QAOA on MaxCut Practice Problems

March 21, 2023

Question 1. Consider the following graph:

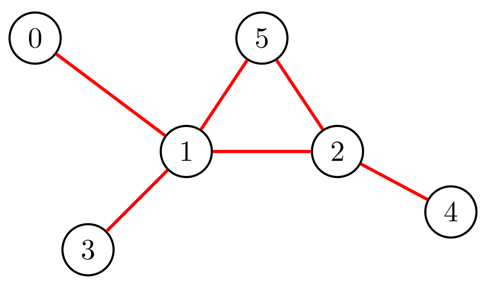

Figure 1: Graph 1

(a) Find the exact max-cut for this graph.

Solution. The exact max-cut for this graph is 5. □

(b) The Goemans-Williamson algorithm gives an approximate solution to the max-cut problem with an approximation ratio of at lease 0.868. Use the Goemans-Williamson algorithm to find an approximate solution to this graph.

Solution. The following code from this video was implemented.

```
In [11]: import cvxpy as cp
         import numpy as np
         from scipy import linalg
```

```
In [4]:  # edges of given graph
         edges = [(0,1),
                  (1,2),
                  (1,3),
                  (1,5),
                  (2,4),
                  (2,5)]

         # creates 6x6 symmetric matrix optimisation variable
         X = cp.Variable((6,6), symmetric=True)

         # creates constraints on X (positive semidefinite & symmetric)
         constraints = [X>>0]
         constraints += [
             X[i,i] == 1 for i in range (6)
         ]

         # algorithm:
         objective = sum(0.5*(1-X[i,j]) for (i,j) in edges)
```

```
In [6]:  prob = cp.Problem(cp.Maximize(objective), constraints)
         prob.solve()

Out[6]:  5.250000797097679
```

Listing 1: Python example

```
import cvxpy as cp
from scipy import linalg

# edges of given graph
edges = [(0,1), (1,2), (1,3), (1,5), (2,4), (2,5)]
# creates 6x6 symmetric matrix optimisation variable
X = cp.Variable((6,6), symmetric=True)
# creates constraints on X (positive semidefinite
# & symmetric)
```

```
constraints = [X>>0]
constraints += [X[i,i] == 1 for i in range (6)]

# algorithm:
objective = sum(0.5*(1-X[i,j]) for (i,j) in edges)
```

The cost was found to be 5.25, yielding a max-cut of 5 for the graph. □

Question 2. Our goal is to derive an analytic expression for the expectation value for $p = 1$ in the Max-Cut problem. Consider the state

$$|\gamma, \beta\rangle = U_B(\beta)U_C(\gamma)|s\rangle,$$

where $|s\rangle = |+, +, \ldots, +\rangle$ is the state where all qubits are initialised to the plus state and $U_B(\beta) = e^{-i\beta B}$ and $U_C(\gamma) = e^{-i\gamma C}$. For an edge (u, v), we want to derive ana analytic expression for $\langle\gamma, \beta|C_{uv}|\gamma, \beta\rangle$, where $C_{uv} = \frac{1}{2}(1 - Z_u Z_v)$.

(a) Show that:

$$e^{i\beta X_u} Z_u e^{-i\beta X_u} = e^{2i\beta X_u} Z_u.$$

Solution. For matrices A, B, C, D, the mixed-product property of the tensor product states that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

X_u and Z_u are tensor products of the identity and σ_x and σ_z respectively where σ_x and σ_z are in the u^{th} position of the product. This means that the σ_{i_u} follows the commutator relationships. We get that

$$\begin{aligned}
e^{i(\beta)X_u} Z_u e^{-i(\beta)X_u} &= (\cos(\beta) + iX_u \sin(\beta))\, Z_u(\cos(\beta) - iX_u \sin(\beta)) \\
&= (\cos(\beta) + iX_u \sin(\beta))(Z_u \cos(\beta) - iZ_u X_u \sin(\beta)) \\
&= \cos^2(\beta)Z_u - i\cos\beta \sin(\beta)Z_u X_u + i\cos(\beta)\sin(\beta)X_u Z_u \\
&\qquad + \sin^2(\beta)X_u Z_u X_u \\
&= \cos^2(\beta)Z_u + \cos(\beta)\sin(\beta)Y_u + \cos(\beta)\sin(\beta)Y_u - \sin^2(\beta)Z_u \\
&= \left(\cos^2(\beta) - \sin^2(\beta)\right) Z_u + \sin(2\beta)Y_u \\
&= \cos(2\beta)Z_u + \sin(2\beta)Y_u \\
&= \cos(2\beta)Z_u + i\sin(2\beta)X_u Z_u \\
&= (\cos 2\beta + iX_u \sin(2\beta))\, Z_u \\
&= e^{2i\beta X_u}
\end{aligned}$$

