CHAPTER 6

The Mathematical Theory of Laser Beam-Splitting Gratings

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1. INTRODUCTION

Beam splitting gratings are a relatively recent development in the long history of diffraction gratings, which dates back to the time of Newton (see Appendix A). Beam splitting gratings are used to split a laser beam into multiple beams for industrial and scientific applications, while diffraction gratings are primarily used in spectroscopy, though they also arise in such applications as wavelength division multiplexing for optical communication systems. The spatial period of beam splitting gratings is typically many wavelengths, while the spatial period of diffraction gratings is of the order of a wavelength. Beam splitting gratings are used in a range of applications, including: parallel processing in laser machining and material processing, sensor systems, interferometry, communication systems, and image processing and gathering systems.

An arbitrary periodic grating will split an incoming beam into a large number of outgoing beams (or orders). For many applications it is desirable to put as much energy as possible into certain orders, while keeping the energy in all of these orders equal to each other (or more generally, in some fixed proportion). If the grating does not absorb any light, we refer to it as a phase grating. In this review we are concerned with the mathematical problem of finding the optimal phase grating that puts as much energy as possible into certain orders, while keeping the energy in these orders in some fixed proportion. This is by no means the only issue involved in designing gratings. For example, this optimization problem does not address the question of how sensitive this design is to errors in the manufacturing process. This question can be addressed by actually building the gratings, or by analyzing a more complex mathematical optimization problem that takes uncertainties into account. However, we will not be concerned with such issues in this review.

Elementary textbooks on physics (Feynman, Leighton, & Sands, 1963; Halliday & Resnick, 1978) show that the angular spacing of the orders in a

diffraction grating depends on the period of the grating, but they typically make no mention of what determines the energy put into the various orders. As discussed in Section 2.1, more specialized texts (Goodman, 1968) show that the energy put into each order depends on the Fourier coefficients of the transmission function of the grating. For the case of one dimensional, or line gratings, the transmission function $h(x)$ of the grating can be written as $h(x) = \mathrm{e}^{\mathrm{i}\phi(x)}$, where $\phi(x)$ is a periodic function. After suitably scaling the problem, the energy in the kth order of a beam passing through this grating can be written as $|a_k|^2$ where a_k is the kth Fourier coefficient of $\mathrm{e}^{\mathrm{i}\phi(x)}$.

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{d}x. \tag{1.1}$$

To facilitate our outline of this review, it is useful to have a specific example of a beam splitting problem in mind. As an example, Gori et al. in (Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, & Gentili, 1998) considered the problem of designing a one dimensional grating to split a beam into three beams of equal intensity. They referred to such a grating as a triplicator. This can be expressed mathematically as follows.

Example 1 (*The Triplicator*). *Find a real 2π periodic phase function $\phi(x)$ such that the Fourier coefficients a_k of $\mathrm{e}^{\mathrm{i}\phi(x)}$ satisfy $|a_1| = |a_0| = |a_{-1}|$, and that puts as much energy as possible into these three modes.*

In the notation used throughout this review, we are interested in finding the phase $\phi(x)$ such that we maximize the energy

$$E = \sum_{k\in K} |a_k|^2 = |a_{-1}|^2 + |a_0|^2 + |a_1|^2, \tag{1.2}$$

subject to the constraint that

$$|a_i| = |a_j| \quad \forall i, j \in K. \tag{1.3}$$

Here a_k are the Fourier coefficients of $\mathrm{e}^{\mathrm{i}\phi(x)}$, and K is a set of integers specifying which modes we are attempting to put energy into. For the triplicator, the set K is the set of integers -1, 0, and 1. That is,

$$k \in K \quad \text{iff } k = -1, 0 \text{ or } 1. \tag{1.4}$$

If we have a phase function that equalizes the energy in all of the modes $a_k, k \in K$, the efficiency of the grating is given by

$$\eta = \frac{\sum_{k \in K} |a_k|^2}{\sum_{k=-\infty}^{\infty} |a_k|^2}. \tag{1.5}$$

In Section 2.1 we review the results from (Gori, 1997) that show that, as with any non-trivial beam splitting problem, it is not possible to solve this problem perfectly. That is, we cannot put one hundred per cent of our energy into these three modes while keeping the energy in these modes the same. Thus, we must solve this as a Constrained Optimization Problem. For the triplicator, this problem was first solved semi-analytically (requiring the numerical determination of a single parameter) in (Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, & Gentili, 1998).

In Section 2.2 we discuss three optimization problems that are relevant to solving the laser beam splitting problem. We call the first of these optimization problems the Constrained Optimization Problem. For the triplicator, this solves the problem that we have already stated of determining $\phi(x)$ so that we maximize the energy E in the modes $a_k, k \in K$, while keeping the energy in these modes the same. Short of discussing the sensitivity of manufacturing such a grating, this is the true optimization problem we would like to solve.

Though the Constrained Optimization Problem is the real problem we would like to solve, in Section 2.2 we discuss two simpler problems that approximate it. We refer to these as the Least Squares Optimization Problem, and the Minimum Variance Optimization Problem. The Minimum Variance Optimization Problem was introduced in (Prongué, Herzig, Dändliker, & Gale, 1992), but was called the Efficiency Optimization Problem by those authors. The solutions to these two simpler problems do not typically produce beams where the intensities of the relevant orders are the same. However, these problems can be used to find a good initial guess to the Constrained Optimization Problem, which can then be solved by using an iterative process such as Newton's method.

In Section 3 we discuss the results in (Romero & Dickey, 2007b) where they show that the Least Squares Optimization Problem can be used to give a bound to the efficiency of the Constrained Optimization Problem. These results are similar to those used by (Krackhardt, Mait, & Streibl, 1992) where they use a theorem due to Wyrowski (1991) to determine an upper bound. We believe there has been some confusion caused in the literature by the fact that in (Wyrowski, 1991) the results were presented in

a more general setting, rather than just for beam splitting, and furthermore the results as stated in (Wyrowski, 1991) are not correct. In particular, in that paper the efficiency (which should be dimensionless) is given as a quantity that has the dimensions of either length or area (depending on whether we are doing one dimensional or two dimensional beam splitting). In Section 3 we present a corrected form of the bound given by Wyrowski.

In Section 4 we present the results for the simplest beam splitting problem, the problem of splitting a beam into two beams of equal intensity. We present the results for this problem because it is the only non-trivial beam splitting problem that has a completely analytical solution.

The papers (Dammann & Gortler, 1971) and (Dammann & Klotz, 1977) are the earliest published works on beam splitting. They considered binary gratings where the phase $\phi(x)$ was restricted to taking only the values 0 or π. In general, gratings where the phase takes on only two values are referred to as Dammann gratings. In Section 5 we give an overview of the work that has been done on Dammann gratings. In order to unify this work we introduce some of our own notation, and prove theorems that have not been stated by workers in the field.

In Section 6 we return to the problem of finding optimal one dimensional phase gratings where the phase $\phi(x)$ can take on arbitrary values. The authors of (Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, & Gentili, 1998) were the first to use methods from the calculus of variations to solve this problem for the special case of the triplicator. In (Romero & Dickey, 2007b) this work was extended to apply to arbitrary problems. In this review we slightly change the notation used in that paper. The results in (Romero & Dickey, 2007b) are equivalent to showing that the optimal phase function $\phi(x)$ can always be written as

$$e^{i\phi(x)} = h(x, \underline{\alpha}, \underline{\mu}) = \frac{s(x, \underline{\alpha}, \underline{\mu})}{|s(x, \underline{\alpha}, \underline{\mu})|} \tag{1.6}$$

where

$$s(x, \underline{\alpha}, \underline{\mu}) = \sum_{k \in K} \mu_k e^{i\alpha_k} e^{ikx}. \tag{1.7}$$

In Equation (1.7) $\underline{\alpha}$ is a collection of phases $\alpha_k, k \in K$, and $\underline{\mu}$ is a collection of weights $\mu_k, k \in K$. Here the parameters $\underline{\alpha}$ and $\underline{\mu}$ must be adjusted so that the constraint on the equality of the amplitude of the Fourier coefficients $a_k, k \in K$ is satisfied, and so that we are in fact at an optimal solution. We emphasize that in the expression for $s(x, \underline{\alpha}, \underline{\mu})$ we are only

summing over values of k in the set K (the set of modes we are trying to put energy into).

Although determining the optimal grating using the form given in Equations (1.6) and (1.7) requires the numerical determination of the constants α_k and μ_k, it reduces the original infinite dimensional problem of determining all of the Fourier coefficients of $\phi(x)$, to a finite dimensional problem. For this reason we say that this is a semi-analytical solution to the beam splitting problem. For the case of the triplicator, there is only one parameter to adjust in order to find the optimal solution.

The fact that the optimal grating can be defined using Equations (1.6) and (1.7) makes it possible to succinctly communicate results to other workers in the field. For example, in Tables 5 and 6 we have given enough information so that anyone who can write a code to evaluate a sum of trigonometric functions can see what the optimal phase function looks like. In the same way, Tables 7 and 8 give similar results for two dimensional gratings.

All of the discussion so far has been concerned with one dimensional (line) gratings. In Sections 7 and 8 we discuss two dimensional beam splitting. In a two dimensional grating, the transmission function $h(x, y) = e^{i\phi(x,y)}$ is periodic on a two dimensional lattice. By this we mean that the function $h(x, y)$ is periodic in two linearly independent directions. For example, a grating that is periodic on a square lattice satisfies $h(x+2\pi m, y+2\pi n) = h(x, y)$ for all integers m and n. When a laser beam passes through such a grating it gets split up into a two dimensional array of beams. If we try to choose the phase $\phi(x, y)$ so that the energy in various orders of this diffracted beam have the same amplitude, and we maximize the energy in these orders, we get the natural extension of the one dimensional beam splitting problems.

As a preliminary to our discussion of two dimensional gratings, in Section 7 we review some of the background material concerning Fourier series on two dimensional lattices. In particular, we discuss how to expand functions that are periodic on a hexagonal lattice. Such lattices are useful for such problems as splitting a beam into six beams that are symmetrically placed at the vertices of a regular hexagon, or for seven beam splitting where we include a beam in the centre of the hexagon.

In the early work on two dimensional gratings authors used separable gratings where the transmission function $h(x, y)$ was assumed to be of the form $h(x, y) = h_1(x)h_2(y)$. Since then, numerical work has been done dropping this assumption. In Section 8 we discuss the results in (Romero & Dickey, 2007a) where they show that, as in the case of one dimensional gratings, the optimal transmission function has the two dimensional equivalent of the form given in Equations (1.6) and (1.7). Several examples are given that show that by using this general theory

one can get considerably higher efficiencies than if one assumes that the grating is separable.

The two dimensional results give an interesting illustration of the concept of symmetry breaking. That is, just because the statement of our optimization problem is symmetrical does not mean that the optimal solution has to be symmetrical. For example, when considering one dimensional gratings, if the function $\phi(x)$ is symmetrical ($\phi(x) = \phi(-x)$), then the Fourier coefficients of $e^{i\phi(x)}$ must satisfy $a_k = a_{-k}$. If we are trying to equalize the energy in a set of symmetrically placed modes, this might suggest that we can look for the optimal grating under the assumption that the function $\phi(x)$ is symmetrical. However, the optimal grating does not have to be symmetrical. If the optimal solution has a lower symmetry than the statement of the problem, we say that this solution has broken the symmetry of the problem.

For the case of one dimensional continuous gratings, we are not aware of any examples of symmetry breaking. However, in Section 5 we see that this is not the case for Dammann gratings. Similarly, for the case of two dimensional gratings, we present several examples of problems where the symmetry is broken. For example, the problem of splitting a beam into nine beams using a square grating has what is known as four fold symmetry (see Section 7). By this we mean that the problem looks the same if we rotate it by $\pi/2$ radians. Similar to the one dimensional case, this might suggest that we look for optimal solutions under the assumption that the phase $\phi(x, y)$ also has four fold symmetry. However, as we see, the optimal solution is given by a phase function that has only two fold symmetry. That is, it looks the same if we rotate it by π radians, but not by $\pi/2$ radians. The efficiency of the grating that has two fold symmetry is .9322, whereas the efficiency of the grating with four fold symmetry is only .8458.

For readers interested in numerical results, we note here that results concerning the efficiencies of the various problems can be found in the tables. Except for Table 4 (which is based on the results from (Prongué, Herzig, Dändliker, & Gale, 1992)), we have given enough information so that the reader should be able to verify the efficiencies from the information in the table, and from equations in this review that are referenced in the caption of the table. Tables 1 and 2 are a slight modification of a table given in (Krackhardt, Mait, & Streibl, 1992) where the upper bound on the efficiencies for one dimensional gratings (discussed in Section 3) are given. Table 3 is an abridged form of the table given in (Mait, 1997) giving the efficiencies for one dimensional Dammann gratings. Tables 5 and 6 give the efficiencies for the Constrained Optimization Problem for one dimensional gratings. Finally Tables 7 and 8

summarize results for two dimensional beam splitting problems on square and hexagonal lattices.

We conclude the summary of this review by noting that we do not discuss such numerical techniques as simulated annealing or the Iterative Fourier Transform Algorithm. We refer the reader to other articles such as (Feldman & Guest, 1989; Ripoll, Kettunen, & Herzig, 2004; Turunen, Vasara, Westerholm, Jin, & Salin, 1988) for such matters.

2. THE BASIC MATHEMATICAL PROBLEM

2.1. Reduction to a Problem in Fourier Analysis

Throughout this review we limit ourselves to scalar diffraction theory. Though the authors of the paper (Tervo & Turunen, 2000) use vector diffraction theory to achieve efficiencies of 100 per cent using polarization effects. we will not consider such systems here.

The energy and phase of the various orders of beams emerging from a diffraction grating are directly related to the Fourier components of the diffraction grating. Goodman's book on Fourier optics, (Goodman, 1968) gives this result for some specific examples assuming the Fraunhofer approximation. Though the Fraunhofer approximation is most likely an excellent approximation in most applications of beam splitting, we believe it is worth noting that the same result can be derived without making either the Fresnel or Fraunhofer approximation. This derivation is given in Appendix A of Romero and Dickey (2007b).

Though the formulations for one and two dimensional gratings are very similar, for clarity we will first formulate the problems for one dimensional gratings.

Suppose we have a lossless one dimensional grating with spatial periodicity d.[1] Using a coordinate system (x, y, z), we suppose the grating lies in the plane $z = 0$, and that the transmission function of the grating is independent of the y coordinate. This one dimensional grating is characterized by a d periodic function $\phi(x) = \phi(x + d)$ that changes the phase of an input beam at x by the amount $\phi(x)$. Diffraction theory shows that a uniform beam travelling in the z direction will get split up into a large number of beams after passing through the grating. The amplitude and phase of these beams are determined by the Fourier coefficients of $e^{i\phi(x)}$.

$$a_k = \frac{1}{d}\int_{-d/2}^{d/2} e^{i\phi(x)} e^{-ik(\frac{2\pi}{d})x}\,dx \quad \text{for } k = 0 \pm 1, \pm 2, \ldots. \tag{2.1}$$

[1] By suitable rescaling, we could, without loss of generality, set d to some convenient number such as 1 or 2π. Since the choice of a convenient d is not universal in the literature, we will not specify a particular value at this time. However, whenever doing actual calculations, we set $d = 2\pi$.

In particular, the intensity I_k of the kth order is given by

$$I_k = |a_k|^2. \tag{2.2}$$

Actually, diffraction theory predicts that we only get a finite number of these terms, since the remaining terms represent evanescent waves that decay after a few wavelengths of propagation. The upper bound on k is related to the ratio d/λ where d is the spacing of the grating, and λ is the wavelength of the incoming light. The equation for the amplitude of the different orders assumes that the illumination is uniform. For the case of beam-like illumination, the output will be the convolution of the plane wave result with the Fourier transform of the beam (see Appendix A of Romero and Dickey (2007b)). Assuming the spatial extent of the beam is large compared to the grating pitch, the uniform beam results are nearly applicable in this case as well.

The following theorem, of relevance to the subject at hand, was proved by Gori in the Appendix of Gori (1997). We give a slightly simplified version of the proof.

Theorem 1. *Let $\phi(x)$ be a real d periodic function. Either $e^{i\phi(x)}$ has a single non-zero Fourier component, or it has an infinite number of non-zero Fourier components.*

Proof. Suppose

$$e^{i\phi(x)} = \sum_{k=m}^{n} a_k e^{ik(\frac{2\pi}{d})x} \tag{2.3}$$

where $m < n$, $a_m \neq 0$, and $a_n \neq 0$. We have

$$e^{i\phi(x)}e^{-i\phi(x)} = 1 = \sum_{k=m}^{n}\sum_{l=m}^{n} a_k \overline{a_l} e^{i(k-l)(\frac{2\pi}{d})x}. \tag{2.4}$$

In order for this equality to exist, the term on the right must contain no Fourier modes other than the DC term. However, the maximum value of $k - l$ in the exponent of $e^{i(k-l)(\frac{2\pi}{d})x}$ is uniquely obtained when $k = n$ and $l = m$. It follows that the coefficient multiplying $e^{i(n-m)(\frac{2\pi}{d})x}$ is given by $a_n\overline{a}_m$, and is clearly non-zero. It follows that $e^{i\phi(x)}$ cannot have the assumed form if $m < n$. This is allowable if $m = n$, in which case $e^{i\phi(x)}$ consists of a single Fourier mode. □

2.2. Three Optimization Problems

Theorem 1 shows that it would not be possible to solve any but the most trivial of our beam shaping problems with 100 per cent efficiency. This is

because it is not possible to find a phase grating that puts 100 per cent of the diffracted energy into a finite set of orders (yet more than one). For this reason we must be content with optimizing the energy in the given orders. This leads to the following optimization problem.

Optimization Problem 1 (*The Constrained Optimization Problem*). *Given a set of indices K, find the d periodic function $\phi(x)$ that maximizes the normalized energy e_{CO} in the modes $k \in K$*

$$e_{CO}(\phi) = \frac{\sum\limits_{k\in K} |a_k|^2}{\sum\limits_{k=-\infty}^{\infty} |a_k|^2}, \tag{2.5}$$

subject to the constraint that the amplitude of the coefficients a_k for $k \in K$ are all the same. Here the coefficients a_k are the Fourier components of $e^{i\phi(x)}$. We will denote $\phi_{CO}(x)$ as the phase that optimizes $e_{CO}(\phi)$, and define the efficiency η_{CO} as

$$\eta_{CO} = e_{CO}(\phi_{CO}). \tag{2.6}$$

We have stated this optimization problem so the set of indices K is arbitrary. However, we are typically interested in two cases. In the first, the set of indices K consists of an odd number of indices $(0, \pm 1, \pm 2, \pm 3, \ldots, \pm M)$. The other case is where K consists of an even number of indices $(\pm 1, \pm 3, \ldots, \pm(2M-1))$. We will elaborate on this second case in Section 2.3. We could have stated this optimization problem so that the intensities of the desired Fourier coefficients are not necessarily equal, but in specified proportions. Essentially all of the general results stated in this review apply to this more general problem, However it simplifies the notation if we assume that we are trying to equalize the energy in a set of modes. The vast majority of problems of interest are concerned with this case.

Most of this review is concerned with solving this optimization problem without applying any constraints on $\phi(x)$ other than that it be real and periodic. However, the original work on this problem was done by Dammann and Gortler (1971) who restricted $\phi(x)$ to be binary. We will review this problem in some detail in Section 5.

