OPTIMAL CONTROL OF DC PENSION PLAN MANAGEMENT UNDER TWO INCENTIVE SCHEMES

Lin He[a], Zongxia Liang[b *], Yang Liu[b *] and Ming Ma[b *]

[a]China Financial Policy Research Center, School of Finance, Renmin University of China, Beijing, 100872, China

[b *]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Abstract. Since the late 1990s, performance fee arrangement has been approved as a managerial incentive in DC pension plan management to motivate managers. However, the fact that managers may take undue risk for the larger performance fees and thus reduce members' utility has been a subject of debate. As such, this study investigates the optimal risk-taking policies of DC pension fund managers under both the single management fee scheme and a mixed scheme with a lower management fee, as well as an additional performance fee. The analytical solutions are derived by using the duality method and concavification techniques in a singular optimization problem. The results show the complex risk-taking structures of fund managers and recognize the win-win situation of implementing performance-based incentives in DC pension plan management. Under the setting of geometric Brownian motion asset price dynamics and constant relative risk aversion utility, the optimal risk investment proportion shows a peak-valley pattern under the mixed scheme. Further, the manager gambles for gain when fund wealth is low and time to maturity is short. As opposed to the existing literature, this study found that the risk-taking policy is more conservative when fund wealth is relatively large. Furthermore, the utilities of the manager and members could both be improved by appropriately choosing the performance fee rate.

MSC(2010): 91B30, 91G10.

JEL Classifications: G11, G22, G32, C61.

Keywords: Management fee; Performance fee; Incentive scheme; Duality method; Concavification techniques; DC pension plan; CRRA utility.

* Corresponding author.
Email: helin@ruc.edu.cn(L.He), zliang@math.tsinghua.edu.cn(Z.Liang), yang-liu16@mails.tsinghua.edu.cn(Y.Liu), mam14@mails.tsinghua.edu.cn(M.Ma)

1. Introduction

Traditionally, the managerial incentive of defined contribution (DC) pension plan managers is the single management fee arrangement, which is charged according to the value of the fund. However, managers typically perform poorly and cannot keep up with benchmarks under a scheme with lower incentives. Hawthorne(1986)[14] points out that fund managers have been paid exorbitant management fees compared to their inferior performances. A similar phenomenon has been widely studied for hedge and mutual fund management. Gomez-Mejia and Wiseman(1997)[9] show that managers under the single management fee scheme suffer from a lack of incentive in exerting greater efforts towards better performance.

It is widely believed that a performance-based fee may create greater incentives for a manager to exert more efforts in terms of investment. As such, the optimal risk-taking policies under different incentive schemes have been widely studied in the literature on fund management. Stoughton (1993)[26] and Li and Tiwari(2009)[19] find that the asymmetry incentive alters the risk attitudes of fund managers, and moral hazard arises. Stark(1987)[25] and Elton et al.(2003)[5] study the impacts of incentive schemes on risk-taking attitudes and fund performances from theoretical and empirical perspectives, respectively. Foster and Young (2010)[7] use empirical data to show that an incentive arrangement increases risk allocation as well as fund volatility under the mixed scheme of both management and performance fees. The geometric Brownian motion asset price dynamics and expected utility theory framework are the most frequently used settings to model the fund process and study the risk-taking policies of fund managers. Gerber and Shiu (2013)[8] review the applications of geometric Brownian motion in modeling pension fund dynamics. Carpenter(2000)[3] and Hodder and Jackwerth (2007)[16] identify the positive relationship between risk taking and managerial incentives under these settings. Further, Bichuch and Sturm(2014)[2] use a general semi-martingale to depict risk asset price dynamics. He and Kou (2018)[15] propose a framework based on cumulative prospect theory (CPT) to study the relationship between risk taking and performance fees in hedge funds. They successfully obtain optimal trading strategies in closed form under the first-loss scheme. As such, in this paper, we follow the settings of geometric Brownian motion asset price dynamics and constant relative risk aversion (CRRA) utility to study the optimal risk-taking policies of fund managers under two incentive schemes.

In the late 1990s, performance fee arrangement was approved as managerial incentive in DC pension plan management. However, the fact that the performance fee could motivate managers to gamble at the expense of fund members has been argued. As safety is the primary objective in pension fund management, the performance fee scheme has not been implemented in most DC pension funds. As such, the questions are whether the performance fee scheme always increases risk and decreases utility and whether it should be implemented in DC fund management. We use stochastic optimal control methods to study the risk-taking policies of DC pension fund managers under the mixed scheme of both management and performance fees. Furthermore, we compare the utilities of both fund managers and members under the two incentive schemes. In other words, we establish more complex risk-taking structures of managers with respect to time and fund wealth compared to the literature. We also recognize the win-win situation of implementing performance-based incentives in DC pension plan management. On one hand, in the mixed scheme, the gain of the manager is the piecewise function of terminal accumulation. That is, the property of nonlinear gain damages the concavity of the utility function, which results in a singular optimization problem. Restricted by the optimization methods for the singularity problem, the complex risk-taking behaviors of DC pension fund managers with respect to time and fund wealth have not been well studied. On the other hand, for DC pension funds, the contribution process is an important determinant of removing the self-financing and martingale properties of the fund wealth process. To solve these problems, we add auxiliary labor capital processes to our model, so that the total wealth process exhibits the self-financing property, as in Kwak et al.(2011)[18], Pirvu and Zhang(2012)[24], and Guan and Liang(2014)[10]. For simplicity, we assume the contribution process to be non-random. If the contribution process is stochastic, the problem could be solved by replacing labor capital processes with similar stochastic auxiliary processes. Therefore, the singular optimization on fund wealth equals the singular optimization on total wealth. Finally, we use the duality method and concavification techniques (cf. Karatzas et al.(1991)[17], He and Kou (2018)[15], Lin et al.(2017)[20], and Guan and Liang(2016)[12]) to solve the singular optimization problem under the martingale framework. We also establish optimal risk-taking policies from the terminal payoff by completeness of the market.

The contribution of this paper is threefold. First, we obtain the closed forms of optimal risk-taking policies of DC pension fund managers under both the single management fee and mixed schemes, with a lower management fee as well as an additional performance fee.

Second, under the setting of geometric Brownian motion asset price dynamics and CRRA utility, we show several innovative results. The results reveal more complex risk-taking structures with respect to time and fund wealth. The optimal proportion allocated to the risky asset has a peak-valley pattern under the mixed scheme. The manager gambles for gain when fund wealth is low and the time to maturity is short. Meanwhile, the risk-taking policy is more conservative when fund wealth is relatively large, which is different from what has been observed in the existing literature.

Finally, as members passively follow the risk-taking policies of the manager, we numerically evaluate the liquidation probabilities and utility changes of the members after the scheme transformation. The results are more optimistic compared to the literature. The liquidation probability under the mixed scheme is relatively higher than under the single incentive scheme, but is still limited. Meanwhile, liquidation will not happen before the terminal time. Furthermore, the utilities of the manager and members could both be improved by choosing an appropriate performance fee rate. As such, this is a win-win situation for both to implement the mixed scheme.

The remainder of this paper is organized as follows. In Section 2, we establish the optimal control problem from the perspective of the fund manager under the two managerial incentive schemes. Section 3 utilizes duality methods and concavification techniques to derive general solutions. In Section 4, the closed-form solution of the optimal risk-taking policies and expected utilities are derived. Section 5 numerically analyzes the impacts of incentive schemes and performance fee rates on the risk-taking behaviors of fund managers. Furthermore, we compare liquidation probabilities and utility changes for both the manager and members under the two incentive schemes. The conclusions are drawn in the last section.

2. **Stochastic optimal control problems of two incentive schemes**

In DC pension fund management, contributions are continuously paid to the fund and the manager dynamically allocates fund wealth to risky and risk-free assets to achieve objectives. First, we establish the dynamics of the risky and risk-free assets.

2.1. **Financial market.** DC pension fund wealth could be allocated on the financial market during the accumulation phase to increase fund value and provide higher old-care utility to members. The financial market consists of two assets: one risky asset and one risk-free asset, whose return rate is constant. The $\mathbb{P}$-augmented probability space $(\Omega, \mathcal{F}, \mathbb{P})$, constructed on the financial market, is generated by Brownian motion $\{W_t\}_{0\leq t\leq T}$, where T is the terminal time. The equipped filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ is defined as by $\mathcal{F}_t = \sigma\{W_u : u \leq t\}$.

The price of the risk-free asset $\{R_t\}_{t\geqslant 0}$ satisfies the following differential equation:

$$dR_t = rR_t dt, \tag{2.1}$$

where the interest rate is constant, $r > 0$. The price of the risky asset $\{S_t\}_{t\geqslant 0}$ is a geometric Brownian motion, satisfying the following stochastic differential equation (SDE):

$$\begin{aligned} dS_t &= S_t\big(\mu dt + \sigma dW_t\big) \\ &= S_t\big[rdt + \sigma(dW_t + \theta dt)\big], \end{aligned} \tag{2.2}$$

where $\sigma > 0$ is the volatility of the risky asset. $\theta = \frac{\mu - r}{\sigma} > 0$ is the market price of risk and μ the expected return of the risky asset.

The financial market is complete and there exists one and only one pricing kernel. For the case of an incomplete market, readers could refer to Cochrane(2005)[4]. We define the state price density process $\{\xi_t\}_{0\le t\le T}$ as follows:

$$\xi_t \triangleq \exp[-(r+\frac{1}{2}\theta^2)t-\theta W_t],\ 0\le t\le T.$$

2.2. **Fund wealth process.** During the accumulation phase of the DC pension plan, fund wealth is dynamically allocated to the risky and risk-free assets, and the risk allocation amount is the control variable in the model. Moreover, contributions are continuously paid to the fund, which cancels the self-financing property of the fund wealth process. We add an auxiliary labor capital process to change the original problem into an equivalent self-financing problem, as in Han and Hung(2012)[13] and Guan and Liang(2016)[11]. Here, the contribution rate is constant for simplicity, as in Ngwira and Gerrard(2007)[23]. Consequently, the constructed total wealth process exhibits the self-financing property, and the optimization of the fund wealth equals the optimization of total wealth by simple substitutions. In the stochastic contribution rate case, by adding the similar auxiliary stochastic process, united market (i.e., the product of the financial and labor capital markets) completeness and self-financing property could also be obtained. Therefore, the mathematical difficulty is not added essentially.

