (1) V is a finite set of nodes; (2) $E \subseteq V \times V$ is a set of edges, in which (v, v') denotes an edge from v to v'; (3) f_V is a function defined on V such that for each node $v \in V$, $f_V(v)$ is a positive rational number; and similarly, (4) f_E is a function defined on E.

Substrate networks. A *substrate network* (SN) is a weighted directed graph $G_S = (V_S,\ E_S,\ f_{V_S},\ f_{E_S})$. where (1) V_S and E_S denote the set of substrate nodes and the set of physical links (directly connected), respectively; and (2) the functions f_{V_S} and f_{E_S} denote resource capacities on the nodes (*e.g.,*CPU) and links (*e.g.,* bandwidth and latency), respectively.

Virtual networks. A *virtual network* (VN) is specified as a weighted directed graph $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$, where (1) V_P and E_P denote virtual nodes and links, and (2) f_{V_P} and f_{E_P} are functions defined on V_P and E_P in the same way as in substrate networks, respectively.

Example 2: The SN depicted in Fig. 1(b) is a weighted graph G_S, where (1) the node set is $\{\mathsf{a}, \mathsf{b}, ..., \mathsf{f}\}$; (2) the edges include the directed edges in the graph; (3) the weights associated with nodes indicate CPU capacities; and (4) the weights of edges denote bandwidth or latency capacities. Figure 1(a) shows a VN, where (1) the node set is $\{\mathsf{VM}_1,\ \mathsf{VM}_2,\ \mathsf{VM}_3\}$; (2) the edge set is $\{(\mathsf{VM_i}, \mathsf{VM_j}) \mid \mathsf{i}, \mathsf{j} = 1, 2, 3\}$; (3) $f_{V_P}(\mathsf{VM}_1) = 66$, $f_{V_P}(\mathsf{VM}_2) = 20$, $f_{V_P}(\mathsf{VM}_3) = 30$; and (4) the function f_{E_P} is given by the edge labels. As will be seen when we define the notion of VN requests, the labels indicate requirements on deploying the VN in an SN. □

Paths. A *path* ρ from node u_0 to u_n in an SN G_S is denoted as $(u_0, u_1, \ldots, u_n)$, where (a) $u_i \in V_S$ for each $i \in [0, n]$, (b) there exists an edge $e_i = (u_{i-1}, u_i)$ in E_S for each $i \in [1, n]$, and moreover, (c) for all $i,\ j \in [0, n]$, if $i \neq j$, then $u_i \neq u_j$. We write $e \in \rho$ if e is an edge on ρ, *i.e.,* e is e_i for some $i \in [1, n]$. When it is clear from the context, we also use ρ to denote the set of edges on the path, *i.e.,* $\{e_i \mid i \in [1, n]\}$.

2.2 Virtual Network Mapping

Virtual network mapping (VNM) from a VN G_P to an SN G_S is specified in terms of a node mapping, an edge mapping and a VN request. The VN request imposes constraints on the node mapping and edge mapping, defining their semantics. We next define these notions.

A *node mapping* from G_P to G_S is a pair (g_V, r_V) of functions, where g_V maps the set V_P of virtual nodes in G_P to the set V_S of substrate nodes in G_S, and for each v in V_P, if $g_V(v) = u$, $r_V(v, u)$ is a positive number. Intuitively, function r_V specifies the amount of resource of the substrate node u that is allocated to the v.

For each edge (v, v') in G_P, we use $P(v, v')$ to denote the set of paths from $g_V(v)$ to $g_V(v')$ in G_S. An *edge mapping* from G_P to G_S is a pair $(g_E,\ r_E)$ of functions such that for each edge $(v, v') \in E_P$, $g_E(v, v')$ is a subset of $P(v, v')$, and r_E attaches a positive number to each pair (e, ρ) if $e \in E_P$ and $\rho \in g_E(e)$. Intuitively, $r_E(e, \rho)$ is the amount of resource of the physical path ρ allocated to the virtual link e.

VN requests. A VN *request* to an SN G_S is a pair $(G_P, \mathcal{C})$, where G_P is a VN, and $\mathcal{C}$ is a set of constraints such that for a pair $((g_V, r_V), (g_E, r_E))$ of node and edge mappings from G_P to G_S, each constraint in $\mathcal{C}$ has one of the forms below:

(1) for each $v \in V_P$, $f_{V_P}(v) \ \leq\ r_V(v, g_V(v))$;

(2) for each $u \in V_S$, $f_{V_S}(u) \ \geq\ \mathsf{sum}(N(u))$, where $N(u)$ is $\{\!|r_V(v, u) \mid v \in V_P,\ g_V(v) = u|\!\}$, a bag (an unordered collection of elements with repetitions) determined by virtual nodes in G_P hosted by u;

(3) for each $e \in E_P$, $f_{E_P}(e)$ op $\mathsf{agg}(Q(e))$, where $Q(e)$ is $\{\!|r_E(e, \rho) \mid \rho \in g_E(e)|\!\}$, a bag collecting physical paths ρ that instantiate e; here op is either the comparison operator $\leq$ or $\geq$, and $\mathsf{agg}()$ is one of the aggregation functions min, max and sum;

(4) for each $e' \in E_S$, $f_{E_S}(e') \geq \mathsf{sum}(M(e'))$, where $M(e')$ is $\{\!|r_E(e, \rho) \mid e \in E_P,\ \rho \in g_E(e),\ e' \in \rho|\!\}$, a bag collecting those virtual links that are instantiated by a physical link ρ containing e';

(5) for each $e \in E_P$ and $\rho \in g_E(e)$, $r_E(e, \rho)$ op $\mathsf{agg}(U(\rho))$ where $U(\rho)$ is $\{\!|f_{E_S}(e') \mid e' \in \rho|\!\}$), a bag of all edges on a physical path that instantiate e.

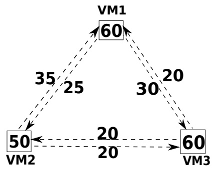
Constraints in a VN request are classified as follows.

<u>*Node constraints*</u>: Constraints of form (1) or (2). Intuitively, a constraint of form (1) assures that when a virtual node v is hosted by a substrate node u, u must provide adequate resource. A constraint of form (2) asserts that when a substrate node u hosts (possibly multiple) virtual nodes, u must have sufficient capacity to accommodate all those virtual nodes. When u hosts at most one virtual node, *i.e.,* if node sharing is not allowed, then $|N(u)| \leq 1$, where we use $|N(u)|$ to denote the number of virtual nodes hosted by u.

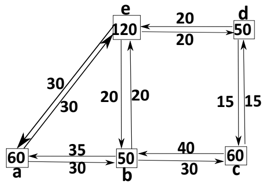
<u>*Edge constraints*</u>: Those constraints of of form (3), (4) or (5). A constraint of form (3) assures that when a virtual link e is mapped to a set of physical paths in the SN, those physical paths taken together satisfy the requirements (on bandwidths or latencies) of e. We denote by $|Q(e)|$ the number of physical paths to which

Table 2: Various VN requests

Constraints	C1	C2	C3	C4	C5
VMP ($\mathsf{VMP_{(NS)}}$)	✓	✓ $\|N(u)\| \leq 1$ (resp. $\|N(u)\| \geq 0$)	×	×	×
$\mathsf{VNM_P}$ ($\mathsf{VNM_{P(NS)}}$)	✓	✓ $\|N(u)\| \leq 1$ (resp. $\|N(u)\| \geq 0$)	op:'$\leq$'; agg:'max', $\|Q(e)\|=1$	✓	op:'$\leq$'; agg:'min'
$\mathsf{VNE_{SP}}$ ($\mathsf{VNE_{SP(NS)}}$)	✓	✓ $\|N(u)\| \leq 1$ (resp. $\|N(u)\| \geq 0$)	op:'$\leq$'; agg:'sum', $\|Q(e)\|=1$	✓	op:'$\leq$'; agg:'min'
$\mathsf{VNE_{MP}}$ ($\mathsf{VNE_{MP(NS)}}$)	✓	✓ $\|N(u)\| \leq 1$ (resp. $\|N(u)\| \geq 0$)	op:'$\leq$'; agg:'sum', $\|Q(e)\| \geq 1$	✓	op:'$\leq$'; agg:'min'
$\mathsf{VNM_L}$ ($\mathsf{VNM_{L(NS)}}$)	✓	✓ $\|N(u)\| \leq 1$ (resp. $\|N(u)\| \geq 0$)	op:'$\geq$'; agg:'sum', $\|Q(e)\|=1$	×	op:'$\geq$'; agg:'sum'



(a) VN Request



(b) SN

Figure 2: VN request and SN for case study

e is mapped. A constraint of form (4) asserts that for each physical link e', it must have sufficient bandwidth to accommodate those of all the virtual links that are mapped to some physical path containing e'. A constraint of form (5) assures that when a virtual link e is mapped to a set of paths, for each ρ in the set, the resource of ρ allocated to e may not exceed the capacities of the physical links on ρ.

VNM. We say that a VN request $(G_P, \mathcal{C})$ can be *mapped to* an SN G_S, denoted by $G_P \rhd_{\mathcal{C}} G_S$, if there exist a pair $((g_V, r_V), (g_E, r_E))$ of node and edge mappings from G_P to G_S such that all the constraints of $\mathcal{C}$ are satisfied, *i.e.,* the functions g_V and g_E satisfy all the inequalities in $\mathcal{C}$.

The VNM *problem* is to determine, given a VN request $(G_P, \mathcal{C})$ and an SN G_S, whether $G_P \rhd_{\mathcal{C}} G_S$.

Case study. All those VNM requirements in Section 1 can be expressed in this model, by treating VN request as a graph pattern and SN as a weighted graph. These are summarized in Table 1 where ✓and × indicate whether the corresponding constraints are needed or not, respectively. Below we illustrate a few cases.

Case 1: Virtual machine placement. VMP can be expressed as a VN request in which only node constraints are present. It is to find an injective mapping (g_V, r_V) from virtual nodes to substrate nodes (hence $|N| \leq 1$) that satisfies the node constraints, while imposing no constraints on edge mapping.

Case 2: Priority mapping. VNM_P can be captured as a VN request specified as $(G_P, \mathcal{C})$, where $\mathcal{C}$ consists of (a) node constraints of forms (1) and (2), and (b) edge constraints of form (3) when op is $\leq$ and agg is max, and form (5) when op is $\leq$ and agg is min. It is to find an injective node mapping (g_V, r_V) and an edge mapping (g_E, r_E) such that for each virtual link e, $g_E(e)$ is a single path (hence $|Q(e)| = 1$). Moreover, it requires the capacity of each virtual node v not to exceed the capacity of the substrate node that hosts v. When a virtual link e is mapped to a physical path ρ, the *bandwidth* of each edge on ρ is no less than that of e, *i.e.,* ρ suffices to serve the connection with the highest priority.

Example 3: Consider the VN given in Fig. 1(a) and the SN of Fig. 1(b). Constraints for priority mapping can be defined as described above, using the node and edge labels (on bandwidths) in Fig. 1(a). There exists a priority mapping from the VN to the SN. Indeed, one can map $\mathsf{VM_1}, \mathsf{VM_2}$ and $\mathsf{VM_3}$ to b, a and d, respectively, and map the virtual links to the shortest physical paths uniquely determined by the node mapping, *e.g.,* $(\mathsf{VM_1}, \mathsf{VM_2})$ is mapped to (b, a). □

Case 3: Single-path VN embedding. A $\mathsf{VNE_{SP}}$ request can be specified as $(G_P, \mathcal{C})$, where $\mathcal{C}$ consists of (a) node constraints of forms (1) and (2), and (b) edge constraints of form (3) when op is $\leq$ and agg is sum, and edge constraints of forms (4) and (5) when op is $\leq$ and agg is min. It differs from $\mathsf{VNM_P}$ in that for each physical link e', it requires the *bandwidth* of e' to be no less than the sum of bandwidths of all those virtual links that are instantiated via e'.

Similarly, multi-path VN embedding ($\mathsf{VNE_{MP}}$) can be expressed as a VN request. It is the same as $\mathsf{VNE_{SP}}$ except that a virtual link e can be mapped to *a set* $g_E(e)$ of physical paths. When taken together, the paths in $g_E(e)$ provide sufficient bandwidth required by e.

When node sharing is allowed in $\mathsf{VNE_{SP}}$, *i.e.,* for single-path embedding with node sharing ($\mathsf{VNE_{SP(NS)}}$), a VN request is specified similarly. Here a substrate node u can host multiple virtual nodes (hence $|N(u)| \geq 0$) such that the sum of the capacities of all the virtual nodes does not exceed the capacity of u. Along the same lines, one can also specify multi-path VN embedding with node sharing ($\mathsf{VNE_{MP(NS)}}$).

Example 4: Consider the VN of Fig. 2(a), and the SN of Fig. 2(b). There is a $\mathsf{VNE_{SP}}$ from the VN to the SN, by mapping $\mathsf{VM_1}, \mathsf{VM_2}, \mathsf{VM_3}$ to a, b, e, respectively, and mapping the VN edges to the shortest paths in the SN determined by the node mapping, *e.g.,* from $(\mathsf{VM_1}, \mathsf{VM_2})$ to (a, b). There is also a multi-path embedding $\mathsf{VNE_{MP}}$ from the VN to the SN, by mapping $\mathsf{VM_1}, \mathsf{VM_2}$ and $\mathsf{VM_3}$ to a, c and e, respectively. For the virtual links, $(\mathsf{VM_1}, \mathsf{VM_2})$ can be mapped to the physical path (a, b, c), $(\mathsf{VM_1}, \mathsf{VM_3})$ to (a, e), and $(\mathsf{VM_1}, \mathsf{VM_3})$ to two paths $\rho_1 = (\mathsf{e}, \mathsf{b}, \mathsf{c})$ and $\rho_2 = (\mathsf{e}, \mathsf{d}, \mathsf{c})$ with $r_E((\mathsf{VM_1}, \mathsf{VM_3}), \rho_1) = 5$ and $r_E((\mathsf{VM_1}, \mathsf{VM_3}), \rho_2) = 15$; similarly for other virtual links.

Table 3: VN **Request for** $\mathsf{VNM_L}$

Requirements placed on VN links
$(\mathsf{VM}_1, \mathsf{VM}_2) = 50,\ (\mathsf{VM}_2, \mathsf{VM}_1) = 55$
$(\mathsf{VM}_1, \mathsf{VM}_3) = 120,\ (\mathsf{VM}_3, \mathsf{VM}_1) = 120$
$(\mathsf{VM}_2, \mathsf{VM}_3) = 60,\ (\mathsf{VM}_3, \mathsf{VM}_2) = 60$

One can verify that the VN of Fig. 2(a) allows no more than one virtual node to be mapped to the same substrate node in Fig. 2(b). However, if we change the bandwidths of the edges connecting a and e in SN from 30 to $f_{V_S}(\mathsf{a}, \mathsf{e}) = 40$ and $f_{V_S}(\mathsf{e}, \mathsf{a}) = 50$, then there exists a mapping from the VN to the SN that supports node sharing. Indeed, in this setting, one can map both $\mathsf{VM}_1, \mathsf{VM}_2$ to e and map VM_3 to a; and map the virtual edges to the shortest physical paths determined by the node mapping; for instance, both $(\mathsf{VM}_1, \mathsf{VM}_3)$ and $(\mathsf{VM}_2, \mathsf{VM}_3)$ can be mapped to (e, a). □

Case 4: Latency constrained mapping. A $\mathsf{VNM_L}$ request is expressed as $(G_P, \mathcal{C})$, where $\mathcal{C}$ consists of (a) node constraints of forms (1) and (2), and (b) edge constraints of form (3) when op is $\geq$ and agg is min, and of form (5) when op is $\geq$ and agg is sum. It is similar to $\mathsf{VNE_{SP}}$ except that when a virtual link e is mapped to a physical path ρ, it requires ρ to satisfy the *latency* requirement of e, *i.e.,* the sum of the latencies of the edges on ρ does not exceed that of e.