We take the point of view that the Constrained Optimization Problem we have just stated is the true optimization problem we want to solve. However, there are two related optimization problems that approximate this problem. We will refer to them as the Least Squares Optimization, and Minimum Variance Optimization problems. The Least Squares Optimization Problem was defined in (Romero & Dickey, 2007b) and

the minimum variance optimization problem was defined in (Prongué, Herzig, Dändliker, & Gale, 1992).[2] Both of these problems are simpler to solve than the Constrained Optimization Problem. One of the main uses of these simpler problems is that they can be used to give good initial guesses for the Constrained Optimization Problem, which can then be solved using techniques such as Newton's method or a gradient descent method.

The least squares problem is motivated by the following reasoning. In an ideal solution to the Constrained Optimization Problem (with perfect efficiency), we would have a function $\phi(x)$ such that the Fourier coefficients $a_k, k \in K$ of $e^{i\phi(x)}$ all have the same amplitude, and the rest of the Fourier coefficients are zero. This would imply that we could write

$$e^{i\phi(x)} = \lambda s(x, \underline{\alpha}), \tag{2.7}$$

where

$$s(x, \underline{\alpha}) = \sum_{k \in K} e^{ik(\frac{2\pi}{d})x} e^{i\alpha_k}. \tag{2.8}$$

Here $\underline{\alpha}$ is a vector containing the phases α_k of the coefficients $a_k = \lambda e^{i\alpha_k}$, and the constant λ is given by (due to Parseval's equality)

$$\lambda^2 = \frac{1}{N}. \tag{2.9}$$

where N is the number of indices in K.

One approach to the least squares problem could be to try to find a function $\phi(x)$, and coefficients α_k such that we minimize the distance squared $\|e^{i\phi(x)} - \lambda s(x, \underline{\alpha})\|^2$ between the functions $e^{i\phi(x)}$ and $\lambda s(x, \underline{\alpha})$. Here we are using the notation

$$\|f(x)\|^2 = \frac{1}{d} \int_{-d/2}^{d/2} |f(x)|^2 dx. \tag{2.10}$$

The actual least squares optimization problem is a slight variant of this problem. First, rather than minimizing $\|e^{i\phi(x)} - \lambda s(x, \underline{\alpha})\|^2$, we maximize $1 - \|e^{i\phi(x)} - \lambda s(x, \underline{\alpha})\|^2$. Second, we do not specify the parameter λ, but determine it as part of the maximization problem.

[2] Though they referred to it as the Efficiency Optimization Problem.

Optimization Problem 2 (*The Least Squares Optimization Problem*). *Given a set of indices K find the d periodic function $\phi(x)$, the phases $\alpha_k, k \in K$, and a real positive number λ such that we maximize the quantity*

$$e_{LS}(\phi, \lambda, \underline{\alpha}) = 1 - \|e^{i\phi(x)} - \lambda s(x, \underline{\alpha})\|^2, \tag{2.11}$$

where

$$s(x, \underline{\alpha}) = \sum_{k \in K} e^{ik(\frac{2\pi}{d}x)} e^{i\alpha_k}. \tag{2.12}$$

Let $\phi_{LS}(x)$, λ_{LS}, and $\underline{\alpha}_{LS}$ be the values that maximize e_{LS}. We define

$$\eta_{LS} = e_{LS}(\phi_{LS}, \lambda_{LS}, \underline{\alpha}_{LS}). \tag{2.13}$$

The motivation for this problem comes from the fact that if we succeed in finding a solution $(\phi_{LS}(x), \underline{\alpha}_{LS}, \lambda_{LS})$ that makes η_{LS} close to unity, then $e^{i\phi_{LS}(x)}$ must have almost all of its energy in the desired modes, and they must be in nearly the right relative amplitudes. In (Romero & Dickey, 2007b) it was shown that the optimal efficiency for this Least Squares Optimization Problem is the same as the upper bound on the efficiency given in (Krackhardt, Mait, & Streibl, 1992) using the corrected form of a theorem due to Wyrowski. See Section 3 for a further discussion of this point.

In (Herzig, Prongué, & Dändliker, 1990) and (Prongué, Herzig, Dändliker, & Gale, 1992) they define the following optimization problem.

Optimization Problem 3 (*The Minimum Variance Optimization Problem*). *Given a set of indices K find the phases $\alpha_k, k \in K$, such that we minimize*

$$V(\underline{\alpha}) = \frac{1}{d} \int_{-d/2}^{d/2} \left(I(x, \underline{\alpha}) - I_0\right)^2 dx \tag{2.14}$$

where

$$I_0 = \frac{1}{d} \int_{-d/2}^{d/2} I(x, \underline{\alpha}) dx, \tag{2.15}$$

$$I(x, \underline{\alpha}) = |s(x, \underline{\alpha})|^2 \tag{2.16}$$

and

$$s(x, \underline{\alpha}) = \sum_{k \in K} e^{i\alpha_k} e^{ik(\frac{2\pi}{d})x}. \tag{2.17}$$

Once we have found the phases $\underline{\alpha}_{MV}$ that minimize the variance $V(\underline{\alpha})$, we define the efficiency as

$$\eta_{MV} = \frac{\sum_{k\in K} |c_k|^2}{\sum_{k=-\infty}^{\infty} |c_k|^2} \tag{2.18}$$

where c_k is the kth Fourier component of

$$e^{i\psi(x)} = \frac{s(x, \underline{\alpha}_{MV})}{|s(x, \underline{\alpha}_{MV})|}. \tag{2.19}$$

In (Prongué, Herzig, Dändliker, & Gale, 1992) they use the phases from this optimization problem as an input to what they call the uniformity optimization problem. In that problem they assume that the phases are the same as for the minimum variance problem, but they adjust other parameters to make the amplitudes of the Fourier components come out in the right proportions. Their uniformity optimization problem is close to, but not identical to, the Constrained Optimization Problem.

In the review in (Gori, 1997), Gori uses this optimization problem (though never giving it a name) to give a heuristic proof that the phase of the middle term in the triplicator should be $\pi/2$. See Section 6.

In order to solve the minimum variance optimization problem we only need to adjust a finite number of parameters, the phases $\alpha_k, k \in K$. As stated, the other two optimization problems require us to find a d periodic function $\phi(x)$, which is equivalent to adjusting an infinite number of parameters (its Fourier coefficients). However, in (Romero & Dickey, 2007b) it was shown that both of these problems can be reduced to finite dimensional optimization problems. Though all three of these optimization problems can be reduced to finite dimensional problems, the least squares and minimum variance problems are simpler to solve since they involve roughly half the number of unknowns as the Constrained Optimization Problem (see Section 6).

2.3. Problems with an Even Number of Modes

Many authors discuss beam splitting problems where they try to maximize the energy that is put into an even number of modes. Often times they assume that the reader knows precisely what they mean by that. However, there are two distinct ways this term is used in the literature, so we believe we should make it clear what we mean by this in this review.

In (Killat, Rabe, & Rave, 1982) they consider the problem of maximizing the energy in an even number of modes $k = \pm 1, \pm 2, \pm 3, \ldots, \pm M$. Since

the spacing between the modes $k = 1$ and $k = -1$ is twice that between the other modes, this does not result in equally spaced beams. More typically when authors consider problems with an even number of modes they are concerned with the modes $k = \pm 1, \pm 3, \pm 5, \ldots, \pm(2M-1)$. When producing beams for manufacturing, it would be a nuisance if the grating produced beams with $k = 0, \pm 2, \ldots$. For this reason, we could further specify that all of the even modes up to order $2M-2$ must vanish. It is not a priori clear if the optimal solution, where we require that these modes vanish, is necessarily optimal when we do not require that these modes vanish.

We could further require not only that the even modes up to order $2M-2$ vanish, but that all of the even modes vanish (Morrison, 1992). Once again, it is not a priori clear that the optimal solution to this problem is guaranteed to be the optimal solution to the previous problem. However, the theory explained in (Romero & Dickey, 2007b) and in Section 6 shows that the solution to the Constrained Optimization Problem for an even number of modes, where we require that all even modes vanish, is the same as the solution where we do not impose this constraint.

In this review when we consider beam splitting problems with an even number of modes we assume that we are concerned with the problem of putting as much energy as possible into the modes $(\pm 1, \pm 3, \ldots \pm(2M-1))$, where we also assume that no energy is put into any modes a_k with k even. The following lemma is a variant of a lemma presented in (Morrison, 1992) in his discussion of Dammann gratings with an even number of modes. This characterizes periodic functions that do not have any even Fourier coefficients.

Lemma 1. *A d periodic function $f(x)$ has no even Fourier coefficients if and only if we have $f(x + d/2) = -f(x)$.*

Proof. If $f(x)$ has no even Fourier coefficients, we can write down the Fourier series for $f(x)$, and verify that $f(x + d/2) = -f(x)$. On the other hand, suppose that $f(x+d/2) = -f(x)$. The kth Fourier coefficient of $f(x)$ can be written as

$$f_k = \frac{1}{d} \int_{-d/2}^{d/2} \mathrm{e}^{-\mathrm{i}kx(\frac{2\pi}{d})} f(x) \mathrm{d}x. \tag{2.20}$$

Since $f(x)\mathrm{e}^{-\mathrm{i}kx(\frac{2\pi}{d})}$ is a d periodic function, we can shift this by any amount without changing its integral over the interval $-d/2$ to $d/2$. That is, for any value of s we have

$$f_k = \frac{1}{d} \int_{-d/2}^{d/2} \mathrm{e}^{-\mathrm{i}k(x+s)(\frac{2\pi}{d})} f(x+s) \mathrm{d}x. \tag{2.21}$$

If we set $s = d/2$, and use our assumption that $f(x + d/2) = -f(x)$, we get $f_k = -\mathrm{e}^{-\mathrm{i}k\pi} f_k$. If k is even, this implies that $f_k = 0$. □

When computing the Fourier coefficients of functions with an even number of modes it can be convenient to use the following lemma, which we state without proof.

Lemma 2. *Let $f(x)$ be a d periodic function that satisfies $f(x + d/2) = -f(x)$. All of the even Fourier coefficients vanish, and the odd Fourier coefficients can be computed using*

$$f_k = \frac{2}{d} \int_{-d/4}^{d/4} e^{-ikx(\frac{2\pi}{d})} f(x) \mathrm{d}x \quad k = odd. \tag{2.22}$$

2.4. Two Dimensional Gratings

The optimization problems discussed so far in this section were all stated in terms of one-dimensional gratings. In a two dimensional grating, the phase function ϕ is a function of two variables which we refer to as x and y. We can define almost identical optimization problems for two dimensional gratings. We will wait until we actually discuss two dimensional gratings in Section 8 before doing this.

3. AN UPPER BOUND ON THE EFFICIENCY

Our primary concern is in solving the Constrained Optimization Problem as stated in Section 2.2. However, it is useful to have a simple procedure for giving an upper bound on this efficiency. In (Romero & Dickey, 2007b) this upper bound is obtained by solving the Least Squares Optimization Problem. Before discussing this view of the upper bound, we discuss this upper bound based on arguments given in (Wyrowski, 1991) and (Krackhardt, Mait, & Streibl, 1992).

3.1. Some Basic Theorems

In (Krackhardt, Mait, & Streibl, 1992) they use a theorem due to Wyrowski (1991) that allows them to compute the upper bound on the efficiency. However, as they point out in their paper, the result by Wyrowski is able to yield efficiencies greater than 100 per cent. Though not pointed out in (Krackhardt, Mait, & Streibl, 1992) this is due to the fact that there are several errors in the paper by Wyrowski, and the eventual theorem as stated. is not correct.[3] In particular, the result is not dimensionally

[3]It should be noted that the result is correct if one limits oneself to intervals of unit length or area.

correct. That is, an expression for the efficiency (a dimensionless quantity) is given that has the dimensions of either length or area (depending on whether we are working with one or two dimensional problems). We believe that much of the confusion comes from the fact that the theorem is stated in very general terms, and it is not clear if the author is dealing with Fourier series or Fourier transforms. In the course of his proof he assumes that the input signal has both unit magnitude, and unit energy (on the infinite interval), which is not possible. However, by proving two different versions of the theorem, one for Fourier series, and one for Fourier integrals, Wyrowski's results can easily be patched up. We will do that in this section.

In order to discuss Wyrowski's results in a way that makes the transition from Fourier series (splitting) to Fourier integrals (beam shaping) simple we use a slightly generalized form of our Constrained Optimization Problem that includes a real positive function $p(x)$ in our input, and that does not require that the desired Fourier coefficients to all have the same magnitude. That is, a_k will no longer be the kth Fourier coefficient of $e^{i\phi(x)}$, but it will be the kth Fourier coefficient of $p(x)e^{i\phi(x)}$. Also, we no longer require that $|a_k|$ is constant for $k \in K$, but require that $|a_k| = c\gamma_k$ where γ_k is a positive number.

Optimization Problem 4 (*The Generalized Constrained Optimization Problem*). *Given a real positive function $p(x)$, a set of indices K, and relative amplitudes γ_k, $k \in K$, find the d periodic function $\phi(x)$ that maximizes the normalized energy e_{CO} in the modes $k \in K$*

$$e_{CO}(\phi) = \frac{\sum_{k\in K} |a_k|^2}{\sum_{k=-\infty}^{\infty} |a_k|^2}, \tag{3.1}$$

subject to the constraint that the amplitude of the coefficients a_k for $k \in K$ are in the right relative proportion. That is, for some constant c we have $|a_k| = c\gamma_k$. Here the coefficients a_k are the Fourier components of $p(x)e^{i\phi(x)}$.

The Constrained Optimization Problem stated in Section 2 is a special case of this problem where we set $p(x) = 1$, and $\gamma_k = 1$.

The results in (Krackhardt, Mait, & Streibl, 1992) using a corrected form of Wyrowski's theorem show that a bound on η_{CO} can be obtained as follows. Define the function

$$s(x, \underline{\alpha}) = \sum_{k\in K} \gamma_k e^{i\alpha_k} e^{ik(\frac{2\pi}{d})x}. \tag{3.2}$$

Here $\underline{\alpha}$ is a vector containing the phases α_k of the relevant Fourier coefficients $a_k, k \in K$. Using this notation, we have the following theorem.

Theorem 2. *With $s(x, \underline{\alpha})$ defined as in Equation (3.2), the efficiency η_{CO} of the Generalized Constrained Optimization Problem satisfies*

$$\eta_{CO} \leq \eta_W \tag{3.3}$$

where

$$\eta_W = \max_{\underline{\alpha}} R(\underline{\alpha}) \tag{3.4}$$

and

$$R(\underline{\alpha}) = \frac{\left(\int_{-d/2}^{d/2} p(x)|s(x, \underline{\alpha})|\mathrm{d}x\right)^2}{\int_{-d/2}^{d/2} |s(x, \underline{\alpha})|^2 \mathrm{d}x \int_{-d/2}^{d/2} p^2(x)\mathrm{d}x}. \tag{3.5}$$

Finding η_W is a simpler problem than finding η_{CO} since it is a finite dimensional optimization problem. That is, we only have a finite number of parameters $\alpha_k, k \in K$ to adjust. This is in contrast to the Constrained Optimization Problem where we have a function $\phi(x)$ to find, which is equivalent to adjusting an infinite number of parameters. In (Romero & Dickey, 2007b) the calculus of variations is used to show that the Constrained Optimization Problem can also be reduced to a finite dimensional problem, but there are twice as many parameters to adjust as in the least squares optimization problem.

We now give the corrected form of Wyrowski's theorem, which allows us to prove Theorem 2.

Theorem 3. *Given a collection of integers K, and Fourier coefficients $a_k, k \in K$ let*

$$f(x) = \sum_{k \in K} a_k \mathrm{e}^{\mathrm{i}k\left(\frac{2\pi}{d}\right)x}. \tag{3.6}$$

Let $p(x) > 0$ and $\phi(x)$ be real functions with period d, such that the Fourier coefficients b_k of $g(x) = p(x)\mathrm{e}^{\mathrm{i}\phi(x)}$ satisfy $b_k = ca_k$, for $k \in K$ and some constant c. Under these assumptions we have

$$\frac{\sum_{k \in K} |b_k|^2}{\sum_{k=-\infty}^{\infty} |b_k|^2} \leq \frac{\left(\int_{-d/2}^{d/2} p(x)|f(x)|\mathrm{d}x\right)^2}{\int_{-d/2}^{d/2} |f(x)|^2 \mathrm{d}x \int_{-d/2}^{d/2} p^2(x)\mathrm{d}x}. \tag{3.7}$$

We prove this theorem in Appendix B, where we also give a version of the theorem that applies to Fourier integrals.

Theorem 3 implies Theorem 2. This follows from the fact that if $\phi_{co}(x)$ is the phase that maximizes the energy in the generalized Constrained Optimization Problem, and c_k are the Fourier coefficients of $p(x)e^{i\phi_{co}(x)}$, then for some constant c we have $c_k = c\gamma_k e^{i\beta_k}$ for $k \in K$, where β_k is the phase of c_k. If we denote $\underline{\beta}$ as the vector of phases $\beta_k, k \in K$, then Theorem 3 implies that

$$\eta_{co} = \frac{\sum\limits_{k\in K} |c_k|^2}{\sum\limits_{k=-\infty}^{\infty} |c_k|^2} \leq R(\underline{\beta}). \tag{3.8}$$

In this expression $R(\underline{\beta})$ is as in Equation (3.5) with $s(x, \underline{\beta}) = f(x)$ and $f(x)$ is defined as in Equation (3.6).

Since η_W is obtained by maximizing $R(\underline{\alpha})$ over all values of $\underline{\alpha}$, we clearly must have $R(\underline{\beta}) \leq \eta_W$¡ and hence $\eta_{co} \leq \eta_W$.

The optimization problem stated in Equation (3.4) does not have any clear intuitive meaning. However, in (Romero & Dickey, 2007b) the inequality in Equation (3.3) was proven by showing that the efficiency of the Least Squares Optimization Problem is always greater than that of the Constrained Optimization Problem, and by showing that the expression for η_W in Equation (3.4) is the same as η_{LS} in the Least Squares Optimization Problem. We summarize this is a theorem.

Theorem 4. *The efficiency obtained from the Least Squares Optimization Problem (see Section 2) is the same as the efficiency η_W in Equation (3.4)*

The results in ths section apply to two dimensional beam splitting problems as well.

3.2. Numerical Results

In the discussion in the last subsection, the results were presented quite generally, where K was a set of indices. In practice we are typically concerned with two cases. In the first case K consists of the $N_{modes} = 2M + 1$ indices $K = (0, \pm 1, \pm 2, \ldots, \pm M)$. Here we are trying to put the energy into an odd number of modes. In the second case K consists of the $N_{modes} = 2M$ indices $K = (\pm 1, \pm 3, \ldots, \pm(2M - 1))$. Here we are trying to put the energy into an even number of modes. In (Krackhardt, Mait, & Streibl, 1992) they computed η_{LS} for N_{modes} ranging between 2 and 25. They give both the efficiencies η and the phases α_k used in computing these efficiencies. Their results were tabulated in table III in (Krackhardt, Mait, & Streibl, 1992).

We would like to take the opportunity to note that a table of values for η_{LS} for N_{modes} odd and less than or equal to 11, was given in (Romero & Dickey, 2007b), but we have found that this table has numerous errors in it. In Table 1 we have recomputed the values of η_{LS} for N_{modes} odd and less than or equal to 15. We also give the phases α_k needed to obtain these values, and have listed the values of the efficiencies given in (Krackhardt, Mait, & Streibl, 1992). For the most part we agree with the results given in (Krackhardt, Mait, & Streibl, 1992), but there are a few discrepancies. When there is disagreement, our results are larger, with the exception of $N_{modes} = 13$, where they report an efficiency of 98.59, and we give 98.57.

