Optimization Problem in standard form

minimize $f_0(x)$

s.t. $f_i(x) \le 0$, $i = 1, \dots, m$

$h_i(x) = 0$, $i = 1, \dots, p$

$p^* = \begin{cases} \infty & \text{infeasible} \\ -\infty & \text{Unbounded below} \end{cases}$

*Optimal

- Locally Optimum

minimize $f_0(z)$

s.t. $f_i(z) \le 0$, $h_i(z) = 0$. $\|z - x\|_2 \le R$. $\exists R > 0$

* Convex Optimization

minimize $f_0(x)$

s.t. $f_i(x) \le 0$. $a_i^T x = b_i$ *written as $Ax = b$

- $f_0, f_1, \dots f_m$ are convex. equality constraints are affine
- Quasiconvex if f_0 is quasiconvex

Th. Any local optimal point of cvxopt problem is global optimal

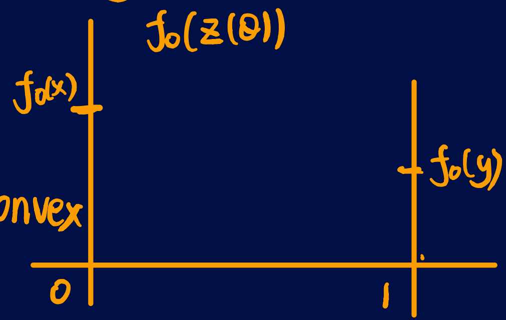
Pf. suppose x locally optimal & y is globally optimal

$f_0(y) < f_0(x)$

consider $z = \theta y + (1-\theta)x$

$f_0(z) < \theta f_0(y) + (1-\theta) f_0(x)$ ~ Convex

But $f_0(z) > f_0(x)$ Contradicts!



1. Optimal Criterion for differentiable f_0

x is optimal iff. $\nabla f_0(x)^T(y-x) \geq 0$ for all feasible y

Unconstrained: $x \in \mathrm{dom} f_0$. $\nabla f_0(x) = 0$

Equality-constrained: $Ax = b$. $\nabla f_0(x) + A^T v = 0$. $\exists v$

Nonnegative orthant: $x \succeq 0$, $\begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$

$\nabla f_0(x)^T(y-x) \geq 0$. $Ay = b$ & $Ax = b$

Thus $\nabla f_0(x)^T z \geq 0$. $z \in N(A)$

But $-z \in N(A)$, too $\Rightarrow \nabla f_0(x)^T z = 0$

$\Rightarrow \nabla f_0(x) \in N(A)^{\perp} = R(A^T)$

Thus $\nabla f_0(x) = A(-v) \Rightarrow \nabla f_0(x) + Av = 0$

2. Equivalent Constraints

- Eliminating Equalities

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{s.t. } f_i(x) \leq 0 \,.\, i = 1, \ldots m \\ &\quad\quad Ax = b \end{aligned} \iff \begin{aligned} &\text{minimize } f_0(Fz + x_0) \\ &\text{s.t. } f_i(Fz + x_0) \leq 0 \end{aligned}$$

where $Ax = b \iff x = Fz + x_0$

- Introducing Equalities

$$\begin{aligned} &\text{minimize } f_0(A_0 x + b_0) \\ &\text{s.t. } f_i(A_i x + b_i) \leq 0 \end{aligned} \iff \begin{aligned} &\text{minimize } f_0(y_0) \\ &\text{s.t. } f_i(y_i) \leq 0 \\ &\quad\quad y_i = A_i x + b_i \end{aligned}$$

- Introducing Slack Variables ☆

$$\text{minimize } f_0(x) \quad \Longleftrightarrow \quad \text{minimize } f_0(x)$$
$$\text{s.t. } a_i^T x \le b_i \qquad\qquad \text{s.t. } a_i^T x + s_i = b_i$$
$$s_i \ge 0$$

- Epigraph form [Boring]

$$\begin{aligned} &\text{minimize } t \\ &\text{s.t. } f_0(x) - t \le 0 \\ &\qquad f_i(x) \le 0, \ i = 1, \dots, m \\ &\qquad Ax = b \end{aligned}$$

- Partial Minimization

$$\text{minimize } f_0(x_1, x_2) \quad \Longleftrightarrow \quad \text{minimize } \tilde{f}_0(x_1)$$
$$\text{s.t. } f_i(x_1) \le 0 \qquad\qquad \text{s.t. } f_i(x_1) \le 0$$
$$\text{where } \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$$

3. Quasiconvex

For f_0 quasiconvex. $\exists \phi_t$ s.t.

