

Math 154: Probability Theory, HW 6

DUE MARCH 6, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. TRYING TO PUT EVERYTHING INTO THE LENS OF A MARTINGALE

1.1. An alternative characterization of conditional expectation. Take X_1, \dots, X_N, Y a set of random variables. Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be any continuous function. Show that

$$\begin{aligned}\mathbb{E}[f(X_1, \dots, X_N) \cdot Y] &= \mathbb{E}\{\mathbb{E}[f(X_1, \dots, X_N) \cdot Y | X_1, \dots, X_N]\} \\ &= \mathbb{E}\{f(X_1, \dots, X_N) \mathbb{E}[Y | X_1, \dots, X_N]\}.\end{aligned}$$

It turns out that $\mathbb{E}[Y | X_1, \dots, X_N]$ is the only random variable which depends only on X_1, \dots, X_N for which this is true for all continuous $f : \mathbb{R}^N \rightarrow \mathbb{R}$. Hence, this is another definition of conditional expectation.

1.2. Law of large numbers, martingale style. It turns out independence is not crucial for the law of large numbers to hold, and that a martingale is really the underlying structure in a lot of cases. Let us see why.

Let $(M_N)_{N \geq 0}$ be a martingale with respect to the filtration generated by some sequence $(X_n)_{n \geq 0}$. We will assume $\sup_{N \geq 0} \mathbb{E}|M_{N+1} - M_N|^2 < \infty$ and $M_0 = 0$.

(1) Using $M_N = \sum_{k=0}^{N-1} (M_{k+1} - M_k)$, show that

$$\mathbb{E}|M_N|^2 = \sum_{k=0}^{N-1} \mathbb{E}|M_{k+1} - M_k|^2 \leq CN$$

for some constant $C > 0$. (*Hint:* it may help to show that if $j < k$, then

$$\mathbb{E}[(M_{k+1} - M_k)(M_{j+1} - M_j)] = \mathbb{E}\{(M_{j+1} - M_j)\mathbb{E}[M_{k+1} - M_k | X_1, \dots, X_k]\} = 0.$$

To show this, it may help to use Problem 1.1 and the martingale property.)

(2) Show that $\mathbb{P}[|N^{-1}M_N| \geq \varepsilon] \leq CN^{-1}\varepsilon^{-2}$ for any $\varepsilon > 0$ and for some constant $C > 0$. (*Hint:* how does one control the tail probability using a second moment?)

(3) Suppose now that X_n are mean 0 and variance 1. Define $Y_N = \sum_{n=1}^N X_n$ and $Y_0 = 0$. Show that $\mathbb{P}[|N^{-1}Y_N| \geq \varepsilon] \leq CN^{-1}\varepsilon^{-2}$ for some constant $C > 0$. (This is the law of large numbers as classically stated, e.g. as in class.)

(4) There is no need to get this right or wrong; you will be given credit for any type of guess. Suppose that $\mathbb{E}|M_{N+1} - M_N|^2 = 1$ for every $N \geq 0$. What do you think the distribution of $N^{-1/2}M_N$ converges to as $N \rightarrow \infty$? (We never defined what it meant for a distribution to converge, so use an intuitive “definition”.)