1 matching

theorem 1.1 *(Tutte 1947). A graph G has a 1-factor* $\iff$ *o(G-S) ≤ |S| for every S ⊆ V(G).*

corollary 1 *(Peterson 1891) Every 3-regular graph with no cut-edge has a 1-factor.*

definition 1 $deiect_G = \max_{S \subseteq V(G)} o(G-S) - |S|$

theorem 1.2 *(Berge-Tutte Formula, Berge 1958). The maximum number of vertices saturated by a matching in G is n(G) -defectG = min S ⊆ V(G) (n(G) -(o(G-S)-|S|))*

2 Connectivity

definition 2 *A separating set or vertex cut of a graph G is a set S V(G) s.t. G−S has more than one component.*

definition 3 *The connectivity of G, written κ(G), is the minimum size of a vertex set S s.t. G-S is disconnected or has only one vertex. A graph G is k-connected if κ(G) ≥ k, i.e., G-S is connected with at least two vertices for every k -1-vertex set S.*

lemma 1 *κ(G) = n(G)-1* $\iff$ *G contains Kn(G), i.e., every vertex is adjacent to every other vertex.*

definition 4 *A disconnecting set of edges is a set F ⊆ E(G) s.t. G-F has more than one component.*

definition 5 *A graph is k-edge-connected if every disconnecting set has at least k edges. The edge-connectivity of G, written κ'(G), is the minimum size of a disconnecting set or equivalently, the maximum k s.t. G is k-edge-connected*

fact 1 *Every edge cut is a disconnecting set.*

fact 2 *Every minimal disconnecting set of edges is an edge cut.*

definition 6 $\delta(G) = minimumvertexdegreeofG$

theorem 2.1 *(Whitney 1932).* $\kappa(G) \le \kappa'(G) \le \delta(G)$ *for every graph G.*

theorem 2.2 *If G is a 3-regular graph of order more than 2, then kappa(G) = κ'(G).*

definition 7 *Two paths are internally disjoint if they do not have common internal vertices. Two x,y-paths are edge-disjoint if they do not share any edges*

theorem 2.3 *(Whitney) A graph G having at least 3 vertices is 2-connected* $\iff$ *for every distinct u,v ∈ V(G) there exist internally disjoint u,v-paths in G.*

lemma 2 *(Expansion Lemma) If G is k-connected, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.*

theorem 2.4 *For a graph G of order at least 3, the following conditions are equivalent and characterize 2-connected graphs.*

- *G is connected and has no cut-vertex.*
- *For all x,y ∈ V(G) there are internally disjoint x,y-paths.*
- *For all x,y ∈ V(G) there is a cycle through x and y.*
- *δ(G) ≥ 1 and every pair of edges in G lies on a common cycle.*

definition 8 *A subdivision of an edge e = xy is a replacement of e with path x,z,y where z is a new vertex.*

corollary 2 *If G is 2-connected, then the graph G' obtained by subdivision of an edge e =xy of G is 2-connected.*

definition 9 *An ear of a graph G is a nontrivial path in G whose endpoints have degree at least 3 and all its internal vertices have degree 2.*

definition 10 *A decomposition of a graph is a list of subgraphs s.t. every edge appears in exactly one subgraph in the list.*

definition 11 *An ear decomposition of G is a decomposition P0,...,Pk s.t. P0 is a cycle of length at least 3 and Pi for i ∈ [k] is an ear of P0 ∪ · · ·∪Pi*

theorem 2.5 *(Whitney) AgraphGis2-connected* $\iff$ *it has a near decomposition.Furthermore,every cycle of length at least 3 in a 2-connected graph G is the initial cycle in some ear decomposition.*

definition 12 *Given x,y ∈ V(G), a set S ⊆ V(G)-{x,y} is an x,y-separator or x,y-cut if G-S has no x,y-path.*

definition 13 $\kappa(x, y)$ *: the minimum size of an x,y-cut.* $\lambda(x, y)$ *: the maximum size of a set of pairwise internally disjoint x,y-paths.*

For nonadjacent vertices x,y, clearly $\kappa(x, y) \ge \lambda(x, y)$ *In fact,* $\kappa(x, y) = \lambda(x, y)$.

theorem 2.6 *(Menger's Theorem for vertex) If x,y are nonadjacent vertices of a graph G, then* $\kappa(x, y) = \lambda(x, y)$.

lemma 3 *x,y are nonadjacent vertices. There are some k internally disjoint x,y-paths, and removing some k vertices disconnects x from y. Then* $\kappa(x,y) = \lambda(x,y) = k$.

