

1 matching

theorem 1.1 (Tutte 1947). A graph G has a 1-factor $\iff o(G-S) \leq |S|$ for every $S \subseteq V(G)$.

corollary 1 (Peterson 1891) Every 3-regular graph with no cut-edge has a 1-factor.

definition 1 $\text{defect}_G = \max_{S \subseteq V(G)} o(G-S) - |S|$

theorem 1.2 (Berge-Tutte Formula, Berge 1958). The maximum number of vertices saturated by a matching in G is $n(G) - \text{defect}_G = \min_{S \subseteq V(G)} (n(G) - (o(G-S) - |S|))$

2 Connectivity

definition 2 A separating set or vertex cut of a graph G is a set $S \subseteq V(G)$ s.t. $G-S$ has more than one component.

definition 3 The connectivity of G , written $\kappa(G)$, is the minimum size of a vertex set S s.t. $G-S$ is disconnected or has only one vertex. A graph G is k -connected if $\kappa(G) \geq k$, i.e., $G-S$ is connected with at least two vertices for every $k-1$ -vertex set S .

lemma 1 $\kappa(G) = n(G)-1 \iff G$ contains $K_n(G)$, i.e., every vertex is adjacent to every other vertex.

definition 4 A disconnecting set of edges is a set $F \subseteq E(G)$ s.t. $G-F$ has more than one component.

definition 5 A graph is k -edge-connected if every disconnecting set has at least k edges. The edge-connectivity of G , written $\kappa'(G)$, is the minimum size of a disconnecting set or equivalently, the maximum k s.t. G is k -edge-connected

fact 1 Every edge cut is a disconnecting set.

fact 2 Every minimal disconnecting set of edges is an edge cut.

definition 6 $\delta(G)$ = minimum vertex degree of G

theorem 2.1 (Whitney 1932). $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ for every graph G .

theorem 2.2 If G is a 3-regular graph of order more than 2, then $\kappa(G) = \kappa'(G)$.

definition 7 Two paths are internally disjoint if they do not have common internal vertices. Two x,y -paths are edge-disjoint if they do not share any edges

theorem 2.3 (Whitney) A graph G having at least 3 vertices is 2-connected \iff for every distinct $u,v \in V(G)$ there exist internally disjoint u,v -paths in G .

lemma 2 (Expansion Lemma) If G is k -connected, and G' is obtained from G by adding a new vertex y with at least k neighbors in G , then G' is k -connected.

theorem 2.4 For a graph G of order at least 3, the following conditions are equivalent and characterize 2-connected graphs.

- G is connected and has no cut-vertex.
- For all $x,y \in V(G)$ there are internally disjoint x,y -paths.
- For all $x,y \in V(G)$ there is a cycle through x and y .
- $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle.

definition 8 A subdivision of an edge $e = xy$ is a replacement of e with path x,z,y where z is a new vertex.

corollary 2 If G is 2-connected, then the graph G' obtained by subdivision of an edge $e = xy$ of G is 2-connected.

definition 9 An ear of a graph G is a nontrivial path in G whose endpoints have degree at least 3 and all its internal vertices have degree 2.

definition 10 A decomposition of a graph is a list of subgraphs s.t. every edge appears in exactly one subgraph in the list.

definition 11 An ear decomposition of G is a decomposition P_0, \dots, P_k s.t. P_0 is a cycle of length at least 3 and P_i for $i \in [k]$ is an ear of $P_0 \cup \dots \cup P_i$

theorem 2.5 (Whitney) A graph G is 2-connected \iff it has a near decomposition. Furthermore, every cycle of length at least 3 in a 2-connected graph G is the initial cycle in some ear decomposition.

definition 12 Given $x,y \in V(G)$, a set $S \subseteq V(G) - \{x,y\}$ is an x,y -separator or x,y -cut if $G-S$ has no x,y -path.

definition 13 $\kappa(x,y)$: the minimum size of an x,y -cut. $\lambda(x,y)$: the maximum size of a set of pairwise internally disjoint x,y -paths.