Example 5: One can verify that there is no latency mapping of the VN shown in Fig. 1(a) to the SN in Fig. 1(b). However, if we change the constraints on the virtual links of the VN request with the setting shown in Table 3, then there exists a mapping from the VN to the SN. Indeed, one can map $\mathsf{VM}_1, \mathsf{VM}_2$, VM_3 to c, b, a, respectively, and map the edges to the shortest physical paths determined by the node mapping, *e.g.,* from $(\mathsf{VM}_1, \mathsf{VM}_3)$ to $(\mathsf{c}, \mathsf{b}, \mathsf{a})$. □

3. COMPLEXITY AND APPROXIMATION

We next study fundamental issues associated with virtual network mapping. We first establish the complexity bounds of the VNM problem in various settings, from PTIME to NP-complete. We then introduce a cost metric for virtual network mapping, formulate optimization problems based on the function, and finally, prove the complexity bounds and approximation hardness of the optimization problems.

3.1 The Complexity of VNM

We provide an upper bound for the VNM problem in the general setting, by showing it is in NP. We also show that the problem is in PTIME when only node constraints are present. However, when node sharing or edge constraints are imposed, it becomes NP-hard, even when both virtual and substrate networks are directed acyclic graphs (DAGs). These tell us that node sharing and edge constraints make our lives harder.

Theorem 1: *The virtual network mapping problem is (1) in* NP *regardless of what constraints are present; (2) in* PTIME *when only node constraints are present, without node sharing,* i.e., VMP *is in* PTIME*; however, (3) it becomes* NP*-complete when node sharing is requested,* i.e., $\mathsf{VMP_{(NS)}}$, $\mathsf{VNM_{P(NS)}}$, $\mathsf{VNM_{L(NS)}}$, $\mathsf{VNE_{SP(NS)}}$ *and* $\mathsf{VNE_{MP(NS)}}$ *are all* NP*-complete; and (4) it is* NP*-complete in the presence of edge constraints;* i.e., $\mathsf{VNM_P}$, $\mathsf{VNM_L}$, $\mathsf{VNE_{SP}}$ *and* $\mathsf{VNE_{MP}}$ *are intractable.*

All the NP*-hardness results remain intact even both* VNs *and* SNs *are* DAGs*.* □

Proof sketch: We sketch the proofs blow.

(1) To show the upper bound, we first provide a generic NP algorithm for cases without multi-path edge mapping, *i.e.,* all except $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$. We then propose a specific NP algorithm for $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$. Given a VN request and an SN, the generic NP algorithm first guesses a node mapping function g_V and an edge function g_E such that $g_E(v, v')$ contains only one path from $P(v, v')$, for each $(v, v') \in E_P$. It then checks whether there exist r_V and r_E such that (g_V, r_V) and (g_E, r_E) make node and edge mappings that satisfy the constraints in the VN request. The existence of r_V can be checked in $O(|V_P|)$-time for node constraints. For r_E, with g_V and g_E, the checking can be formulated as a linear programming problem with $r_E(e, \rho)$ as its variables, which can be solved in time polynomial in $|V_P|$ [32]. Thus the checking can be done in PTIME. Note that the guessed certificate $(g_V,\, g_E)$ is of polynomial size and thus the algorithm is in NP. For $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$, we first guess a node mapping function g_V solely as the certificate, and then checks whether there exist r_V, g_E and r_E such that (g_V, r_V) and (g_E, r_E) satisfies the constraints carried by the VN request. The existence of r_V is checked exactly the same as the generic NP algorithm above. For the existence of edge mapping (g_E, r_E), we reduce the checking problem to the fractional multi-commodity flow problem such that there exists a fractional multi-commodity flow iff there exists an edge mapping in accord with the guessed node mapping g_V. The latter problem can be solved by a polynomial-time linear-programming algorithm [18], in which the variables are $r_E(e_P, e_S)$ (for all edges e_p in VN and e_S in SN). Since the guessed certificate is of polynomial size and the checking can be done in PTIME, this is an NP algorithm for $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$.

(2) It suffices to show that virtual machine placement (VMP) is in PTIME without node sharing, since VMP allows node constraints of both form 1 and form 2. We develop a PTIME algorithm to check whether there exists a VMP from a VN request $(G_P, \mathcal{C})$ to an SN G_S. The algorithm first builds a bipartite graph G_B based on G_P and G_S. We show that there exists a VMP satisfying constraints in $\mathcal{C}$ iff there is a maximum bipartite matching of G_B. The algorithm finds such a matching

if it exists, in $O((|G_P| + |G_S|)^3)$ time.

(3) In contrast, we show that virtual machine placement with node sharing ($\mathsf{VMP}_{(\mathsf{NS})}$) is NP-hard, by reduction from the Bin Packing problem, which is NP-complete (cf. [25]). The proof constructs a VN and an SN as DAGs. This suffices since $\mathsf{VMP}_{(\mathsf{NS})}$ is a special case of $\mathsf{VNM}_{\mathsf{P(NS)}}$, $\mathsf{VNM}_{\mathsf{L(NS)}}$, $\mathsf{VNE}_{\mathsf{SP(NS)}}$ and $\mathsf{VNE}_{\mathsf{MP(NS)}}$, when edge constraints are absent.

(4) We show that $\mathsf{VNM}_{\mathsf{P}}$, $\mathsf{VNM}_{\mathsf{L}}$ and $\mathsf{VNE}_{\mathsf{SP}}$ are also NP-hard by reduction from X3C, Subgraph Isomorphism and Partition, respectively, which are known to be NP-complete (cf. [25]). The intractability of $\mathsf{VNE}_{\mathsf{MP}}$ is also verified by reduction from Partition. The reductions use only DAGs for VNs and SNs. □

3.2 Approximation of Optimization Problems

In practice, one typically wants to find a VNM mapping with "the lowest cost". This highlights the need for introducing a function to measure the cost of a mapping and studying its corresponding optimization problems.

A Cost Function. Consider an SN $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$, and a VN request $(G_P, \mathcal{C})$, where $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$. Assume there is a positive number associated with all nodes v and links e in G_S, denoted by $w(v)$ and $w(e)$, respectively, that indicates the price of the resources in the SN. Given a pair $((g_V, r_V), (g_E, r_E))$ of node and edge mappings from $(G_P, \mathcal{C})$ to G_S, its *cost* $c((g_V, r_V), (g_E, r_E))$ is defined as follows:

$$\sum_{v \in V_P} h_V(g_V, r_V, v) \cdot c(g_V(v)) + \sum_{e' \in E_S} h_E(g_E, r_E, e') \cdot c(e'),$$

where (a) $h_V(g_V,\ r_V,\ v) = r_V(v,\ g_V(v))/f_{V_S}(g_V(v))$, (b) $h_E\ (g_E, r_V, e') = \sum_{e \in E_P, \rho \in g_E(e), e' \in \rho} r_E(e, \rho)/f_{E_S}(e')$ when the resource of physical links is bandwidth, and (c) when latency is concerned, $h_E(g_E, r_V, e')$ is 1 if there exists $e \in E_P$ such that $e' \in g_E(e)$, and 0 otherwise.

Here functions h_V and h_E specify the cost of the mapping based on its resource allocations (*i.e.,* r_V and r_E), and function $w()$ measures the unit cost of substrate network resources. Intuitively, h_V indicates that the more CPU resource is allocated from a substrate node, the higher the cost it incurs; similarly for h_E when bandwidth is concerned. When latency is considered, the cost of the edge mapping is determined only by g_E, whereas the resource allocation function r_E is irrelevant.

The cost function is motivated by economic models of network virtualization [16]. It is justified by Web hosting and cloud storage [11], which mainly sell CPU powers or storage services of nodes, and by virtual network mapping, which also sells bandwidth of links [17]. It is also to serve cloud provision in virtualized data center networks [26], for which dynamic routing strategy (latency) is critical while routing congestion (bandwidth allocation) is considered secondary.

Minimum Cost Mapping. We now introduce optimization problems for virtual network mapping.

The *minimum cost mapping* problem is to find, given a VN request and an SN, a mapping $((g_V, r_V), (g_E, r_E))$ from the VN to the SN such that its cost based on the function above is minimum among all such mappings.

The decision problem for minimum cost mapping is to decide, given a constant K, a VN request and an SN, whether there is a mapping $((g_V, r_V), (g_E, r_E))$ from the VN to the SN such that its cost is no larger than K.

We shall refer to the minimum cost mapping problem and its decision problem interchangeably in the sequel.

Example 6: Consider the SN $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$ shown in Fig. 2(b), and the VN depicted in Fig. 2(a). Assume that the cost function $c()$ is set to be the same as f_{V_S} for the nodes and as f_{E_S} for the links in the SN, *i.e.,* the cost of a substrate node is the same as its CPU capacity, and the cost of a physical link is the same as its bandwidth capacity or latency. Consider the multi-path embedding from the VN to the SN described in Example 4. Then the cost of the node mapping is $\frac{60}{60} \times 60 + \frac{50}{60} \times 60 + \frac{60}{120} \times 120 = 170$, while the cost of its edge mapping is $(\frac{25}{30} \times 30) \times 2 + (\frac{35}{40} \times 40 + \frac{35}{35} \times 35) + (\frac{20}{30} \times 30 + \frac{30}{30} \times 30) + ((\frac{5}{20} \times 20 + \frac{5}{30} \times 30) + (\frac{5}{40} \times 40 + \frac{5}{20} \times 20)) + ((\frac{15}{20} \times 20 + \frac{15}{15} \times 15) + (\frac{15}{15} \times 15 + \frac{15}{20} \times 20)) = 250$. Putting these together, the total cost is 420. Consider the latency mapping given in Example 5. We can compute its cost along the same lines as above, except here that the cost of each edge (h_E) is either 1 or 0. One can easily verify that the cost of this mapping is $(66 + 20 + 30) + (40 + 45 + 50 + 50) = 301$. □

Complexity and Approximation. We next study the minimum cost mapping problem for all cases given in Table 1. Having seen Theorem 1, it is not surprising that this problem is intractable in most cases. This motivates us to study their approximation, to find an efficient algorithm with performance guarantees.

Unfortunately, the problem is hard to approximate in most cases. To see these, we first briefly illustrate approximation classes (see [10, 19] for details).

(1) The *class* NPO is the set of all NP optimization problems. For any problem in NPO, its corresponding decision problem is in NP [10]. It is *NP-hard to optimize* if its decision version is NP-complete. A NPO-complete problem is NP-hard to optimize, and is among the hardest optimization problems.

(2) A minimization problem (*e.g.,* minimum cost mapping) admits a r-approximation if there exists a PTIME algorithm such that for each instance, it produces a solution of value at most $r \cdot \mathsf{opt}$, where r is a constant greater than 1 and opt is the value of the optimal solution; similarly for the case when r is a polynomial.

A problem is in APX if there exists a constant $r > 1$ such that it admits a r-approximation. A problem is APX-complete if all problems in the class APX can be

Table 4: Summary of complexity results

Problems	Complexity	Approximation
VMP	PTIME	– –
$\mathsf{VMP_{(NS)}}$	NP-complete	APX-hard
$\mathsf{VNM_P}$, $\mathsf{VNM_{P(NS)}}$	NP-complete	NPO-complete
$\mathsf{VNM_L}$, $\mathsf{VNM_{L(NS)}}$	NP-complete	NPO-complete
$\mathsf{VNE_{SP}}$, $\mathsf{VNE_{SP(NS)}}$	NP-complete	NPO-complete
$\mathsf{VNE_{MP}}$, $\mathsf{VNE_{MP(NS)}}$	NP-complete	NPO-complete

reduced to it, preserving the approximation ratio.

The results below tell us that when node sharing is requested or edge constraints are present, minimum cost mapping is beyond reach in practice for approximation.

Theorem 2: *The minimum cost mapping problem is*
(1) in PTIME *for* VMP *without node sharing; however, when node sharing is requested,* i.e., *for* $\mathsf{VMP_{(NS)}}$*, it becomes* NP*-complete and is* APX*-hard;*
(2) NP*-complete and* NPO*-complete for* VNM_P, $\mathsf{VNE_{SP}}$, $\mathsf{VNE_{MP}}$, VNM_L, $\mathsf{VNM_{P(NS)}}$, $\mathsf{VNE_{SP(NS)}}$, $\mathsf{VNE_{MP(NS)}}$ *and* $\mathsf{VNM_{L(NS)}}$*; and*
(3) APX*-hard when there exists a unique node mapping in the presence of edge constraints. In particular,* $\mathsf{VNM_P}$ *does not admit* $\ln(|V_P|)$*-approximation, unless* P = NP.

All the NPO*-hardness results remain intact even when both* VNs *and* SNs *are* DAGs. □

Proof sketch: Proofs of the intractability results are similar to their counterparts in Theorem 1. Below we focus on approximation. The proofs use *approximation preserving reduction* (AP-reduction), which preserves all approximation classes, and *linear reduction* (L-reduction), which preserves classes such as APX [10].

(1) We develop a cubic-time algorithm to find a minimum cost VMP, leveraging the algorithm for the linear assignment problem [33]. This also shows that the minimum cost mapping problem for VMP is in PTIME.

When node sharing is allowed, we reduce from the minimum generalized assignment problem to minimum cost mapping for VMP. Since the former is APX-hard [14], so is the minimum cost mapping problem.

(2) We show that VNM_P, $\mathsf{VNE_{SP}}$, $\mathsf{VNE_{MP}}$, VNM_L ($\mathsf{VNM_{P(NS)}}$, $\mathsf{VNE_{SP(NS)}}$, $\mathsf{VNE_{MP(NS)}}$, $\mathsf{VNM_{L(NS)}}$) are NPO-complete by AP-reductions from the Minimum Weighted 3SAT problem [10], which is known to be NPO-complete. The reduction uses VN and SN as DAGs.

(3) The first statement is verified by L-reduction from the Minimum Steiner Tree problem, which is known to be APX-complete [10]. The second statement is by L-*reduction* from the Minimum Directed Steiner Tree problem to the minimum cost mapping problem for $\mathsf{VNM_P}$ with node mapping uniquely determined in the presence of edge constraints; the former does not admit $\ln(k)$-approximation [13]. □

We summarize the complexity results in Table 4.

4. COMPUTING MINIMUM COST VNM

Theorem 2 tells us that any efficient algorithms for computing minimum cost VNM are necessarily heuristic. We next develop a greedy algorithm to find minimum cost priority mappings ($\mathsf{VNM_P}$), with node sharing or not. Given a VN request $(G_P, \mathcal{C})$, an SN G_S, and a cost function $c()$, the algorithm finds a mapping $((g_V, r_V), (g_E, r_E))$ from G_P to G_S such that it satisfies the node and edge constraints in $\mathcal{C}$ and moreover, the cost $c((g_V, r_V), (g_E, r_E))$ is minimized, if such a mapping exists. To the best of our knowledge, this is the first algorithm for computing $\mathsf{VNM_P}$.

