Introduction to General Relativity

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Very much in-progress, these notes are intended to cover a single semester introductory course on General Relativity. They draw mainly from Schutz [1] (an excellent undergraduate introduction), Carroll [2] (one of the best modern graduate texts, takes a geometric approach well-adapted to physicists), and Weinberg [3] (the holy text, follows the old, coordinate-based approach, but contains every calculation you could ask for), adapted to my own personal take. Additional resources one might find useful include Wald [4] (a comprehensive, modern text for physicists, but heavier on the math), Misner Thorne and Wheeler [5] (a legendary omnibus), and Spivak [6] (the truly epic five-volume treatise on pure differential geometry).

As background, I assume at least some upper-year Special Relativity, Newtonian mechanics, Electrodynamics, and Quantum Mechanics. Since time is limited in such a course, and GR is such a rich field, I include in appendices many interesting diversions that, while not crucial to completing the course, are well worth exploring at some point.

Throughout, I will use the only sensible choice of units, in which the speed of light is numerically 1, and so it will be omitted almost everywhere. When we get there, we will also use the correct (mostly-plus) metric. With the exception of chapter 1.1, spacetime vectors are written with unaltered letters and are indexed by Greek letters, while spatial vectors are written in bold and are indexed by Latin letters.

Introduction

General Relativity is easily in contention for the title of "most elegant theory" in Physics. The entire theory rests on two compact, yet extremely powerful and meaningful sets of equations. The first set are the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\kappa T_{\mu\nu}.$$

What makes this set of equations so elegant is that without knowing the details of either side, the equation graphically represents the core idea of GR:

$$\text{Geometry} = \text{Matter}.$$

Spacetime—the substrate of all physics—is not set in stone, it is dynamic and its dynamics are intimately tied to the matter that lives within it.

Another way of looking at the Einstein Field Equations is that they are the equations of motion for the geometry of spacetime. Closely related to that is the second crucial set of equations in this course, the geodesic equations:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\dot{\gamma}^\mu + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}\dot{\gamma}^\rho}{\mathrm{d}s}\frac{\mathrm{d}\dot{\gamma}^\sigma}{\mathrm{d}s} = 0.$$

The geodesic equations are the equations of motion of point-like matter in curved space, and are the relativistic version of Newton's second law[1]. These two sets of equations of motion are coupled to each other, and of course they are. The geometry of space reacts to the properties of the matter within it, but the behaviour of the matter in space reacts to the shape of the space it lives in, which influences the geometry of space, etc. This recursive relationship just reinforces the identity of spacetime and matter.

The purpose of these notes is to go from a background of elementary mechanics and special relativity to an understanding of these equations, and the physical consequences thereof. To do so, most of our time will be spent learning the beautiful language of differential geometry, and unpacking this sleek, compact tensor notation. A single course is nowhere near enough time to fully explore the enormous landscape of this theory, so our goal will be to establish the fundamental tools required to be able to learn any further topics with relative ease (many of these topics will be teased, but not explored throughout).

[1]N.B. As written, the equations only represent motion in a gravitational field, there are additional terms present when other forces are involved.

Chapter 1

The Geometry of Special Relativity

1.1 Minkowski Space

Lightning SR

Special Relativity is a beautiful thing. Two postulates, empirically determined, fix the fundamental substrate of (inertial) physics in a way that is almost impossible for our animal brains to believe. The first postulate goes back as far as Galileo, a convenient framing of which is:

Postulate 1: The Principle of Relativity

There exists a special class of coordinate systems (called ***inertial coordinate systems***) in which the laws of physics take the same mathematical form.

This is clearly true from experience; everything you know about how Nature works is equally true when you're at home as it is when you're at the library, at the park, on a mountain, etc. We call these systems **inertial** because they are the systems in which Newton's first law is manifest. They are special in that they are *clean*, there are no confounding external influences. The cockpit of a plane performing aerial acrobatics, for example, is good example of a *non*-inertial frame, as any experiment you conduct in that environment would be very hard to reproduce indeed. Perhaps a less intuitive non-inertial frame is the one you're in now; the ground pushing up on you is an external force that augments your measurements of the fundamental laws of physics.

The first postulate applies equally well to Newton's laws as it does to Einstein's, the difference comes with the second postulate:

Postulate 2: Invariance of the Speed of Light

There exists a finite speed **c** whose value is measured to be the same in every inertial coordinate system (numerically, this is the speed of light in a classical vacuum).

This is the unshakable conclusion drawn from the failure of experiments to detect the luminiferous aether, and it is this second postulate that completely changes the landscape. Hopefully you'll recall from your earlier introduction to Special Relativity that these postulates lead to the famous Lorentz transformations. For any two collinear inertial observers, their coordinate systems $\mathcal{O}$, and

$\mathcal{O}'$, are related by:

$$t' = \gamma t - \gamma\beta x, \quad \text{and} \tag{1.1.1}$$
$$x' = \gamma x - \gamma\beta t, \tag{1.1.2}$$

where it is assumed the collinear axis is the x axis. The relativistic factor is defined as:

$$\gamma := \frac{1}{\sqrt{1-\beta^2}}, \tag{1.1.3}$$

where $\beta := v/c$ is the normalized relative velocity of the frames.

Aside 1: Natural Units

In fact, it is always only useful to talk of velocities relative to c, to the point where it is really just clumsy to keep around the factor of c that is always there. Instead, what we'll do is work in a more *natural* system of units where the numerical value of c "happens" to be 1. For instance, we may define a unit of time, the du, such that $1\,\mathrm{du} = 1\,m/(299792458\,m/s)$, and then measure all time intervals in du[a]. Of course, this strategy would work equally well for units of spatial separation; the important thing is that we work with a system of units that makes the relation between distances in time and space absurdly simple, so we never have to think about it again.

[a] Common practice in the field, see e.g. http://www.peebleslab.com/20

Now you will have seen all the quirky business with length contraction and time dilation, and various and sundry silly paradoxes that arise from forcing our naive animal viewpoint on a special relativistic world. As a prelude to the general theory, I will instead focus on the important mathematical characteristics of **Minkowski space** (the mathematical name for the spacetime of special relativity).

There are two keys lessons to learn from special relativity. The first should be familiar now, that time is not a parameter, it is a *place*. Instead of time, we must parameterize physics that takes place in spacetime by some arbitrary external parameter (typically chosen to be something physically meaningful, like a particle's proper time, or proper distance). The second lesson is that as a coordinate, time is a little bit unusual. We can make this statement more concrete; first note that the Lorentz transformations (1.1.1) are linear transformations, so we can write those equations more succinctly:

$$\vec{w}' = \Lambda \vec{w}, \tag{1.1.4}$$

where $\vec{w} = (t, x)^T$, $\vec{w}' = (t', x')^T$ are vectors in (two-dimensional) spacetime, and

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}. \tag{1.1.5}$$

is the matrix of the Lorentz transformation.

Besides the obvious translations in time and position, the Lorentz transformations (1.1.4) define the only sense in which vectors in (two-dimensional) Minkowski space can be considered equivalent. Think about an ordinary 2-d plane: if I have vectors $\vec{u} = (y, z)$ and $\vec{u}' = (y', z')$ that are related by the action of a linear operator (matrix) O, $\vec{u}' = O\vec{u}$, they are considered equivalent if it is true that $(\vec{u}')^T\vec{u} = \vec{v}^T O^T O\vec{v}$, or in other words, if $O^T O = 1$. A simple calculation suffices to show that this is *not* the case with the Lorentz transformations:

$$\Lambda^T \Lambda \neq 1. \tag{1.1.6}$$

Of course, (1.1.6) is just the statement that a Lorentz transformation is not an orthogonal transformation, and this should come as no surprise, the Lorentz transformations are not rotations[1] (and more directly, the matrix (1.1.4) is symmetric—$\Lambda^T = \Lambda$—but the inverse transformation is obtained by $\beta \to -\beta$, $\gamma \to \gamma$, which flips the sign on the off-diagonal elements). Fortunately though, there is an equivalent statement: Lorentz transformations define an equivalence class of vectors that satisfy: for $\vec{w}' = \Lambda\vec{w}$

$$(\vec{w}')^T \eta \vec{w}' = \vec{w}^T \Lambda^T \eta \Lambda \vec{w}, \tag{1.1.7}$$

where η is a matrix called the **Minkowski metric tensor**, defined by:

$$\eta := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.1.8}$$

(You should calculate for yourself that (1.1.7) is true given (1.1.8)). We'll have a lot more to say about η later, but for now the important thing is that Minkowski spacetime, the fundamental canvas of all of physics, preserves a generalized notion of the length of a vector:

$$-t^2 + x^2 = \text{const.} \tag{1.1.9}$$

And of course, nothing was special about the x-direction, so we can immediately generalize all of this discussion to the full 3+1 (space+time) dimensional spacetime of the real world:

$$-t^2 + x^2 + y^2 + z^2 = \text{const.} \tag{1.1.10}$$

and just as importantly,

$$\eta := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.1.11}$$

A Closer Look at η

The metric tensor η is a crucial component of Minkowski space, and fundamentally what makes it different from Euclidean space. We've been looking at it as a matrix, and while that will continue to be a useful thing to do some times, it is more useful to look at it as an operator, specifically as a tool to measure the length of vectors. Mathematically, Euclidean spacetime is an *inner product* space, it is a vectorspace that comes equipped with the inner product $\langle \cdot, \cdot \rangle$ that defines the dot product between vectors. For vectors $\vec{v} = v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z$ and $\vec{w} = w_x\hat{e}_x + w_y\hat{e}_y + w_z\hat{e}_z$, we define $\langle \vec{v}, \vec{w} \rangle := v_x w_x + v_y w_y + v_z w_z$, and the unit vectors are defined to be orthonormal, $\langle \hat{e}_i, \hat{e}_j \rangle := \delta_{ij}$. (Note: the Kronecker δ can be represented by the identity matrix, the matrix left invariant by orthogonal matrices). The Euclidean inner product is a true inner product, it satisfies the following properties (Ex: check for yourself that they are all true).

[1] Well, not *ordinary* rotations—see the surprising similarities in 1A.1

$\langle \cdot, \cdot \rangle$:

- Bilinear: The Euclidean inner product is linear in "both slots," e.g., $\langle \vec{v}+\vec{w}, \vec{r}\rangle = \langle \vec{v}, \vec{r}\rangle + \langle \vec{w}, \vec{r}\rangle$, and $\langle \vec{v}, \vec{w}+\vec{r}\rangle = \langle \vec{v}, \vec{w}\rangle + \langle \vec{v}, \vec{r}\rangle$, etc.

- Non-Degenerate: The Euclidean product satisfies that only the zero vector is the zero vector. That is, if $\langle \vec{v}, \vec{w}\rangle = 0$ for all $\vec{w}$, then $\vec{v} = \vec{0}$.