It should be noted that in (Krackhardt, Mait, & Streibl, 1992) they do not assume that the function $s(x, \underline{\alpha})$ defined in Equation (3.2) is symmetric. That is, they do not assume that $\alpha_k = \alpha_{-k}$. The phases they give in their table do not satisfy this condition. However, we have found that the optimal solution for all values of N_{modes} odd that we have computed can be obtained by assuming that $s(x, \underline{\alpha})$ is symmetric. One might conjecture that the functions $s(x, \underline{\alpha})$ resulting from the phases reported in (Krackhardt, Mait, & Streibl, 1992) could be made symmetric by appropriately shifting the function, but we have found that this is not the case. We have checked their results in a number of cases using their phases, and found that they do in fact give the values they report. Thus, it appears that one can actually get optimal solutions that are symmetric as well as non-symmetric that give the same efficiencies.

In Table 2 we reproduce the efficiencies given in (Krackhardt, Mait, & Streibl, 1992) when N_{modes} is even, and also give values of η_{LS} that we have computed assuming $s(x, \underline{\alpha})$ is symmetric. For the case of N_{modes} even, we see that the efficiencies under the constraint that $s(x, \underline{\alpha})$ is symmetric are a bit less than those when this constraint is dropped. In Table 2 we also list the values of α_k needed to obtain these efficiencies. In table III of Krackhardt, Mait, and Streibl (1992) they also give the values of α_k needed to achieve these efficiencies, as well as giving the results for up to 24 modes.

It should be noted that if we assume that $s(x, \underline{\alpha})$ in Equation (3.2) is even ($\alpha_k = \alpha_{-k}$), then we will have

$$s(x, \underline{\alpha}) = \sum_{k=0}^{M-1} a_{2k+1} \cos\left((2k+1)\frac{2\pi}{d}x\right). \tag{3.9}$$

This implies that $s(\pm d/4, \underline{\alpha}) = 0$. Typically (unless the derivative of $s(x, \underline{\alpha})$ also vanishes at these values) this means that $|s(x, \underline{\alpha})|$ will be non differentiable at $x = \pm d/4$. This makes the integral in the numerator of $R(\alpha)$ in Theorem 2 difficult to compute numerically, unless we break the

TABLE 1 The values of η_{LS} for splitting a beam into N_{Modes} with N_{modes} odd. The phases α_k are the phases used in Equations (3.2), (3.4) and (3.5) to obtain the function $\phi(x)$ (we are also using $\gamma_k = 1$). We only give the phases for the positive indices since we have $\alpha_{-k} = \alpha_k$. The numbers η_{krack} are from the paper (Krackhardt, Mait, & Streibl, 1992)

N_{modes}	η_{LS}	η_{krack}	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
3	93.81	93.81	0.	$\pi/2$						
5	96.28	96.28	0.	$\pi/2$	π					
7	97.53	97.52	0.	5.285	1.954	0.730				
9	99.34	99.33	0.	3.833	5.538	6.146	1.371			
11	98.38	97.61	0.	3.465	4.550	5.912	5.638	1.265		
13	98.57	98.59	0.	4.774	6.354	4.745	2.915	1.410	6.278	
15	98.21	98.21	0.	2.415	4.222	0.883	2.753	2.938	3.782	4.821

TABLE 2 The values of η_{LS} for splitting a beam into N_{Modes} for N_{modes} even. The results η_{LS} assume that the phase function is even. We include the values of the constant α_k needed to generate $\phi(x)$ using Equations (3.2), (3.4) and (3.5) (we are also using $\gamma_k = 1$). We also include the results η_{krack} from table III of Krackhardt, Mait, and Streibl (1992) where $\phi(x)$ is not assumed to be symmetric. In that table they also give the values of α_k needed to obtain these efficiencies

N_{modes}	η_{LS}	η_{krack}	α_1	α_3	α_5	α_7	α_9	α_{11}	α_{13}
2	81.06	81.06	0.						
4	91.94	92.69	0.	4.425					
6	91.41	91.46	0.	1.107	3.196				
8	96.12	96.23	0.	0.724	3.548	5.364			
10	95.79	97.40	0.	0.126	4.941	2.683	0.739		
12	95.93	96.82	0.	4.639	3.654	5.544	3.680	1.735	
14	96.80	97.98	0.	.190	2.944	1.567	1.513	4.880	2.583

integral up into several pieces. It should be noted that this complication goes away if we use Lemma 2 to compute the Fourier coefficients. If the optimal solution is almost, but not quite, symmetric the numerics get to be a bit tricky (though by no means insurmountable).

4. TWO BEAM SPLITTING

In this section we discuss the problem of splitting a beam into two beams of equal intensity such that the energy in each of the two beams is as great as possible. We believe this problem merits special attention since it is the only beam splitting problem that can be solved completely analytically. Our discussion is along the lines of (Gori, 1997). Throughout this section

we will assume that we have appropriately scaled our grating so that $d = 2\pi$.

We begin by explicitly stating the problem.

Example 2 (*Two Beam Splitting Problem*). *Find the real 2π periodic function $\phi(x)$ such that we maximize the energy*

$$E = \frac{|a_1|^2 + |a_{-1}|^2}{\sum_{k=-\infty}^{\infty} |a_k|^2} \tag{4.1}$$

subject to the constraint that

$$|a_1|^2 = |a_{-1}|^2 , \tag{4.2}$$

where a_k are the Fourier coefficients of $e^{i\phi(x)}$.

The following lemma simplifies the search for the function $\phi(x)$.

Lemma 3. *Without loss of generality the Fourier coefficients a_1 and a_{-1} associated with the optimal phase $\phi(x)$ in the two beam splitting problem can be assumed to be real and positive.*

Proof. For any $\phi_0(x)$, let b_k be the kth Fourier coefficient of $e^{i\phi_0(x)}$. Let

$$\phi(x) = \alpha + \phi_0(x + \beta). \tag{4.3}$$

It is a simple exercise to show that amplitude of the Fourier coefficients of $e^{i\phi(x)}$ are the same as those of $e^{i\phi_0(x)}$, and that the kth Fourier coefficient of $e^{i\phi(x)}$ is given by

$$a_k = b_k e^{i(\alpha + k\beta)}. \tag{4.4}$$

Clearly we can always choose α and β so that the phases of a_1 and a_{-1} are both zero. That is, they are real and positive. □

Using this lemma we can prove the following theorem.

Theorem 5. *The optimal phase $\phi(x)$ in the two beam splitting problem can be chosen to be binary, and such that $\cos(\phi(x)) = \operatorname{sgn}(\cos(x))$. The optimal efficiency is given by $\eta = \frac{8}{\pi^2}$.*

The rest of this subsection is devoted to proving this theorem. Lemma 3 implies that we can look for the optimal solution assuming that a_1 and a_{-1}

are real and positive. Since they are real and positive, the constraint that $|a_1| = |a_{-1}|$ can be replaced by the much stricter constraint that $a_1 = a_{-1}$.

We now write down explicit expressions for the coefficients a_1 and a_{-1}.

$$a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(\phi(x)-x)}\mathrm{d}x = \alpha_1 + i\beta_1 \tag{4.5}$$

where

$$\alpha_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(\phi(x))\cos(x) + \sin(\phi(x))\sin(x))\,\mathrm{d}x \tag{4.6}$$

and

$$\beta_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin(\phi(x))\cos(x) - \cos(\phi(x))\sin(x))\,\mathrm{d}x. \tag{4.7}$$

Similarly, we have

$$a_{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(\phi(x)+x)}\mathrm{d}x = \alpha_{-1} + \mathrm{i}\beta_{-1} \tag{4.8}$$

where

$$\alpha_{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(\phi(x))\cos(x) - \sin(\phi(x))\sin(x))\,\mathrm{d}x \tag{4.9}$$

and

$$\beta_{-1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\sin(\phi(x))\cos(x) + \cos(\phi(x))\sin(x))\,\mathrm{d}x. \tag{4.10}$$

In order to have a_1 and a_{-1} be real, and equal to each other, we must have $\alpha_1 = \alpha_{-1}$, and $\beta_1 = \beta_{-1} = 0$. It follows that we must have

$$\beta_{-1} - \beta_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\phi(x))\sin(x)\mathrm{d}x = 0 \tag{4.11}$$

$$\beta_1 + \beta_{-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\phi(x))\cos(x)\mathrm{d}x = 0 \tag{4.12}$$

$$\alpha_1 - \alpha_{-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\phi(x))\sin(x)\mathrm{d}x = 0. \tag{4.13}$$

Assuming these conditions hold, in order to make a_1 and a_{-1} as large as possible we would like to maximize the quantity

$$I = \alpha_1 + \alpha_{-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\phi(x))\cos(x)\mathrm{d}x. \tag{4.14}$$

Thus, the optimal solution for two beam splitting tries to maximize I in Equation (4.14) subject to the constraints that Equations (4.11)–(4.13) hold. However, if we maximize I in Equation (4.14) without regard to our constraints, we find that we can maximize I by choosing $\phi(x)$ so that

$$\cos(\phi(x)) = \mathrm{sgn}\,(\cos(x))\,. \tag{4.15}$$

This is equivalent to setting

$$\phi(x) = \begin{cases} 0 & |\phi| < \pi/2 \\ \pi & |\phi| \geq \pi/2. \end{cases} \tag{4.16}$$

Here we are only defining $\phi(x)$ on the interval $(-\pi, \pi)$. The function $\phi(x)$ in Equation (4.16) maximizes I without applying any constraints. However, it also happens to satisfy all of the constraints (4.11)–(4.13). In particular, it satisfies the constraint (4.11) because $\cos(\phi(x))$ is an even function and the integral of any even function with $\sin(x)$ vanishes. It satisfies the other two constraints because $\sin(\phi(x)) = 0$ everywhere.

A simple calculation now shows that we have

$$a_1 = a_{-1} = \frac{2}{\pi}. \tag{4.17}$$

It follows that the efficiency η is given by

$$\eta = |a_1|^2 + |a_{-1}|^2 = \frac{8}{\pi^2} \approx .8106. \tag{4.18}$$

5. DAMMANN GRATINGS

5.1. Introduction

In Section 2.2 we stated the Constrained Optimization Problem without imposing any constraints on the phase $\phi(x)$. However, much of the original work on beam splitting was devoted to finding beam splitting gratings where the phase $\phi(x)$ was constrained to take on only two distinct

values. Such binary phase gratings are now commonly referred to as Dammann gratings, which is the subject of this section.

Though the pioneering work on what are now known as "Dammann gratings" was given by Dammann and Gortler (1971) and Dammann and Klotz (1977), it is more convenient to begin our discussion with the more general problem discussed by Killat, Rabe, and Rave (1982). In this problem we are looking for a lossless binary grating that maximizes the energy in a set of diffracted beams such that the energy in each of these beams is the same. Throughout this section we assume that we have appropriately scaled the grating so that its period d is equal to 2π.

Consider the function $h(x, \mathbf{z}, \alpha)$ that is 2π periodic, has unit amplitude, and has two values of the phase. Since multiplying any function by e^{ia} does not change the amplitude of its Fourier coefficients, without loss of generality, we will assume that the two phases of $h(x, \mathbf{z}, \alpha)$ are given by $\pi/2 - \alpha/2$, and $-\pi/2 + \alpha/2$. In our notation, the vector $\mathbf{z}$ contains the N step points where the phase changes

$$\mathbf{z}^T = (z_0, z_1, \ldots, z_{N-1}). \tag{5.1}$$

Here it is assumed that $-\pi \leq z_k \leq \pi$, and $z_k \leq z_{k+1}$. In what follows we will also assume without loss of generality that N is even. A few examples should convince the reader that if N were odd then we would automatically have an additional step point at $x = -\pi$ that we have not included, and the inclusion of this step point would now make N even.

In our discussion of the function $h(x, \mathbf{z}, \alpha)$ it will be useful to use the function $w(x, \mathbf{z})$ that is 2π periodic, and that is 1 on the intervals where $h(x, \mathbf{z}, \alpha)$ has phase $\pi/2 - \alpha/2$, and 0 on the intervals where $h(x, \mathbf{z}, \alpha)$has phase $-\pi/2 + \alpha/2$ (see Figure 1). In particular, we set

$$w(x, \mathbf{z}, \alpha) = \begin{cases} 1 & \text{if } x \in I_k, k = \text{even} \\ 0 & \text{if } x \in I_k, k = \text{odd}. \end{cases} \tag{5.2}$$

Here the intervals I_k are defined as

$$x \in I_k \quad \text{iff } z_k \leq x \leq z_{k+1}. \tag{5.3}$$

In defining the interval I_{N-1} we use the additional point $z_N = z_0 + 2\pi$ that is the periodic extension of z_0.

The transmission function $h(x, \mathbf{z}, \alpha)$ can be written in terms of the functions $w(x, \mathbf{z})$. We state this as a lemma.

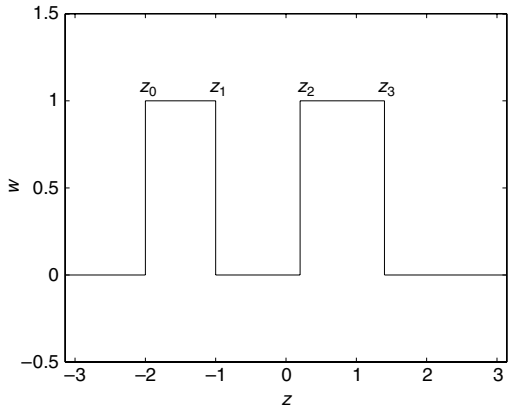

FIGURE 1 The function $w(x, \mathbf{z})$ defined in Equation (5.2) is a 2π periodic function that alternates between 1 and 0, on the intervals $I_k = (z_k, z_{k+1})$

Lemma 4. *The transmission function of a binary phase grating with step points* $\mathbf{z}$ *and phases* $\pi/2 - \alpha/2$, *and* $-\pi/2 + \alpha/2$ *can be written as*

$$h(x, \mathbf{z}, \alpha) = \mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha/2} w(x, \mathbf{z}) - i\mathrm{e}^{\mathrm{i}\alpha/2} (1 - w(x, \mathbf{z})) \tag{5.4}$$

where $w(x, \mathbf{z})$ *is defined as in Equations* (5.2) and (5.3).

Proof. This is a direct consequence of the definition of the functions $w(x, \mathbf{z}, \alpha)$ and $h(x, \mathbf{z}, \alpha)$. □

In the rest of this section we use the notation $b_k(\mathbf{z}, \alpha)$ for the Fourier coefficients of $h(x, \mathbf{z}, \alpha)$.

$$b_k(\mathbf{z}, \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}kx} h(x, \mathbf{z}, \alpha) \mathrm{d}x. \tag{5.5}$$

In (Killat, Rabe, & Rave, 1982) they are concerned with the following optimization problem.

Optimization Problem 5 (*The Dammann Beam Splitting Problem (Odd Number of Modes)*). *Find the vector of step points* $\mathbf{z}$, *and the phase* α *such that we maximize*

$$E_M(\mathbf{z}, \alpha) = \sum_{k=-M}^{M} |b_k(\mathbf{z}, \alpha)|^2 \tag{5.6}$$

subject to the constraint that

$$|b_0(\mathbf{z}, \alpha)| = \left|b_j(\mathbf{z}, \alpha)\right| \quad \textit{for } j = \pm 1, \pm 2, \ldots, \pm M. \tag{5.7}$$

Here $b_k(\mathbf{z}, \alpha)$ is the kth Fourier coefficient of the function $h(x, \mathbf{z}, \alpha)$ as defined in Equation (5.4).

In this problem we are trying to equalize the energy in an odd number of Fourier modes. In (Killat, Rabe, & Rave, 1982) they also studied the problem of equalizing the energy in an even number of modes obtained by omitting the DC term b_0. That is, maximizing the energy in the modes $b_k, k = \pm 1, \pm 2, \ldots \pm M$, subject to the constraint that the amplitudes of these Fourier coefficients are the same. This problem does not lead to equally spaced beams since the gap between the modes $k = 1$ and $k = -1$ is twice that between the other modes. Other authors (Mait, 1990; Morrison, 1992) consider the problem of equalizing the energy in an even number of modes by only including the odd Fourier modes in the problem. That is, by maximizing the energy in the modes $b_k, k = \pm 1, \pm 3, \ldots, \pm 2M - 1$ subject to the constraint that the amplitude of each of these modes is the same.

In the original papers (Dammann & Gortler, 1971) and (Dammann & Klotz, 1977) they considered a special case of this problem. In particular, they assumed that the phase jumped between 0 and π. In our notation, this is equivalent to (after multiplying h by $e^{-i\pi/2}$) setting $\alpha = 0$. Furthermore, Dammann et al. assumed that the number of step points N was given by $N = 2M$ (where $2M + 1$ is the number of Fourier modes of interest), and that the step points z_k were distributed symmetrically about the origin. That is, that $z_k = -z_{M-1-k}$. By adjusting the step points z_k at which the phase changed, they were able to get the amplitude of the $2M + 1$ Fourier coefficients $b_k, k = -M, M$ to be the same.

In (Krackhardt & Streibl, 1989) they kept the same framework as in the original papers by Dammann (assuming symmetrically placed points and $N = 2M$), but systematically examined the cases up to $M = 10$, where there are 21 Fourier modes of the same amplitude. Walker and Jahns (1990) extended Dammann's ideas to include multi-level gratings that have more than two values of the phase. In the works by Turunen, Vasara, Westerholm, Jin, and Salin (1988) and Feldman and Guest (1989) they discuss numerical techniques such as simulated annealing and the IFTA (Iterative Fourier Transform Algorithm) for finding optimal Dammann gratings. In (Doskolovich, Soifer, Alessandretti, Perlo, & Repetto, 1994) they present a technique for finding good initial guesses that can be fed to an iterative algorithm.

In the original papers by Dammann et al., they considered two dimensional problems, but limited themselves to cases where the

transmission function $h(x, y)$ is separable. That is, it can be written as $h(x, y) = h_1(x)h_2(y)$. This effectively reduced the problem to a one dimensional problem. Part of the motivation for choosing $\alpha = 0$ in the original papers by Dammann et al. was that in this case the function $h(x, y) = h_1(x, y)h_2(x, y)$ is also binary. This would not be the case for $\alpha \neq 0$. In the papers (Mait, 1989; Turunen, Vasara, Westerholm, & Salin, 1989) they generalized Dammann's ideas to non-separable gratings. They divided up a unit cell of a two dimensional grating into rectangular blocks, and required that the phase of the transmission function was constant inside each one of these blocks.

Though in this survey we use the term Dammann grating to refer to binary phase gratings, the term is sometimes used to refer to gratings where the transmission function has constant amplitude, and the phase jumps between a small number of values. This definition of Dammann gratings would include multilevel gratings as discussed by Walker and Jahns (1990), and Miller, Taghizadeh, Turunen, and Ross (1993). Recently several papers have been written using the term 'Circular Dammann Grating' (Wen & Chung, 2008; Zhao & Chung, 2006; Zhou, Jia, & Liu, 2003). These are actually not gratings, and would more appropriately be called Fresnel zone plates. In our discussion of Dammann gratings that follows we will limit our discussion to one dimensional binary gratings where the phase jumps between two values.

