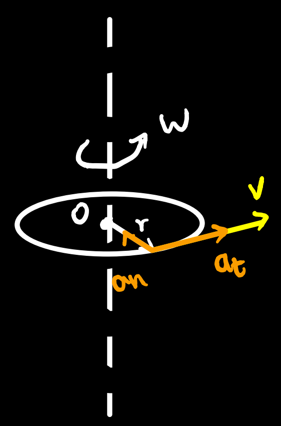
What is Robot Dynamics about?

- To establish the relationship of Motion. Force & Moments

1. Basics

· Rotation about a fixed axis



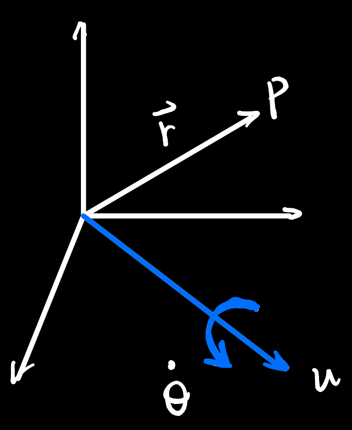
$v = \omega \times r$

$a = \frac{dv}{dt} = \frac{d\omega}{dt} \times r + \omega \times \frac{dr}{dt}$

$= \alpha \times r + \omega \times (\omega \times r) = a_t + a_n$

Magnitude of acceleration $a = \sqrt{a_t^2 + a_n^2}$

· Rotation of a vector



$\vec{\omega} = \vec{u} \cdot \dot{\theta}$

$\frac{d}{dt}\vec{r} = \vec{\omega} \times \vec{r}$ i.e. velocity

* Thus we can see. According to a frame i, j, k.

$\frac{d}{dt} i = \omega \times i$. $\frac{d}{dt} j = \omega \times j$. $\frac{d}{dt} k = \omega \times k$

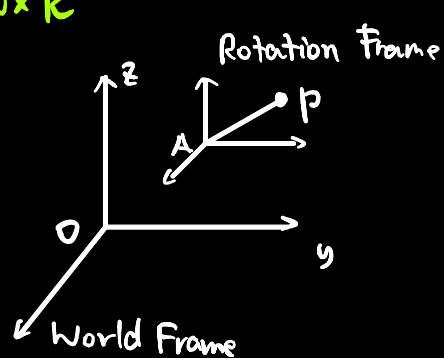
$\dot{R} = \frac{d}{dt}(i \ j \ k) = \omega \times R$

· General Motion

~ Regard as Rotation + Translation

$v_p = v_A + \omega \times r$

$a_B = a_A + \alpha \times r + \omega \times (\omega \times r)$



2. Introduction to Dynamics

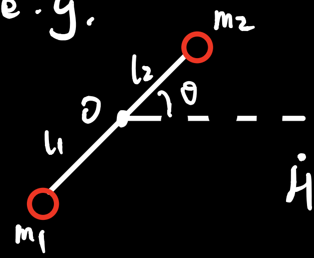
- Linear Momentum & Angular Momentum

$L = m\vec{V}$. Thus we have $F = m\vec{a} = \dot{L}$

$H_o = r \times m\vec{v}$ * We should have reference point O

$\dot{H}_o = \dot{r} \times m\vec{v} + r \times m\dot{\vec{v}} = r \times m\dot{\vec{v}} = r \times F = M_o$

e.g.

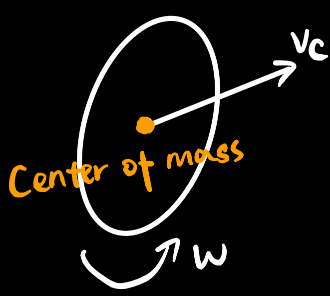


$H_o = l_1 \times m_1(\dot{\theta} \times l_1) + l_2 \times m_2(\dot{\theta} \times l_2)$

$\dot{H}_o = M \Rightarrow \frac{d}{dt}(m_1 l_1^2 \dot{\theta} + m_2 l_2^2 \dot{\theta}) = m_1 g l_1 c_\theta - m_2 g l_2 c_\theta$

$$\Rightarrow \ddot{\theta} = \frac{(m_1 l_1 - m_2 l_2) c_\theta}{m_1 l_1^2 + m_2 l_2^2}$$

$\Rightarrow$ Now. For a rigid body



$\vec{L} = m\vec{V}_c$

$\vec{H}_c = I\vec{\omega}$ * Here I ~ Inertia tensor matrix

3. Lagrange Formulation of Robot Dynamics

- Based on the viewpoint of energy & work principle α. Kinetic Energy

$$K = \frac{1}{2}mv^2 \xrightarrow{\text{Particles}} K = \frac{1}{2}\sum m_i v_i^2 \xrightarrow[\text{Body}]{\text{Rigid}} K = \int_V \frac{1}{2}\dot{r}^T \dot{r}\, dm$$

$$= \frac{1}{2}m v_c^2 + \frac{1}{2}\omega^T I \omega$$

β. Potential Energy

$$U = mgz \xrightarrow{\text{Particles}} U = \sum m_i g z_i \xrightarrow[\text{Body}]{\text{Rigid}} U = mgz_c$$

