

Math 154: Probability Theory, HW 9

DUE APRIL 16, 2024 BY 9AM

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. GETTING OUR HANDS ON BROWNIAN MOTION

1.1. **A computation.** Consider the integral $\int_0^t \mathbf{B}_s^2 ds$.

- (1) Compute $\mathbb{E} \int_0^t \mathbf{B}_s^2 ds$.
- (2) Compute $\mathbb{E} |\int_0^t \mathbf{B}_s^2 ds|^2$. (*Hint:* as in class, square the integral to get a double integral over $0 \leq r \leq s \leq t$. For $r \leq s$, it may then help to write $\mathbf{B}_s^2 \mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r + \mathbf{B}_r)^2 \mathbf{B}_r^2 = (\mathbf{B}_s - \mathbf{B}_r)^2 \mathbf{B}_r^2 + 2(\mathbf{B}_s - \mathbf{B}_r) \mathbf{B}_r^3 + \mathbf{B}_r^4$. Now use independence of increments and knowledge of the distribution of increments.)
- (3) Deduce the variance of $\int_0^t \mathbf{B}_s^2 ds$.

1.2. **Brownian Gambler's ruin (*Hint: use optional stopping!*)** Let \mathbf{B} be Brownian motion, and fix $a, b > 0$. Let $\tau_{a,b}$ be the first time τ such that $\mathbf{B}_\tau \in \{-a, b\}$.

- (1) Find the probability that $\mathbf{B}_{\tau_{a,b}} = a$.
- (2) Compute $\mathbb{E} \tau_{a,b}$.

1.3. **Moment generating function of Gaussians, Brownian motion style.** Consider the process $\mathbf{M}_t := \exp \{ \lambda \mathbf{B}_t - \mu t \}$, where $\lambda, \mu \in \mathbb{R}$.

- (1) Fix $\lambda \in \mathbb{R}$. For which $\mu = \mu(\lambda) \in \mathbb{R}$ does \mathbf{M} satisfy the martingale property? ($\mu(\lambda)$ will depend on λ)
- (2) Fix $\lambda \in \mathbb{R}$. Show that $\mathbb{E} \mathbf{M}_1 = 1$. In what follows, we will always take \mathbf{M}_t for this choice of $\mu = \mu(\lambda)$.
- (3) Deduce that if $Z \sim N(0, 1)$, then $\mathbb{E} e^{\lambda Z} = e^{\lambda^2/2}$. (*Hint:* recall $\mathbf{B}_1 \sim N(0, 1)$.)

1.4. **Ergodicity of the OU process.** Suppose X_t is an OU process with initial condition X_0 , that is $dX_t = -X_t dt + d\mathbf{B}_t$, where \mathbf{B}_t is a Brownian motion.

- (1) Show that $N(0, 1)$ is an invariant distribution for the OU process (see the notes for what this means).
- (2) Let Z_t be an OU process with initial condition $Z_0 \sim N(0, 1)$. That is, $dZ_t = -Z_t dt + d\mathbf{B}_t$, where \mathbf{B} is the *same* Brownian motion from above. Define $Y_t = X_t - Z_t$. Show that $Y_t = Y_0 e^{-t}$ for all $t \geq 0$. Deduce that $Y_t \rightarrow 0$ as $t \rightarrow \infty$. (*Hint:* compute the differential equation solved by Y_t using the SDEs for X_t, Z_t ; you can use that any solution to $f'(t) = -f(t)$ is given by $f(t) = f(0)e^{-t}$.)

1.5. Brownian bridge. The Brownian bridge is a “Brownian motion conditioned to hit 0 at time 1”. The point of this exercise is to make this precise in a more natural way.

Let $\{z_k\}_{k=1}^\infty$ be a collection of i.i.d. $N(0, 1)$ random variables. For any $N > 0$, define

$$\mathbf{Z}_t^{(N)} := \sum_{k=1}^N \frac{z_k \sqrt{2}}{k\pi} \sin(k\pi t).$$

Show that $\mathbf{Z}_0^{(N)} = \mathbf{Z}_1^{(N)} = 0$. Show that $\mathbb{E}\mathbf{Z}_t^{(N)} = 0$ and that

$$\mathbb{E}|\mathbf{Z}_t^{(N)} - \mathbf{Z}_t^{(M)}|^2 \rightarrow_{N,M \rightarrow \infty} 0.$$