Proof of GeLSA Theory

(1)Process of the LSA algorithm:

For $x_i, y_i \in R; i \in \{0,1,2,\dots,n\}; D \in N$;

Obtain:

$$\max_{-D\le d\le D}\left(\max_{0\le j_1\le j_2\le n}\left|\sum_{i=j_1}^{j_2} x_{i+d}y_i\right|\right);$$

$$x_{-D} = x_{-D+1} = \cdots = x_{-1} = x_{n+1} = x_{n+2} = \cdots = x_{n+D} = 0;$$

$$y_{-D} = y_{-D+1} = \cdots = y_{-1} = y_{n+1} = y_{n+2} = \cdots = y_{n+D} = 0;$$

Set the following variables:

$$\alpha_d = \max_{0\le j_1\le j_2\le n}\left|\sum_{i=j_1}^{j_2} x_{i+d}y_i\right|;$$

$$H_1 = \max_{-D\le d\le D}\alpha_d;$$

(2)Process of the GeLSA algorithm:

For $k \in \{0,1,2,\dots,n\}, -D \le d \le D, d \in Z, D \in N$;

$$P_{(k,d)} = \max\left(P_{(k-1,d)} + x_{k+d}y_k, 0\right); P_{(0,d)} = 0;$$

$$N_{(k,d)} = \min\left(N_{(k-1,d)} + x_{k+d}y_k, 0\right); N_{(0,d)} = 0;$$

$$x_{-D} = x_{-D+1} = \cdots = x_{-1} = x_{n+1} = x_{n+2} = \cdots = x_{n+D} = 0;$$

$$y_{-D} = y_{-D+1} = \cdots = y_{-1} = y_{n+1} = y_{n+2} = \cdots = y_{n+D} = 0;$$

Set the following variables:

$$\beta_d = \max_{k\in\{0,1,2,\dots,n\}}\left(P_{(k,d)}, -N_{(k,d)}\right);$$

$$H_2 = \max_{-D\le d\le D}\beta_d\,;$$

(3)To prove the correctness of the GeLSA algorithm, it is sufficient to demonstrate that $H_1 = H_2$, which is equivalent to showing:$\alpha_d = \beta_d, d \in Z, -D \leq d \leq D$;

Firstly, we verify the case when $d = 0$ that is $\alpha_0 = \beta_0$;

This serves as the initial step in our verification process.

$\alpha_0 = \max_{0 \leq j_1 \leq j_2 \leq n} \left|\sum_{i=j_1}^{j_2} x_i y_i\right|$;

$P_{(k,0)} = \max(P_{(k-1,0)} + x_k y_k, 0); P_{(0,0)} = 0$;

$N_{(k,0)} = \min(N_{(k-1,0)} + x_k y_k, 0); N_{(0,0)} = 0$;

$\beta_0 = \max_{k \in \{0,1,2,\ldots,n\}} (P_{(k,0)}, -N_{(k,0)})$;

Clearly, based on the derivation formulas of $P_{(k,0)}$ and $N_{(k,0)}$, it can be concluded that both functions are the result of summing a continuous segment of $x_i y_i$. Therefore, $\alpha_0 \geq \beta_0$.

(4)We now proof: $\alpha_0 \leq \beta_0$ that is :$N_{(k,0)} \leq \alpha_0 = \max_{0 \leq j_1 \leq j_2 \leq n} \left|\sum_{i=j_1}^{j_2} x_i y_i\right| \leq P_{(k,0)}$;

To supplement, we can observe that: $N_{(k,0)} \leq 0 \leq P_{(k,0)}$ are two polygonal chains.

where the zeros of $N_{(k,0)}$ and $P_{(k,0)}$ are:

$0 = r_0 < r_1 < r_2 < \cdots r_g \leq n; 0 = q_0 < q_1 < q_2 < \cdots q_l \leq n$;

$\varphi(j_1, j_2) = \forall_{0 \leq j_1 \leq j_2 \leq n} \sum_{i=j_1}^{j_2} x_i y_i$,represents a curve from j_1 to j_2;

It suffices to prove that: $N_{(j_2,0)} \leq \varphi(j_1, j_2) \leq P_{(j_2,0)}$ In other words, the curve $\varphi(j_1, j_2)$lies between the two curves $N_{(k,0)}$ and $P_{(k,0)}$;

(5)We proof in hear: $\sum_{i=r_s+1}^{j_2} x_i y_i = P_{(j_2,0)} \geq \varphi(j_1, j_2) = \forall_{0 \leq j_1 \leq j_2 \leq n} \sum_{i=j_1}^{j_2} x_i y_i$;

