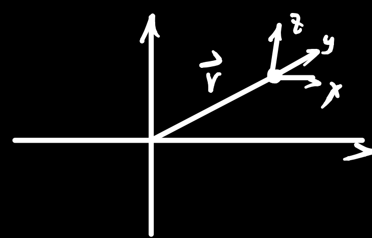
§3. Jacobian & Differential Kinematics

- Linear Velocity

$$\vec{v} = \frac{d\vec{r}}{dt}$$



- Angular Velocity

Velocity of Rotation

$$\vec{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \dot{\phi}\vec{u} + \dot{\theta}\vec{b}$$

Axis vectors

*** Jacobian** WHY: $\dot{\vec{x}} = J\dot{\vec{\theta}}$. $\dot{\vec{\theta}} = (J^T J)^{-1} J^T \dot{\vec{x}}$ if J rank n

which is the inverse kinematics!

α. Forward Kinematics

$$\vec{x} = \vec{f}(\vec{\theta})$$

* x is 3-d in R^2 (x, y, θ) & 6-d in R^3 (pos, ori)

$$\dot{\vec{x}} = \frac{\partial \vec{f}(\vec{\theta})}{\partial \vec{\theta}} \dot{\vec{\theta}} \triangleq J(\vec{\theta})\dot{\vec{\theta}}$$

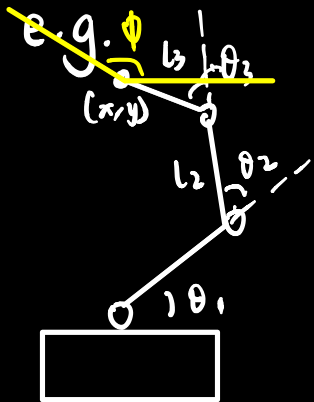
where

$$\frac{\partial \vec{f}(\vec{\theta})}{\partial \vec{\theta}} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \cdots & \frac{\partial f_1}{\partial \theta_n} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \cdots & \frac{\partial f_2}{\partial \theta_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial \theta_1} & \frac{\partial f_m}{\partial \theta_2} & \cdots & \frac{\partial f_m}{\partial \theta_n} \end{bmatrix}$$

$$\frac{\partial x}{\partial t} = J\frac{\partial \theta}{\partial t} \iff \delta\vec{x} = J(\vec{\theta})\,\delta\vec{\theta}$$

J establish the relationship

e.g.



$$\vec{x} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos\theta_{12} + l_3\cos\theta_{123} \\ l_1\sin\theta_1 + l_2\sin\theta_{12} + l_3\sin\theta_{123} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix}$$

$J(\vec{\theta})$ can be easily calculated

- General Method

$$^{0}T_{end} = \begin{bmatrix} R(\vec{\theta}) & \vec{p}(\vec{\theta}) \\ 0_{1\times 3} & 1 \end{bmatrix}$$

Then $\dot{\vec{x}} = \begin{bmatrix} \dfrac{\partial \vec{p}(\vec{\theta})}{\partial \theta} \\ B(\vec{\theta}) \end{bmatrix} \dot{\vec{\theta}}$

* $B(\theta)$ builds $w = B(\theta)\cdot\dot{\vec{\theta}}$

⇒ How to Calculate $B(\vec{\theta})$

$$\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} B_1(\theta), B_2(\theta) \cdots B_n(\theta) \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \vdots \\ \dot\theta_n \end{bmatrix}$$

