扩展卡尔曼滤波EKF

如果将置信度和噪声限制为高斯分布，并且对运动模型和观测进行线性化，计算贝叶斯滤波中的积分（以及归一化积），即可得到扩展卡尔曼滤波（EKF）。

为了推导EKF，首先假设 $\boldsymbol{x}_{k}$ 的置信度函数限制为高斯分布：

$$p\left(\boldsymbol{x}_{k} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k}, \boldsymbol{y}_{0: k}\right)=\mathcal{N}\left(\hat{\boldsymbol{x}}_{k}, \hat{\boldsymbol{P}}_{k}\right)$$

其中，$\hat{\boldsymbol{x}}_{k}$ 为均值，$\hat{\boldsymbol{P}}_{k}$ 为协方差。

并且假设噪声变量 $\boldsymbol{w}_{k}$ 和 $\boldsymbol{n}_{k}$ 也是高斯分布的：

$$\begin{array}{l}\boldsymbol{w}_{k} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{Q}_{k}\right) \\\boldsymbol{n}_{k} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{R}_{k}\right)\end{array}$$

对于以下运动和观测模型：

$$\begin{aligned}&\boldsymbol{x}_{k}=f\left(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k},\boldsymbol{w}_{k}\right) \\&\boldsymbol{y}_{k}=g\left(\boldsymbol{x}_{k}, \boldsymbol{n}_{k}\right)\end{aligned}$$

由于 $\boldsymbol{f}(\cdot)$ 和 $\boldsymbol{g}(\cdot)$ 是非线性函数，所以我们需要对其进行线性化。**在当前状态的均值处**展开，对运动和观测模型进行线性化：

$$\boldsymbol{f}\left(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}, \boldsymbol{w}_{k}\right) \approx \check{\boldsymbol{x}}_{k}+\boldsymbol{F}_{k-1}\left(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1}\right)+\boldsymbol{w}_{k}^{\prime}$$

$$\boldsymbol{g}\left(\boldsymbol{x}_{k}, \boldsymbol{n}_{k}\right) \approx \check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right)+\boldsymbol{n}_{k}^{\prime}$$

其中：

- $\check{\boldsymbol{x}}_{k}=\boldsymbol{f}\left(\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{v}_{k}, \mathbf{0}\right)$
- $\boldsymbol{F}_{k-1}=\left.\frac{\partial \boldsymbol{f}\left(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}, \boldsymbol{w}_{k}\right)}{\partial \boldsymbol{x}_{k-1}}\right|_{\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{v}_{k}, \mathbf{0}}$
- $\boldsymbol{w}_{k}^{\prime}=\left.\frac{\partial \boldsymbol{f}\left(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}, \boldsymbol{w}_{k}\right)}{\partial \boldsymbol{w}_{k}}\right|_{\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{v}_{k}, \mathbf{0}} \boldsymbol{w}_{k}$
- $\check{\boldsymbol{y}}_{k}=\boldsymbol{g}\left(\check{\boldsymbol{x}}_{k}, \mathbf{0}\right)$
- $\boldsymbol{G}_{k}=\left.\frac{\partial \boldsymbol{g}\left(\boldsymbol{x}_{k}, \boldsymbol{n}_{k}\right)}{\partial \boldsymbol{x}_{k}}\right|_{\check{\boldsymbol{x}}_{k}, \mathbf{0}}$
- $\boldsymbol{n}_{k}^{\prime}=\left.\frac{\partial \boldsymbol{g}\left(\boldsymbol{x}_{k}, \boldsymbol{n}_{k}\right)}{\partial \boldsymbol{n}_{k}}\right|_{\check{\boldsymbol{x}}_{k}, \mathbf{0}} \boldsymbol{n}_{k}$