Previous algorithms for computing VNM (*e.g.,* [40]) typically consists of two stages. It first finds a candidate node mapping (g_V, r_V), and then checks whether it is *valid*, *i.e.,* whether it admits a corresponding edge mapping (g_E, r_E); if so, it computes (g_E, r_E) by traversing the entire SN. If the (g_V, r_V) is not valid, the entire process has to start all over again. Hence a mapping is often found only after repeated trials and failures. This hinders the scalability of the algorithms.

In contrast, we unify the processes of computing node mappings and edge mappings. During the process of building a node mapping, we check whether the (partial) mapping found so far is valid, *i.e.,* we do not wait for a node mapping to be completed before starting the validation process. To efficiently validate a partial node mapping and build its corresponding edge mapping, we use an auxiliary graph structure for SN G_S. In addition, we minimize the VN pattern G_P. Obviously, the smaller G_P is, the less costly the mapping process is.

Below we first present auxiliary structures for SN (Section 4.1) and then develop an algorithm for minimizing VN patterns (Section 4.2). Finally, leveraging the auxiliary structures and minimization, we present algorithm for minimum cost $\mathsf{VNM_P}$ (Section 4.3).

4.1 Auxiliary Graphs: Unifying Mappings

Given a weighted directed graph $G(V, E, f_V, f_E)$, its *auxiliary graph* $G_{aux}(V_a, E_a, f_{V_a}, f_{E_a}, P_{E_a})$ is a weighted directed graph such that (1) $V_a = V$, $f_{V_a} = f_V$, (2) edge $(u, v) \in E_a$ iff there exists a path from u to v in G, (3) $f_{E_a}(u, v) = \max\{\min\{\rho\} \mid \rho$ is a path from u to v in $G\}$, where $\min\{\rho\} = \min\{f_E(e) \mid e$ is an edge on $\rho\}$ in G, and (4) $P_{E_a}(u, v)$ is a path ρ in G with $\min\{\rho\} = f_{E_a}(u, v)$.

One can easily verify the following for priority mappings, which justifies the need for auxiliary graphs.

Proposition 3: *Consider a* VN $G_P(V_P, E_P, f_{V_P}, f_{E_P})$ *and an* SN $G_S(V_S, E_S, f_{V_S}, f_{E_S})$*. For any node mapping* (g_V, r_V)*, with the auxiliary graph of* G_S *it takes* $O(|E_P|)$ *time to determine whether* (g_V, r_V) *is valid, and to compute a corresponding edge mapping.* □

Algorithm. We next present an algorithm, referred to as compAuxGraph, for building auxiliary graphs. Given

Input: A weighted directed graph $G(V, E, f_V, f_E)$.
Output: An auxiliary graph G_{aux}.

1. $G_{aux}(V_a, E_a, f_{V_a}, f_{E_a}, P_{E_a}) := (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$;
2. **for each** node v in G **do**
3. $\quad G_{aux} :=$ updateAuxGraph(G_{aux}, G, v);
4. Remove edges (u, u') from G_{aux} having $f_{E_a}(u, u') = 0$;
5. **return** G_{aux}.

Procedure updateAuxGraph (G_{aux}, G, v)

Input: Auxiliary graph G_{aux}, graph G, and node v.
Output: Updated G_{aux} by incorporating v.

1. **for each** node u in V_a **do**
2. $\quad E_a := E_a \cup \{(u, v), (v, u)\}$;
3. $\quad$ Assign(v, u, G_{aux}); Assign(u, v, G_{aux});
4. **for each** edge (u, u') in G_{aux} having $u, u' \in V_a$ **do**
5. $\quad h := \min\{f_{E_a}(u, v), f_{E_a}(v, u')\}$;
6. $\quad$ **if** $f_{E_a}(u, u') < h$ **then**
7. $\qquad f_{E_a}(u, u') := h$;
8. $\qquad P_{E_a}(u, u') := P_{E_a}(u, v) + P_{E_a}(v, u')$;
9. $V_a := V_a \cup \{v\}$; $f_{V_a}(v) := f_V(v)$;
10. **return** G_{aux};

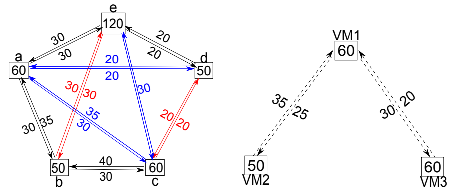
Figure 3: Algorithm compAuxGraph

a weighted directed graph G, it returns the auxiliary graph G_{aux} of G, as shown in Fig. 3.

Algorithm compAuxGraph starts from an empty G_{aux} (line 1) and iteratively adds nodes to G_{aux} by calling procedure updateAuxGraph (lines 2-3). As will be seen shortly, it may add an edge (u, u') to G_{aux}; when $f_{E_a}(u, u') = 0$, it indicates that there exists no path from u to u' in G. Such edges are removed form G_{aux} (line 4) and the auxiliary graph is returned (line 5).

Given a node v in G and the auxiliary graph $G_{aux}(V_a, E_a, f_{V_a}, f_{E_a}, P_{E_a})$ of the subgraph of G such that $v \notin V_a$, procedure updateAuxGraph returns the auxiliary graph of the subgraph of G with nodes $V_a \cup \{v\}$. It works as follows. For each node u in G_{aux}, updateAuxGraph adds two new edges (v, u) and (u, v) to E_a, and assigns their weights $f_{E_a}(u, v)$, $f_{E_a}(v, u)$ and paths $P_{E_a}(u, v)$ and $P_{E_a}(v, u)$ by calling procedure assign (omitted duo to the lack of space; lines 1-3). For each new edge (v, u), weight $f_{E_a}(v, u)$ is either $f_E(v, u)$ (if there exists an edge (v, u) in G), or $\max\{\min\{f_{E_a}(v, u'), f_E(u', u)\}\}$ for all nodes u' in V_a such that (v, u') is an edge in G. Moreover, $P_{E_a}(v, u)$ is either edge (v, u), or a path consisting of (v, u') followed by $P_{E_a}(u', u)$; similarly for the new edge (u, v). Then the weights and paths of existing edges are updated (lines 4-8). For each edge (u, u'), the triangle with edges $(u, u'), (u, v)$ and (v, u') is considered to find weight h. If $h > f_{E_a}(u, u')$, $f_{E_a}(u, u')$ is changed to h (line 7), and $P_{E_a}(u, u')$ is changed to the concatenation of $P_{E_a}(u, v)$ and $P_{E_a}(v, u')$ (line 8). Finally, node v is added to G_{aux} (line 9), and the updated auxiliary graph is returned in the end (line 10).

Example 7: For the SN of Fig. 2(b), the auxiliary graph constructed by compAuxGraph is shown in Fig. 4(a). Note that the bandwidths on edges between b and e, c and d are larger than they are in the SN of Fig. 2(b), since they are updated by procedure updateAuxGraph (lines 5-7). Moreover, there are now



(a) Auxiliary graph (b) Minimized VN

Figure 4: Examples of auxiliary graphs and minimizing VNs

new edges with positive bandwidth directly connecting a and d, a and c, b and d, c and e. For each edge (u, v), the auxiliary graph also records the path with the maximum bandwidth among all paths connecting u and v in SN. Taking edges (b, e) and (c, d) as examples, $P(b, e) = (b, a, e)$, $P(e, b) = (e, a, b)$, $P(c, d) = (c, b, e, d)$ and $P(d, c) = (d, e, a, b, c)$. Note that paths (c, b, a, e, d) and (d, e, b, c) also carry the maximum bandwidth in the SN for edges (c, d) and (d, c), respectively, but compAuxGraph only records one of them since it already suffices to derive the existence of an edge mapping. $\Box$

Correctness & complexity. One can verify the following.

Lemma 4: *In* updateAuxGraph*, (1) for any new edge (u, v), its weight and path are not affected by updating existing edges; and (2) for any existing edge (u, u'), it suffices to consider the triangle with edges $(u, u'), (u, v)$ and (v, u') for updating its weight and path.* $\Box$

Lemma 4 shows that updateAuxGraph always produces an auxiliary graph $G_{aux}(V_a, E_a, f_{V_a}, f_{E_a}, P_{E_a})$ for the subgraph of G with nodes V_a only. From this the correctness of compAuxGraph follows immediately.

Algorithm compAuxGraph is in $O(|V|^3)$ time since procedure updateAuxGraph takes $O(|V_a|^2)$ time, and it is called $|V|$ times in total. Note that $|V_a| \le |V|$.

4.2 Minimizing Virtual Network Patterns

We next show how to minimize VNs.

Equivalence. Given two VNs $G_{P_1}(V_P, E_{P_1}, f_{V_P}, f_{E_{P_1}})$ and $G_{P_2}(V_P, E_{P_2}, f_{V_P}, f_{E_{P_2}})$, we say that G_{P_1} is *equivalent* to G_{P_2}, denoted by $G_{P_1} \equiv G_{P_2}$, if for any SN G_S and cost function $c()$, there exists a VNM from G_{P_1} to G_S iff there exists another VNM from G_{P_2} to G_S with the same cost.

We can minimize an VN G_P in cubic-time:

Theorem 5: *There exists a cubic-time algorithm that, given any* VN G_P*, finds an equivalent* VN G_P^m *of* G_p *such that for any* $G'_P \equiv G_P$*,* G_P^m *has no more edges than* G'_P. $\Box$

We next present an algorithm for minimizing VNs denoted by minVN and shown in Fig. 5. Given a VN G_P, it returns a minimized equivalent VN G_P^m.

Given G_P, algorithm minVN first computes the auxiliary graph G'_P of G_P (line 1), which has an empty path set since the path information is not needed here. Starting from an empty VN G_P^m (line 2), the algorithm

Input: A virtual network $G_P(V_P, E_P, f_{V_P}, f_{E_P})$.
Output: A minimized equivalent VN G_P^m.

1. $G'_P(V_P, E'_P, f_{V_P}, f_{E'_P}, \emptyset)$:= compAuxGraph(G_P);
2. $G_P^m(V_P^m, E_P^m, f_{V_P^m}, f_{E_P^m})$:= $(\emptyset, \emptyset, \emptyset, \emptyset)$;
3. **for each** node v in G'_P **do**
4. $\quad G_P^m$:= UpdateVN (G_P^m, v, G'_P);
5. **return** G_P^m.

Procedure UpdateVN (G_P^m, v, G'_P).
Input: VN G_P^m, node v, and auxiliary graph G'_P.
Output: Updated G_P^m by incorporating v.

1. $V_P^m := V_P^m \cup \{v\}$; $f_{V_P^m}(v) := f_{V_P}(v)$;
2. **for each** node $u \neq v$ in V_P^m **do**
3. $\quad$ **if** $(v, u) \in E'_P$ **and** there is no $u' \in V_P^m$ such that $(u', u) \in E'_P$ and $(v, u') \in E_P^m$
4. $\quad$ **then** $E_P^m := E_P^m \cup \{(v, u)\}$; $f_{E_P^m}(v, u) := f_{E'_P}(v, u)$;
5. $\quad$ **if** $(u, v) \in E'_P$ **and** there is no $u' \in V_P^m$ such that $(u, u') \in E'_P$ and $(u', v) \in E_P^m$
6. $\quad$ **then** $E_P^m := E_P^m \cup \{(u, v)\}$; $f_{E_P^m}(u, v) := f_{E'_P}(u, v)$;
7. **return** G_P^m;

Figure 5: Algorithm minVN for minimizing VNs

iteratively adds nodes to G_P^m, one at a time by calling procedure updateVN (lines 3-4). Finally, the minimized VN G_P^m is returned (line 5).

We next present procedure updateVN. Given a node v in G'_P and the minimized VN $G_P^m(V_P^m, E_P^m, f_{V_P^m}, f_{E_P^m})$ of the subgraph of G with nodes V_P^m, where $v \notin G'_P$, it returns the minimized VN G_P^m of the subgraph of G with nodes $V_P^m \cup \{v\}$.

More specifically, procedure updateVN first adds node v to G_P^m (line 1). It then adds edges to G_P^m that connect node v with other nodes in G_P^m (lines 2-6). An edge (v, u) is added to E_P^m only if there exists no node u' such that there is a path from u' to u in G_P^m ($(u', u) \in E'_P$) and (v, u') is an edge in G_P^m (lines 3-4); similarly for edge (u, v) (lines 5-6). Finally, the updated G_P^m is returned (line 7).

Example 8: Consider the VN shown in Fig. 2(a) for priority mapping. Given the VN, procedure minVN derives from it an equivalent yet simpler VN, as shown in Fig. 4(b). Observe the following. (1) There exist no edges (VM_2, VM_3) and (VM_3, VM_2) in Fig. 4(b), as opposed to Fig. 2(a). This is because (VM_2, VM_3) (resp. (VM_3, VM_2)) is entailed by edges (VM_2, VM_1) and (VM_1, VM_3) (resp. edges (VM_3, VM_1) and (VM_1, VM_3)), and hence, can be left out. (2) The edge constraints in Fig. 4(b) are different from their counterparts shown in Fig. 2(a). $\square$

Correctness & complexity. One can verify the following.

Lemma 6: *For any* VN G_P*, procedure* updateVN *returns an* VN G_P^m *such that there exists a unique path from node u to v in G_P^m if and only if there exists a path from node u to v in the* VN G_P. $\square$

By Lemma 6, we can show that procedure updateVN produces a minimized VN $G_P^m(V_P^m, E_P^m, f_{V_P^m}, f_{E_P^m})$ for the subgraph of G_P with nodes V_P^m only. From this the correctness of algorithm minVN immediately follows.

Observe the following. (1) Algorithm compAuxGraph runs in $O(|V_P|^3)$ time. (2) Procedure updateVN takes $O(|V_P^m|^2)$ time, and it is called $|V_P|$ times in total. Hence, algorithm minVN runs in $O(|V_P|^3)$ time.

These complete the proof of Theorem 5 and provide an algorithm for minimizing VN patterns.

4.3 Computing Minimum Cost Priority Mappings

We are now ready to present our algorithm for computing priority mappings, denoted by compVNM and shown in Fig. 6. Given a VN request $(G_p, \mathcal{C})$, an SN G_S, and a cost function $c()$, the algorithm finds a low cost VNM $((g_V, r_V), (g_E, r_E))$ from G_P to G_S if there exists one. As will be seen shortly, it uses a predefined non-negative integer k to control the level of backtracks, which is typically small (no more than 3).

As remarked earlier, the algorithm employs two optimization strategies to reduce search space. (1) It removes redundant edge constraints from VN G_P, via algorithm minVN (line 2). (2) It constructs the auxiliary graph G_{aux} of SN G_S with algorithm compAuxGraph, to validate a node mapping without traversing the entire G_S (line 3). The algorithm builds a low cost node mapping by inspecting nodes one by one, via procedure backTrackMap (lines 4-6). It unifies the processes of building node mappings and edge mappings: it checks whether the partial node mapping found so far is valid ($(g_V, r_V, S) \neq$ null, line 6). If so, it finds the corresponding edge mapping by calling procedure identifyEdgeMap (omitted; line 7). With G_{aux}, the edge mapping can be found in $O(|E_P^m|)$ time (Lemma 4). A VNM is finally returned if there exists one (line 8).