For nonadjacent vertices x,y , clearly $\kappa(x,y) \geq \lambda(x,y)$ In fact, $\kappa(x,y) = \lambda(x,y)$.

theorem 2.6 (Menger's Theorem for vertex) If x,y are nonadjacent vertices of a graph G , then $\kappa(x,y) = \lambda(x,y)$.

lemma 3 x,y are nonadjacent vertices. There are some k internally disjoint x,y -paths, and removing some k vertices disconnects x from y . Then $\kappa(x,y) = \lambda(x,y) = k$.

definition 14 Given $x,y \in V(G)$ $\kappa'(x,y)$: the minimum number of edges whose deletion makes y unreachable from x $\lambda'(x,y)$: the maximum size of a set of pairwise edge-disjoint x,y -paths Clearly $\kappa'(x,y) \geq \lambda'(x,y)$

In fact, $\kappa'(x,y) = \lambda'(x,y)$

lemma 4 x,y are distinct vertices. There are some k pairwise edge-disjoint x,y -paths, and removing some k edges disconnects x from y . Then $\kappa'(x,y) = \lambda'(x,y) = k$.

lemma 5 $\kappa'(G) = \min_{x,y} \lambda'(x,y)$. Equivalently, for every $k \in [n(G)-1]$, G is k -edge-connected $\iff \lambda'(x,y) \geq k$ for every distinct $x,y \in V(G)$.

theorem 2.7 (Menger's Theorem for edge) If x,y are distinct vertices of a graph G , then $\kappa'(x,y) = \lambda'(x,y)$.

lemma 6 Deletion of an edge reduces connectivity $\kappa(G)$ by at most 1

lemma 7 $\kappa(G) = \min(n(G)-1, \min_{x,y} \lambda(x,y))$. Equivalently, for every $k \in [n(G)-1]$, G is k -connected $\iff \lambda(x,y) \geq k$ for every distinct $x,y \in V(G)$.

definition 15 Given a set U of vertices and a vertex $x \notin U$, an x,U -fan is a set of paths from x to U s.t. any two of them share only the vertex x .

lemma 8 (Fan Lemma, Dirac 1960) A graph G is k -connected \iff it has at least $k+1$ vertices and, for every choice of x,U with $|U| \geq k$, it has an x,U -fan of size k .

definition 16 When f is a feasible flow in a network N , an f -augmenting path is a s,t -path P in the underlying graph G s.t. for each $e \in E(P)$

- if P follows e in the forward direction, then $f(e) < c(e)$
- if P follows e in the backward direction, then $0 < f(e)$

Let $\epsilon(e) = c(e) - f(e)$ when e is forward on P , and let $\epsilon(e) = f(e)$ when e is backward on P . The tolerance of P is $\min_{e \in E(P)} \epsilon(e)$.

lemma 9 If P is an f -augmenting path with tolerance z , then changing flow by $+z$ on edges followed forward by P and by $-z$ on edges followed backward on P produces a feasible flow f' with $\text{val}(f') = \text{val}(f) + z$.

definition 17 In a network, a s/t cut $[S,T]$ consists of the edges from a source set S to a sink set T , where S,T partition the set of vertices with $s \in S, t \in T$. The capacity of the cut $[S,T]$, written $\text{cap}(S,T)$, is the total of the capacities on the edges with tail in S and head in T .

definition 18

- net flow out of $U :=$ sum of net flow from vertices in U to vertices not in U .
- net flow into $U :=$ sum of net flow from vertices not in U to vertices in U .

lemma 10 If f is a feasible flow and $[S,T]$ is an s/t cut, then the net flow out of S and net flow into T equal $\text{val}(f)$.

corollary 3 (Weak duality, Max-Flow Min-Cut inequality) If f is a feasible flow and $[S,T]$ is an s/t cut, then $\text{val}(f) \leq \text{cap}(S,T)$. Thus $\max_f \text{val}(f) \leq \min_{[S,T]} \text{cap}(S,T)$.

corollary 4 If there is a flow f and an s/t cut $[S,T]$ s.t. $\text{cap}(S,T) = \text{val}(f)$, then f is a maximum flow and $[S,T]$ is a minimum s/t cut.

If there is a flow f and an s/t cut $[S,T]$ s.t. $\text{cap}(S,T) = \text{val}(f)$, then f is a maximum flow and $[S,T]$ is a minimum s/t cut.

theorem 2.8 (Max-Flow Min-Cut Theorem, Ford and Fulkerson 1956) In every network N , the maximum value of a feasible flow equals the minimum value of a s/t cut.

theorem 2.9 (Integrity Theorem) If all capacities in a network are integers, then there is a maximum flow f assigning integral flow to every edge. Furthermore, f can be partitioned into flows of unit value along paths from s to t .

lemma 11 The maximum flow in N found by Ford-Fulkerson or Edmonds-Karp algorithm corresponds to a maximum matching in G .

theorem 2.10 (Menger's Theorem for edge in digraphs). If s,t are vertices of a digraph D , then $\kappa'(s,t) = \lambda'(s,t)$.

theorem 2.11 (Menger's Theorem for vertex in digraphs). If s,t are nonadjacent vertices of a digraph D , then $\kappa(s,t) = \lambda(s,t)$.