definition 14 *Given x,y ∈ V(G) κ'(x,y): the minimum number of edges whose deletion makes y unreachable from x λ'(x,y): the maximum size of a set of pairwise edge-disjoint x,y-paths Clearly κ'(x,y) ≥ λ'(x,y)*

In fact, κ'(x,y) = λ'(x,y)

lemma 4 *x,y are distinct vertices. There are some k pairwise edge-disjoint x,y-paths, and removing some k edges disconnects x from y. Then κ'(x,y) = λ'(x,y) = k.*

lemma 5 *κ'(G) = minx,y λ'(x,y). Equivalently, for every k ∈ [n(G) - 1], G is k-edge-connected* $\iff$ *λ'(x,y) ≥ k for every distinct x,y ∈ V (G).*

theorem 2.7 *(Menger's Theorem for edge) If x,y are distinct vertices of a graph G, then* $\kappa'(x, y) = \lambda'(x, y)$.

lemma 6 *Deletion of an edge reduces connectivity κ(G) by at most 1*

lemma 7 *κ(G) = minn(G)-1,minx,y λ(x,y). Equivalently, for every k ∈ [n(G) - 1], G is k-connected* $\iff$ $\lambda(x, y) \ge k$ *for every distinct x,y ∈ V(G).*

definition 15 *Given a set U of vertices and a vertex x ∉ U, an x,U-fan is a set of paths from x to U s.t. any two of them share only the vertex x.*

lemma 8 *(Fan Lemma, Dirac 1960) A graph G is k-connected* $\iff$ *it has at least k +1 vertices and, for every choice of x,U with |U| ≥ k, it has an x,U-fan of size k.*

definition 16 *When f is a feasible flow in a network N, an f-augmenting path is a s,t-path P in the underlying graph G s.t. for each e ∈ E(P) 1. if P follows e in the forward direction, then f(e) < c(e) 2. if P follows e in the backward direction, then 0 < f(e) Let ϵ(e) = c(e) -f(e) when e is forward on P, and let ϵ(e) = f(e) when e is backward on P. The tolerance of P is* $min_{e \in E(P)}\ \epsilon(e)$.

lemma 9 *If P is an f-augmenting path with tolerance z, then changing flow by +z on edges followed forward by P and by −z on edges followed backward on P produces a feasible flow f' with val(f')=val(f)+z.*

definition 17 *In a network, a s/t cut [S,T] consists of the edges from a source set S to a sink set T, where S,T partition the set of vertices with s ∈ S,t ∈ T. The capacity of the cut [S,T], written cap(S,T), is the total of the capacities on the edges with tail in S and head in T.*

definition 18

- *net flow out of U:= sum of net flow from vertices in U to vertices not in U. net flow into*
- *U:= sum of net flow from vertices not in U to vertices in U.*

lemma 10 *If f is a feasible flow and [S,T] is an s/t cut, then the net flow out of S and net flow into T equal val(f).*

corollary 3 *(Weak duality, Max-Flow Min-Cut inequality) If f is a feasible flow and [S,T] is an s/t cut, then val(f) ≤ cap(S,T). Thus* $max_f val(f) \le min_{[S,T]} cap(S,T)$.

corollary 4 *If there is a flow f and an s/t cut [S,T] s.t. cap(S,T)=val(f), then f is a maximum flow and [S,T] is a minimum s/t cut.*

If there is a flow f and an s/t cut [S,T] s.t. cap(S,T) = val(f), then f is a maximum flow and [S,T] is a minimum s/t cut.

theorem 2.8 *(Max-FlowMin-Cut Theorem,Ford and Fulkerson 1956) In every network N, the maximum value of a feasible flow equals the minimum value of a s/tcut.*

theorem 2.9 *(IntegrityTheorem) If all capacities in a network are integers, then there is a maximum flow f assigning integral flow to every edge.Furthermore,f can be partitioned into flows of unit value a long paths from s to t.*

lemma 11 *The maximum flow in N found by Ford-Fulkerson or Edmonds-Karp algorithm corresponds to a maximum matching in G.*

theorem 2.10 *(Menger's Theorem for edge in digraphs). If s,t are vertices of a digraph D, then κ'(s,t) = λ'(s,t).*

theorem 2.11 *(Menger's Theorem for vertex in digraphs). If s,t are nonadjacent vertices of a digraph D, then κ(s,t) = λ(s,t).*