$0 = r_0 < r_1 < r_2 < \cdots r_g \leq n$; are the zeros of $P_{(k,0)}$;

We can obtain: $P_{(j_2,0)} = \begin{cases} \sum_{i=r_s+1}^{j_2} x_i y_i, & r_s + 1 \leq j_2 < r_{s+1} ; \\ 0, & j_2 = r_{s+1} ; \end{cases}$

$\sum_{i=r_s+1}^{h} x_i y_i > 0, \; r_s + 1 \leq h < r_{s+1} ; \sum_{i=r_s+1}^{r_{s+1}} x_i y_i \leq 0$;

Situation(1): $r_s + 1 \leq j_1 \leq j_2 < r_{s+1}$

$\varphi(j_1, j_2) = \underset{0\le j_1\le j_2\le n}{\forall} \sum_{i=j_1}^{j_2} x_i y_i = \sum_{i=r_s+1}^{j_2} x_i y_i - \sum_{i=r_s+1}^{j_1-1} x_i y_i$;

Because $\sum_{i=r_s+1}^{j_1-1} x_i y_i \geq 0$,so: $\varphi(j_1, j_2) \leq \sum_{i=r_s+1}^{j_2} x_i y_i = P_{(j_2,0)}$;

Situation(2): $r'' < j_1 < r' < \cdots < r_s + 1 \leq j_2 < r_{s+1}$

We can obtain: $\sum_{i=r''+1}^{j_1-1} x_i y_i + \sum_{i=j_1}^{r'} x_i y_i + \sum_{i=r'+1}^{r_s} x_i y_i + \sum_{i=r_s+1}^{j_2} x_i y_i =$

$\sum_{i=r''+1}^{j_1-1} x_i y_i + \sum_{i=j_1}^{r'} x_i y_i + \sum_{i=r'+1}^{r_s} x_i y_i + P_{(j_2,0)}$;

and: $\sum_{i=r''+1}^{j_1-1} x_i y_i \geq 0, \sum_{i=r''+1}^{r_s} x_i y_i \leq 0$; so, $\sum_{i=j_1}^{r_s} x_i y_i \leq 0$;

So: $\varphi(j_1, j_2) = \sum_{i=j_1}^{j_2} x_i y_i = \sum_{i=r''+1}^{r_s} x_i y_i - \sum_{i=r''+1}^{j_1-1} x_i y_i + P_{(j_2,0)} \leq P_{(j_2,0)}$;

Situation(3): $j_2 = r_{s+1}, P_{(j_2,0)} = 0$;

We may derive: $\varphi(j_1, j_2) = \sum_{i=j_1}^{r_s} x_i y_i + \sum_{i=r_s+1}^{r_{s+1}} x_i y_i = \sum_{i=r''+1}^{r_{s+1}} x_i y_i - \sum_{i=r''+1}^{j_1-1} x_i y_i \leq 0 = P_{(j_2,0)}$;

In summary situation(1-3), the collective evidence demonstrates that

$$\sum_{i=r_s+1}^{j_2} x_i y_i = P_{(j_2,0)} \geq \varphi(j_1, j_2) = \underset{0\le j_1\le j_2\le n}{\forall} \sum_{i=j_1}^{j_2} x_i y_i \text{ ;}$$

By an analogous argument, we may prove that $N_{(k,0)} \leq \varphi(j_1, j_2)$;

So: $N_{(j_2,0)} \leq \varphi(j_1, j_2) \leq P_{(j_2,0)}$; In other words, the curve

$\varphi(j_1, j_2)$lies between the two curves $N_{(k,0)}$ and $P_{(k,0)}$;

So, $\alpha_0 = \max_{0\le j_1\le j_2\le n} \left|\sum_{i=j_1}^{j_2} x_i y_i\right| \leq \max_{k\in\{0,1,2,\ldots,n\}} \left(P_{(k,d)}, -N_{(k,d)}\right)$; that is $\alpha_0 \leq \beta_0$;

(5)Therefore, based on (3)-(5), we can conclude that $\alpha_0 = \beta_0$;

The same process can be used to prove $\alpha_d = \beta_d, d \in Z, -D \leq d \leq D, d \in Z$;

So $(The\ result\ of\ LSA) H_1 = \max_{-D\le d\le D} \alpha_d = \max_{-D\le d\le D} \beta_d = H_2 (The\ result\ of\ GeLSA)$;

Therefore, since the computational results of LSA and GeLSA are identical, the correctness of GeLSA is verified.