The articles by Mait (1990, 1997) and Gori (1997) are good general references for the subject of Dammann gratings. Table 3 is a slight modification (using our notation and fewer decimal places) of a table presented by Mait (1997). Here η_{ub} is an upper bound on the efficiency given by Mait, and η is the actual efficiency achieved. We would think that η_{ub} would be the efficiency obtained using the Least Squares Optimization Problem discussed in Sections 2 and 3, but it does not seem to be. In the table N_{modes} is the number of modes, and the points z_k are the step points in phase. Since Mait used the interval $(-1/2, 1/2)$, to get our step points, these values need to be multiplied by 2π. In the notation we have been using, $N_{modes} = 2M + 1$, but Mait also included values where N_{modes} was even.

Using general purpose tools, such as the IFTA and simulated annealing, it is possible to make a table such as Table 3 without having to explain certain of its features. The purpose of the next few sub-sections is to understand some theoretical properties of Dammann gratings that allow us to appreciate certain features in this table. For example:

- If N_{modes} is odd Table 3 has two entries, one where α is constrained to be zero, and one where it is not. The table does not give two such values when N_{modes} is even.

TABLE 3 A reproduction of the table given by Mait in (Mait, 1997). Here η_{CO} gives the efficiency for splitting a beam into N_{Modes} number of beams using binary phase gratings. The quantity η_{ub} is an upper bound on the efficiency used in the paper by Mait, but it is not clear precisely how this was obtained. Since in the paper by Mait, the interval was defined as $(-1/2, 1/2)$, the step points z_k must be multiplied by 2π in order to agree with the step points in our notation. Once we know the step points, the transmission function $h(x)$ is given by Equations (5.2)–(5.4)

N_{modes}	$\pi - 2\alpha$	η_{ub}	η	z_0	z_1	z_2	z_3	z_4	z_5
2	π	81.06	81.06	−.250	.250				
3	π	68.74	66.42	−.368	.368				
	2.008	93.82	86.52	−.250	.250				
4	π	72.05	70.64	−.054	.054				
5	π	83.80	77.39	−.368	−.020	.020	.368		
	2.993	87.20	77.38	−.471	−.133	.133	.489		
6	π	85.28	82.45	−.302	−.122	.116	.496		
7	π	83.07	78.63	−.338	−.237	.237	.469		
	2.473	89.62	84.48	−.430	−.215	.215	.439		
8	π	83.06	74.55	−.428	−.182	.179	.294		
9	π	80.57	70.26	−.281	−.158	−.078	.124	.189	.500
	2.535	87.74	80.78	−.352	−.174	−.134	−.059	.359	.500
10	π	83.31	74.40	−.476	−.249	−.002	.119	.269	.342
11	π	82.11	78.40	−.364	−.296	−.153	.084	.167	.500
	2.589	89.03	84.44	−.413	−.282	−.155	−.046	.217	.500
12	π	86.16	77.96	−.381	−.335	−.050	.173	.275	.418

- For $N_{modes} \leq 5$, the step points listed in the table are all symmetrical. For $N_{modes} \geq 6$ the step points are not symmetrical.
- The number of step points as a function of N_{modes} in Table 3 is a bit mysterious. There is no clear pattern.

The first of these points is easily addressed by the theory, which shows that setting $\alpha \neq 0$ decreases the efficiency if the DC term is not included in the set of modes that we are trying to put energy into (see Theorem 6). The second of these points gives an example of symmetry breaking (see Section 5.5). On a certain level the third point is easily explained. That is, we can at least explain how many step points are needed to satisfy the constraints. However, on a deeper level it is not an easy matter to understand the numbers of step points listed in Table 3. For the case of $N_{modes} = 2$, the results in Section 4 show that using more step points does not improve the efficiency. For the other cases, numerical evidence suggests that the number of step points used in the table might actually be optimal, though no proof is given.

5.2. The Role of α

In this subsection we prove a simple lemma that allows us to understand why Table 3 only has entries for $\alpha = 0$ when the number of modes are even. Throughout the rest of our discussion of Dammann gratings it will be convenient to use the notation

$$a_k(\mathbf{z}) = b_k(\mathbf{z}, 0) \tag{5.8}$$

where $b_k(\mathbf{z}, \alpha)$ is defined in Equation (5.5). The following lemma helps us understand several theoretical points about Dammann gratings.

Lemma 5. *With $a_k(\mathbf{z})$ and $b_k(\mathbf{z}, \alpha)$ defined as in Equations (5.5) and (5.8), we have the identities*

$$|b_k(\mathbf{z}, \alpha)|^2 = \begin{cases} \cos^2(\alpha/2)\,|a_k(\mathbf{z})|^2 & k \neq 0 \\ \cos^2(\alpha/2)\,|a_k(\mathbf{z})|^2 + 1 - \cos^2(\alpha/2) & k = 0. \end{cases} \tag{5.9}$$

Proof. If $k \neq 0$, then the kth Fourier coefficient $w_k(\mathbf{z})$ of $w(x, \mathbf{z})$ is the negative of that of $1 - w(x, \mathbf{z})$. This follows from that fact that the two functions add up to a constant, and the kth Fourier coefficient of the constant function vanishes. From Equations (5.4) and (5.5) it follows that for $k \neq 0$

$$\begin{aligned} b_k(\mathbf{z}, \alpha) &= \mathrm{i}e^{-\mathrm{i}\alpha/2} w_k(\mathbf{z}) + \mathrm{i}e^{\mathrm{i}\alpha/2} w_k(\mathbf{z}) = 2\mathrm{i}\cos(\alpha/2) w_k(\mathbf{z}) \\ &= \cos(\alpha/2) b_k(\mathbf{z}, 0). \end{aligned} \tag{5.10}$$

The lemma for $k \neq 0$ follows from taking the norm squared of both sides of this equation. An almost identical argument holds when $k = 0$, but in this case we need to use the fact that the zero Fourier coefficients of $w(x, \mathbf{z})$ and $(1 - w(x, \mathbf{z}))$ add up to unity, not to zero. □

Theorem 6. *In problems where we are not concerned with the energy put into the DC component, the maximum efficiency of any binary phase grating that has phases $\pi/2 - \alpha/2$, and $-\pi/2 + \alpha/2$, is obtained by setting $\alpha = 0$.*

Proof. If we have a set of step points in the phase such that the energy in a set of modes not including the DC mode are equal, then Lemma 5 shows that the energy in these modes will be equal no matter what value of α we choose. The lemma shows that in order to maximize the energy in these modes we should set $\alpha = 0$. □

This theorem explains why in Table 3 we only have entries for $\alpha = 0$ when N_{modes} is even. The above argument does not hold when N_{modes} is odd, and the DC term is one of the modes of interest.

Lemma 5 can shed some light on the entries for $N_{modes} = 2$ and $N_{modes} = 3$ in Table 3. When $N_{modes} = 2$, we have the two beam splitting problem described in Section 4. In this case the optimal solution has no energy in the mode b_0. This can be considered as a solution to a Constrained Optimization Problem involving the three modes a_1, a_{-1}, and a_0, where we require that the magnitudes of a_1 and a_{-1} are the same, and that the mode a_0 vanishes. This solution has energy $4/\pi^2$ in each of the modes a_1 and a_{-1}. If we use the same step points, but use a phase α, we can equalize the energy in the DC component with those in the other two components if we choose α so that

$$1 - \cos^2(\alpha/2) = \cos^2(\alpha/2)\frac{4}{\pi^2}. \tag{5.11}$$

A brief calculation shows that this yields the same value for α as given for the solution with $N_{modes} = 3$. If we compute the corresponding efficiency η, we get the same value as for $N_{modes} = 3$.

5.3. Some Properties of Dammann Transmission Functions

We now present a few simple lemmas that have been used by many researchers in the field. These lemmas are simple enough that many researchers barely mention them, though they play an important part in their work. We believe they are worth stating explicitly. We remind the reader that $b_k(\mathbf{z}, \alpha)$ refers to the kth Fourier coefficient of $h(x, \mathbf{z}, \alpha)$ as defined in Equation (5.4).

Lemma 6. *The Fourier coefficients $b_k(\mathbf{z}, \alpha)$ satisfy $|b_k(\mathbf{z}, \alpha)| = |b_{-k}(\mathbf{z}, \alpha)|$.*

Proof. If we can prove this lemma for the case of $\alpha = 0$, Lemma 5 shows that it is true for all values of α. The lemma follows from the fact that $h(x, \mathbf{z}, 0)$ is a purely imaginary function, and that if c_k are the Fourier coefficients of a purely imaginary function, then $c_k = -\overline{c_{-k}}$, and hence $|c_k| = |c_{-k}|$. □

Lemma 7. *We have $|b_k(\mathbf{z}, -\alpha)| = |b_k(\mathbf{z}, \alpha)|$.*

Proof. This follows immediately from Lemma 5. □

The next lemma states the the magnitude of $b_k(\mathbf{z}, \alpha)$ does not change if we shift all of the step points by the same amount.

Lemma 8. *Let $\mathbf{e}^T = (1, 1, 1, \ldots, 1, 1)$, then $|b_k(\mathbf{z} + s\mathbf{e}, \alpha)| = |b_k(\mathbf{z}, \alpha)|$ for all values of s.*

Proof. If we shift any periodic function by an amount s, that merely multiplies the kth Fourier component by e^{iks} and hence does not change its magnitude. □

The next few lemmas involve reflecting the step points about the origin. In order to clearly state these lemmas we need to define what we mean by the reflection operator R.

Definition 1. *The reflection operator R sends a sequence of step points $\mathbf{z}^T$ into a sequence of step points $\hat{\mathbf{z}}^T$ such that $R\mathbf{z} = \hat{\mathbf{z}}$ where*

$$\hat{\mathbf{z}}^T = (-z_{N-1}, -z_{N-2}, \ldots, -z_1, -z_0). \tag{5.12}$$

Lemma 9. *Assuming the number of step points N is even, the function $w(x, \mathbf{z})$ defined in Equation (5.2) satisfies $w(x, R\mathbf{z}) = w(-x, \mathbf{z})$.*

Proof. Let $R\mathbf{z} = \hat{\mathbf{z}} = (\hat{z}_0, \hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{N-1})$. We have $\hat{z}_k = -z_{N-1-k}$. The function $w(x, \hat{\mathbf{z}})$ is 1 if $\hat{z}_k \le x \le \hat{z}_{k+1}$ for k even, and zero if $\hat{z}_k \le x \le \hat{z}_{k+1}$ for k odd. Hence the function $w(-x, \hat{\mathbf{z}})$ will be 1 provided $\hat{z}_k \le -x \le \hat{z}_{k+1}$ for k even. this can be written as $-\hat{z}_{k+1} \le x \le -\hat{z}_k$ for k even, Which can be written as $z_{N-1-k-1} \le x \le z_{N-1-k}$, for k even. However, if N and k are even then $N - 1 - k - 1$ will also be even, hence this is the same as requiring that $z_j \le x \le z_{j+1}$ for j even. Although our proof has been concerned with intervals where $w(x, \hat{\mathbf{z}})$ was one, similar arguments hold for intervals where it is zero. □

The next lemma is an immediate consequence of this last lemma.

Lemma 10. *The function $h(x, \mathbf{z}, \alpha)$ satisfies $h(x, R\mathbf{z}, \alpha) = h(-x, \mathbf{z}, \alpha)$.*

Lemma 11. *We have $|b_k(R\mathbf{z}, \alpha)| = |b_k(\mathbf{z}, \alpha)|$.*

Proof. This follows from the fact that for any periodic function $f(x)$, the kth Fourier coefficient of $f(x)$ is the same as the $-k$th Fourier coefficient of $f(-x)$. From this fact, and Lemma 10, we know that $b_k(R\mathbf{z}, \alpha) = b_{-k}(\mathbf{z}, \alpha)$. Our lemma now follows from Lemma 6. □

The following lemma is useful when proving that a critical point to the symmetrized optimization problem is also a critical point to the full optimization problem.

Lemma 12. *Let $F(\mathbf{z})$ be a function that is symmetric with respect to the reflection operator R. That is, we have $F(R\mathbf{z}) = F(\mathbf{z})$. Let $\mathbf{z}_0$ be a symmetric vector that satisfies $R\mathbf{z}_0 = \mathbf{z}_0$. We have*

$$\frac{\partial F(\mathbf{z}_0)}{\partial z_k} + \frac{\partial F(\mathbf{z}_0)}{\partial z_{N-1-k}} = 0. \tag{5.13}$$

Proof. Since $F(\mathbf{z})$ is symmetric with respect to R, for any vectors $\mathbf{z}_0$ and $\mathbf{a}$ and scalar s we have $F(\mathbf{z}_0+s\mathbf{a}) = F(R\mathbf{z}_0+sR\mathbf{a})$. If $\mathbf{z}_0$ is a symmetric vector this implies that we must have $F(\mathbf{z}_0 + s\mathbf{a}) = F(\mathbf{z}_0 + sR\mathbf{a})$ for all values of s. If we take the derivative of this last expression with respect to s, and evaluate it at $s = 0$, we get that $\nabla F(\mathbf{z}_0)\cdot\mathbf{a} = \nabla F(\mathbf{z}_0)\cdot R\mathbf{a}$ for all $\mathbf{a}$, and hence $\nabla F(\mathbf{z}_0)\cdot(\mathbf{a} - R\mathbf{a}) = 0$. This implies the result in the lemma. For example, if $\mathbf{a}^T = (1, 0, \ldots, 0)$, then $R\mathbf{a}^T = (0, 0, \ldots, 0, -1)$, and $\nabla F(\mathbf{z}_0)\cdot(\mathbf{a} - R\mathbf{a}) = 0$, implies that $\frac{\partial F(\mathbf{z}_0)}{\partial z_0} + \frac{\partial F(\mathbf{z}_0)}{\partial z_{N-1}} = 0$. □

5.4. Formulation Using Lagrange Multipliers

In this section we discuss how to use the method of Lagrange multipliers to write down a system of equations that a locally optimal solution must satisfy. Writing down this system of equations helps us better understand how many step points N are necessary in order to solve a beam splitting problem with the number of modes given by N_{modes}. Using Newton's method, this system of equations can be used to quickly locate locally optimal solutions once we have a good initial guess.

When formulating our Constrained Optimization Problem we need to take into account the shift invariance property stated in Lemma 8. Due to this shift invariance, if we have an optimal (or locally optimal) solution $(\mathbf{z}, \alpha)$, then there will be other optimal (or locally optimal) solutions arbitrarily close to this one (just shifted versions). The implicit function theorem (Apostol, 1974) shows that if such a multiplicity of solutions exists, then our Jacobian must be singular. If our Jacobian is singular, such methods as Newton's method cannot be used to iteratively solve our equations. For this reason it is necessary to begin by removing the arbitrariness caused by the shift invariance from our equations.

Due to this shift invariance, if $(z_0, z_1, \ldots, z_{N-1})$ is an optimal set of step points, then so is $(z_0 + a, z_1 + a, \ldots, z_{N-1} + a)$ for any value of a. By appropriately choosing a we could clearly make $z_0 = -\pi$. This would successfully get rid of the arbitrariness caused by the shift invariance, giving us one fewer parameter to adjust. However, it can be useful to eliminate the shift invariance in a way that allows us to have symmetrical solutions to the equations. For example, we could also choose a so that the sum of $z_k + a$ is zero. Hence, without loss of generality, we could assume that our optimal step points add up to zero. Such a constraint would allow symmetrically placed step points to be candidates for optimal solutions.

More generally we could account for the shift invariance by requiring a condition of the form

$$\mathbf{z}^T\mathbf{b} = c \tag{5.14}$$

where $\mathbf{b}$ is a vector such that $\mathbf{e}^T\mathbf{b} \neq 0$ where $\mathbf{e}^T = (1, 1, 1, 1 \ldots, 1)$. The constraint $z_0 = -\pi$ and the constraint that the sum of the break points add up to zero are both of this form. We could use Equation (5.14) to eliminate one of the variables in $\mathbf{z}$. However, we will merely consider Equation (5.14) as an additional constraint that needs to be satisfied.

At this stage it is worth referring back to Table 3. When N_{modes} is odd, we have $N_{modes} = 2M+1$. Due to Lemma 6, the $2M$ constraints in Equation (5.7) are equivalent to the M constraints

$$|b_0(\mathbf{z}, \alpha)| = |b_k(\mathbf{z}, \alpha)| \quad k = 1, 2, 3, \ldots, M. \tag{5.15}$$

The Equations (5.14) and (5.15) give us $M+1$ equations to satisfy. In order to have enough variables to adjust to satisfy these equations, the number of step points N must satisfy $N \geq M+1$. If we have exactly $N = M+1$, then we have no further parameters to adjust. If $N > M+1$, then we need to adjust some of our parameters to assure that we are at a local optimum. All of the entries in Table 3 satisfy this inequality. The entry with $N_{modes} = 11$ has $N = M+1$, and hence this is an example where we have no free parameters to adjust

We summarize these results in the following lemma.

Lemma 13. *In the Dammann beam-splitting problem for an odd number of modes, the number of break points N must satisfy the inequality $N \geq M+1$, where $2M+1$ is the number of modes that we are putting the energy into.*

We will begin by formulating our optimization problem assuming that α is given. We will comment on what happens when we adjust α later in this subsection.

Our Constrained Optimization Problem is to maximize the quantity $E_M(\mathbf{z}, \alpha)$, as defined in Equation (5.6), subject to the constraints in Equations (5.14) and (5.15). Using the method of Lagrange multipliers, this motivates the introduction of the function $P(\mathbf{z}, \alpha, \underline{\lambda})$

$$\begin{aligned} P(\mathbf{z}, \alpha, \underline{\lambda}) = &\sum_{k=-M}^{M} |b_k(\mathbf{z}, \alpha)|^2 - \sum_{k=1}^{M} \lambda_k \left(|b_k(\mathbf{z}, \alpha)|^2 - |b_0(\mathbf{z}, \alpha)|^2 \right) \\ &- \lambda_{M+1} \left(\mathbf{z}^T\mathbf{b} - c \right). \end{aligned} \tag{5.16}$$

The first sum on the right hand side of this equation is the quantity $E_M(\mathbf{z})$ in Equation (5.6) that we are trying to maximize. The second sum is associated with the constraints from Equation (5.15), and the last term is from the constraint in Equation (5.14). The theory of Lagrange multipliers (Apostol, 1974; Courant, 1947) implies the following lemma.

Lemma 14. *Any locally optimal solution to the Dammann Beam Splitting problem must satisfy the following $N + M + 1$ equations.*

$$\frac{\partial P}{\partial z_k} = 0 \quad k = 1, N \tag{5.17}$$

$$\frac{\partial P}{\partial \lambda_k} = 0 \quad k = 1, M + 1. \tag{5.18}$$

These equations involve $N + M + 1$ unknowns consisting of the N unknowns in $\mathbf{z}$ and the $M + 1$ unknowns in $\underline{\lambda}$.

The solutions to the Equations (5.17) and (5.18) do not necessarily give locally optimal solutions. In particular, they could give local minima, or saddle points solutions. We call any solution to these equations critical points of our optimization problem.

We now briefly discuss the role of the parameter α in our optimization problem. It would be tempting to include α as an independent variable in our equations that we want to solve. If we did this we would include an additional equation $\frac{\partial P}{\partial \alpha} = 0$ in our system of equations. However, the inclusion of such an equation is based on the assumption that the optimal value of α occurs at an interior point of our domain of feasibility. Here we are using the term domain of feasibility to denote the range of values of α where there are any solutions at all (not necessarily optimal) that satisfy our constraints. In most cases this domain of feasibility is not known a priori, but must be found by solving systems of non-linear equations.

In practice we have found that the optimal solutions seem to always lie on the boundary of the domain of feasibility. Though it would be possible to write down an alternative condition requiring that we are on this edge, we will not go into this detail here.

