Progress report: Optimization for Beaming Splitting

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Review

Research Targets and Steps:

1 Figure out the basic **design and optimization principle** for **polarization-sensitive elements**

2 **Separate light’s polarization information** with **polarization-sensitive metasurface**

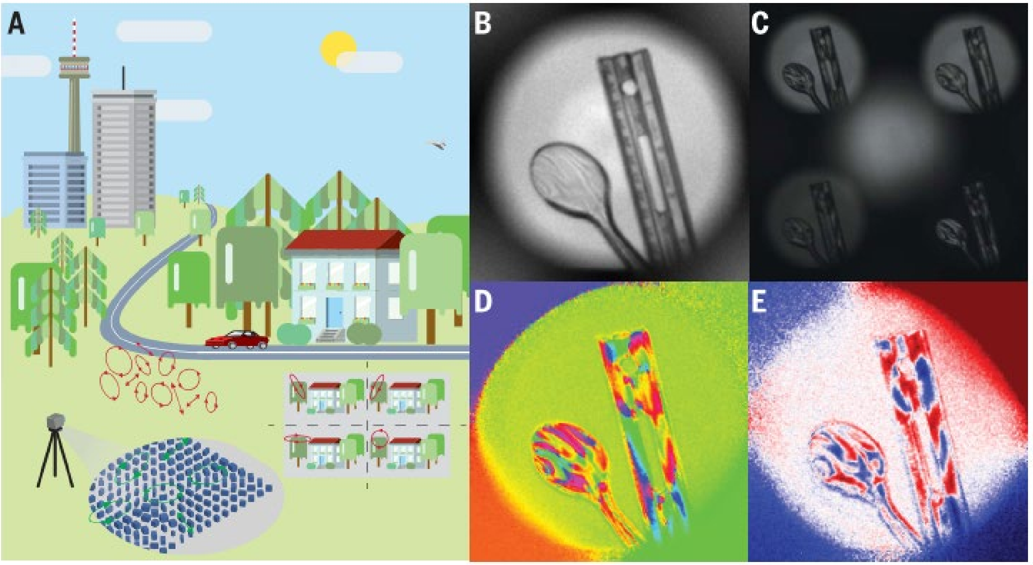
3 Look to **further applications** based on **polarization information**

e.g. hyperspectral imaging, stress analysis, remote sensing …

METASURFACES

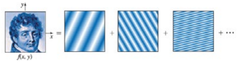
Matrix Fourier optics enables a compact full-Stokes polarization camera

Noah A. Rubin[1], Gabriele D’Aversa[1,2], Paul Chevalier[1], Zhujun Shi[3], Wei Ting Chen[1], Federico Capasso[1*]



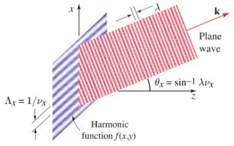
Review

1 Scalar Fourier Optics



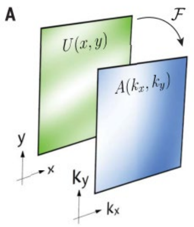
- complex scalar **transmittance function**

$$U(x,y) = t(x,y)E_0(x,y)$$



- **plane wave expansion**

$$\begin{cases} U(x,y) = \displaystyle\iint_{-\infty}^{+\infty} A(k_x,k_y)e^{-i(k_x x+k_y y)}d\frac{k_x}{2\pi}d\frac{k_y}{2\pi} \\ A(k_x,k_y) = \displaystyle\iint_{-\infty}^{+\infty} U(x,y)e^{i(k_x x+k_y y)}dxdy \end{cases}$$



→ **xy coordinate space** v.s. **k wave vector space**

- requirements: **paraxial incident**

2 Vector Fourier Optics

- Matrix transmittance function (**local Jones Matrix**) $\tilde{J}(x,y)$
- vector valued incident light (**local Jones Vector**) $|E_0(x,y)\rangle$

$$|U(x,y)\rangle = \tilde{J}(x,y)|E_0(x,y)\rangle$$

- **plane wave expansion**

$$\begin{cases} |U(x,y)\rangle = \tilde{J}(x,y)|E_0(x,y)\rangle = \displaystyle\iint_{-\infty}^{+\infty} |A(k_x,k_y)\rangle\, e^{-i(k_x x+k_y y)}d\frac{k_x}{2\pi}d\frac{k_y}{2\pi} \\ |A(k_x,k_y)\rangle = \displaystyle\iint_{-\infty}^{+\infty} \tilde{J}(x,y)|E_0(x,y)\rangle e^{i(k_x x+k_y y)}dxdy \end{cases}$$

