Math 154: Probability Theory, HW 2

Due Feb 6, 2024 by 9am

Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.

1. Some practice

1.1. **Poisson and binomial distributions show up everywhere.** Let X and Y be independent Poisson random variables with parameters λ and μ, respectively.

(1) By computing the pmf of $X+Y$, show that $X+Y$ is a Poisson random variable with parameter $\lambda+\mu$
(2) By computing $\mathbb{P}(X=k|X+Y=n)$, show that $\mathbb{P}(X=k|X+Y=n)=p(k)$, where $p(k)$ is the pmf for a Binomial distribution (with parameters that you must compute).

1.2. **What?** Suppose X is a geometric random variable. Show that $\mathbb{P}(X=n+k|X>n)=\mathbb{P}(X=k)$ for any integers $k,n\geqslant 1$. (This is called the "memorylessness" property.)

1.3. **For some reason, probabilists like urns.** An urn contains N balls, b of which are blue and $r=N-b$ of which are red. Let us randomly take n of the N balls (without replacement). If R is the number of red balls drawn, explain briefly why

$$\mathbb{P}(R=k)=\frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}.$$

This is the *hypergeometric distribution*. Now, take the limit $b,N,r\to\infty$ and suppose $\frac{r}{N}\to p$ (and thus $\frac{b}{N}\to 1-p$). (In words, keep a constant fraction of blue and red balls.) Compute the limit of $\mathbb{P}(R=k)$ as $N\to\infty$, i.e. confirm that

$$\mathbb{P}(R=k)\to\binom{n}{k}p^k(1-p)^{n-k}. \tag{1.1}$$

(This is saying that in the limit of infinitely many balls, sampling with and without replacement is the same, as long as the number of samples n and the proportion of colors is fixed.) (*Hint*: it may help to prove $\binom{X}{y}\to\frac{1}{y!}X^y$ as $X\to\infty$ and y is fixed, i.e. not large.)

2. Some lemmas (and another urn)

2.1. **The "layer-cake formula" (and an application).**

(1) Suppose X is a discrete random variable that takes values in the non-negative integers. Show that $\mathbb{E}(X)=\sum_{n=0}^{\infty}\mathbb{P}(X>n)$.
(2) An urn contains b blue and r red balls. Balls are removed from the urn at random one-by-one. Compute the expected number of turns that we must take until the first red balls is drawn. (You should get $\frac{b+r+1}{r+1}$, but show your work.)

2.2. **Maximum disorder.** Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with parameters $p_1, \ldots, p_n \in [0,1]$, respectively. Define $Y = X_1 + \ldots X_n$.

(1) Show that $\mathbb{E}(Y) = \sum_{k=1}^n p_k$ and $\mathrm{Var}(Y) = \sum_{k=1}^n p_k(1-p_k)$.

(2) Suppose we fix the value of $\mathbb{E}(Y)$ (to be, say, E). Show that the choice of $p_1, \ldots, p_n$ which maximizes $\mathrm{Var}(Y)$ satisfies $p_1 = \ldots = p_n$. (This part has nothing random in it. You can do it by Lagrange multipliers or by plugging in $p_n = E-(p_1+\ldots+p_{n-1})$ into the variance formula and maximizing over $n-1$ variables without any constraints by using calculus.)

2.3. **An old friend, the covariance matrix.** Let $X_1, \ldots, X_n$ be (possibly dependent) random variables. Define the matrix $\mathrm{Cov}(\mathbf{X})$ as an $n \times n$ matrix with entries $\mathrm{Cov}(\mathbf{X})_{ij} = \mathrm{Cov}(X_i, X_j)$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be the vector with entries $X_1, \ldots, X_n$.

(1) Show that for any $\mathbf{v} = (v_1, \ldots, v_n)$, we have $\mathbf{v}\mathrm{Cov}(\mathbf{X})\mathbf{v}^T = \mathrm{Var}(\mathbf{v} \cdot \mathbf{X}) \geqslant 0$.

(2) Show that $\mathrm{Cov}(\mathbf{X})$ is invertible if and only if the following is satisfied:
- If $\mathbf{v} \cdot \mathbf{X}$ is a constant random variables, then $\mathbf{v}$ is the zero vector.

(*Hint*: what condition on the null-space is equivalent to a square matrix being invertible?)