We have found that all of the solutions with $\alpha \neq 0$ in Table 3 can be found by starting with the solution given by $\alpha = 0$, and then using continuation on α. That is, we change α by a small amount using the solution for the previous value of α as the new guess for the next value of α. As we change α, we eventually reach the limits of the domain of feasibility, and our equations start becoming singular. The optimal solutions in this table are always found at the edge of this domain of feasibility.

A technique such as simulated annealing can relatively quickly give answers that are close to optimal, but can take a long time to improve upon these crudely optimal solutions. A good strategy is to use a technique such as simulated annealing to find likely candidates for globally optimal solutions, feed these solutions as an initial guess to Newton's method, and then let Newton's method rapidly converge to a local (or possibly global optimum). The method of Lagrange multipliers

allows us to write down a system of equations that can be solved by Newton's method.

5.5. The Effect of Symmetry

As long as the constraint in Equation (5.14) is chosen so that it is symmetric with respect to the reflection operator R, the function $P(\mathbf{z}, \alpha, \underline{\lambda})$ defined in Equation (5.16) will be invariant with respect to the reflection operator R (due to Lemma 11). It is possible to define a symmetrized optimization problem where we require the break points $\mathbf{z}$ be symmetric with respect to this reflection operator. We now show that any critical point of this symmetrized problem will be a critical point of the full optimization problem. That is, we will prove the following lemma.

Lemma 15. *If* $\mathbf{z}$ *is a critical point of the symmetrical optimization problem (where the break points are required to be symmetric), then* $\mathbf{z}$ *will also be a critical point of the full optimization problem.*

To prove the lemma, note that in our symmetrized optimization problem we would replace the Equation (5.17) by the equations

$$\frac{\partial P(\mathbf{z}_0)}{\partial z_k} - \frac{\partial P(\mathbf{z}_0)}{\partial z_{M-1-k}} = 0. \tag{5.19}$$

These equations take into account the fact that for the symmetrized equations the point z_k is the negative of z_{N-1-k}. Hence, we are trying to maximize $P(z_0, z_1, z_2, \ldots - z_2, -z_1, -z_0, \alpha, \underline{\lambda})$. If we take the derivative of this with respect to z_0, z_1, etc, we get equations of the form (5.19). Once we have solved these symmetrized equations, we end up with a symmetrical solution $\mathbf{z}_0$ that satisfies the Equation (5.19) as well as the Equation (5.18). Since $P(\mathbf{z}, \alpha, \underline{\lambda})$ is symmetrical with respect to reflections, Lemma 12 implies that we also have

$$\frac{\partial P(\mathbf{z}_0)}{\partial z_k} + \frac{\partial P(\mathbf{z}_0)}{\partial z_{M-1-k}} = 0. \tag{5.20}$$

Together, Equations (5.19) and (5.20) imply that we must have $\frac{\partial P}{\partial z_k} = 0$, and hence our symmetrical solution is in fact a solution to the full system of equations.

Although any critical point of the symmetrized problem is a critical point of the full problem, the converse does not hold. Just because the equations are invariant under the reflection operator R, it does not mean that the global optimal solution is symmetric under this reflection operator. However, this may be the case in certain situations. Looking

at Table 3 we see that this is the case when $N_{modes} \leq 5$. The solutions for $N_{modes} > 5$ are examples of solutions where this symmetry has been broken.

The results in this sub-section make it easier to understand how the results in (Dammann & Gortler, 1971; Dammann & Klotz, 1977; Krackhardt & Streibl, 1989), where they assumed that the solutions were symmetrical and that $N = 2M$, fit into the general picture. In that case, these authors adjust M points and their symmetrical counterparts in order to satisfy the M Equation (5.7). In this case there are no additional parameters to adjust. However, this could also be considered as a case where we have $N = 2M > M + 1$, and hence there are additional parameters to adjust (assuming $M > 1$). This now gives us a Constrained Optimization Problem that can be solved using Equations (5.17) and (5.18). Any particular critical point of this Constrained Optimization Problem may or may not be symmetrical. If it happens to be symmetrical, then the solution to the Equations (5.17) and (5.18) decouple from each other. That is, we could first solve for the break points $\mathbf{z}$ using Equation (5.18), and then for the Lagrange multipliers using Equation (5.17). However, in this case, it would be unnecessary to solve for the Lagrange multipliers.

5.6. Problems with an Even Number of Modes

Recall the discussion in Section 2.3 concerning problems with an even number of modes. As in (Killat, Rabe, & Rave, 1982), we could try to maximize the energy in the modes $k = \pm 1, \pm 2, \pm 3, \ldots, \pm M$, subject to the constraint that the energy in each of these modes is the same. This does not result in equally spaced beams.

We could also try to maximize the energy in the modes with $k = \pm 1, \pm 3, \pm 5, \ldots \pm 2M - 1$, subject to the constraint that the energy in each of these modes is the same. As in (Morrison, 1992) we could further specify that all of the even Fourier modes vanish. The solutions in Table 3 are in fact solutions to this last optimization problem.

6. ONE DIMENSIONAL CONTINUOUS GRATINGS

As pointed out by Herzig, Prongué, and Dändliker (1990), though Dammann gratings yield arrays with good uniformity, the efficiencies are not high. This motivated these and other authors to look into techniques for improving efficiencies by using continuous profiles for the phase transmission function. Since it is not possible to control the modulation depth of the etches perfectly, the mathematically optimal profile cannot always be satisfactorily approximated in practice. Nevertheless, in this section, we concern ourselves with finding the continuous phase that

gives the mathematically optimal efficiency as stated in the Constrained Optimization Problem in Section 2.2.

The main goal of this section is to discuss Theorem 7 in Section 6.2 which gives the general form of the solution to the Constrained Optimization Problem. This form was first given by Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, and Gentili (1998) for the case of the triplicator, and later for the general case by Romero and Dickey (2007b). Before proceeding we will briefly review other approaches that have been used to find sub-optimal solutions to the Constrained Optimization Problem.

Boivin (1998) found solutions to the triplicator problem by assuming that the phase $\phi(x)$ of the grating was proportional to some fixed profile $\psi(x)$. That is, he assumed that $\phi(x) = \lambda\psi(x)$, where $\psi(x)$ was chosen to have a rectangular, triangular, or sinusoidal form. By adjusting λ he was able to make the amplitude of the first three Fourier coefficients $a_k, k = -1, 1$ of $e^{i\phi(x)}$ all have the same amplitude. By doing this he got efficiencies of 75, 72, and 81 per cent for the rectangular, triangular, and sinusoidal gratings, respectively.

Daly, Hodson, and Hutley (1991) approximated the phase $\phi(x)$ of the grating using a finite number of Tchebychev polynomials. Rather than trying to find solutions with optimal efficiency, they put the efficiency at a fixed level (90 per cent). They then solved an over-determined system of non-linear equations to determine the coefficients in the Tchebychev polynomial expansion such that the desired efficiency was achieved, and all the relevant modes had the same intensity. They applied this technique to the problem of splitting a beam into 15 orders. When they actually built this grating, the efficiency was only slightly lower than their predictions, but the uniformity of the beams was (by their own admission) disappointing. From looking at the profiles for the phase in (Prongué, Herzig, Dändliker, & Gale, 1992) (discussed later in this section) and (Daly, Hodson, & Hutley, 1991), it does not appear that the grating in (Daly, Hodson, & Hutley, 1991) would be harder to manufacture. It is possible that the poor performance of the grating in (Daly, Hodson, & Hutley, 1991) compared to (Prongué, Herzig, Dändliker, & Gale, 1992) was due not to mathematical problems with the phase, but due to the actual implementation of the manufacturing process.

Another approach to improving the efficiency was given by Hatakoshi and Nakamura (1993). They looked for phases that were divided up into five piecewise linear segments in each period of the grating. They considered three and five beam splitting problems. The answers they got for these problems are much better than for the case of Dammann gratings. Their answers are suboptimal, but only about 2 per cent below the optimal

values. They also manufactured these gratings, and were satisfied with their results, though the analysis of their results was not quantitative.

Herzig, Prongué, and Dändliker (1990) determined continuous gratings by solving what we called the Minimum Variance Optimization problem in Section 2.2. The solution to this problem allowed them to find efficient gratings that did not produce perfectly uniform beams. Prongué, Herzig, Dändliker, and Gale (1992) showed how to take the solutions from the Minimum Variance Optimization Problem and, by making small adjustments to this solution, end up with a grating that does achieve uniformity. They called this the Uniformity Optimization Problem. The answers obtained in this way are not mathematically optimal, but they are very close to being optimal. Prongué, Herzig, Dändliker, and Gale (1992) built a grating for producing 9 uniform beams, and got reasonably good agreement between the theory and the mathematical predictions. We will discuss this work further in the next subsection.

6.1. The Uniformity Optimization Problem

Though the solutions given by Prongué, Herzig, Dändliker, and Gale (1992) are suboptimal, in practice they are extremely close to being optimal. Furthermore, the procedure for obtaining these solutions is similar to the procedure for obtaining the optimal solutions. For this reason we believe it is worth elaborating on their procedure.

To simplify the presentation, we limit our discussion to the case where we are trying to put as much energy as possible into the $2M + 1$ modes $a_k, k = -M, M$. We will assume that the problem has been normalized so that the grating has period $d = 2\pi$. To find a nearly optimal grating, they begin by solving the Minimum Variance Optimization Problem described in Section 2.2 (though they call it the Efficiency Optimization Problem). That is, they try to find the phases $\underline{\alpha} = (\alpha_{-M}, \alpha_{-M+1}, \ldots, \alpha_{M-1}, \alpha_M)$ in the function

$$s(x, \underline{\alpha}) = \sum_{k=-M}^{M} e^{i\alpha_k} e^{ikx} \tag{6.1}$$

such that they minimize the variance $V(\underline{\alpha})$, as defined in Equation (2.14).

Prongué, Herzig, Dändliker, and Gale (1992) state that the phases obtained by solving the Least Squares Optimization Problem are identical to those obtained by solving the Minimum Variance Optimization Problem. This is nearly but not precisely true. It holds for the cases of 3 and 5 beam splitting where the optimal phases for both problems are given by $(\alpha_0, \alpha_{\pm 1}) = (0, \pi/2)$ and $(\alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}) = (0, \pi/2, \pi)$. However, we find that for the case of 7 beam splitting

TABLE 4 The values of $\eta_{efficiency}$ and $\eta_{uniformity}$ given in (Prongué, Herzig, Dändliker, & Gale, 1992) for splitting a beam into N_{Modes} number of beams

N_{Modes}	$\eta_{efficiency}$	$\eta_{uniformity}$
3	.949	.926
5	.980	.921
7	.980	.968
9	.994	.993
11	.988	.977
13	.993	.963

the optimal phases for the Least Squares Optimization Problem are given by $(\alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}, \alpha_{\pm 3}) = (0., 5.28, 1.95, .73)$, while the optimal phases for the Minimum Variance Optimization Problem are given by $(\alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}, \alpha_{\pm 3}) = (0., 5.19, 1.93, .68)$. These phases are close, but not identical.

Once they have found the phases $\underline{\alpha}$ that solve the Minimum Variance Optimization Problem, they keep these phases fixed, and define a function

$$s(x, \underline{\alpha}, \underline{\mu}) = \sum_{k=-M}^{M} \mu_k e^{i\alpha_k} e^{ikx}. \tag{6.2}$$

They now use an iterative procedure such as Newton's method to adjust the weightings μ_k so that the Fourier coefficients a_k of

$$h(x, \underline{\alpha}, \underline{\mu}) = \frac{s(x, \underline{\alpha}, \underline{\mu})}{|s(x, \underline{\alpha}, \underline{\mu})|} \tag{6.3}$$

satisfy $|a_k| = |a_0|$, for $k = -M, M$. They call this second procedure the Uniformity Optimization step. The efficiencies obtained by this procedure are given in Table 4. With the exception of the case $N_{modes} = 13$, to the number of decimal places reported, the efficiencies reported for the Efficiency Optimization Problem are identical to the optimal values given in Table 5.

In the next few subsections we show that the optimal phase function for the Constrained Optimization Problem, has the form given in Equation (6.3), but in that case we also need to adjust the phases as well as the weightings μ_k. That is, the phases must be solved along with the weightings, they cannot be assumed to be the same as those for the Minimum Variance (or Least Squares) Optimization Problems.

TABLE 5 Optimum efficiencies for splitting a beam into an odd number of beams. We also list the values of α_k and μ_k in Equation (6.6) needed to obtain these. Our solutions have $\alpha_k = \alpha_{-k}$, and $\mu_k = \mu_{-k}$, as well as $\alpha_0 = 0$, and $\mu_0 = 1$. If $N_{modes} = 2M + 1$, the vectors $\underline{\alpha}$ and $\underline{\mu}$ contain the values $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M)$, and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_M)$. The optimal phase functions for $N_{modes} = 3, 11$ are given in Figure 2

N_{modes}	η_{LS}	η_{CO}	$\underline{\alpha}$ and $\underline{\mu}$
3	93.81	92.56	$\underline{\alpha} = \pi/2$ $\underline{\mu} = 1.329$
5	96.28	92.12	$\underline{\alpha} = (-\pi/2, \pi)$ $\underline{\mu} = (.459, .899)$
7	97.53	96.84	$\underline{\alpha} = (-.984, 1.891, .748)$ $\underline{\mu} = (1.289, 1.463, 1.249)$
9	99.34	99.28	$\underline{\alpha} = (.720, 5.567, 3.033, 1.405)$ $\underline{\mu} = (.971, .964, .943, 1.029)$
11	98.38	97.71	$\underline{\alpha} = (.311, 4.492, 2.847, 5.546, 4.406)$ $\underline{\mu} = (1.207, 1.297, 1.483, 1.427, 1.275)$
13	98.57	97.53	$\underline{\alpha} = (2.308, 4.345, 1.517, 1.692, 0.066, 6.243)$ $\underline{\mu} = (0.912, 0.968, 0.806, 0.923, 1.099, 1.027)$
15	98.21	97.29	$\underline{\alpha} = (2.625, 4.534, 0.970, 2.983, 3.328, 4.070)$ $\underline{\mu} = (4.945, 1.116, 1.463, 0.930, 1.114, 1.466, 1.359, 1.211)$

6.2. Solutions Using the Calculus of Variations

Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, and Gentili (1998) were the first to use the calculus of variations to solve the Constrained Optimization Problem. They solved the 'triplicator ' problem of using a phase grating to split a beam into three equal parts. Using techniques from the calculus of variations they were able to show that the optimal phase in the triplicator problem could be written as

$$\phi(x) = \tan^{-1}\left(a \sin(\phi(x))\right) \tag{6.4}$$

where a is a parameter that needs to be determined numerically. They found that $a = 2.65718$ gave the optimal efficiency of $\eta_{co} = .92556$. Using similar techniques Borghi, Cincotti, and Santarsiero (2000) solved the problem of splitting a beam into two beams that are not necessarily of the same intensity.

Romero and Dickey (2007b) reproduced the results in both Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, and Gentili (1998) and Borghi, Cincotti, and Santarsiero (2000) as an example of the more general theory. Using the calculus of variations and the theory of Lagrange multipliers they proved the following theorem (using slightly different notation than in this review).

Theorem 7. *The optimal solution to the Constrained Optimization Problem must have the form*

$$h(x, \underline{\alpha}, \underline{\mu}) = \mathrm{e}^{\mathrm{i}\phi(x)} = \frac{s(x, \underline{\alpha}, \underline{\mu})}{|s(x, \underline{\alpha}, \underline{\mu})|} \tag{6.5}$$

where

$$s(x, \underline{\alpha}) = \sum_{k \in K} \mu_k \mathrm{e}^{\mathrm{i}\alpha_k} \mathrm{e}^{\mathrm{i}kx}. \tag{6.6}$$

Here α_k and μ_k are parameters that need to be adjusted in order to obtain the optimal solution.

It should be noted that the calculus of variations only gives a necessary condition for the form of the optimal solution. However, this theorem shows that we can reduce the infinite dimensional problem of finding the function $\phi(x)$, to the finite dimensional problem of determining the parameters α_k, and μ_k. These parameters must be chosen so that the Fourier coefficients $a_k, k \in K$ of $\mathrm{e}^{\mathrm{i}\phi(x)}$ all have the same magnitude, and so that we are at a local maximum of the energy in the desired modes

Note that the form for the transmission function given in Equation (6.5) is the same form as in Equation (6.3) that is obtained by Prongué et al. in their Uniformity Optimization Problem. However, for the true optimum the phases $\underline{\alpha}$ and the weighting $\underline{\mu}$ must be obtained simultaneously.

When the general theory is applied to the triplicator, it is found that $\alpha_1 = \alpha_{-1} = \pi/2$, $\alpha_0 = 0$, $\mu_0 = 1$, and $\mu_1 = \mu_{-1} = \mu$. This shows that the optimal phase satisfies

$$\phi(x) = \tan^{-1}\left(2\mu \cos(x)\right). \tag{6.7}$$

Numerical calculations show that $\mu = 1.32859$. This is equivalent to the solution presented in (Gori, Santarsiero, Vicalvi, Borghi, Cincotti, Di Fabrizio, & Gentili, 1998) if we shift the solution by $\pi/2$.

6.3. Numerical Calculations

In order to determine the function $h(x, \underline{\alpha}, \underline{\mu})$ in Equation (6.5), one needs to solve for both the parameters $\underline{\mu}$ and $\underline{\alpha}$. Similar to the procedure carried

out by Prongué, Herzig, Dändliker, and Gale (1992), we can compute the parameters $\underline{\alpha}$ by solving one of the simpler problems described in Section 2.2 such as the Least Squares Optimization Problem or the Minimum Variance Optimization Problem. In practice we have found that near optimal values of the phases in these simpler optimization problems can be found by systematically searching over the parameter space, randomly generating guesses, or using a technique such as simulated annealing. Once we have found these phases, we can use these as initial guesses for the Constrained Optimization Problem. This problem can then be solved using Newton's method.

Romero and Dickey (2007b) showed how to write down a system of equations for determining the parameters $\underline{\alpha}$ and $\underline{\mu}$ in the function $s(x, \underline{\alpha}, \underline{\mu})$. First of all they note that adding an arbitrary constant to each of the phases does not change the amplitude of any of the Fourier coefficients of $h(x, \underline{\alpha}, \underline{\mu})$. Furthermore, shifting the function $s(x, \underline{\alpha}, \underline{\mu})$ by an amount x_0 does not change the amplitude of any of the Fourier coefficients of $h(x, \underline{\alpha}, \underline{\mu})$, but it changes the phase of its kth Fourier coefficient by kx_0. It follows that, by combining a shift in x and adding a constant to the phases, we can always set any two of the phases α_k to zero. Furthermore, multiplying all of the coefficients $\underline{\mu}$ in Equation (6.5) by a nonzero value does not change $h(x, \underline{\alpha}, \underline{\mu})$, hence, without loss of generality, we can set one of the coefficients μ_k equal to unity.

If there are N numbers in the set K of relevant Fourier coefficients, there are $2N$ parameters $\underline{\mu}$ and $\underline{\alpha}$, but we can arbitrarily set two of the α_k to zero, and one of the μ_k to one, so we have $2N - 3$ parameters to adjust. The requirement that all of the relevant Fourier coefficients have the same amplitude gives us $N-1$ equations. We now describe a way for prescribing $N - 2$ additional equations.

For arbitrary values of the phases α_k, the phases of the Fourier coefficients a_k of $h(x, \underline{\alpha}, \underline{\mu})$ will not be the same as α_k. However, in §4 of Romero and Dickey (2007b) they prove the following theorem.

