Linear Algebra

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CHAPTER

ONE

SYSTEMS OF LINEAR EQUATIONS

1.1 Linear Equation

Definition 1 *Generally, we define a* ***linear equation*** *in the* n *variables* $x_1, x_2, \cdots, x_n$ *to be one that can be expressed in the form* $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. *The terms* $a_1, a_2, \cdots, a_n$ *are called* ***coefficients****, and* $x_1, x_2, \cdots, x_n$ *are called* ***unknowns****. If* $b = 0$*, the linear equation is called* ***homogeneous****.*

Example 2 *Find all linear equations.*

$$
\begin{array}{ll}
(1)\ x + 3y = 7 & (5)\ x_1 - 2x_2 - 3x_3 + x_4 = 0 \\
(2)\ \dfrac{1}{2}x - y + 3z = -1 & (6)\ x_1 + x_2 + \cdots + x_n = 1 \\
(3)\ x + 3y^2 = 4 & (7)\ 3x + 2y - xy = 5 \\
(4)\ \sin x + y = 0 & (8)\ \sqrt{x_1} + 2x_2 + x_3 = 1
\end{array}
$$

Solution:

The equation (1), (2), (5), and (6) are linear equations.

Note: $ax + by = c$ represents a straight line in *xy* plane; $ax + by + cz = d$ represents a plane in *xyz* space.

1.2 System of Linear Equations

The system of linear equations (sometimes called linear system) consists of one or more linear equations and can be expressed in the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\tag{1.1}
$$

For example,

$$
\begin{aligned}
5x + y &= 3 \\
2x - y &= 4
\end{aligned}
\tag{1.2}
$$

is a linear system in two unknowns, and

$$
\begin{aligned}
4x_1 - x_2 + 3x_3 &= -1 \\
3x_1 + x_2 + 9x_3 &= -4
\end{aligned}
\tag{1.3}
$$

is a linear system in three unknowns.

1.3 Solutions for Linear Systems

Suppose $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ ($s_1, s_2, \cdots, s_n$ are real numbers) make each equation in Eq (1.1) holds true, then the **ordered n-tuple** $(s_1, s_2, \cdots, s_n)$ is a solution of Eq (1.1). For example, $(1, -2)$ is the solution of Eq (1.2) and $(1, 2, -1)$ is a solution of Eq (1.3).

Note: Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

In general, we say that a linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

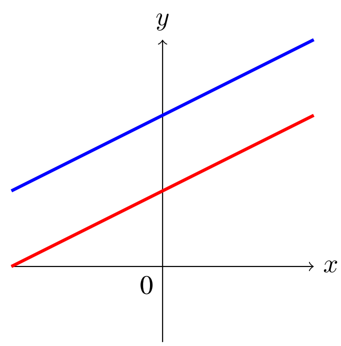

Fig. 1.1: No solution

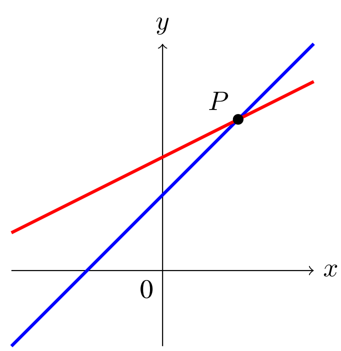

Fig. 1.2: One solution

Example 3 *Solve the linear systems*

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 5 \end{aligned}$$

Solution:

We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields

$$\begin{aligned} x - y &= 1 \\ 3y &= 3 \end{aligned}$$

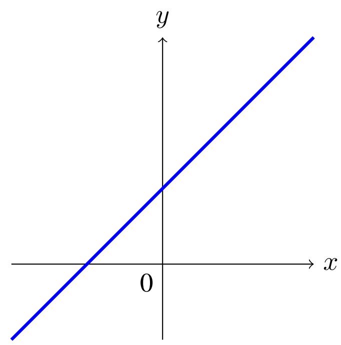

Fig. 1.3: Infinitely many solutions (coincident lines)

From the second equation we obtain $y = 1$, and by substituting it in the first equation we obtain $x = 2$. Thus, the system has the unique solution

$$\begin{aligned} x &= 2 \\ y &= 1 \end{aligned}$$

Example 4 *Solve the linear systems*

$$\begin{aligned} x + y &= 1 \\ 3x + 3y &= 9 \end{aligned}$$

Solution:

We can eliminate x from the second equation by adding -3 times the first equation to the second. This yields

$$\begin{aligned} x + y &= 1 \\ 0 &= 6 \end{aligned}$$

The second equation is contradictory, hence this linear system has no solution.

