MATH 154 Homework 1 Solutions: Spring 2024

1. (a) There are two die; choose one to roll a 6 with probability $\frac{1}{6}$. The other needs to roll a non-6 with probability $\frac{5}{6}$. Thus, the probability of exactly one six is $2 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{18}$

   (b) The probability of a single roll being odd is $\frac{1}{2}$. Due to independence, the probability of two rolls both being odd is $\frac{1}{4}$.

   (c) With two die, we can sum to 4 by rolling 1,3; 2,2; 3,1. Since each of the 36 combinations are equally likely, the probability of rolling a cumulative 4 is $\frac{1}{12}$

   (d) We can get a sum divisible by 3 by summing two numbers that are 0 mod 3, or one that is 1 mod 3 and one that is 2 mod 3. Adding probabilities and using the product rule, we get

$$\frac{1}{3} \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

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2. (a) If heads appears for the first time on the nth throw, tails were flipped on the first $n-1$ throws. The probability is thus

$$(1-p)^{n-1}p$$

   (b) If n is odd, the probability is 0. If n is even, the probability is

$$\binom{n}{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}$$

   since we choose $\frac{n}{2}$ flips to be heads and assign the appropriate probabilities.

   (c) Using similar logic as above, the probability of exactly two heads is

$$\binom{n}{2} p^2 (1-p)^{n-2}$$

   (d) Using complimentary counting, probability of at least two heads is 1 minus the probability of zero or one, giving

$$1 - \binom{n}{0}(1-p)^n - \binom{n}{1}p(1-p)^{n-1}$$

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3. The principle of inclusion and exclusion gives that

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

However, we also have that

$$\mathbb{P}(A \cup B) = \mathbb{P}(\text{exactly one of } A, B) + \mathbb{P}(A \cap B).$$

Thus,

$$\mathbb{P}(\text{exactly one of } A, B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$\mathbb{P}(\text{exactly one of } A, B) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).$$

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4. (a) By the definition of conditional probability,

$$\mathbb{P}(A^c|B^c \cap C^c)\mathbb{P}(B^c|C^c)\mathbb{P}(C^c) = \frac{\mathbb{P}(A^c \cap B^c \cap C^c)}{\mathbb{P}(B^c \cap C^c)} \cdot \frac{\mathbb{P}(B^c \cap C^c)}{\mathbb{P}(C^c)} \cdot \mathbb{P}(C^c) = \mathbb{P}(A^c \cap B^c \cap C^c).$$

By DeMorgan's Law,

$$\mathbb{P}(A^c \cap B^c \cap C^c) = \mathbb{P}((A^c \cup B^c \cup C^c)^c)$$

Thus, we have

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= 1 - \mathbb{P}((A^c \cup B^c \cup C^c)^c) \\ &= 1 - \mathbb{P}(A^c|B^c \cap C^c)\mathbb{P}(B^c|C^c)\mathbb{P}(C^c).\end{aligned}$$

(b) No. Imagine you have two coins, one fair, one that always returns heads. Choose a coin at random. Let A be the event that the first toss is H, and B be the event that the second is H. Let C be the event that the fair coin is chosen. Note that $\mathbb{P}(A|C) = \mathbb{P}(B|C) = \frac{1}{2}$, with $\mathbb{P}(A \cap B|C) = \frac{1}{4} = \mathbb{P}(A|C)\mathbb{P}(B|C)$. However,

$$\mathbb{P}(A) = \mathbb{P}(A|C)\mathbb{P}(C) + \mathbb{P}(A|C^c)\mathbb{P}(C^c) = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}.$$

$\mathbb{P}(B) = \frac{3}{4}$ similarly. However,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap B|C)\mathbb{P}(C) + \mathbb{P}(A \cap B|C^c)\mathbb{P}(C^c) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot 1 \cdot \frac{1}{2} = \frac{5}{8}$$

and conditional independence does not imply independence.

(c) No. Suppose you roll a die. Let A be an event of rolling a 1 or 2; B be rolling 2,4,6; C be rolling 1,4. We have $\mathbb{P}(A) = \frac{1}{3}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(A \cup B) = \frac{1}{6} = \mathbb{P}(A)\mathbb{P}(B)$. However, we have $\mathbb{P}(A|C) = \frac{1}{2}, \mathbb{P}(B|C) = \frac{1}{2}, \mathbb{P}(A \cup B|C) = 0 \neq \mathbb{P}(A|C)\mathbb{P}(B|C)$. Thus, independence does not imply conditional independence.

Grading:

(a) 4 points:

- 0 points - no correct steps
- 2 points - some correct steps
- 4 points - all correct steps

(b) 3 points:

- 0 points - incorrect answer
- 1 point - correct answer, incorrect counterexample
- 3 points - correct answer, correct counterexample

(c) 3 points:

- 0 points - incorrect answer
- 1 point - correct answer, incorrect counterexample
- 3 points - correct answer, correct counterexample

5. By the definition of conditional probability,

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(A_j \cap B)}{\mathbb{P}(B)}.$$

Using the definition of conditional probability in the numerator and the law of total probability in the denominator, we get

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j)\mathbb{P}(A_j)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}.$$

Grading:

(a) +3 points - initial step of using definition of conditional probability

(b) +3 points - using definition of conditional probability to transform numerator

(c) +4 points - using LOTP to transform denominator

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6. Let $|A| = a, |B| = b, |A \cap B| = c$. If A and B are independent,

$$\frac{c}{p} = \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \frac{a}{p}\frac{b}{p} \Rightarrow cp = ab.$$

Since p is prime and $p|ab$, $p|a$ or $p|b$, implying that at least one of a or b is 0 or p, or, equivalently, at least one of A or B is $\emptyset$ or Ω.