Theorem 8. *Let $\underline{\alpha}$ and $\underline{\mu}$ be values of the parameters in Equation (6.6) that give an optimal solution to the Constrained Optimization Problem. Then the phase of any Fourier coefficient $a_k, k \in K$ of $h(x, \underline{\alpha}, \underline{\mu})$ defined in Equation (6.5) must be the same as the phase α_k in the function $h(x, \underline{\alpha}, \underline{\mu})$.*

If we apply this theorem for each of the $N - 2$ phases we are adjusting, it gives us $N - 2$ equations. When combined with the constraints that the relevant Fourier coefficients all have the same amplitude, this gives us $2N - 3$ equations in $2N - 3$ unknowns.

In Tables 5 and 6 we give the results for the Constrained Optimization Problem for the number of modes up to 15. Figures 2 and 3 show the optimal phase functions associated with some of these problems.

TABLE 6 Optimum efficiencies for splitting a beam into an even number of beams. Here the modes are given by $k = \pm 2m + 1, m = 1, M$ We also list the values of α_k and μ_k in Equation (6.6) needed to obtain these. Our solutions have $\alpha_k = \alpha_{-k}$, and $\mu_k = \mu_{-k}$, as well as $\alpha_1 = 0$, and $\mu_1 = 1$. If $N_{modes} = 2M$, the vectors $\underline{\alpha}$ and $\underline{\mu}$ contain the values $\underline{\alpha} = (\alpha_3, \ldots, \alpha_{2M-1})$, and $\underline{\mu} = (\mu_3, \ldots, \mu_{2M-1})$. The optimal phase function for $N_{modes} = 4$ is given in Figure 3

N_{modes}	η_{LS}	η_{CO}	$\underline{\alpha}$ and $\underline{\mu}$
4	91.94	91.19	$\underline{\alpha} = 4.438$ $\underline{\mu} = .523$
6	91.41	88.17	$\underline{\alpha} = (0.863, 3.069)$ $\underline{\mu} = (0.274, 0.487)$
8	96.12	95.94	$\underline{\alpha} = (0.724, 3.668, 5.367)$ $\underline{\mu} = (0.560, 0.601, 0.544)$
10	95.79	92.69	$\underline{\alpha} = (0.152, 4.683, 2.681, 0.651)$ $\underline{\mu} = (0.598, 0.412, 0.211, 0.546)$
12	95.93	95.36	$\underline{\alpha} = (4.562, 3.704, 5.465, 3.448, 1.725)$ $\underline{\mu} = (0.523, 0.424, 0.509, 0.586, 0.538)$
14	96.80	96.34	$\underline{\alpha} = (0.235, 2.906, 1.661, 1.521, 4.847, 2.527)$ $\underline{\mu} = (0.430, 0.471, 0.419, 0.505, 0.511, 0.545)$

In numerically computing the solution to the Constrained Optimization Problem it is convenient to get an initial guess by solving either the Minimum Variance or the Least Squares Optimization Problem. Either of these give agood initial guess for the phases α_k. One can then input these phases as an initial guess for Newton's method to solve the Constrained Optimization Problem. The following theorem was proven in (Romero & Dickey, 2007b). This both characterizes the solutions to the Least Squares Optimization Problem, and shows that in certain cases these solutions are identical to those of the Constrained Optimization Problem.

Theorem 9. *The solution to the Least Squares Optimization problem has the form given in Equations* (6.5) and (6.6) *except that* $\mu_k = 1$. *Let* $h(x, \underline{\alpha})$ *be the optimal transmission function for the Least Squares Optimization Problem. If the Fourier coefficients* $a_k, k \in K$ *of* $h(x, \underline{\alpha})$ *all have the same magnitude, then* $h(x, \underline{\alpha})$ *is also the solution to the Constrained Optimization Problem.*

7. TWO DIMENSIONAL FOURIER SERIES

When designing a phase element to do two dimensional beam splitting we could theoretically design the element without making any assumptions

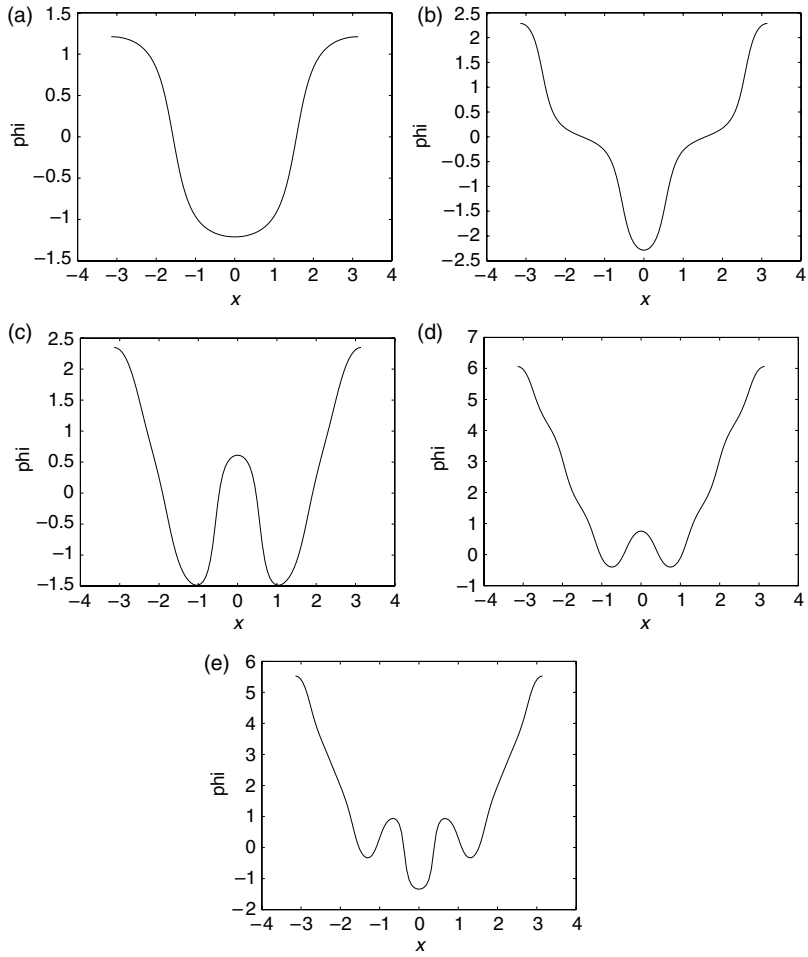

FIGURE 2 Plots of the optimal phase functions for one dimensional beam splitting. (a)$N_{Modes} = 3$ (b) $N_{Modes} = 5$ (c) $N_{Modes} = 7$ (d) $N_{Modes} = 9$ (e) $N_{Modes} = 11$.

about the spatial periodicity of the element. That is, we could solve the general problem in diffractive optics (Bryngdahl & Wyrowski, 1993) of designing an element (not assumed to be periodic) that splits an incoming beam into a set of prescribed outgoing beams. We do not consider such approaches here, but assume that we, a priori, specify a particular periodic structure for our grating. We then optimize the grating under the assumption that the phase $\phi(x, y)$ of the grating has the specified periodicity. We will be particularly concerned with the two cases where the phase function is periodic on a square lattice, or on a hexagonal lattice.

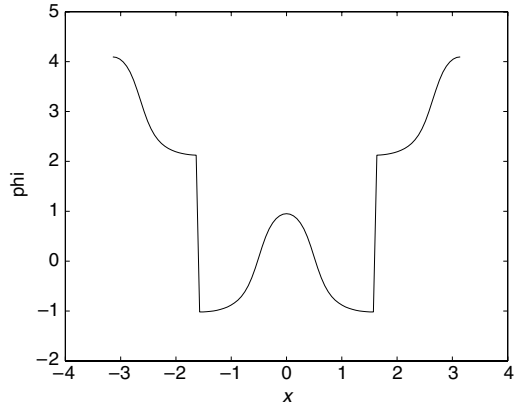

FIGURE 3 A plot of the optimal phase function for one dimensional four beam splitting.

The results presented in Section 6 carry over almost directly to the two dimensional case. However, in order to apply the results, one needs to be familiar with two dimensional Fourier series on non-orthogonal lattices. An expanded version of this section, can be found in (Romero & Dickey, 2007a).

7.1. Two Dimensional Fourier Series on General Lattices

Here we are concerned with expanding two dimensional functions $f(x, y)$ that are periodic on an arbitrary lattice. We will use the notation

$$\mathbf{x}^T = (x, y) \tag{7.1}$$

to represent a point in two dimensional space, and

$$\mathbf{m}^T = (m_1, m_2) \tag{7.2}$$

to give the integers m_1 and m_2 used to define a lattice point or a term in a Fourier series.

The periodicity of a two dimensional periodic function $f(\mathbf{x})$ is specified by giving two linearly independent vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ such that

$$f(\mathbf{x} + m_1\mathbf{p}_1 + m_2\mathbf{p}_2) = f(\mathbf{x}) \quad \forall \text{ integers } m_1, m_2. \tag{7.3}$$

It is assumed that we have chosen the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ so that they are minimal. That is, so that that any translation $\mathbf{p}$ that leaves the function

$f(\mathbf{x})$ invariant can be written as $\mathbf{p} = m_1\mathbf{p}_1 + m_2\mathbf{p}_2$, where m_1 and m_2 are integers.

For a rectangular lattice, the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ may be chosen so they are orthogonal to each other. The theory of Fourier series on such rectangular lattices should be well known to most readers of this review. Our main goal is to familiarize the reader with the results for hexagonal lattices, where the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are not orthogonal to each other.

Since the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent, we can always find two vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ that are bi-orthogonal to the vectors $\mathbf{p}_i$. That is, we have

$$\mathbf{q}_1 \cdot \mathbf{p}_2 = \mathbf{q}_2 \cdot \mathbf{p}_1 = 0 \tag{7.4}$$

$$\mathbf{q}_1 \cdot \mathbf{p}_1 = \mathbf{q}_2 \cdot \mathbf{p}_2 = 2\pi. \tag{7.5}$$

Due to these bi-orthogonality relations, it is straightforward to see that for any vector $\mathbf{m}$ of integers, the function

$$g_{\mathbf{m}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{q}_{\mathbf{m}}} \tag{7.6}$$

is periodic on our lattice. That is, we have $g_{\mathbf{m}}(\mathbf{x} + \mathbf{p}_{\mathbf{m}'}) = g_{\mathbf{m}}(\mathbf{x})$ for all translations $\mathbf{p}_{\mathbf{m}'}$ of the lattice. Here we are using the notation

$$\mathbf{p}_{\mathbf{m}} = m_1\mathbf{p}_1 + m_2\mathbf{p}_2 \tag{7.7}$$

$$\mathbf{q}_{\mathbf{m}} = m_1\mathbf{q}_1 + m_2\mathbf{q}_2. \tag{7.8}$$

Any function $f(\mathbf{x})$ that is periodic on the lattice generated by $\mathbf{p}_1$ and $\mathbf{p}_2$ can be represented as a Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathrm{e}^{\mathrm{i}\mathbf{q}_{\mathbf{m}}\cdot\mathbf{x}}. \tag{7.9}$$

The Fourier coefficients $a_{\mathbf{m}}$ satisfy

$$a_{\mathbf{m}} = \frac{1}{A}\int_{\Omega} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{q}_{\mathbf{m}}\cdot\mathbf{x}} d\mathbf{x}, \tag{7.10}$$

where Ω is a unit cell of the lattice, and A is the area of this unit cell. A unit cell is a region such that when we apply all of the translations of the lattice to this region, it fills two dimensional space without any overlapping. The unit cell of any lattice can always be taken to be a parallelogram generated by the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. However, there are infinitely many other unit cells.

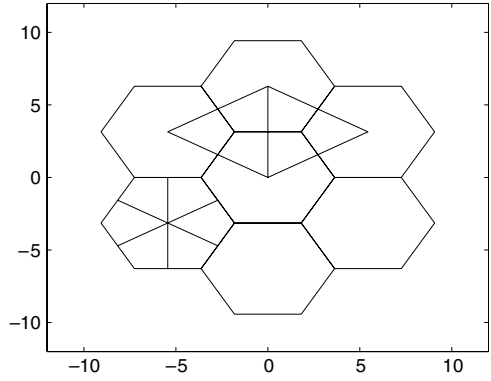

FIGURE 4 This figure illustrates that either a single rhombus or a single hexagon can serve as the unit cell for a hexagonal lattice. Note that each of the 'kites' in the hexagon is equivalent to one and only one 'kite' in the rhombus by a translation of the lattice.

For the case of a square lattice the simplest unit cell is a square. For a hexagonal lattice there are two particularly simple unit cells. The first is a hexagon, the second is a rhombus (see Figure 4). As with any unit cell, these two unit cells have the same area as each other. In Figure 4 we have divided the rhombus and one of the hexagons up into six "kites", A careful examination of this figure shows that each "kite" in the rhombus is equivalent (by a translation in the lattice) to one and only one "kite" in the hexagon. That is, each kite in the rhombus can be obtained by translating a "kite" in the hexagon by an integer multiple of the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. This shows that we can use either a hexagon or a rhombus for the unit cell in the tiling with hexagons. We find it simplest to use the rhombus as the unit cell when actually carrying out calculations, since in the coordinate system based on $\mathbf{p}_1$ and $\mathbf{p}_2$ our region of integration is now a square.

The next two lemmas give the vectors $\mathbf{p}_i$ and $\mathbf{q}_i$ for the case of square and hexagonal lattices.

Lemma 16. *For a function that is periodic on a square lattice, the vectors* $\mathbf{p}_i$ *and* $\mathbf{q}_i$ *can be written as*

$$\mathbf{p}_1^T = (2\pi, 0), \quad \mathbf{p}_2^T = (0, 2\pi) \tag{7.11}$$

$$\mathbf{q}_1^T = (1, 0), \quad \mathbf{q}_2^T = (0, 1). \tag{7.12}$$

It should be noted that for a rectangular lattice, the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthogonal to each other, and hence $\mathbf{q}_i, i = 1, 2$ points in the same direction as $\mathbf{p}_i, i = 1, 2$. However, if $\mathbf{p}_1$ is bigger than $\mathbf{p}_2$ in magnitude,

then $\mathbf{q}_2$ will be bigger than $\mathbf{q}_1$ in magnitude. For optical problems this implies that if the spacing is bigger in the x direction, then the spacing of the diffracted beams will be smaller in the x direction.

Lemma 17. *For a function that is periodic on a hexagonal lattice, the vectors* $\mathbf{p}_i$ *and* $\mathbf{q}_i$ *are given by*

$$\mathbf{p}_1^T = 2\pi(\sqrt{3}/2, -1/2), \quad \mathbf{p}_2^T = 2\pi(\sqrt{3}/2, 1/2) \tag{7.13}$$

$$\mathbf{q}_1^T = (1/\sqrt{3}, -1), \quad \mathbf{q}_2^T = (1/\sqrt{3}, 1). \tag{7.14}$$

7.2. Comments on Two Dimensional Symmetry

When discussing results for two dimensional beam splitting the matter of the symmetry of the gratings we are using arises. A full discussion of the symmetry of the grating would involve discussing the full set of isometries (distance preserving transformations) that leave the grating invariant. This would include a discussion of the translations, rotations, reflections, and glide reflections that leave the grating invariant. However, in describing the results here we will only concern ourselves with the translational and rotational symmetry of the grating. We will give a brief summary of what we mean by this.

The translational symmetry of the pattern is defined by the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. The totality of translations that leave the pattern invariant can also be represented by the lattice of points $\mathbf{p_m}$ for all integer pairs $\mathbf{m}$. We can think of the lattice as the set of all points that look just like some original point $\mathbf{x}_0$. It should be noted that the translational symmetry is completely defined by the lattice, but the lattice tells us nothing about the rotational or reflectional symmetry of the pattern.

A figure is said to have n fold rotational symmetry about a point $\mathbf{x}_0$ if, when we rotate the figure by $2\pi/n$ about $\mathbf{x}_0$, the figure looks the same. For example, a rectangle has two-fold symmetry about its centre, an equilateral triangle has three-fold symmetry about its centre, and a square has four-fold symmetry about its centre. We can similarly talk about functions having n-fold symmetry.

Definition 2. *A function* $f(\mathbf{x})$ *is said to have n-fold symmetry about the origin if* $f(\mathbf{R}_{2\pi/n}\mathbf{x}) = f(\mathbf{x})$ *where* $\mathbf{R}_{2\pi/n}$ *is a rotation matrix representing a rotation by* $2\pi/n$ *about the origin.*

For a one dimensional Fourier series, the Fourier coefficients a_k of $f(x)$ are related to the Fourier coefficients a'_k of $f(-x)$ by the relation $a'_k = a_{-k}$. This is a well known result, but it is good to see how to derive this result in a way that carries over to the case of two dimensional Fourier series. For any function $g(x)$ the integral of $g(x)$ over the interval $(-\pi, \pi)$ is the same

as the integral of $g(-x)$ over the same interval. If we apply this principle to the function $g(x) = f(x)\mathrm{e}^{-\mathrm{i}kx}$, this shows that the the Fourier coefficients a_k of $f(x)$ are related to the coefficients a'_k of $f(-x)$ by $a'_k = a_{-k}$.

Similar relations hold for Fourier coefficients of two dimensional functions. Here we are primarily concerned with how the Fourier coefficients change when we rotate the function $f(\mathbf{x})$. Suppose $g(\mathbf{x})$ is a periodic function defined on a square lattice. The integral of this function over a unit cell will be the same as the integral of $g(\mathbf{R}_{\pi/2}\mathbf{x})$ integrated over the unit cell. If we let $g(\mathbf{x}) = f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{q}_\mathbf{m}\cdot\mathbf{x}}$, then we can prove the following lemma.

Lemma 18. *Let $f(\mathbf{x})$ be a function that is periodic on the square lattice, with Fourier coefficients $a_\mathbf{m}$. Let $\mathbf{R}_{2\pi/n}$ be a rotation by $2\pi/n$, where $n = 2$ or 4. The Fourier coefficients $a'_\mathbf{m}$ of $f(\mathbf{R}_{2\pi/n}\mathbf{x})$ satisfy $a'_\mathbf{m} = a_{\mathbf{m}'}$, where $\mathbf{m}' = R^T_{2\pi/n}\mathbf{m}$.*

If the function $f(\mathbf{x})$ has four-fold symmetry, then $f(\mathbf{R}_{\pi/2}\mathbf{x}) = f(\mathbf{x})$, and the last lemma shows, for example, that the Fourier coefficients $a_{1,0}$, $a_{-1,0}$, $a_{0,1}$, and $a_{0,-1}$ are all equal to each other.