The dynamics of fund wealth $\{X_t\}_{0\le t\le T}$ are uniquely determined by the control policy $\pi=\{\pi_t\}_{0\le t\le T}$, and the initial value $X_0=0$, as follows:

$$dX_t = rX_tdt+\pi_t\sigma(dW_t+\theta dt)+cdt, \tag{2.3}$$

where c is the contribution rate and π_t represents the total amount allocated to the risky asset at time $t\in[0,T]$. We denote $\mathcal{V}[0,T]$ as the set of all the admissible policies. Control policy π is called an admissible strategy if

(i) π is an $\{\mathcal{F}_t\}_{t\ge 0}$-adapted $\mathbb{R}$-valued process and $\int_0^T \pi_t^2dt<\infty$, $a.s.$;
(ii) There exists a unique solution of the SDE (2.3), denoted by X;
(iii) $X_T\ge M:=\rho G$.

The first two conditions are common in the literature (cf. Lin et al. (2017)[20]), and guarantee the existence and finiteness of X. The third condition comes from the safety requirement. As the DC pension fund is the primary component of an old-age security system, fund wealth should be protected from significant losses. Therefore, the liquidation level should be included in DC pension fund management, which is the proportion ρ of benchmark $G = G(T)$ at time T, and $0 < \rho < 1$. Benchmark $G(T)$ is the expected accumulation of contributions until time T under the reasonable accumulation rate: that is,

$$G(T) = \int_0^T ce^{\bar{r}(T-s)}ds = \frac{c}{\bar{r}}(e^{\bar{r}T} - 1), \tag{2.4}$$

where $\bar{r}$ is the accumulation rate. Obviously, $G(T)$ is the benchmark for the availability of the performance fee.

2.3. **Two incentive schemes.** As the objectives of the DC pension fund manager and members differ due to the principal-agent relationship, managerial incentives are implemented to motivate the manager to exert greater efforts to gain profit. Traditionally, the fund manager charges a management fee as reward, which is a fixed proportion of terminal fund wealth. According to the literature, the traditional incentive scheme is less efficient compared to other schemes, and the performance of the manager is thus poorer.

Under the single management fee scheme, the gain $\Theta_1(X_T)$ of the manager is

$$\Theta_1(X_T) = k_1 X_T, \tag{2.5}$$

where k_1 is the rate of the management fee and X_T is the terminal fund wealth.

It is well known that the performance fee may create a greater incentive for the manager. As an asymmetric incentive scheme rewards gain and does not penalize loss, the manager may prefer to take more risk and gamble for gain under the mixed scheme. As such, implementing the mixed incentive scheme in DC pension funds is still a controversial issue. Under the mixed scheme, the gain $\Theta_2(X_T)$ of the manager is

$$\Theta_2(X_T) = \begin{cases} k_2 X_T + k_3(X_T - G), & X_T \geq G, \\ k_2 X_T, & \rho G \leq X_T < G, \end{cases} \tag{2.6}$$

where k_2 is the rate of the management fee under the mixed scheme, which is smaller than the rate under the single management fee scheme, that is, $k_2 < k_1$. k_3 is the rate of the performance fee. Obviously, a higher k_3 generates a higher incentive for the manager. The availability of the performance fee is determined by benchmark $G = G(T)$. If fund wealth is larger than the benchmark, the fund manager will receive the basic management fee and the additional performance fee as rewards. Conversely, the manager could receive only the basic management fee as reward.

The manager's gains under the two incentive schemes are shown in Fig.1. Under the single management fee scheme, the gain of the manager

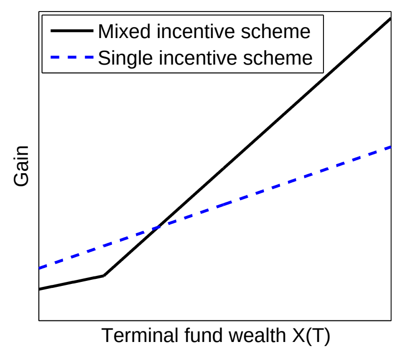

FIGURE 1. The gains of the fund manager under the two incentive schemes.

is linear with respect to fund wealth. Under the mixed scheme, the gain of the manager is nonlinear, and the singularity yields the complexity in establishing optimal policies.

2.4. **Utility.** As the objectives of the fund manager and members are to maximize the utilities of their respective gains, there are disputes in terms of their heterogenous objectives. We thus study the optimal control problem from the manager's perspective. The members passively follow the policies of the manager. This phenomenon could be observed for most DC pension funds, as members lack the ability to monitor the behavior of the manager. In China, most retirement funds are managed by pension management companies. Meanwhile, members are unable to participate in the investment decision processes. For this reason, the opacity of information has exacerbated the moral hazard problem. Therefore, it is important to design more efficient incentive schemes to motivate managers to exert greater effort for productivity.

The objective function of the fund manager is

$$\mathbb{E}[U_i(X_T)], \tag{2.7}$$

where $U_i = \widehat{U} \circ \Theta_i$. $\widehat{U}(\cdot)$ is the manager's utility function, and we choose CRRA utility for simplicity. $\Theta_i(\cdot)$ measures the gain of the manager under incentive scheme i, which is the function of terminal wealth X_T. For a unified notation, we do not distinguish between U_1 and U_2 and omit their subscripts. Therefore, the stochastic control problem is formalized as follows:

$$u(X_0) = \sup_{\pi \in \mathcal{V}[0,T]} \{\mathbb{E}[U(X_T)]\}. \tag{2.8}$$

To provide an explicit expression of U, we set

$$\widehat{U}(x) = \frac{x^p}{p}, \tag{2.9}$$

where $p \in (-\infty, 0) \bigcup (0, 1)$ is the diminishing sensitivity parameter, which measures the risk aversion degree of the manager. The objective functions under the two managerial incentive schemes are expressed by

$$U_1(x) = \frac{k_1^p x^p}{p}, \quad x \geq \rho G \tag{2.10}$$

and

$$U_2(x) = \begin{cases} \dfrac{(k_2 x + k_3(x - G))^p}{p}, & x \geq G, \\ \dfrac{k_2^p x^p}{p}, & \rho G \leq x < G. \end{cases} \tag{2.11}$$

Furthermore, to measure the utility of members, we assume that the preferences of members are represented by the CRRA utility function with a different diminishing sensitivity parameter:

$$\widehat{I}(x) = \frac{x^q}{q}. \tag{2.12}$$

Usually, the risk aversion degree of members is higher than that of the manager, that is, $q \in (-\infty, 0) \bigcup (0, 1)$ and $q < p$. The utilities of members under the two incentive schemes are

$$I_1(x) = \frac{(1 - k_1)^q x^q}{q}, \quad x \geq \rho G \tag{2.13}$$

and

$$I_2(x) = \begin{cases} \dfrac{((1 - k_2 - k_3)x + k_3 G)^q}{q}, & x \geq G, \\ \dfrac{(1 - k_2)^q x^q}{q}, & \rho G \leq x < G. \end{cases} \tag{2.14}$$

We assume that the policies are controlled by the manager, and the utility of members is achieved by passively following the optimal policies of the manager.

3. General solution

According to the above analysis, the continuous contribution injection process eliminates the self-financing property of the fund wealth process. Drawing from Merton(1969)[21], Han and Hung(2012) [13], and Guan and Liang(2016)[11], we add an auxiliary labor capital process to change the original problem into an equivalent self-financing one. Therefore, the martingale and duality methods can be used in the self-financing optimization, and the optimal risk-taking policies can be derived from the terminal payoff through the completeness of the market.

3.1. **Labor capital.** We define $H(t)$ as the labor capital per unit contribution at time t, that is,

$$H(t) = \int_t^T e^{-r(s-t)} ds = \frac{1}{r}(1 - e^{-r(T-t)}).$$

$H(t)c$ is the present value of the future contributions, representing the labor capital at time t. Total wealth Y_t at time t is:

$$Y_t = X_t + H(t)c,$$

which is compounded of fund wealth and labor capital. Therefore, the dynamics of Y_t satisfy the following SDE:

$$\begin{cases} dY_t = rY_t dt + \pi_t \sigma (dW_t + \theta dt), \\ Y_0 = \frac{1}{r}(1 - e^{-rT})c. \end{cases} \tag{3.1}$$

Obviously, the total wealth process $\{Y_t\}_{0 \le t \le T}$ is a self-financing process, and the stochastic optimal control problem could be solved under the martingale framework. Furthermore, the labor capital $\{H(t)c\}_{0 \le t \le T}$ is a deterministic process and the optimization problem on $\{X_t\}_{0 \le t \le T}$ could be easily transformed into an optimization problem on $\{Y_t\}_{0 \le t \le T}$. Particularly, labor capital terminates at the time of retirement, and $X_T = Y_T$ is valid. Furthermore, the optimal risk-taking policies could be derived from the optimal terminal payoff by the completeness of the market. We carry out the analysis based on total wealth to avoid the impacts of the life-cycle effect, and the results are more robust, as shown by Merton(1969,1971)[21, 22].

