Optimization with Newton's Method

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2$$

At what point does f attain a minimum?

$$\begin{aligned}
\nabla f(x_1, x_2) &= \begin{bmatrix} 2x_1^3 - 2x_1x_2 + x_1 - 1 \\ -x_1^2 + x_2 \end{bmatrix} = 0 \\
&\Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases} \\
H_f &= \begin{bmatrix} 6x_1^2 - 2x_2 + 1 & -2x_1 \\ -2x_1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \\
&\Rightarrow \lambda_1 = -3, \lambda_2 = 2
\end{aligned}$$

Perform one iteration of Newton's method for minimizing f using as starting point $\mathbf{x_0} = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$

$$\begin{aligned}
x_1 &= x_0 - H_f^{-1}(x_0)\nabla f(x_0) \\
x_1 &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 21 & -4 \\ -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ -2 \end{bmatrix} \\
x_1 &= \begin{bmatrix} 1.8 \\ 3.2 \end{bmatrix}
\end{aligned}$$

In what sense is this a good step?

Convergence rate is good: for minimization it is normally quadratic.

In what sense is this a bad step?

It is reliable only if starting point is close enough to solution to converge.