Def. $f: \mathbb{R}^n \to \mathbb{R}$ is convex. if $\mathrm{dom} f$ is a convex set

& $f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$

* f is concave if $-f$ is convex

e.g. ① Affine functions $ax + b$ on $\mathbb{R}$

② Exponential e^{ax} $\forall a \in \mathbb{R}$

③ Powers x^{α} on $\mathbb{R}_{++}$ for $\alpha \geq 1$ or $\alpha \leq 0$

④ Powers of absolute value $|x|^p$ on $\mathbb{R}$ for $p \geq 1$

⑤ Negative Entropy $x \log x$ on $\mathbb{R}_{++}$

Examples on $\mathbb{R}^n$: • Affine Functions $f(x) = a^T x + b$

• Norms $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$

on $\mathbb{R}^{m \times n}$: • Affine Functions $f(X) = tr(A^T X) + b$

• Spectral Norm $f(X) = \sigma_{max}(X) = \sqrt{\lambda_{max}(X^T X)}$

1. Restriction of Convex Function

$f: \mathbb{R}^n \to \mathbb{R}$ is convex iff. function $g: \mathbb{R} \to \mathbb{R}$

$g(t) = f(x + tv)$ $\mathrm{dom}\, g = \{ t \mid x + tv \in \mathrm{dom} f \}$

is convex for $\forall x \in \mathrm{dom} f$, $v \in \mathbb{R}^n$

e.g. $f: S^n \to \mathbb{R}$ with $f(X) = \log \det(X)$. $\mathrm{dom} f = S^n_{++}$

$g(t) = \log(\det[X + tV])$

$$= \log\left[\det\left(X^{\frac{1}{2}}(I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})X^{\frac{1}{2}}\right)\right]$$

$$= \log\det X + \log\det(I + tX^{-\frac{1}{2}}VX^{-\frac{1}{2}})$$

$$= \log\det X + \sum_{i=1}^{n} \underline{\log(1 + t\lambda_i)} \quad \text{Concave!}$$

2. Extended-value extension

E-v extension $\tilde{f}$ of f is

$$\tilde{f}(x) = f(x), \quad x \in \operatorname{dom} f. \quad \tilde{f}(x) = \infty. \quad x \notin \operatorname{dom} f$$

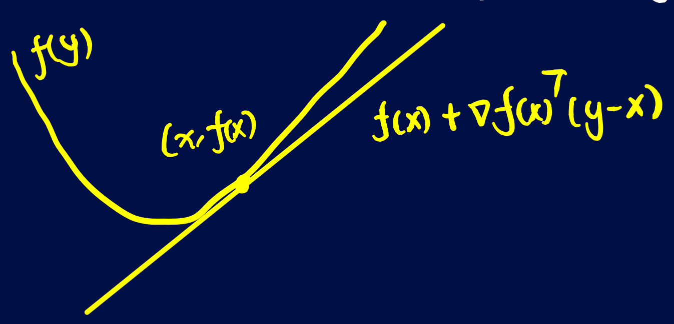
3. First-order Conditions

If f is differentiable. Gradient

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n}\right) \text{ exists at each } x \in \operatorname{dom} f$$

1st order condition: differentiable f with convex dom is Convex iff.

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{* Taylor Expansion}$$



4. Second-order Condition

f is convex iff. $\nabla^2 f(x) \succeq 0$ *Heissian Matrix

e.g. ① Quadratic Func. $f(x) = \frac{1}{2} x^T P x + q^T x + r$

$\nabla^2 f = P$. convex if $P \succeq 0$

② Least-Square $f(x) = \|Ax - b\|_2^2$

$\nabla^2 f(x) = 2A^T A$. convex!