3.2. **Terminal optimization.** Using the martingale properties, the stochastic singular optimization problem (2.8), in which the element of $\mathcal{V}[0,T]$ is a stochastic process, can be transformed into optimization at terminal time T:

$$\sup_{Y_T\in\mathcal{V}(Y_0)}\{\mathbb{E}[U(Y_T)]\}, \tag{3.2}$$

where

$$\mathcal{V}(Y_0)\triangleq\left\{Y_T : Y_T \text{ is } \mathcal{F}_T\text{-measurable},\ \mathbb{E}[\xi_T Y_T]=Y_0;\ Y_T\geq M,\ Y_T^2<+\infty,\ a.s.\right\}. \tag{3.3}$$

Proposition 3.1. *We have*

$$\mathcal{V}(Y_0)=\{X_T:\ \exists\,\pi\in\mathcal{V}[0,T],\ \textit{s.t. } X \textit{ is the solution of SDE(2.3) and } X_T\geq M\}.$$

Thus,

$$u(X_0)=\sup_{Y_T\in\mathcal{V}(Y_0)}\{\mathbb{E}[U(Y_T)]\}.$$

Proof. It is easy to verify $\mathbb{E}[\xi_T X_T]=X_0+H(0)c$ and $X_T=Y_T$. Therefore,

$$\{X_T:\ \exists\,\pi\in\mathcal{V}[0,T],\ \text{s.t. } X \text{ is the solution of SDE(2.3) and } X_T\geq M\}\subseteq\mathcal{V}(Y_0).$$

On the other hand, for any $Y_T\in\mathcal{V}(Y_0)$, using the martingale representation theorem, there exists an $\mathcal{F}$-adapted process A satisfying

$$\xi_T Y_T=\mathbb{E}[\xi_T Y_T]+\int_0^T A_t dW_t \ \text{ and } \int_0^T A_t^2 dt<\infty, a.s.$$

Letting $\pi_t=(\sigma\xi_t)^{-1}(A_t+\theta\xi_t X_t)$, $X=\{X_t\triangleq Y_t-H(t)c, t\in[0,T]\}$ be the unique solution of SDE (2.3) and $X_T\geq M$, implies that

$$\mathcal{V}(Y_0)\subseteq\{X_T:\ \exists\,\pi\in\mathcal{V}[0,T],\ \text{s.t. } X \text{ is the solution of SDE(2.3) and } X_T\geq M\}.$$

□

3.3. **Duality method.** As the nonlinear gain property damages the concavity of the utility function under the mixed scheme, we explore the duality method to solve the problem. By establishing the envelope of the singular utility function, we determine optimal terminal fund wealth and, finally, find the optimal risk-taking policies according to the payoff generated from the envelope. For the unified method, we do not distinguish between U_1 and U_2 and omit their subscripts in this subsection. The conjugate function of U is

$$V(y) = \sup_{x \geq M}[U(x) - yx]. \tag{3.4}$$

We denote $\mathcal{Y}(y)$ as the conjugate point of y with respect to U, that is,

$$\mathcal{Y}(y) \triangleq \arg\sup_{x \geq M}[U(x) - yx].$$

$\mathcal{Y}(\cdot)$ is a single-valued function, except on finitely many points. From the theory of convex analysis (cf. Fang and Xing(2013) [6]), we know $V(y)$ is differentiable and $V'(y) = -\mathcal{Y}(y)$.

Lemma 3.1. *$u(X_0)$ is upper bounded by*

$$\inf_{\nu > 0}\{\nu Y_0 + \mathbb{E}[V(\nu \xi_T)]\}. \tag{3.5}$$

Proof. For any $\nu > 0$, $X_T \in \mathcal{V}(Y_0)$, we have

$$V(\nu \xi_T) \geq U(Y_T) - \nu \xi_T Y_T, a.s..$$

Taking expectation and supremum over $\mathcal{V}(Y_0)$, we obtain

$$\mathbb{E}[V(\nu \xi_T)] \geq \sup_{Y_T \in \mathcal{V}(Y_0)} \mathbb{E}[U(Y_T)] - \nu Y_0.$$

Therefore, the upper bound of $u(X_0)$ is

$$\inf_{\nu > 0}\{\mathbb{E}[V(\nu \xi_T)] + \nu Y_0\}.$$

□

From the mathematical viewpoint, the existence of the upper bound is provided by the weak duality. If we choose an element of $\mathcal{V}(Y_0)$ attaining the upper bound, then the duality gap is eliminated and, consequently, this element is the optimum of the terminal optimization. Theorem 3.1 below shows the respective optimal policy and optimal fund wealth.

We prove the general results of the optimal wealth process and optimal policies in Theorem 3.1. In Section 4, we verify the corresponding conditions to confirm the validity of Theorem 3.1 and establish closed-form optimal risk-taking policies.

Theorem 3.1. *For every case in Section 4:*

(1) If $\rho < \frac{\frac{1}{r}(1-e^{-rT})}{\frac{1}{\bar{r}}(1-e^{-\bar{r}T})} e^{(r-\bar{r})T}$ (or equivalently, $Me^{-rT} < Y_0$), ν^* is uniquely determined by

$$\mathbb{E}[\xi_T \mathcal{Y}(\nu^* \xi_T)] = Y_0.$$

(2) The optimum of the optimization problem (3.2) is

$$Y_T^* = \mathcal{Y}(\nu^* \xi_T). \tag{3.6}$$

(3) The optimal fund wealth process is

$$X_t^* = f(t, \xi_t) - H(t)c,$$

where $f(t,\xi)$ is given by (3.8) below.

(4) If $f(t,\xi)$ is C^2 continuous with respect to ξ, the optimal policy is given by

$$\pi_t^* = -\frac{\theta}{\sigma}\frac{\partial f}{\partial \xi}(t, \xi_t)\xi_t. \tag{3.7}$$

Proof.

(1) We verify the existence and uniqueness of ν^*. Denoting

$$R(\nu) \triangleq \mathbb{E}[\xi_T \mathcal{Y}(\nu^* \xi_T)],$$

we know

- $R(\nu)$ is continuous, since there exists no atom for ξ_T;
- $R(\nu)$ is strictly decreasing on $(0, +\infty)$, as $\mathcal{Y}$ is strictly decreasing;
- $\lim_{\nu \downarrow 0} R(\nu) = +\infty$;
- $\lim_{\nu \uparrow +\infty} R(\nu) = Me^{-rT} = \frac{\rho c}{\bar{r}}(1 - e^{-\bar{r}T})e^{(\bar{r}-r)T}$, and $Y_0 = H(0) = \frac{c}{r}(1 - e^{-rT})$ by (2.4).

For $\rho < \frac{\frac{1}{r}(1-e^{-rT})}{\frac{1}{\bar{r}}(1-e^{-\bar{r}T})} e^{(r-\bar{r})T}$, we have

$$\lim_{\nu \uparrow +\infty} R(\nu) < Y_0 < \lim_{\nu \downarrow 0} R(\nu),$$

which means there exists a unique $\nu^* > 0$ so that $R(\nu^*) = Y_0$.

(2) We verify the optimality and feasibility of $Y_T^* = \mathcal{Y}(\nu^* \xi_T)$.

- Optimality. ν^* satisfies the optimum condition of (3.5):

$$\mathbb{E}[\xi_T V'(\nu^* \xi_T)] + Y_0 = -\mathbb{E}[\xi_T \mathcal{Y}(\nu^* \xi_T)] + Y_0 = 0.$$

- Feasibility. We have $\mathbb{E}[\xi_T Y_T^*] = \mathbb{E}[\xi_T \mathcal{Y}(\nu^* \xi_T)] = Y_0$. Furthermore, $\mathcal{Y}$ is defined as larger than M. $\mathcal{Y}(\nu^* \xi_T) < +\infty$, *a.s.*. Therefore, we verify $Y_T^* \in \mathcal{V}(Y_0)$.

(3) We construct the optimal fund wealth process satisfying $X_T^* = Y_T^*$. Using the martingale method (Karatzas et al. (1991)[17]), we have

$$X_t^* = \xi_t^{-1} \mathbb{E}[\xi_T Y_T^* | \mathcal{F}_t] - H(t)c.$$

Substituting (3.6) into the above equation, we have

$$X_t^* = \mathbb{E}[Z_{t,T}\mathcal{Y}(\nu^*\xi_t Z_{t,T})|\mathcal{F}_t] - H(t)c,$$

where $Z_{t,T} = \frac{\xi_T}{\xi_t} = \exp[-(r+\frac{\theta^2}{2})(T-t) - \theta(W_T - W_t)]$ and we use the fact that $Z_{t,T}$ is independent of $\mathcal{F}_t$.

We define

$$f(t,\xi) \triangleq \mathbb{E}[Z_{t,T}\mathcal{Y}(\nu^*\xi Z_{t,T})], \tag{3.8}$$

which indicates $X_t^* = f(t,\xi_t) - H(t)c$.

(4) If $f(t,\xi)$ is C^2 continuous with respect to ξ (we verify its validity for each case of Section 4), applying Itô's formula to X_t^*, yields

$$dX_t^* = (\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \xi}\xi_t r + \frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}\xi_t^2\theta^2 - H'(t)c)dt + (-\frac{\partial f}{\partial \xi}\xi_t\theta dW_t).$$

We compare the last equation to SDE (2.3) and derive the optimal policy as:

$$\pi_t^* = -\frac{\theta}{\sigma}\frac{\partial f}{\partial \xi}(t,\xi_t)\xi_t,$$

which will be explicitly expressed in the next section.

Next, referring to the proof of Lin et al. (2017)[20], we prove the admissibility of π^*. By Proposition 3.1, we have $\pi_t^* = (\sigma\xi_t)^{-1}(A_t + \theta\xi_t X_t^*)$, where

$$\int_0^T A_t^2 dt < \infty, \quad a.s..$$

Because ξ_t and X_t^* are continuous almost surely processes,

$$\max_{0\le t\le T}(\xi_t^{-2}) < \infty, \quad \max_{0\le t\le T}(X_t^*)^2 < \infty, \quad a.s..$$

Therefore,

$$\int_0^T (\pi_t^*)^2 dt \le 2\sigma^{-2}\max_{0\le t\le T}(\xi_t^{-2})\int_0^T A_t^2 dt + 2(\frac{\theta}{\sigma})^2 T\max_{0\le t\le T}(X_t^*)^2 < \infty,$$

almost surely, which implies π^* is admissible.

□

The optimal policy and optimal fund wealth are expressed by an implicit-form function f and the density process $\{\xi_t\}_{0\le t\le T}$. In economics, ν^* is the marginal utility of one extra unit of fund wealth at the initial time. Mathematically, ν^* is determined by V, which is the conjugate function of U. Therefore, f is only affected by utility U, that is, the incentive scheme. According to the properties of SDE, there is a one-to-one mapping between the density process and solution of SDE (2.3). As such, at any time $t \in [0,T]$, π_t is a function of ξ_t and X_t^*, that is, the optimal policy is a feedback control.

4. **Optimal Control Policies and Value Functions**

Here, we provide the explicit expressions of the optimal policies, optimal fund wealth, and the corresponding value functions under two incentive schemes.

