Math 154: Probability Theory, HW 9

DUE APRIL 16, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. GETTING OUR HANDS ON BROWNIAN MOTION

1.1. **A computation.** Consider the integral $\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.

(1) Compute $\mathbb{E}\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.
(2) Compute $\mathbb{E}|\int_0^t \mathbf{B}_s^2 \mathrm{d}s|^2$. (*Hint*: as in class, square the integral to get a double integral over $0 \leqslant r \leqslant s \leqslant t$. For $r \leqslant s$, it may then help to write $\mathbf{B}_s^2\mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r + \mathbf{B}_r)^2\mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r)^2\mathbf{B}_r^2 + 2(\mathbf{B}_s - \mathbf{B}_r)\mathbf{B}_r^3 + \mathbf{B}_r^4$. Now use independence of increments and knowledge of the distribution of increments.)
(3) Deduce the variance of $\int_0^t \mathbf{B}_s^2 \mathrm{d}s$.

1.2. **Brownian Gambler's ruin (*Hint*: use optional stopping!)** Let $\mathbf{B}$ be Brownian motion, and fix $a, b > 0$. Let $\tau_{a,b}$ be the first time τ such that $\mathbf{B}_\tau \in \{-a, b\}$.

(1) Find the probability that $\mathbf{B}_{\tau_{a,b}} = -a$.
(2) Compute $\mathbb{E}\tau_{a,b}$.

1.3. **Moment generating function of Gaussians, Brownian motion style.** Consider the process $\mathbf{M}_t := \exp\{\lambda \mathbf{B}_t - \mu t\}$, where $\lambda, \mu \in \mathbb{R}$.

(1) Fix $\lambda \in \mathbb{R}$. For which $\mu = \mu(\lambda) \in \mathbb{R}$ does $\mathbf{M}$ satisfy the martingale property? ($\mu(\lambda)$ will depend on λ)
(2) Fix $\lambda \in \mathbb{R}$. Show that $\mathbb{E}\mathbf{M}_1 = 1$. In what follows, we will always take $\mathbf{M}_t$ for this choice of $\mu = \mu(\lambda)$.
(3) Deduce that if $Z \sim N(0,1)$, then $\mathbb{E}e^{\lambda Z} = e^{\lambda^2/2}$. (*Hint*: recall $\mathbf{B}_1 \sim N(0,1)$.)

1.4. **Ergodicity of the OU process.** Suppose X_t is an OU process with initial condition X_0, that is $\mathrm{d}X_t = -X_t \mathrm{d}t + \sqrt{2}\mathrm{d}\mathbf{B}_t$, where $\mathbf{B}_t$ is a Brownian motion.

(1) Show that $N(0,1)$ is an invariant distribution for the OU process (see the notes for what this means).
(2) Let Z_t be an OU process with initial condition $Z_0 \sim N(0,1)$. That is, $\mathrm{d}Z_t = -Z_t + \mathrm{d}\mathbf{B}_t$, where $\mathbf{B}$ is the *same* Brownian motion from above. Define $Y_t = X_t - Z_t$. Show that $Y_t = Y_0 e^{-t}$ for all $t \geqslant 0$. Deduce that $Y_t \to 0$ as $t \to \infty$. (*Hint*: compute the differential equation solved by Y_t using the SDEs for X_t, Z_t; you can use that any solution to $f'(t) = -f(t)$ is given by $f(t) = f(0)e^{-t}$.)

1.5. **Brownian bridge.** The Brownian bridge is a "Brownian motion conditioned to hit 0 at time 1". The point of this exercise is to make this precise in a more natural way.

Let $\{z_k\}_{k=1}^{\infty}$ be a collection of i.i.d. $N(0,1)$ random variables. For any $N > 0$, define

$$\mathbf{Z}_t^{(N)} := \sum_{k=1}^{N} \frac{z_k\sqrt{2}}{k\pi}\sin(k\pi t).$$

Show that $\mathbf{Z}_0^{(N)} = \mathbf{Z}_1^{(N)} = 0$. Show that $\mathbb{E}\mathbf{Z}_t^{(N)} = 0$ and that

$$\mathbb{E}|\mathbf{Z}_t^{(N)} - \mathbf{Z}_t^{(M)}|^2 \to_{N,M\to\infty} 0.$$