- Positive-Definite: The overlap between two non-zero Euclidean vectors is always positive, $\langle \vec{v}, \vec{w}\rangle > 0$ for all non-zero $\vec{v}$ and $\vec{w}$.

The metric tensor η is the corresponding way we measure vectors in Minkowski space. It can be written as a matrix as above, but it is typically more useful to write it as a function like the Euclidean inner product, $\eta(\cdot, \cdot)$. Viewed this way, for vectors $\vec{v} = v_t\hat{e}_t + v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z$ and $\vec{w} = w_t\hat{e}_t + w_x\hat{e}_x + w_y\hat{e}_y + w_z\hat{e}_z$, we define $\eta(\vec{v}, \vec{w}) := -v_tw_t + v_xw_x + v_yw_y + v_zw_z$, and the unit vectors are defined to be orthonormal, $\eta(\hat{e}_\mu, \hat{e}_\nu) := \eta_{\mu\nu}$ (where $\eta_{\mu\nu}$ are the elements of the matrix (1.1.11)). However, that minus sign shakes things up a bit. The Minkowski metric tensor is *not* a proper inner product, like the Euclidean inner product. Instead, it satisfies the following properties (Ex: verify these properties).

$\eta(\cdot, \cdot)$:

- Bilinear: The Minkowski metric tensor is linear in "both slots," e.g., $\langle \vec{v}+\vec{w}, \vec{r}\rangle = \langle \vec{v}, \vec{r}\rangle + \langle \vec{w}, \vec{r}\rangle$, and $\langle \vec{v}, \vec{w}+\vec{r}\rangle = \langle \vec{v}, \vec{w}\rangle + \langle \vec{v}, \vec{r}\rangle$, etc.

- Non-Degenerate: The Minkowski metric tensor satisfies that only the zero vector is the zero vector. That is, if $\langle \vec{v}, \vec{w}\rangle = 0$ for all $\vec{w}$, then $\vec{v} = \vec{0}$.

- **NOT Positive-Definite**: The overlap between two non-zero Minkowski vectors is *not necessarily* positive, $\eta(\vec{v}, \vec{w})$ can be negative, positive, or vanish for all non-zero $\vec{v}$ and $\vec{w}$.

Dividing Spacetime

The fact that the overlap of two Minkowski vectors can be non-positive is responsible for a lot of physics. We can classify vectors as follows:

1. **Timelike**: If $\eta(\vec{v}, \vec{v}) < 0$

2. **Spacelike**: If $\eta(\vec{v}, \vec{v}) > 0$

3. **Lightlike (or Null)**: If $\eta(\vec{v}, \vec{v}) = 0$

Causally-connected events must be either timelike or lightlike, it must be possible for something to connect the spatial distances in *less than or equal to* the amount of time it would take a light beam (i.e., $(t_f - t_i)^2$ must be bigger than $(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2$). Also notice that since Lorentz transformations preserve the form of the Minkowski metric tensor, it is impossible to find an inertial frame where a vector changes its nature (e.g., there is no way to boost to a frame where a timelike, causally connected sequence becomes spacelike and acausal).

1.2 Minkowski Vectors

The Peculiar Nature of Displacement

So we have established some idea of Minkowski space as basically a 4D Cartesian space where the dot product on vectors is a little bit funky, but what vectors exactly? Later on, it will be important that we were precise in talking about the structure of Minkowski space, because one of the big points of GR is that the fundamental structure of spacetime is *different* when gravity is taken into consideration.

Spacetime is the canvas of physics, it's where all the stuff happens. Mathematically, that means we can think of spacetime as a collection of points (t, x, y, z). A tuple like that is conventionally called an "event," but don't be fooled by the name, on their own these points are meaningless, just a collection of numbers. Physically, what matters is paths of particles, or scalar or vector fields, things that map say a trajectory $(\gamma : s \in (0,1) \mapsto (t(s), x(s), y(s), z(s)))$, or a topography $(\phi : x \in \mathbb{R}^4 \mapsto \mathbb{R})$. These external constructions are the real physical things, and their motion is typically governed by their well-defined derivatives, tangents, gradients, etc.

Having said that, Minkowski spacetime (and incidentally, also the ordinary Euclidean space we've learned about since high school) is special in that it readily admits an additional structure—Minkowski space is an Affine space. What's an Affine space, you ask? An Affine space is basically a collection of points that comes with a built-in vectorspace. Think of two events in spacetime, p and q. On their own, they're physically meaningless, and you certainly wouldn't give any thought to adding them, or scaling them (what's $2 \times$ [the location of the Eiffel Tower]?). Ah, but what if you subtract them? Well then you've got something kind of familiar, you've got a displacement vector. The notion of a displacement vector is a fluke, it comes about because the points of spacetime happen to make up the set $\mathbb{R}^4$, which is conveniently also the quintessential example of a 4D vectorspace. And since $\mathbb{R}^4$ the Affine space and $\mathbb{R}^4$ the vectorspace are described so similarly, it is almost impossible *not* to conflate the two, to the point that the whole derivation of Minkowski space you've seen in terms of rods and cones *explicitly* relied on displacement vectors (what else could $(\Delta t, \Delta x, \Delta y, \Delta z)$ be?)[2]. Now of course, there's nothing wrong with that per se, but it is crucial we remember the distinction between the barebones spacetime points which fundamentally serve as the canvas for physics, and the displacement vectors that just happen to have a natural relationship to the way we usually describe spacetime points (because spoiler alert: the first thing we're going to do in GR is rip apart that relationship).

A good way to convince yourself of the fragility of displacement vectors is to think about polar coordinates. Take two points in the plane written in polar coordinates, what is the displacement vector between them? You can't simply take the difference between their radii and angles, there's no simple mechanism to compare points. In fact, the way to get a "displacement" between points in polar coordinates is first to convert to Cartesian coordinates, perform the calculation, then convert back to polar coordinates. In physics, we might think about this in terms of symmetry. Spacetime in Cartesian coordinates in fully translationally symmetric, but polar coordinates select out a point (in fact, you have to remove the origin since the angle is not defined there). Affine spaces are basically the mathematical name for translationally symmetry spaces, and intuitively as physicists we know that *most* things are not translationally symmetric.

[2] A good way to demonstrate this is to draw a point on the board with some random coordinate, and point out how natural it is for us as physicists to *also* go ahead and draw an origin. In shifting our thinking to GR, we have to resist that urge

Aside 2: Matrices and Indices

With the Kronecker δ and the Minkowski metric tensor, we've now used both an index representation (δ_{ij}, $\eta_{\mu\nu}$) and a matrix representation (e.g., (1.1.11)). In this course, the fundamental thing will be the index definition, matrices are just a convenience for visualizing what we will eventually call rank-2 tensors. Notice also that we wrote the indices as subscripts (as usual). Next up, we will also work with indices as superscripts, and it will be important that we pay attention to which index goes where.

Vector Variance

With that out of the way, lets look at a few different types of vectors that tend to arise in physics. Let's consider a displacement vector, let's consider the tangent of some curve $\gamma : U \in \mathbb{R} \to \mathbb{R}^4_M$, and let's consider the gradient of some scalar field $\phi : \mathbb{R}^4_M \to \mathbb{R}$. To be super clear, we'll separate out the vectorspaces involved, so even though they're all copies of $\mathbb{R}^4$, we'll say the displacement vectors live in $\mathbb{R}^4_D$, while the spacetime itself is $\mathbb{R}^4_M$ (remember, *not* a vectorspace), and the gradient of the curve will live in $\mathbb{R}^4_{T^*}$ ('T' for tangent space, asterisk for dual—more on that in the next chapter).

First up, displacements. A displacement vector is just a vector we're all familiar with, we can write it in terms of some basis as:

$$\Delta x = \sum_{\mu=0}^{3} \Delta x^\mu \hat{e}_\mu := \Delta x^\mu \hat{e}_\mu. \tag{1.2.1}$$

Aside 3

And here seems as good a time as any to introduce Einstein notation. We see sums of the form (1.2.1) a *lot* in relativity, so it's more convenient to assume a sum and drop the symbol. For our safety though, we establish a couple of rules with this. First, the terms are only summed if there is an index that is repeated, once as a subscript and once as a superscript. Second, when the index is a Greek letter the sum is over space and time indices (0 to 3), but when it is a Latin letter, the sum is only over the spatial indices (1 to 3—trust me, this will be handy). We're also using the notation that $(t, x, y, z) := (x^0, x^1, x^2, x^3)$, and that $\hat{e}_\mu$ denotes a unit vector in the x^μ direction (it may be more familiar to see these as $\hat{x}, \hat{y}$, etc., but $\hat{e}_\mu$ is the standard notation in the field since it generalizes easily to higher dimensions). It will also be convenient to visually separate spacetime vectors (we'll call them four-vectors) from purely spatial vectors (which we'll call three-vectors). Since we use spacetime vectors so much, we won't use any special notation, like the simple Δx above. Three-vectors are less common, so we can spare some familiar notation for them, and we'll write them in boldface. For example, the four-vector Δx can be written $\Delta x = \Delta t \, \hat{t} + \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is the three-vector $\boldsymbol{\Delta} := \Delta x \, \hat{x} + \Delta y \, \hat{y} + \Delta z \, \hat{z}$.

In the expression (1.2.1), it is very important to make some distinctions clear. Δx is a vector, an abstract object. Similarly, the set $\{\hat{e}_\mu\}$ is a set of vectors (abstract objects) that are linearly independent, orthogonal, and of unit length. This means that $\eta(\hat{e}_\mu, \hat{e}_\nu) = \eta_{\mu\nu}$ (this is how we *define* orthonormality in Minkowski space). The *components* $\{\Delta x^\mu\}$ are simply numbers, by themselves they're meaningless. Physically, we can think of the vectors as things with dimension, like length—e.g., $[\hat{e}_\mu] = [L]$. Now imagine we re-define what we mean by the x direction, say scale it up by a factor of 1000. This is the same as changing units from m to km. The abstract four-vector Δx doesn't care that we did that, because our choice of coordinates was just for our own convenience, so it must be the case that the *x-component* of Δx should be scaled down by a factor of 1000 (e.g., a displacement of $500m$ is the same as a displacement of $0.5km$). In general, if we redefine our coordinates by any Lorentz transformation[3] (which includes spatial rotations) Λ so that

$$\overline{e}_\mu := \Lambda_\mu{}^\nu \hat{e}_\nu, \tag{1.2.2}$$

the components of any displacement vector must be redefined by the inverse map,

$$\overline{\Delta x}^\mu := \Delta x^\nu (\Lambda^{-1})_\nu{}^\mu. \tag{1.2.3}$$

Any vector whose components transform as the components of Δx under a coordinate transformation is said[4] to be **contravariant**, because its components *contra-vary* with changes of basis.