- requirements: **paraxial incident**

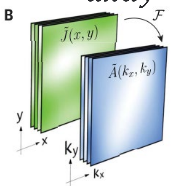
3 Matrix Fourier Optics

- special case 1:

normal, uniform incident

$$\tilde{J}(x,y)|E_0(x,y)\rangle \to \tilde{J}(x,y)|E_0\rangle$$

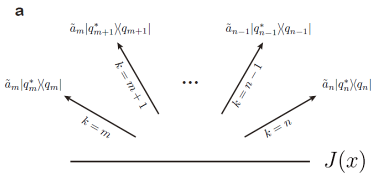
$$\begin{cases} \tilde{J}(x,y) = \displaystyle\iint_{-\infty}^{+\infty} \tilde{A}(k_x,k_y)e^{-i(k_x x+k_y y)}d\frac{k_x}{2\pi}d\frac{k_y}{2\pi} \\ \tilde{A}(k_x,k_y) = \displaystyle\iint_{-\infty}^{+\infty} \tilde{J}(x,y)e^{i(k_x x+k_y y)}dxdy \end{cases}$$



- special case 2:

normal, uniform incident periodic grating

$$\begin{cases} \tilde{J}(x,y) = \displaystyle\sum_{\vec{k}} \tilde{J}_{\vec{k}}\, e^{-i(k_x x+k_y y)} \\ \tilde{J}_{\vec{k}} = \displaystyle\iint_{-\infty}^{+\infty} \tilde{J}(x,y)e^{i(k_x x+k_y y)}dxdy \end{cases}$$



4 Beam Splitting → Parallel Polarization Analyzer

- **multiplexing** in **k space: polarization analyzer**

$$\tilde{J}(x,y) = \sum_{\vec{k}\in\{l\}} \tilde{J}_{\vec{k}}\, e^{-i(k_x x+k_y y)}, \qquad \text{where} \quad \tilde{J}_{\vec{k}} = a_k|p_k\rangle\langle q_k|$$

- **realization** in **xy space: pp phase metasurface**

$$\tilde{J}(x,y) = R(\theta(x,y))\begin{bmatrix} e^{i\varphi_x(x,y)} & 0 \\ 0 & e^{i\varphi_y(x,y)} \end{bmatrix} R(-\theta(x,y))$$

- **conflict between two space:**

can only **ideally** separate **conjugate polarized light**

→allow light **to leak into other orders**

Analytical Problem → **Optimization Problem**

Optimization for beam splitting

1) 1D Scalar Grating Optimization (As Introduction)

1 1D Scalar Diffraction Grating

· **diffraction orders** decide **diffraction angle**

Fourier coefficient decides **amplitude** and **phase**

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\tilde{t}(x)dx$$

· **traditional diffraction gratings** realize beam splitting:

e.g. 2 beams splittings:

$$t(x) = \frac{1}{\sqrt{2}}(1 + e^{ikx}) \quad T(x) = t^*(x)t(x) = 1 + \cos kx \quad \int_0^d T(x)dx = 1$$

· **problem**: **local gains is hard to realize**

→compress t(x) to ensure no local gains

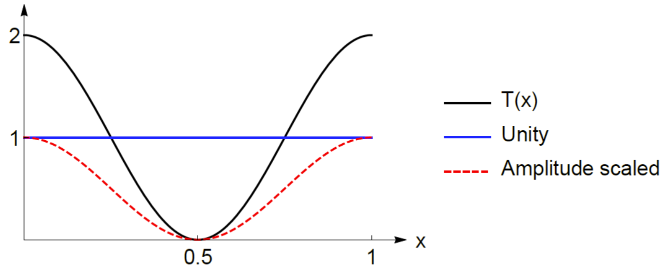
→**no-gain gratings**

· **new problem**: **too low efficiency**

2 orders: $\int_0^d T(x)dx = 1/2$ N orders: $\eta = 1/N$

→look for no gain & no loss gratings

→**phase-only gratings**



2 1D Scalar Phase-only Gratings

· **phase-only gratings**: $\tilde{t}(x) = e^{i\phi(x)}$

· **problem:**

can only precisely separate **1 beam** without leaking

proof:

$$e^{i\phi(x)} = \sum_{k=m}^{n} a_k e^{ikx} \rightarrow e^{-i\phi(x)}e^{i\phi(x)} = 1 = \sum_{k=m}^{n}\sum_{\ell=m}^{n} a_\ell^* a_k e^{i(k-\ell)x}$$