3 Planar graph

definition 19 *A curve is the image of a continuous map from [0,1] to* R^2*. A polygonal curve is a curve composed of finitely many line segments. It is a polygonal u,v-curve when it starts at u and ends at v.*

definition 20 *A drawing of a graph G is a function f defined on V(G)∪E(G) that assigns each vertex v a point f(v) in the plane and assigns each edge with endpoints u,v a polygonal f(u),f(v)-curve. The images of vertices are distinct. A point in f(e) ∩ f(e') that is not a common endpoint is a crossing.*

definition 21 *A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G. A plane graph is a particular planar embedding of a planar graph. A curve is closed if its first and last points are the same. It is simple if it has no repeated points except possibly first=last. An open set in the plane is a set U ⊆R2 s.t. for every $p \in U$, all points within some small distance from p belong to U. A region is an open set U that contains a polygonal u,v-curve for every pair of $u,v \in U$. The faces of a plane graph are the maximal regions of the plane that contain no point used in the embedding. The length l(F) of a face F in a plane graph G is the total length of the closed walk(s) in G bounding the face F.*

theorem 3.1 *(RestrictedJordanCurveTheorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces,each having C as boundary.*

proposition 1 *If l(Fi) denotes the length of face Fi in a plane graph G, then $\sum_i l(F_i) = 2e(G)$*

theorem 3.2 *(Euler1758) If a connected plane graph G has exactly n vertices,e edges,and f faces, then n-e+f=2.*

corollary 5 *All planar embeddings of a connected planar graph G have the same number of faces.*

theorem 3.3 *If G is a simple planar graph with at least 3 vertices, then e(G)≤3n(G)-6. Also if G is triangle-free, thene(G)≤2n(G)-4.*

corollary 6 *Every simple planar graph G has $\delta(G) \leq 5$*

lemma 12 *A graph embeds in the plane $\iff$ it embeds on a sphere.*

definition 22 *Informally, a regular polyhedron is a solid whose boundary consists of regular polygons of the same length k, with the same number of faces d meeting at each vertex. A Platonic solid is a convex regular polyhedron.*

corollary 7 *There are at most 5 platonic solids.*

definition 23 *A maximal planar graph G is a simple planar graph s.t. adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.*

A triangulation is a simple plane graph where every face boundary is a 3-cycle.

proposition 2 *Let n≥ 3.For a simple n-vertex plane graph G, the following are equivalent. (A)G has 3n-6 edges. (B)G is a triangulation. (C)G is a maximal plane graph.*

definition 24 *A subdivision of an edge $e = xy$ is a replacement of e with path x,z,y where z is a new vertex.*

A graph H is a subdivision of a graph G if H can be obtained from G by a sequence of subdivisions of edges.

fact 3 *Subdividing edges does not affect planarity.*

proposition 3 *If a graph G has a subgraph that is a subdivision of K5 or K3,3, then G is not planar.*

corollary 8 *Petersen graph is not planar.*

theorem 3.4 *(Kuratowski 1930) A graph is planar $\iff$ it does not contain a subdivision of K5 nor K3,3*

definition 25 *A graph H is a minor of a graph G if H can be obtained from G by a sequence of these operations in any order: 1. deleting an edge 2. contracting an edge*

theorem 3.5 *(Wagner's Theorem) A graph G is planar $\iff$ it does not contain K5 nor K3,3 as minors.*

definition 26 *A graph is outerplanar if it has an embedding with every vertex on the boundary of the unbounded face. An outerplane graph is such an embedding of an outerplanar graph.*

theorem 3.6 *A graph G is outerplanar $\iff$ G does not contain subdivisions of K4 nor K2,3.*

4 Coloring

definition 27 *α(G): maximum size of independent set*

α'(G): maximum size of matching β(G): minimum size of vertex cover β'(G): minimum size of edge cover

proposition 4 $\chi(G) \geq \max\{\omega(G), \frac{n(G)}{\alpha(G)}\}$

proposition 5 $\chi(G) \leq \Delta(G) + 1$

definition 28 *A graph G is k-degenerate if for every subgraph H of G, $\delta(H) \leq k$, i.e., every subgraph of G has a vertex with degree at most k*

proposition 6 *A graph G is k-degenerate $\iff$ V(G) can be ordered v1,v2,...,vn s.t. for every i ∈ {2,3,...,n} vertex vi has at most k neighbors in {v1,v2,...,vi-1}.*

proposition 7 *Every k-degenerate graph G is k+1-colorable.*

corollary 9 *Every planar graph G is 6-colorable.*

theorem 4.1 *(Mycielski's construction) From a k-chromatic triangle-free graph G, Mycielski's construction produces a k+1-chromatic triangle-free graph G .*

theorem 4.2 *(Brooks 1941) If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$*

theorem 4.3 *(Appel-Haken-Koch 1977) Every planar graph is 4-colorable.*

definition 29 *A k-edge-coloring of G is a labeling $f : E(G) \to [k]$. The labels are colors. The edges of one color form a color class. A k-edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring. The edge-chromatic number $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable. G is k-edge-chromatic $\iff \chi'(G) = k$.*