4.1. **The single incentive scheme.** We derive the value function and the optimal risk-taking policy of the fund manager under the single management fee scheme.

First, the expressions of the conjugate function V_1 and conjugate point $\mathcal{Y}_1$ are, respectively:

$$V_1(y) = \begin{cases} \frac{1-p}{p}(\frac{y}{k_1})^{\frac{p}{p-1}}, & 0 < y < k_1^p M^{p-1}, \\ -My + \frac{(k_1 M)^p}{p}, & y \geq k_1^p M^{p-1} \end{cases} \tag{4.1}$$

and

$$\mathcal{Y}_1(y) = \begin{cases} k_1^{\frac{-p}{p-1}} y^{\frac{1}{p-1}}, & 0 < y < k_1^p M^{p-1}, \\ M, & y \geq k_1^p M^{p-1}. \end{cases} \tag{4.2}$$

Second, for explicit expressions of the optimal policy and the optimal fund wealth, we need to express f in Theorem 3.1 explicitly. Substituting (4.2) into the definition of f, we have

$$f(t,\xi_t) = \left(\Phi(d_{1,t}) + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}\right)Me^{-r(T-t)},$$

where Φ is the cumulative distribution function of a standard normal random variable $\mathcal{Z}$. Moreover, for any $t \in [0,T]$, $d_{1,t}$ and $d_{2,t}$ are functions of ξ_t and t:

$$d_{1,t} \triangleq \frac{\ln\left(\frac{k_1^p M^{p-1}}{\nu_1^* \xi_t}\right) + (r - \frac{\theta^2}{2})(T-t)}{-\theta\sqrt{T-t}}, \quad d_{2,t} \triangleq -d_{1,t} + \frac{\theta\sqrt{T-t}}{1-p}.$$

Note that $f(t,\xi)$ is sufficiently smooth with respect to ξ. Therefore, Theorem 3.1 is valid, and the optimal fund wealth at time t is

$$X_t^{1*} = \left(\Phi(d_{1,t}) + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}\right)Me^{-r(T-t)} - H(t)c. \tag{4.3}$$

It can be verified that

$$\Phi(d_{1,t}) + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})} > 1, \quad \forall t < T.$$

The optimal total wealth Y_t^{1*} is strictly greater than $Me^{-r(T-t)}$ before the terminal time. Thus, $X_t^{1*} > Me^{-r(T-t)} - H(t)c$ is always valid before the terminal time. Consequently, optimal fund wealth will not be liquidated before the terminal time.

For the explicit expression of the optimal policy, we have to differentiate f with respect to ξ_t:

$$\begin{aligned}
&-\frac{\partial f}{\partial \xi_t}(t,\xi_t)\xi_t \\
=&-\Phi'(d_{1,t})Me^{-r(T-t)}\frac{1}{\theta\sqrt{T-t}} \\
&+\Phi'(d_{2,t})k_1^{\frac{-p}{p-1}}(\nu_1^*\xi_t)^{\frac{1}{p-1}}e^{\frac{p}{1-p}r(T-t)}e^{\frac{-p}{(1-p)^2}\frac{\theta^2}{2}(T-t)}\frac{1}{\theta\sqrt{T-t}} \\
&+\Phi(d_{2,t})k_1^{\frac{-p}{p-1}}(\nu_1^*\xi_t)^{\frac{1}{p-1}}e^{\frac{p}{1-p}r(T-t)}e^{\frac{-p}{(1-p)^2}\frac{\theta^2}{2}(T-t)}\frac{1}{1-p} \\
=&-\Phi'(d_{1,t})Me^{-r(T-t)}\frac{1}{\theta\sqrt{T-t}}+\Phi'(d_{1,t})Me^{-r(T-t)}\frac{1}{\theta\sqrt{T-t}} \\
&+\Phi(d_{2,t})k_1^{\frac{-p}{p-1}}(\nu_1^*\xi_t)^{\frac{1}{p-1}}e^{\frac{p}{1-p}r(T-t)}e^{\frac{-p}{(1-p)^2}\frac{\theta^2}{2}(T-t)}\frac{1}{1-p} \\
=&\frac{1}{1-p}(f(t,\xi_t)-\Phi(d_{1,t})Me^{-r(T-t)}).
\end{aligned}$$

Therefore, the optimal policy at time t is

$$\pi_t^{1*} = \frac{\theta}{(1-p)\sigma}\Bigg(\underbrace{X_t^{1*}+H(t)c}_{\text{total wealth}} - \underbrace{\Phi(d_{1,t})Me^{-r(T-t)}}_{\text{deposit term}}\Bigg). \qquad (4.4)$$

Being different from the classical Merton model, the optimal risky investment amount is similarly formed, except that a deposit term is subtracted from total wealth. This deposit term specifically ensures that the optimal fund wealth cannot be liquidated before the terminal time.

Finally, we calculate the value functions of the manager and of the members by passively following the optimal policies, respectively:

$$
\begin{aligned}
&\mathbb{E}[U_1(X_T^{1*})] \\
=&\mathbb{E}\left[U_1\left(M\mathbf{1}_{\left\{\nu_1^*\xi_T\geq k_1^p M^{p-1}\right\}}+k_1^{\frac{-p}{p-1}}(\nu_1^*\xi_T)^{\frac{1}{p-1}}\mathbf{1}_{\left\{\nu_1^*\xi_T<k_1^p M^{p-1}\right\}}\right)\right] \\
=&\mathbb{E}\left[\frac{(k_1M)^p}{p}\mathbf{1}_{\left\{\nu_1^*Z_{0,T}\geq k_1^p M^{p-1}\right\}}+\frac{1}{p}\left(\frac{\nu_1^*Z_{0,T}}{k_1}\right)^{\frac{p}{p-1}}\mathbf{1}_{\left\{\nu_1^*Z_{0,T}<k_1^p M^{p-1}\right\}}\right] \\
=&\frac{(k_1M)^p}{p}\Phi(d_{0,0})+\frac{1}{p}\left(\frac{\nu_1^*}{k_1}\right)^{\frac{p}{p-1}}\Phi(d_{2,0})e^{\frac{p}{1-p}rT}e^{\frac{p}{(1-p)^2}\frac{\theta^2}{2}T}
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
&\mathbb{E}[I_1(X_T^{1*})] \\
=&\mathbb{E}\left[\frac{((1-k_1)M)^q}{q}\mathbf{1}_{\left\{\nu_1^*Z_{0,T}\geq k_1^p M^{p-1}\right\}}\right. \\
&\left.+\frac{1}{q}\left(\frac{1-k_1}{k_1}\right)^p\left(\frac{\nu_1^*Z_{0,T}}{k_1}\right)^{\frac{q}{p-1}}\mathbf{1}_{\left\{\nu_1^*Z_{0,T}<k_1^p M^{p-1}\right\}}\right] \\
=&\frac{((1-k_1)M)^p}{p}\Phi(d_{0,0}) \\
&+\frac{1}{q}\left(\frac{1-k_1}{k_1}\right)^q\left(\frac{\nu_1^*}{k_1}\right)^{\frac{q}{p-1}}\Phi(d_{2,0}^q)e^{\frac{q}{1-p}rT}e^{\frac{q-pq+q^2}{(1-p)^2}\frac{\theta^2}{2}T},
\end{aligned}
\tag{4.6}
$$

where

$$
d_{0,0}=\frac{\ln\left(\frac{k_1^pM^{p-1}}{\nu_1^*}\right)+(r+\frac{\theta^2}{2})T}{-\theta\sqrt{T}},\ d_{2,0}^q=-(d_{0,t}+\frac{q}{p-1}\theta\sqrt{T}).
$$

The expressions above include marginal utility ν_1^*. Here, we display the equation that ν_1^* satisfies:

$$
\begin{aligned}
&f(0,1)=X_0+H(0)c \\
\Leftrightarrow&\left(\Phi(d_{1,0})+\Phi(d_{2,0})\frac{\Phi'(d_{1,0})}{\Phi'(d_{2,0})}\right)Me^{-rT}=X_0+H(0)c,
\end{aligned}
$$

where $d_{1,0},d_{2,0}$ are functions of ν_1^*.

Summarizing the above calculations, Theorem 4.1 gives the closed-forms of the optimal policy and optimal fund wealth under the single incentive scheme.

Theorem 4.1. *Under the single incentive scheme, the optimal policy and optimal fund wealth are, respectively,*

$$
\pi_t^{1*}=\frac{\theta}{(1-p)\sigma}\left(X_t^{1*}+H(t)c-\Phi(d_{1,t})Me^{-r(T-t)}\right)
$$

and

$$X_t^{1*} = \left(\Phi(d_{1,t}) + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}\right)Me^{-r(T-t)} - H(t)c,$$

where $d_{1,t}$ *and* $d_{2,t}$ *are functions of* t *and* ξ_t*:*

$$d_{1,t} = \frac{\ln\left(\frac{k_1^p M^{p-1}}{\nu_1^* \xi_t}\right) + (r - \frac{\theta^2}{2})(T-t)}{-\theta\sqrt{T-t}}, \quad d_{2,t} = -d_{1,t} + \frac{\theta\sqrt{T-t}}{1-p}.$$

4.2. **The mixed incentive scheme.** Here, we derive the optimal utility and optimal risk-taking policy of the fund manager under the mixed scheme. It is notable that V_2 depends on the relationship among ρ, k_2, and k_3. From the analysis, there are two heterogenous cases distinguished by the relationship between ρ and $\frac{\frac{p}{1-p}k_3(k_2+k_3)^{\frac{1}{p-1}}}{k_2^{\frac{p}{p-1}} - (k_2+k_3)^{\frac{p}{p-1}}}$.