Aside 4: A note on indices

Normally indices will refer to components of vectors (and similar objects). The unit vectors $\hat{e}_\mu$ are a special case though, here the subscript μ is the *name* of the vector. We can write it in components as $\hat{e}_\mu = \delta_\mu{}^\nu \hat{e}_\nu$, so the components of $\hat{e}_\mu$ are $[\delta_\mu]^\nu$. The abstract basis vectors $\hat{e}_\mu$ carry the dimension (like metres), while $\delta_\mu{}^\nu$ is just a 1 or a 0.

As an example, consider the curve $\gamma : U \in \mathbb{R} \to \mathbb{R}^4_M$, where $U = [0, 1]$ and $\gamma : s \mapsto (x^0(s), x^1(s), x^2(s), x^3(s))$. The derivative of this curve can be evaluated at any point s_0 to be $\dot{\gamma}(s_0) = (\dot{x}^0(s_0), \dot{x}^1(s_0), \dot{x}^2(s_0), \dot{x}^3(s_0))$. If we think of the derivatives $\dot{x}^\mu$ as infinitessimal displacements

$$\dot{x}^\mu(s_0) := \lim_{h \to 0} \frac{x^\mu(s+h) - x^\mu(s)}{h}\Big|_{s=s_0}, \tag{1.2.4}$$

then clearly this tangent vector is also *contravariant* with respect to changes of the basis vectors. Pop quiz: is $\gamma(s_0)$ also contravariant? Nope! It is *not* a vector at all, remember? It is merely a point in spacetime, it has no directionality, no vectorial nature of any kind. We *had* to take its derivative to turn it into a vector, because a tangent vector is nothing but an infinitessimal displacement, which *is* a vector quantity. (Note: tangent vectors will be very important later on, stay tuned).

Finally, we consider the gradient of a scalar function $\phi(x)$. From ordinary calculus, this can be written $\nabla\phi = (\partial_0\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi)$, which defines the notation $\partial_\mu := \frac{\partial}{\partial x^\mu}$. Now, from what we've seen above, we should also say that this is contravariant, but this time that would clash with convention! The idea here is that contra/covariance of vectors is established with respect to a preferred vectorspace, the space of displacement vectors. Intuitively, if the tuple of components

[3]PIN: Put a pin in this; what follows will actually apply generally to any linear transformation of the basis vectors, but for now we restrict to Lorentz transformations. We will explain why when we look at curved spaces and relax this requirement.

[4]By physicists. There are some areas of math where the convention is actually the opposite.

of a displacement vector $(\Delta x^0, \ldots)$ is transformed by some Lorentz transformation Λ^{-1}, then it makes sense that a tuple that looks like $(1/\Delta x^0, \ldots)$ should transform under the action of the inverse matrix Λ. So for any change of basis for the vector $\nabla\phi$, we *define* that change to be the inverse of a change on the basis of the vector Δx, and so we say that the components of $\nabla\phi$ are *covariant*. The contents of the next section will make this sound less arbitrary (the gradient space is mathematically dual to the tangent vector space).

In fact, this preferred vectorspace business is actually what physicists *mean* by dimensionful quantities; we define the unit of displacement to be the metre m, so when we measure the rate of change of something with respect to a displacement, we measure it in inverse metres m^{-1}. Maybe more intuitively, we think of distances in time as measured in seconds s, and rates of change (with respect to displacements in time) as being measured in s^{-1}. Of course, no one held a gun to our heads and told us to base our system on displacement. In principle, there's nothing wrong with choosing gradients to be the fundamental quantity of interest, measuring things in terms of Hertz say, and then thinking about displacements in time as being rates of change with respect to some gradient—i.e., measuring time in Hz^{-1}. The important thing about conventions is that once you set them, you stick to them for ever and ever, so since the physicists of old based their lives around displacements, we will follow suit.

Aside 5

The change of basis transformations we just discussed don't change vectors themselves, just their representations in terms of a given basis. This sort of transformation is called *passive*, since it changes the lens we use to look at things, but not the things themselves. A transformation that actually changes one vector into another is called *active*, and is essentially what physics is all about, solving equations of the form $Mv = w$.

1.3 Minkowski Co-Vectors

Co-vectors as operators

There is another (better) way to think about covariant vectors, like the gradient. First, it helps to think about the dot product of two vectors in a more general way. Instead of matrix multiplication

$$v \cdot w = v^T \eta \, w, \tag{1.3.1}$$

think about η as a function $\eta(\cdot,\cdot)$ that takes two vectors as input and outputs a real number:

$$\eta(v, w) = v \cdot w = v^T \eta \, w. \tag{1.3.2}$$

Now, if we take away the second vector, we have an operator that is parameterized by v and acts on all other vectors. In other words, for any vector v, the operator $\omega_v = \eta(v, \cdot)$ is a function that takes one vector as input and spits out a real number. This can be represented as a row vector if necessary:

$$\omega_v = \eta(v, \cdot) = v^T \eta = \begin{pmatrix} -v_0 & v_1 & v_2 & v_3 \end{pmatrix}, \tag{1.3.3}$$

or better yet, in terms of a basis of orthonormal covariant vectors:

$$\omega_v = \eta_{\mu\nu} v^\nu \hat{e}^\mu =: (\omega_v)_\mu \hat{e}^\mu. \tag{1.3.4}$$

Here, the basis vectors $\hat{e}^\mu$ are uniquely defined by the condition $\hat{e}^\mu(\hat{e}_\nu) = \delta^\mu_\nu$ (N.B., here again, the superscript on $\hat{e}^\mu$ is the *name* of the covector, not an index, just like for the basis vectors for contravariant vectors). ω_v is mathematically a vector, but it is called a *dual vector* since it is in a sense dual to the vector v with respect to the metric. The key point here is that dual vectors are covariant vectors (also called covectors). To see this, recall that the overlap of two vectors must be independent of our choice of basis, so we must have

$$\begin{aligned}\omega_v(w) &= \left(\overline{\omega}_0, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\right)\begin{pmatrix}\overline{w}_0\\ \overline{w}_1\\ \overline{w}_2\\ \overline{w}_3\end{pmatrix},\\ &= \left(\overline{\omega}_0, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\right)\Lambda^{-1}\begin{pmatrix}w_0\\ w_1\\ w_2\\ w_3\end{pmatrix},\end{aligned}$$

which implies the components of the dual vector must transform as $(\overline{\omega}_v)_\mu = (\omega_v)_\nu \Lambda_\mu{}^\nu$, which is the definition of a covariant vector.

Two important points. First, it is worthwhile writing the action of action of a dual vector on a vector in components:

$$\omega_v(w) = (\omega_v)_\mu \hat{e}^\mu(w^\nu \hat{e}_\nu) = (\omega_v)_\mu w^\nu \hat{e}^\mu(\hat{e}_\nu) = (\omega_v)_\mu w^\nu \delta^\mu_\nu = (\omega_v)_\mu w^\mu = \eta_{\mu\rho} v^\rho w^\mu = \eta(v, w) \quad (1.3.5)$$

as it should be. Second, the pseudo-inner product on the dual vector space is not actually η but instead the *inverse* matrix η^{-1}. To see this, consider that if we want covectors to really be dual to vectors, we should want it to be the case that the notion of lengths is preserved. That is, if we have two covectors ω_v and ω_w corresponding to the vectors v and w, then we want it to be the case that the scalar product between ω_v and ω_w satisfies:

$$\omega_v \cdot \omega_w = ?^{\mu\nu}(\omega_v)_\mu(\omega_w)_\nu = \eta_{\rho\sigma} v^\rho w^\sigma. \quad (1.3.6)$$

We can find the mystery matrix $?^{\mu\nu}$ by following our nose:

$$\begin{aligned}?^{\mu\nu}(\omega_v)_\mu(\omega_w)_\nu &= ?^{\mu\nu}\eta_{\mu\rho} v^\rho \eta_{\nu\sigma} w^\sigma\\ &= ?^{\mu\nu}\eta_{\mu\rho}\eta_{\nu\sigma} v^\rho w^\sigma.\end{aligned} \quad (1.3.7)$$

Then if the mystery matrix satisfies $?^{\mu\nu} = (\eta^{-1})^{\mu\nu}$, we have $(\eta^{-1})^{\mu\nu}\eta_{\mu\rho} = \delta^\nu_\rho$ and

$$\omega_v \cdot \omega_w = \delta^\nu_\rho \eta_{\nu\sigma} v^\rho w^\sigma = \eta_{\rho\sigma} v^\rho w^\sigma, \quad (1.3.8)$$

so the natural metric tensor associated with the dual space is the inverse of the Minkowski metric, η^{-1}.

Aside 6

In full generality, for any vectorspace V, there is a complementary vectorspace $\widetilde{V}$ made up of all functions $f \in \widetilde{V}$ such that $f : V \to \mathbb{R}$. When a vectorspace has the additional structure of a bilinear form (like the metric η), there is a natural isomorphism between V and $\widetilde{V}$ given by $v \mapsto \eta(v, \cdot)$.

To help drive home the idea of dual vectors, it's good to cover a couple of familiar examples. First, ordinary row vectors are already dual vectors, just for Euclidean space. The metric on Euclidean space is just the identity matrix, so if we take a Euclidean column vector v, then the dual vector is $\omega_v = \mathbb{I}(v, \cdot) = v^T \mathbb{I} = v^T$. This is the more natural way we would shmush two column vectors together, Minkowski space basically just amounts to throwing a minus sign onto the first component (general curved spaces will be more complicated).

A less obvious example you will have seen comes to us from quantum mechanics. In Dirac notation, kets are the vectors, and bras are the dual vectors. Pictorially, this is pretty obvious—$\langle\psi|$ shmushed with $|\phi\rangle$ is a (complex) number. More concretely, the vectors in quantum mechanics are square-integrable functions, and (mod-)squared integration is the inner product: $\langle\psi|\phi\rangle = \int \mathrm{d}V \psi^* \phi$, so if ϕ is the vector $|\phi\rangle$, then $\int \mathrm{d}V \psi^* \cdot$ is the dual vector $\langle\psi|$.

Schutz has an excellent way of visualizing dual vectors. While vectors can be thought of as little arrows scattered about the plane, dual vectors are like topographical marks on that plane. Like topographical gradients, dual vectors change the sense in which vectors' lengths are interpreted. An arrow laid across a steep part of the topographical map means something quite different from the same arrow laid across a shallow part. Fundamentally, this all boils down to the notion that dual vectors represent rates with respect to quantities of interest (s^{-1} vs s, etc.).

