CS 450-HW#3-Q1

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October 14, 2017

General Eigenvalue Knowledge

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

1. What is the characteristic polynomial of A?

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (1-\lambda) \times (1-\lambda) - 4 &= 0 \\ \lambda^2 - 2\lambda - 3 &= 0 \end{aligned}$$

The characteristic polynomial of A is $\lambda^2 - 2\lambda - 3 = 0$

2. What are the roots of the characteristic polynomial of A?

$$\begin{aligned} \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda+1)(\lambda-3) &= 0 \\ \lambda_1 &= 3 \\ \lambda_2 &= -1 \end{aligned}$$

The roots of characteristic polynomial is 3 and -1.

3. What are the eigenvalues of A?

The eigenvalues of A are the roots of characteristic polynomial: 3 and -1.

4. What are the eigenvectors of A?

Calculate the eigenvector for $\lambda_1 = 3$:

$$
\begin{aligned}
Av_1 &= \lambda_1 v_1 \\
\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{1_1} \\ v_{1_2} \end{bmatrix} &= \begin{bmatrix} 3v_{1_1} \\ 3v_{1_2} \end{bmatrix} \\
v_{1_1} &= 2v_{1_2} \\
v_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}
$$

Calculate the eigenvector for $\lambda_2 = -1$:

$$
\begin{aligned}
Av_1 &= \lambda_1 v_1 \\
\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_{2_1} \\ v_{2_2} \end{bmatrix} &= \begin{bmatrix} -v_{2_1} \\ -v_{2_2} \end{bmatrix} \\
v_{2_1} &= -2v_2 \\
v_2 &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\end{aligned}
$$

The normalized eigenspace $s_\lambda = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}$

5. Perform one iteration of power iteration on A, using $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ as starting vector.

$$
\begin{aligned}
x_1 &= Ax_0 \\
x_1 &= \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
x_1 &= \begin{bmatrix} 5 \\ 2 \end{bmatrix}
\end{aligned}
$$

The vector after first power iteration on A without normalization is $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

6. To what eigenvector of A will power iteration ultimately converge?

Power iteration (without normalization) will ultimately converge to $v = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where $k \in R$.

7. What eigenvalue estimate is given by the Rayleigh quotient, using the vector $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$?

$$\begin{aligned} \lambda &= \frac{x^T A x}{x^T x} \\ &= \frac{7}{2} = 3.5 \end{aligned}$$

Rayleigh quotient gives estimated eigenvalue $\lambda_{estimate} = 3.5$

8. To what eigenvector of A would the inverse iteration ultimately converge?

Inverse iteration will ultimately converge to $v = \begin{bmatrix} \frac{-2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix}$.

9. What eigenvalue of A would be obtained if inverse iteration were used with shift $\sigma = 2$?

$$\begin{aligned} \frac{1}{\lambda_1 - \sigma} &= 1 \\ \frac{1}{\lambda_2 - \sigma} &= -\frac{1}{3} \end{aligned}$$

By inverse iteration with shift, eigenvalue$\lambda = 3$ will still be obtained.

10. If QR iteration were applied to A, to what form would it converge: diagonal or triangular? Why?

It will converge to an upper triangular matrix. The symmetry are preserved by QR iteration. Since the original A is asymetric, the converging form cannot be diagonal.