4.2.1. *Case 1:*

$$\rho \geq \frac{\frac{p}{1-p}k_3(k_2+k_3)^{\frac{1}{p-1}}}{k_2^{\frac{p}{p-1}} - (k_2+k_3)^{\frac{p}{p-1}}}. \tag{4.7}$$

There exists a unique line passing through $(M, U_2(M))$ and tangent to U_2, such as in Fig.2. Denoting G' as the point of tangency, the concave envelope of U_2 is:

$$U_2^{**}(x) = \begin{cases} \dfrac{(k_2 x + k_3(x-G))^p}{p}, & x \geq G', \\ U_2(M) + \dfrac{U_2(G') - U_2(M)}{G' - M}(x - M), & M < x < G'. \end{cases} \tag{4.8}$$

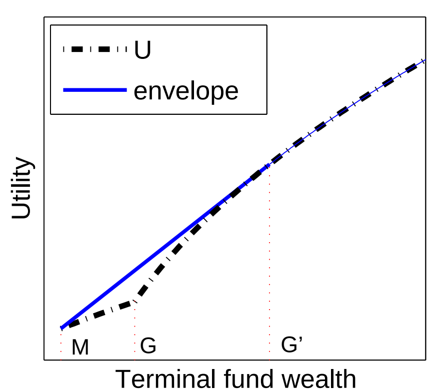

FIGURE 2. The concave envelope of the optimal utility in the mixed scheme.

Because the calculation is similar to the single incentive scheme, we omit the complicated calculations and only show the results. The optimal fund wealth at time t is:

$$
\begin{aligned}
X_t^{2*} =& e^{-r(T-t)}\Bigg(\frac{k_3 G}{k_2+k_3} + \Phi(g_{1,t})\Bigg(M - \frac{k_3 G}{k_2+k_3}\Bigg) \\
&+ \Phi(g_{2,t})\frac{\Phi'(g_{1,t})}{\Phi'(g_{2,t})}\Bigg(G' - \frac{k_3 G}{(k_2+k_3)}\Bigg)\Bigg) - H(t)c.
\end{aligned}
\tag{4.9}
$$

For any $t \in [0,T]$, we define

$$
g_{1,t} = \frac{\ln\left(\frac{U_2'(G')}{\nu_2^* \xi_t}\right) + (r - \frac{\theta^2}{2})(T-t)}{-\theta\sqrt{T-t}}, \quad g_{2,t} = -g_{1,t} + \frac{\theta\sqrt{T-t}}{1-p}.
$$

Note that $f(t,\xi_t) = X_t^{2*} + H(t)c$ is sufficiently smooth with respect to ξ_t. Therefore, Theorem 3.1 is valid. As such, the optimal policy at time t is

$$
\begin{aligned}
\pi_t^{2*} =& \frac{\theta}{(1-p)\sigma}\Bigg(\underbrace{X_t^{2*} + H(t)c}_{\text{total wealth}} \\
&- \underbrace{\left(\Phi(g_{1,t}) + (1-\Phi(g_{1,t}))\frac{k_3}{\rho(k_2+k_3)}\right) M e^{-r(T-t)}}_{\text{deposit term}} \\
&+ \underbrace{(1-p)\Phi'(g_{1,t})\frac{e^{-r(T-t)}}{\theta\sqrt{T-t}}\left(G' - M\right)}_{\text{risk seeking term}}\Bigg).
\end{aligned}
\tag{4.10}
$$

We can verify that the deposit term guarantees that fund wealth cannot be liquidated before the terminal time. Unlike in the single management fee case, there exists a risk-seeking term, generated by the incentive of the performance fee. The risk-seeking term could be large when time t approaches T and fund wealth is low. Under this circumstance, the manager prefers a more aggressive policy to have the opportunity of obtaining the performance fee. However, the risk-taking policy is more conservative when fund wealth is large, which is more favorable for members. In fact, the linear envelope exists only under the singular optimization of the mixed scheme, and the potential gambling risk-taking policy exists only under the mixed scheme. Therefore, the liquidation risks under the two incentive schemes cannot be controlled, as in He and Kou[15]. However, the mixed scheme provides utility improvement opportunities, which will be further discussed in Section 5.

In Fig.2, the linear envelope begins at the liquidation level and is tangent to utility function U at G', which we call the turning point. According to the risk-seeking term in (4.10), the optimal policy is to allocate a large amount of wealth to the risky asset and gamble for the chance of obtaining the performance fee between the liquidation level and the turning point, when time to maturity is short. Thus, terminal fund wealth will either be liquidated or become larger than the turning point. Beyond the turning point, the optimal utility function is the power function, which is similar to the one under the single incentive scheme. In this circumstance, the management fee is rewarding enough, so that the manager need not endure more risk. Generally, G' is the turning point of the manager's attitude changing from risk seeking to risk averse.

Finally, we calculate the value functions of the manager and of the members by passively following the policies.

$$\begin{aligned}\mathbb{E}[U_2(X_T^{2*})] = &\frac{(k_2M)^p}{p}\Phi(g_{0,0})+\\ &\frac{1}{p}(\frac{\nu_2^*}{k_2+k_3})^{\frac{p}{p-1}}\Phi(g_{2,0})e^{\frac{p}{1-p}rT}e^{\frac{p}{(1-p)^2}\frac{\theta^2}{2}T}\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\mathbb{E}[I_2(X_T^{2*})] = &\frac{((1-k_2)M)^q}{q}\Phi(g_{0,0})+\\ &\mathbb{E}\left[\frac{1}{q}\left((1-k_2-k_3)\left(\frac{\nu_2^* e^{-(r+\frac{\theta^2}{2})T-\theta\sqrt{T}\mathcal{Z}}}{(k_2+k_3)^p}\right)^{\frac{1}{p-1}}+\frac{k_3G}{k_2+k_3}\right)^q \mathbf{1}_{\{\mathcal{Z}>g_{0,0}\}}\right],\end{aligned} \tag{4.12}$$

where

$$g_{0,0} = \frac{\ln\left(\frac{U_2'(G')}{\nu_2^*}\right)+(r+\frac{\theta^2}{2})T}{-\theta\sqrt{T}}.$$

Marginal utility ν_2^* can be obtained from the following equation:

$$\begin{aligned}&e^{rT}(X_0+H(0)c)\\ =&\frac{k_3G}{k_2+k_3}+\Phi(g_{1,0})\left(M-\frac{k_3G}{k_2+k_3}\right)+\Phi(g_{2,0})\frac{\Phi'(g_{1,0})}{\Phi'(g_{2,0})}\left(G'-\frac{k_3G}{k_2+k_3}\right),\end{aligned}$$

where $g_{1,0}, g_{2,0}$ are functions of ν_2^*.

4.2.2. *Case 2:*

$$\rho < \frac{\frac{p}{1-p}k_3(k_2+k_3)^{\frac{1}{p-1}}}{k_2^{\frac{p}{p-1}}-(k_2+k_3)^{\frac{p}{p-1}}}, \tag{4.13}$$

There exists a unique $C_2 \in [M, G]$ and $G' > G$, so that

$$U_2^{**}(x) = \begin{cases} \dfrac{(k_2 x + k_3(x-G))^p}{p}, & x \geq G', \\ U_2(C_2) + \dfrac{U_2(G') - U_2(C_2)}{G' - C_2}(x - C_2), & C_2 < x < G', \\ \dfrac{k_2^p x^p}{p}, & x \leq C_2. \end{cases} \tag{4.14}$$

This case rarely appears, because the supervision requires that the liquidation level should be high enough. Therefore, the condition (4.13) is generally not satisfied for the realistic values of parameters. Nevertheless, we show the optimal fund wealth and optimal policy for the integrality of our discussion:

$$\begin{aligned} X_t^{2*} =& e^{-r(T-t)}\Bigg[\frac{k_3 G}{k_2+k_3} + \Phi(g_{2,t})\left(\frac{\Phi'(g_{1,t})}{\Phi'(g_{2,t})}(G' - \frac{k_3 G}{k_2+k_3}) - \frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\right) \\ & -\Phi(g_{1,t})\frac{k_3 G}{k_2+k_3} + \Phi(d_{1,t})M + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\Bigg] - H(t)c \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \pi_t^{2*} =& \frac{\theta}{(1-p)\sigma}\Bigg(\underbrace{X_t^{2*} + H(t)c}_{\text{total wealth}} \\ & - \underbrace{\left(\Phi(d_{1,t}) + (1 - \Phi(g_{1,t}))\frac{k_3}{\rho(k_2+k_3)}\right) M e^{-r(T-t)}}_{\text{deposit term}} \\ & + \underbrace{(1-p)\Phi'(g_{1,t})\frac{e^{-r(T-t)}}{\theta\sqrt{T-t}}\big(G' - \frac{\Phi'(g_{2,t})}{\Phi'(g_{1,t})}\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\big)}_{\text{risk seeking term}}\Bigg). \end{aligned} \tag{4.16}$$

Note that $f(t, \xi_t) = X_t^{2*} + H(t)c$ is sufficiently smooth with respect to ξ_t. Thus, Theorem 3.1 is valid for the derivation of π_t^*.

The value function of the manager is:

$$\begin{aligned} & \mathbb{E}[U_2(X_T^{2*})] \\ =& \frac{(k_2 M)^p}{p}\Phi(d_{0,0}) + \frac{1}{p}\exp\left(\frac{p}{1-p}rT + \frac{p}{(1-p)^2}\frac{\theta^2}{2}T\right) \\ & \times\left(\big(\frac{\nu_2^*}{k_2+k_3}\big)^{\frac{p}{p-1}}\Phi(g_{2,0}) + \big(\frac{\nu_2^*}{k_2}\big)^{\frac{p}{p-1}}(\Phi(d_{2,0}) - \Phi(g_{2,0}))\right), \end{aligned}$$

and the value function of members by passively following the policies is:

$$
\begin{aligned}
&\mathbb{E}[I_2(X_T^{2*})] = \frac{((1-k_2)M)^q}{q}\Phi(d_{0,0}) + \\
&\frac{1}{q}\left(\frac{1-k_2}{k_2}\right)^q \left(\frac{\nu_2^*}{k_2}\right)^{\frac{q}{p-1}} (\Phi(d_{2,0}^q) - \Phi(g_{2,0}^q)) e^{\frac{q}{1-p}rT + \frac{q-pq+q^2}{(1-p)^2}\frac{\theta^2}{2}T} + \\
&\mathbb{E}\left[\frac{1}{q}\left((1-k_2-k_3)\Big(\frac{\nu_2^* e^{-(r+\frac{\theta^2}{2})T - \theta\sqrt{T}\mathcal{Z}}}{(k_2+k_3)^p}\Big)^{\frac{1}{p-1}} + \frac{k_3 G}{k_2+k_3}\right)^q \mathbf{1}_{\{\mathcal{Z} > g_{0,0}\}}\right],
\end{aligned}
$$

where

$$
g_{2,0}^q = -(g_{0,0} + \frac{q}{p-1}\theta\sqrt{T}).
$$

Theorem 4.2 summarizes the results of both cases.