3 Planar graph

definition 19 A curve is the image of a continuous map from $[0,1]$ to \mathbb{R}^2 . A polygonal curve is a curve composed of finitely many line segments. It is a polygonal u,v -curve when it starts at u and ends at v .

definition 20 A drawing of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge with endpoints u,v a polygonal $f(u),f(v)$ -curve. The images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a crossing.

definition 21 A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of G . A plane graph is a particular planar embedding of a planar graph. A curve is closed if its first and last points are the same. It is simple if it has no repeated points except possibly first=last. An open set in the plane is a set $U \subseteq \mathbb{R}^2$ s.t. for every $p \in U$, all points within some small distance from p belong to U . A region is an open set U that contains a polygonal u,v -curve for every pair of $u,v \in U$. The faces of a plane graph are the maximal regions of the plane that contain no point used in the embedding. The length $l(F)$ of a face F in a plane graph G is the total length of the closed walk(s) in G bounding the face F .

theorem 3.1 (RestrictedJordanCurveTheorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary.

proposition 1 If $l(F_i)$ denotes the length of face F_i in a plane graph G , then $\sum_i l(F_i) = 2e(G)$

theorem 3.2 (Euler1758) If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n-e+f=2$.

corollary 5 All planar embeddings of a connected planar graph G have the same number of faces.

theorem 3.3 If G is a simple planar graph with at least 3 vertices, then $e(G) \leq 3n(G)-6$. Also if G is triangle-free, then $e(G) \leq 2n(G)-4$.

corollary 6 Every simple planar graph G has $\delta(G) \leq 5$

lemma 12 A graph embeds in the plane \iff it embeds on a sphere.

definition 22 Informally, a regular polyhedron is a solid whose boundary consists of regular polygons of the same length k , with the same number of faces d meeting at each vertex. A Platonic solid is a convex regular polyhedron.

corollary 7 There are at most 5 platonic solids.

definition 23 A maximal planar graph G is a simple planar graph s.t. adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a simple plane graph where every face boundary is a 3-cycle.

proposition 2 Let $n \geq 3$. For a simple n -vertex plane graph G , the following are equivalent. (A) G has $3n-6$ edges. (B) G is a triangulation. (C) G is a maximal plane graph.

definition 24 A subdivision of an edge $e = xy$ is a replacement of e with path x,z,y where z is a new vertex.

A graph H is a subdivision of a graph G if H can be obtained from G by a sequence of subdivisions of edges.

fact 3 Subdividing edges does not affect planarity.

proposition 3 If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is not planar.

corollary 8 Petersen graph is not planar.

theorem 3.4 (Kuratowski 1930) A graph is planar \iff it does not contain a subdivision of K_5 nor $K_{3,3}$

definition 25 A graph H is a minor of a graph G if H can be obtained from G by a sequence of these operations in any order: 1. deleting an edge 2. contracting an edge

theorem 3.5 (Wagner's Theorem) A graph G is planar \iff it does not contain K_5 nor $K_{3,3}$ as minors.

definition 26 A graph is outerplanar if it has an embedding with every vertex on the boundary of the unbounded face. An outerplane graph is such an embedding of an outerplanar graph.

theorem 3.6 A graph G is outerplanar \iff G does not contain subdivisions of K_4 nor $K_{2,3}$.

4 Coloring

definition 27 $\alpha(G)$: maximum size of independent set

$\alpha'(G)$: maximum size of matching $\beta(G)$: minimum size of vertex cover $\beta'(G)$: minimum size of edge cover

proposition 4 $\chi(G) \geq \max\{\omega(G), \frac{n(G)}{\alpha(G)}\}$

proposition 5 $\chi(G) \leq \Delta(G) + 1$

definition 28 A graph G is k -degenerate if for every subgraph H of G , $\delta(H) \leq k$, i.e., every subgraph of G has a vertex with degree at most k

proposition 6 A graph G is k -degenerate \iff $V(G)$ can be ordered v_1, v_2, \dots, v_n s.t. for every $i \in \{2, 3, \dots, n\}$ vertex v_i has at most k neighbors in $\{v_1, v_2, \dots, v_{i-1}\}$.

proposition 7 Every k -degenerate graph G is $k+1$ -colorable.

corollary 9 Every planar graph G is 6-colorable.

theorem 4.1 (Mycielski's construction) From a k -chromatic triangle-free graph G , Mycielski's construction produces a $k+1$ -chromatic triangle-free graph G .

theorem 4.2 (Brooks 1941) If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$

theorem 4.3 (Appel-Haken-Koch 1977) Every planar graph is 4-colorable.

definition 29 A k -edge-coloring of G is a labeling $f: E(G) \rightarrow [k]$. The labels are colors. The edges of one color form a color class. A k -edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is k -edge-colorable if it has a proper k -edge-coloring. The edge-chromatic number $\chi'(G)$ of a loopless graph G is the least k such that G is k -edge-colorable. G is k -edge-chromatic $\iff \chi'(G) = k$.

