ECON 640 Stationary Time Series

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1 Introduction to Time Series and Stochastic Processes

1.1 Definitions

A time series is a sequence of observations indexed over time:

$$\{X_t\}, \quad t = 1, 2, \ldots, n$$

A stochastic process is a mapping:

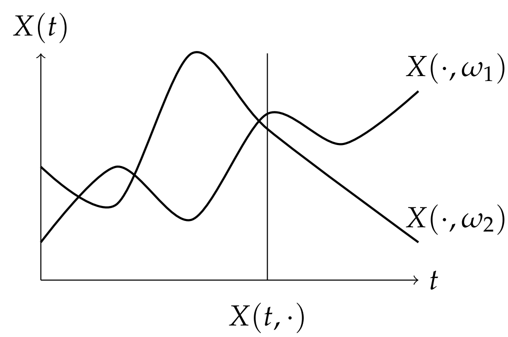
$$X : (T \times \Omega) \to \mathbb{R}$$

where:

- T: Time domain.
- Ω: Sample space.

For a fixed t, $X(t, \omega)$ represents the sample paths. A single path is $X(t, \cdot)$.

1.2 Graphical Representation of a Sample Path



1.3 Important Properties of Time Series

- **Stationarity:** A time series $\{X_t\}$ is stationary if the joint distribution is invariant under time shifts:
$$P(X_{t_1}, X_{t_2}, \ldots, X_{t_k}) = P(X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_k+\tau})$$
- **Ergodicity:** A time series is ergodic if:
$$X_t \quad \text{and} \quad X_{t+k} \quad \text{are independent as } k \to \infty.$$

2 Stationary Processes in the Time Domain

2.1 Auto-correlation Function (ACF)

The auto-correlation function measures the correlation between X_t and X_{t+k} for a lag k:

$$\gamma(k) = \mathbb{E}(X_t X_{t+k})$$

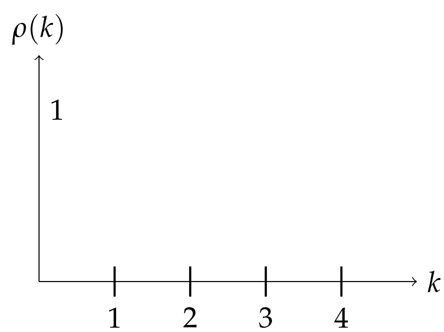
The normalized version is the auto-correlation:

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

2.2 White Noise

A white noise process $\{\epsilon_t\}$ is defined as:

$$\epsilon_t \sim \mathrm{WN}(0, \sigma^2), \quad \text{with } \rho(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$



3 Moving Average (MA) Processes

3.1 Definition of MA(q)

An MA(q) process is defined as:

$$X_t = \epsilon_t + \beta_1 \epsilon_{t-1} + \cdots + \beta_q \epsilon_{t-q}, \quad \epsilon_t \sim \mathrm{WN}(0, \sigma^2).$$

3.2 Derivation of ACF for MA(q)

- For $k > q$: $\rho(k) = 0$, as there are no terms beyond lag q.
- For $k \leq q$:

$$\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i \beta_{i+k} \sigma^2}{\sum_{i=0}^{q} \beta_i^2 \sigma^2}.$$

3.3 Special Case: MA(1)

$$X_t = \epsilon_t + \beta_1 \epsilon_{t-1},$$
$$\gamma(0) = \mathbb{E}[X_t^2] = \sigma^2(1 + \beta_1^2),$$
$$\gamma(1) = \mathbb{E}[X_t X_{t-1}] = \sigma^2 \beta_1.$$

The auto-correlation is:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\beta_1}{1 + \beta_1^2}.$$

4 Auto-Regressive Processes (AR)

4.1 Definition

An AR(p) process is defined as:

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + \epsilon_t, \quad \epsilon_t \sim \mathrm{WN}(0, \sigma^2).$$

4.2 Special Case: AR(1)

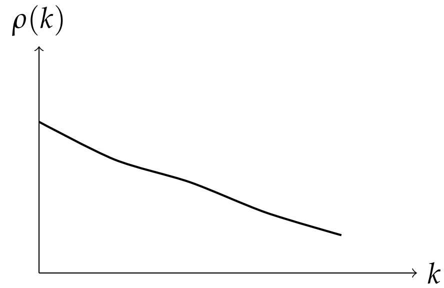
For an AR(1) process:

$$X_t = \alpha X_{t-1} + \epsilon_t$$

where $|\alpha| < 1$, the process is stationary with:

$$\mathbb{E}(X_t) = 0, \quad \text{and} \quad \mathrm{Var}(X_t) = \frac{\sigma^2}{1 - \alpha^2}.$$

