

**1 matching**

**theorem 1.1** (Tutte 1947). A graph  $G$  has a 1-factor  $\iff o(G-S) \leq |S|$  for every  $S \subseteq V(G)$ .

**corollary 1** (Peterson 1891) Every 3-regular graph with no cut-edge has a 1-factor.

**definition 1**  $delect_G = \max_{S \subseteq V(G)} o(G-S) - |S|$

**theorem 1.2** (Berge-Tutte Formula, Berge 1958). The maximum number of vertices saturated by a matching in  $G$  is  $n(G) - defect_G = \min_{S \subseteq V(G)} (n(G) - (o(G-S) - |S|))$

**2 Connectivity**

**definition 2** A separating set or vertex cut of a graph  $G$  is a set  $S \subseteq V(G)$  s.t.  $G-S$  has more than one component.

**definition 3** The connectivity of  $G$ , written  $\kappa(G)$ , is the minimum size of a vertex set  $S$  s.t.  $G-S$  is disconnected or has only one vertex. A graph  $G$  is  $k$ -connected if  $\kappa(G) \geq k$ , i.e.,  $G-S$  is connected with at least two vertices for every  $k-1$ -vertex set  $S$ .

**lemma 1**  $\kappa(G) = n(G) - 1 \iff G$  contains  $K_n(G)$ , i.e., every vertex is adjacent to every other vertex.

**definition 4** A disconnecting set of edges is a set  $F \subseteq E(G)$  s.t.  $G-F$  has more than one component.

**definition 5** A graph is  $k$ -edge-connected if every disconnecting set has at least  $k$  edges. The edge-connectivity of  $G$ , written  $\kappa'(G)$ , is the minimum size of a disconnecting set or equivalently, the maximum  $k$  s.t.  $G$  is  $k$ -edge-connected

**fact 1** Every edge cut is a disconnecting set.

**fact 2** Every minimal disconnecting set of edges is an edge cut.

**definition 6**  $\delta(G) = \text{minimum vertex degree of } G$

**theorem 2.1** (Whitney 1932).  $\kappa(G) \leq \kappa'(G) \leq \delta(G)$  for every graph  $G$ .

**theorem 2.2** If  $G$  is a 3-regular graph of order more than 2, then  $\kappa(G) = \kappa'(G)$ .

**definition 7** Two paths are internally disjoint if they do not have common internal vertices. Two  $x,y$ -paths are edge-disjoint if they do not share any edges

**theorem 2.3** (Whitney) A graph  $G$  having at least 3 vertices is 2-connected  $\iff$  for every distinct  $u,v \in V(G)$  there exist internally disjoint  $u,v$ -paths in  $G$ .

**lemma 2** (Expansion Lemma) If  $G$  is  $k$ -connected, and  $G'$  is obtained from  $G$  by adding a new vertex  $y$  with at least  $k$  neighbors in  $G$ , then  $G'$  is  $k$ -connected.

**theorem 2.4** For a graph  $G$  of order at least 3, the following conditions are equivalent and characterize 2-connected graphs.

- $G$  is connected and has no cut-vertex.
- For all  $x,y \in V(G)$  there are internally disjoint  $x,y$ -paths.
- For all  $x,y \in V(G)$  there is a cycle through  $x$  and  $y$ .
- $\delta(G) \geq 1$  and every pair of edges in  $G$  lies on a common cycle.

**definition 8** A subdivision of an edge  $e = xy$  is a replacement of  $e$  with path  $x,z,y$  where  $z$  is a new vertex.

**corollary 2** If  $G$  is 2-connected, then the graph  $G'$  obtained by subdivision of an edge  $e = xy$  of  $G$  is 2-connected.

**definition 9** An ear of a graph  $G$  is a nontrivial path in  $G$  whose endpoints have degree at least 3 and all its internal vertices have degree 2.

**definition 10** A decomposition of a graph is a list of subgraphs s.t. every edge appears in exactly one subgraph in the list.

**definition 11** An ear decomposition of  $G$  is a decomposition  $P_0, \dots, P_k$  s.t.  $P_0$  is a cycle of length at least 3 and  $P_i$  for  $i \in [k]$  is an ear of  $P_0 \cup \dots \cup P_i$

**theorem 2.5** (Whitney) A graph  $G$  is 2-connected  $\iff$  it has a near decomposition. Furthermore, every cycle of length at least 3 in a 2-connected graph  $G$  is the initial cycle in some ear decomposition.

**definition 12** Given  $x,y \in V(G)$ , a set  $S \subseteq V(G) - \{x,y\}$  is an  $x,y$ -separator or  $x,y$ -cut if  $G-S$  has no  $x,y$ -path.