We next present procedure backTrackMap. Given a new node v, a node set S for which mappings are already identified, a node set $backS$, and non-negative integers i and k, it expands the mapping for S by including v. If v cannot be mapped to a substrate node, it backtracks and searches other nodes, along the same lines as [30]. The backtrack depth is bounded by k. It uses i to keep track of the current backtrack depth, and $backS$ to record the set of nodes backtracked. In contrast to [30] that has to traverse the entire G_s, we reduce the search space by inspecting only virtual nodes in the minimized VN G_P^m, and by checking edge constraints using auxiliary graph G_{aux}.

More specifically, if the current backtrack depth $i > k$, then the procedure returns null since a node mapping cannot be found (line 1). Otherwise, it checks whether there is a node u to which node v can be mapped (lines 3-4). It uses procedure Valid (omitted), which checks whether the (partial) node mapping admits an edge mapping by inspecting the edge constraints in G_{aux}. If not, node v may be mapped to a node $g_V(v')$ to which node v' is already mapped (line 6), and proce-

Input: An SN G_S, a VN request $(G_P, \mathcal{C})$, a cost function $c()$, and a positive integer k.
Output: A low cost mapping from G_P to G_S.

1. $(g_V, r_V) := (\emptyset, \emptyset)$; $\quad S := \emptyset$;
2. $G_P^m(V_P^m, E_P^m, f_{V_P^m}, f_{E_P^m}) := \mathsf{minVN}(G_P)$;
3. $G_{aux}(V_a, E_a, f_{V_a}, f_{E_a}) := \mathsf{compAuxGraph}(G_S)$;
4. **for each** v in V_{P_m} **do**
5. $\quad (g_V, r_V, S) := \mathsf{backTrackMap}(v, S, \emptyset, 0, k)$;
6. $\quad$ **if** $(g_V, r_V, S) =$ null **then return** null;
7. $(g_E, r_E) := \mathsf{identifyEdgeMap}(g_V, r_V, G_{aux})$;
8. **return** $((g_V, r_V), (g_E, r_E))$.

Procedure $\mathsf{backTrackMap}(v, S, backS, i, k)$
Input: Node v, node sets S and $backS$, non-negative integers i and k.
Output: Updated node mapping (g_V, r_V).

1. **if** $i > k$ **then return** null;
2. **if** there exists u in G_{aux} with $\mathsf{Valid}(v, u, S) = \mathsf{true}$ **then**
3. $\quad g_V(v) := u$; $\quad r_V(v) := f_{V_P^m}(v)$; $\quad S := S \cup \{v\}$;
4. $\quad$ **return** (g_V, r_V, S);
5. **for each** $v' \in S \setminus backS$ **do**
6. $\quad$ **if** $\mathsf{Valid}(v, g_V(v'), S \setminus \{v'\})$ **then**
7. $\quad\quad g_V(v) := g_V(v')$; $\quad r_V(v) := f_{V_P^m}(v)$; $\quad S := S \cup \{v\} \setminus \{v'\}$;
8. $\quad$ **if** $\mathsf{backTrackMap}(v', S, backS \cup \{v\}, i+1, k)$ **then**
9. $\quad\quad$ **return** (g_V, r_V, S);
10. $\quad S := S \setminus \{v\} \cup \{v'\}$; $\quad g_V(v') := g_V(v)$;
11. **return** null;

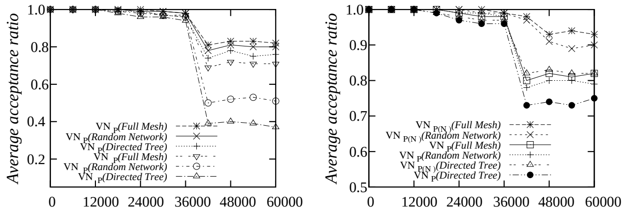
Figure 6: Algorithm compVNM for priority mappings

dure backTrackMap is called recursively to find a mapping node for node v' (line 8). Such nodes v' are checked (lines 5-9), with their information backed up (line 7) and restored later (line 10). If a valid node mapping cannot be found, null is returned (line 11).

Example 9: Consider the VN request and SN shown in Fig. 2. Assume a cost function $c()$ for the SN such that (1) for nodes a, b and c, their costs are the same as their node capacities; (2) for d and e, their costs are ten times of their node capacities; and (3) the cost of each physical link in the SN is its edge capacity.

We show below how compVNM finds a priority mapping from the VN to the SN. Algorithm compVNM first computes the minimized VN and the auxiliary graph G_{aux} of the SN, as shown in Fig. 4. It then finds mappings for nodes VM_1, VM_2 and VM_3 in the minimized VN by calling procedure backTrackMap and by leveraging G_{aux}. It starts with VM_1 and maps it to SN node a via backTrackMap such that (VM_1, a) is a valid node mapping for the subgraph of V_P^m with node VM_1 only. It then invokes backTrackMap and maps VM_2 to SN node c such that (VM_1, a) and (VM_3, c) make a valid node mapping for the the subgraph of V_P^m with nodes VM_1 and VM_3. Similarly, a candidate mapping node b is found for VM_2. No backtrack is needed in backTrackMap for all these nodes. Then compVNM identifies edge mappings by using the auxiliary graph G_{aux}. It maps virtual edges to those paths recorded in G_{aux}, *e.g.,* $(\mathsf{VM}_1, \mathsf{VM}_2)$ is mapped to $P(a, b)$ (see Example 7). Finally the mapping is completed and returned. □

Complexity. Algorithm compVNM is in $O(|V_S|^3 + |V_P|^{(k+1)}\ |E_P|(|V_P| + |V_S|) + |V_P|^3)$ time, where $|V_S|$, $|V_P|$, $|E_P|$ are the number of nodes in G_S, the number of nodes and edges in G_P, respectively. Indeed, procedures compAuxGraph, minVN and backTrackMap take $O(|V_S|^3)$ time, $O(|V_P|^3)$ time and $O(|V_P|^k|E_P|(|V_P| + |V_S|))$ time, respectively. Here k is a *predefined constant*. We found that a small k (usually no more than 3) typically suffices, as will be verified by the experimental study presented in the next section.



(a) Vary time t from 0 to 60000 (b) Vary time t from 0 to 60000

Figure 7: Mapping quality over time

Remark. One can extend algorithm compVNM for priority mappings with node sharing, denoted by $\mathsf{compVNM_{NS}}$, by simply allowing multiple virtual nodes in G_P to be mapped to the same node in G_S in Valid.

5. EXPERIMENTAL STUDY

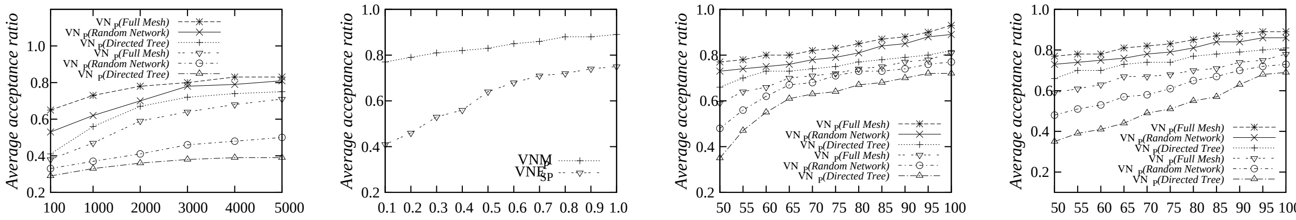
We next present an experimental study of our techniques for computing virtual network priority mappings ($\mathsf{VNM_P}$). We conducted two sets of experiments to evaluate: (1) the effectiveness of $\mathsf{VNM_P}$ vs. conventional virtual network embedding ($\mathsf{VNM_{SP}}$) and (2) the efficiency of our algorithms.

Experimental setting. We used the following datasets.

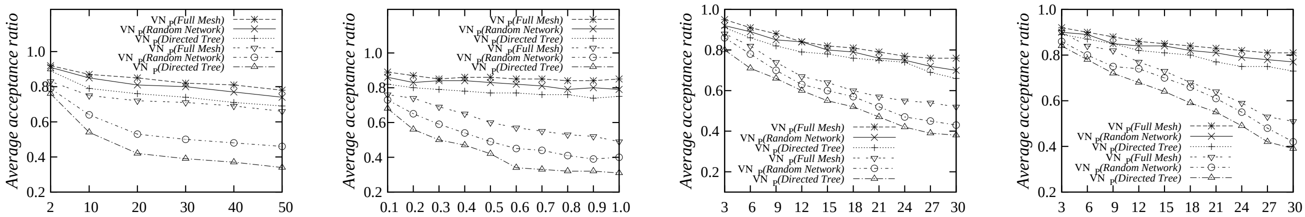
Substrate networks (SNs). We used three types of substrate networks, as found in real life. (a) Directed tree networks, in which for any two nodes u and v, there exists an edge (u, v) iff there exists an edge (v, u), and the network becomes a tree if the two edges between any two nodes are merged into one. (b) Full mesh networks, in which for any two nodes u and v, there exists an edge (u, v). (c) Random networks, in which for any two nodes u and v, there exists an edge (u, v) with probability p. Directed tree networks and full mesh networks were constructed by adopting real-life network topologies (http://en.wikipedia.org/wiki/Network_topology).

We also developed a graph generator to produce these networks, controlled by the following parameters: (a) the number n_S of nodes, (b) the node capacity w_{V_S}, (c) the edge capacity w_{E_S}, and (d) the probability p_S (for random networks only).

VN requests. VN requests arrive in a Poisson process with an average of λ requests per time unit, commonly adopted by network community [17, 30, 40]. Each one has exponentially many distributed lifespan with an average of u time units. The VNs were randomly produced by the same graph generator for substrate networks, and they were controlled by four parameters: (a) the num-



(a) Vary n_S from 100 to 5000 (b) Vary p_S from 0.1 to 1.0 (c) Vary w_{V_S} from 50 to 100 (d) Vary w_{E_S} from 50 to 100



(e) Vary n_P from 2 to 50 (f) Vary p_P from 0.1 to 1.0 (g) Vary w_{V_P} from 3 to 30 (h) Vary w_{E_P} from 3 to 30

Figure 8: Mapping quality over SN and VN

Table 5: Experimental parameters

	VN Requests	SNs
Number of nodes	n_P	n_S
Edge probability	p_P	p_S
Node capacity	w_{V_P}	w_{V_S}
Edge capacity	w_{E_p}	w_{E_S}

ber n_P of nodes, (b) the virtual node capacity w_{V_P}, (c) the edge capacity w_{E_P}, and (d) the probability p_P.

Algorithms. We have implemented the following algorithms, all in C++. (a) Algorithms compVNM and compVNM$_{\mathsf{NS}}$ for computing VNM$_{\mathsf{P}}$, without node sharing and with node sharing; (b) Algorithms SubIso [30], ViNE [17] and RW-SP [15] for computing VNE$_{\mathsf{SP}}$; (c) Algorithm ViNE$_{\mathsf{NS}}$ that extends ViNE for computing VNE$_{\mathsf{SP}}$ with node sharing; and (d) the graph generator for producing VNs and SNs.

The experiments were run on a machine with Intel Core i7 860 CPU and 16GB of memory. All the experiments were repeated over 5 times and the average is reported here.

Experimental results. We next report our findings. In all experiments, for VN requests, we fixed $\lambda = 0.02$ and $u = 1000$, which were decided based on the substrate networks considered, and had little impact on the quality and efficiency tests. We also fixed the backtrack depth $k = 3$ for compVNM and compVNM$_{\mathsf{NS}}$. We adopted algorithm ViNE for VNE$_{\mathsf{SP}}$ when comparing the mapping quality of VNM$_{\mathsf{P}}$ with VNE$_{\mathsf{SP}}$. We summarize the tested factors in Table 5.

Exp-1: Mapping quality. In the first set of experiments, we evaluated (1) the mapping quality of VNM$_{\mathsf{P}}$ vs. VNE$_{\mathsf{SP}}$; and (2) the impact of node sharing. We used the *average acceptance ratio* (AR), a quality measure commonly adopted by the network community [17,30,40], to evaluate the mapping quality. Given a time stamp t, AR is defined as:

$$\mathsf{AR}(t) = \#\mathsf{validVNs}(t) \ / \ \#\mathsf{arrivedVNs}(t),$$

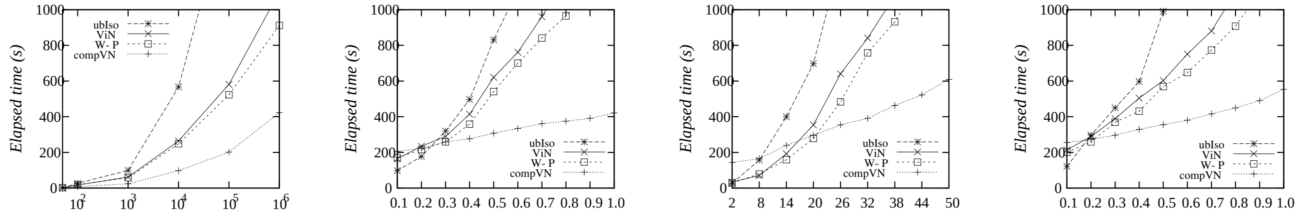
where $\#\mathsf{validVNs}(t)$ denotes the number of VN requests that are fulfilled until time t, and $\#\mathsf{arrivedVNs}(t)$ denotes the total number of VN requests arrived until time t, respectively. Intuitively, $\mathsf{AR}(t)$ is the ratio of VNs successfully mapped during time interval $[0, t]$.

(1) We first evaluated the impact of time t on AR. For VN requests, we fixed $p_P = 0.5$, n_P in [2, 50], and w_{V_P}, w_{E_P} in [3, 30]. For SNs, we fixed $n_S = 5000$, and w_{V_S}, w_{E_S} in [50, 100]. We took $n_S = 5000$ since *medium-size ISPs* have about *500* nodes only [30]. We varied t from 0 to $60,000$.

Figure 7(a) shows the AR of VNM$_{\mathsf{P}}$ and VNE$_{\mathsf{SP}}$ over directed tree, full mesh and random networks. We found the following. (i) In all the cases, the AR decreases *w.r.t.* t, and becomes stable when t is about $42,000$. This is because initially there exists no workload in the SNs; the SNs are fully loaded when t reaches $42,000$, since only a certain amount of work could be handled by the SNs. (ii) The AR of VNM$_{\mathsf{P}}$ is consistently higher than that of VNE$_{\mathsf{SP}}$ (in the range of $[11\%, 39\%]$) in all the cases. (iii) The impact of network topologies on the AR of VNM$_{\mathsf{P}}$ is much smaller than that of VNE$_{\mathsf{SP}}$. Indeed, the stable AR for VNM$_{\mathsf{P}}$ is in the range of [76%, 82%], while for VNE$_{\mathsf{SP}}$, it is around 37% and 71% on directed tree and full mesh networks, respectively.

Figure 7(b) shows the AR of VNM$_{\mathsf{P}}$, with node sharing or not, over directed tree, full mesh and random networks. The results show the following. (i) Node sharing consistently improves the AR for priority mappings (in the range of [8%, 11%]). Indeed, the AR on full mesh networks is over 93% with node sharing, as opposed to 82% without node sharing. (ii) Node sharing also improves the AR for VNE$_{\mathsf{SP}}$ (we did not report it here due to the space constraint). This shows that the idea of node sharing is generic, and can be employed by other virtual network mapping models.