$$\rightarrow \begin{aligned} a_m^* a_n &= 0 \\ a_m^* a_{n-1} + a_{m+1}^* a_n &= 0 \\ a_m^* a_{n-2} + a_{m+1}^* a_{n-1} + a_{m+2}^* a_n &= 0 \\ &\vdots \\ a_m^* a_{m+2} + a_{m+1}^* a_{m+3} + \ldots + a_{n-2}^* a_n &= 0 \\ a_m^* a_{m+1} + a_{m+1}^* a_{m+2} + \ldots + a_{n-1}^* a_n &= 0 \\ \sum_{k=n}^{m} |a_k|^2 &= 1 \end{aligned}$$

→ all coefficients must be zero except for a_n

→allow light **to leak into other orders**

Analytical Problem → **Optimization Problem**

Reference: Supplementary Materials for “Matrix Fourier optics enables a compact full-Stokes polarization camera”

Reference

Outline:

Optimization for Parallel Polarization Analyzer

- **1) 1D Scalar Grating**
  - Diffraction Grating
    - Traditional Diffraction Grating
    - No-gain Diffraction Grating
  - Phase-only Grating
    - Conflict Between x-space and k-space
    - Phase-only Grating Optimization
      - The Constrained Optimization Problem
      - The Least Squares Optimization Problem
      - The Minimum Variance Optimization Problem
    - Upper Bound on the Efficiency
      - The Generalized Constrained Optimization Problem
      - Numerical Results
  - Special Case
    - Non-trivial Analytical Solution: Two Beam Spliting
    - New Restrictions on Gratings: Binary Gratings
    - The Effect of Symmetry: Even Orders and Odd Orders Gratings
    - One Dimensional Continuous Gratings
- **2) 2D Scalar Grating**
  - Two Dimensional Fourier Series on General Lattices
  - Two Dimensional Symmetry
  - Two Dimensional Beam Splitting
    - The General Theory
    - Symmetry and Symmetry Breaking
    - Examples Using Square Gratings
      - Four Beam Splitting Using Square Gratings
      - Five Beam Splitting Using Square Gratings
      - Nine Beam Splitting With Square Gratings
      - Examples Using Hexagonal Gratings
      - Six Beam Splitting
      - Seven Beam Splitting with Hexagonal Gratings
  - Two Dimensional Continuous Gratings
- **3) 2D Matrix Grating**
  - Parallel Polarization Analysis by Matrix Gratings
    - Tetrahedron Grating
    - Cctahedron grating
  - Parallel Polarization Analysis by Linearly Birefringent Matrix Gratings
    - Linear Birefringence Implies |pk> = |qk∗>
    - Relation to Past Work on "Metasurface Polarization Optics"
    - A Unitary, Linearly Birefringent Matrix Grating for Polarization Analysis Can Only Have Two Orders Acting as Analyzers
  - Optimization-based Design Of Matrix Gratings For Parallel Polarization Analysis
    - Development Of An Optimization Scheme
      - Parameterization Of The Grating Unit Cell
      - Optimization
    - Results for 11-element Gratings
      - Progression of the Optimization
      - Optimized Results
    - Conversion to Metasurface Design

Today's report

Optimization for beam splitting

1) 1D Scalar Grating Optimization

2 1D Scalar Phase-only Gratings

- **Three Optimization Problems**

Optimization Problem 1 *(The Constrained Optimization Problem). Given a set of indices K, find the d periodic function $\phi(x)$ that maximizes the normalized energy e_{CO} in the modes $k \in K$*

$$e_{CO}(\phi) = \frac{\sum_{k \in K} |a_k|^2}{\sum_{k=-\infty}^{\infty} |a_k|^2}, \tag{2.5}$$

subject to the constraint that the amplitude of the coefficients a_k for $k \in K$ are all the same. Here the coefficients a_k are the Fourier components of $e^{i\phi(x)}$. We will denote $\phi_{CO}(x)$ as the phase that optimizes $e_{CO}(\phi)$, and define the efficiency η_{CO} as

$$\eta_{CO} = e_{CO}(\phi_{CO}). \tag{2.6}$$

Optimization Problem 2 *(The Least Squares Optimization Problem). Given a set of indices K find the d periodic function $\phi(x)$, the phases $\alpha_k, k \in K$, and a real positive number λ such that we maximize the quantity*

$$e_{LS}(\phi, \lambda, \underline{\alpha}) = 1 - \|e^{i\phi(x)} - \lambda s(x, \underline{\alpha})\|^2, \tag{2.11}$$

where

$$s(x, \underline{\alpha}) = \sum_{k \in K} e^{ik(\frac{2\pi}{d}x)} e^{i\alpha_k}. \tag{2.12}$$