Loops are excluded because they are incident to themselves. However, we allow multiple edges and they affect edge-coloring.

proposition 8 $\chi'(G) \geq \Delta(G)$

$\chi'(G) \leq 2\Delta(G) - 1$

theorem 4.4 *The Petersen graph is not 3-edge-colorable. Thus, it is 4-edge-chromatic.*

theorem 4.5 *(Konig 1916) If G is bipartite, then $\chi'(G) = \Delta(G)$*

5 Hamiltonian cycle

definition 30 *c(G) = number of components of G.*

proposition 9 *If G is Hamiltonian, then $c(G\ \text{-}S) \leq |S|$ for all $\emptyset \neq S \subseteq V(G)$.*

corollary 10 *Every Hamiltonian graph is 2-connected.*

theorem 5.1 *(Dirac 1952) If G is a simple n-vertex graph with $n \geq 3, \delta(G) \geq n/2$, then G is Hamiltonian.*

lemma 13 *(Ore 1960) Let G be a simple graph. If x, y are distinct nonadjacent vertices of G with deg(x) + deg(y) $\geq$ n(G), then G is Hamiltonian $\iff$ G + xy is Hamiltonian.*

corollary 11 *Let G be a simple graph. If deg(u)+deg(v) $\geq$ n(G) for every nonadjacent vertices u,v, then G is Hamiltonian.*

definition 31 *The Hamiltonian closure of a graph G, denoted C(G), is the graph with vertex set V (G) obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n(G), until no such pair remains.*

theorem 5.2 *(Bondy-Chvatal 1976) A simple n-vertex graph is Hamiltonian $\iff$ its closure is Hamiltonian.*

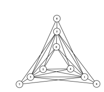
lemma 14 *The closure of G is well-defined, i.e., the order in which to add edges does not affect the resulting graph.*

theorem 5.3 *Petersen's graph is not Hamiltonian.*

6 HW

hw 1 *problem 1:*

Prove that graph G shown below is not Hamiltonian by finding a nonempty $S \in V(G)$ such that c(G - S) > |S|.



We find that for S = {1, 3, 5}, c(G-S) = 5 > |S| = 3. Thus G is not Hamiltonian.

problem2:

Graph G is simple. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. Hint: induct on n(G).

sol:Prove by induction on n(G). For $n(G) = 1$, the statement holds because $\chi(G) = \chi(G) = 1$.

Suppose that $n(G) = k > 1$. Consider an arbitrary vertex $v \in V(G)$. We have $|N_G(v)| + |N_{\overline{G}}(v)| = k - 1$. By inductive hypothesis, $\chi(G - v) + \chi(\overline{G - v}) \leq k$.

Case 1: *If $\chi(G-v)+\chi(\overline{G-v}) = k > k-1$, then either $\chi(G-v) > |N_G(v)|$ or $\chi(\overline{G-v}) > |N_{\overline{G}}(v)|$. This is because otherwise we have $\chi(G-v) \leq |N_G(v)|$ and $\chi(\overline{G-v}) \leq |N_{\overline{G}}(v)|$, implying $\chi(G-v) + \chi(\overline{G-v}) \leq |N_G(v)| + |N_{\overline{G}}(v)| = k-1$, a contradiction. Without loss of generality assume that $\chi(G-v) > |N_G(v)|$. We can then extend an optimal coloring f' of $G-v$ to an optimal coloring f of G by coloring v with some color in $[\chi(G-v)]$ not used by any neighbor of v in G, which exists because $\chi(G-v) > |N_G(v)|$. Thus $\chi(G) = \chi(G-v)$, and $\chi(G) + \chi(\overline{G}) \leq \chi(G-v) + \chi(\overline{G-v}) + 1 \leq k + 1$.*

Case 2: *If $\chi(G-v) + \chi(\overline{G-v}) < k$, then we extend an optimal coloring of $G - v$ to a coloring of G by coloring v with a new color, and extend an optimal coloring of $\overline{G-v}$ to a coloring of $\overline{G}$ by coloring v with a new color. Thus $\chi(G) + \chi(\overline{G}) \leq 1 + 1 + k - 1 = k + 1$.*