(2) To evaluate the impact of SNs on AR, we fixed $t =$



(a) Vary n_S from 10^2 to 10^6 (b) Vary p_S from 0.1 to 1.0 (c) Vary n_P from 2 to 50 (d) Vary p_P from 0.1 to 1.0

Figure 9: Mapping efficiency and scalability evaluation

60,000, and VN with $n_P = 50$, $p_P = 0.5$ and $w_{V_P} = w_{E_P} = 30$, and we varied n_S from 100 to 5,000, p_S from 0.1 to 1.0, and w_{V_S}, w_{E_S} from 50 to 100. The test of p_S can be conducted on SNs of random networks only.

The results are reported in Figures 8(a), 8(b), 8(c) and 8(d). (i) These figures show that AR increases *w.r.t.* n_S, p_S, w_{V_S} and w_{E_S}. This is because of the following. (a) The larger n_S is, there are more nodes in the SNs, and the larger p_S is, there are more links in the SNs. (b) The lager w_{V_P} and w_{E_P} are, there are larger capacities in the nodes and links of the SNs, respectively. Hence, the SN can handle more requests when any of these four factors is increased, and therefore, so does AR. (ii) The AR of VNM_P is consistently higher than the AR of VNE_SP in all the cases, up to 37%. (iii) The AR of VNM_P is less sensitive to network topologies than the AR of VNE_SP.

(3) To evaluate the impact of VN requests on AR, we fixed $t = 60,000$, and SNs with $n_S = 5000$, $p_S = 0.5$ and $w_{V_S} = w_{E_S} = 100$, while varying n_P from 2 to 50, p_P from 0.1 to 1.0, and w_{V_S}, w_{E_S} from 3 to 30. Again the test of p_P can be conducted on the VNs of random networks only.

The results are shown in Figures 8(e), 8(f), 8(g) and 8(h). (i) The results tell us that AR decreases *w.r.t.* n_P, p_P, w_{V_P} and w_{E_P}. Indeed, (a) the larger n_P is, more machines are requested by the VNs; (b) the larger p_P is, more links are demanded; and (c) the lager w_{V_P} and w_{E_P} are, more capacities are required. As a result, AR decreases with the increase of any of these four factors, which makes the VN requests harder to fulfill. (ii) The AR of VNM_P is consistently higher than the AR of VNE_SP in all the cases, up to 33%. (iii) The AR of VNM_P is less sensitive to network topologies than that of VNE_SP.

Exp-2: Mapping efficiency. In this set of experiments, we evaluated the efficiency of our algorithm compVNM for VNM_P vs. algorithms SubIso [30], ViNE [17] and RW-SP [15] for VNE_SP. We used large random networks in the experiments. Due to the space constraint, we do not report the impact of node and edge capacities w_{V_P} and w_{E_P} on VNs, and w_{V_S} and w_{E_S} on SNs, since these factors have little impact on the efficiency, as shown by the complexity analysis.

(1) To evaluate the impact of SNs, we fixed VN requests with $n_V = 25$, $p_P = 0.5$, $w_{V_P} = w_{E_P} = 30$, and varied n_S from 10^2 to 10^6 and p_S from 0.1 to 1.0, respectively. The results are shown in Figures 9(a) and 9(b), respectively.

(2) To evaluate the impact of VNs, we fixed VNs with $n_S = 500,000$, $p_S = 0.5$, $w_{V_S} = w_{E_S} = 100$, and varied n_V from 2 to 50 and p_V from 0.1 to 1.0, respectively. The results are reported in Figures 9(c) and 9(d), respectively.

These results show the following. (i) As expected, the running time of all these algorithms increases with the increase of n_S, p_S, n_P and p_P. (ii) Algorithm compVNM is efficient: it took only around 420s for SNs with *1 million* nodes. (ii) It outperforms the other three algorithms for VNE_SP in almost all the cases. Indeed, compVNM is about twice faster than the other algorithms. While it took compVNM less that 600s for $n_S = 10^6$, $p_S = 1.0$, $n_P = 50$ or $p_P = 1.0$ in Figures 9(a), 9(b), 9(c) and 9(d), respectively, the other algorithms took at least 900s, or could not run to completion.

Summary. From these experimental results we find the following. (a) Priority mapping (VNM_P) proposed in this work is able to find high-quality mappings, and has higher acceptance ratio than previous models (*e.g.,* VNE_SP), typically up from 11% to 39%. (b) Priority mapping is less sensitive to network topologies. (c) The introduction of node sharing improves mapping quality, typically in the range [8%, 11%]. (d) Our algorithm for computing priority mapping is efficient, *e.g.,* it took 400s for SNs with 10^6 nodes, and it substantially outperforms previous algorithms for VNE_SP.

6. CONCLUSION

We have proposed a model to express various VN requests commonly found in practice, based on graph pattern matching. We have also established a number of intractability and approximation hardness results in various practical VNM settings. These are among the first efforts to settle fundamental problems for virtual network mapping. For intractable VNM cases, we have developed the first algorithms for priority mapping, a

VNM problem identified in this work that is important in emerging applications. We have experimentally verified that the algorithms are effective and efficient, using real-life and synthetic data. These results not only provide a foundation for network virtualization commonly used in data centers and IaaS cloud computing, among other things, but are also useful to the study of graph pattern matching for, *e.g.,* querying social networks.

Several extensions are targeted for future work. We are currently developing further optimization techniques for VNM, and are experimentally verifying the techniques with large SNs. We are also studying other practical quality functions for virtual network mapping, beyond mapping costs. In addition, we are exploring techniques for processing VN requests in the uniform model for different applications, as well as their use in graph pattern matching.

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Appendix A: Proofs

Proof of Theorem 1

Given a VN request (G_P, $\mathcal{C}$) in which G_P = (V_P, E_P, f_{V_P}, f_{E_P}) and an SN G_S = (V_S, E_S, f_{V_S}, f_{E_S}). We prove results of theorem 1 as follows.

Proof:

(1) To show the upper bound, we first provide a generic NP algorithm for cases without multi-path edge mapping, *i.e.,* all except $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$. We then propose a specific NP algorithm for $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$. Given a VN request and an SN, the generic NP algorithm first guesses a node mapping function g_V and an edge function g_E such that $g_E(v, v')$ contains only one path from $P(v, v')$, for each $(v, v') \in E_P$. It then checks whether there exist r_V and r_E such that (g_V, r_V) and (g_E, r_E) make node and edge mappings that satisfy the constraints in the VN request. The existence of r_V can be checked in $O(|V_P|)$-time for node constraints. For r_E, with g_V and g_E, the checking can be formulated as a linear programming problem with $r_E(e, \rho)$ as its variables, which can be solved in time polynomial in $|V_P|$ [32]. Thus the checking can be done in PTIME. Note that the guessed certificate (g_V, g_E) is of polynomial size and thus the algorithm is in NP. For $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$, we first guess a node mapping function g_V solely as the certificate, and then checks whether there exist r_V, g_E and r_E such that (g_V, r_V) and (g_E, r_E) satisfies the constraints carried by the VN request. The existence of r_V is checked exactly the same as the generic NP algorithm above. For the existence of edge mapping (g_E, r_E), we reduce the checking problem to the fractional multi-commodity flow problem such that there exists a fractional multi-commodity flow iff there exists an edge mapping in accord with the guessed node mapping g_V. The latter problem can be solved by a polynomial-time linear-programming algorithm [18], in which the variables are $r_E(e_P, e_S)$ (for all edges e_p in VN and e_S in SN). Since the guessed certificate is of polynomial size and the checking can be done in PTIME, this is an NP algorithm for $\mathsf{VNE_{MP}}$ and $\mathsf{VNE_{MP(NS)}}$.

Below is the details about the generic NP algorithm. The specific NP algorithm works similarly. The algorithm consists three steps:

(i) Guess a node mapping function g_V and an edge mapping function g_E of VN on the SN.

(ii) Check whether there exists r_V and r_E such that (g_V, r_V) and (g_E, r_E) make node and edge mappings that satisfy the constraints set $\mathcal{C}$. It is sure that the existence of r_V can be checked in $O(|V_P|)$-time. We now claim that the checking of edge constraints can be formulated as a linear (rational number) programming problem with $r_E(e, \rho)$ as its variables.

(a) For constraints of form 3, if agg = "max", then the constraints can be formulated as a set of inequality with each one in the form $f_{E_S}(e)$ op $r_E(e, \rho)$ ($\forall$ e $\in$ $g_E(e)$). While if agg = "sum", the constraints can be formulated as one inequality in the form $f_{E_S}(e)$ op $\sum\limits_{\forall e \in g_E(e)} \{r_E(e, \rho)\}$.

(b) For constraints of form 4, 5, the formulations are similar to constraints of form 3. For example, constraints in form 4 can be formulated similarly to form 3 with settings op = "$\geq$" and agg = "sum".

As linear (rational number) programming problem can be done in time polynomial in $|V_P|$ [32], the existence of edge mapping can be check in PTIME.

Thus the checking can be done in PTIME, *i.e.,* the above procedures form into an NP algorithm for VNM problem.

(2)We next propose a PTIME algorithm to check whether there exists a VMP from a VN request (G_P,$\mathcal{C}$) to an SN G_S with only node constraints and without node sharing. Let G_P = (V_P, E_P, f_{V_P}, f_{E_P}) and G_S = (V_S, E_S, f_{V_S}, f_{E_S}). The algorithm consists of three steps:

(a) It builds a bipartite graph $G_B = (N_L, N_R, E_B)$, where $N_L = V_P$, $N_R = V_S$ and $E = \{(u,u') \mid u \in V_P, u' \in V_S, f_{V_P}(u) \leq f_{V_S}(u')\}$.

(b) It finds a maximum bipartite matching of G_B (*i.e.,* an injective function f_B from N_L to N_R) in $O(|E_B|(|N_L|+|N_R|))$ time [18]. If such f_B does not exist, it returns 'NO', as one can verify that there is no VMP from the VN request to the SN in this case.

(c) Otherwise, it constructs a VMP from the VN request to the SN satisfying constraints specified by $\mathcal{C}$, such that $g_V(v) = f_B(v)$, $r_V(v,\ g_V(v)) = f_{V_P}(v)$ (for $\forall v \in V_P$). The calculated functions g_V and r_V satisfy constraints of form 1 for $f_{V_P}(v) = r_V(v, g_V(v))$, and those of form 2 since $\mathsf{sum}(N(u))$ is 0 or $f_{V_P}(v)$ (where $g_V(v) = f_B(v) = u$).

(3) To prove all cases with node sharing are NP-complete, we have only to show that the virtual machine placement with node sharing ($\mathsf{VMP_{(NS)}}$) is NP-complete, as the latter one is a special case of the other cases such as $\mathsf{VNM_{P(NS)}}$, $\mathsf{VNM_{L(NS)}}$, $\mathsf{VNE_{SP(NS)}}$ and $\mathsf{VNE_{MP(NS)}}$, when edge constraints are absent.

We conduct a reduction from the Bin Packing problem. Recall that the instance of Bin Packing problem consists of a finite set $U = \{u_1,\ u_2,\ \ldots,\ u_n\}$ of items, a size $s(u) \in Z^+$ for each $u \in U$, a positive integer bin capacity B and a positive integer K which denotes the number of bins. The Bin Packing problem is to ask whether there is a partition of U in to disjoint sets U_1, $\ldots$, U_k such that the sum of the sizes of the items in each U_i is B or less. We called a partition of U a valid partition, if the constraints are satisfied, *i.e.,* items can be partitioned and packed by the K bins. Given an arbitrary instance $\mathsf{I_B}$ of the Bin Packing problem, we construct an instance $\mathsf{I_M}$ of $\mathsf{VMP_{(NS)}}$ such that there exists a valid partition of U in $\mathsf{I_B}$ for instance $\mathsf{I_B}$ iff there exists an mapping for $\mathsf{I_M}$. We give constructions of $\mathsf{I_M}$ below:

(a) VN request $G_P = (V_P,\, E_P,\, f_{V_P},\, f_{E_P})$ and $\mathcal{C}$:

V_P : $\{v_1,\, v_2,\, \ldots,\, v_n\}$;

E_P : $\varnothing$;

f_{V_P} : $\forall i \in \mathbb{N}^+, i \leq n,\; f_{V_P}(v_i) = s(u_i)$;

f_{E_P} : $\varnothing$;

$\mathcal{C}$: Constraints of form 1 and form 2;

(b) SN $G_S = (V_S,\, E_S,\, f_{V_S},\, f_{E_S})$:

V_S : $\{v'_1,\, v'_2,\, \ldots,\, v'_K\}$

E_S : $\varnothing$;

f_{V_S} : $\forall j \in \mathbb{N}^+, j \leq K,\; f_{V_P}(v_j) = B$;

f_{E_S} : $\varnothing$;

It is obviously that the above construction of $\mathsf{I_M}$ can be finished in PTIME, and there is a valid partition of U of instance $\mathsf{I_B}$, iff there is a VMP of instance $\mathsf{I_M}$, from the VN request to the SN with satisfying the constraints.

For one thing, if there is a valid partition of U in $\mathsf{I_B}$, there must be a VMP for $\mathsf{I_M}$. We present the VMP (g_V, r_V) as follows:

g_V : $\forall i \in \mathbb{N}^+, i \leq n$, if u_i is partitioned into U'_k, then $g_V(v_i) = v'_{k'}$;

r_V : $\forall i \in \{1, 2, \cdots, n\}$, $r_V(v_i) = f_{V_P}(v)$;

As in $\mathsf{I_B}$, $\sum\limits_{\forall i, u_i \to U_{k'}} \{s(u_i)\} \leq B$, thus the VMP for $\mathsf{I_M}$ satisfies the constraints of form 1 and 2.

For another thing, if there is a VMP for instance $\mathsf{I_M}$, *i.e.,* there exists a node mapping (g_V, r_V) from the VN to the SN constructed above and satisfies the constraints, then there must exists a valid partition of U in instance $\mathsf{I_B}$. We present the partition as follows: $\forall i \in \{1, 2, \cdots, n\}$, if $g_V(v_i) = v'_{k'}$ ($k' \in \{1, 2, \cdots, K\}$), we partition the item u_i to the set $U_{k'}$ which will be packed by the k'th bin. It is obviously that the partition of U is a valid partition for $\mathsf{I_B}$ for a substrate node to which some virtual nodes are mapped must have enough capacity to accommadate those virtual nodes, which leads the validity of the partition for $\mathsf{I_B}$.

Thus VMP with node sharing is NP-hard as Bin Packing is NP-complete [32]. Because it is already proved in NP, it is a NP-complete problem.

(4) We next prove that $\mathsf{VNM_{SP}}$, $\mathsf{VNM_L}$ and $\mathsf{VNM_P}$ are NP-complete problem by reducing from PARTITION, Subgraph Isomorphism and X3C problems respectively.

(a) We first show that the $\mathsf{VNE_{SP}}$ is NP-complete by reducing from PARTITION, which is a NP-complete problem [32].

