Critical Points in Two Dimensions

Determine the critical points of each of the following functions and characterize each as a minimum, maximum, or saddle point. Also determine whether each function has a global minimum or maximum on $\mathbb{R}^2$.

1. $f(x,y) = x^2 - 4xy + y^2$

$$\begin{aligned}
\nabla f(x,y) &= \begin{bmatrix} 2x - 4y \\ -4x + 2y \end{bmatrix} = 0 \\
&\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \\
H_f &= \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} \\
(2-\lambda)^2 - 16 &= 0 \\
\lambda_1 &= -2 \\
\lambda_2 &= 6
\end{aligned}$$

The critical point of this function is $(0,0)$. It is a saddle point. Since f does not go infinite when x and y approach infinite, f is not coersive, and hence f does not have a global minimum or maximum on $\mathbb{R}^2$.

2. $f(x,y) = x^4 - 4xy + y^4$

$$\begin{aligned}
\nabla f(x,y) &= \begin{bmatrix} 4x^3 - 4y \\ -4x + 4y^3 \end{bmatrix} = 0 \\
&\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \\
\mathit{or} &\Rightarrow \begin{cases} x = -1 \\ y = -1 \end{cases} \\
\mathit{or} &\Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \\
H_f &= \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix} \\
(0,0) &\Rightarrow \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \Rightarrow \lambda = \pm 4 \\
(1,1), (-1,-1) &\Rightarrow \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 8 \\ \lambda_2 = 16 \end{cases}
\end{aligned}$$

The critical point $(0,0)$ is a saddle point;
The critical point $(1,1)$ and $(-1,-1)$ are minimum.
This function is coersive, since x^4+y^4 has dominant order over $-4xy$. Therefore it has global minimum.

3. $f(x,y) = 2x^3 - 3x^2 - 6xy(x-y-1)$

$$\nabla f(x,y) = \begin{bmatrix} 6x^2 - 6x - 12xy + 6y^2 + 6y \\ -6x^2 + 12xy + 6x \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$or \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$$or \Rightarrow \begin{cases} x = 0 \\ y = -1 \end{cases}$$

$$or \Rightarrow \begin{cases} x = -1 \\ y = -1 \end{cases}$$

$$H_f = \begin{bmatrix} 12x - 6 - 12y & -12x + 12y + 6 \\ -12x + 12y + 6 & 12x \end{bmatrix}$$

$$(0,0) \Rightarrow \begin{bmatrix} -6 & 6 \\ 6 & 12 \end{bmatrix} \Rightarrow \lambda = 3 \pm 3\sqrt{13}$$

$$(1,0) \Rightarrow \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 18 \end{cases}$$

$$(0,-1) \Rightarrow \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix} \Rightarrow \lambda = 3 \pm 3\sqrt{5}$$

$$(-1,-1) \Rightarrow \begin{bmatrix} -6 & 6 \\ 6 & -12 \end{bmatrix} \Rightarrow \lambda = -9 \pm 3\sqrt{5}$$

The critical point $(0,0)$ and $(0,-1)$ is a saddle point;
The critical point $(1,0)$ is minima;
The critical point (-1,-1) is maxima.
This function is not coersive. Therefore it has no global minima or maxima.

4. $f(x, y) = (x - y)^4 + x^2 - y^2 - 2x + 2y + 1$

$$\begin{aligned}
\nabla f(x, y) &= \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix} = 0 \\
&\Rightarrow \begin{cases} x = k \\ y = k \end{cases} \quad k \in R \\
H_f &= \begin{bmatrix} 12(x-y)^2 + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix} \\
x = y \Rightarrow & \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = \pm 2
\end{aligned}$$

There are infinite critical points as long as $x = y$, and they are all saddle points.

The function is not coersive since x and y can cancel out when they both go to infinite. Therefore it has no global minima or maxima.