Let $\phi_{LS}(x)$, λ_{LS}, and $\underline{\alpha}_{LS}$ be the values that maximize e_{LS}. We define

$$\eta_{LS} = e_{LS}(\phi_{LS}, \lambda_{LS}, \underline{\alpha}_{LS}). \tag{2.13}$$

Optimization Problem 3 *(The Minimum Variance Optimization Problem). Given a set of indices K find the phases $\alpha_k, k \in K$, such that we minimize*

$$V(\underline{\alpha}) = \frac{1}{d} \int_{-d/2}^{d/2} \left(I(x, \underline{\alpha}) - I_0\right)^2 dx \tag{2.14}$$

where

$$I_0 = \frac{1}{d} \int_{-d/2}^{d/2} I(x, \underline{\alpha}) dx, \tag{2.15}$$

$$I(x, \underline{\alpha}) = |s(x, \underline{\alpha})|^2 \tag{2.16}$$

and

$$s(x, \underline{\alpha}) = \sum_{k \in K} e^{i\alpha_k} e^{ik(\frac{2\pi}{d})x}. \tag{2.17}$$

Once we have found the phases $\underline{\alpha}_{MV}$ that minimize the variance $V(\underline{\alpha})$, we define the efficiency as

$$\eta_{MV} = \frac{\sum_{k \in K} |c_k|^2}{\sum_{k=-\infty}^{\infty} |c_k|^2} \tag{2.18}$$

where c_k is the kth Fourier component of

$$e^{i\psi(x)} = \frac{s(x, \underline{\alpha}_{MV})}{|s(x, \underline{\alpha}_{MV})|}. \tag{2.19}$$

Problem 1:
Targeted

Problem 2&3:
Not completely the same
But easier to solve
Can be used for test solutions

Reference: L. A. Romero, F. M. Dickey, Journal of the Optical Society of America A 24, 2280 (2007).
L. A. Romero, F. M. Dickey, Progress in Optics 54, 319 (2010).

Optimization for beam splitting

1) 1D Scalar Grating Optimization

2 1D Scalar Phase-only Gratings

- **Upper Bound on the Efficiency**

mainly obtained by solving **the Least Squares Optimization Problem**

also **many other methods**:

Krackhardt, U., Mait, J. N., & Streibl, N. (1992). Applied Optics, 31(1), 27–37

Romero, L. A., & Dickey, F. M. (2007b). Journal of the Optical Society of America A, 24(8), 2280–2295.

Wyrowski, F. (1991). Optics Letters, 16, 1917.

TABLE 1 The values of η_{LS} for splitting a beam into N_{Modes} with N_{modes} odd. The phases α_k are the phases used in Equations (3.2), (3.4) and (3.5) to obtain the function $\phi(x)$ (we are also using $\gamma_k = 1$). We only give the phases for the positive indices since we have $\alpha_{-k} = \alpha_k$. The numbers η_{krack} are from the paper (Krackhardt, Mait, & Streibl, 1992)

N_{modes}	η_{LS}	η_{krack}	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
3	93.81	93.81	0.	$\pi/2$						
5	96.28	96.28	0.	$\pi/2$	π					
7	97.53	97.52	0.	5.285	1.954	0.730				
9	99.34	99.33	0.	3.833	5.538	6.146	1.371			
11	98.38	97.61	0.	3.465	4.550	5.912	5.638	1.265		
13	98.57	98.59	0.	4.774	6.354	4.745	2.915	1.410	6.278	
15	98.21	98.21	0.	2.415	4.222	0.883	2.753	2.938	3.782	4.821

TABLE 2 The values of η_{LS} for splitting a beam into N_{Modes} for N_{modes} even. The results η_{LS} assume that the phase function is even. We include the values of the constant α_k needed to generate $\phi(x)$ using Equations (3.2), (3.4) and (3.5) (we are also using $\gamma_k = 1$). We also include the results η_{krack} from table III of Krackhardt, Mait, and Streibl (1992) where $\phi(x)$ is not assumed to be symmetric. In that table they also give the values of α_k needed to obtain these efficiencies