Theorem 4.2. *Under the mixed incentive scheme, the optimal policy and optimal fund wealth are*

$$
\begin{aligned}
\pi_t^{2*} =& \frac{\theta}{(1-p)\sigma}\Bigg(X_t^{2*} + H(t)c \\
&- \left(\Phi(d_{1,t}) + (1-\Phi(g_{1,t}))\frac{k_3}{\rho(k_2+k_3)}\right) M e^{-r(T-t)} \\
&+ (1-p)\Phi'(g_{1,t})\frac{e^{-r(T-t)}}{\theta\sqrt{T-t}}\Big(G' - \frac{\Phi'(g_{2,t})}{\Phi'(g_{1,t})}\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\Big)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
X_t^{2*} =& e^{-r(T-t)}\Bigg[\frac{k_3 G}{k_2+k_3} + \Phi(g_{2,t})\left(\frac{\Phi'(g_{1,t})}{\Phi'(g_{2,t})}(G' - \frac{k_3 G}{k_2+k_3}) - \frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\right) \\
&- \Phi(g_{1,t})\frac{k_3 G}{k_2+k_3} + \Phi(d_{1,t})M + \Phi(d_{2,t})\frac{\Phi'(d_{1,t})}{\Phi'(d_{2,t})}M\Bigg] - H(t)c,
\end{aligned}
$$

where $g_{1,t}$*,* $g_{2,t}$*,* $d_{1,t}$*,* $d_{2,t}$ *are*

$$
g_{1,t} = \frac{\ln\left(\frac{U_2'(G')}{\nu_2^*\xi_t}\right) + (r - \frac{\theta^2}{2})(T-t)}{-\theta\sqrt{T-t}}, \quad g_{2,t} = -g_{1,t} + \frac{\theta\sqrt{T-t}}{1-p},
$$

$$
d_{1,t} = \begin{cases} g_{1,t}, & \textit{Case 1}, \\ \dfrac{\ln\left(\frac{k_2^p M^{p-1}}{\nu_1^*\xi_t}\right) + (r - \frac{\theta^2}{2})(T-t)}{-\theta\sqrt{T-t}}, & \textit{Case 2}, \end{cases}
$$

and

$$
d_{2,t} = \begin{cases} g_{2,t}, & \textit{Case 1}, \\ -d_{1,t} + \dfrac{\theta\sqrt{T-t}}{1-p}, & \textit{Case 2}. \end{cases}
$$

5. Numerical examples

Here, we compare the optimal risk-taking policies of the DC pension fund manager and numerically evaluate the utility changes of both the manager and members under the two incentive schemes.

First, we set reasonable values for the parameters in the model. According to the empirical data on the financial market, parameter values are chosen as $r = 0.05, \mu = 0.17, \sigma = 0.3$. Thus, $\theta = 0.4$. Furthermore, $X_0 = 0$ and $c = 1$ represent the one-unit contribution rate. The duration of the accumulation phase of the DC pension plan lasts for $T = 20$ years. The accumulation rate is the rate for calculating expected accumulation. Therefore, it should be comparable with some well-known indexes. Obviously, the accumulation rate should be higher than the risk-free interest rate and lower than the return of the stock index, that is, $\bar{r} > r$ is valid. We use $\bar{r} = 0.06$ in the numerical analysis. ρ is the liquidation level. The safety requirement is the primary constraint of the DC pension plan, and $\rho = 0.7$. Moreover, condition $\rho \geq \frac{\frac{p}{1-p} k_3 (k_2+k_3)^{\frac{1}{p-1}}}{k_2^{\frac{p}{p-1}} - (k_2+k_3)^{\frac{p}{p-1}}}$ is usually valid in practice when the liquidation level is relatively high. The diminishing sensitivity parameter of the manager is $p = 0.5$, which is compatible with the parameter in He and Kou (2018)[15]. Since the members are more risk averse than the manager, their diminishing sensitivity parameter is $q = 0.3$.

Under the incentive fee, the management fee is within the threshold of $1\% - 2\%$ of the fund wealth and is charged annually. In this model, the management fee is charged once, according to the terminal fund wealth, and its rate should be relatively higher. We use $k_1 = 5\%$ in the single management fee scheme. In the mixed scheme, the rate of the management fee should be lower, and $k_2 = 3\%$. Moreover, the additional performance fee is included. According to the practice in the fund management industry, the performance fee ranges from 5% to 30% according to historical performance. In this paper, we test $k_3 = 5\%, 10\%$ and 20%, respectively, to study the impacts of performance fee rates on the optimal policies.

5.1. **Optimal risk investment proportion.** In Fig.3, we show the optimal proportion allocated to risky asset π_t^*/Y_t^* with respect to total wealth Y_t^* under the single management fee scheme and the mixed scheme of the performance fee rate, as $k_3 = 5\%, 10\%$ and 20%. The periods to the performance fee payment are $0.05, 1, 5$ and 20 years, respectively.

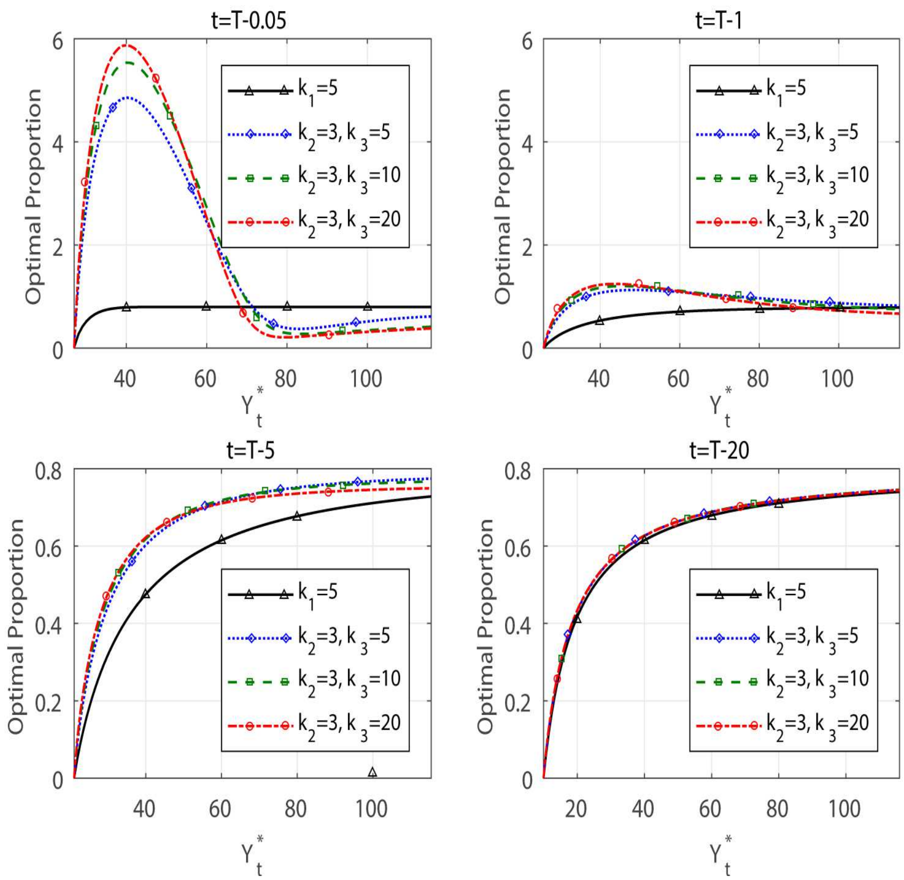

FIGURE 3. The optimal proportion allocated in the risky asset π_t^*/Y_t^*, with respect to total wealth Y_t^*.

If the risk investment proportion is calculated on fund wealth basis, the results are unstable because of the life-cycle effect. As such, the risk investment proportion is calculated on the basis of the total wealth. Therefore, the optimal proportion in the risky asset approaches the Merton line under the single incentive scheme, as per Merton(1969,1971)[21, 22]. In the 0.05 years to maturity case, that is, close to the performance fee payment, the optimal risky proportion reveals a peak-valley pattern within the interval between the liquidation level and the turning point under the mixed incentive scheme. When fund wealth is close to the liquidation level, the manager tries to avoid liquidation by allocating less to the risky asset. Moreover, the manager prefers to allocate a substantial proportion to the risky asset to gamble for gain when fund wealth is lower. In this circumstance, the higher performance fee rate results in a more aggressive risk-taking policy, which is common sense. Being different from the literature, when wealth is relatively large, the risk investment proportion is even smaller under the mixed scheme. Furthermore, the higher performance fee rate results in a more conservative risk-taking policy when wealth is beyond the turning point. By the diminishing marginal utility of the manager, the optimal policy becomes conservative when the gain is solid and the performance fee is reliable.

In the 1 year to maturity case, gambling behaviors are more moderate than in the 0.05 year case. Here, the manager has a longer time to gain and higher chance of receiving the performance fee. As such, the manager needs not take undue risk at the early stage. In the 5 and 20 year to maturity cases, the optimal risk investment proportions under the mixed scheme are compatible with, but slightly higher than, the single management fee scheme. The results in Fig.3 show that the incentive scheme transformation increases gambling behaviors when fund wealth is low and the time to maturity is short. However, the terminal risk-taking policies could be more conservative by well managing the fund during the preliminary stages of the accumulation phase. These complex risk-taking behaviors are the main findings of the study.

As the incentive scheme transformation alters the optimal risk-taking policies of the fund manager, this also changes the welfare of both the manager and members. In the following, we explore two categories of criteria to evaluate the impacts of the incentive scheme transformation.

5.2. **Liquidation probability.** There are two categories of criteria to evaluate the transformation: risk and utility changes. The literature on risk measures includes the Basel Committee using Value at Risk (VaR) and Artzner et al.(1999)[1] using conditional VaR as risk measures. Furthermore, the probability of liquidation is often used as risk measurement, as in He and Kou (2018)[15]. Fig.4 shows the liquidation probabilities with respect to the total wealth Y_t^* under the single management fee and mixed schemes of the performance fee rates as $k_3 = 5\%, 10\%$ and 20%, respectively. The periods to the performance fee payment are $0.05, 1, 5$ and 20 years, respectively.