□

(b) Show that:

$$\begin{aligned}
e^{i\beta B} Z_u Z_v e^{-i\beta B} &= e^{2i\beta X_u} Z_u e^{2i\beta X_v} Z_v \\
&= \cos^2(2\beta)Z_u Z_v + \cos(2\beta)\sin(2\beta)(Z_u Y_v + Y_u Z_v) + \sin^2(2\beta)Y_u Y_v
\end{aligned}$$

Solution. First, consider Z_1 and Z_2 acting on two qubits. Then

$$\begin{aligned}
(Z_1 \otimes \mathbb{1})(\mathbb{1} \otimes Z_2) &= Z_1\mathbb{1} \otimes \mathbb{1}Z_2 \\
&= \mathbb{1}Z_1 \otimes Z_2\mathbb{1} \\
&= (\mathbb{1} \otimes Z_2)(Z_1 \otimes \mathbb{1}).
\end{aligned}$$

This means that Z_i commutes with Z_j for $i \neq j$. Fix two nodes, u and v. We get that

$$\begin{aligned}
e^{i\beta B} Z_u Z_v e^{-i\beta B} &= \prod_n e^{i\beta X_n} Z_u Z_v \prod_m e^{-i\beta X_m} \\
&= e^{i\beta X_u} Z_u e^{-i\beta X_u} e^{i\beta X_v} Z_v e^{-i\beta X_v} \\
&= e^{2i\beta X_u} Z_u e^{2i\beta X_v} Z_v.
\end{aligned}$$

For the second part, we get

$$\begin{aligned}
e^{2i\beta X_u} Z_u e^{2i\beta X_v} Z_v &= (\cos(2\beta) + iX_u \sin(2\beta))\, Z_u (\cos(2\beta) + iX_v \sin(2\beta))\, Z_v \\
&= (\cos(2\beta) + iX_u \sin(2\beta))\, Z_u (\cos(2\beta) Z_v + i\sin(2\beta) X_v Z_v) \\
&= (\cos(2\beta) + iX_u \sin(2\beta))(\cos(2\beta) Z_u Z_v + i\sin(2\beta) Z_u X_v Z_v) \\
&= (\cos(2\beta) + iX_u \sin(2\beta))(\cos(2\beta) Z_u Z_v + \sin(2\beta) Z_u Y_v) \\
&= \cos^2(2\beta) Z_u Z_v + \cos(2\beta)\sin(2\beta) Z_u Y_v \\
&\qquad + i\sin(2\beta)\cos(2\beta) X_u Z_u Z_v + i\sin^2(2\beta) X_u Z_u Y_v \\
&= \cos^2(2\beta) Z_u Z_v + \cos(2\beta)\sin(2\beta) Z_u Y_v \\
&\qquad + \sin(2\beta)\cos(2\beta) Y_u Z_v + \sin^2(2\beta) Y_u Y_v \\
&= \cos^2(2\beta) Z_u Z_v + \cos(2\beta)\sin(2\beta)(Z_u Y_v + Y_u Z_v) + \sin^2(2\beta) Y_u Y_v.
\end{aligned}$$

□

(c) To evaluate $\langle \gamma, \beta | C_{uv} | \gamma, \beta \rangle$, we need to evaluate the four terms $\left\langle s \left| e^{i\gamma C} Z_u Z_v e^{-i\gamma C} \right| s \right\rangle$, $\left\langle s \left| e^{i\gamma C} Z_u Y_v e^{-i\gamma C} \right| s \right\rangle$, $\left\langle s \left| e^{i\gamma C} Y_u Z_v e^{-i\gamma C} \right| s \right\rangle$ and $\left\langle s \left| e^{i\gamma C} Y_u Y_v e^{-i\gamma C} \right| s \right\rangle$.

Show that:

- $\left\langle s \left| e^{i\gamma C} Z_u Z_v e^{-i\gamma C} \right| s \right\rangle = 0$
- $\left\langle s \left| e^{i\gamma C} Z_u Y_v e^{-i\gamma C} \right| s \right\rangle = -\sin\gamma \cos^e \gamma$
- $\left\langle s \left| e^{i\gamma C} Y_u Z_v e^{-i\gamma C} \right| s \right\rangle = -\sin\gamma \cos^d \gamma$
- $\left\langle s \left| e^{i\gamma C} Y_u Y_v e^{-i\gamma C} \right| s \right\rangle = (\cos\gamma)^{d+e-2f} \left(1 - \cos^f 2\gamma\right)/2,$

where d and e are the number of neighbours for nodes u and v minus 1. f is the number of triangles (nodes that are connected to both u and v).

