ECON 640 Key Algebra and Statistics Techniques You Should Know

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1. Law of Iterated Expectation (LIE) $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X))$, the inner expectation is the conditional expectation of Y, given X; the outer expectation is based on the distribution of X.
   Proof of LIE (discrete case):

$$\begin{aligned} E(Y) &= \sum_j y_j(p(Y = y_j) \\ &= \sum_j y_j(\sum_i P(Y = y_j, X = x_i)) \\ &= \sum_j y_j \left(\sum_i P(Y = y_j | X = x_i) P(X = x_i) \right) \\ &= \sum_i \left(\sum_j y_j P(Y = y_j | X = x_i) \right) P(X = x_i) \end{aligned}$$

2. then if $\mathbb{E}(u|X) = 0 \implies$ three important corollaries:
   - $\mathbb{E}(u) = 0$ because $\mathbb{E}(u|X) = 0 \to \mathbb{E}(\mathbb{E}(u|X)) = \mathbb{E}(u) = 0$
   - $\mathbb{E}(uX) = 0$ because $\mathbb{E}(u|X) = 0 \to \mathbb{E}(\mathbb{E}(u|X)) = \mathbb{E}(u) = 0$
   - u and X are uncorrelated, why?
     $Cov(u, X) = \mathtt{E}((\mathtt{u} - \bar{\mathtt{u}})(\mathtt{X} - \bar{\mathtt{X}})) = \mathtt{E}(\mathtt{uX}) - \mathtt{E}(\mathtt{u})\mathtt{E}(\mathtt{X})$
     since both $\mathtt{E}(\mathtt{u}) = \mathtt{0}, \mathtt{E}(\mathtt{ux}) = \mathtt{0} \to Cov(u, X) = 0$

3. powerful scalar notation:

$$\begin{aligned} &\sum_i ((x_i - \bar{x})(y_i - \bar{y})) \\ &= \sum_i (x_i - \bar{x})(y_i) \\ &= \sum_i (x_i)(y_i - \bar{y}) \\ &= \sum_i x_i y_i - n\bar{x}\bar{y} \end{aligned}$$

4. hence,

$$\sum_i (x_i - \bar{x}) \\ = \left(\sum_i x_i^2 \right) - n\bar{x}^2$$

5. $Var(X+Y) = Var(X) + Var(Y) + 2COV(X,Y)$
   $Var(X-Y) = Var(X) + Var(Y) - 2COV(X,Y)$, because $Var(Y) = Var(-Y)$
   $Var(X \cdot Y) = E(Y)^2 Var(X) + Var(Y) E(X)^2$
   $Var(constant \cdot X) = constant^2 \cdot Var(X)$

Linear Algebra Basics Before OLS

This is a short set of notes on the linear algebra your students should know before learning OLS estimators.

1. Scalars, Vectors, and Matrices

- **Scalar**: a single number, e.g. $a \in \mathbb{R}$.
- **Vector**: a column of numbers, e.g. $x \in \mathbb{R}^n$.
- **Matrix**: a rectangular array of numbers, e.g. $A \in \mathbb{R}^{m \times n}$.
- Matrix multiplication: if A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

2. Transpose Rules

$$(AB)^\top = B^\top A^\top, \quad (A^\top)^\top = A, \quad a^\top b = b^\top a \text{ (scalar)}$$

3. Inner Products

If $a, b \in \mathbb{R}^n$, then:

$$a^\top b = \sum_{i=1}^{n} a_i b_i$$

Geometric meaning:

$$a^\top b = \|a\| \, \|b\| \cos\theta$$

4. Special Matrices

- **Identity**: I_n satisfies $AI_n = I_n A = A$.
- **Diagonal**: only diagonal entries can be nonzero.
- **Symmetric**: $A = A^\top$.

5. Rank and Invertibility

- **Rank**: number of linearly independent columns (or rows).
- A square matrix A is invertible iff its rank equals the number of columns.
- For regression, we need $X^\top X$ to be invertible (columns of X must be independent).

6. Projection Intuition

OLS can be seen as projecting y onto the space spanned by the columns of X. Residuals are orthogonal to the regressors.

$$P = X(X^\top X)^{-1} X^\top, \quad M = I - P, \quad X^\top \hat{u} = 0$$

7. Matrix Differentiation (optional)

$$\frac{\partial(a^\top X)}{\partial X} = a, \quad \frac{\partial(X^\top a)}{\partial X} = a$$

$$\frac{\partial(X^\top A X)}{\partial X} = (A + A^\top)X = 2AX \quad \text{if } A \text{ is symmetric}$$

This is all the minimal linear algebra needed to understand OLS formulas.