**definition 13**  $\kappa(x,y)$  : the minimum size of an  $x,y$ -cut.  $\lambda(x,y)$  : the maximum size of a set of pairwise internally disjoint  $x,y$ -paths.

For nonadjacent vertices  $x,y$ , clearly  $\kappa(x,y) \geq \lambda(x,y)$  In fact,  $\kappa(x,y) = \lambda(x,y)$ .

**theorem 2.6** (Menger's Theorem for vertex) If  $x,y$  are nonadjacent vertices of a graph  $G$ , then  $\kappa(x,y) = \lambda(x,y)$ .

**lemma 3**  $x,y$  are nonadjacent vertices. There are some  $k$  internally disjoint  $x,y$ -paths, and removing some  $k$  vertices disconnects  $x$  from  $y$ . Then  $\kappa(x,y) = \lambda(x,y) = k$ .

**definition 14** Given  $x,y \in V(G)$   $\kappa'(x,y)$ : the minimum number of edges whose deletion makes  $y$  unreachable from  $x$   $\lambda'(x,y)$ : the maximum size of a set of pairwise edge-disjoint  $x,y$ -paths Clearly  $\kappa'(x,y) \geq \lambda'(x,y)$

In fact,  $\kappa'(x,y) = \lambda'(x,y)$

**lemma 4**  $x,y$  are distinct vertices. There are some  $k$  pairwise edge-disjoint  $x,y$ -paths, and removing some  $k$  edges disconnects  $x$  from  $y$ . Then  $\kappa'(x,y) = \lambda'(x,y) = k$ .

**lemma 5**  $\kappa'(G) = \min_{x,y} \lambda'(x,y)$ . Equivalently, for every  $k \in [n(G) - 1]$ ,  $G$  is  $k$ -edge-connected  $\iff \lambda'(x,y) \geq k$  for every distinct  $x,y \in V(G)$ .

**theorem 2.7** (Menger's Theorem for edge) If  $x,y$  are distinct vertices of a graph  $G$ , then  $\kappa'(x,y) = \lambda'(x,y)$ .

**lemma 6** Deletion of an edge reduces connectivity  $\kappa(G)$  by at most 1

**lemma 7**  $\kappa(G) = \min(n(G) - 1, \min_{x,y} \lambda(x,y))$ . Equivalently, for every  $k \in [n(G) - 1]$ ,  $G$  is  $k$ -connected  $\iff \lambda(x,y) \geq k$  for every distinct  $x,y \in V(G)$ .

**definition 15** Given a set  $U$  of vertices and a vertex  $x \notin U$ , an  $x,U$ -fan is a set of paths from  $x$  to  $U$  s.t. any two of them share only the vertex  $x$ .

**lemma 8** (Fan Lemma, Dirac 1960) A graph  $G$  is  $k$ -connected  $\iff$  it has at least  $k+1$  vertices and, for every choice of  $x,U$  with  $|U| \geq k$ , it has an  $x,U$ -fan of size  $k$ .

**definition 16** When  $f$  is a feasible flow in a network  $N$ , an  $f$ -augmenting path is a  $s,t$ -path  $P$  in the underlying graph  $G$  s.t. for each  $e \in E(P)$  1. if  $P$  follows  $e$  in the forward direction, then  $f(e) < c(e)$  2. if  $P$  follows  $e$  in the backward direction, then  $0 < f(e)$  Let  $\epsilon(e) = c(e) - f(e)$  when  $e$  is forward on  $P$ , and let  $\epsilon(e) = f(e)$  when  $e$  is backward on  $P$ . The tolerance of  $P$  is  $\min_{e \in E(P)} \epsilon(e)$ .

**lemma 9** If  $P$  is an  $f$ -augmenting path with tolerance  $z$ , then changing flow by  $+z$  on edges followed forward by  $P$  and by  $-z$  on edges followed backward on  $P$  produces a feasible flow  $f'$  with  $\text{val}(f') = \text{val}(f) + z$ .

**definition 17** In a network, a  $s/t$  cut  $[S,T]$  consists of the edges from a source set  $S$  to a sink set  $T$ , where  $S,T$  partition the set of vertices with  $s \in S, t \in T$ . The capacity of the cut  $[S,T]$ , written  $\text{cap}(S,T)$ , is the total of the capacities on the edges with tail in  $S$  and head in  $T$ .

**definition 18** • net flow out of  $U :=$  sum of net flow from vertices in  $U$  to vertices not in  $U$ . net flow into

- $U :=$  sum of net flow from vertices not in  $U$  to vertices in  $U$ .