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7. (a)

$$\begin{aligned}
\mathbb{P}(N = 2|S = 4) &= \frac{\mathbb{P}(S = 4|N = 2)\mathbb{P}(N = 2)}{\mathbb{P}(S = 4)} \\
&= \frac{\frac{1}{12} \cdot \frac{1}{4}}{\sum_{i=1}^{\infty} \mathbb{P}(S = 4|N = i)\mathbb{P}(N = i)} \\
&= \frac{\frac{1}{48}}{\sum_{i=1}^{4} \mathbb{P}(S = 4|N = i)\mathbb{P}(N = i)} \\
&= \frac{\frac{1}{48}}{\frac{1}{6} \cdot \frac{1}{2} + \frac{1}{12} \cdot \frac{1}{4} + \frac{3}{6^3} \cdot \frac{1}{8} + \frac{1}{6^4} \cdot \frac{1}{16}} \\
&= \frac{\frac{1}{12}}{\frac{2197}{20736}} \\
&= \frac{432}{2197}.
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathbb{P}(S=4|N \text{ even}) &= \frac{\mathbb{P}(S=4|N=2)\mathbb{P}(N=2)+\mathbb{P}(S=4|N=4)\mathbb{P}(N=4)}{\mathbb{P}(N \text{ even})} \\
&= \frac{\frac{1}{12}\cdot\frac{1}{4}+\frac{1}{6^4}\cdot\frac{1}{16}}{\frac{\frac{1}{4}}{1-\frac{1}{4}}} \\
&= \frac{433}{6912}
\end{aligned}
$$

(c) By Bayes Rule,

$$
\begin{aligned}
\mathbb{P}(N=2|S=4,F=1) &= \frac{\mathbb{P}(S=4|N=2,F=1)\mathbb{P}(N=2|F=1)}{\mathbb{P}(S=4|F=1)} \\
&= \frac{\frac{1}{6}\cdot\mathbb{P}(N=2)}{\sum_{i=1}^{\infty}\mathbb{P}(S=4|F=1,N=i)\mathbb{P}(N=i)} \\
&= \frac{\frac{1}{6}\cdot\frac{1}{4}}{\sum_{i=1}^{4}\mathbb{P}(S=4|F=1,N=i)\mathbb{P}(N=i)} \\
&= \frac{\frac{1}{24}}{0\cdot\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{4}+\frac{2}{36}\cdot\frac{1}{8}+\frac{1}{6^3}\cdot\frac{1}{16}} \\
&= \frac{144}{169}.
\end{aligned}
$$

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8. Let X be the event that there is an open road from A to B, Y be the event that there is an open road from B to C, and Z be the event that there is no open route from A to C. We have

$$
\mathbb{P}(X|Z)=\frac{\mathbb{P}(Z|X)\mathbb{P}(X)}{\mathbb{P}(Z)}=\frac{\mathbb{P}(Y^c)\mathbb{P}(X)}{\mathbb{P}(X^c)+\mathbb{P}(Y^c)-\mathbb{P}(X^c\cap Y^c)}=\frac{p^2\cdot(1-p^2)}{p^2+p^2-p^4}=\frac{1-p^2}{2-p^2}.
$$

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9. Since 0 is even, $p_0=1$. Consider $n-1$ tosses for $n\geq 1$. If there were an odd number of H tossed in the first $n-1$ tosses, the nth toss must be a head for the total count to be even after n tosses. In other words, we contribute a term

$$
\mathbb{P}(n \text{ head})\mathbb{P}(\text{odd H after } n-1)=p(1-p_{n-1})
$$

If there were an even number of H tossed in the first $n-1$ tosses, the nth toss must be a tail for the total count to be even aftter n tosses. In other words, we contribute a term

$$
\mathbb{P}(n \text{ tail})\mathbb{P}(\text{even H after } n-1)=(1-p)p_{n-1}.
$$

Thus, we indeed have that

$$
p_n=p(1-p_{n-1})+(1-p)p_{n-1}.
$$

Rearranging, we get

$$p_n = p + (1-2p)p_{n-1} \Rightarrow p_n - \frac{1}{2} = (1-2p)(p_{n-1} - \frac{1}{2}).$$

Define the sequence $a_n = p_n - \frac{1}{2}$. Since $p_0 = 1, a_0 = \frac{1}{2}$. The above gives the relation

$$a_n = (1-2p)a_{n-1},$$

which is simply a geometric series. Thus, a closed form for $a_n = \frac{1}{2}(1-2p)^n \Rightarrow p_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n$. Plugging in this closed form expression also suffices for proof.

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10. The crux of the passenger problem is that, regardless of how all $n-1$ passengers behave beforehand, the last passenger will only have two potential seats to choose from; the first person's seat or their assigned seat. To see why, note that if the ith person's seat was open, the ith person *must* sit there, but the emptiness of that seat contradicts the fact that the ith person is already seated. Thus, the last passenger will always only have those two potential seats, regardless of n. Symmetry gives the answer $\frac{1}{2}$.