Solution. Fix two nodes, u and v. To show the first equation, we start with

$$\begin{aligned}
\left\langle s \left| e^{i\gamma C} Z_u Z_v e^{-i\gamma C} \right| s \right\rangle &= \left\langle s \left| \prod_\alpha e^{i\gamma C_\alpha} Z_u Z_v \prod_\alpha e^{-i\gamma C_\alpha} \right| s \right\rangle \\
&= \left\langle s \left| \prod_{\langle jk \rangle} e^{i\gamma C_{\langle jk \rangle}} Z_u Z_v \prod_{\langle jk \rangle} e^{-i\gamma C_{\langle jk \rangle}} \right| s \right\rangle \\
&= \left\langle s \left| \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2}(1 - Z_j Z_k)} Z_u Z_v \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2}(1 - Z_j Z_k)} \right| s \right\rangle \\
&= \left\langle s \left| \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2}} e^{-\frac{i\gamma}{2} Z_j Z_k} Z_u Z_v \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2}} e^{\frac{i\gamma}{2} Z_j Z_k} \right| s \right\rangle
\end{aligned}$$

Since the expectation value remains invariant under a global phase shift, we can remove all factors of $e^{\frac{i\gamma}{2}}$ and $e^{-\frac{i\gamma}{2}}$. Next, since any terms not involving u or v commute with Z_u and Z_v, we can move them around so that they cancel with their conjugates. We then get that

$$\begin{aligned}
&= \left\langle s \left| e^{-\frac{i\gamma}{2} Z_u Z_v} Z_u Z_v e^{\frac{i\gamma}{2} Z_u Z_v} \right| s \right\rangle \\
&= \left\langle s \left| \left(\cos\left(\frac{\gamma}{2}\right) - i\sin\left(\frac{\gamma}{2}\right) Z_u Z_v\right) Z_u Z_v \left(\cos\left(\frac{\gamma}{2}\right) + i\sin\left(\frac{\gamma}{2}\right) Z_u Z_v\right) \right| s \right\rangle \\
&= \left\langle s \left| \left(\cos\left(\frac{\gamma}{2}\right) - i\sin\left(\frac{\gamma}{2}\right) Z_u Z_v\right) \left(\cos\left(\frac{\gamma}{2}\right) Z_u Z_v + i\sin\left(\frac{\gamma}{2}\right) (Z_u Z_v)^2\right) \right| s \right\rangle \\
&= \left\langle s \left| \left[\cos^2\left(\frac{\gamma}{2}\right) + i\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) Z_u Z_v - i\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) Z_u Z_v + \sin^2\left(\frac{\gamma}{2}\right) (Z_u Z_v)^2\right] Z_u Z_v \right| s \right\rangle
\end{aligned}$$

Since Z_u and Z_v commute, $(Z_u Z_v)^2 = Z_u^2 Z_v^2 = 1$. We get that

$$\langle s | Z_u Z_v | s \rangle = 0$$

since Z gates collapses $|+\rangle$.

For the second equation, we get

$$\left\langle s \left| e^{i\gamma C} Z_u Y_v e^{-i\gamma C} \right| s \right\rangle = \left\langle s \left| \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2}} e^{-\frac{i\gamma}{2} Z_j Z_k} Z_u Y_v \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2}} e^{\frac{i\gamma}{2} Z_j Z_k} \right| s \right\rangle$$

Again, we can ignore the global phase shift. Then

$$= \left\langle s \left| \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2} Z_j Z_k} Z_u Y_v \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2} Z_j Z_k} \right| s \right\rangle$$