The PARTITION problem is to judge, when given a finite set C and a "size" $s(a) \in \mathbb{Z}^+$ for each $a \in C$, whether there exists a subset $C' \subseteq C$ such that $\sum\limits_{a \in C'} s(a) = \sum\limits_{a \in C - C'} s(a)$. Given an instance $\mathsf{I_P}$ of the PARTITION problem, *i.e.,* a finite set $C = \{a_1,\, a_2,\, \ldots,\, a_n\}$ and weight function $s(a_i) \in \mathbb{Z}^+$, we construct an instance $\mathsf{I_M}$ of $\mathsf{VNM_{SP}}$, such that there exists a partition of $\mathsf{I_P}$ iff there is a mapping of $\mathsf{I_M}$. We present $\mathsf{I_M}$ corresponding to $\mathsf{I_P}$, as follows:

(1) The VN request (G_P, $\mathcal{C}$) in which $G_P = (V_P,\, E_P,\, f_{V_P},\, f_{E_P})$ is defined as follows:

V_P : $\{v_1,\, v_2,\, \ldots,\, v_n,\, T_P\}$;

E_P : $\{(T_P,\, v_i) \mid \forall i \in \mathbb{Z}^+, i \leq n\}$;

f_{V_P} : $\forall v_i \in V_P$, $f_{V_P}(v_i) = z_i$, $f_{V_P}(T_P) = z_{n+1}$, in which $\forall i, j \in \{1, 2, \cdots, n\}$, if $i > j$, then $z_i > z_j$ ($z_i,\, z_j \in \mathbb{Z}^+$);

f_{E_P} : $\forall i \in \{1, 2, \cdots, n\}$, $f_{V_P}(T_P, v_i) = s(a_i)$;

(2) The SN $G_S = (V_S,\, E_S,\, f_{V_S},\, f_{E_S})$ is defined as follows:

V_S : $\{u^L_1,\, u^L_2,\, \ldots,\, u^L_n,\, u^R_1,\, u^R_2,\, \ldots,\, u^R_n,\, T_S\}$;

E_S : $\{(u^L_{i+1},\, u^L_i) \mid \forall i \in \{1, \ldots, n-1\}\} \cup \{(T_S,\, u^L_n)\} \cup \{(T_S,\, u^R_n)\} \cup \{(s^R_{i+1},\, s^R_i) \mid \forall i \in \{1,\, 2,\, \ldots,\, n-1\}\}$;

f_{V_S} : $f_{V_S}(u^L_i) = f_{V_S}(u^R_i) = f_{V_P}(v_i) = z_i$ ($\forall i \in \{1,\, 2,\, \ldots,\, n\}$), $f_{V_S}(T_S) = z_{n+1}$;

f_{E_S} : $f_{E_S}(T_S, u^L_n) = f_{V_S}(T_S, u^R_n) = \frac{A_0}{2}$, in which $A_0 = \sum\limits_{i=1}^{n} s(a_i)$. For any other edge $e \in E_S$, $f_{E_S}(e) = +\infty$;

It is obviously that the above construction can be finished in PTIME. We now verify that instance $\mathsf{I_B}$ has a partition iff $\mathsf{I_M}$ has a single-path embedding (g_V, r_V, g_E, r_E).

In fact, if there is a partition of $\mathsf{I_B}$, then one can verify that the following mapping is a valid single-path embedding:

g_V : for each item $a_i \in C$, if $a_i \in C'$ in the partition of $\mathsf{I_B}$, then $g_V(a_i) = u^L_i$; otherwise $g_V(v_i) = u^R_i$;

r_V : for each $v \in V_P$, $r_V(v) = f_{V_P}(v)$;

g_E : $\forall i \in \{1, 2, \cdots, n\}$, $g_E(T_P, v_i) = (T_S,\, g_V(v_i))$;

r_E : for each edge (T_P, v_i) in E_P, $r_E((T_P,\, v_i),\, g_E(T_P,\, v_i)) = f_{E_S}(T_P, v_i)$;

For the other direction, when there exists a single-path embedding (g_V, r_V, g_E, r_E) for instance $\mathsf{I_M}$, one can verify that the following subset C' of C in instance $\mathsf{I_B}$ forms a valid partition: $\forall i \in \{1,\, 2,\, \ldots,\, n\}$, if $g_V(v_i) = u^L_i$, then $a_i \in C'$.

Thus the single-path embedding problem ($\mathsf{VNE_{SP}}$) is NP-hard. As we've already proved that it is in NP, thus $\mathsf{VNE_{SP}}$ is NP-complete.

(b) We then show that $\mathsf{VNM_L}$ is NP-complete. In fact, the Subgraph Isomorphism problem is a special case of

$\mathsf{VNM_L}$ when the latency requirements on virtual links and latency on physical links are all the same, *e.g.,* a constant a. As Subgraph Isomorphism is NP-hard [32], we get the result that $\mathsf{VNM_L}$ is NP-hard. Thus it NP-complete as we've already showed that it is in NP.

(c) We next conduct a reduction from the NP-complete problem X3C (.cf [32]) to show that $\mathsf{VNM_P}$ is NP-hard.

Recall that the instance I_C of the X3C problem is: given a finite set $S = \{x_1, x_2, \ldots, x_{3q}\}$, and a collection C of 3-element subsets of X, *i.e.,* $C = \{C_1, C_2, \ldots, C_n\}$ of which $C_i = \{x_{i1}, x_{i2}, x_{i3}\}$ ($\forall j \in \{1, 2, 3\}$, $x_{ij} \in S$). The X3C problem is to determine whether C contains an *exact cover* for X, *i.e.,* whether there exists a subset $C' \subseteq C$ such that every element x_i of X occurs in exactly one member of C'.

Given a instance $\mathsf{I_C}$ of the X3C problem, we now construct an instance of VNM_P problem, denoted by $\mathsf{I_P}$, *i.e.,* a VN and an SN, such that there is a mapping (g_V, r_V, g_E, r_E) of VN to SN with satisfying the priority mapping constraints of $\mathcal{C}$ iff there is an *exact cover* of $\mathsf{I_C}$. We present the constructed instance $\mathsf{I_P}$ by the following:

(1) VN $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$:

V_P : $\{v_{11}, v_{12}, v_{13}, v_{21}, \ldots, v_{q1}, v_{q2}, v_{q3}, v_1^C, v_2^C, \ldots, v_q^C\}$;

E_P : $\{(v_i^C, v_{ij}) \mid \forall i \in \{1, 2, \ldots, q\}, j \in \{1, 2, 3\}\}$;

f_{V_P} : $\forall v \in V_P$, $f_{V_P}(v) = 0$;

f_{E_P} : $\forall e \in E_P$, $f_{E_P}(e) = a$, in which a is a positive constant;

(2) SN $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$:

V_S : $\{u_{11}, u_{12}, u_{13}, u_{21}, \ldots, u_{q1}, u_{q2}, u_{q3}, u_1^C, u_2^C, \ldots, u_n^C\}$;

E_S : $\{(u_i^C, u_{i[1,2,3]} \mid \forall i \in \{1, 2, \ldots, q\})\}$;

f_{V_S} : $\forall u \in V_S$, $f_{V_S}(u) = +\infty$;

f_{E_S} : $\forall e \in E_S$, $f_{E_S}(e) = a$;

It is clear that the construction of $\mathsf{I_P}$ can be finished in PTIME. One can find that, v_i^C in V_P can be only mapped to one node in the set $\{u_j^C \mid \forall j \in \{1, 2, \ldots, n\}\}$, and v_{ik} in V_P can be only mapped to u_{jl} in V_S ($k, l \in \{1, 2, 3\}$). In fact, G_P encodes an *exact cover* of $\mathsf{I_C}$ and G_S encodes $\mathsf{I_C}$. We next show that there exists a priority mapping (g_V, r_V, g_E, r_E) iff there exits an *exact cover* of $\mathsf{I_C}$.

On one hand, if there is a priority mapping (g_V, r_V, g_E, r_E) of VN to the SN in instance $\mathsf{I_P}$, then we claim that $\{g_V(v_i^C) \mid \forall i \in \{1, 2, \cdots, q\}\}$ is an *exact cover* of $\mathsf{I_C}$. For if it is not an *exact cover*, we have $|\{g_V(v_{ik}^C) \mid \forall i \in \{1, 2, \cdots, q\}, k \in \{1, 2, 3\}\}| < |\{v_{ik}^C \mid \forall i \in \{1, 2, \cdots, q\}, k \in \{1, 2, 3\}\}|$. This is against the definition of the injection g_V.

On the other hand, if there is an *exact cover* of $\mathsf{I_C}$, we claim that the *exact cover* also gives an priority mapping (g_V, r_V, g_E, r_E) for instance $\mathsf{I_P}$. We use $S^C = \{v_{j_k}^C \mid k \in \{1, 2, \cdots, q\}$ and $j_k \in \{1, 2, \cdots, n\} \subseteq C = \{C_1, C_2, \ldots, C_n\}\}$ to denote the *exact cover* of I_C. Then we have the priority mapping as follows:

g_V : $\forall i \in \{1, 2, \cdots, q\}$, $k \in \{1, 2, 3\}$, $g_V(v_i^C) = u_{j_i}^C$, $g_V(v_{ik}) = u_{j_i k}$;

r_V : $\forall v \in V_P$, $r_V(v) = 0$;

g_E : $\forall (v_1, v_2) \in E_P$, $g_E(v_1, v_2) = (g_V(v_1), g_V(v_2))$;

r_E : $\forall e \in E_P$, $r_E(e) = f_{V_P}(e)$.

It is clear that (g_V, r_V, g_E, r_E) defines a priority mapping of VN on the SN of instance $\mathsf{I_P}$.

Thus the instance $\mathsf{I_P}$ of $\mathsf{VNM_P}$ has a priority mapping iff the instance $\mathsf{I_C}$, which is NP-complete [32], has an *exact cover*. This follows that the $\mathsf{VNM_P}$ is NP-hard. Along with the result that it is in NP, $\mathsf{VNM_P}$ is NP-complete.

□

Proof for theorem 2

Given a VN request $(G_P, \mathcal{C}$ in which $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$, an SN $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$ and a *cost function* $c : V_S \cup E_S \to \mathbb{Z}^+$ associates a positive number with each substrate node and link in G_S. We then prove results of theorem 2 as follows.

Proof:

(1) We first prove VMP without node sharing is in NP.

We provide an polynomial time (cubic-time) algorithm to find minimum cost VMP of a VN request on an SN, without node sharing and the edge constraints are absent. The algorithm consists three steps:

(a) Add $|V_S|$-$|V_P|$ meta nodes to the set V_P and denote the augmented virtual nodes set as V_P'. (If $|V_P| \geq |V_S|$, then return 'NO').

(b) Attach costs $x(u,v)$ to each node pair $(u,v) \in V_P' \times V_S$: when $u \in V_P$, $v \in V_S$ and $f_{V_P}(v) \leq f_{V_S}(u)$, $x(u,v) = \frac{f_{V_P}(u)}{f_{V_S}(v)} c(v)$; else $x(u,v) = \mathsf{M}$, where $\mathsf{M} = \sum_{\forall v \in V_S} (c(v))$.

(c) Assign the $|V_S|$ meter nodes and virtual nodes to the $|V_S|$ substrate nodes under the settings of the linear assignment problem, which aims to assign m object to another m object with minimizing the total assignment cost and can be solved in $O(|V_S|^3)$ time [33].

(d) Assume the the minimum assignment of the assignment problem in (c) is denoted by an injection f_B, if the total cost of assignment f_B is larger than $\mathsf{M}(|V_S| - |V_P| + 1)$, then outputs 'NO' for there is no VMP of the VN on SN, otherwise outputs the minimum VMP denoted by a node mapping (g_V, r_V) as: $g_V(v) = f_B(v)$ and $r_V(v) = f_{V_P}(v)$ $(\forall v \in V_P)$.

Thus the above algorithm finds a minimum VMP of a VN on an SN in $O(|V_S|^3)$-time if there exists one.

We next show that it becomes NP-complete and APX-hard to approximate when node sharing is requested.

The NP-complete result is obviously. First, as proved in theorem 1, it is NP-hard to decide whether there exists a VMP of a VN on an SN, thus it is NP-hard to decide whether there exists a VMP such that its cost is smaller than a constant K when consider costs on the substrate nodes and links. As it is in NP as stated in theorem 1, it is NP-complete.

We next prove that it is APX-hard to approximate. We conduct a L-reduction from minimum RGAP (restricted generalized assignment problem), which can be proved to be APX-hard by adopting the work about the GAP (generalized assignment problem) in [14, 37], to verify that the minimum cost for $\mathsf{VMP}_{(NS)}$ is APX-hard.

We give the description of minimum GAP as follows. Given a pair $(\mathcal{B}, \mathcal{S})$ where $\mathcal{B} = \{b_1, b_2, \cdots, b_m\}$ is a set of m bins and $\mathcal{S} = \{a_1, a_2, \cdots, a_n\}$ is a set of n items. Each bin $b_j \in \mathcal{B}$ has a capacity $c_B(b_j)$, and for each item a and bin b, we are given a size $s(a, b)$ and a budget $w(a, b)$ specifying the capacity requirement of items and the cost of packing a by bin b. The minimum GAP is to find a feasible packing for $\mathcal{S}$ in $\mathcal{B}$ and minimize the budget of the packing while packing. Here the budget of a packing is the sum of the budget of each item and consider instances that possess a feasible packing. RGAP is a restricted version of GAP satisfying:

(a) $\forall i \le m, j \le n$, $\frac{w(a_i,b_j)}{s(a_i,b_j)}$ depends only on j; (We use $w_s(b_j)$ to denote $\frac{w(a_i,b_j)}{s(a_i,b_j)}$.)

(b) $\forall i, j \le n$, $a \in \mathcal{S}$, $s(a, b_i) = s(a, b_j)$. (We use $s(a)$ to denote $s(a, b_i)$.)

We below L-reduce from RGAP to prove that $\mathsf{VMP}_{(\mathsf{NS})}$ is APX-hard.

Recall a L-reduction from A and B (two optimization problems) consists of two polynomial time computable functions R and S such that:

i) (R: $I_A \to I_B$) If $x \in I_A$ (set of instances of A), and $OPT(x)$ is the measurement of the optimum solution of instance x, then $R(x)$ is an instance of B with its measurement of the optimum solution satisfing $OPT(R(x)) \le \alpha OPT(x)$, in which α is a positive constant;

ii) (S: $SOL_A(R(x)) \to SOL_B(x)$) If s is any feasible solution of $R(x)$, then $S(s)$ is a feasible solution of x such that $|OPT(x) - c(S(s))| \le \beta|OPT(R(x)) - c(s)|$, in which c denotes the measurement of instances and β is a positive constant.

The key property of L-reduction is that it preserves approximability: If there is a L-reduction (R, S) from A to B with constance α and β, and there is polynomial time ϵ-approximation algorithm for B ($\epsilon > 1$), then there is a PTIME $\alpha\beta(\epsilon - 1)$-approximation algorithm for A. We'll use the contrapositive proposition to prove approximation-hardness.

