$\min f(x)$.

where $f(x)$ convex. $2^{nd}$ continuous differentiable

- Produce sequence of points $x^{(k)} \in \operatorname{dom} f$

  with $f(x^{(k)}) \to p^*$ where $p^* = \inf_x f(x)$

## 1. Initial point & Sublevel set

Starting Point $\begin{cases} x^{(0)} \in \operatorname{dom} f \\ \text{sublevel set } S = \{x \mid f(x) \leq f(x^{(0)})\} \text{ is closed} \end{cases}$

Strong convexity $\iff \nabla^2 f(x) \succeq mI$ . $\exists m > 0$

- Implication

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2} \|x - y\|_2^2$$

we minimize rhs for $y$

$$\Rightarrow f(x) - p^* \leq \frac{1}{2m} \|\nabla f(x)\|_2^2$$ useful as stopping criterion

## 2. Descent Methods

Search Direction

$$x^{(k+1)} = x^{(k)} + \underline{t}^{(k)} \Delta x^{(k)} \quad \text{with } f(x^{(k+1)}) < f(x^{(k)})$$

Step size

- From convexity. we know $\nabla f(x)^T \Delta x < 0$

① Linear Search Type

$$t = \operatorname*{argmin}_{t > 0} f(x + t\Delta x)$$ ~ Exact Line Search

⇒ Backtracking line search

$t=1$ at start. repeat $t := \beta t$ until

$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$$

② Gradient Descent Method

$\Delta x = -\nabla f(x)$

- Stopping Criterion $\|\nabla f(x)\|_2 \le \varepsilon$

③ Steepest Descent Method

$\Delta x_{nsd} = \arg\min \{\nabla f(x)^T v \mid \|v\| = 1\}$ ~ Most steep downhill

or $\Delta x_{sd} = \|\nabla f(x)\|_* \Delta x_{nsd}$

3. Newton Step

$\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$

- $x + \Delta x_{nt}$ minimizes second-order approximation

$$\hat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

- Newton Decrement

$$\lambda(x) = \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{\frac{1}{2}}$$

gives a measure of proximity of $x$ to $x^*$

Newton's method

1. Compute $\Delta x_{nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$, $\lambda^2 = \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$
2. quit if $\frac{\lambda^2}{2} \leq \varepsilon$
3. choose step size $t$ by $x := x + t\Delta x_{nt}$