Example 5 *Solve the linear systems*

$$\begin{aligned} 4x - 2y &= 2 \\ 2x - y &= 1 \end{aligned}$$

Solution:

We can eliminate x from the second equation by adding $-1/2$ times the first equation to the second. This yields

$$\begin{aligned} 4x - 2y &= 2 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and hence can be omitted. Thus, the solution of the system are those values of x that satisfy the equation

$$4x - 2y = 2$$

Then we obtain can express the solution by two parametric equations

$$\begin{aligned} x &= \frac{1}{2} + \frac{1}{2}t \\ y &= t \end{aligned}$$

Example 6 *Solve the linear systems*

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution:

This equation can be solved by inspection, since the second and third equations are multiples of the first. Thus, there is only one restriction for x, y and z

$$x - y + 2z = 5$$

Then we can obtain the solution in parametric form

$$\begin{aligned} x &= 5 + r - 2s \\ y &= r \\ z &= s \end{aligned}$$

1.4 Augmented Matrices and Elementary Row Operations

Consider the general form of linear system in Eq (1.1),

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

if we take all coefficients and constants out and put them in an array, we will have the **augmented matrix** of this linear system,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.

2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called **elementary row operations** on a matrix.

Example 7 *Use elementary row operations and the augmented matrix to solve the linear system*

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

Solution:

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first row to the second row and add -3 times the first row to the third row yield

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second row by $1/2$ yields

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second row to the third row yields

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

Multiply the third row by -2 yields

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second row to the first yields

$$\begin{bmatrix} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-11/2$ times the third row to the first and $7/2$ times the third row to the second yield

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus, the solution is

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

CHAPTER

TWO

GAUSSIAN ELIMINATION

2.1 Echelon Forms

Definition 8 (row echelon form) *A matrix that have the following properties is in **row echelon form**:*

1. *If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **leading 1**.*
2. *If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.*
3. *In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.*

Definition 9 (reduced row echelon form) *A row echelon row form matrix is in **reduced row echelon form** if each column that contains a leading 1 has zeros everywhere else in that column.*

Definition 10 (leading variables and free variables) *In a linear system, the variables corresponding to the leading 1's in the augmented matrix, we call these the **leading variables**. The remaining variables are called **free variables**.*

Example 11 *Are the following matrices in row echelon form? If so, are they in reduced row echelon form?*

1) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5) $\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6) $\begin{bmatrix} 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

7) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

8) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 12 *Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3 and x_4 has been reduced by elementary row operations to*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Find the solution of the linear system.

Example 13 *In each part, suppose that the augmented matrix for a linear system in the unknowns* x, y, *and* z *has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.*

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *(b)* $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *(c)* $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2.2 Elimination Methods

We have just seen how easy it is to solve a system of linear equations once its augmented matrix is in reduced row echelon form. Now we will give a step-by-step elimination procedure that can be used to reduce any matrix to reduced row echelon form.

Example 14 *Reduce the following matrix to reduced row echelon form.*

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Solution:

1. *Locate the leftmost column that does not consist entirely of zeros.*
2. *Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.*
3. *If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $\frac{1}{a}$ in order to introduce a leading 1.*
4. *Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.*
5. *Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row echelon form.*
6. *Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.*

Definition 15 *Step 1-5 produces a row echelon form and is called* ***Gaussian elimination****. Step 1-6 produces a reduced row echelon form and is called* ***Gaussian-Jordan elimination****.*

Example 16 *Solve by Gauss–Jordan elimination.*

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

2.3 Homogeneous Linear Systems

Definition 17 *A system of linear equations is said to be* ***homogeneous*** *if the constant terms are all zero; that is, the system has the form*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{2.1}$$

Definition 18 *Since the m-tuple* $(0, 0, \cdots, 0)$ *is a solution of Eq (2.1), every homogeneous system of linear equations is consistent. This solution is called the* ***trivial solution****; if there are other solutions, they are called* ***nontrivial solutions****.*

Note: Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

Example 19 *Use Gauss–Jordan elimination to solve the homogeneous linear system*

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Theorem 20 *If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has* $n - r$ *free variables.*

Theorem 21 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

There are three facts about row echelon forms and reduced row echelon forms:

- Every matrix has a unique reduced row echelon form.
- Row echelon forms are not unique.
- Although row echelon forms are not unique, the reduced row echelon form and all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions. Those are called the **pivot positions** of A. A column that contains a pivot position is called a **pivot column** of A.