Calculating Interpolants

Given the three data points $(-1,1),(0,0),(1,1)$, determine the interpolating polynomial of degree two.

Show that the three representations give the same polynomial.

Be sure to show enough work to demonstrate that you understand how to get the answers. Results without sufficient work will be given little or no credit.

Using the monomial basis

$$\phi_j(t) = t^{j-1}$$

$$Ax = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$

$$p_2(t) = x_1 + x_2 t + x_3 t^2 = t^2$$

Using the Lagrange basis

$$l_j(t) = \prod_{k=1,k\neq \mathrm{j}}^{n} (t - t_k) / \prod_{k=1,k\neq \mathrm{j}}^{n} (t_j - t_k), j = 1, 2, ..., n$$

$$l_j(t_i) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \quad i, j = 1, 2, ..., n$$

$$\begin{aligned} p_2(t) =& y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)} \\ =& \frac{t(t-1)}{2} + \frac{t(t+1)}{2} = t^2 \end{aligned}$$

Using the Newton basis

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 & t_3 - t_1 & (t_3 - t_1)(t_3 - t_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 1 \end{cases}$$

$$p_2(t) = x_1 + x_2(t - t_1) + x_3(t - t_1)(t - t_2)$$

$$p_2(t) = 1 - (t + 1) + (t + 1)t = t^2$$