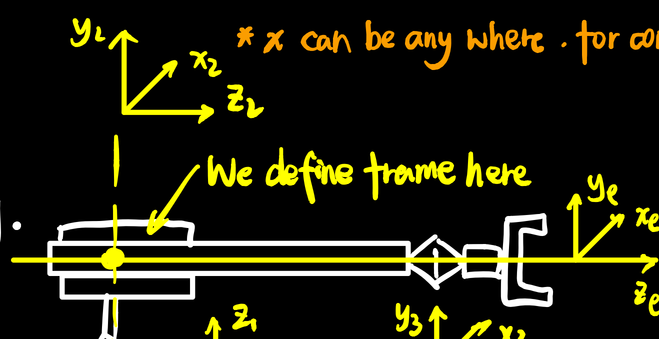
Direction

Numerical

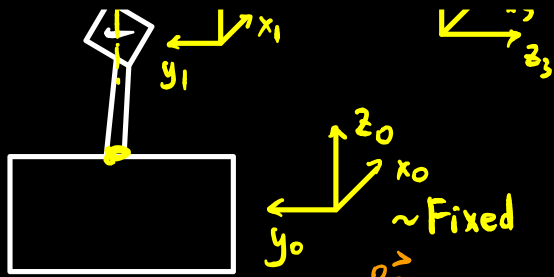
$$\vec{w} = \sum_{i=1}^{n} \xi_i \, {}^{0}\vec{z}_i \dot{\theta}_i \qquad \xi_i = \begin{cases} 0 & \text{prismatic} \\ 1 & \text{revolt} \end{cases}$$

* x can be any where . for convenience here

e.g.



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0



3	0	0	0	θ_3
4	0	0	l_3	0

$$
{}^0T_1 = \begin{array}{cccc} & & {}^0\vec{z}_1 & \\ \left[\begin{matrix} c_1 \\ s_1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} -s_1 \\ c_1 \\ 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}} \atop 0 & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}\right] \end{array}
\qquad
{}^0T_2 = \begin{array}{cccc} & & {}^0\vec{z}_2 & \\ \left[\begin{matrix} c_1 \\ s_1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} & \boxed{\begin{matrix} s_1 \\ -c_1 \\ 0 \end{matrix}} \atop 0 & \left.\begin{matrix} d_2 s_1 \\ -d_2 c_1 \\ 0 \\ 1 \end{matrix}\right] \end{array}
$$

$$
{}^0T_3 = \begin{array}{cccc} & & {}^0\vec{z}_3 & \\ \left[\begin{matrix} c_1 c_3 \\ s_1 c_3 \\ -s_3 \\ 0 \end{matrix}\right. & \begin{matrix} -c_1 s_3 \\ -s_1 s_3 \\ c_3 \\ 0 \end{matrix} & \boxed{\begin{matrix} s_1 \\ -c_1 \\ 0 \end{matrix}} \atop 0 & \left.\begin{matrix} d_2 s_1 \\ -d_2 c_1 \\ 0 \\ 1 \end{matrix}\right] \end{array}
\qquad
{}^0T_4 = \begin{bmatrix} c_1 c_3 & -c_1 s_3 & s_1 & (d_2+l_3)s_1 \\ s_1 c_3 & -s_1 s_3 & -c_1 & -(d_2+l_3)c_1 \\ -s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
J(\theta) = \left[\begin{matrix} (d_2+l_3)c_1 \\ (d_2+l_3)s_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}\right. \quad \begin{matrix} s_1 \\ -c_1 \\ 0 \\ \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \end{matrix} \quad \left.\begin{matrix} 0 \\ 0 \\ 0 \\ s_1 \\ -c_1 \\ 0 \end{matrix}\right]
$$

*Prismatic Joints do nothing to angular

B. for Inverse Kinematics

$\dot{\theta} = J^{-1} \cdot \dot{x}$ if $J(\theta)$ is invertible

*by checking determinant

- How to ensure: avoid the singular workspace e.g. Robot Velocity Control

$$\dot{\theta} = -KJ^{-1}\Delta x \quad \text{where } \Delta x = x - x_d$$

$$J^{-1}\dot{x} = -KJ^{-1}\Delta x \Rightarrow \dot{x} = -k\Delta x.$$

$$\dot{x} = (\dot{x} - \dot{x}_d) = \Delta\dot{x} \Rightarrow \Delta\dot{x} = -k\Delta x.$$

$$\Delta x = Ae^{-kt} \text{ which indicates } \Delta x \to 0$$

- Improvement $\dot{\theta} = -KJ^{T}\Delta x$. ~ Avoiding singularity

- At each joints. we should generate τ & f

$$\vec{f}^{T} \cdot \delta\vec{x} = \vec{\tau}^{T}\delta\vec{\theta} \Rightarrow \vec{\tau} = \vec{J}^{T}(\theta)\vec{f}$$

* Jacobian also build the relationship between the force & time