Index Gymnastics

With an understanding of how vectors and covectors are related by the metric, the practical consequence for the working physicist is that we now have a way of switching between upper and lower indices. That is, the dual vector with components ω_μ can be understood as the dual to the vector with components ω^μ by defining $\omega_\mu := \eta_{\mu\nu}\omega^\nu$. To go the other way, we define the components of the inverse metric η^{-1} to be $\eta^{\mu\nu}$, so that we may identify the components of a vector v^μ with the components of a dual vector v_μ by $v^\mu := \eta^{\mu\nu} v_\nu$. These mappings are known as "**raising**" and "**lowering**" indices, and will become as natural as addition and multiplication when we get into the meat of GR. Incidentally, mathematicians refer to index raising and lowering as the musical isomorphisms, and represent the process with the musical "sharp" and "flat" symbols respectively.

1.4 A Little Physics

Heavelocities

My aim is to get us to gravitational physics as soon as possible, but to avoid getting bogged down in abstract math, it helps to place some of this in the context of actual physics.

Let's start with the simplest possible physical thing in Minkowski space: a single massive particle in its rest frame. Recall that we *define* a particle's proper time τ to be the time elapsed in its rest frame. Turning that around, we can say that the parameter that governs a particle's motion is its proper time, and its trajectory in its rest frame is simply $x(\tau) = (\tau, 0, 0, 0)^T$. Then the particle's *four-velocity* is the tangent to this curve, given by $U(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau) = (1, 0, 0, 0)^T$ (notice the length of this vector is $U \cdot U = \eta(U, U) = -1$, a property of massive particles). Need some convincing that this makes sense as a velocity? Easy peasy, we'll just boost out of the particle's rest frame. Here's where we put what we've learned into practice. First, we write the *general abstract vector* U in terms of the basis vectors associated with the particle's rest frame:

$$U = \delta_0^\mu \, \overline{e}_\mu. \tag{1.4.1}$$

Here δ_0^μ is the usual Kronecker delta symbol, and we are denoting the unit basis vectors *in the particle's rest frame* as $\overline{e}_\mu$ (N.B., Schutz denotes this by putting a bar over the index, but I prefer to make more clear that the vectors themselves are different). Now if we wish to look at this vector from the perspective of a frame that is moving at velocity β in, say, the x-direction with respect to this one, the basis vectors of the boosted frame will be related to $\overline{e}_\mu$ by a Lorentz transformation $\Lambda(\beta)$ (where we explicitly express the velocity-dependence of the transformation). So we write:

$$\hat{e}_\mu = (\Lambda(\beta))^\nu{}_\mu \overline{e}_\nu, \tag{1.4.2}$$

where

$$\Lambda(\beta) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.4.3}$$

Then the *components* must *contra-vary* with this boost, so we write:

$$\overline{U}^\mu = (\Lambda(-\beta))^\mu{}_\nu U^\nu = (\Lambda(\beta))^\mu{}_\nu \delta_0^\nu = \begin{bmatrix} \gamma \\ \gamma\beta \\ 0 \\ 0 \end{bmatrix}. \tag{1.4.4}$$

(Ex: Verify that the *vector* U is unchanged by this transformation. That is, verify explicitly that $\overline{U}^\mu \overline{e}_\mu = U^\nu \hat{e}_\nu$). Recall that $\gamma := 1/\sqrt{1-\beta^2}$, so that for $\beta \ll 1$, we have $\gamma \sim 1$ and $\gamma\beta \sim \beta$, so the components of the four-velocity U in the boosted frame are $\overline{U}^\mu \sim (1, \beta, 0, 0)^T$, justifying the name.

In particle mechanics, the incredibly important quantity *momentum* is related to velocity by $p = m\beta$. Continuing our analogy, we define the *four-momentum* to be the four-vector $p = mU$. In the particle's rest frame, this has components $p^\mu = (m, 0, 0, 0)^T$, while in a boosted frame (along the x-axis, say), it has components $\overline{U}^\mu = (m\gamma, m\gamma\beta, 0, 0)^T$. Notice that to leading order in β, we have in the boosted frame that $\overline{p}^0 = m\gamma \sim m + \frac{1}{2}m\beta^2 + \ldots$ which has the form "rest mass + kinetic energy" (Ex: verify this). This leads to the identification of the energy $E := p^0$, and linear momentum $\vec{p}^i := p^i$. Notice that the length of the vector $p^2 = -m^2 = -E^2 + \vec{p}^2$ is independent of inertial reference frame (by definition), hence why it is often called a particle's *invariant mass*.

Light Velocities

Speaking of invariant mass, what about invariant massless things, like photons? Our discussion above started with a timelike vector ($U \cdot U = -1 < 0$), so it's worth asking, what would lightlike and spacelike four-velocities and momenta represent? Well in the first case, the name is kind of a spoiler. Lightlike vectors satisfy $U \cdot U = 0$, or in the case of momentum, $p \cdot p = -m^2 = 0$—a simple example of such a vector would be $U = (1, 1, 0, 0)$ (ex. show that the norm of this vector vanishes). What would *not* be an example of such a vector? Obviously the one we started with, $U = (1, 0, 0, 0)$. Recall that these are *disjoint* classes of vectors, there is no Lorentz transformation that will take you from a lightlike to a timelike vector and vice versa. This leads to the important following statement: *there is no rest frame for massless particles.* The curse of the photon, to maintain its velocity in everyone else's frame it must give up a rest frame of its own. Fortunately, since all of us studying physics have mass (that I know of...), it won't be important that we grapple with this fact of life, just keep it handy when looking at vectors.

Imaginary Velocities

Having explored the meaningful cases, there's one more class of vectors to think about, spacelike vectors. These must satisfy $U \cdot U > 0$, or in the case of momentum, $p \cdot p = -m^2 > 0 \implies m^2 < 0$. While this class of vectors is not so relevant when talking about physical particles (for the same reason as the null vectors above, no perspective shift can take us from physically relevant $m^2 > 0$ vectors to these), it is interesting to note that they do have some sort of meaning in particle physics. Weinberg has a good short discussion of why (see chapter 1, section 13 in [3]), the takeaway of which is that special relativity demands the existence of anti-particles (particles of the same mass but opposite charge of a given particle) because the uncertainty principle allows particles to traverse spacelike displacements.

Accelerating Ahead

The four-velocity and four-momentum just defined are applicable to *unaccelerated* or *inertial* bodies, hence why we were able to find an inertial rest frame. In fact, while we defined them as tangents to a curve, we didn't need to take that infinitessimal limit, the four-velocity and four-momentum of inertial particles are just as easily defined in terms of displacement vectors. For any two proper times τ_1 and τ_2 (where $\tau_2 > \tau_1$), we have in the particle's rest frame $\Delta x = (\tau_2 - \tau_1, 0, 0, 0)$, and so the velocity defined by $\Delta x / \Delta \tau = (1, 0, 0, 0)$. This is *unique* to inertial observers in Minkowski space.

In general, particles may be subject to external forces (e.g., an electron in a magnetic field) in which case the tangent to the particle's trajectory will *vary* along its path through spacetime. Now you can't find a globally defined (displacement) vector that represents the particle's velocity, but you can do the next best thing, you can find a vector-*field* for the particle's velocity. A vectorfield is pretty much exactly what it sounds like, it is a function that takes a point in spacetime (just a place, not a vector, remember) and turns it into a vector, and it can turn *different* points in to *different* vectors—mathematically[5], $v : M \to V$. In practice, this means the components of the vector take on a dependence on position. For example, if $U = (1, 0, 0, 0)$ is a vector, then $U(x) = (t^2, 0, 0, 0)$ is a vectorfield, or if we were dealing with a particle tracing a path through spacetime, we would write $U(x(s)) = (x^0(s), x^1(s), x^2(s), x^3(s))$.

[5]PIN: Put a pin in this definition too. Here the vectorfield maps points in M to the same vectorspace V, but in GR it will map different points to *different* vectorspaces. Mathematically, $v : M \to TM$, where TM is the tangent bundle of the manifold that contains *all* of the tangent vectorspaces at every point.

The quintessential vectorfields in classical mechanics are again velocity and momentum, and now also acceleration and force. These are defined as one would expect, $a = \frac{d}{d\tau}U(x(\tau))$ and $f = \frac{d}{d\tau}p(x(\tau))$. Another four-vectorfield you are familiar with comes from electromagnetism. EM has a special history with relativity, it was the first Lorentz-covariant theory developed (and that covariance was a coincidence). In fact, it turns out that the electric scalar potential and the magnetic vector potential naturally fit together into a four-vector $A = (\phi, \vec{A})$ known as the electromagnetic four-potential. In four-vector (and tensor) notation, Maxwell's equations simplify greatly, but that is beyond the scope of this course.

ICRFs

One important note to make about the four-velocity field of an accelerated particle. While it is not possible to find a single inertial "rest" frame for an accelerated particle, it is important to remember that accelerated particles are still massive particles, and are good observers of Nature. As a result, it must always be possible to *instantaneously* find a rest frame, even for an accelerated particle. This frame, only defined for a specific value of τ, is called the **Instantaneous Co-Moving Rest Frame** of the particle[6]. This actually imposes a fairly stringent restriction on the types of paths a massive particle can trace out in spacetime. A general result is that at every proper time τ, there exists a frame such that $U(\tau) \cdot U(\tau) = -1$ (these frames are not the same from τ to τ, but the dot product is Lorentz-invariant, so you can move between frames with impunity). The four-acceleration therefore satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(U(\tau) \cdot U(\tau)\right) = 2U(\tau) \cdot \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = 2U(\tau) \cdot a(\tau) = 0. \tag{1.4.5}$$

That is, the four-acceleration of physical particles (and I stress this result is *only* valid for physical particles) is always perpendicular to their instantaneous four-velocity. (Ex. prove the first equality).

1.5 Tensors

The four-momentum p of a particle is sufficient to describe everything about its energy and momentum. However, what we're working towards is a theory of gravity, and that necessarily requires thinking about more than one thing. In fact, for typical gravitational scenarios, we're interested in very macroscopic quantities of things, so knowing a single particle's four-momentum is not extraordinarily helpful. Instead, we'll need to construct more complicated objects to describe N particles' energy-momenta, as well as their interactions.

In fact, this need to expand from one-dimensional vectors should be familiar. Look no further than quantum mechanics for an example why: The quantum state of a system of more than one particle is a weird, giant, coupled Hilbert space, $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots$. For instance, a three-dimensional particle-in-a-box is described by three independent particle-in-a-boxes shmushed (multiplied) together. What we need is general a way to build up more complicated structures out of the relatively simple ones we've come to know and love[7], and the most general (nice) way to do that is through the **tensor product**.

[6] Your text, as well as some others, calls this a "momentarily" co-moving rest frame. I choose "instantaneous" because I feel it is more descriptive. Neither of these are usually mentioned in the literature, so use whatever makes the most sense to you.