给定过去的状态和最新输入，则当前状态 $\boldsymbol{x}_{k}$ 的统计学特性为：

$$\boldsymbol{x}_{k} \approx \check{\boldsymbol{x}}_{k}+\boldsymbol{F}_{k-1}\left(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1}\right)+\boldsymbol{w}_{k}^{\prime}$$

$$E\left[\boldsymbol{x}_{k}\right] \approx \check{\boldsymbol{x}}_{k}+\boldsymbol{F}_{k-1}\left(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1}\right)+\underbrace{E\left[\boldsymbol{w}_{k}^{\prime}\right]}_{0}$$

$$E\left[\left(\boldsymbol{x}_{k}-E\left[\boldsymbol{x}_{k}\right]\right)\left(\boldsymbol{x}_{k}-E\left[\boldsymbol{x}_{k}\right]\right)^{\mathrm{T}}\right] \approx \underbrace{E\left[\boldsymbol{w}_{k}^{\prime} \boldsymbol{w}_{k}^{\prime\mathrm{T}}\right]}_{\boldsymbol{Q}_{k}^{\prime}}$$

$$p\left(\boldsymbol{x}_{k} \mid \boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}\right) \approx \mathcal{N}\left(\check{\boldsymbol{x}}_{k}+\boldsymbol{F}_{k-1}\left(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1}\right), \boldsymbol{Q}_{k}^{\prime}\right)$$

给定当前状态，则当前观测 $\check{\boldsymbol{y}}_{k}$ 的统计学特性为：

$$\boldsymbol{y}_{k} \approx \check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right)+\boldsymbol{n}_{k}^{\prime}$$

$$E\left[\boldsymbol{y}_{k}\right] \approx \check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right)+\underbrace{E\left[\boldsymbol{n}_{k}^{\prime}\right]}_{0}$$

$$E\left[\left(\boldsymbol{y}_{k}-E\left[\boldsymbol{y}_{k}\right]\right)\left(\boldsymbol{y}_{k}-E\left[\boldsymbol{y}_{k}\right]\right)^{\mathrm{T}}\right] \approx \underbrace{E\left[\boldsymbol{n}_{k}^{\prime} \boldsymbol{n}_{k}^{\prime \mathrm{T}}\right]}_{\boldsymbol{R}_{k}^{\prime}}$$

$$p\left(\boldsymbol{y}_{k} \mid \boldsymbol{x}_{k}\right) \approx \mathcal{N}\left(\check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right), \boldsymbol{R}_{k}^{\prime}\right)$$

由贝叶斯滤波器有：

$$\begin{aligned}& \underbrace{p\left(\boldsymbol{x}_{k} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k}, \boldsymbol{y}_{0: k}\right)}_{\text {后验置信度 }} \\=& \eta \underbrace{p\left(\boldsymbol{y}_{k} \mid \boldsymbol{x}_{k}\right)}_{\text {利用 } \boldsymbol{g}(\cdot) \text { 进行更新 }} \int \underbrace{p\left(\boldsymbol{x}_{k} \mid \boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}\right)}_{\text {利用 } \boldsymbol{f}(\cdot) \text { 进行预测 }} \underbrace{p\left(\boldsymbol{x}_{k-1} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k-1}, \boldsymbol{y}_{0: k-1}\right)}_{\text {先验置信度 }} \mathrm{d} \boldsymbol{x}_{k-1}\end{aligned}$$