Then we present the L-reduction (R, S, α, β) from minimum RGAP to minimum cost mapping for $\mathsf{VMP_P}$ as follows:

(a) R: $I_{\mathsf{RGAP}} \to I_{\mathsf{VMP}}$. We present function R by construct an instance of $\mathsf{VMP}_{(\mathsf{NS})}$ corresponding to an arbitrary instance of RGAP in the form above:

VN : $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$
- $V_P = \{v_1, \ldots, v_n\}$;
- $f_{V_P}(v_i) = s(a_i)$ ($\forall i \le n$);
- $E_P = \varnothing$; $f_{E_P} = \mathsf{NULL}$;

SN : $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$
- $V_S = \{u_1, u_2, \ldots, u_m\}$
- $f_{V_S}(u_i) = c_B(b_i)$ ($\forall i \in \{1, 2, \cdots, m\}$), ;
- $E_S = \varnothing$; $f_{E_P} = \mathsf{NULL}$.

c : $c(u_i) = w_s(b_i) \cdot c_B(b_i)$ ($\forall b_i \in \mathcal{B}$);

(b) S: $SOL_{\mathsf{VMP}}(f(I_{P_A})) \to SOL_{P_A}(I_{P_A})$. We define the function S by the following. For any feasible solution (g'_V, r'_V), function S derives a packing for instance of RGAP by packing item a_i to bin b_j if $g'_V(v_i) = u_j$ ($\forall i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$);

Intuitively, R and S constructs a bijection from instance of minimum RGAP to instance of minimum cost mapping for VMP. Moreover, the cost of mapping a node of VN to the SN is equal to the budge of packing the corresponding item to the corresponding bin. Specifically, leveraging the additional property of RGAP, the cost of mapping node v_i in V_P is mapped to u_j in V_S is $c_M(v_i) = \frac{f_{V_P}(v_i)}{f_{V_S}(u_j)} \cdot c(u_j) = \frac{s(a_i)}{c_B(b_j)} \cdot w_s(b_j) c_B(b_j)$ $= w(a_i, b_j)$, *i.e.*, the cost of the node mapping in the instance of minimum cost mapping for VMP is equal to the budget of bin packing in the instance of minimum RGAP. Thus the above reduction ensures the following:

1. function R asserts that for each feasible solution of I_{RGAP}, there is a corresponding feasible solution for I_{VMP}; Moreover, the optimum solution of I_{RGAP} corresponds to the optimum solution of I_{VMP} by f;
2. function S ensures that for each feasible solution of I_{VMP}, there is a corresponding feasible solution for I_{RGAP}. If y is the optimum (minimum) solution of $f(I_{\mathsf{VMP}})$, then $S(y)$ is the optimum (maximum) solution of I_{RGAP};
3. For each feasible solution s_1 of I_{RGAP}, the budget of s_1 is equal to the mapping cost of the corresponding feasible solution s_2 for I_{VMP}.

Thus R and S satisfy the constraints of the L-reduction with α=β=1. As minimum RGAP is APX-hard, so is the minimum cost mapping for $\mathsf{VMP_P}$.

(2) We first show the NP-compete results of minimum cost mapping with edge constraints by reduction from

X3C problem. Then we propose an AP-reduction from minimum weight 3-SAT problem to show the NPO-completeness of those optimization problems.

As showed in theorem 1, it is NP-hard to deciding whether there exists a VNM with edge constraints, thus it is also NP-hard to deciding whether ethere exists a VNM with bounded weight (decision version of the minimum cost mapping problem). Besides, it is in NP for the NP algorithm for deciding whethere there exits a VNM with edge constraints, when given a VN and an SN, can be naturally adopted here as the checking of the weight of a VNM can be finished in polynomial time. Thus the minimum cost mapping problem is NP-complete for VNM_P, VNE_{SP}, VNE_{MP}, VNM_L, $\mathsf{VNM}_{P(NS)}$, $\mathsf{VNE}_{MP(NS)}$ and $\mathsf{VNM}_{L(NS)}$, which are all VNM cases with edge constraints.

Moreover, we next show that minimum cost mapping problem is NPO-complete in the sense of approximation-hardnss, which means that there exists no polynomial time approximation algorithm when consider the edge constraints. We verify it by conducting a AP-reduction from the MW-3SAT (short for minimum weight 3SAT) problem which is a known NPO-complete problem.

First, recall that an instance ϕ of MW-3SAT is of the form $C_1 \wedge C_2 \cdots \wedge C_n$ where all variables in ϕ are x_1, ..., x_m with nonnegative weights w_1, w_2, ..., w_n, each clause C_j ($j \in [1, n]$) is a Boolean formula of the form $y_{j_1} \vee y_{j_2} \vee y_{j_3}$, and moreover, for $i \in [1, 3]$, y_{j_i} is either $x_{p_{ji}}$ or $\overline{x_{p_{ji}}}$ for $p_{ji} \in [1, m]$, in which $x_{p_{ji}}$ denotes the occurrence of a variable in the literal i of clause C_j. A solution is a truth assignment τ t othe variables $x_i (i \in [1, m])$ that satisfies ϕ. The MW-3SAT problem is to find the minimum solution, *i.e.*, minimize $\sum_{\forall i \in [1,m]} x_i \cdot w(x_i)$ where the Boolean values TRUE and FALSE are identified with 1 and 0, respectively.

We next briefly introduce the AP-reduction. Given two optimization problems P_1 and P_2 in NPO, recall that an AP-reduction consists of two functions f and g, and a positive constant $\alpha > 1$ subject to the following constraints (We use I_P to denote the set of all instances of NPO problem P, use $SOL_P(x)$ to denote the set of feasible solutions of instance x of P, use $\mathsf{R}_P(x, s)$ to denote the relative approximation factor of solution s of instance x, *i.e.*, $\max(\frac{m(x,s)}{OPT(x)}, \frac{OPT(x)}{m(x,s)})$ where $m(x, s)$ denotes the value of feasible solution s of instance x and $OPT(x)$ denotes the value of optimum value of instance x.) :

1. For any instance $x \in I_{P_1}$ and any rational $r > 1$, $R(x, r) \in I_{P_2}$.
2. For any instance $x \in I_{P_1}$ and for any rational $r > 1$, if $SOL_{P_1}(x) \neq \varnothing$, then $SOL_{P_2}(R(x, r)) \neq \varnothing$.
3. For any instance $x \in I_{P_1}$, for any rational $r > 1$, and for any $y \in SOL_{P_2}(R(x, r))$, $S(x, y, r) \in SOL_{P_1}(x)$.
4. R and S are computable in time polynomial for any fixed rational r.
5. For any instance $x \in I_{P_1}$, for any rational $r > 1$, and for any $y \in SOL_{P_2}(R(x, r))$, $R_{P_2}(R(x, r), y) \leq r$ implies $R_{P_1}(x, S(x, y, r)) \leq 1 + \alpha(r - 1)$.

We now construct an AP-reduction from the MW-3SAT problem to the minimum cost mapping problem with edge constraints.

(I) function S. Given an instance ϕ of the MW-3SAT problem, we construct VN and SN as follows:

VN: $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$ is defined as follows.
- $V_P = \{C_1^P, \cdots, C_n^P, X_1, \cdots, X_m^P\}$;
- $E_P = \{(X_{p_{j1}}^P, C_j), (X_{p_{j2}}^P, C_j), (X_{p_{j3}}^P, C_j) \mid \forall i \in [1, m], \forall j \in [1, n]\}$;
- $f_{V_P} : V_P \to \{0\}$, *i.e.*, there is node capacity constraints on virtual nodes of the VN;
- f_{E_P}: For each edge $e \in E_P$, $f_{E_P}(e) = a$ where a is a constant.

SN: $G_S = (V_S, E_S, f_{V_S}, f_{E_S})$ is defined as follows.
- $V_S = \{X_{T1}^S, X_{F1}^S, \cdots, X_{Tm}^S, X_{Fm}^S, 0_1, \cdots, 7_1, \cdots, 0_n, \cdots, 7_n\}$;
- E_S contains 7×3 edges for each clause $C_j = y_{j_1} \vee y_{j_2} \vee y_{j_3}$ of ϕ ($j \in [1, n]$), and there are in total $21n$ edges in E_S. We represent the truth assignments of clause C_j in terms of 8 nodes $C_j(\rho)$, where ρ ranges over all truth assignment of variables $x_{p_{ji}} (i = 1, 2, 3)$. Each node $C_j^S(\rho)$ is a three-bit constant $y_{j_1} y_{j_2} y_{j_3}$. For each truth assignment ρ of $x_{p_{j1}}$, $x_{p_{j2}}$ and $x_{p_{j3}}$ that maks C_j true, E_P consists of the following edges: $(X_{Tp_{jk}}, C_j^S(\rho))$ if $\rho(X_{p_{jk}}) = \mathsf{true}$, or $(X_{Fp_{jk}}, C_j^S(\rho))$ if $\rho(X_{p_{jk}}) = \mathsf{false}$, where $k \in [1, 3]$.
- $f_{V_S} : V_S \to +\infty$;
- f_{E_S}: For each edge $e \in E_S$, $f_{E_S}(e) = a$.

c: For each edge $e \in E_S$, if $e = (X_{Tj}^S, C_j^R(\rho))$ for some $j \in [1, m]$, then $f_{E_S}(e) = w(x_j)$, otherwise $(e = (X_{Fj}^S, C_j^R(\rho)))$ $f_{E_S}(e) = 0$. For each node $v \in V_P$, $c(v) = 0$.

Intuitively, VN encodes the instance ϕ of MW-3SAT. Node $X_i^P (i \in [1, m])$ denotes variable x_i, and node $C_j^P (j \in [1, n])$ denotes clause C_j^P. Edge (X_i^P, C_j^P) encodes that variable x_i apprears in clause C_j, *i.e.*, x_i is one of the three variables $x_{p_{ji}} (i \in [1, 3])$. SN encodes the truth assignments of the variables that satisfy thaclauses in the instance ϕ of MW-3SAT. Node X_{Ti}^S ($i \in [1, m]$, resp. X_{Fi}^S) means assigning variable x_i a true (resp. false) value. Nodes $\{0_j, \cdots, 7_j\}$ denotes $C_j(\rho)$, which are denoted as a three-bit constant w.r.t the truth assignments of the three variables in clause C_j.

The above construction of R guarantees that (a) node $X_i^P (i \in [1, m])$ in VN is mapped to either node

X^S_{Ti}(true) or X^S_{Fi}(false) of G_2 and (b) node C^P_j ($j \in [1, n]$) in VN is mapped to one of the nodes $\{0_j, \dots, 7_j\}$ of SN.

(II) function S: We next present function S which maps a solution of instance $R(x)$ for the MW-3SAT problem to a solution of instance x for the minimum cost mapping problem with edge constraints.

A solution s of instance $R(x)$ for the MW-3SAT problem is a mapping (g_V, g_E, r_V, r_E) from the VN to SN constructed above. We define a truth assignment for instance x with respect to the mapping as follows. For each variable x_i ($i \in [1, m]$), $\rho(x_i) = \mathsf{true}$ if $g_V(X^P_i) = X^S_{Ti}$ and $\rho(x_i) = \mathsf{false}$ if $g_V(X^R_i) = X^S_{Fi}$.

(III) α: The constant α here is defined to be 1.

We below verify that (R, S, α) defines an AP-reduction from the NPO-complete problem MW-3SAT to the minimum cost mapping problem with edge constraints.

1). It is obviously that the functions R and S, *i.e.,* the above constructions, are finished in polynomial time.
2). For any instance ϕ of MW-3SAT, $R(\phi)$ is an instance of the minimum cost mapping problem with edge constraints (*i.e.,* a VN and SN, with additional cost and constraints on edges) as constructed above.
3). For any instance ϕ of MW-3SAT, if $SOL_{\mathsf{MW-3SAT}}(\phi) \neq \varnothing$, *i.e.,* there is a truth assignment ρ that makes ϕ true, then $SOL_{\mathsf{VNM}}(R(\phi)) \neq \varnothing$. In fact we can present a valid VNM (g_V, g_E, r_V, r_E) for $R(\phi)$ as follows: (1) for each $i \in [1, m]$, $g_V(X^R_i) = X^S_{Ti}$ if $\rho(x_i) = true$, and $g_V(X^R_i) = X^S_{Fi}$ if $\rho(x_i) = false$, (2) for each $j \in [1, n]$, $g_V(C^R_j) = C^S_j(\rho)$ defined as above, (3) for each edge $e = (u, v)$ in VN, $g_E(e)$ is defined as the unique path (edge) in SN from $g_V(u)$ to $g_V(v)$, (4) for each node $v \in V_P$, $r_V(v, g_V(v)) = f_{V_P}(v)$, and (5) for each edge $e \in E_P$, $r_E(e, g_E(e)) = f_{E_P}(e)$. It is easy to verify that the VNM (g_V, r_V, r_V, r_E) is a feasible VNM from VN to SN.
4). As the SN constructed by R encodes all the truth assignments of instance of ϕ for the MW-3SAT problem, function S indeed transfors solution of $R(\phi)$ to solution of ϕ, which means $S(s) \in SOL_{\mathsf{MW-3SAT}}(\phi)$.
5). From the definition of cost function for SN, one can verify that (1) for any instance ϕ of MW-3SAT, if ρ is the optimum solution for ϕ (*i.e.,* minimum weight truth assignment), then the weight of the ρ is equal to the optimum solution of $R(\phi)$, *i.e.,* the minimum cost mapping, and (2) for any feasible solution s of $R(\phi)$, the mapping cost of s is equal to the weight of the truth assignment of $S(s)$.

With the two observations, for any instance ϕ of MW-3SAT, for any feasible mapping s of $R(\phi)$, $\mathsf{R}_{\mathsf{VNM}}(R(\phi), s) = \mathsf{R}_{\mathsf{MW-3SAT}}(\phi, S(s))$.

Thus (R, S, 1) forms an AP-reduction from MW-3SAT to the minimum cost mapping problem with edge constraints. As MW-3SAT problem is NPO-complete and VNM is in NP, minimum cost mapping problem with edge constraints is NPO-complete for cases VNM_P, VNE_{SP}, VNE_{MP}, VNM_L, $\mathsf{VNM}_{P(NS)}$, $\mathsf{VNE}_{MP(NS)}$ and $\mathsf{VNM}_{L(NS)}$,

(3) The APX-hard result is obvious as finding the minimum cost mapping with determined node mapping contains the minimum Steiner Tree problem as a special case, which naturally implies a L-reduction. As the minimum Steiner Tree is APX-hard, so is the minimum cost mapping problem for this special case. More specifically, by conducting L-reductions from the minimum directed steiner tree [13] problem to the minimum cost mapping with determined node mapping for $\mathsf{VNM_P}$, we show get an approximation bound for the latter problem.

We next show the approximation-hardness of VNM_P with determined node mapping by a L-reduction from the minimum DST (directed steiner tree) problem.

Recall that an instance I_{DST} consists of a directed graph $G_D = (V_D, E_D)$ (assume $V_D = \{v_1, v_2, ..., v_m\}$, $E_D = \{e_1, e_2, ..., e_n\}$), a root node $r \in V_D = \{v_1, v_2, ..., v_m\}$, a set of terminal nodes $X = \{t_1, t_2, ..., t_k\}$, and a cost function w that attaches a positive number to each of the edges in E_D. The minimum DST problem is to find the minimum weight arborescence rooted at r and spanning all the vertices in X, *i.e.,* r should have a path to every vertex in X.

The L-reduction from the minimum DST problem to the minimum cost mapping problem for VNM_P with node mapping determined consists of function R, function S and two positive constant α, β defined as follows:

I) function R: R maps instance of DST to instance of minimum cost mapping problem for $\mathsf{VNM_P}$ with node mapping uniquely determined. Given any arbitrary instance I_{DST} of the minimum DST problem, we present R by constructing corresponding instance $I_{\mathsf{VNM_P}}$ of minimum cost mapping problem for VNM_P with node mapping uniquely determined, *i.e.,* a VN, an SN, and a cost function c of $I_{\mathsf{VNM_P}}$.

