# Math 154: Probability Theory, HW 3

DUE FEB 3, 2024 BY 9AM

*Remember, if you are stuck, take a look at the lemmas/theorems/examples from class, and see if anything looks familiar.*

## 1. ALL OF THESE PROBLEMS REQUIRE AT LEAST A LITTLE THOUGHT

**1.1. Some magic in the Gaussian.** Suppose $X \sim N(0,1)$.

(1) Show that
$$xe^{-\frac{x^2}{2}} = -\frac{d}{dx}e^{-\frac{x^2}{2}}$$
(2) Take any smooth function $f : \mathbb{R} \to \mathbb{R}$. Show that
$$\mathbb{E}Xf(X) = \mathbb{E}f'(X),$$
provided that both sides converge absolutely (when written as integrals). This is often known as *Gaussian integration by parts*. (*Hint*: the hint is in the name.)
(3) Show that for any integer $k \geqslant 0$, we have $\mathbb{E}X^{2k+1} = 0$.
(4) Show that for any integer $k \geqslant 0$, we have $\mathbb{E}X^{2k} = (2k-1)!!$, where $(2k-1)!! := (2k-1)(2k-3)\ldots 1$. (*Hint*: use part (2) with $f(X) = X^{2k-1}$, and induct on $k$.)

**1.2. Another fact about the Gaussian distribution.** Let $X \sim N(0, \sigma^2)$ for some $\sigma > 0$. Take any $\lambda \in \mathbb{R}$. Show that
$$\mathbb{E}e^{\lambda X} = e^{\frac{\lambda^2\sigma^2}{2}}.$$
(*Hint*: you may want to use the completing-the-square formula $a^2 - 2ba = (a-b)^2 - b^2$ after you write out what the expectation on the LHS is as an integral on $\mathbb{R}$.) Give another proof of $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = \sigma^2$ by differentiating both sides of this identity (once and twice) and setting $\lambda = 0$.

**1.3. How does one sample from a distribution?** Suppose $X$ is a continuous random variable, so that $\mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} p(u)du$. Suppose $p$ is smooth and $p(u) > 0$ for all $u \in \mathbb{R}$.

(1) Show that the distribution of the random variable
$$F(X) = \int_{-\infty}^{X} p(u)du$$
is the uniform distribution on $[0,1]$. (Here, we evaluate the top limit of the integral at the random variable $X$. *Hint*: it is not important to know what its inverse exactly is.)
(2) Show that the random variable $-\log F(X)$ has p.d.f given by $e^{-x}$.

1.4. **What?** Suppose $X$ is an exponential random variable (i.e. it has the exponential distribution). Show that $\mathbb{P}(X > s + x | X > s) = \mathbb{P}(X > x)$ for any $x, s \geqslant 0$. (*Hint*: you can use the fact that the only function $g(\cdot)$ which satisfies $g(s+t) = g(s)g(t)$ for $s, t \geqslant 0$ and $g(0) = 1$ has the form $g(s) = e^{\mu s}$ for some $\mu \in \mathbb{R}$.)

1.5. **To the right or to the left?** Let $X$ have variance $\sigma^2$, and write $m_k = \mathbb{E}X^k$. Define the *skewness* of (the distribution of) $X$ to be $\operatorname{skw}(X) = \frac{\mathbb{E}(X-m_1)^3}{\sigma^3}$. (This measures how much to the left/right the graph of the pdf is.)

(1) Show that $\operatorname{skw}(X) = \frac{m_3 - 3m_1m_2 + 2m_1^3}{\sigma^3}$
(2) Let $X_1, \ldots, X_n$ be i.i.d. copies of $X$ (i.e. they are independent and have the same distribution). Set $S_n = X_1 + \ldots + X_n$. Using the following, show $\operatorname{skw}(S_n) = \frac{\operatorname{skw}(X_1)}{\sqrt{n}}$.
   - Compute $\operatorname{Var}(S_n)$ in terms of $\operatorname{Var}(X_1)$ using the i.i.d. property of $X_1, \ldots, X_n$.
   - Show that $\mathbb{E}S_n = n\mathbb{E}X_1$.
   - Letting $m = \mathbb{E}X_1$, show that $\mathbb{E}(S_n - \mathbb{E}S_n)^3 = \sum_{i,j,k=1}^n \mathbb{E}[(X_i - m)(X_j - m)(X_k - m)]$.
   - Using independence, i.e. that $\mathbb{E}[\prod_{i=1}^n f_i(W_i)] = \prod_{i=1}^n \mathbb{E}[f_i(W_i)]$ for any functions $f_1, \ldots, f_n$ and any independent random variables $W_1, \ldots, W_n$, show that $\mathbb{E}[(X_i - m)(X_j - m)(X_k - m)] = 0$ unless $i, j, k$ are all the same. (Note that for any random variable $Y$, $\mathbb{E}(Y - \mathbb{E}(Y)) = 0$.)
   - Deduce that $\mathbb{E}(S_n - \mathbb{E}S_n)^3 = n\mathbb{E}(X_1 - \mathbb{E}X_1)^3$.
   - Now compute $\operatorname{skw}(S_n) = \frac{\mathbb{E}(S_n - \mathbb{E}S_n)^3}{\operatorname{Var}(S_n)^{3/2}}$ in terms of $\operatorname{skw}(X_1)$.
(3) Suppose $X \sim \operatorname{Bern}(n, p)$. Show that $\operatorname{skw}(X) = \frac{1-2p}{\sqrt{p(1-p)}}$ by direct computation.
(4) Suppose $X \sim \operatorname{Bin}(n, p)$. Show that $\operatorname{skw}(X) = \frac{1-2p}{\sqrt{np(1-p)}}$, so that it vanishes as $N \to \infty$. (In particular, this shows that averaging a bunch of random variables can reduce skewness.)

1.6. **Some more computations.** Keep the notation in the setting of Problem 1.5. Define the *kurtosis* of $X$ by $\operatorname{kur}(X) = \frac{\mathbb{E}(X-m_1)^4}{\sigma^4}$. (This is kind of like a variance, but it tells you a little more about the shape of the graph of the pdf.)

(1) Show that if $X \sim N(\mu, \sigma^2)$, then $\operatorname{kur}(X) = 3$. Notice how this is much simpler! (It does not depend on the parameters of the distribution.)
(2) Let $X_1, X_2$ be i.i.d. $N(0,1)$. Define $S = X_1 + X_2$. *Without using the fact that* $X_1 + X_2 \sim N(0, 2)$, show that $\operatorname{kur}(S) = 3$. (In particular, use $\operatorname{kur}(S) = \frac{\mathbb{E}(S - \mathbb{E}S)^4}{\operatorname{Var}(S)^2}$.)