- $\phi_t(x)$ is convex for fixed t
- $f_0(x) \le t \Longleftrightarrow \phi_t(x) \le 0$

Now $\phi_t(x) \le 0$. $f_i(x) \le 0$. For fixed t. ~ Convex

If feasible $t \ge p^*$

else $t \le p^*$

4. Linear Programming

$$\begin{aligned} \text{minimize } & c^T x + d \\ \text{s.t. } & Gx \preceq h \\ & Ax = b \end{aligned}$$

- Feasible set is a polyhedron!

e.g. choose quantities $x_1, \cdots x_n$ of n food

- one unit food j costs c_j. contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

$$\begin{aligned} \text{minimize } & c^T x \\ \text{s.t. } & Ax \succeq b, \quad x \succeq 0 \end{aligned}$$

$$\Rightarrow \text{s.t. } \begin{bmatrix} -A \\ I \end{bmatrix} x + \begin{bmatrix} -b \\ 0 \end{bmatrix} \preceq 0$$

$$\text{minimize } \max_{i=1 \ldots m} (a_i^T x + b)$$

$$\Leftrightarrow \text{minimize } t \quad \text{s.t. } a_i^T x + b_i \leq t, \quad i = 1, \cdots, m$$

- Linear-fractional program

$$\begin{aligned} \text{minimize } & f_0(x) \qquad \text{where } f(x) = \frac{c^T x + d}{e^T x + f} \\ \text{s.t. } & Gx \preceq h \\ & Ax = b. \end{aligned}$$

$$\Leftrightarrow \begin{aligned} \text{minimize } & c^T y + dz \\ \text{s.t. } & Gy \preceq hz, \quad z \geq 0 \end{aligned}$$

$$Ay = bz$$
$$e^T y + fz = 1$$

5. Quadratic Programming

minimize $\frac{1}{2}x^T P x + q^T x + r$ $\quad P \in S_+^n \Rightarrow$ Convex

s.t. $Gx \preceq h$

$Ax = b$

$\Rightarrow$ Quadratically Constrained Quadratic Program (QCQP)

minimize $\frac{1}{2}x^T P_0 x + q_0^T x + r_0$

s.t. $\frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0 \quad i=1, \cdots, m$

$Ax = b$

$\Rightarrow$ Second-order Cone Program (SOCP)

minimize $f^T x$

s.t. $\|A_i x + b_i\|_2 \leq c_i^T x + d_i$

$Fx = g$

6. Geometric Programming

- Monomial function

$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. $\mathrm{dom} f = R_{++}^n$ $(c > 0)$

$a_i \in R$

- Posynomial function

$$f(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}, \quad \mathrm{dom} f = \mathbb{R}^n_{++}$$

Now, GP

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{s.t.} \quad f_i(x) \le 1, \quad i = 1, \cdots m \\ &\qquad\quad h_i(x) = 1, \quad i = 1, \cdots p \end{aligned}$$

with f_i posynomial & h_i monomial

Take variable $y_i = \log x_i$

Then $\log f(e^{y_1}, \dots e^{y_n}) = a^T y + b \quad (b = \log c)$

<Monomial>

7. Generalized Inequality Constraints

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{s.t.} \quad f_i(x) \preceq_K 0, \quad i = 1, \dots, m. \\ &\qquad\quad Ax = b \end{aligned}$$

$f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$. K_i - convex

- Conic form Problem

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{s.t.} \quad Fx + g \preceq_K 0 \\ &\qquad\quad Ax = b \end{aligned}$$

8. Semidefinite Program (SDP)

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{s.t.} \quad x_1F_1 + x_2F_2 + \cdots + x_nF_n + G \leq 0 \ , \ F_i, G \in S^k \\ &\qquad\quad Ax = 0 \end{aligned}$$

9. Vector Optimization

$$\begin{aligned} &\text{minimize} \quad f_0(x) \qquad \sim f_0 : R^n \to R^q \\ &\text{s.t.} \quad f_i(x) \leq 0 \quad i = 1, \cdots m \\ &\qquad\quad Ax = b \quad i = 1, \cdots, p \end{aligned}$$

$\begin{cases} \text{Optimal : Minimum value of a set} \\ \text{Pareto Optimal : Minimal value of a set} \end{cases}$