According to the deposit terms in (4.4) and (4.10), liquidation will not happen before the terminal time. In Fig.4, we show liquidation probabilities at the terminal time.

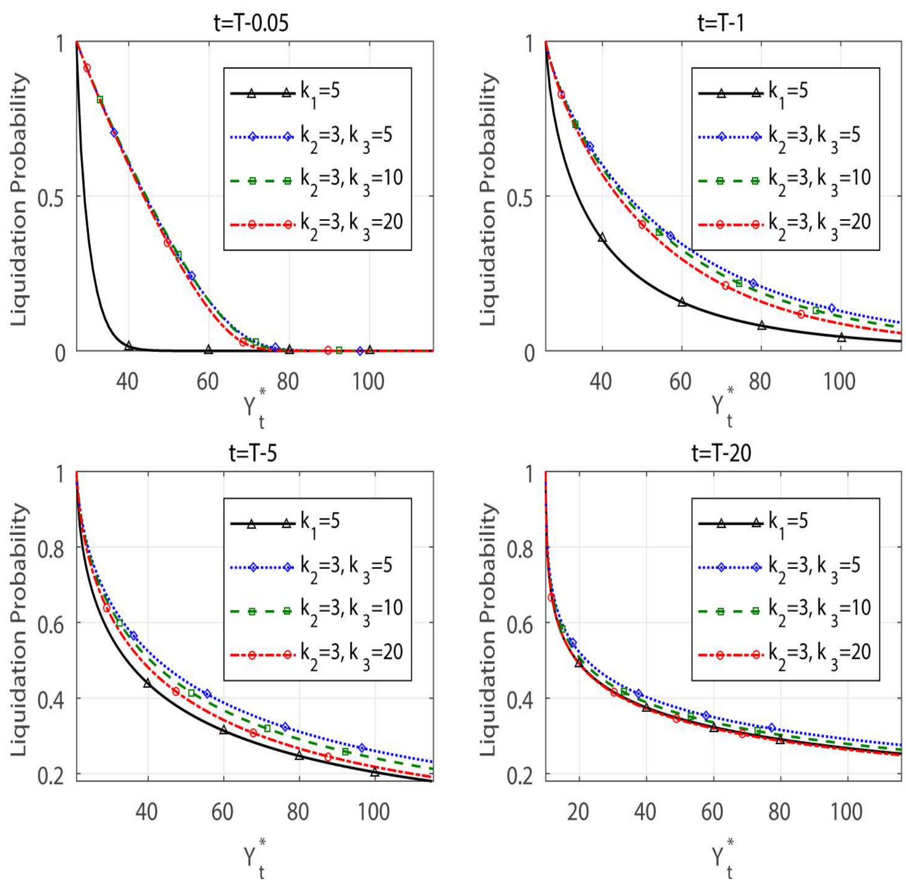

FIGURE 4. The liquidation probability with respect to the total wealth Y_t^*.

The figure shows that, when time to maturity is short, the liquidation probability under the mixed scheme is relatively higher than under the single incentive scheme when fund wealth is below the turning point. Under the single incentive scheme, liquidation probability decreases dramatically with respect to the increase in fund wealth. Conversely, the liquidation probability decreases moderately under the mixed scheme. In this circumstance, the manager will lose the entire performance fee if he/she does not take more risk to raise fund wealth above the benchmark. This gambling behavior will result in either liquidation or a fund wealth larger than the benchmark. In the cases with longer time to maturity, the differences in the liquidation probabilities between the two schemes are relatively moderate, as there are more follow-up contributions and investment flexibilities. Therefore, it is unnecessary to increase risk taking substantially at this stage. Interestingly, a higher performance fee rate results in a lower liquidation probability. When the reward rate of the fund manager is higher, the manager could obtain a relatively higher performance fee, and investment policies should be more conservative due to effect of diminishing marginal utility. Moreover, according to the discussions on (4.10), liquidation will not happen before the terminal time. The results are more optimistic compared to the literature.

5.3. **Utilities.** To compare utilities between the two schemes, we explore two approaches. One is to study the marginal utility of the fund manager for one extra unit of fund wealth. The other is sensitivity analysis on the utility changes of the manager and members after scheme transformation.

5.3.1. *Marginal utility.* Fig.5 shows the marginal utility of the fund manager for one extra unit of fund wealth, with respect to total wealth Y_t^* in the single management fee scheme, and the mixed scheme of the performance fee rates as $k_3 = 5\%, 10\%$ and 20%. The periods to the performance fee payment are $0.05, 1, 5$ and 20 years, respectively.

The figure shows that the marginal utility of one extra unit of fund wealth is higher under the mixed incentive scheme than under the single incentive scheme. The extra fund wealth improves the utility of the manager by directly increasing the amount of the reward under both schemes. Specifically, under the mixed scheme, the extra fund wealth also increases the probability of obtaining the performance fee, thus expanding the admissible domain with less aggressive policies. Furthermore, the one extra unit of fund wealth generates the highest marginal

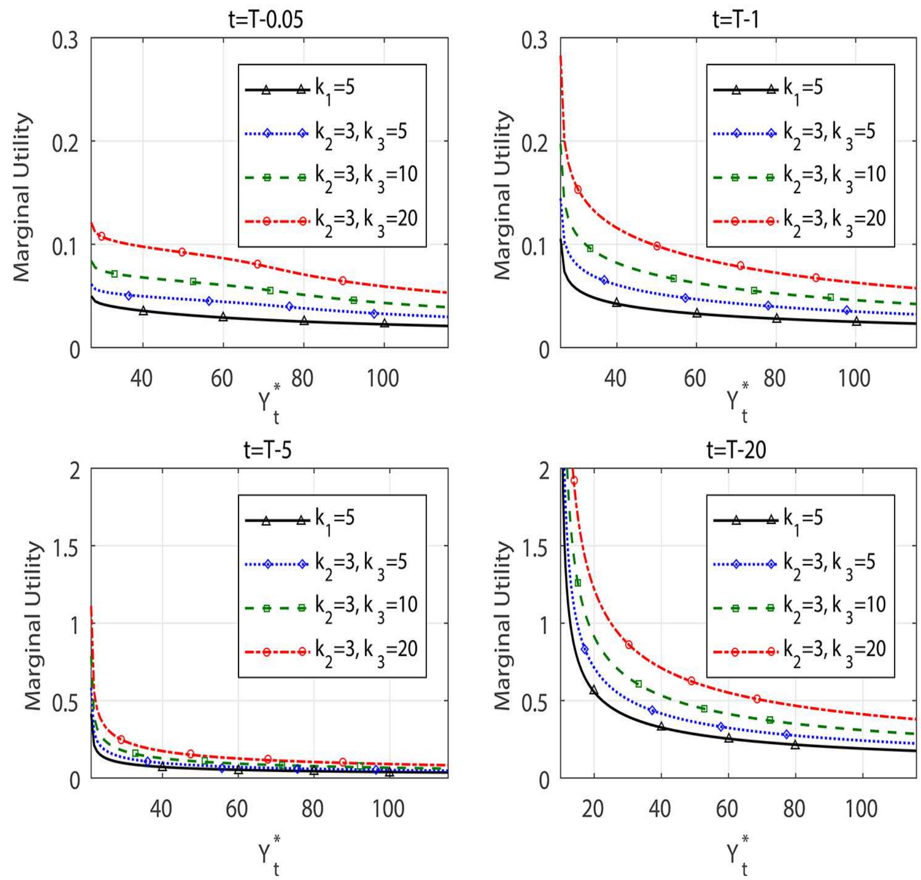

FIGURE 5. The marginal utility of the fund manager for one extra unit of fund wealth, with respect to total wealth Y_t^*.

utility for the manager under the highest performance fee rate case. The higher performance fee rate represents a higher amount of gain, which is common sense when wealth is large. However, when fund wealth is low, a higher performance fee rate results in more aggressive risk-taking behaviors. This gambling leads to higher probabilities for larger gains. Therefore, the marginal utility is relatively higher in this case. Moreover, the marginal utility improvement decreases as we approach the terminal time. The former accumulation is vital for DC pension fund management, as there is time for the manager to implement follow-up adaptive policies. According to the above results, for a higher performance fee rate, a well-managed fund will increase the manager's utility dramatically.

5.3.2. *Utility changes.* A controversial issue of the performance fee is that it motivates the manager to take more risk and gamble at the expense of fund members. Therefore, we should comprehensively study how the utilities of the manager and members change when the scheme changes. The utilities of members are defined in (2.13) and (2.14), and are derived in (4.6) and (4.12). The utilities of the manager are expressed by (4.5) and (4.11). As the members lack the ability to monitor the behaviors of the fund manager, they passively follow the optimal policies of the manager. The utilities of members are calculated on the basis of Monte Carlo methods.

In Fig.6, we consider the impacts of four important parameters on utility changes. We believe that the liquidation level ρ, market price of risk θ, accumulation rate $\bar{r}$, and management fee rate k_2 are the key parameters that affect utility changes. Obviously, k_3 plays a vital role in deciding utilities. As such, we treat k_3 as a variable parameter. $\underline{k_3}$ is the minimal performance fee rate, under which the manager's utility will be improved after scheme transformation. $\overline{k_3}$ is the maximal performance fee rate, under which the members' utility will be improved after the transformation. Obviously, if there exists an interval $[\underline{k_3}, \overline{k_3}]$ in which the performance fee rate lies, the utilities of the manager and members will improve simultaneously. Therefore, it is a win-win situation for them to switch to the mixed scheme by appropriately choosing incentive levels.

In Fig.6, the star line represents $\overline{k_3}$ of members and the circle line $\underline{k_3}$ of the manager. The results show that the subsets of the performance fee rates between $\underline{k_3}$ and $\overline{k_3}$ are not empty. Therefore, if the incentive performance fee is fixed within the subset, the utilities of both the manager and members could be improved.