**lemma 10** If  $f$  is a feasible flow and  $[S,T]$  is an  $s/t$  cut, then the net flow out of  $S$  and net flow into  $T$  equal  $\text{val}(f)$ .

**corollary 3** (Weak duality, Max-Flow Min-Cut inequality) If  $f$  is a feasible flow and  $[S,T]$  is an  $s/t$  cut, then  $\text{val}(f) \leq \text{cap}(S,T)$ . Thus  $\max_f \text{val}(f) \leq \min_{[S,T]} \text{cap}(S,T)$ .

**corollary 4** If there is a flow  $f$  and an  $s/t$  cut  $[S,T]$  s.t.  $\text{cap}(S,T) = \text{val}(f)$ , then  $f$  is a maximum flow and  $[S,T]$  is a minimum  $s/t$  cut.

If there is a flow  $f$  and an  $s/t$  cut  $[S,T]$  s.t.  $\text{cap}(S,T) = \text{val}(f)$ , then  $f$  is a maximum flow and  $[S,T]$  is a minimum  $s/t$  cut.

**theorem 2.8** (Max-Flow Min-Cut Theorem, Ford and Fulkerson 1956) In every network  $N$ , the maximum value of a feasible flow equals the minimum value of a  $s/t$  cut.

**theorem 2.9** (Integrity Theorem) If all capacities in a network are integers, then there is a maximum flow  $f$  assigning integral flow to every edge. Furthermore,  $f$  can be partitioned into flows of unit value along paths from  $s$  to  $t$ .

**lemma 11** The maximum flow in  $N$  found by Ford-Fulkerson or Edmonds-Karp algorithm corresponds to a maximum matching in  $G$ .

**theorem 2.10** (Menger's Theorem for edge in digraphs). If  $s,t$  are vertices of a digraph  $D$ , then  $\kappa'(s,t) = \lambda'(s,t)$ .

**theorem 2.11** (Menger's Theorem for vertex in digraphs). If  $s,t$  are nonadjacent vertices of a digraph  $D$ , then  $\kappa(s,t) = \lambda(s,t)$ .

**3 Planar graph**

**definition 19** A curve is the image of a continuous map from  $[0,1]$  to  $\mathbb{R}^2$ . A polygonal curve is a curve composed of finitely many line segments. It is a polygonal  $u,v$ -curve when it starts at  $u$  and ends at  $v$ .

**definition 20** A drawing of a graph  $G$  is a function  $f$  defined on  $V(G) \cup E(G)$  that assigns each vertex  $v$  a point  $f(v)$  in the plane and assigns each edge with endpoints  $u,v$  a polygonal  $f(u),f(v)$ -curve. The images of vertices are distinct. A point in  $f(e) \cap f(e')$  that is not a common endpoint is a crossing.

**definition 21** A graph is planar if it has a drawing without crossings. Such a drawing is a planar embedding of  $G$ . A plane graph is a particular planar embedding of a planar graph. A curve is closed if its first and last points are the same. It is simple if it has no repeated points except possibly first=last. An open set in the plane is a set  $U \subseteq \mathbb{R}^2$  s.t. for every  $p \in U$ , all points within some small distance from  $p$  belong to  $U$ . A region is an open set  $U$  that contains a polygonal  $u,v$ -curve for every pair of  $u,v \in U$ . The faces of a plane graph are the maximal regions of the plane that contain no point used in the embedding. The length  $l(F)$  of a face  $F$  in a plane graph  $G$  is the total length of the closed walk(s) in  $G$  bounding the face  $F$ .

**theorem 3.1** (RestrictedJordanCurveTheorem) A simple closed polygonal curve  $C$  consisting of finitely many segments partitions the plane into exactly two faces, each having  $C$  as boundary.

**proposition 1** If  $l(F_i)$  denotes the length of face  $F_i$  in a plane graph  $G$ , then  $\sum_i l(F_i) = 2e(G)$

**theorem 3.2** (Euler1758) If a connected plane graph  $G$  has exactly  $n$  vertices,  $e$  edges, and  $f$  faces, then  $n - e + f = 2$ .

**corollary 5** All planar embeddings of a connected planar graph  $G$  have the same number of faces.

**theorem 3.3** If  $G$  is a simple planar graph with at least 3 vertices, then  $e(G) \leq 3n(G) - 6$ . Also if  $G$  is triangle-free, then  $e(G) \leq 2n(G) - 4$ .

