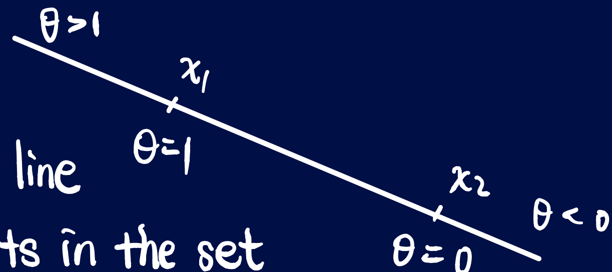
1. Affine Set

line through x_1 & x_2 : $x = \theta x_1 + (1-\theta) x_2,\ \theta \in R$

Def.

Affine set : contains the line through any 2 distinct points in the set



e.g. $\{x \mid Ax = b\}$

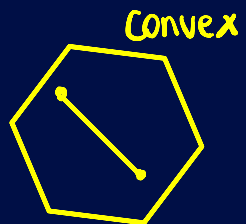
where $\begin{cases} Ax_1 = b \\ Ax_2 = b \end{cases}$ $\quad A\theta x_1 + A(1-\theta)x_2 = (\theta + 1 - \theta) b = b$ ✓ In the set!

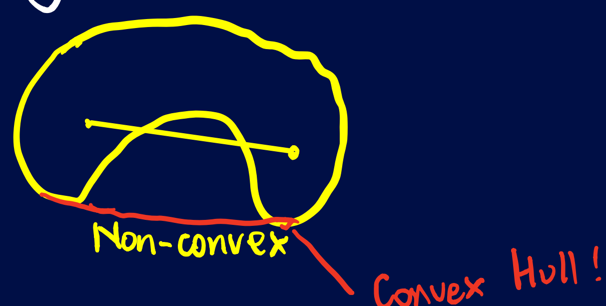
2. Convex set

line segment through x_1 & x_2 : $x = \theta x_1 + (1-\theta) x_2,\ \theta \in [0,1]$

Convex set : contains line segment through 2 distinct points

e.g.





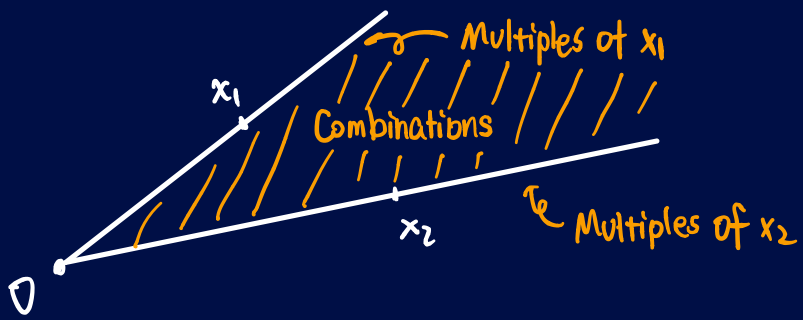
3. Convex combination

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \text{ with } \sum_{i=1}^{k} \theta_i = 1 .\ \theta_i \geq 0$$

4. Convex hull : set of all convex combinations of points in S

5. Convex Cone

~Conic combination: $x = \theta_1 x_1 + \theta_2 x_2$ where $\theta_i \geq 0$



6. Hyperplanes & Halfspaces

Hyperplane (* Affine & Convex): set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

Halfspace (* Convex): set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$

7. Euclidean balls & Ellipsoids

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$ * Constrained Version

$$E = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$, P: symmetric positive def.

$$= \{x_c + Au \mid \|u\|_2 \leq 1\}$$ * $P = r^2 I$. Euclidean Ball!

8. Norm balls & Cones

- Norm: a function $\|\cdot\|$ satisfy that

$$\begin{cases} \|x\| \geq 0 \text{ \& } \|x\| = 0 \text{ iff } x = 0 \\ \|tx\| = |t|\,\|x\| \text{ for } t \in R \\ \|x+y\| \leq \|x\| + \|y\| \end{cases}$$

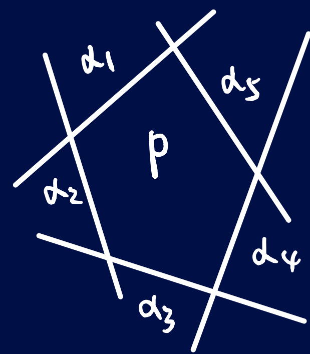
- Norm ball with center x_c & radius r : $\{x \mid \|x - x_c\| \leq r\}$
- Norm Cone : $\{(x,t) \mid \|x\| \leq t\}$

* Both norm ball & cone are convex!