将线性化之后的运动和观测模型代入到贝叶斯滤波器中，有：

$$\begin{array}{l}\underbrace{p\left(\boldsymbol{x}_{k} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k}, \boldsymbol{y}_{0: k}\right)}_{\mathcal{N}\left(\hat{\boldsymbol{x}}_{k}, \hat{\boldsymbol{P}}_{k}\right)}=\eta \underbrace{p\left(\boldsymbol{y}_{k} \mid \boldsymbol{x}_{k}\right)}_{\mathcal{N}\left(\check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right), \boldsymbol{R}_{k}^{\prime}\right)} \\\times \int \underbrace{p\left(\boldsymbol{x}_{k} \mid \boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}\right)}_{\mathcal{N}\left(\check{\boldsymbol{x}}_{k}+\boldsymbol{F}_{k-1}\left(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1}\right), \boldsymbol{Q}_{k}^{\prime}\right)} \underbrace{p\left(\boldsymbol{x}_{k-1} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k-1}, \boldsymbol{y}_{0: k-1}\right)}_{\mathcal{N}\left(\hat{\boldsymbol{x}}_{k-1}, \hat{\boldsymbol{P}}_{k-1}\right)} \mathrm{d} \boldsymbol{x}_{k-1}\end{array}$$

p(yk ∣ xk)×∫N(xˇk+Fk−1(xk−1−x^k−1),Qk′)

p(xk ∣ xk−1,vk)N(x^k−1,P^k−1)

p(xk−1 ∣ xˇ0,v1:k−1,y0:k−1)dxk−1

将服从高斯分布的变量传入到非线性函数中，积分之后仍然服从高斯分布：

p (x k ∣ x ˇ 0 , v 1 : k , y 0 : k) N (x ^ k , P ^ k) = η p (y k ∣ x k) N (y ˇ k + G k (x k − x ˇ k) , R k ′) × ∫ p (x k ∣ x k − 1 , v k) p (x k − 1 ∣ x ˇ 0 , v 1 : k − 1 , y 0 : k − 1) d x k − 1 N (x ˇ k , F k − 1 P ^ k − 1 F k − 1 T + Q k ′)

$$\begin{array}{l}\underbrace{p\left(\boldsymbol{x}_{k} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k}, \boldsymbol{y}_{0: k}\right)}_{\mathcal{N}\left(\hat{\boldsymbol{x}}_{k}, \hat{\boldsymbol{P}}_{k}\right)}=\eta \underbrace{p\left(\boldsymbol{y}_{k} \mid \boldsymbol{x}_{k}\right)}_{\mathcal{N}\left(\check{\boldsymbol{y}}_{k}+\boldsymbol{G}_{k}\left(\boldsymbol{x}_{k}-\check{\boldsymbol{x}}_{k}\right), \boldsymbol{R}_{k}^{\prime}\right)} \\ \times \underbrace{\int p\left(\boldsymbol{x}_{k} \mid \boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}\right) p\left(\boldsymbol{x}_{k-1} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k-1}, \boldsymbol{y}_{0: k-1}\right) \mathrm{d} \boldsymbol{x}_{k-1}}_{\mathcal{N}\left(\check{\boldsymbol{x}}_{k}, \boldsymbol{F}_{k-1} \hat{\boldsymbol{P}}_{k-1} \boldsymbol{F}_{k-1}^{\mathrm{T}}+\boldsymbol{Q}_{k}^{\prime}\right)}\end{array}$$

N(x^k,P^k)

p(xk ∣ xˇ0,v1:k,y0:k) = ηN(yˇk+Gk(xk−xˇk),Rk′)

p(yk ∣ xk)×N(xˇk,Fk−1P^k−1Fk−1T+Qk′)

∫ p(xk ∣ xk−1,vk)p(xk−1 ∣ xˇ0,v1:k−1,y0:k−1)dxk−1

再利用高斯概率密度函数的归一化积的性质，有：

p (x k ∣ x ˇ 0 , v 1 : k , y 0 : k) N (x ^ k , P ^ k) = η p (y k ∣ x k) ∫ p (x k ∣ x k − 1 , v k) p (x k − 1 ∣ x ˇ 0 , v 1 : k − 1 , y 0 : k − 1) d x k − 1 N (x ˇ k + K k (y k − y ˇ k) , (1 − K k G k) (F k − 1 P ^ k − 1 F k − 1 T + Q k ′))