**corollary 6** Every simple planar graph  $G$  has  $\delta(G) \leq 5$

**lemma 12** A graph embeds in the plane  $\iff$  it embeds on a sphere.

**definition 22** Informally, a regular polyhedron is a solid whose boundary consists of regular polygons of the same length  $k$ , with the same number of faces  $d$  meeting at each vertex. A Platonic solid is a convex regular polyhedron.

**corollary 7** There are at most 5 platonic solids.

**definition 23** A maximal planar graph  $G$  is a simple planar graph s.t. adding any non-loop edge not parallel to any edge of  $G$  results in a nonplanar graph.

A triangulation is a simple plane graph where every face boundary is a 3-cycle.

**proposition 2** Let  $n \geq 3$ . For a simple  $n$ -vertex plane graph  $G$ , the following are equivalent. (A)  $G$  has  $3n - 6$  edges. (B)  $G$  is a triangulation. (C)  $G$  is a maximal plane graph.

**definition 24** A subdivision of an edge  $e = xy$  is a replacement of  $e$  with path  $x, z, y$  where  $z$  is a new vertex.

A graph  $H$  is a subdivision of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of subdivisions of edges.

**fact 3** Subdividing edges does not affect planarity.

**proposition 3** If a graph  $G$  has a subgraph that is a subdivision of  $K_5$  or  $K_{3,3}$ , then  $G$  is not planar.

**corollary 8** Petersen graph is not planar.

**theorem 3.4** (Kuratowski 1930) A graph is planar  $\iff$  it does not contain a subdivision of  $K_5$  nor  $K_{3,3}$

**definition 25** A graph  $H$  is a minor of a graph  $G$  if  $H$  can be obtained from  $G$  by a sequence of these operations in any order: 1. deleting an edge 2. contracting an edge

**theorem 3.5** (Wagner's Theorem) A graph  $G$  is planar  $\iff$  it does not contain  $K_5$  nor  $K_{3,3}$  as minors.

**definition 26** A graph is outerplanar if it has an embedding with every vertex on the boundary of the unbounded face. An outerplane graph is such an embedding of an outerplanar graph.

**theorem 3.6** A graph  $G$  is outerplanar  $\iff G$  does not contain subdivisions of  $K_4$  nor  $K_{2,3}$ .

#### 4 Coloring

**definition 27**  $\alpha(G)$ : maximum size of independent set

$\alpha'(G)$ : maximum size of matching  $\beta(G)$ : minimum size of vertex cover  $\beta'(G)$ : minimum size of edge cover

**proposition 4**  $\chi(G) \geq \max\{\omega(G), \frac{n(G)}{\alpha(G)}\}$

**proposition 5**  $\chi(G) \leq \Delta(G) + 1$

**definition 28** A graph  $G$  is  $k$ -degenerate if for every subgraph  $H$  of  $G$ ,  $\delta(H) \leq k$ , i.e., every subgraph of  $G$  has a vertex with degree at most  $k$

**proposition 6** A graph  $G$  is  $k$ -degenerate  $\iff V(G)$  can be ordered  $v_1, v_2, \dots, v_n$  s.t. for every  $i \in \{2, 3, \dots, n\}$  vertex  $v_i$  has at most  $k$  neighbors in  $\{v_1, v_2, \dots, v_{i-1}\}$ .

**proposition 7** Every  $k$ -degenerate graph  $G$  is  $k+1$ -colorable.

**corollary 9** Every planar graph  $G$  is 6-colorable.

**theorem 4.1** (Mycielski's construction) From a  $k$ -chromatic triangle-free graph  $G$ , Mycielski's construction produces a  $k+1$ -chromatic triangle-free graph  $G$ .

**theorem 4.2** (Brooks 1941) If  $G$  is a connected graph other than a clique or an odd cycle, then  $\chi(G) \leq \Delta(G)$

**theorem 4.3** (Appel-Haken-Koch 1977) Every planar graph is 4-colorable.

**definition 29** A  $k$ -edge-coloring of  $G$  is a labeling  $f : E(G) \rightarrow [k]$ . The labels are colors. The edges of one color form a color class. A  $k$ -edge-coloring is proper if incident edges have different labels; that is, if each color class is a matching. A graph is  $k$ -edge-colorable if it has a proper  $k$ -edge-coloring. The edge-chromatic number  $\chi'(G)$  of a loopless graph  $G$  is the least  $k$  such that  $G$  is  $k$ -edge-colorable.  $G$  is  $k$ -edge-chromatic  $\iff \chi'(G) = k$ .

Loops are excluded because they are incident to themselves. However, we allow multiple edges and they affect edge-coloring.