First, the larger liquidation level ρ results in higher $\overline{k_3}$ and slightly higher $\underline{k_3}$. When the liquidation level is higher, members are better protected from large losses. Technically, the turning point G' to conservative investment policies is lower and the risk investment proportions could be stabilized at a lower level. This increases the utility improvement ability of members. Thus, members could afford a higher performance fee rate to maintain the same utility, as under the single incentive scheme.

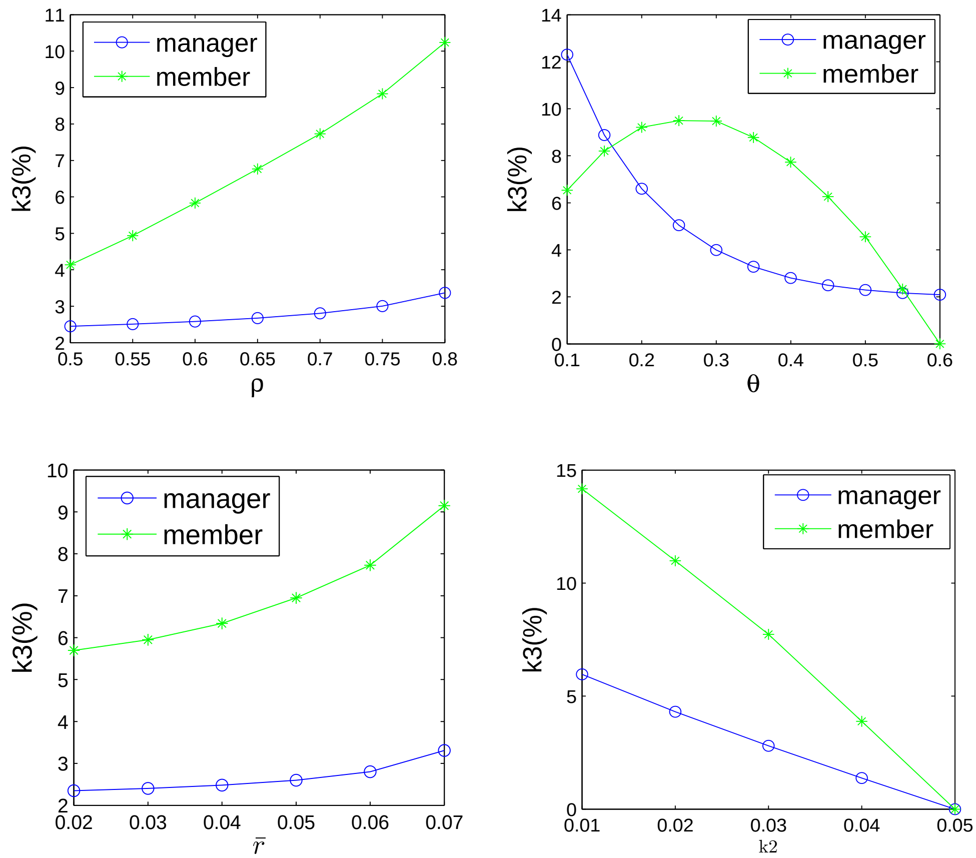

FIGURE 6. The impacts of ρ, θ, $\bar{r}$, and k_2 on the utility changes after the scheme transformation.

Second, the larger market price of risk θ first increases $\overline{k_3}$ and then decreases $\overline{k_3}$. Conversely, $\underline{k_3}$ keeps decreasing. When the market risk price is small, the manager has to gamble for the performance fee, and the utility improvement abilities of both members and the manager are low because of the instability and low probability of gain. As the market risk price increases, fund wealth increases more effectively, and the manager adopts more conservative investment policies. Therefore, the utility improvement abilities of both the members and manager increase. However, if the market risk price is too high, gambling risk-taking behaviors arise correspondingly, which makes the transformation unfavorable for members.

Third, a higher accumulation rate $\bar{r}$ increases $\overline{k_3}$ and $\underline{k_3}$ simultaneously. The higher $\bar{r}$ increases the benchmark of the availability of the performance fee. Technically, the turning point G' decreases correspondingly. Thus, the manager's optimal policy is to gamble within a smaller range of fund wealth, and the investment policy could be stabilized at this lower wealth level. This increases the utility improvement ability of members after the transformation. As such, $\overline{k_3}$ increases with respect to $\bar{r}$. Obviously, the higher benchmark reduces the utility improvement ability of the manager. Thus, a higher $\underline{k_3}$ is required to maintain the same utility as under the single incentive scheme.

Finally, $\overline{k_3}$ decreases with respect to performance fee rate k_2. The utility improvement ability of members is directly reduced as a higher management fee is charged by the manager. Conversely, a lower $\underline{k_3}$ is required, as the manager already obtains a sufficiently large management fee.

In conclusion, there is a wide range of performance fee rates, so that the utilities of the members and manager could both be improved after the transformation. A performance fee rate around 5% seems to fit all scenarios with reasonable parameters. Therefore, considering the variability of exogenous parameters, it remains a win-win situation for the manager and members to switch to the mixed scheme.

6. **Conclusions**

In this paper, we derive closed-form optimal risk-taking policies of DC pension fund managers and evaluate the utility changes for both the manager and members after incentive scheme transformation. The results reveal complex risk-taking behaviors. The optimal risk investment proportion shows a peak-valley pattern between the liquidation level and turning point under the mixed scheme. This reflects the fact that more aggressive risk-taking behaviors happen only when fund wealth is low and the time to maturity is short. Correspondingly, the liquidation probability under the mixed scheme is relatively higher than that under the single incentive scheme. However, liquidation will not happen before the terminal time. Different from the literature, the risk-taking policy is more conservative under the mixed scheme when fund wealth is relatively large. Moreover, the utilities of both the manager and members could be improved when the performance fee rate is appropriately chosen. As such, it is a win-win situation to implement the performance based incentive arrangement into DC pension plan management.

References

[1] Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber and David D. Heath. 1999. Coherent Measures of Risk. Mathematical Finance 9: 203-228.

[2] Bichuch, Maxim and Stephan Sturm. 2014. Portfolio Optimization under Convex Incentive Schemes. Finance and Stochastics 18: 873-915.

[3] Carpenter, Jennfer N. 2000. Does Option Compensation Increase Managerial Risk Appetite? Journal of Finance 55: 2311-2331.

[4] Cochrane, John H. 2005. Asset Pricing. Princeton University Press.

[5] Elton, Edwin J., Martin. J. Gruber and Christopher R. Blake. 2003. Incentive Fees and Mutual Funds. The Journal of Finance 58: 779-804.

[6] Fang, Shu-Cheng and Wen-Xun Xing. 2013. Linear Conic Optimization. Science Press, Beijing.

[7] Foster, Dean P. and H. Peyton Young. 2010. Gaming Performance Fees by Portfolio Managers. The Quarterly Journal of Economics 125: 1435-1458.

[8] Gerber, Hans U. and Elias S. W. Shiu. 2013. Geometric Brownian Motion Models for Assets and Liabilities: From Pension Funding to Optimal Dividends. North American Actuarial Journal 7: 37-51.

[9] Gomez-Mejia, Luis and Robert M. Wiseman. 1997. Refraining Executive Compensation: An Assessment and Outlook. Journal of Management 23: 291-374.

[10] Guan, Guo-Hui and Zong-Xia Liang. 2014. Optimal Management of DC Pension Plan in a Stochastic Interest Rate and Stochastic Volatility Framework. Insurance: Mathematics and Economics 57: 58-66.

[11] Guan, Guo-Hui and Zong-Xia Liang. 2016. A Stochastic Nash Equilibrium Portfolio Game between Two DC Pension Funds. Insurance: Mathematics and Economics 70: 237-244.

[12] Guan, Guo-Hui and Zong-Xia Liang. 2016. Optimal Management of DC Pension Plan under Loss Aversion and Value-at-Risk Constraints. Insurance: Mathematics and Economics 69: 224-237.

[13] Han, Nan-Wei and Mao-Wei Hung. 2012. Optimal Asset Allocation for DC Pension Plans under Inflation. Insurance: Mathematics and Economics 51: 172-181.

[14] Hawthorne, Fran. 1986. The Dawning of Performance Fees. Institutional Investor 20: 139-146.

[15] He, Xue-Dong and Steven Kou. 2018. Profit Sharing in Hedge Funds. Mathematical Finance 28: 50-81.

[16] Hodder, James E. and Jens Carsten Jackwerth. 2007. Incentive Contracts and Hedge Fund Management. Journal of Financial and Quantitative Analysis 42: 811-826.

[17] Karatzas, Ioannis, John P. Lehoczky, Steven. E. Shreve and Gan-Lin Xu. 1991. Martingale and Duality Methods for Utility Maximization in Incomplete Markets. SIAM Journal on Control and Optimization 29: 702-730.

[18] Kwak, Minsuk, Yong Hyun Shin and U. Jin Choi. 2011. Optimal Investment and Consumption Decision of a Family with Life Insurance. Insurance: Mathematics and Economics 48: 176-188.

[19] Li, C. Wei and Ashish Tiwari. 2009. Incentive Contract in Delegated Portfolio Management. The Review of Financial Studies 22: 4681-4714.

[20] Lin, Hong-Can, David Saunders and Cheng-Guo Weng. 2017. Optimal Investment Strategies for Participating Contracts. Insurance: Mathematics and Economics 73: 137-155.

[21] Merton, Robert C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. The Review of Economics and Statistics 51: 247-257.

[22] Merton, Robert C. 1971. Optimum Consumption and Portfolio Rules in A Continuous Time Model. Journal of Economic Theory 3: 373-413.

[23] Ngwira, Bernard and Russell Gerrard. 2007. Stochastic Pension Fund Control in the Presence of Poisson Jumps. Insurance: Mathematics and Economics 40: 283-292.

[24] Pirvu, Traian A. and Hua-Yue Zhang. 2012. Optimal Investment, Consumption and Life Insurance under Mean-Reverting Returns: The Complete Market Solution. Insurance: Mathematics and Economics 51: 303-309.

[25] Stark, Laura T. 1987. Performance Incentive Fees: An Agency Theoretic Approach. The Journal of Financial and Quantitative Analysis 22: 17-32.

[26] Stoughton, Neal M. 1993. Moral Hazard and the Portfolio Management Problem. Journal of Finance 48: 2009-2018.