9. Polyhedra

elementwise less than for vectors

Solution set of $Ax \preceq b$, $Cx = d$



~ Intersection of half spaces & hyperplanes

10. Positive semi-definite Cone

- S^n ~ set of symmetric $n \times n$ matrices
- $S_+^n = \{X \in S^n \mid X \succeq 0\}$: positive semidefinite * $z^T S_+^n z \geq 0$
- $S_{++}^n = \{X \in S^n \mid X \succ 0\}$: positive definite

e.g. $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2$

11. Operations that maintain convexity

α. Intersection

Convex sets S_1 & S_2. Now $x \in S_1 \cap S_2 = S$

$\forall x_1, x_2 \in S$. $x_1 \in S_1$, $x_2 \in S_1$

Thus $\theta x_1 + (1-\theta) x_2 \in S_1$

Also. $x_1 \in S_2$, $x_2 \in S_2$ $\Rightarrow$ $\theta x_1 + (1-\theta) x_2 \in S_2$

Thus $\theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2 = S$

β. Affine Functions

suppose $f: R^n \to R^m$ is affine ($f(x) = Ax + b$)

Then $S \subseteq R^n$ convex $\Rightarrow$ $f(S) \subseteq R^m$ convex

Also $C \subseteq R^m$ convex $\Rightarrow$ $f^{-1}(C) \subseteq R^n$ convex * $f^{-1}(C) = \{x \mid f(x) \in C\}$

* x_1 x_2 $\in S$. Then $\theta x_1 + (1-\theta) x_2 \in S$

$f(x_1) = Ax_1 + b$ & $f(x_2) = Ax_2 + b$

Then $\theta f(x_1) + (1-\theta) f(x_2) = \theta A x_1 + (1-\theta) x_2 + b$

$= A[\theta x_1 + (1-\theta) x_2] + b$.

Since $\theta x_1 + (1-\theta) x_2 \in S$. regard it as x_0

Then we got $Ax_0 + b \in f(S)$. Convex !

e.g. $\{x \mid x_1A_1 + \cdots + x_mA_m \preceq B\}$

Let affine func. $f(x) = B - \sum_{i=1}^{m} x_iA_i$

Then $f^{-1}(S_+^n) = \{x \mid x_1A_1 + \cdots + x_mA_m \preceq B\}$

γ. Perspective & Linear-fractional Function

$P: R^{n+1} \to R^n$. $P(x, t) = \frac{x}{t}$. $\mathrm{dom}\, P = \{(x,t) \mid t > 0\}$

Linear-fractional func.

$f: R^n \to R^m$. $f(x) = \frac{Ax+b}{c^Tx+d}$ $\mathrm{dom}\, f = \{x \mid c^Tx + d > 0\}$

e.g. $f(x) = \frac{1}{x_1 + x_2 + 1} x$

ξ. Generalized Inequalities

Def. a convex cone $K \subseteq R^n$ is proper cone if

$\begin{cases} K \text{ is closed (Contains boundary)} \\ K \text{ is solid (has nonempty interior)} \\ K \text{ is pointed (contains no line)} \end{cases}$

$\Rightarrow$ Generalized inequality

$x \preceq_K y \iff y - x \in K$

* interior

$x \prec_K y \iff y - x \in \mathrm{int}\, K$

e.g. $K = \mathbb{R}^n_+$. $x \preceq_{\mathbb{R}^n_+} y \iff x_i \le y_i$

$K = S^n_+$ $X \preceq_{S^n_+} Y \iff Y - X$ positive semidefinite

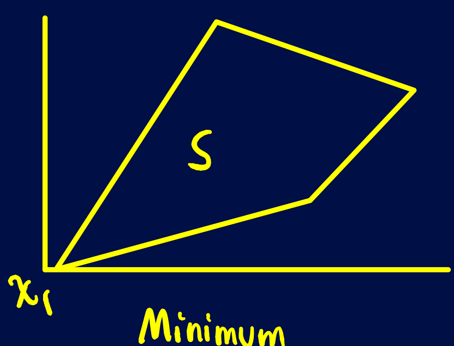
* by default $\preceq$ means $\preceq_{S^n_+}$