N_{modes}	η_{LS}	η_{krack}	α_1	α_3	α_5	α_7	α_9	α_{11}	α_{13}
2	81.06	81.06	0.						
4	91.94	92.69	0.	4.425					
6	91.41	91.46	0.	1.107	3.196				
8	96.12	96.23	0.	0.724	3.548	5.364			
10	95.79	97.40	0.	0.126	4.941	2.683	0.739		
12	95.93	96.82	0.	4.639	3.654	5.544	3.680	1.735	
14	96.80	97.98	0.	.190	2.944	1.567	1.513	4.880	2.583

N	η_{phase}	η_{loss}
2	0.8106	0.500
3	0.9256	0.3333
4	0.9119	0.2500
5	0.9212	0.2000
6	0.8817	0.1667
7	0.9684	0.1429

Table S1. Comparison of diffraction efficiency for gratings that implement the optimized phase-only gratings defined by Eqs. S8 and S9 (η_{phase}) and those that implement loss-only modulation ($\eta_{\text{loss}} = 1/N$, as in Fig. S1) as a function of N, the number of diffraction orders into which light is directed. This assumes equal intensity on these orders for simplicity.

Optimization for beam splitting

1) 1D Scalar Grating Optimization

2 1D Scalar Phase-only Gratings · **Special Case:**

① two beam splitting

the only beam splitting problem that can be solved **completely analytically**

order: +1 -1

result: $\cos(\phi(x)) = \mathrm{sgn}\,(\cos(x)) \rightarrow \phi(x) = \begin{cases} 0 & |\phi| < \pi/2 \\ \pi & |\phi| \geq \pi/2 \end{cases}$

$\rightarrow a_1 = a_{-1} = \frac{2}{\pi} \quad \eta = |a_1|^2 + |a_{-1}|^2 = \frac{8}{\pi^2} \approx .8106$

Reference:

Gori, F. (1997). Diffractive optics: An introduction.

② binary grating/Dammann grating

- add constraints on ϕ(x): binary
- we can significantly improve the efficiency of the grating by breaking the symmetry of the beam splitting problem

TABLE 3 A reproduction of the table given by Mait in (Mait, 1997). Here η_{CO} gives the efficiency for splitting a beam into N_{Modes} number of beams using binary phase gratings. The quantity η_{ub} is an upper bound on the efficiency used in the paper by Mait, but it is not clear precisely how this was obtained. Since in the paper by Mait, the interval was defined as $(-1/2, 1/2)$, the step points z_k must be multiplied by 2π in order to agree with the step points in our notation. Once we know the step points, the transmission function $h(x)$ is given by Equations (5.2)–(5.4)

N_{modes}	$\pi - 2\alpha$	η_{ub}	η	z_0	z_1	z_2	z_3	z_4	z_5
2	π	81.06	81.06	−.250	.250				
3	π	68.74	66.42	−.368	.368				
	2.008	93.82	86.52	−.250	.250				
4	π	72.05	70.64	−.054	.054				
5	π	83.80	77.39	−.368	−.020	.020	.368		
	2.993	87.20	77.38	−.471	−.133	.133	.489		
6	π	85.28	82.45	−.302	−.122	.116	.496		
7	π	83.07	78.63	−.338	−.237	.237	.469		
	2.473	89.62	84.48	−.430	−.215	.215	.439		
8	π	83.06	74.55	−.428	−.182	.179	.294		
9	π	80.57	70.26	−.281	−.158	−.078	.124	.189	.500
	2.535	87.74	80.78	−.352	−.174	−.134	−.059	.359	.500
10	π	83.31	74.40	−.476	−.249	−.002	.119	.269	.342
11	π	82.11	78.40	−.364	−.296	−.153	.084	.167	.500
	2.589	89.03	84.44	−.413	−.282	−.155	−.046	.217	.500
12	π	86.16	77.96	−.381	−.335	−.050	.173	.275	.418

Reference:

Dammann, G. H. (1971). Optics Communications, 3(5), 312–315.
Dammann, H., & Klotz, (1977). Optica Acta, 24(4), 505–515.

③ even/odd order gratings

- Using the symmetry requirement, let some order be zero

Lemma 1. *A d periodic function $f(x)$ has no even Fourier coefficients if and only if we have $f(x + d/2) = -f(x)$.*

Lemma 2. *Let $f(x)$ be a d periodic function that satisfies $f(x + d/2) = -f(x)$. All of the even Fourier coefficients vanish, and the odd Fourier coefficients can be computed using*

$$f_k = \frac{2}{d} \int_{-d/4}^{d/4} e^{-ikx\left(\frac{2\pi}{d}\right)} f(x)\mathrm{d}x \quad k = odd. \tag{2.22}$$

Reference:

Killat, U., & Rave, W. (1982). Fiber and Integrated Optics, 4(2), 159–167.