We can apply similar arguments to functions that are periodic on a hexagonal lattice. If $g(\mathbf{x})$ is any such function, then the integral of $g(\mathbf{x})$ is the same as the integral of $g(\mathbf{R}_{\pi/3}\mathbf{x})$ over a unit cell. If we let $\mathbf{g}(\mathbf{x}) = f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{q}_\mathbf{m}\cdot\mathbf{x}}$, we can see how the Fourier coefficients of a function rotated by $\pi/3$ radians are related to those of the original function. This result is given in (Romero & Dickey, 2007a), so we summarize the results without proof. In order to state the lemma we need to define the matrix

$$\mathbf{S}_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{7.15a}$$

as well as

$$\mathbf{S}_3 = \mathbf{S}_6^2 \tag{7.15b}$$

and

$$\mathbf{S}_2 = \mathbf{S}_6^3. \tag{7.15c}$$

Lemma 19. *Let $f(\mathbf{x})$ be a function that is periodic on the hexagonal lattice, with Fourier coefficients $a_\mathbf{m}$. Let $\mathbf{R}_{2\pi/n}$ be a rotation by $2\pi/n$ radians ($n = 2, 3,$ or 6), then the Fourier coefficients $a'_\mathbf{m}$ of the function $f(\mathbf{R}_{2\pi/n}\mathbf{x})$ are given by $a'_\mathbf{m} = a_{\mathbf{m}'}$, where $\mathbf{m}' = \mathbf{S}_n\mathbf{m}$ and the matrices $\mathbf{S}_n$ are defined as in Equation (7.15)*

As an example of ths lemma, we see that if the function $f(\mathbf{x})$, has six-fold symmetry, then its Fourier coefficients must satisfy

$$a_{1,0} = a_{1,1} = a_{0,1} = a_{-1,0} = a_{-1,-1} = a_{0,-1}. \tag{7.16}$$

8. TWO DIMENSIONAL BEAM SPLITTING

In the original papers on Dammann gratings Dammann and Gortler (1971) and (Dammann & Klotz, 1977) considered two dimensional gratings, but limited themselves to gratings that were separable. That is, where the transmission function $h(x, y)$ could be written as $h(x, y) = h_1(x)h_2(y)$. The papers by Mait (1989) and Turunen, Vasara, Westerholm, and Salin (1989) extended this work to include non-separable gratings. In these papers they still were considering binary gratings, and they divided up the unit cell into rectangular blocks, assigning a phase to each block. The papers by Vasara, Taghizadeh, Turunen, Westerholm, Noponen, Ichikawa, Miller, Jaakkola, and Kuisma (1992), Mait (1990), Barton, Blair, and Taghizadeh (1997), Blair, Lüpken, Taghizadeh, and Wyrowski (1997) and Zhou and Liu (1995) took a similar approach.

In this section we present the theory in (Romero & Dickey, 2007a) which extends the results for one dimensional continuous gratings to two dimensional non-separable gratings.

8.1. The General Theory

In (Romero & Dickey, 2007a) they showed how to use the calculus of variations to solve the general beam splitting problem in two dimensions. These results are a direct extension of the results for one dimensional beam splitting problems. The results can be stated quite generally for all types of lattices. In particular, suppose we are trying to equalize the energy in the modes $\mathbf{k} \in \mathbf{K}$, where $\mathbf{K}$ is a collection of integer pairs. As in Section 7, each pair of coefficients in $\mathbf{K}$ corresponds to a Fourier coefficient

$$\mathbf{q}_{\mathbf{k}} = m_1\mathbf{q}_1 + m_2\mathbf{q}_2 \tag{8.1}$$

where

$$\mathbf{k} = (m_1, m_2). \tag{8.2}$$

For a square lattice, the vectors $\mathbf{q}_k$ are given by Equation (7.12). For a hexagonal lattice they are given by Equation (7.14).

Similar to the one dimensional case, the calculus of variations, along with the theory of Lagrange multipliers, shows that the optimal transmission function can be written as

$$h(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = \mathrm{e}^{\mathrm{i}\phi(\mathbf{x})} = \frac{s(\mathbf{x}, \underline{\alpha}, \underline{\mu})}{|s(\mathbf{x}, \underline{\alpha}, \underline{\mu})|} \tag{8.3}$$

where

$$s(\mathbf{x}, \underline{\alpha}) = \sum_{\mathbf{k} \in \mathbf{K}} \mu_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\alpha_{\mathbf{k}}} \mathrm{e}^{\mathrm{i}\mathbf{q}_{\mathbf{k}} \cdot \mathbf{x}}. \tag{8.4}$$

As with the one dimensional case this reduces what was an infinite dimensional optimization problem, to a finite dimensional optimization problem where we need to determine the coefficients $\mu_{\mathbf{k}}$, and $\alpha_{\mathbf{k}}$. We need to determine these coefficients so that all of the Fourier coefficients $a_{\mathbf{k}}$ for $\mathbf{k} \in \mathbf{K}$ have the same amplitude, and so that we are also at a local maximum.

In the one dimensional case, by multiplying the transmission function by an arbitrary unit amplitude constant, and by suitably shifting the function, we could arbitrarily set two of the phases α_k to zero. In the two dimensional case similar results hold. However, in this case we can shift the function in two linearly independent directions so we can set three of the phases α_k to zero. As in the one dimensional case, multiplying all of the coefficients μ_k by the same real constant does not change the transmission function $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$, and hence we can arbitrarily set one of the μ_k to unity.

Identical to the one dimensional problem it can be shown (Romero & Dickey, 2007a) that in general the phases of the Fourier coefficients a_k of $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ will not be the same as the phases $\alpha_{\mathbf{k}}$. However, if we are at a local maximum this will be the case. If the set $\mathbf{K}$ contains N different Fourier coefficients, this can be used to get a system of $2N-4$ equations in $2N-4$ unknowns. In particular, once we have set three of the phases α_k to zero, and one of the coefficients μ_k to unity, we have $2N-4$ coefficients to adjust. The requirement that all of the Fourier coefficients $a_{\mathbf{k}}$ for $\mathbf{k} \in \mathbf{K}$ have the same amplitude, gives us $N-1$ equations. The requirement that the phases of these coefficients are the same as the phases α_k used to generate $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$, gives us $N-3$ equations. All together this gives us $2N-4$ equations in $2N-4$ unknowns. This system of equations can be solved using Newton's method.

Once again, a good initial guess can be obtained by solving the Least Squares Optimization Problem. For this problem we have roughly half

the number of parameters to adjust. The optimal solution to this problem arises from setting all of the μ_k to unity, and adjusting the parameters α_k until we maximize $R(\underline{\alpha})$ where

$$R(\underline{\alpha}) = \frac{\left(\int_\Omega |s(\mathbf{x}, \underline{\alpha})| \mathrm{d}x\mathrm{d}y\right)^2}{\int_\Omega |s(\mathbf{x}, \underline{\alpha})|^2 \mathrm{d}x\mathrm{d}y \int_\Omega \mathrm{d}x\mathrm{d}y}. \tag{8.5}$$

Once we have found the phases that optimize this solution, we use these phases, along with the values $\mu_k = 1$, as an initial guess to Newtons method. This procedure is found to work well in practice.

8.2. Symmetry and Symmetry Breaking

Suppose we want to design a square grating that puts equal energy into the four modes $a_{\pm 1,0}$, and $a_{0,\pm 1}$ while maximizing the energy in these modes. Ignoring the other diffraction orders, if we were to look at the intensity of the diffraction pattern resulting from such a grating, it would have four-fold symmetry. That is, if we were to rotate the pattern by $\pi/2$ radians it would look exactly the same. This might suggest that we could look for the optimal transmission function $h(\mathbf{x})$ by assuming that it had four-fold symmetry. In particular, by assuming that

$$h(\mathbf{R}_{\pi/2}\mathbf{x}) = h(\mathbf{x}) \tag{8.6}$$

where $\mathbf{R}_{\pi/2}$ is a matrix representing a rotation by $\pi/2$ degrees about the origin.

Lemma 18 shows that if we design a grating such that $h(\mathbf{x})$ has four-fold symmetry, as in Equation (8.6), then we are guaranteed that the energy in the four modes $a_{0,\pm 1}$ and $a_{\pm 1,0}$ will all be the same. Assuming such a symmetry for $h(\mathbf{x})$ would greatly simplify the search for an optimal grating. However, as we shall see, making this assumption considerably reduces the efficiency of such gratings. This statement applies not only to this specific problem, but to other problems that have four-fold symmetry. A similar statement also applies when we consider hexagonal gratings.

Numerical calculations show that the optimal grating for the problem described in the last few paragraphs (maximizing the energy in the modes $a_{\pm 1,0}$ and $a_{0,\pm 1}$) has two-fold symmetry. That is, we have

$$h(\mathbf{x}) = h(\mathbf{R}_\pi \mathbf{x}) \tag{8.7}$$

where $\mathbf{R}_\pi$ is a rotation by π radians about the origin.

If we look for optimal solutions that have a certain symmetry, the optimal transmission function will still have the form in Equation (8.3),

but the coefficients $\mu_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}$ will have the same symmetry as the transmission function. For the case of square gratings we would consider functions that either have four-fold symmetry, two-fold symmetry, or no rotational symmetry at all. Suppose the function $h(\mathbf{x})$ satisfies $h(\mathbf{x}) = h(\mathbf{R}\mathbf{x})$ (where $\mathbf{R}$ could represent a rotation matrix of $\pi/2$ radians for the case of four- fold symmetry, or 180 degrees for the case of two-fold symmetry). As in Lemma 18 the Fourier coefficients of functions with such symmetry satisfy

$$a_{\mathbf{k}} = a_{\mathbf{k}'} \quad \text{where } \mathbf{k}' = \mathbf{R}^T\mathbf{k}. \tag{8.8}$$

For example, assuming we have two-fold symmetry we would have $a_{0,1} = a_{0,-1}$, and $a_{1,0} = a_{-1,0}$.

Typically we can arbitrarily specify the phases of three of the Fourier components of the transmission function $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$. This arises from the fact that we can adjust the phases by multiplying $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ by a unit amplitude number, and by shifting the function in two linearly independent directions. When we use the fact that the grating has rotational symmetry to simplify the problem, this is no longer the case. In this case we can only specify the phase of one Fourier coefficient (and its symmetric counterparts).

For example, suppose we are trying to solve the problem of putting equal energy into the five modes $a_{0,0}$, $a_{\pm 1,0}$, and $a_{0,\pm 1}$. If we assume that the grating has two-fold symmetry, and this two-fold centre of symmetry is at the origin, then we can assume that $a_{1,0} = a_{-1,0}$, and $a_{0,1} = a_{0,-1}$. Under this assumption, this will give us three phases α_k to be determined. However, in this case we are no longer at liberty to shift the grating around so as to change the phases of these Fourier coefficients. This is because if we shifted the grating by an arbitrary amount the origin would no longer be a centre of two-fold symmetry, and hence the equalities $a_{1,0} = a_{-1,0}$ and $a_{0,1} = a_{0,-1}$ would no longer hold. However, we can still multiply the whole function $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ by an arbitrary unit modulus number without changing the amplitude of any of its Fourier coefficients. It follows that, in this case, we can arbitrarily set the phases of one of our Fourier coefficients (and its symmetric counterpart) to be anything we want it to be.

8.3. Examples Using Square Gratings

8.3.1. Four Beam Splitting Using Square Gratings

We consider the problem of splitting a beam into four equal beams using a square grating. In particular, we try to equalize the energy in the four modes $a_{\pm 1,0}$, $a_{0,\pm 1}$, If we assume that the grating has four-fold symmetry,

then the phases α_k and μ_k must all be equal. This requires that the function $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$, used in determining the transmission function $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ in Equation (8.3), has the form

$$s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{\mathrm{i}y} + \mathrm{e}^{-\mathrm{i}y} = 2\cos(x) + 2\cos(y). \tag{8.9}$$

In this case we know the phases α_k and the parameters μ_k, and hence there is no need to optimize anything. In (Romero & Dickey, 2007a) it was shown that the efficiency of this grating is given by

$$\eta_{CO} = \frac{64}{\pi^4} \approx .658. \tag{8.10}$$

This is the square of the efficiency for doing one dimensional two beam splitting. Though the function $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ in Equation (8.9) does not appear to be separable, it is if we use a coordinate system that is rotated by $\pi/4$ radians. If we had tried to equalize the energy in the modes $a_{\pm 1, \pm 1}$, we would get the same efficiency, and the transmission function would then clearly be separable in our original coordinate system.

If, instead of assuming that the grating has four-fold symmetry, we merely assume that it has two-fold symmetry, the Fourier coefficients must satisfy $a_{\mathbf{m}} = a_{-\mathbf{m}}$. This implies that the optimal grating must have the form

$$s(s, \underline{\alpha}, \mu) = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x} + \mu \mathrm{e}^{\mathrm{i}(y+\alpha)} + \mu \mathrm{e}^{\mathrm{i}(-y+\alpha)}. \tag{8.11}$$

Here we have used the fact that we can arbitrarily set the phase of the coefficients of $\mathrm{e}^{\pm \mathrm{i}x}$ equal to zero, and the coefficients μ_k, associated with these, equal to unity. In (Romero & Dickey, 2007a) it was proven that the optimal solution to this problem is the same as the solution to the Least Squares Optimization Problem. It was shown that the optimal values of α and μ are given by

$$\alpha = \pi/2, \quad \mu = 1. \tag{8.12}$$

It was shown that the efficiency for this grating is given by

$$\eta_{CO} = \eta_{LS} \approx .9179. \tag{8.13}$$

This is considerably greater than the efficiency where we assumed that the grating has four-fold symmetry. This grating is not separable.

When it is not assumed that the grating has any rotational symmetry, it appears that the optimal solution is still the one that has two-fold rotational symmetry.

8.3.2. Five Beam Splitting Using Square Gratings

In (Romero & Dickey, 2007a) they also considered the problem of splitting a beam into five beams of equal intensity. In particular, putting the energy equally into the modes $a_{0,0}$, $a_{\pm 1,0}$, and $a_{0,\pm 1}$.

If the grating is assumed to have four-fold symmetry, then the Fourier coefficients $a_{\pm 1,0}$, and $a_{0,\pm 1}$ must all be the same. Hence, the function $s(\mathbf{x}, \alpha, \mu)$ used in generating the optimal transmission function $h(\mathbf{x}, \alpha, \mu)$ in Equation (8.3) must have the form

$$s(\mathbf{x}, \alpha, \mu) = \mu e^{i\alpha} + e^{ix} + e^{-ix} + e^{iy} + e^{-iy}. \tag{8.14}$$

Numerical calculations show that the optimal values for α and μ are given by

$$\mu \approx .40314, \quad \alpha = \pi/2. \tag{8.15}$$

This gives an efficiency of

$$\eta_{CO} \approx .7629. \tag{8.16}$$

If we require that the grating has two-fold symmetry, then for any integer pair $\mathbf{m}$ we must have $a_{\mathbf{m}} = a_{-\mathbf{m}}$. In this case the optimal value of the transmission function $h(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ in Equation (8.3) is generated by the function $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ where

$$s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = \mu_2 e^{i\alpha_2} + e^{ix} + e^{-ix} + \mu_1 e^{i(y+\alpha_1)} + \mu_1 e^{i(-y+\alpha_1)}. \tag{8.17}$$

Numerical calculations show that

$$\mu_1 \approx 1.1928, \tag{8.18a}$$
$$\mu_2 \approx .7192, \tag{8.18b}$$
$$\alpha_1 = \pi/2, \tag{8.18c}$$
$$\alpha_2 = 0. \tag{8.18d}$$

This gives an efficiency of

$$\eta_{CO} \approx .8433. \tag{8.19}$$

As with the case of four beam splitting this is significantly larger than the efficiency given in Equation (8.16) where the grating was assumed to have four-fold symmetry.

8.3.3. Nine Beam Splitting With Square Gratings

In (Romero & Dickey, 2007a) they considered the problem of splitting a beam into 9 beams of equal magnitude. In particular, they were trying to put the energy equally into the modes $a_{0,0}$, $a_{\pm 1,0}$, $a_{0,\pm 1}$, $a_{\pm 1,\mp 1}$, and $a_{\pm 1,\pm 1}$. Nine beam splitting could be obtained by assuming that $h(x, y) = h_1(x)h_1(y)$, where h_1 is the transmission function for the triplicator. In this case the efficiency would be the square of the efficiency for the triplicator, and hence would be

$$\eta_{1D} = (.92556)^2 = .8456. \tag{8.20}$$

If the grating is assumed to have four-fold symmetry, the optimal solution is found to be this separable solution. In (Romero & Dickey, 2007a) they considered the case where the grating has two-fold symmetry. This requires that

$$a_{\mathbf{m}} = a_{-\mathbf{m}}. \tag{8.21}$$

To further simplify matters they also assumed that the grating had some reflectional symmetry. If the grating had a line of reflection about $x = 0$, then it must also have a line of reflection about $y = 0$. In this case the Fourier components must satisfy

$$a_{1,-1} = a_{1,1}. \tag{8.22}$$

If we make this assumption, then we can solve the Constrained Optimization Problem, and find that, once again, the optimal solution is the same as the separable solution.

However, if we assume that the grating has reflectional symmetry about lines inclined at 45 degrees to the horizontal and vertical, this implies that the $a_{1,0} = a_{0,1}$. In this case the optimal transmission function has the form

$$\begin{aligned} s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = 1 + \mu_1 g_1(\mathbf{x}, \underline{\alpha}) + \mu_2 \left(\mathrm{e}^{\mathrm{i}(x+y+\alpha_2)} + \mathrm{e}^{\mathrm{i}(-x-y+\alpha_2)}\right) \\ + \mu_3 \left(\mathrm{e}^{\mathrm{i}(x-y+\alpha_3)} + \mathrm{e}^{\mathrm{i}(-x+y+\alpha_3)}\right) \end{aligned} \tag{8.23}$$

where

$$g_1(\mathbf{x}, \underline{\alpha}) = \mathrm{e}^{\mathrm{i}(x+\alpha_1)} + \mathrm{e}^{\mathrm{i}(-x+\alpha_1)} + \mathrm{e}^{\mathrm{i}(y+\alpha_1)} + \mathrm{e}^{\mathrm{i}(-y+\alpha_1)}. \tag{8.24}$$

Numerical calculations show that

$$\alpha_1 = 0 \tag{8.25a}$$
$$\alpha_2 = 2.103 \tag{8.25b}$$
$$\alpha_3 = 4.1806 \tag{8.25c}$$
$$\mu_1 = 1.379 \tag{8.25d}$$
$$\mu_2 = 1.111 \tag{8.25e}$$
$$\mu_3 = 1.111 \tag{8.25f}$$

and that the efficiency is given by

$$\eta_{CO} = .9327. \tag{8.26}$$

This is considerably better than the result where the grating was assumed to be separable.

8.4. Examples Using Hexagonal Gratings

We will consider two examples of two dimensional splitting using hexagonal gratings. The first example involves trying to optimize the energy in 6 symmetrically placed modes In the second example we try to maximize the energy in these modes plus the DC mode. Both of these problems are solved assuming that the energy in all of these modes is the same.

The six symmetrically placed modes are given by $a_{1,0}$, $a_{-1,0}$, $a_{0,1}$, $a_{0,-1}$, $a_{1,1}$, and $a_{-1,-1}$. The fact that these modes are symmetrical follows from the fact that all of these modes can be written as $a_{\mathbf{m}}$ where

$$\mathbf{m} = \mathbf{S}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad k = 0, 5 \tag{8.27}$$

where $\mathbf{S}$ is defined as in Equation (7.15). Once we know that this is true, Lemma 19 shows that these modes are symmetrically placed. In particular, if the pattern has six-fold symmetry, then the Fourier coefficients of each of these modes will be the same.

Using the definition of $\mathbf{q}_1$ and $\mathbf{q}_2$ in Equation (7.14), we can compute the complex exponentials $e^{\mathbf{q}_{\mathbf{m}} \cdot \mathbf{x}}$ that go along with the coefficients $a_{\mathbf{m}}$.

