1. Norm Approximation

min $\|Ax-b\|$ where $A \in R^{m\times n}$ with $m \geq n$

* $x^* = \underset{x}{\text{argmin}} \|Ax-b\| \sim dist(b, R(A))$

Interpretation

- Geometric : Ax^* is point in $R(A)$ closest to b
- Estimation : Linear measurement

$$y = Ax + v$$

given $y=b$, best guess of x

- Optimal Design : $Ax \sim$ output of design.

$x^* \sim$ design that best approximates desired result b.

e.g.

① Least-squares . $x^* = (A^TA)^{-1}A^Tb$ if rank$(A) = n$

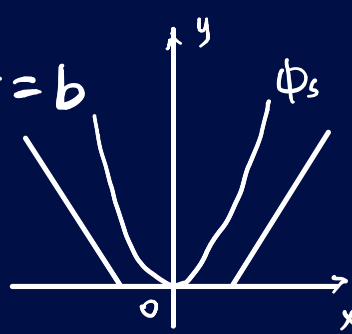
② sum of absolute residuals $(\|\cdot\|_1)$

$$\min \ \mathbb{1}^T y \quad s.t. \ -y \preceq Ax-b \preceq y$$

- Generalization

$$\min \ \phi(r_1) + \cdots \phi(r_m) \quad s.t. \ r = Ax-b$$

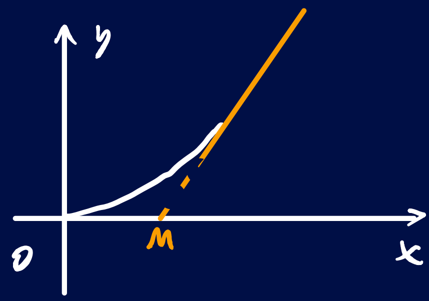
$\phi: R \to R$, convex penalty function

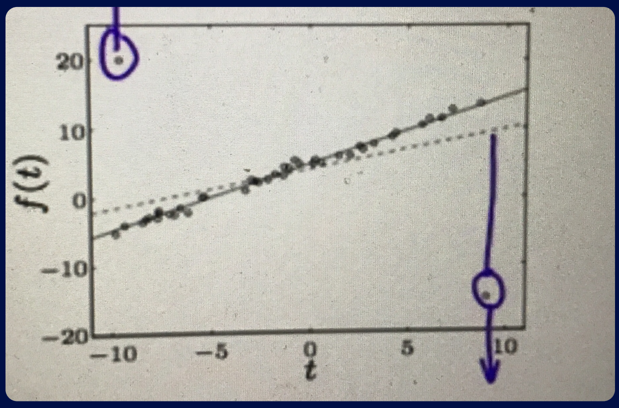


*Different ϕ can help you design the opt.

- Huber penalty function

$$\phi_{hub}(u) = \begin{cases} u^2 & |u| \le M \\ M(2|u| - M) & |u| > M \end{cases}$$





- Dash Line : Least Square
- Dark Line : Huber Penalty

2. Least-norm problems

$\min \|x\|$ s.t. $Ax = b$ where $A \in R^{m \times n}$, $m \le n$

3. Regularized Approximation

$\min (\|Ax - b\|, \|x\|)$

*Robust Approximation : $Ax \approx b$ with error in A

$(A+\Delta)x \approx b$ has smaller error if x is small!!!

① Scalarized : $\min \|Ax - b\| + \gamma \|x\|$

② Tikhonov Regularization : $\min \|Ax - b\|_2^2 + \delta \|x\|_2^2$

4. Optimal Input Design

$$y(t) = \sum_{\tau=0}^{t} h(\tau) \underbrace{u(t-\tau)}_{\text{Input}}, \quad t = 0, 1, \dots, N$$

$$J_{track} = \sum_{t=0}^{N}[y(t) - \underbrace{y_{des}(t)}_{\text{desired trajectory}}]^2 \quad \sim \text{tracking}$$

$$J_{mag} = \sum_{t=0}^{N} u(t)^2 \quad \sim \text{Magnitude}$$

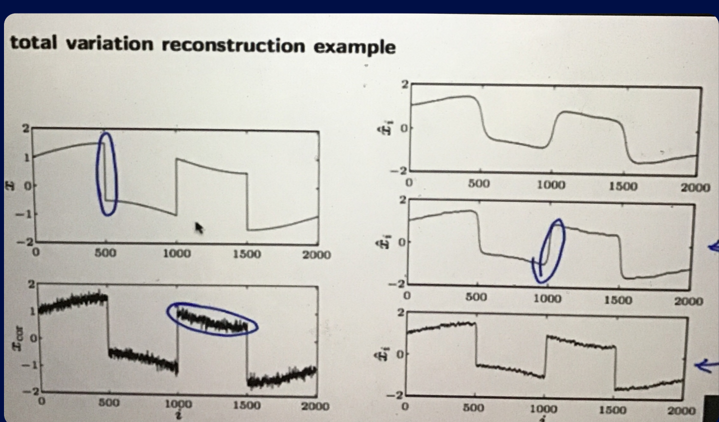
$$J_{der} = \sum_{t=0}^{N-1}[u(t+1) - u(t)]^2 \quad \sim \text{Input Variation}$$

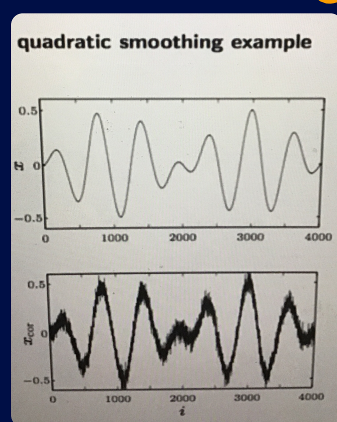
$$\min J_{track} + \delta J_{der} + \eta J_{mag}$$

① Signal Reconstruction

$$\min(\|\hat{x} - \underbrace{x_{cor}}_{\text{Corrupted}}\|_2 , \underbrace{\phi(\hat{x})}_{\text{smoothing Regularizer}})$$

$$\phi_{quad}(\hat{x}) = \sum_{i=1}^{n-1}(\hat{x}_{i+1} - \hat{x}_i)^2 \quad \sim \text{Quadratic Smoothing}$$





Doing Low pass filter make it corrupted

5. Robust Approximation

minimize $\|Ax - b\|$ with $\boxed{\text{uncertain}}$ A

- Stochastic : A is random. $\min E\|Ax - b\|$
- Worst-case : set $\mathcal{A}$ of possible values of A. $\min \sup_{A \in \mathcal{A}} \|Ax - b\|$

α. Stochastic Robust LS

$A = \bar{A} + U$ where U random $E[U] = 0$. $E[U^T U] = P$

$$\min E\|(\bar{A}+U)x - b\|_2^2$$

$$E\|Ax-b\|_2^2 = E\|\bar{A}x - b + Ux\|_2^2$$

$$= \|\bar{A}x - b\|_2^2 + E x^T U^T U x$$

$$= \|\bar{A}x - b\|_2^2 + x^T P x$$

* Quadratic Regularization