[7] Love not necessarily required.

Take two vectorspaces V and W, and define an operator $\bigotimes$ that maps $V \times W$ to a brand new vectorspace $V \otimes W$. All we really need from this operator is that it preserves the linearity of the original vectorspaces, so it must be "linear in both slots", like an inner product, only this time it maps to a new vector instead of a number. It's easiest to see how this works by first looking at basis vectors. Say $V = \text{span}\{\hat{e}_i\}$ and $W = \text{span}\{\hat{f}_i\}$, then $V \otimes W = \text{span}\{\hat{e}_i \otimes \hat{f}_j\}$. The objects $\hat{e}_i \otimes \hat{f}_j$ are just individual vectors with a long (descriptive) name, and there are $\dim(V) \times \dim(W)$ of them. A general vector in the tensor product space $V \otimes W$ therefore looks like:

$$T = T^{\mu\nu} \hat{e}_\mu \otimes \hat{f}_\nu. \tag{1.5.1}$$

A couple of important points here: 1. any two vectors v and w can be tensored together, the result being a tensor $v \otimes w = v^\mu w^\nu \hat{e}_\mu \hat{f}_\nu$, and 2. *not* every tensor is the tensor product of two vectors (ex. prove $T = \hat{e}^0 \otimes \hat{f}^0 + \hat{e}^1 \otimes \hat{f}^1 \neq v \otimes w$ for any $v \in V\,, w \in W$). This second point is maybe less intuitive, but is extremely important, it is the reason the tensor product is so powerful. The tensor product space is much larger than its constituent vectorspaces individually, and in that extra space is room for a lot of physics. Again, quantum mechanics furnishes an excellent example of this point. When two or more quantum particles are studied together, the Hilbert space of the whole system is the tensor product of their individual Hilbert spaces. When you can think of a state as being $|\psi\rangle \otimes |\phi\rangle$, it is possible to think of the two particles individually, but the full (tensor product) space has room for more complicated states—*entangled* states—that don't look like that, and force us to lose our ability to understand the particles as existing separately.

So now that we have some sense of what tensor products are, and what the tensor product space is, we can write down some things to do with them, and conventions we will follow. First, our tensors will be defined in terms of a select pair of vectorspaces. For now, we'll make these the space of displacement vectors D and its dual space D^*. We then define the following:

- The **rank** of a tensor is the number of copies of D and D^* that make up the tensor product space to which it belongs. For example, if T is a tensor in the space $D \otimes D \otimes D^* \otimes D^*$ the it is a rank-4 tensor[8]. Equivalently, if we write it in component form:

  $$T = T^{\mu\nu}{}_{\rho\lambda}\, \hat{e}_\mu \otimes \hat{e}_\nu \otimes \hat{e}^\rho \otimes \hat{e}^\lambda, \tag{1.5.2}$$

  then the rank of the tensor is the number of indices it carries.

- The **type** of a tensor is more descriptive. If a tensor lives in a space that is M copies of D and N copies of D^*, we call it a type-$\begin{pmatrix} M \\ N \end{pmatrix}$ tensor. As above, we can get this information from the components as well. According to convention, we write indices for contravariant vectors in superscript, and indices for covariant vectors in subscript, so a type-$\begin{pmatrix} M \\ N \end{pmatrix}$ tensor has M upper indices and N lower indices. Notice that in this way, we can consider contravariant vectors to be type-$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ rank-1 tensors, while covariant vectors are type-$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ rank-1 tensors. It is also convenient to define scalars to be rank-0 tensors.

Next, let's go over some of the things we can do with tensors.

[8]For physicists. There are some areas of mathematics where the rank of a tensor is defined as the fewest number of basis vectors it takes to express it.

- Since tensor product spaces are vectorspaces, it follows that we can take the shmush any two tensors together into a bigger tensor. If T is a type-$\binom{M}{N}$ tensor, and R is a type-$\binom{P}{Q}$ tensor, then $T \otimes R$ is a rank-$(M+N+P+Q)$, type-$\binom{M+P}{N+Q}$ tensor.

- Interestingly, we can also find a natural way to *reduce* the rank of a tensor, as long as it has one of each type of index. It might have always seemed a bit precarious having vectors live so close to things that eat vectors, and we can exploit that. Whenever we so choose, we can take a type-$\binom{M}{N}$ tensor and turn it into a type-$\binom{M-1}{N-1}$ tensor by letting one of the dual vectors eat one of the vectors. For example, if we have $T = T^{\mu\nu\dots}{}_{\rho\sigma\dots}\, \hat{e}_\mu \hat{e}_\nu \cdots \hat{f}^\rho \hat{f}^\sigma \cdots$, we can "contract" say the ν and σ indices by using $\hat{f}^\sigma(\hat{e}_\nu) = \delta^\sigma_\nu$ to get a new tensor $T' = T^{\mu\nu\dots}{}_{\rho\nu\dots}\, \hat{e}_\mu \cdots \hat{f}^\rho \cdots$. Note: this is also sometimes referred to as taking the trace across the indices ν and σ, since this is a generalized notion of the trace from linear algebra (to see why, notice that a rank-2 type-$\binom{1}{1}$ tensor can be represented as a matrix, and the contraction of its two indices is $M^i{}_i$, exactly the usual trace).

- As alluded to earlier, we can also play a little gymnastics with our indices. The multilinearity of the tensor product is really a beautiful thing, any linear thing we could do with our separate vectorspaces we can *also* do to the vectorspaces sitting inside a tensor product. In particular, we can also use the metric tensor to raise or lower indices on a general tensor. In index notation, for $T^{\mu\nu}{}_{\rho\sigma}$, we "lower the index ν" by defining a new tensor by with components $T^\mu{}_{\nu\rho\sigma} := \eta_{\nu\lambda} T^{\mu\lambda}{}_{\rho\sigma}$. Similarly, for the same tensor, we can "raise the index ρ" by defining a new tensor with components $T^{\mu\nu\rho}{}_\sigma := \eta^{\rho\lambda} T^{\mu\nu}{}_{\lambda\sigma}$. (Note, if it helps, you can think of this as a two-step operation: 1. take the tensor product with the metric, 2. contract the desired index with one of the metric's indices).

Lastly, a word of caution: order matters. The object $v \otimes w$ is in general *not* the same thing as $w \otimes v$. In fact, this is such an important point, that we use special symbols when there is a symmetry or anti-symmetry. In the mathematical literature, you will find the symmetric tensor product denoted $\odot$, while the anti-symmetric product is $\wedge$ (the anti-symmetric "wedge" product is actually very useful in physics, and while not so important for this course, will be covered in an appendix at some point). In index notation, no special symbols are employed, it's generally just stated that something is (anti)symmetric on two specific indices. Sometimes, however, it is handy to symmetrize or anti-symmetrize a given tensor, in which case we use a special notation around the indices. For example, $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ represents the symmetrization of the tensor $T_{\mu\nu}$, while $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$ represents the anti-symmetrization of the same tensor.

Aside 7: Tensors as operators

In some more modern circles, it is common to think of all tensors as functions that map vectors and covectors to a single number. How is this possible? Simply by recognizing that it's "duals all the way down." That is, while we had defined the dual space V^* as the space of functions that take vectors in V to real numbers, we can also recognize that the original vectorspace is actually the dual of the dual space, $V = V^{**}$. For any vector $v \in V$, we can define the map $v : V^* \to \mathbb{R}$ by $v(\omega) = \omega(v)$. In this way, we can release V from its burden as a special vectorspace and live in a magical world where everybody is a function. While this is very useful from a formal perspective, it is not so important for a basic derivation of GR, so we won't pursue it further.

The Metric Tensor Really is a Tensor!

One last note: we've been calling η the metric "tensor," it is time to justify the name. Recall that one of the ways we have been thinking about η is as a bilinear form on contravariant vectors, that is that $\eta(v, w) \in \mathbb{R}$ and is linear in both slots. Knowing what we now know, another way to express something like that would be as a type-$\binom{0}{2}$ tensor:

$$\eta = \eta_{\mu\nu}\hat{e}^\mu \otimes \hat{e}^\nu. \tag{1.5.3}$$

When presented with two vectors $v = v^\mu \hat{e}_\mu$ and $w = w^\mu \hat{e}_\mu$, we find:

$$\begin{aligned}
\eta(v, w) &= \eta_{\mu\nu}(\hat{e}^\mu \otimes \hat{e}^\nu)(v^\rho \hat{e}_\rho, w^\sigma \hat{e}_\sigma), \\
&= \eta_{\mu\nu} v^\rho w^\sigma \hat{e}^\mu(\hat{e}_\rho)\hat{e}^\nu \hat{e}_\sigma, \\
&= \eta_{\mu\nu} v^\rho w^\sigma \delta^\mu_\rho \delta^\nu_\sigma, \\
&= \eta_{\mu\nu} v^\mu w^\nu,
\end{aligned} \tag{1.5.4}$$

exactly as we wanted. Similarly, we can define the metric inverse as a type-$\binom{2}{0}$ tensor, $\eta^{-1} := \eta^{\mu\nu}\hat{e}_\mu \hat{e}_\nu$ (show that this also leads to $v^* \cdot w^* = \eta^{\mu\nu} v_\mu w_\nu$). It is also a useful calculation to show that the mixed metric $\eta^\mu{}_\nu$ is just the identity δ^μ_ν (hint: use the inverse metric to raise an index on the metric).

Tensor Equations

And this brings us to the point of tensors, and why it's so important that we phrase physics in terms of them. Tensors are geometric objects, so when we write equations in the form $T = P$, where T and P are tensors (including scalars, vectors, and covectors), the equations are valid in *all* equivalent coordinate systems. This is also why it's so important that we specify what vectorspaces we are using to construct our tensors, and what constitute equivalent coordinate systems. For example, consider a rank-3 tensor given by $T = U \otimes U^* \otimes U^*$, or in components[9]:

$$T^\mu{}_{\nu\lambda} = U^\mu U_\nu U_\lambda, \tag{1.5.5}$$

where U is the four-velocity of an unaccelerated particle. This equation might seem intimidating—a rank-3 tensor in 3+1 dimensions has $4^3 = 64$ independent components!—but knowing it is valid in *all* inertial coordinates greatly simplifies the calculation. In the rest frame of the unaccelerated

[9] Here Wald [4] would use his "abstract index notation" and use Latin letters for the indices to indicate that this is a proper tensor equation in index form, as opposed to say (1.5.6), which is only valid in a particular coordinate system. I feel it is just as good to make clear notes in the margin, or in the variable names, so I will follow the more common convention of always using Greek letters, and let the meaning be clear from context.

particle, its four-velocity is simply $U^\mu = (1,0,0,0)$ and the covariant form $U_\mu = (-1,0,0,0)$, so we can say that:

$$T^\mu{}_{\nu\lambda} = \begin{cases} 1, & \text{if} \quad \mu = \nu = \lambda = 0, \\ 0, & \text{else} \end{cases} \quad , \qquad \text{(Rest frame)} \tag{1.5.6}$$

and from there, the components can be retrieved in any other inertial frame by appropriate Lorentz transformations:

$$(T_{\text{Gen}})^\mu{}_{\nu\lambda} = \Lambda^\mu{}_\alpha \Lambda_\nu{}^\beta \Lambda_\lambda{}^\gamma \, (T_{\text{Rest}})^\alpha{}_{\beta\gamma} = \Lambda^\mu{}_0 \Lambda_\nu{}^0 \Lambda_\lambda{}^0. \tag{1.5.7}$$

Here again we'll emphasize that the appearance of Lorentz transformations on the RHS of (1.5.7) is a direct consequence of our choice to work exclusively in flat, Cartesian Minkowski space. Had we been working in Cartesian Euclidean space, for example, the matrices would have been orthogonal instead, or in a more general space, they would have been the Jacobian of the coordinate transformation between reference frames (but more on that shortly).