γ. Work

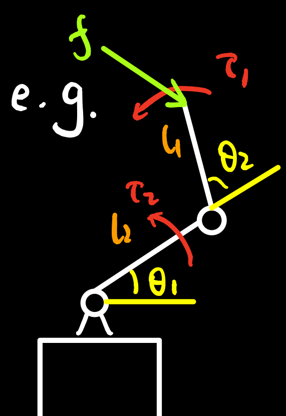
$$W = f^T r \xrightarrow[\text{Force}]{\text{Time-vary}} W = \int f^T(t)\, dr = \Delta(K+U)$$

⇒ Lagrange Equation

Generalized Coordinate $\vec{q}$ *Dim = DOF

$$\delta W = \sum f_i^T \delta r_i \quad \text{where } F = \frac{\partial}{\partial(\partial q)}(\delta W)$$

e.g.



$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\delta W = \tau_1 \delta\theta_1 + \tau_2 \delta\theta_2 + f^T \delta x$$

$$= \tau_1 \delta\theta_1 + \tau_2 \delta\theta_2 + f^T J \delta q$$

*Actually, we can see generalized Force

$$F = \frac{\partial}{\partial \partial q} \delta W = f^T J + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

q - Generalized Coordinate

K - Kinetic Energy

U - Potential Energy

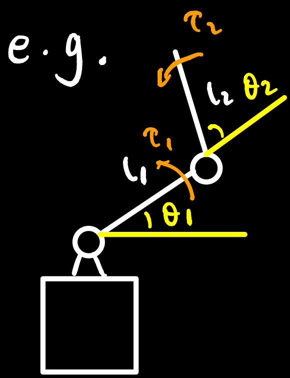
F - Generalized Force

*Def. Lagrangian $L = K - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

~ Dynamics of the system

e.g.



Here Generalized Coordinate $q = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$

$$k_1 = \frac{1}{2} I_0 \dot{\theta}_1^2$$

$$k_2 = \underbrace{\frac{1}{2} m_2 V_{c_2}^2}_{\text{Given by } l_1} + \boxed{\frac{1}{2} I_{c_2} \dot{\theta}_2^2}$$

l_2's rotation!

We know $\begin{cases} x_{c_2} = l_1 c_1 + \frac{l_2}{2} c_{12} \\ y_{c_2} = l_1 s_1 + \frac{l_2}{2} s_{12} \end{cases}$

$$\Rightarrow V_{c_2}^2 = \dot{x}_{c_2}^2 + \dot{y}_{c_2}^2$$

$$= l_1^2 \dot{\theta}_1^2 + \frac{l_2^2}{4}(\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$+ l_1 l_2 c_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)$$

$K = K_1 + K_2$.

$$U = m_1 g y_{c_1} + m_2 g y_{c_2} = m_1 g \frac{l_1}{2} s_1 + m_2 g \left(l_1 s_1 + \frac{l_2}{2} s_{12}\right)$$

$$L = K - U = \frac{1}{2} I_0 \dot{\theta}_1^2 + \frac{1}{2} I_{c_2} \dot{\theta}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{l_2^2}{8} m_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$+ \frac{1}{2} m_2 l_1 l_2 c_1 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g \frac{l_1}{2} c_1 - m_2 g \left(l_1 c_1 + \frac{l_2}{2} c_{12} \right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (I_0 + m_2 l_1^2)\dot{\theta}_1 + m_2 l_1 l_2 c_2 \left(\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\right) + (\dot{\theta}_1 + \dot{\theta}_2)\left(\frac{m_2}{4} l_2^2 + I_{c_2}\right)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 g \frac{l_2}{2} c_{12} - \frac{1}{2} m_2 l_1 l_2 s_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} m_2 l_1 l_2 c_2 \dot{\theta}_1 + \left(\frac{1}{4} m_2 l_2^2 + I_{c_2}\right)(\dot{\theta}_1 + \dot{\theta}_2)$$

Generalized Force $F = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$

Then Lagrange Equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} I_{0_1} + m_2 l_1^2 + \frac{1}{4} m_2 l_2^2 + I_{c_2} + m_2 l_1 l_2 c_2 & \frac{1}{4} m_2 l_2^2 + I_{c_2} + \frac{1}{2} m_2 l_1 l_2 c_2 \\ \frac{1}{4} m_2 l_2^2 + I_{c_2} + \frac{1}{2} m_2 l_1 l_2 c_2 & \frac{1}{4} m_2 l_2^2 + I_{c_2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_2 l_1 l_2 s_2 \left(\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\right)\dot{\theta}_2 \\ 0.5 m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} (0.5 m_1 + m_2) l_1 g c_1 + 0.5 m_2 l_2 g c_{12} \\ 0.5 m_2 l_2 g c_{12} \end{bmatrix}$$

* Denote as $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = H(q)\ddot{q} + \left(\frac{1}{2}\dot{H}(q) + S(q,\dot{q})\right)\dot{q} + G(q)$

$H(q)$: Inertia Matrix of the robot

$G(q)$: Gravity Force

$S(q, \dot{q})$: Centrifugal & Coridis forces