Now, note that Y_v anticommutes with any term containing Z_v in the product on the right. Elements containin Z_v are nodes in the neighbourhood of v, denoted $N(v)$. Consider a product $Y_v(\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2} Z_v Z_w)$ for some w. We have that $Y_v(\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2} Z_v Z_w) = (\cos\frac{\gamma}{2} Y_v + i\sin\frac{\gamma}{2} y_v Z_v Z_w) = (\cos\frac{\gamma}{2} Y_v - i\sin\frac{\gamma}{2} Z_v Y_v Z_w) = (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2} Z_v Z_w) Y_v$. Continuing our derivation gives

$$\begin{aligned}
&= \left\langle s \left| \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2} Z_j Z_k} \prod_{\langle jk \rangle \setminus N(v)} e^{\frac{i\gamma}{2} Z_j Z_k} \prod_{w \in N(v)} e^{-\frac{i\gamma}{2} Z_w Z_v} Z_u Y_v \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v)} e^{-\frac{i\gamma}{2} Z_w Z_v} \prod_{w \in N(v)} e^{-\frac{i\gamma}{2} Z_w Z_v} Z_u Y_v \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v)} e^{-i\gamma Z_w Z_v} Z_u Y_v \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v) \setminus u} e^{-i\gamma Z_w Z_v} \cdot e^{-i\gamma Z_u Z_v} Z_u Y_v \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v) \setminus u} (\cos(\gamma) - i\sin(\gamma)\, Z_w Z_v)(\cos(\gamma) - i\sin(\gamma) Z_u Z_v)\, Z_u Y_v \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v) \setminus u} (\cos(\gamma) - i\sin(\gamma)\, Z_w Z_v)(\cos(\gamma) Z_u Y_v - i\sin(\gamma) Z_u Z_v Z_u Y_v) \right| s \right\rangle \\
&= \left\langle s \left| \prod_{w \in N(v) \setminus u} (\cos(\gamma) - i\sin(\gamma)\, Z_w Z_v)(i\cos(\gamma) Z_u X_v Z_v - \sin(\gamma) X_v) \right| s \right\rangle
\end{aligned}$$

If we expand this expression, we find that terms containing a Z gate disappear in the calculation of the expectation value since Z gates applied to the $|+\rangle$ state gives 0. This simplifies to

$$\langle s | -\cos^e(\gamma)\sin(\gamma) X_v | s \rangle = -\cos^e(\gamma)\sin(\gamma).$$

By symmetry, the third equation follows the same steps.

Lastly, we look at the fourth equation.

$$\begin{aligned}
\left\langle s \left| e^{i\gamma C} Y_u Y_v e^{-i\gamma C} \right| s \right\rangle &= \left\langle s \left| \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2}(1 - Z_j Z_k)} Y_u Y_v \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2}(1 - Z_j Z_k)} \right| s \right\rangle \\
&= \left\langle s \left| \prod_{\langle jk \rangle} e^{-\frac{i\gamma}{2} Z_j Z_k} Y_u Y_v \prod_{\langle jk \rangle} e^{\frac{i\gamma}{2} Z_j Z_k} \right| s \right\rangle
\end{aligned}$$

To simplify this expression, we can only commute the terms from the right over to the left if they do not have Z_u or Z_v, that is, if they do not belong to set $E \setminus (N(u) \cup N(v))$ where E is the set of all possible edges. Terms containing Z_u or Z_v anticommute. The term containing $Z_u Z_v$ anticommutes twice and so remains the same since $Y_u Y_v Z_u Z_v =$

$Y_uZ_uY_vZ_v = (-Z_uY_u)(-Z_vY_v) = Z_uY_uZ_vY_v = Z_uZ_vY_uY_v$. We end up with this manipulation

$$\begin{aligned}
&= \left\langle s \middle| \prod_{\langle jk\rangle} e^{-\frac{i\gamma}{2}Z_jZ_k} \cdot \prod_{\langle jk\rangle\in E\setminus(N(u)\cup N(v))} e^{\frac{i\gamma}{2}Z_jZ_k} \right. \\
&\qquad \left. \cdot \prod_{w\in N(u)\setminus v} e^{-\frac{i\gamma}{2}Z_uZ_w} \cdot \prod_{w\in N(v)\setminus u} e^{-\frac{i\gamma}{2}Z_vZ_w} \cdot e^{\frac{i\gamma}{2}Z_uZ_v} Y_uY_v \middle| s \right\rangle \\
&= \left\langle s \middle| \left(\prod_{w\in N(u)\setminus v} e^{-\frac{i\gamma}{2}Z_uZ_w} \right)^2 \left(\prod_{w\in N(v)\setminus u} e^{-\frac{i\gamma}{2}Z_vZ_w} \right)^2 Y_uY_v \middle| s \right\rangle \\
&= \left\langle s \middle| \prod_{w\in N(u)\setminus v} e^{-i\gamma Z_uZ_w} \prod_{w\in N(v)\setminus u} e^{-i\gamma Z_vZ_w} Y_uY_v \middle| s \right\rangle \\
&= \left\langle s \middle| \prod_{w\in N(u)\setminus v} (\cos(\gamma) - i\sin(\gamma)Z_uZ_w) \prod_{w\in N(v)\setminus u} (\cos(\gamma) - i\sin(\gamma)Z_vZ_w) \cdot Y_uY_v \middle| s \right\rangle
\end{aligned}$$