④ 1D continues gratings

- Dammann gratings: relatively low efficiency
- The Uniformity Optimization Problem
- Solutions Using the Calculus of Variations
- Numerical Calculations

→ **final result for 1D scalar grating optimization**

Reference:

Romero, L. A., & Dickey, F. M. (2007b), 24(8), 2280–2295.

Optimization for beam splitting

Final Result for 1D Scalar Grating Optimization

TABLE 5 Optimum efficiencies for splitting a beam into an odd number of beams. We also list the values of α_k and μ_k in Equation (6.6) needed to obtain these. Our solutions have $\alpha_k = \alpha_{-k}$, and $\mu_k = \mu_{-k}$, as well as $\alpha_0 = 0$, and $\mu_0 = 1$. If $N_{modes} = 2M + 1$, the vectors $\underline{\alpha}$ and $\underline{\mu}$ contain the values $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M)$, and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_M)$. The optimal phase functions for $N_{modes} = 3, 11$ are given in Figure 2

N_{modes}	η_{LS}	η_{CO}	$\underline{\alpha}$ and $\underline{\mu}$
3	93.81	92.56	$\underline{\alpha} = \pi/2$ $\underline{\mu} = 1.329$
5	96.28	92.12	$\underline{\alpha} = (-\pi/2, \pi)$ $\underline{\mu} = (.459, .899)$
7	97.53	96.84	$\underline{\alpha} = (-.984, 1.891, .748)$ $\underline{\mu} = (1.289, 1.463, 1.249)$
9	99.34	99.28	$\underline{\alpha} = (.720, 5.567, 3.033, 1.405)$ $\underline{\mu} = (.971, .964, .943, 1.029)$
11	98.38	97.71	$\underline{\alpha} = (.311, 4.492, 2.847, 5.546, 4.406)$ $\underline{\mu} = (1.207, 1.297, 1.483, 1.427, 1.275)$
13	98.57	97.53	$\underline{\alpha} = (2.308, 4.345, 1.517, 1.692, 0.066, 6.243)$ $\underline{\mu} = (0.912, 0.968, 0.806, 0.923, 1.099, 1.027)$
15	98.21	97.29	$\underline{\alpha} = (2.625, 4.534, 0.970, 2.983, 3.328, 4.070)$ $\underline{\mu} = (4.945, 1.116, 1.463, 0.930, 1.114, 1.466, 1.359, 1.211)$

TABLE 6 Optimum efficiencies for splitting a beam into an even number of beams. Here the modes are given by $k = \pm 2m + 1$, $m = 1, M$ We also list the values of α_k and μ_k in Equation (6.6) needed to obtain these. Our solutions have $\alpha_k = \alpha_{-k}$, and $\mu_k = \mu_{-k}$, as well as $\alpha_1 = 0$, and $\mu_1 = 1$. If $N_{modes} = 2M$, the vectors $\underline{\alpha}$ and $\underline{\mu}$ contain the values $\underline{\alpha} = (\alpha_3, \ldots, \alpha_{2M-1})$, and $\underline{\mu} = (\mu_3, \ldots, \mu_{2M-1})$. The optimal phase function for $N_{modes} = 4$ is given in Figure 3

N_{modes}	η_{LS}	η_{CO}	$\underline{\alpha}$ and $\underline{\mu}$
4	91.94	91.19	$\underline{\alpha} = 4.438$ $\underline{\mu} = .523$
6	91.41	88.17	$\underline{\alpha} = (0.863, 3.069)$ $\underline{\mu} = (0.274, 0.487)$
8	96.12	95.94	$\underline{\alpha} = (0.724, 3.668, 5.367)$ $\underline{\mu} = (0.560, 0.601, 0.544)$
10	95.79	92.69	$\underline{\alpha} = (0.152, 4.683, 2.681, 0.651)$ $\underline{\mu} = (0.598, 0.412, 0.211, 0.546)$
12	95.93	95.36	$\underline{\alpha} = (4.562, 3.704, 5.465, 3.448, 1.725)$ $\underline{\mu} = (0.523, 0.424, 0.509, 0.586, 0.538)$
14	96.80	96.34	$\underline{\alpha} = (0.235, 2.906, 1.661, 1.521, 4.847, 2.527)$ $\underline{\mu} = (0.430, 0.471, 0.419, 0.505, 0.511, 0.545)$