1.6 A Little More Physics

While we've mostly discussed a single particle, or sometimes a scalar field so far, we're moving towards a theory of gravity, which is necessarily a very macroscopic theory, so we need to be able to describe not just particles but macroscopic objects, and distributions of particles. Fundamentally, it is of course possible to describe a macroscopic system in terms of a collection $\{(p_n)^\mu\}$ of four-momentum vectors, but an equivalent (and better) strategy is to think about four-momentum *flux*, the amount of four-momentum vectors p^μ flowing through a given 3-surface, defined by a covector ω_ν. This must necessarily be described not by a four-vector, but rather by a rank-2 tensor, $T^{\mu\nu}$, known as the **Stress-Energy Tensor**. The components refer to the flux of the μ-component of four-momentum through a surface defined by the dual vector $\hat{e}^\nu$.

Unfortunately, we immediately hit a snag. Fluxes are defined as a rate of change of *densities*, but densities are more complicated in relativity. The easy way to see this is to think of some density as a quantity Q divided by some volume $V = \Delta x \Delta y \Delta z$, but volume is a *frame-dependent* concept, lengths in different frames are contracted relative to other frames, and that follows through to the (pre-relativity) definition of volume. In order to make densities precise and applicable to special and general relativity, we need more tools from differential geometry than we have time to cover in this course. What we'll do instead as a workaround is follow Carroll [2] and manually construct the one type example of stress-energy that we'll actually need for physical applications later, and in the next chapter we will use a variational principle to define general stress-energy tensors without the need to build them from first principles. I'll just need you to trust me that there exists a stress-energy tensor that is exactly equivalent to the collection $\{(p_n)^\mu\}$ of n-particles' four-momentum.

In this course, our emphasis is on deriving the field equations of GR, and applying them to 3 physical situations: gravitational radiation, black holes, and cosmology. Interestingly, the first two situations can be covered without even describing any matter-energy content[10]. The third situation, Cosmology, can be understood to first approximation by describing a very simple kind of matter, a perfect fluid, so we will only construct a stress-energy tensor for that.

[10] Strictly speaking, black holes are described by a non-zero $T_{\mu\nu}$, but all of the physics we understand takes places in regions where $T_{\mu\nu}$ does indeed vanish.

Fluids are a kind of very macroscopic matter, where individual particles are so small and inconsequential that it makes more sense to think of fields, where the field values at each point represent a statistical average of the properties of the large number of particles contained within an infinitessimal region around that point. Perfect fluids are a special class of fluids characterized by having no shear, or viscous forces, or heat conduction (basically all the same thing). We'll explain what those properties mean in a moment, first we'll examine the simplest fluid possible: dust.

Dust

In relativity, dust refers to a macroscopic collection of non-interacting, stationary, homogeneously and isotropically distributed particles. Of course, "stationary" is a frame-dependent concept, but here we mean there exists a rest frame in which all particles in the fluid are at rest. Such a distribution is very simple to characterize in its rest frame, one only needs to know the density n of particles in space. Note that this is explicitly defined *in the rest frame* of the dust, so although it is a "density" it is an invariant quantity, much like how mass is defined in a particle's rest frame, and so is an invariant quantity.

In its rest frame, the macroscopic dust distribution can be assigned a single four-velocity. Since all of its constituent particles are at rest, the rest-frame four-velocity of a dust field is simply $U^\mu = (1,0,0,0)^T$, the same as it was for a point-particle. The only flow of particles per unit volume is in the timelike direction, so we define the **number-flux four-vector** as:

$$N^\mu = nU^\mu. \tag{1.6.1}$$

(Notice that (1.6.1) is a tensor equation, so it is valid in all equivalent reference frames). This is almost what we're looking for, but we're really interested in energy, not just particle number, so still in the rest frame, observe that there is no kinetic energy in sight (no spatial momentum) so the only energy available is the rest mass of each particle in the dust field (and for simplicity, we take the dust to be composed of only a single species of particle). In the rest frame of the dust, we therefore have:

$$T^{00} = mN^0 = mnU^0 \qquad \text{(rest frame)}. \tag{1.6.2}$$

Now we get to see some immediate return on our investment of defining tensors as geometric, coordinate-free objects. Equation (1.6.2) is not *quite* a tensor equation, there's a rank-2 tensor on the left-hand side, but only a rank-1 tensor on the right-hand side, so so far it's only an equation for a single component in a single reference frame. However, if we could find a second-rank tensor that is equivalent to the right-hand side in the rest-frame of the dust, and for its 00-component, that *would* be a tensor equation, and it would *have* to be equal to the stress-energy tensor in all frames. While this kind of guess-work is not easy as a general rule, it is shockingly easy here. All we need is a tensor whose components are $(1,0,0,0)^T$, and we already have that, it's just U^μ. Therefore, observing that $T^{00} = mnU^0U^0$ is a valid tensor equation, we can write:

$$T^{\mu\nu}_{\text{dust}} = mnU \otimes U = \rho U \otimes U, \tag{1.6.3}$$

where we define the invariant energy density $\rho := nm$ (remember, n and m are both invariant quantities). Next up, a slightly less dull fluid.

Perfect Fluids

Back again in the rest frame of the fluid, this time the particles are allowed some motion, but heavily restricted in the type of motion. As a perfect fluid, it is not allowed any shear forces, or heat conduction, so there can be no linear momentum flux through an orthogonal direction, and there can be no energy flux through any spatial direction. Therefore, in addition to the energy density we already saw with dust, perfect fluids are allowed *one* more parameter, a component of linear momentum flux through a surface defined by a dual vector parallel to that linear momentum—a.k.a., pressure. In components, we must have:

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \qquad \text{(rest frame)}. \tag{1.6.4}$$

(Note: my apologies for re-using the variable p. This is standard practice in the field, you'll just have to use context to differentiate p the four-momentum vector from p the pressure of a perfect fluid). Notice that there is only a single pressure as well; if the fluid had a different pressure in the x and y directions, that would introduce sheer (picture the stress induced on the surface of an ellipse expanding faster along its major axis than its minor axis).

Now we do the same thing, we have to use the tensors we know to put together a tensor expression that takes the form (1.6.4) in the rest frame of the fluid. This time it's a little bit trickier, but one way to make the solution more apparent is write the matrix as a sum:

$$\begin{aligned} \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} &= \rho \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + p \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ &= (\rho + p) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + p \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{1.6.5}$$

where we used the magic of $0 = 1 + (-1)$. Now we have recognizable tensors in both terms ($U \otimes U$ in the first, and η in the second), so we can immediately write:

$$T^{\mu\nu}_{\text{Perfect fluid}} = (\rho + p) U \otimes U + p\eta^{-1}, \tag{1.6.6}$$

which again is valid in all equivalent frames because it is a *tensor* equation. (Recall that tensors form a vectorspace so the sum of two tensors is indeed a valid tensor, and note that we wrote η^{-1} since we need contravariant indicies to match).

Symmetry of the Stress-Energy Tensor

It is always important to track whether or not a tensor is symmetric or anti-symmetric on any of its indices. Take the metric tensor, for example: it is very important that η be symmetric in order to reasonably be considered a sort of inner product, and this imposes strong constraints on the possible values for its components. The stress-energy tensor also turns out to be symmetric.

A detailed, general proof of this statement is somewhat orthogonal to this course, so we refer the reader to the nice explanation in Schutz [1, §4.5]. In lieu of a full explanation, we offer a few short observations: a) the stress-energies of dust and perfect fluids given above are both clearly symmetric tensors, b) $T^{0i} = T^{i0}$ because energy and momentum are the same thing (energy flowing through time is $\rho U = nmU = np$), and c) $T^{ij} = T^{ji}$ must be true so that there is no local torque (e.g., shear in $x - y$ plane induces a torque in the z-plane, so it must be balanced by exactly the same shear in the $y - x$ plane, or else the material would be very odd indeed).

Conservation of Stress-Energy

The stress-energy tensor also satisfies one more property, it is conserved. This is encoded in the statement:

$$\partial_\nu T^{\mu\nu} = 0. \tag{1.6.7}$$

Notice that this is essentially a 3+1-dimensional divergence, and is the statement that energy/momentum can flow in one spacetime dimension as long as that flow *came from* or is *going to* another spacetime dimension. Conservation of stress-energy is not a property of the tensor as a geometric object, it has to be built in by construction, or as a consequence of an underlying symmetry (as in the case when it is derived from an action). This is the property that generalizes the notion of conservation of energy.

To understand the conservation equation, first we have to understand how to define the derivative of a tensor. In Minkowski space this is fairly straightforward (spoiler: this will be more complicated very soon). Take the gradient of the entire tensor:

$$\begin{aligned}\partial T = \partial\left(T^{\mu\nu}\hat{e}_\mu \otimes \hat{e}_\nu\right) &:= (\partial_\lambda T^{\mu\nu})\hat{e}_\mu \otimes \hat{e}_\nu \otimes \hat{e}^\lambda + T^{\mu\nu}\partial\left(\hat{e}_\mu \otimes \hat{e}_\nu\right), \\ &= (\partial_\lambda T^{\mu\nu})\hat{e}_\mu \otimes \hat{e}_\nu \otimes \hat{e}^\lambda. \end{aligned} \tag{1.6.8}$$

The second term in the first line vanishes because of the affine structure of Minkowski space; displacement vectors are defined throughout *all* of spacetime, and as such we choose to describe displacement vectors in terms of constant, orthogonal, Cartesian bases. The end result is that the gradient of a tensor is the tensor product of the gradient of the components of that tensor (the gradient of scalar fields, which we know is a well-defined covector in Minkowski space) with the associated tensor basis vectors. For a general type-$\begin{pmatrix} M \\ N \end{pmatrix}$ Minkowski tensor, the gradient is a well-defined type-$\begin{pmatrix} M \\ N+1 \end{pmatrix}$ Minkowski tensor. The conservation equation (1.6.7) is then just the trace of the gradient of the stress-energy tensor.