Loops are excluded because they are incident to themselves. However, we allow multiple edges and they affect edge-coloring.

proposition 8 $\chi'(G) \geq \Delta(G)$

$\chi'(G) \leq 2\Delta(G) - 1$

theorem 4.4 The Petersen graph is not 3-edge-colorable. Thus, it is 4-edge-chromatic.

theorem 4.5 (Konig 1916) If G is bipartite, then $\chi'(G) = \Delta(G)$

5 Hamiltonian cycle

definition 30 $c(G)$ = number of components of G .

proposition 9 If G is Hamiltonian, then $c(G-S) \leq |S|$ for all $\emptyset \neq S \subseteq V(G)$.

corollary 10 Every Hamiltonian graph is 2-connected.

theorem 5.1 (Dirac 1952) If G is a simple n -vertex graph with $n \geq 3$, $\delta(G) \geq n/2$, then G is Hamiltonian.

lemma 13 (Ore 1960) Let G be a simple graph. If x, y are distinct nonadjacent vertices of G with $\deg(x) + \deg(y) \geq n(G)$, then G is Hamiltonian $\iff G + xy$ is Hamiltonian.

corollary 11 Let G be a simple graph. If $\deg(u) + \deg(v) \geq n(G)$ for every nonadjacent vertices u, v , then G is Hamiltonian.

definition 31 The Hamiltonian closure of a graph G , denoted $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least $n(G)$, until no such pair remains.

theorem 5.2 (Bondy-Chvatal 1976) A simple n -vertex graph is Hamiltonian \iff its closure is Hamiltonian.

lemma 14 The closure of G is well-defined, i.e., the order in which to add edges does not affect the resulting graph.

theorem 5.3 Petersen's graph is not Hamiltonian.

6 HW

hw 1 problem 1:

Prove that graph G shown below is not Hamiltonian by finding a nonempty $S \in V(G)$ such that $c(G-S) > |S|$.



We find that for $S = \{1, 3, 5\}$, $c(G-S) = 5 > |S| = 3$. Thus G is not Hamiltonian.

problem 2:

Graph G is simple. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. Hint: induct on $n(G)$.

sol: Prove by induction on $n(G)$. For $n(G) = 1$, the statement holds because $\chi(G) = \chi(\overline{G}) = 1$.

Suppose that $n(G) = k > 1$. Consider an arbitrary vertex $v \in V(G)$. We have $|N_G(v)| + |N_{\overline{G}}(v)| = k - 1$. By inductive hypothesis, $\chi(G-v) + \chi(\overline{G-v}) \leq k$.

Case 1: If $\chi(G-v) + \chi(\overline{G-v}) = k > k-1$, then either $\chi(G-v) > |N_G(v)|$ or $\chi(\overline{G-v}) > |N_{\overline{G}}(v)|$. This is because otherwise we have $\chi(G-v) \leq |N_G(v)|$ and $\chi(\overline{G-v}) \leq |N_{\overline{G}}(v)|$, implying $\chi(G-v) + \chi(\overline{G-v}) \leq |N_G(v)| + |N_{\overline{G}}(v)| = k-1$, a contradiction. Without loss of generality assume that $\chi(G-v) > |N_G(v)|$. We can then extend an optimal coloring f' of $G-v$ to an optimal coloring f of G by coloring v with some color in $[\chi(G-v)]$ not used by any neighbor of v in G , which exists because $\chi(G-v) > |N_G(v)|$. Thus $\chi(G) = \chi(G-v)$, and $\chi(G) + \chi(\overline{G}) \leq \chi(G-v) + \chi(\overline{G-v}) + 1 \leq k + 1$.

Case 2: If $\chi(G-v) + \chi(\overline{G-v}) < k$, then we extend an optimal coloring of $G-v$ to a coloring of G by coloring v with a new color, and extend an optimal coloring of $\overline{G-v}$ to a coloring of \overline{G} by coloring v with a new color. Thus $\chi(G) + \chi(\overline{G}) \leq 1 + 1 + k - 1 = k + 1$.