- Mean:
  $$\mathbb{E}(X_t) = \alpha \mathbb{E}(X_{t-1}) + \mathbb{E}(\epsilon_t).$$
  Assuming stationarity and $\mathbb{E}(\epsilon_t) = 0$, we get:
  $$\mathbb{E}(X_t) = 0.$$
- Variance:
  $$\mathrm{Var}(X_t) = \alpha^2 \mathrm{Var}(X_{t-1}) + \sigma^2.$$
  Solving for stationarity ($\mathrm{Var}(X_t) = \mathrm{Var}(X_{t-1})$):
  $$\mathrm{Var}(X_t) = \frac{\sigma^2}{1 - \alpha^2}, \quad |\alpha| < 1.$$
- ACF:
  $$\rho(k) = \alpha^k, \quad k \geq 0.$$



4.3 Derivation of AR(1)

An AR(1) process is defined as:

$$X_t = \alpha X_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{WN}(0, \sigma^2).$$

4.4 Using the Lag Operator

Using the lag operator L, we can rewrite the AR(1) process as:

$$X_t - \alpha X_{t-1} = \epsilon_t, \quad \text{or } (1 - \alpha L)X_t = \epsilon_t.$$

Thus:

$$X_t = \frac{\epsilon_t}{1 - \alpha L}.$$

4.5 Expanding the Inverse Operator

Expanding $\frac{1}{1-\alpha L}$ as a geometric series (valid for $|\alpha| < 1$):

$$X_t = \epsilon_t + \alpha\epsilon_{t-1} + \alpha^2\epsilon_{t-2} + \ldots.$$

This can be written as:

$$X_t = \sum_{k=0}^{\infty} \alpha^k \epsilon_{t-k}.$$

Mean and Variance

- **Mean:**

$$\mathbb{E}(X_t) = \mathbb{E}\left(\sum_{k=0}^{\infty} \alpha^k \epsilon_{t-k}\right) = \sum_{k=0}^{\infty} \alpha^k \mathbb{E}(\epsilon_{t-k}).$$

  Since $\mathbb{E}(\epsilon_t) = 0$, we have:

$$\mathbb{E}(X_t) = 0.$$

- **Variance:**

$$\text{Var}(X_t) = \mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \alpha^k \epsilon_{t-k}\right)^2\right].$$

Cross terms vanish due to $\mathbb{E}(\epsilon_t \epsilon_{t-j}) = 0$ for $j \neq 0$. Thus:

$$\text{Var}(X_t) = \sum_{k=0}^{\infty} (\alpha^k)^2 \sigma^2 = \sigma^2 \sum_{k=0}^{\infty} \alpha^{2k}.$$

Using the sum of a geometric series:

$$\text{Var}(X_t) = \frac{\sigma^2}{1-\alpha^2}, \quad \text{for } |\alpha| < 1.$$

Auto-Correlation Function (ACF)

The ACF at lag k is defined as:

$$\rho(k) = \frac{\mathbb{E}[X_t X_{t-k}]}{\text{Var}(X_t)}.$$

For $k = 1$:

$$\mathbb{E}[X_t X_{t-1}] = \mathbb{E}\left[\left(\sum_{j=0}^{\infty} \alpha^j \epsilon_{t-j}\right)\left(\sum_{i=0}^{\infty} \alpha^i \epsilon_{t-1-i}\right)\right].$$

Cross terms vanish except for $\epsilon_{t-j}\epsilon_{t-1-j}$ when $j = i + 1$:

$$\mathbb{E}[X_t X_{t-1}] = \alpha \text{Var}(\epsilon_t) = \alpha\sigma^2.$$

Thus:

$$\rho(1) = \frac{\alpha\sigma^2}{\text{Var}(X_t)} = \frac{\alpha\sigma^2}{\sigma^2/(1-\alpha^2)} = \alpha.$$

For general k:

$$\rho(k) = \alpha^k, \quad \text{for } |\alpha| < 1.$$

Summary of ACF

$$\rho(k) = \begin{cases} 1 & k = 0, \\ \alpha^k & k > 0. \end{cases}$$

The ACF decays exponentially for $|\alpha| < 1$.

4.6 Stationarity Condition

The AR(1) process is stationary if and only if $|\alpha| < 1$.

5 ARMA Processes

5.1 Definition of ARMA(p, q)

An ARMA(p, q) process combines AR and MA components:

$$\phi(L)X_t = \theta(L)\epsilon_t,$$

where $\phi(L)$ and $\theta(L)$ are lag operators:

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p, \quad \theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

5.2 Derivation for ARMA(1, 1)

For ARMA(1, 1):

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}.$$

- Mean:

  $$\mathbb{E}(X_t) = 0 \quad \text{(stationarity assumption)}.$$

- Variance:

  $$\mathrm{Var}(X_t) = \frac{\sigma^2(1+\theta^2)}{1-\phi^2}.$$

- ACF:

  $$\rho(1) = \frac{\phi(1+\theta\phi)}{1+\theta^2}.$$

  Higher lags decay geometrically.