③ Quardratic over Linear $f(x, y) = \frac{x^2}{y}$

$\nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$ convex for $y > 0$

④ log-sum-exp: $f(x) = \log \sum_{k=1}^{n} \exp(x_k)$ convex

$\nabla^2 f(x) = \frac{1}{1^T z} \operatorname{diag}(z) - \frac{1}{(1^T z)^2} z z^T$

easily proved by $v^T \nabla^2 f(x) v \geq 0$ for all v

⑤ Geometric Mean: $f(x) = \left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}$ on R^n_{++} ~ Concave

5. Epigraph & Sublevel set

α-sublevel set of $f: R^n \to R$ *sublevel sets of convex sets are convex

$C_\alpha = \{x \in \operatorname{dom} f \mid f(x) \leq \alpha\}$

epigraph of $f: R^n \to R$:

$$\text{epi } f = \{(x,t) \in R^{n+1} \mid x \in \text{dom} f,\ f(x) \le t\}$$

※: Function f is convex. iff. epi f is convex

6. Jensen's Inequality

If f is convex. $f(E\,z) \le E\,f(z)$ * Expectation

7. Perservate Convex functions

- Nonnegative Multiples
  Sum $f(x) = -\sum \log(b_i - a_i^T x)$
  Composition with affine function $\|Ax+b\|$

- Pointwise Maximum / Supremum

  If $f_1, f_2 \cdots f_m$ are convex. $f(x) = \max\{f_1, f_2, \cdots f_m\}$

- Composition with scalar func.

  composition of $g: R^n \to R$. $h: R \to R$. * ext. value ext.

  $f = h(g(x))$. f is convex $\begin{cases} g \text{ convex } h \text{ convex } \tilde{h}\uparrow \\ g \text{ concave } h \text{ convex } \tilde{h}\downarrow \end{cases}$

  * easily proved by calculating f''

- Vector Composition

  Composition of $g: R^n \to R^k$ & $h: R^k \to R$.

  $f(x) = h(g(x)) = h(g_1, g_2, \cdots g_k)$

  f is convex if $\begin{cases} g \text{ convex. } h \text{ convex } \tilde{h}\uparrow \text{ in each argument} \\ g \text{ concave } h \text{ convex } \tilde{h}\downarrow \cdots \end{cases}$

- Minimization

  $f(x,y)$ is convex in (x,y) & C – convex set

  $$g(x) = \inf_{y \in C} f(x,y) \text{ is convex}$$

- Perspective Mapping

  $g(x,t) = t f\left(\frac{x}{t}\right)$ $\quad \mathrm{dom}\, g = \{(x,t) \mid \frac{x}{t} \in \mathrm{dom} f,\ t > 0\}$

- Conjugate Function

  $$f^*(y) = \sup_{x \in \mathrm{dom} f} \left(y^T x - f(x)\right)$$

8. Quasiconvex

$f: R^n \to R$ is quasiconvex if $\mathrm{dom} f$ convex

& $S_\alpha = \{x \in \mathrm{dom} f \mid f(x) \le \alpha\}$ convex $\forall \alpha$.

e.g. $\sqrt{|x|}$ ~ Quasiconvex

ceil(x) . $\log x$ ~ Quasilinear ($\log x$ on R_{++})

$f(x_1, x_2) = x_1 x_2$ ~ Quasiconcave

linear-fractional $\frac{a^T x + b}{c^T x + d}$ ~ Quasilinear $\mathrm{dom} f = \{x \mid c^T x + d > 0\}$

- Properties

$0 \le \theta \le 1 \Rightarrow f(\theta x + (1-\theta) y) \le \max\{f(x), f(y)\}$

$f(y) \le f(x) \Rightarrow \nabla f(x)^T (y-x) \le 0$

9. Log-concave

positive func. f is log-concave if $\log f$ is concave

$$f(\theta x + (1-\theta)y) \ge f(x)^{\theta} f(y)^{1-\theta} \quad \text{for } 0 \le \theta \le 1$$

- Properties

f is log-concave iff.

$$f(x) \nabla^2 f(x) \preceq \nabla f(x) \nabla f(x)^T$$

$\star\ \nabla^2 f(x) \preceq \dfrac{\nabla f(x) \nabla f(x)^T}{f(x)}$

① product of log-concave is log-concave

② Integration : If $f : R^n \times R^m \to R$ is log-concave

$g(x) = \int f(x,y)\,dy$ is log-concave

10. Generalization

$f : R^n \to R^m$ is K-convex if dom f convex

& $f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta) f(y)$