FIX W INDEXING

To simplify this, first consider the case where u and v share a common edge with one node w. We would be looking for terms that contain $Z_uZ_wZ_vZ_w$ once the above is expanded. Similar to the second and third equation, all other terms do not contribute to the expectation value. The only term that does contribute simplifies as

$$\begin{aligned}
&\left\langle s \middle| \cos(\gamma)^{d+e-2} \left(-i\sin(\gamma)Z_uZ_w\right) \left(-i\sin(\gamma)Z_vZ_w\right) \cdot Y_uY_v \middle| s \right\rangle \\
&\quad = \left\langle s \middle| -\cos(\gamma)^{d+e-2}\sin^2(\gamma)Z_uZ_wZ_vZ_wY_uY_v \middle| s \right\rangle \\
&\quad = \left\langle s \middle| -\cos(\gamma)^{d+e-2}\sin^2(\gamma)Z_uZ_vZ_wZ_wY_uY_v \middle| s \right\rangle \\
&\quad = \left\langle s \middle| -\cos(\gamma)^{d+e-2}\sin^2(\gamma)Z_uY_uZ_vY_v \middle| s \right\rangle \\
&\quad = \left\langle s \middle| -\cos(\gamma)^{d+e-2}\sin^2(\gamma)(-iX_u)(-iX_v) \middle| s \right\rangle \\
&\quad = \left\langle s \middle| \cos(\gamma)^{d+e-2}\sin^2(\gamma)X_uX_v \middle| s \right\rangle \\
&\quad = \cos(\gamma)^{d+e-2}\sin^2(\gamma)
\end{aligned}$$

There are f many of these which correspond to the f triangles containing $\langle uv\rangle$. Since $Z_uZ_{w_i} \cdot Z_uZ_{w_i} = I$, only odd number f will have any terms to contribute to the expecation value. The next higher-order term results from three different pairs $(Z_uZ_{w_i}, Z_vZ_{w_i})$ for $i = 1, 2, 3$. This will have a factor of $\sin^6(\gamma)$. We can generalise this as follows

$$\begin{aligned}
\left\langle s \middle| e^{i\gamma C}Y_uY_ve^{-\imath\gamma C} \middle| s \right\rangle &= \binom{f}{1}\cos^{d+e-2}(\gamma)\sin^2(\gamma) \\
&\qquad + \binom{f}{3}\cos^{d+e-6}(\gamma)\sin^6(\gamma) \\
&\qquad + \binom{f}{5}\cos^{d+e-10}(\gamma)\sin^{10}(\gamma) + \dots \\
&= \sum_{i=1,3,5,\dots}^{f} \binom{f}{i}\cos^{d+e-2i}(\gamma)\sin^{2i}(\gamma) \\
&= \cos^{d+e-2f}(\gamma) \sum_{i=1,3,5,\dots}^{f} \binom{f}{i}(\cos^2(\gamma))^{f-i}(\sin^2(\gamma))^i
\end{aligned}$$

We notice that

$$\begin{aligned}
\sum_{i=1,3,5,\ldots}^{f} \binom{f}{i} b^i a^{f-i} &= \frac{1}{2}\left[2 \sum_{i=1,3,5,\ldots}^{f} \binom{f}{i} b^i a^{f-i}\right] \\
&= \frac{1}{2}\left[\sum_{i=0}^{f} \binom{f}{i} b^i a^{f-i} - \sum_{j=0}^{f} \binom{f}{j} (-1)^j b^j a^{f-j}\right] \\
&= \frac{1}{2}\Big[(a+b)^f - (a-b)^f\Big]
\end{aligned}$$

Using this identity, we continue the simplification from above.

$$\begin{aligned}
&= \cos^{d+e-2f}(\gamma) \cdot \frac{1}{2}\Big[(\cos^2(\gamma) + \sin^2(\gamma))^f - (\cos^2(\gamma) - \sin^2(\gamma))^f\Big] \\
&= \frac{1}{2} \cos^{d+e-2f}(\gamma)\Big[1 - \cos^f(2\gamma)\Big]
\end{aligned}$$

as required. $\square$

References