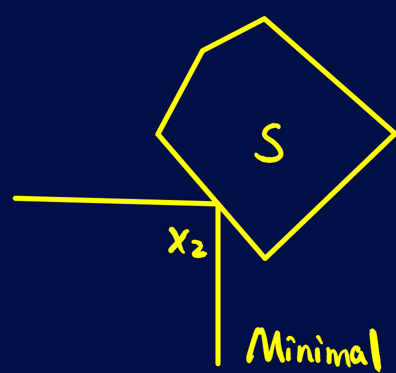
Properties ① $x \preceq_K y$, $u \preceq_K v \Rightarrow x + u \preceq_K y + v$

ω. Minimum & Minimal Elements

$\begin{cases} \text{Minimum element of } S \text{ if } y \in S \Rightarrow x \preceq_K y \text{ for } x \in S \\ \text{Minimal element of } S \text{ if } y \in S,\ y \preceq_K x \Rightarrow y = x,\ x \in S \end{cases}$

e.g. $(K = \mathbb{R}^2_+)$





τ. Seperating hyperplane Theorem

if C, D are disjoint convex sets. $\exists\, a \neq 0$, b.

s.t. $a^T x \le b$ for $x \in C$. $a^T x \ge b$ for $x \in D$

- Supporting Hyperplane Theorem

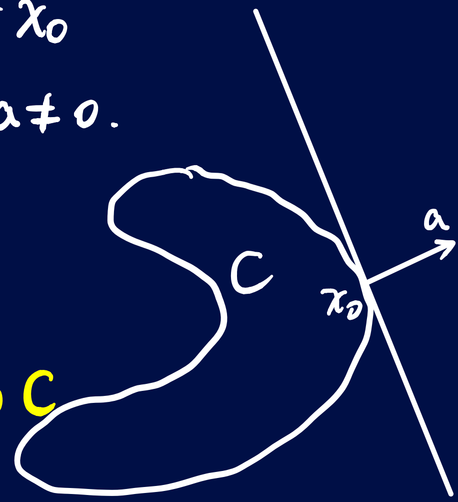
S.H. to set C at boundary point x_0

$$\{x \mid a^T x = a^T x_0\} \text{ where } a \neq 0.$$

& $a^T x \le a^T x_0$ for all $x \in C$

* if C is convex

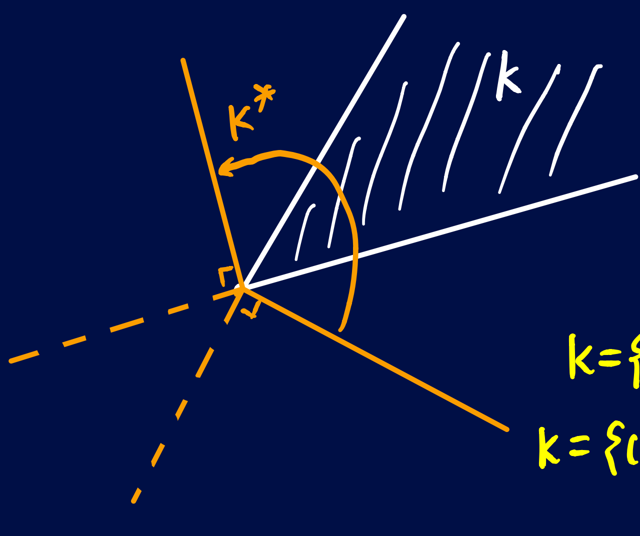
$\exists$ S.H for every boundary point to C



12. More

- Dual cones

$$K^* = \{y \mid y^T x \ge 0 \text{ for } x \in K\}$$



e.g.

$K = \mathbb{R}^n_+$. $K^* = \mathbb{R}^n_+$

$K = S^n_+$. $K^* = S^n_+$

$K = \{(x,t) \mid \|x\|_2 \le t\}$. $K^* = K$

$K = \{(x,t) \mid \|x\|_1 \le t\}$. $K^* = \{(x,t) \mid \|x\|_\infty \le t\}$

※ Dual cones of proper cones are proper

$\Rightarrow$ $y \succeq_{K^*} 0 \iff y^T x \ge 0$ for all $x \succeq_K 0$