ECON 640 Heteroskedasticity

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1 Basics of OLS Variance-Covariance Matrix

$$
\begin{aligned}
y &= X\beta + u \\
\hat{\beta}_{OLS} &= (X'X)^{-1}X'y \\
Var(\hat{\beta}) &= \sigma^2(X'X)^{-1}, \quad \text{if } Var(u_i|X) = \sigma^2
\end{aligned}
$$

1.1 Heteroskedasticity

- Definition: Variance of the error term is not constant across levels of independent variables:

$$Var(u_i|X) = \sigma_i^2 \neq \sigma^2$$

- Common causes: - Data structure - Measurement error - Model misspecification

Then OLS is no longer efficient but still consistent:

$$
\begin{aligned}
\hat{\beta}_{OLS} &= \beta + (X'X)^{-1}X'u \\
\hat{\beta}_{OLS} &\to \beta \\
Var(\hat{\beta}_{OLS}) &= (X'X)^{-1}X'Var(u)X(X'X)^{-1}
\end{aligned}
$$

The corrected variance of beta OLS becomes:

$$Var(\hat{\beta}_{OLS}) = (X'X)^{-1}X'\Omega X(X'X)^{-1}$$

$$Var(\hat{\beta}_{OLS}) = (X'X)^{-1}(\Sigma\sigma_i^2 x_i x_i')(X'X)^{-1}$$

where Ω is the variance-covariance matrix of u, e.g., $\Omega = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$; and x_i is the first of **X** matrix, with dimension 1 by k (i.e., $x_i = (x_{11}, x_{21}, ..., x_{k1})$

2 Generalized Least Squares (GLS)

To address the inefficiency of OLS in the presence of heteroskedasticity, we propose GLS. The GLS estimator is more efficient (i.e., more precise) while still being consistent.

2.1 Derivation of GLS

Consider the model:

$$y = X\beta + u, \quad \text{with } Var(u) = \Omega$$

To transform the problem into a homoskedastic framework, we define $P = \Omega^{-1/2}$, the square root of the inverse of the variance-covariance matrix Ω. Multiply P on both sides of the equation:

$$Py = PX\beta + Pu$$

Let:

$$y^* = Py, \quad X^* = PX, \quad u^* = Pu$$

Thus, the transformed model becomes:

$$y^* = X^*\beta + u^*, \quad \text{with } Var(u^*) = Var(Pu) = P\Omega P' = I$$

This transformation effectively eliminates heteroskedasticity, as the new error term u^* is homoskedastic with variance I.

2.2 Applying OLS to the Transformed Model

The OLS estimator applied to the transformed model is:

$$\hat{\beta}_{GLS} = (X^{*'}X^*)^{-1}X^{*'}y^*$$

Substituting $X^* = PX$ and $y^* = Py$, we can rewrite:

$$\hat{\beta}_{GLS} = \left(X'P'PX\right)^{-1} X'P'Py$$

Since $P = \Omega^{-1/2}$, we have:

$$\hat{\beta}_{GLS} = \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}y$$

Prove:

$$Var(\hat{\beta}_{GLS}) = \left(X'\Omega^{-1}X\right)^{-1} X'$$

Key Challenge in Practice: The variance-covariance matrix Ω is still unknown. To utilize GLS, we first need to estimate $\hat{\Omega}$. While OLS remains unbiased and consistent, it is no longer efficient under heteroskedasticity. However, we can still rely on OLS by correcting its standard errors. Hence, in practice, there are two common approaches:

1. Use Huber-White robust standard errors for OLS.

2. Apply Feasible Generalized Least Squares (FGLS).

3 White Robust Standard Errors

(aka Heteroskedastic Consistent Estimator- HCE)
Even with heteroskedasticity, OLS remains:

- **Unbiased**
- **Consistent**

However, the standard errors must be corrected (we just mimic the variance of $\hat{\beta}_{OLS}$!!:

$$Var(\hat{\beta}) = (X'X)^{-1} \left(\sum_{i=1}^{n} \hat{u}_i^2 x_i x_i' \right) (X'X)^{-1}$$

where u_i is the OLS residual.
This correction is known as the **White robust standard errors**.

4 Feasible GLS (FGLS)

Or, use feasible GLS:

1. Similarly, fit an OLS model to obtain residuals $\hat{u}_i$.
2. Use $\hat{u}_i$ to estimate $\hat{\Omega}$.
3. Apply GLS:

$$\hat{\beta}_{FGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}y$$

4.1 Variance of FGLS Estimator

$$Var(\hat{\beta}_{FGLS}) = (X'\hat{\Omega}^{-1}X)^{-1}$$

5 GLS vs. OLS takeaways

- OLS is unbiased and consistent, but not efficient under heteroskedasticity.
- GLS is BLUE (Best Linear Unbiased Estimator).
- Heteroskedasticity violates the OLS efficiency assumption, but corrections (White standard errors, FGLS) address this issue.
- GLS provides efficient estimates if Ω is known.
- Robust standard errors are simpler to implement and widely used in empirical work.