1) VN: $G_P = (V_P, E_P, f_{V_P}, f_{E_P})$, in which
   - $V_P = \{r^P, t^P_1, ..., t^P_k\}$;
   - $E_P = \{(r^P, t^P_1), (r^P, t^P_2), ..., (r^P, t^P_k)\}$;
   - f_{V_P}: $f_{V_P}(r^P) = 1$, $f_{V_P}(t^P_i) = i$ ($\forall i \in [1, k]$);
   - f_{E_P}: $\forall e \in E_P$, $f_{E_P}(e) = a$, a is a constant;
2) SN: $G_S = (V_S, E_P, f_{V_S}, f_{E_S})$, in which
   - $V_S = \{v^S_1, v^S_2, ..., v^S_m\}$, in which $r^S \in V_S$ and $\{t^S_1, t^S_2, ..., t^S_k\} \subseteq V_S$;

- $E_S = \{e_1^S, e_2^S, ..., e_n^S\}$;
- f_{V_S}: $f_{V_S}(r^S) = 1$, $f_{V_S}(t_i^S) = i$ ($\forall i \in [1,k]$);
- f_{E_P}: $\forall e \in E_S$, $f_{E_S}(e) = a$;

3) c: $\forall i \in [1,n]$, $c(e_i^S) = w(e_i)$.

II) function S: For any instance I_{DST}, $R(I_{\mathsf{DST}})$ is an instance of the minimum cost mapping problem for $\mathsf{VNM_P}$ with node mapping uniquely determined. Here function S maps an arbitrary feasible solution s of instance $R(I_{\mathsf{DST}})$ to a feasible solution of instance I_{DST}.

For any feasible solution s of $R(I_{\mathsf{DST}})$, *i.e.,* an mapping (g_V, g_E, r_V, r_E) in which g_V and r_V are uniquely determined, we present S by constructing the corresponding feasible solution of I_{DST} as follows: for any $e = (r^P, t_i^P) \in E_P$ ($\forall i \in [1,k]$), if $g_E(e) = \rho_e$ in which $\rho_e = \{r^S, v_{p_1}^S, ..., v_{p_q}^S, t_i^S\}$ is a path in VN from r^S to t_i^S, then for instance I_{DST}, the path connecting root node r to terminal node t_i is the path (r, v_{p_1}, ..., v_{p_q}, t_i).

III) positive number α and β:

1). $\alpha = 1$;

2). $\beta = 1$;

We next verify that (R, S, α, β) forms a L-reduction from minimum DST problem to minimum cost mapping problem for $\mathsf{VNM_P}$ with node mapping uniquely determined.

(i). From the construction above, we know that R and S are polynomial time computable.

(ii). For any instance I_{DST} of the minimum DST problem, the weight of the optimum solution of $\mathsf{OPT}(I_{\mathsf{DST}})$ is equal to cost of the optimum solution (*i.e.,* $\mathsf{VNM_P}$) of instance $R(I_{\mathsf{DST}})$ for a) As edges of G_D and SN are one-to-one correspondent, along with the cost (weight) on them, and b) the uniquely determined node mapping g_V in the instance constructed by R ensures that SN encodes G_D and VN encodes r and the terminal nodes set X of the instance I_{EDP}. Thus the restriction i) in the definition of L-reduction established with $\alpha = 1$;

(iii). Similar to (ii), if s is any feasible solution of $R(I_{\mathsf{DST}})$, one can easily verify that $S(s)$ is a feasible solution of I_{DST} such that $|OPT(x) - w(S(s))| = |OPT(R(x)) - c(s)|$, which admits the restriction ii) in the definition of L-reduction with $\beta = 1$; In fact, here $w(S(s)) = c(s)$.

As it is NP-hard to approximate minimum DST problem to a factor better than $\ln k$ where the k is the number of the terminals unless P $\neq$ NP, the minimum cost mapping problem for $\mathsf{VNM_P}$ does not admit $\ln|V_P|$-approximation algorithms even with node mapping uniquely determined. □

Appendix B: Algorithms

Proof for proposition 3

This is intuitive. One can observe the followings.

1) The auxiliary graph G_{aux} used by compVNM records the maximum bandwidth between any two nodes in SN.
2) For any node mapping (g_V, r_V), compVNM determines whether it admits a edge mapping by querying the maximum bandwidth between matched nodes from G_{aux}. This takes $O(|E_P|)$ time as there are at most $|E_P|$ pairs of nodes that requires edge mapping checks.
3) For any node mapping (g_V, r_V) found by compVNM, there exists a valid compatible edge mapping (g_E, r_E).

Proof for lemma 4

Proof: In order the show the correctness of lemma 4, we first present the *augmentation condition* that will be used in the proof.

Augmentation Condition. *In the auxiliary graph constructed by* compAuxGraph*, for any two nodes* u *and* v*, if there exists an*

This condition is necessary for the following two reasons. a) If the bandwidth of any two sides of a triangle are different, then the least one (*e.g.,* carried by edge (u, v)) should be update to the lesser one of the remaining two sides (*e.g.,* (u, w) and (w, v)), for $u \to w \to v$ forms a valid path connecting u and v while possessing larger bandwidth than edge (u, v). b) If the bandwidth carried by the equal sides (*e.g.,* (u, w) and (w, v)) is larger than that carried by the remaining side (*e.g.,* (u, v)), then it turns out that the latter should be updated to that larger bandwidth carried by the equal sides.

The proof consists of three blocks.

(a) *Necessary edges to be considered when updating the newly introduced edges.* It is obviously that the new edges, which are introduced by connecting the new node m to existing partial auxiliary graph G_{aux}, may not carry the largest bandwidth. To update their bandwidth, one only needs to reconfigure the bandwidths on the new edges such that any triangles containing those new edges satisfy the augmentation condition. As those triangles must contain the new node m, those newly introduced edges that connecting m the the partial constructed auxiliary graph G_{aux} is necessary to be considered.

(b) *Necessary and sufficient edges to be considered when updating existing edges in G_{aux} after bandwidths on those new edges involved with m are up to date.* After the bandwidths on the newly introduced edges are updated, those on the old edges in G_{aux} may be outdated. We next show that to update the bandwidth carried on any old edge (i', j') in G_{aux}, one only needs to concern about the triangle that consisting of i', j' and m.

Obviously, it is necessary to check whether the triangle that containing nodes i', j' and m satisfies the augmentation condition, as bandwidths on (m, i') (or (i', m)) may be changed. We then show that after updating those triangles consisting of the new node m and old edges in G_{aux}, all triangles in G_{aux} already meet the augmentation condition. Consider an arbitrary triangle consisting of three nodes i', j' and k' in G_{aux}. For simplicity, we consider undirected graphs here, things remains similar for directed graph. We use $\mathsf{b}(i', j')$ (or $\mathsf{b}(e)$) to denote the bandwidth on edge (i', j') (or e). After updating the bandwidth on edges (m, i'), (m, j') and (m, k'), $\mathsf{b}(i', j') = \min\{\mathsf{b}(m', i'), \mathsf{b}(m', j')\}$, $\mathsf{b}(j', k') = \min\{\mathsf{b}(m', j'), \mathsf{b}(m', k')\}$, $\mathsf{b}(k', i') = \min\{\mathsf{b}(m', k'), \mathsf{b}(m', i')\}$. Without lose of generality, we assume that $\mathsf{b}(m', i') \geq \mathsf{b}(m', j') \geq \mathsf{b}(m', k')$. Then $\mathsf{b}(i', j') = \mathsf{b}(m', j')$, $\mathsf{b}(j', k') = \mathsf{b}(m', k')$, $\mathsf{b}(k', i') = \mathsf{b}(m', k')$. Thus the triangle consisting of nodes i', j' and k' in G^{Aug} satisfies the augmentation condition. This follows that there is no need to update the old edges in G_{aux} after checking and updating all triangles consisting of nodes i, j and m (for any nodes i and j in G_{aux}).

(c) *Sufficient edges to be consider for the newly introduced edges.* With (b), we learn that all triangles in G_{aux} along with the newly introduced node m and associated edges already satisfy the augmentation condition, revealing that those edges concerned in (a) when updating edges connecting m to G_{aux} are sufficient.

From (a) and (c), conjecture (1) in lemma 4 is established. The correctness of conjecture (2) is ensured by (b). □

Proof for theorem 5

Proof: Consider the scenario that adding the new node v to the subgraph G^k of the VN G_P. We denote the auxiliary graph of G^k by G^k_{aux}.

(1) We first show that the updates of existing edges in G^k_{aux} will not affect the determination of the weight and recorded path on the new edge (u, v) (or (v, u)). In fact, existing edge (u, u') needs to be updated if and only if the newly introduced edges (u, v) and (v, u') form into a directed path connecting u and u' and moreover, the weight (bandwidth) carried this path is larger than that on existing edge (u, u'). This is because for any edge (u, u') in G^k_{aux}, it carries the maximum weight (bandwidth) from u to u' in G^k. Once the updates of (u, u') *w.r.t.* (u, v) and (v, u') finished, along with the conjecture (2) of the lemma (For any existing edge (u, u'), it suffices to consider the triangle with edges (u, u'), (u', v) and (v, u) for updating its weights and path), the new edge (u, v) need not to be updated. We next verify the conjecture (2) of the lemma (without leveraging conjecture (1)).

(2) Consider three nodes in G^k_{aux}, u_1, u_2 and u_3. We use $w(e)$ and $w(\rho)$ to denote the weight (bandwidth) of an edge e and a path ρ. After the new edges connecting v and u_1, u_2 and u_3 are updated, $w(u_1, u_2) = \min\{w(u_1, v), w(v, u_2)\}$, $w(u_2, u_3) = \min\{w(u_2, v), w(v, u_3)\}$, $w(u_1, u_3) = \min\{w(u_1, v), w(v, u_3)\}$. We next show that $w(u_1, u_3) \geq \min\{w(u_1, u_2), w(u_2, u_3)\}$. This is intuitive as $\min\{w(u_1, u_2), w(u_2, u_3)\} = \min\{\min\{w(u_1, v), w(v, u_2)\}, \min\{w(u_2, v), w(v, u_3)\}\} = \min\{w(u_1, v), w(v, u_2), w(u_2, v), w(v, u_3)\} \leq \min\{w(u_1, v), w(v, u_3)\}$. Thus it suffices to consider the triangle with edges (u, u'), (u, v) and (v, u'), in order to update the weight and path of the existing edge (u, u').

From (1) and (2), we have the establishment of lemma 5. □

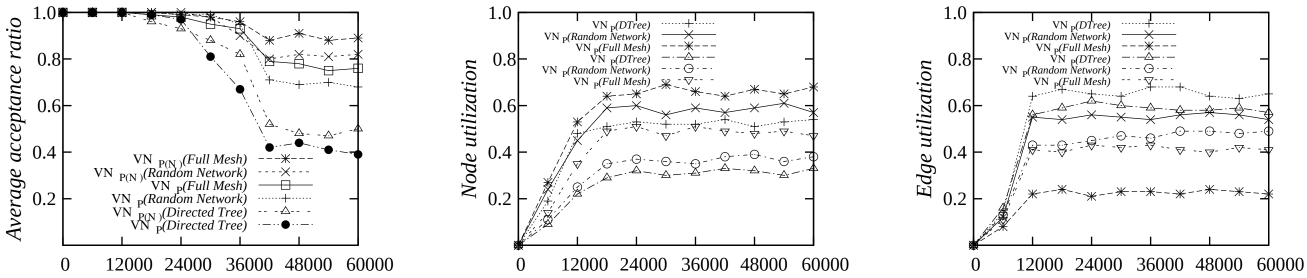
Proof for lemma 6

Proof: (1) First we show that if there exists a path from nodes u to v in VN G_P, then there must exists a unique path from nodes u to v in G^m_P, which is returned by $\mathsf{UpdateVN}$. From the definition of auxiliary graph, we know that there must exits an edge (u, v) in G'_P. From line 3 and line 5 of $\mathsf{UpdateVN}$, one can find that if (u, v) is in G'_P, then there must be a path that carries the same bandwidth in G^m_P. This path is either the edge (u, v) in G^m_P, or a path that transits a intermediate node u', as stated in line 4 and line 6. This verifies the existence of a path from nodes u to v to G^m_P. We need to show the uniqueness of the path by the following. Once the path that connecting u to v is added to G^m_P at some round of the for blocks in line 2 in $\mathsf{UpdateVN}$, paths that from u to v will not introduced as in line 3 and line 5, $\mathsf{UpdateVN}$ finds that there exits u' such that (u, u') is in G'_P and (u', v) is in G^m_P. Thus the path from u to v is unique.

(2) We next show the reverse direction. If there exits a path from u to v in G^m_P, then it must be introduced in some round of the for block in $\mathsf{UpdateVN}$ invoked with parameter u. As there exits a path connecting u to v in $\mathsf{UpdateVN}$, thus the edge (u, v) must be included in G'_P, which means that there is a path connecting u to v in VN G_P (following the definition of auxiliary graph for VN). □

Appendix C: Additional Experiments

We conduct two sets of additional experiments here, to study the impacts of node sharing and resource utilization over time. The settings are the same to the first set of experiments in mapping quality (**Exp-1**), *i.e.,* fixing $p_P = 0.5$, n_P in [2, 50], w_{V_P} and w_{E_P} in [3, 30] for VN requests, $n_S = 5000$, w_{V_S} and w_{E_S} in [50, 100] for SNs,



(a) Vary time t from 0 to 60000 (b) Vary time t from 0 to 60000 (c) Vary time t from 0 to 60000

Figure 10: Additional experiments

and varying t from 0 to 60,000. Results are shown in Fig. 10.

(1) In the first additional experiment, we evaluate the impacts of node sharing on $\mathsf{VNE_{SP}}$. Figure 10(a) shows the AR of $\mathsf{VNE_{SP}}$, with node sharing or not, over directed tree, full mesh and random networks. The results reveals that node sharing consistently improves the AR for $\mathsf{VNE_{SP}}$ (in the range of [12%, 15%]). Indeed, the AR over directed tree is over 53%, as opposed to 38% without node sharing.

(2) In the second set of additional experiments, we evaluate the utilization of nodes and edges. (a) The average resource utilization of substrate nodes is shown in Fig. 10(b). It shows the following. (i) Node utilization of SNs becomes consistent after $t = 24000$. (ii) Node utilization of full mesh is higher than that of random network, followed by directed tree, for both $\mathsf{VNM_P}$ mappings and $\mathsf{VNE_{SP}}$ mappings. (iii) For each type of the three SN topologies, the node utilization of $\mathsf{VNM_P}$ is higher than that of $\mathsf{VNE_{SP}}$. (b) Figure 10(c) shows the average edge utilization. It tells the followings. (i) After $t = 12000$, the average edge utilization becomes consistency, no matter on which kind of SNs. (ii)Priority mapping over directed tree gains the highest edge mapping, however, it is the lowest of all cases while over full mesh. (iii) Generally, the impact of network topologies on the edge utilization for $\mathsf{VNM_P}$ is larger than that for $\mathsf{VNE_{SP}}$.

Conclusion. The additional experiments show the following. (a) Node sharing is also helpful to $\mathsf{VNE_{SP}}$. It improves the AR of $\mathsf{VNE_{SP}}$ on various SNs. (b) The average node utilization of $\mathsf{VNM_P}$ is much higher than that of $\mathsf{VNE_{SP}}$. (c) The impact of network typologies on average edge utilization for $\mathsf{VNM_P}$ is higher that that for $\mathsf{VNE_{SP}}$.