Aside 8: Notation

Derivatives of tensors are going to be pretty common in the next chapter, so for convenience, we define the following notation for tensor components. The simple partial derivative of a tensor is written $\partial_\mu T^\nu{}_\rho =: T^\nu{}_{\rho,\mu}$.

Take the stress-energy of dust, for example. The conservation equation is $\partial(\rho U \otimes U) = 0$, or in components:

$$\partial_\nu T^{\mu\nu}_{\text{Dust}} = \partial_\nu\left(\rho U^\mu U^\nu\right) = 0. \tag{1.6.9}$$

For unaccelerated dust, this is the uninteresting statement that the energy density flow is constant in all spacetime directions. If however, we imagine the dust is say electrically charged and responding to an external electric field[11], then the dust no longer has a constant four-velocity U^μ, so the statement (1.6.9) becomes more interesting.

$$\begin{aligned} & U^\nu \partial_\nu (\rho U^\mu) + U^\mu \partial_\nu (\rho U^\nu) = 0, \\ \implies \quad & U^\nu \partial_\nu (n U^\mu) + U^\mu \partial_\nu (n U^\nu) = 0, \end{aligned} \tag{1.6.10}$$

using that $\rho = mn$ for a single species of particle. The conservation equation is then a statement about flow of particles. In particular, the first term can be rewritten using $\frac{d}{d\tau} = \frac{dx^\mu}{d\tau}\partial_\mu = U^\mu \partial_\mu$ so we have in fact a continuity equation for particle number:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(nU^\mu) + U^\mu \partial_\nu (nU^\nu) = 0, \tag{1.6.11}$$

i.e, the (proper) time rate of change of particles out of the fluid is the divergence of particle number. Of course given the way we've defined fluids, the way for particles to flow out is to be converted or destroyed, and in most cases we will only be interested in fluids that preserve particle type and number (this can be built in by symmetry in a variational approach), so we will in fact separately require:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(nU^\mu) = 0, \qquad \text{and} \tag{1.6.12}$$

$$\partial_\nu (nU^\nu) = 0. \tag{1.6.13}$$

Looking Forward

Fun as that exercise was, in general it is not trivial to guess the form of meaningful tensors, and later on we will define the stress-energy tensor more generally by means of an action. For now though, we have all that we need to describe the physical situations we will study in this course. The three main topics we'll look at are: gravitational waves, black holes, and cosmology. These situations surprisingly parallel what one would find in a course on electromagnetism: sources are respectively none (gravitational radiation ↔ EM radiation), a delta function (black hole ↔ point-charge), and full distributions (perfect fluid matter ↔ dielectric media). But first, we need to "covariantize" gravity, and that is the task we tackle in the next chapter.

1.7 Recap

Let's recap the most important points, and lay out the rules for how to use what we've learned.

- Time is a *place* not a parameter
- Minkowski space is like Euclidean space, but the inner product is not a proper inner product, it is the Minkowski metric tensor η and it is *not* positive-definite
- Displacement vectors are *not* guaranteed to be a thing, they're a peculiarity of Minkowski/Euclidean space

[11] Well, not really of course, an actual electric charge and field would involve potential energy and energy intrinsic to the field, both of which would need to be included in $T^{\mu\nu}$.

- Vectors are abstract objects, not tied to a particular choice of basis. Think of them as carrying units, so that if we write $v = v^\mu \hat{e}_\mu$, then the vector quantities are v and $\{\hat{e}_\mu\}$ so the dimensionalities are $[v] = L$, $[\hat{e}_\mu] = L$, and $[v^\mu] = 0$

- Despite the fact that their existence is fragile, we base our whole lives around displacement vectors, so we say that if the basis vectors of displacement vectors are transformed by a Lorentz transformation, then objects whose components are corrected by the inverse matrix are *contravariant*, while objects whose components are corrected by the original transformation are *covariant*

- Metric induces natural duality, quantities $\leftrightarrow$ rates, vectors $\leftrightarrow$ covectors

- Tensor product to build up more complicated structures

- Macroscopic theory means macroscopic description of matter-energy $\longrightarrow$ the stress-energy tensor $T_{\mu\nu}$.

1A Appendix

1A.1 The Lorentz Group and The Rotation Group

1A.1.1 Some Group Theory

1A.1.2 The Rotation Group

1A.1.3 The Lorentz Group

1A.2 The Wedge Product, Exterior Derivative, and Hodge Dual

Chapter 2

Manifolds, Tangent Vectors, Cotangent Vectors, and Tensors

Special relativity is all well and good, but what's this we keep hearing about gravity? What could that possibly have to do with anything we've just seen? Well of course, the setting of Minkowski space, and removal of time as a parameter, requires a refactoring of our other physical laws, all of which are typically written in what we will soon call a "non-covariant" way. In principle though, that should be a separate topic, a separate course of relativistic electromagnetism, relativistic thermodynamics, relativistic quantum mechanics, etc. Why is the theory of relativistic gravity called the *general* theory of relativity? The answer boils down to a very simple equation:

$$m_I = m_G.$$

This is (a phrasing of) **the Equivalence Principle**: inertial mass (from $F = m_I a$) is equivalent to gravitational mass (from $F_G = -Gm_G M/r^2$). It directly follows from the equivalence principle that no two bodies can exist unaccelerated, and that *the gravitational acceleration of every body is identical, regardless of its gravitational charge (m_G).* That is to say, while you could theoretically construct a relativistic version of Newtonian gravity[1]... why would you? If everything responds the same way to everything else's gravitational field, why bother with a gravitational field at all, why not just map out the geometry that everything will follow?

The absence of an impartial observer really is a fundamental point here as well. How do we construct every other physical theory? Who can impartially observe the electromagentic interaction? Well surely someone without electric charge. Who can observe the nuclear interactions? Again, someone with no nuclear charge. Even thermodynamics ensembles can safely be observed by someone not involved in the ensemble. But in gravity, every potential observer has mass and must partake in the festivities, and the only objects without mass (photons, gluons, maybe gravitons) are cursed to live without an inertial frame of reference.

This is essentially the argument Einstein used to derive the general theory of relativity. Today, it's very common to be introduced to this line of thinking through the gedanken experiment of a person in an accelerating rocket in space (under a constant acceleration, and with no windows in the craft, any experiment the astronaut performs in the rocket will return precisely the same results as it would in a lab on Earth—an isolated, non-rotating Earth, of course). Whichever way you slice it, the statement is that gravity is just the geometry of spacetime, so we have two tasks before us: 1. How do we understand motion in curved spacetime? and 2. What equation describes the geometrical response to matter? The first question just requires us to learn some math, and

[1] And in some practical cases, this is actually useful, cf. Gravitoelectromagnetism, [7].

is the topic of the next couple of chapters. The second question is answered by the Einstein Field Equations. Strictly speaking, the EFEs are an axiom of the theory, but we will at least motivate them as the equations of motion of spacetime geometry in chapter 4, and with the rest of the material explore some of their physical applications.

2.1 Differentiable Manifolds

When two or more massive bodies exist in spacetime, it is impossible for any of them not to accelerate. Phrased another way, it is impossible to associate a global inertial frame with any object when there exists more than one object. Think about it like this, if I try to use the (isolated, non-rotating) Earth as my global coordinate system, I will immediately fail, as no object that I let go of will stay put, they'll all start moving towards the ground (violating Newton's first law). And from the perspective of the ball I drop, the Earth will rapidly approach it, no matter how much it asks it not to. There was a crucial word there though: *global*. Einstein's brilliant insight that led to GR is completely analogous to the brilliant insight that led Newton to his laws of motion. When Newton wanted to study the complicated curved trajectories followed by physical bodies, he broke their motion down into small line segments and invented calculus; when Einstein wanted to study the curved geometry that results from universal gravitation, he broke the big curved universe down into small, flat patches, and (co-)invented differential geometry. Where Newton realized all we really know how to work with are lines, Einstein realized all we really know how to live in is a Cartesian space, so the general strategy of GR is to describe physics as **locally Minkowski**, even in situations where it can't be Minkowski everywhere. The statement that spacetime looks like something *locally* actually has two meanings, one in terms of the global structure (that of a manifold) and another in terms of the setting for vectors (tangent spaces). We'll treat them in turn; first the universe.

Step 1 of GR is constructing spacetime as a *differentiable manifold*. We'll start with the definition, and pick it apart.

Definition 1

A smooth **differentiable manifold** is a topological space[a] M together with an atlas of charts $A = \cup_i(U_i, \varphi_i)$ such that M is fully contained in A, such that the coordinate functions φ_i are C^∞, invertible, and such that the transition functions are also C^∞.

[a] Strictly speaking also Hausdorff and second-countable.

I admit that doesn't immediately sound very helpful, but it's really just mathematical lingo for a very intuitive idea: we understand Cartesian spaces, and while not everything in Nature is Cartesian, everything important can be *covered* in Cartesian spaces. Maybe the best part of this lingo is that the overly-generic mathematical language handily represents the enormous freedom one has to say exactly *how* to cover these lovely things in Cartesian spaces. Let's break it down.

- A topological space M is a bunch of points with a minimum of structure (something about an ability to define limits and whatnot).
- A C^n function is a function whose nth derivative is continuous, so a C^0 function is just a continuous function, while a C^∞—or smooth—function is infinitely differentiable. It is

important we work with smooth functions because we have good empirical evidence that everything physical is smooth, there are no singularities anywhere in Nature.

- A coordinate chart is a map φ from an open subset $U \subset M$ to an open subset of the *topological space* $\mathbb{R}^n$ (note that this is *not* Euclidean space, we do *not* include a vectorspace structure here), see 2.1.1. It is important that φ is both smooth (because Nature is) and invertible because ...
- Whenever two coordinate charts (U, φ) and (V, ψ) overlap, they have to agree on what they're mapping, so in the overlap $U \cap V$, it better be the case that $\psi(\varphi^{-1}) \sim \varphi(\psi^{-1})$.