**proposition 8**  $\chi'(G) \geq \Delta(G)$

$\chi'(G) \leq 2\Delta(G) - 1$

**theorem 4.4** The Petersen graph is not 3-edge-colorable. Thus, it is 4-edge-chromatic.

**theorem 4.5** (Konig 1916) If  $G$  is bipartite, then  $\chi'(G) = \Delta(G)$

#### 5 Hamiltonian cycle

**definition 30**  $c(G)$  = number of components of  $G$ .

**proposition 9** If  $G$  is Hamiltonian, then  $c(G - S) \leq |S|$  for all  $\emptyset \neq S \subseteq V(G)$ .

**corollary 10** Every Hamiltonian graph is 2-connected.

**theorem 5.1** (Dirac 1952) If  $G$  is a simple  $n$ -vertex graph with  $n \geq 3$ ,  $\delta(G) \geq n/2$ , then  $G$  is Hamiltonian.

**lemma 13** (Ore 1960) Let  $G$  be a simple graph. If  $x, y$  are distinct nonadjacent vertices of  $G$  with  $\deg(x) + \deg(y) \geq n(G)$ , then  $G$  is Hamiltonian  $\iff G + xy$  is Hamiltonian.

**corollary 11** Let  $G$  be a simple graph. If  $\deg(u) + \deg(v) \geq n(G)$  for every nonadjacent vertices  $u, v$ , then  $G$  is Hamiltonian.

**definition 31** The Hamiltonian closure of a graph  $G$ , denoted  $C(G)$ , is the graph with vertex set  $V(G)$  obtained from  $G$  by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least  $n(G)$ , until no such pair remains.

**theorem 5.2** (Bondy-Chvatal 1976) A simple  $n$ -vertex graph is Hamiltonian  $\iff$  its closure is Hamiltonian.

**lemma 14** The closure of  $G$  is well-defined, i.e., the order in which to add edges does not affect the resulting graph.

**theorem 5.3** Petersen's graph is not Hamiltonian.

#### 6 HW

**hw 1** problem 1:

Prove that graph  $G$  shown below is not Hamiltonian by finding a nonempty  $S \in V(G)$  such that  $c(G - S) > |S|$ .



We find that for  $S = \{1, 3, 5\}$ ,  $c(G - S) = 5 > |S| = 3$ . Thus  $G$  is not Hamiltonian.

problem 2:

Graph  $G$  is simple. Prove that  $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$ . Hint: induct on  $n(G)$ .

sol: Prove by induction on  $n(G)$ . For  $n(G) = 1$ , the statement holds because  $\chi(G) = \chi(\overline{G}) = 1$ .

Suppose that  $n(G) = k > 1$ . Consider an arbitrary vertex  $v \in V(G)$ . We have  $|N_G(v)| + |N_{\overline{G}}(v)| = k - 1$ . By inductive hypothesis,  $\chi(G - v) + \chi(\overline{G - v}) \leq k$ .

**Case 1:** If  $\chi(G - v) + \chi(\overline{G - v}) = k > k - 1$ , then either  $\chi(G - v) > |N_G(v)|$  or  $\chi(\overline{G - v}) > |N_{\overline{G}}(v)|$ . This is because otherwise we have  $\chi(G - v) \leq |N_G(v)|$  and  $\chi(\overline{G - v}) \leq |N_{\overline{G}}(v)|$ , implying  $\chi(G - v) + \chi(\overline{G - v}) \leq |N_G(v)| + |N_{\overline{G}}(v)| = k - 1$ , a contradiction. Without loss of generality assume that  $\chi(G - v) > |N_G(v)|$ . We can then extend an optimal coloring  $f'$  of  $G - v$  to an optimal coloring  $f$  of  $G$  by coloring  $v$  with some color in  $[\chi(G - v)]$  not used by any neighbor of  $v$  in  $G$ , which exists because  $\chi(G - v) > |N_G(v)|$ . Thus  $\chi(G) = \chi(G - v)$ , and  $\chi(G) + \chi(\overline{G}) \leq \chi(G - v) + \chi(\overline{G - v}) + 1 \leq k + 1$ .

**Case 2:** If  $\chi(G - v) + \chi(\overline{G - v}) < k$ , then we extend an optimal coloring of  $G - v$  to a coloring of  $G$  by coloring  $v$  with a new color, and extend an optimal coloring of  $\overline{G - v}$  to a coloring of  $\overline{G}$  by coloring  $v$  with a new color. Thus  $\chi(G) + \chi(\overline{G}) \leq 1 + 1 + k - 1 = k + 1$ .