This gives us

$$\mathbf{q}_{1,0} \cdot \mathbf{x} = -\mathbf{q}_{-1,0} \cdot \mathbf{x} = x/\sqrt{3} - y \tag{8.28}$$

$$\mathbf{q}_{1,1} \cdot \mathbf{x} = -\mathbf{q}_{-1,-1} \cdot \mathbf{x} = 2x/\sqrt{3} \tag{8.29}$$

$$\mathbf{q}_{0,1} \cdot \mathbf{x} = -\mathbf{q}_{0,-1} \cdot \mathbf{x} = x/\sqrt{3} + y. \tag{8.30}$$

8.4.1. Six Beam Splitting

If we assume that the grating has six-fold symmetry, the general theory shows that the optimal transmission function must be generated from the function $s(\mathbf{x})$ given by

$$s(\mathbf{x}) = e^{i\mathbf{q}_{1,0}\cdot\mathbf{x}} + e^{i\mathbf{q}_{-1,0}\cdot\mathbf{x}} + e^{i\mathbf{q}_{0,1}\cdot\mathbf{x}} + e^{i\mathbf{q}_{0,-1}\cdot\mathbf{x}} + e^{i\mathbf{q}_{1,1}\cdot\mathbf{x}} + e^{i\mathbf{q}_{-1,-1}\cdot\mathbf{x}}. \tag{8.31}$$

This can be written as

$$s(\mathbf{x}) = 2\cos(\mathbf{q}_{1,0} \cdot \mathbf{x}) + 2\cos(\mathbf{q}_{0,1} \cdot \mathbf{x}) + 2\cos(\mathbf{q}_{1,1} \cdot \mathbf{x}). \tag{8.32}$$

This form arises from the fact that in this symmetrical situation all of the relevant Fourier coefficients must not only have the same amplitude, but they must also have the same phase. This phase can arbitrarily be set to zero. If we substitute this form into the expression for $h(\mathbf{x})$, we can compute the Fourier coefficients, and hence the efficiency of the grating. If we do this we find that

$$\eta_{LS} = \eta_{CO} = .7107. \tag{8.33}$$

If instead of assuming that the grating has six-fold symmetry, we use the weaker assumption that it has two-fold symmetry, this requires that the Fourier coefficients satisfy $a_{\mathbf{m}} = a_{-\mathbf{m}}$. The general theory now requires that

$$\begin{aligned} s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = {} & e^{i\mathbf{q}_{1,0}\cdot\mathbf{x}} + e^{i\mathbf{q}_{-1,0}\cdot\mathbf{x}} + \mu_1 e^{i\alpha_1}\left(e^{i\mathbf{q}_{0,1}\cdot\mathbf{x}} + e^{i\mathbf{q}_{0,-1}\cdot\mathbf{x}}\right) \\ & + \mu_2 e^{i\alpha_2}\left(e^{i\mathbf{q}_{1,1}\cdot\mathbf{x}} + e^{i\mathbf{q}_{-1,-1}\cdot\mathbf{x}}\right) \end{aligned} \tag{8.34}$$

this can be written as

$$\begin{aligned} s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = {} & 2\cos(\mathbf{q}_{1,0} \cdot \mathbf{x}) + \mu_1 e^{i\alpha_1}\cos(\mathbf{q}_{0,1} \cdot \mathbf{x}) \\ & + \mu_2 e^{i\alpha_2}\cos(\mathbf{q}_{1,1} \cdot \mathbf{x}). \end{aligned} \tag{8.35}$$

Numerical calculations show that the optimal value is given by

$$\alpha_1 = \pi/2 \quad \alpha_2 = 0 \tag{8.36}$$

$$\mu_1 = .5671 \quad \mu_2 = 1. \tag{8.37}$$

The efficiency is given by

$$\eta_{CO} = .8338. \tag{8.38}$$

When we drop the assumption of the grating having any rotational symmetry at all, we do not improve upon this efficiency. Thus, it appears that the optimal grating has two-fold symmetry.

It is interesting to compare this result to what we would get if we did six beam splitting by crossing a two beam splitter with a three beam splitter. This would give a separable grating that has the efficiency of the product of the efficiencies of each of the one dimensional gratings. This would give an efficiency of $.923 \times 8/\pi^2 = .75$.

8.4.2. Seven Beam Splitting with Hexagonal Gratings

If we assume that the grating has six-fold symmetry, the Fourier coefficients of all of the symmetrically placed modes $a_{1,0}$, $a_{-1,0}$, $a_{0,1}$, $a_{0,-1}$, $a_{1,1}$, and $a_{-1,-1}$, must be the same. This implies that the optimum transmission function is generated by the function $s(\mathbf{x}, \alpha, \mu)$ where

$$s(\mathbf{x}, \alpha, \mu) = \mu e^{i\alpha} + 2\cos(\mathbf{q}_{1,0} \cdot \mathbf{x}) + 2\cos(\mathbf{q}_{1,1} \cdot \mathbf{x}) + 2\cos(\mathbf{q}_{0,1} \cdot \mathbf{x}). \tag{8.39}$$

Here we have used the fact that we can arbitrarily set the phases of all of the 6 symmetrically placed components equal to zero. Numerical calculations show that

$$\mu = .4455, \quad \alpha = 2.494 \tag{8.40}$$

and that the efficiency is given by

$$\eta_{CO} = .8015. \tag{8.41}$$

If instead we assume that the grating has two-fold symmetry, we find that

$$\begin{aligned} s(\mathbf{x}, \underline{\alpha}, \underline{\mu}) = {} & \mu_3 e^{i\alpha_3} + 2\cos(\mathbf{q}_{1,0} \cdot \mathbf{x}) \\ & + \mu_2 e^{i\alpha_2} \cos(\mathbf{q}_{1,1} \cdot \mathbf{x}) + \mu_1 e^{i\alpha_1} \cos(\mathbf{q}_{0,1} \cdot \mathbf{x}). \end{aligned} \tag{8.42}$$

Numerically it is found that

$$(\alpha_1, \alpha_2, \alpha_3) = (\pi/2, \pi/2, 0) \tag{8.43}$$

$$(\mu_1, \mu_2, \mu_3) = (1.3368, 1.3368, .9811). \tag{8.44}$$

The efficiency is given by

$$\eta_{CO} = .9003. \tag{8.45}$$

9. SUMMARY

We have addressed four main topics in this review: (1) Techniques for giving upper bounds on the efficiencies of gratings (Section 3) (2) Dammann gratings (Section 5) (3) One dimensional continuous gratings (Section 6) (4) Two dimensional continuous gratings (Section 8).

The efficiencies for various problems of these four types can be found in the tables. There is enough information in these tables, so the reader can reconstruct the efficiencies from this information, and from the equations in this review that are referenced in the captions of the tables. Tables 1 and 2 are a slight modification of a table given in (Krackhardt, Mait, & Streibl, 1992) where the upper bounds of the efficiencies for one dimensional gratings (discussed in Section 3) are given. Table 3 is an abridged form of the table in (Mait, 1997) giving the efficiencies for one dimensional Dammann gratings. Tables 5 and 6 give the efficiencies for the Constrained Optimization Problem for one dimensional gratings. Finally Tables 7 and 8 summarize results for two dimensional beam splitting problems on square and hexagonal lattices.

A survey of these tables brings up a general question. How does the efficiency of a grating vary as the number of modes increases? Table 3 shows that for Dammann gratings the highest efficiency is obtained for three beam splitting. Tables 5 and 6 show that for one dimensional continuous gratings the efficiencies tend to increase as the number of modes increases. However, the efficiency $\eta_{CO} = .9928$ for 9 beam splitting is the highest for any of the problems listed. A look at Tables 7 and 8 shows that no clear pattern emerges from the small number of problems we have listed.

Another general question concerns the symmetry of these gratings. As we have noted, just because the beam shaping problem has a certain symmetry does not mean that the optimal grating needs to possess that symmetry. This is an important point, since assuming that a grating

TABLE 7 This table summarizes the various problems we have considered for gratings on a square lattice. For each number of modes we give the symmetry of the grating, its efficiency η_{CO}, the equation in the text where the definition of the function $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ can be found, and the equation in the text where the values of the parameters $\underline{\alpha}$ and $\underline{\mu}$ can be found. The phase function $\phi(\mathbf{x})$ is defined from $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ by Equation (8.3)

N_{modes}	Symmetry	η_{CO}	Defining equation	Equation for parameters
4	4 fold	.658	Equation (8.9)	NA
4	2 fold	.9179	Equation (8.11)	Equation (8.12)
5	4 fold	.7629	Equation (8.14)	Equation (8.15)
5	2 fold	.8433	Equation (8.17)	Equation (8.18)
9	4 fold	.8456	separable	same as triplicator
9	2 fold	.9327	Equations (8.23) and (8.24)	Equation (8.18)

TABLE 8 This table summarizes the various problems we have considered for gratings on a hexagonal lattice. For each number of modes we give the symmetry of the grating, its efficiency η_{CO}, the equation in the text where the definition of the function $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ can be found, and the equation in the text where the values of the parameters $\underline{\alpha}$ and $\underline{\mu}$ can be found. The phase function $\phi(\mathbf{x})$ is defined from $s(\mathbf{x}, \underline{\alpha}, \underline{\mu})$ by Equation (8.3)

N_{modes}	Symmetry	η_{CO}	Defining equaiton	Equation for parameters
6	6 fold	.7107	Equation (8.31)	NA
6	2 fold	.8338	Equation (8.34)	Equations (8.36) and (8.37)
7	6 fold	.8015	Equation (8.39)	Equation (8.40)
7	2 fold	.9003	Equation (8.42)	Equations (8.43) and (8.44)

has symmetry greatly simplifies the search for an optimal grating. For one dimensional continuous gratings we know of no example where the optimal grating breaks the symmetry of the original beam splitting problem. However, this is not the case for Dammann gratings and two dimensional beam splitting problems. For two dimensional beam splitting problems Tables 7 and 8 show that we can significantly improve the efficiency of the grating by breaking the symmetry of the beam splitting problem.

In this review we have limited ourselves to discussing the problem of designing optimal beam splitting gratings without regard to manufacturing tolerances. There is no question that manufacturing errors can be an important consideration when designing a grating. However, these processes are continually under improvement, and hence, even if some of the gratings we have discussed are not feasible to make today, we believe they will quite likely become feasible in the near future.

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APPENDIX A

The invention and development of diffraction gratings has a very interesting history. Born and Wolf (1999) attribute the discovery of the diffraction grating to David Rittenhouse in 1785, but the discovery received little attention. Hutley (1982) attributes Rittenhouses discovery to observations of light patterns obtained with a fine silk handkerchief. Rittenhouse was able to repeat this phenomenon by arranging a parallel pattern of fine hairs across two screws provided by a watch maker. Hutley also points out that Sir John Barton frivolously exploited spectral colour by cutting crossed gratings on steel to use in molding fancy metal waistcoat buttons. Both Born and Wolf, and Hutley attribute the invention of the modern diffraction grating to Joseph von Fraunhofer. In 1821 he independently repeated Rittenhouse's experiments with fine wire gratings and produced reflection gratings by diamond ruling of a mirror surface. Great advances were made by Rowland [2,3]. In 1882 at the Johns Hopkins University he developed the ruling engine. This work evolved into the Johns Hopkins University being the principal supplier of fine diffraction gratings up until World War II. The rest is history; the details of the development of the modern diffraction technology are described in Hutleys book and the Diffraction Grating Handbook (Palmer & Loewen, 2002).

Although not widely recognized, the discovery of the diffraction grating dates back to the time of Newton. The diffraction grating appears to have been discovered by mathematician and astronomer James Gregory (Baker, 2002). While at the University of St. Andrews, 1669–1674, Gregory discovered the diffraction grating by observing the pattern produced by passing a beam of light through a bird feather. An article by J. J. O'Connor and E. F. Robertson on the University of St. Andrews, School of Mathematics and Statistics website (O'Connor & Robinson, 2008) gives an interesting and concise history of James Gregory's many accomplishments, including the discovery of the diffraction grating. The article states that Gregory wrote:

> "Let in the sun's rays by a small hole to a darkened house, and at the hole place a feather (the more delicate and white the better for this

purpose), and it shall direct to a white wall or paper opposite to it a number of small circles and ovals (if I mistake them not) whereof one is somewhat white (to wit, the middle which is opposite the sun) and all the rest severally coloured. I would gladly hear Mr Newton's thoughts of it."

Based on this description of what he observed, one can argue that Gregory discovered the beam splitting grating. Certainly the period of the bird feather grating would be many wavelengths, which is compatible with beam splitting gratings.

APPENDIX B

In this appendix we give the proof of Theorem 3 as well as giving a version of the theorem that applies to Fourier integrals. We will use the notation

$$\langle r, q\rangle = \frac{1}{d}\int_{-d/2}^{d/2} r(x)q^{\star}(x)\mathrm{d}x. \tag{B.1}$$

For easy reference, we restate the theorem.

Theorem 10. *Given a collection of integers K, and Fourier coefficients $a_k, k \in K$ let*

$$f(x) = \sum_{k\in K} a_k \mathrm{e}^{\mathrm{i}k(\frac{2\pi}{d})x}. \tag{B.2}$$

Let $p(x) > 0$ and $\phi(x)$ be real functions with period d, such that the Fourier coefficients b_k of $g(x) = p(x)\mathrm{e}^{\mathrm{i}\phi(x)}$ satisfy $b_k = ca_k$, for $k \in K$ and some constant c. Under these assumptions we have

$$\frac{\sum\limits_{k\in K}|b_k|^2}{\sum\limits_{k=-\infty}^{\infty}|b_k|^2} \leq \frac{\left(\int_{-d/2}^{d/2} p(x)|f(x)|\mathrm{d}x\right)^2}{\int_{-d/2}^{d/2}|f(x)|^2\mathrm{d}x\int_{-d/2}^{d/2}p^2(x)\mathrm{d}x}. \tag{B.3}$$

Proof. Let $f(x) = a(x)\mathrm{e}^{\mathrm{i}\psi(x)}$, where $a(x)$ and $\psi(x)$ are the amplitude and phase of $f(x)$. The Fourier coefficients c_k of $h(x) = g(x) - cf(x)$ vanish if $k \in K$, and are equal to b_k if $k \notin K$. From Parsevals theorem we have

$$\langle a, a\rangle = \langle f, f\rangle = \sum_{k\in K}|a_k|^2. \tag{B.4}$$

Since $b_k = ca_k$ for $k \in K$, we have

$$\sum_{k\in K} |b_k|^2 = c^2\langle a, a\rangle. \tag{B.5}$$

Parseval's equality implies that

$$\langle p, p\rangle = \langle g, g\rangle = \sum_{k=-\infty}^{\infty} |b_k|^2 \tag{B.6}$$

and hence from Equation (B.5)

$$\langle p, p\rangle = c^2\langle a, a\rangle + \sum_{k\notin K} |b_k|^2. \tag{B.7}$$

Parseval's equality also implies that

$$\langle h, h\rangle = \sum_{k\notin K} |b_k|^2 \tag{B.8}$$

and hence due to Equation (B.7)

$$\langle h, h\rangle = \langle p, p\rangle - c^2\langle a, a\rangle. \tag{B.9}$$

Explicitly writing out our expression for $\langle h, h\rangle$, we get

$$\langle h, h\rangle = \langle g, g\rangle + c^2\langle f, f\rangle - 2cRe\,(\langle f, g\rangle) \tag{B.10}$$

which can be written as

$$\langle h, h\rangle = \langle p, p\rangle + c^2\langle a, a\rangle - 2c\langle p(x)\cos(\gamma(x)), a(x)\rangle \tag{B.11}$$

where $\gamma(x) = \phi(x) - \psi(x)$. If we equate the expressions in Equations (B.9) and (B.11), and solve for c we find that

$$c = \frac{\langle p(x)\cos(\gamma(x)), a(x)\rangle}{\langle a, a\rangle} \leq \frac{\langle p, a\rangle}{\langle a, a\rangle}. \tag{B.12}$$

Equations (B.5) and (B.6) imply that

$$\frac{\sum\limits_{k\in K} |b_k|^2}{\sum\limits_{k=-\infty}^{\infty} |b_k|^2} = c^2\frac{\langle a, a\rangle}{\langle p, p\rangle}. \tag{B.13}$$

The theorem follows from using our expression for c in Equation (B.12). □

We now prove Wyrowski's theorem when we are using Fourier integrals. The proof is almost identical to that where we were using Fourier series. We now use the notation

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)g^{\star}(x)\mathrm{d}x. \tag{B.14}$$

To avoid confusion, we state which version of the Fourier transform we are using. We will use the transform

$$R(\omega) = \int_{-\infty}^{\infty} r(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x \tag{B.15}$$

and the inverse transform

$$r(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega. \tag{B.16}$$

Parsevals equality gives us

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |R(\omega)|^2\mathrm{d}\omega = \int_{-\infty}^{\infty} |r(x)|^2\mathrm{d}x. \tag{B.17}$$

Theorem 11. *Given a subset I of the real line, and a function $F(\omega)$ defined on I let*

$$f(x) = a(x)\mathrm{e}^{\mathrm{i}\psi(x)} = \frac{1}{2\pi}\int_{I} F(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega. \tag{B.18}$$

Here $a(x)$ is the amplitude and $\psi(x)$ is the phase of $f(x)$. Let $p(x) > 0$ and $\phi(x)$ be real functions such that the Fourier transform $G(\omega)$ of $g(x) = p(x)\mathrm{e}^{\mathrm{i}\phi(x)}$ satisfies $G(\omega) = cF(\omega)$, for $\omega \in I$ and some constant c. Under these assumptions we have

$$\frac{\int_I |G(\omega)|^2\mathrm{d}\omega}{\int_{-\infty}^{\infty} |G(\omega)|^2\mathrm{d}\omega} \leq \frac{(\langle p, a\rangle)^2}{\langle a, a\rangle\langle p, p\rangle}. \tag{B.19}$$

Proof. The transform of $H(\omega)$ of $h(x) = g(x) - cf(x)$ vanishes if $\omega \in I$ and is equal to $G(\omega)$ if $\omega \notin I$. From Parsevals theorem we have

$$\langle a, a\rangle = \frac{1}{2\pi}\int_{I} |F(\omega)|^2\mathrm{d}\omega \tag{B.20}$$

$$\langle p, p\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\mathrm{d}\omega = c^2\langle a, a\rangle + \frac{1}{2\pi}\int_{\omega\notin I} |G(\omega)|^2\mathrm{d}\omega. \tag{B.21}$$

The last equality in Equation (B.21) follows from the fact that $G(\omega) = cF(\omega)$ for $\omega \in I$, and from Equation (B.20). Parseval's equality also implies that

$$\langle h, h\rangle = \frac{1}{2\pi}\int_{\omega\notin I} |G(\omega)|^2 = \langle p, p\rangle - c^2\langle a, a\rangle. \tag{B.22}$$

This last equality has used Equation (B.21). Explicitly writing out our expression for $\langle H, H\rangle$, we get

$$\langle h, h\rangle = \langle p, p\rangle + c^2\langle a, a\rangle - 2c\langle p(x)\cos(\gamma(x)), a(x)\rangle \tag{B.23}$$

where $\gamma(x) = \phi(x) - \psi(x)$. If we equate the expressions in Equations (B.22) and (B.23), and solve for c we find that

$$c = \frac{\langle p(x)\cos(\gamma(x)), a(\omega)\rangle}{\langle a, a\rangle} \leq \frac{\langle p, a\rangle}{\langle a, a\rangle}. \tag{B.24}$$

The theorem follows from using the fact that

$$\frac{\int_{\omega\in I} |G(\omega)|^2}{\int_{-\infty}^{\infty} |G(\omega)|^2} = c^2\frac{\langle a, a\rangle}{\langle p, p\rangle} \tag{B.25}$$

and our expression for c in Equation (B.24). □

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