1. Lagrangian

Standard form problem

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{s.t. } & f_i(x) \le 0,\ i=1,\dots,m \\ & h_i(x) = 0,\ i=1,\dots,p \end{aligned}$$

Def: Lograngian $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ with $\operatorname{dom} L = D \times \mathbb{R}^m \times \mathbb{R}^p$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

Lagrange Multipliers

Lagrange Dual Func. $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in D} L(x, \lambda, \nu) \\ &= \inf_{x \in D} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \end{aligned}$$

* g is concave ! ~ inf. of affine function

- lower bound property : $\lambda \ge 0 \Rightarrow g(\lambda, \nu) \le p^*$

Pf. Suppose $\tilde{x}$ is feasible & $\lambda \ge 0$

$$f_0(\tilde{x}) \ge L(\tilde{x}, \lambda, \nu) \ge \inf_{x \in D} L(x, \lambda, \nu) = g(\lambda, \nu)$$

★ It's providing a lower bound for the problem!

e.g. For LP: minimize $c^T x$

s.t. $Ax = b,\ x \succeq 0$

$$L(x,\lambda,v) = -b^T v + (c + A^T v - \lambda)^T x$$

$$g(\lambda, v) = \inf_x L(x,\lambda,v) = \begin{cases} -b^T v & A^T v - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

※ Dual Problem

Maximize $g(\lambda, v)$

s.t. $\lambda \succeq 0$

~ Finding the best lower bound

α. Weak duality : $d^* \le p^*$

- Always holds

WHY?

β. Strong duality : $d^* = p^*$

- holds for convex problems (usually)

Slater's condition

$\exists \tilde{x}.\ A\tilde{x} < b \Rightarrow p^* = d^*$

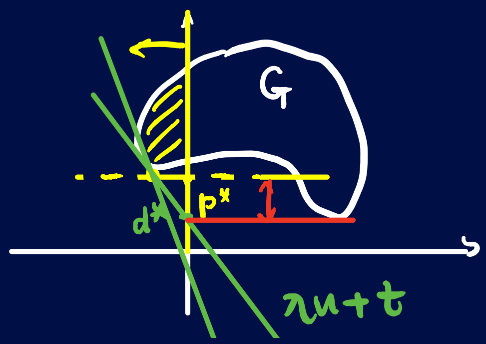
- constraints qualification : guarantee strong duality

e.g. $\min c^T x$ s.t. $Ax \le b$ $\overset{\text{Dual}}{\Longleftrightarrow}$ $\max -b^T \lambda$ s.t. $A^T \lambda + c = 0$, $\lambda \succeq 0$

2. Geometry Interpretation

Constraint $f_1(x) \le 0$

$g(\lambda) = \inf(t + \lambda u)$



3. Complementary Slackness

Assume Strong duality holds.

x^* is primal optimal & (λ^*, v^*) is dual optimal

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, v^*) \\
&= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} v_i^* h_i(x) \right) \\
&= f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} v_i^* h_i(x^*) \\
&= f_0(x^*)
\end{aligned}
$$

* $\lambda_i^* f_i(x^*) = 0$ ~ Complementary Slackness

$$
\begin{cases}
\lambda_i^* > 0 & \text{then } f_i(x^*) = 0 \\
\lambda_i^* = 0 & \text{for } f_i(x^*) < 0
\end{cases}
$$

$\Rightarrow$ KKT Conditions

1. primal constraints : $f_i(x) \le 0$ & $h_i(x) = 0$
2. duality constraints : $\lambda \ge 0$
3. complementary slackness : $\lambda_i f_i(x) = 0$
4. $\nabla_x L = \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} v_i \nabla h_i(x) = 0$

$\Longleftrightarrow$ Optimal in convex optimization

4. Perturbation & Sensitivity Analysis

Originally : $\min f_0(x)$ s.t. $f_i(x) \leq 0$, $h_i(x) = 0$ $\iff$ $\max g(\lambda, \nu)$ s.t. $\lambda \succeq 0$

$$\begin{aligned} &\min f_0(x) & & & \max\ & g(\lambda,\nu) \\ &\text{s.t. } f_i(x) \leq 0 & &\iff & \text{s.t. }\ & \lambda \succeq 0 \\ &\phantom{\text{s.t. }} h_i(x) = 0 & & & & \end{aligned}$$

Relaxation

Perturbed :

$$\begin{aligned} &\min f_0(x) & & & \max\ & g(\lambda,\nu) - u^T\lambda - v^T\nu \\ &\text{s.t. } f_i(x) \leq u_i & &\iff & \text{s.t. }\ & \lambda \succeq 0 \\ &\phantom{\text{s.t. }} h_i(x) = v_i & & & & \end{aligned}$$

$$* \; p^*(u,v) \geq g(\lambda^*, \nu^*) - u^T\lambda^* - v^T\nu^*$$

$$= p^*(0,0) - u^T\lambda^* - v^T\nu^*$$

Sensitivity Interpretation

λ_i^* ↑ p ↑↑↑ if we tighten constraint i

↓ p ↓ if we loosen constraint i

ν_i^* the same to analyze by ± & largeness

5. Duality & Reformulation

$$\begin{array}{ccc} \mathcal{P} & \longleftrightarrow & \mathcal{P}' \\ \updownarrow & & \updownarrow \\ \mathcal{D} & \cancel{\longleftrightarrow} & \mathcal{D}' \end{array}$$

e.g. $\min f_0(Ax+b)$

Dual func. useless

$$g = \inf_x L(x) = \inf_x f_0(Ax+b) = p^*$$

Reformulated:

$$\begin{aligned} &\min f_0(y) & & & \max\ & b^T\nu - f_0^*(\nu) \\ &\text{s.t. } Ax+b-y=0 & &\xLeftrightarrow{D} & \text{s.t. }\ & A^T\nu = 0 \end{aligned}$$

$$g(\nu) = \inf_{x,y}\left(f_0(y) - \nu^T y + \nu^T A x + b^T\nu\right)$$

$$= \begin{cases} -f_0^*(v) + b^T v & A^T v = 0 \\ -\infty & \text{otherwise} \end{cases}$$

6. Generalized Inequalities

$$\min f_0(x)$$

$$\text{s.t. } f_i(x) \preceq_{K_i} 0$$

$$h_i(x) = 0 \quad \Rightarrow \quad L(x, \lambda_1 \cdots \lambda_m, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} v_i^T h_i(x)$$

e.g. SDP: $\min c^T x$

$$\text{s.t. } x_1 F_1 + \cdots + x_n F_n \preceq G$$

$$L(x, Z) = c^T x + \mathrm{tr}\left(Z(x_1 F_1 + \cdots + x_n F_n - G)\right)$$

$$g(Z) = \inf_x L(x, Z) = \begin{cases} -\mathrm{tr}(GZ) & . \ \mathrm{tr}(F_i Z) + c_i = 0 \\ -\infty & \text{otherwise} \end{cases}$$