$$\begin{aligned}\underbrace{p\left(\boldsymbol{x}_{k} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k}, \boldsymbol{y}_{0: k}\right)}_{\mathcal{N}\left(\hat{\boldsymbol{x}}_{k}, \hat{\boldsymbol{P}}_{k}\right)} =& \underbrace{\eta p\left(\boldsymbol{y}_{k} \mid \boldsymbol{x}_{k}\right) \int p\left(\boldsymbol{x}_{k} \mid \boldsymbol{x}_{k-1}, \boldsymbol{v}_{k}\right) p\left(\boldsymbol{x}_{k-1} \mid \check{\boldsymbol{x}}_{0}, \boldsymbol{v}_{1: k-1}, \boldsymbol{y}_{0: k-1}\right) \mathrm{d} \boldsymbol{x}_{k-1}}_{\mathcal{N}\left(\check{\boldsymbol{x}}_{k}+\boldsymbol{K}_{k}\left(\boldsymbol{y}_{k}-\check{\boldsymbol{y}}_{k}\right),\left(\mathbf{1}-\boldsymbol{K}_{k} \boldsymbol{G}_{k}\right)\left(\boldsymbol{F}_{k-1} \hat{\boldsymbol{P}}_{k-1} \boldsymbol{F}_{k-1}^{\mathrm{T}}+\boldsymbol{Q}_{k}^{\prime}\right)\right)}\end{aligned}$$

N(x^k,P^k)

p(xk ∣ xˇ0,v1:k,y0:k) =N(xˇk+Kk(yk−yˇk),(1−KkGk)(Fk−1P^k−1Fk−1T+Qk′))

ηp(yk ∣ xk)∫ p(xk ∣ xk−1,vk)p(xk−1 ∣ xˇ0,v1:k−1,y0:k−1)dxk−1

其中， K k $\boldsymbol{K}_{k}$ Kk 为卡尔曼增益。比较上面式子的左右两侧，有：

预测

x ˇ k = f (x ^ k − 1 , v k , 0) $\check{\boldsymbol{x}}_{k} =\boldsymbol{f}\left(\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{v}_{k}, \mathbf{0}\right)$ xˇk = f(x^k−1, vk, 0)

P ˇ k = F k − 1 P ^ k − 1 F k − 1 T + Q k ′ $\check{\boldsymbol{P}}_{k} =\boldsymbol{F}_{k-1} \hat{\boldsymbol{P}}_{k-1} \boldsymbol{F}_{k-1}^{\mathrm{T}}+\boldsymbol{Q}_{k}^{\prime}$ Pˇk = Fk−1P^k−1Fk−1T + Qk′

卡尔曼增益

K k = P ˇ k G k T (G k P ˇ k G k T + R k ′) − 1 $\boldsymbol{K}_{k} =\check{\boldsymbol{P}}_{k} \boldsymbol{G}_{k}^{\mathrm{T}}\left(\boldsymbol{G}_{k} \check{\boldsymbol{P}}_{k} \boldsymbol{G}_{k}^{\mathrm{T}}+\boldsymbol{R}_{k}^{\prime}\right)^{-1}$ Kk = PˇkGkT(GkPˇkGkT + Rk′)−1

更新

x ^ k = x ˇ k + K k (y k − g (x ˇ k , 0)) 更新量 $\hat{\boldsymbol{x}}_{k} =\check{\boldsymbol{x}}_{k}+\boldsymbol{K}_{k} \underbrace{\left(\boldsymbol{y}_{k}-\boldsymbol{g}\left(\check{\boldsymbol{x}}_{k}, \mathbf{0}\right)\right)}_{\text {更新量 }}$ x^k = xˇk + Kk更新量

(yk − g(xˇk, 0))

P ^ k = (1 − K k G k) P ˇ k $\hat{\boldsymbol{P}}_{k} =\left(\mathbf{1}-\boldsymbol{K}_{k} \boldsymbol{G}_{k}\right) \check{\boldsymbol{P}}_{k}$ P^k = (1 − KkGk)Pˇk