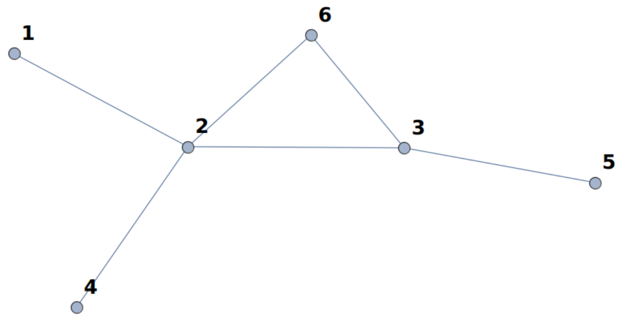
Practice questions.

1. **Max-cut** Consider the following graph.



   (a) Find the exact max-cut for this graph.

   (b) Goemans–Williamson algorithm gives an approximate solution to the max-cut problem with an approximation ratio of at least 0.868. Use the Goemans–Williamson algorithm to find an approximate solution to this graph.

   (c) Consider QAOA with depth $p = 1$ ansatz state of the form $|\gamma, \beta\rangle = U_B(\beta)U_C(\gamma)|s\rangle$. Plot the expectation value of the cost C

$$F_1(\gamma, \beta) = \langle \gamma, \beta | C | \gamma, \beta \rangle \tag{1}$$

   as a function of γ and β. Find the angles that maximises F_1. Find the variance of the cost function $\langle C^2 \rangle - \langle C \rangle^2$ at this point. Find the probability of finding the optimal cut at this point.

   (d) Find the approximate QAOA solution to the last exercise with depth $p = 2$.

   (e) When the depth is large, it becomes hard to evaluate the cost classically. Instead, we can prepare the quantum state $|\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle$ and evaluate the cost on a quantum computer. Simulate this with depth $p = 2$ to find the approximate QAOA solution..

   (f) Repeat the last problem with $p = 3$ and $p = 4$.

2. **Level-1 QAOA expectation value.** Our goal is to derive an analytic expression for the expectation value for $p = 1$ in the Max-Cut problem. Consider the state

$$|\gamma, \beta\rangle = U_B(\beta)U_C(\gamma)|s\rangle \,, \tag{2}$$

   where $|s\rangle = |+, +, \ldots, +\rangle$ is state where all qubits are initialised to the plus state and $U_B(\beta) = e^{-i\beta B}$ and $U_C(\gamma) = e^{-i\gamma C}$. For an edge (u, v), we want derive an analytic expression for $\langle \gamma, \beta | C_{uv} | \gamma, \beta \rangle$, where $C_{uv} = \frac{1}{2}(1 - Z_u Z_v)$.

   (a) Show that:

$$e^{i\beta X_u} Z_u e^{-i\beta X_u} = e^{2i\beta X_u} Z_u \tag{3}$$

   (b) Show that:

$$\begin{aligned} e^{i\beta B} Z_u Z_v e^{-i\beta B} &= e^{2i\beta X_u} Z_u e^{2i\beta X_v} Z_v & (4) \\ &= \cos^2(2\beta) Z_u Z_v + \cos(2\beta)\sin(2\beta)(Z_u Y_v + Y_u Z_v) + \sin^2(2\beta) Y_u Y_v \,. & (5) \end{aligned}$$

   (c) To evaluate $\langle \gamma, \beta | C_{uv} | \gamma, \beta \rangle$, we need to evaluate the four terms $\langle s | e^{i\gamma C} Z_u Z_v e^{-i\gamma C} | s \rangle$, $\langle s | e^{i\gamma C} Z_u Y_v e^{-i\gamma C} | s \rangle$, $\langle s | e^{i\gamma C} Y_u Z_v e^{-i\gamma C} | s \rangle$ and $\langle s | e^{i\gamma C} Y_u Y_v e^{-i\gamma C} | s \rangle$.

   Show that:

   - $\langle s | e^{i\gamma C} Z_u Z_v e^{-i\gamma C} | s \rangle = 0$

- $\langle s| e^{i\gamma C} Z_u Y_v e^{-i\gamma C} |s\rangle = -\sin\gamma \cos^d \gamma$
- $\langle s| e^{i\gamma C} Y_u Z_v e^{-i\gamma C} |s\rangle = -\sin\gamma \cos^e \gamma$
- $\langle s| e^{i\gamma C} Y_u Y_v e^{-i\gamma C} |s\rangle = (\cos\gamma)^{d+e-2f}(1 - \cos^f 2\gamma)/2$,

where d and e are the number of neighbours for nodes u and v minus 1. f is the number of triangles (nodes that are connected to both u and v).

(d) Put everything together to get an analytic expression for $\langle\gamma, \beta| C_{uv} |\gamma, \beta\rangle$. Plot a contour plot of this expectation value as a function of β and γ for problem 1 and compare the results with question 1.