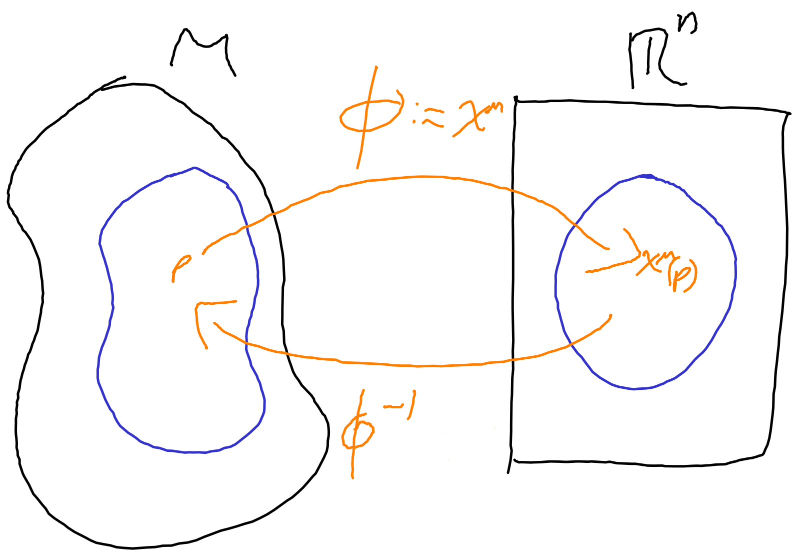

Figure 2.1.1: The coordinate functions on a manifold. The coordinate charts formalize the intuitive notion of coordinates for a given point, $x^\mu(p)$.

Always best to put this in context. Consider the very simple manifold $\mathbb{R}^2 - L$, where L is the origin and positive x-axis. There are loads of ways to cover this manifold in charts, but one way we're very familiar with is polar coordinates, so we can say the "coordinate map" $\phi : M \to V \subset \mathbb{R}^2$ looks like $\phi(p) = (r, \theta)$. As physicists, we're maybe more familiar with going the other way; we visualize the "polar" manifold in terms of $\phi^{-1} : V \to M$ with $\phi^{-1}(r, \theta) = (r \cos\theta, r \sin\theta)$, called a **paramaterization** of the manifold patch. In practice, we need both directions: the fundamental things we need to work with are coordinate charts, but we need to know what they look like to use them, and to know what they look like we need to know how to represent points on the manifold, but a parameterization of points on the manifold is just the inverse of a coordinate chart. Confused? Good. An example should sort us out, lets see what a simple coordinate transformation means in this language.

Again considering $\mathbb{R}^2 - L$, this time we'll look at two charts, the polar coordinates $\overline{x}^\mu(p)$ and Cartesian coordinates, $x^\mu(p)$, and we'll imagine taking a point p in Cartesian coordinates and changing it into polar coordinates (see figure 2.1.2). The Cartesian map is exactly what it sounds like, $(x^{-1})^\mu(x,y) = (x,y) \leftrightarrow x^\mu(x,y) = (x,y)$, while the polar map is as given above. A change of coordinates is a map between $\mathbb{R}^2_{\text{Cart}}$ and $\mathbb{R}^2_{\text{Polar}}$ given by function composition (and in general only defined on the overlap of the patches on the manifold, but they coincide in this example). That is, changing out outlook on the manifold means taking all the points we had mapped with Cartesian coordinates, un-mapping them, then mapping them with the new function, polar coordinates: $\overline{x}^\mu((x^{-1})^\nu) : \mathbb{R}_{\text{Cart}} \to \mathbb{R}_{\text{Polar}}$. Explicitly, for a point (x,y) in $\mathbb{R}_{\text{Cart}}$, we have $\overline{x}^\mu((x^{-1})^\nu(x,y)) = \overline{x}^\mu(x,y) = (r,\theta)$ where r and θ satisfy:

$$\begin{aligned} x = r\cos\theta, &\quad \text{and} \quad y = r\sin\theta, \\ \implies r = \sqrt{x^2+y^2}, &\quad \text{and} \quad \theta = \operatorname{atan}(y/x). \end{aligned} \tag{2.1.1}$$

so we have $\overline{x}^\mu((x^{-1})^\nu(x,y)) = (\sqrt{x^2+y^2}, \operatorname{atan}(y/x))$. Very importantly, notice that the coordinate transformations $\overline{x}^\mu((x^{-1})^\nu)$ are *not* usually linear functions—a major departure from the lovely linear Lorentz transformations from chapter 1! (Ex: perform this calculation again, but now instead of regular Cartesian coordinates, use the rotated coordinates $x^\mu(x+y, x-y) = (x,y)$).

Aside 9: No Vectors!

Here is a good place to really emphasize the point that so far there are *no* vectors in sight. Sure, $\mathbb{R}_{\text{Cart}}$ again has the accidental structure of an Affine space, but $\mathbb{R}_{\text{Polar}}$ *definitely* does not (try it out, take a putative "vector" $(0, 3\pi/2)$ and add it to itself, you'll get $(0, 3\pi)$, and since $3\pi > 2\pi$ it is *not* in the space). This is the *norm* for manifolds, not the exception, so we simply have to give up any notion of a general displacement vector.

In differential geometry, the best non-trivial example is always a sphere. Consider an ordinary 2D sphere (called the two-sphere, or S^2). Clearly it is most convenient to label points on the sphere with polar coordinates, (θ, ϕ), but this is actually harder than it looks. First, there is the problem of open sets; charts on manifolds need to be defined on open sets, and need to be one-to-one, so for angular coordinates, we always have to exclude a whole line of longitude in order to keep the coordinates well-defined, so it takes a *minimum* of two charts to cover the sphere. The other issue is in finding a good mapping to $\mathbb{R}^2$ (and back). Even though the coordinates (θ, ϕ) look nice, we have to remember they represent points on a sphere, so the maps to and from $\mathbb{R}^2$ must reflect that. We might come back to this later, but it is a good idea to look at some texts (e.g., [6]) for the "stereographic projection" representation of the 2-sphere. For now though, we'll direct you to the appendix 2A.1 for a couple more examples, and here we'll just observe two important lessons the 2-sphere tells us about general manifolds: a) a complete description of the two-sphere requires at least two coordinate patches, and b) the simplest coordinates are not always the simplest coordinates.

This is the first meaning of spacetime being "locally Minkowski," that you know how to translate whatever weird coordinates you're using into sensible 3+1-dimensional Cartesian coordinates and back again (even if you can't do it the same way everywhere). The next step is to figure out what vectors mean on a manifold, and this is why we so strongly emphasized the unique affine structure of Minkowski space earlier. Again, a collection of points is generally *not* an affine space

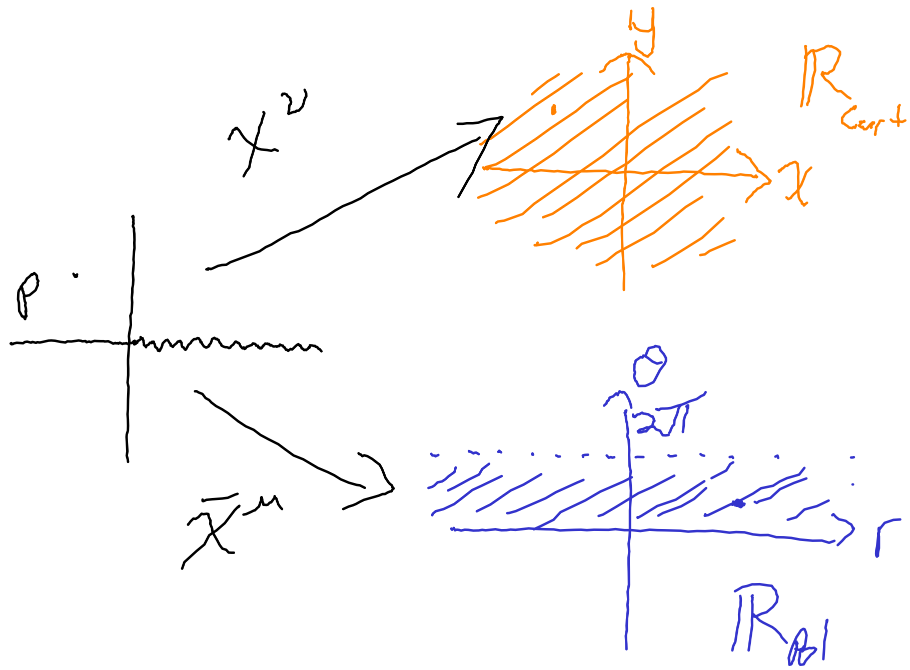

Figure 2.1.2: The manifoldy way of changing from Cartesian to polar coordinates on $\mathbb{R}^2 - L$. The wiggly line on the manifold is the removed positive x-axis, the hatching on the target spaces indicates the range of the coordinate maps, the dashed line in $\mathbb{R}^2_{\text{Polar}}$ is the line $\theta = 2\pi$ and is not included in the target space (and neither is the line $\theta = 0$), and the points in each plane represent respectively p, $x^\nu(p)$, and $\overline{x}^\mu(p)$.

like Minkowski space was, so there is in general no natural notion of displacement vectors. Think about the two-sphere again; what happens if I take the point $(\pi/2, \pi/2)$ and add it to itself? Well that would be a pole, and we said poles can't be contained in the simple (θ, ϕ) coordinate chart. Try as you might, you can't come up with a consistent vector space structure on just the coordinate charts of a manifold. However, since vectors are crucial to Nature, we're going to have to find some way of adapting them to curvy spaces.

2.2 Tangent Vectors, Cotangent Vectors, and Tensors

To figure out how to construct some sort of vector on general spacetime manifolds, we have to invoke a second notion of "locally Minkowski." On the one hand, spacetime is locally Minkowski if we can understand its points in terms of mappings to Cartesian coordinates, but on the other hand we can also think about a space being approximately Minkowski if it looks like Minkowski space when we zoom in *really, really* close. That is, maybe we can't define general displacement vectors in a general spacetime, but if we look at really, really, infinitessimally small displacement vectors? Well surely those must be valid, after all, we were able to study physics for a long time before GR came around.

At any point on the manifold of spacetime, a human-sized ideal observer can be plopped and establish what they think of as a Minkowski coordinate system, complete with its affine structure of displacement vectors. *Physical* things exist on worldlines that pass through that point, and from the observer's perspective, they can compute displacements along those worldlines and play with them as we did in chapter 1. From a global perspective though, those "displacement" vectors are really infinitessimally small, and so are identified with the **tangent vectors** to the worldlines that pass through the point. In this way, the only vectors that survive the transition from SR to GR are the tangent vectors to worldlines, e.g.: $v_p = \frac{d}{ds}\gamma(s)|_p$ is a valid vector (where $\gamma : \mathbb{R} \to M$ is a curve along the manifold, s is the parameter of the curve, and p is a point on that manifold). This leads us to define:

Definition 2: Tangent Space

At any point p in the spacetime manifold M, the set of all tangent vectors to curves forms a vectorspace called the **Tangent Space at p**, denoted T_pM. In general, the tangent spaces at different points p and q are entirely different vectorspaces T_pM and T_qM (see figure 2.2.1), and vectors in one cannot interact with vectors in the other.

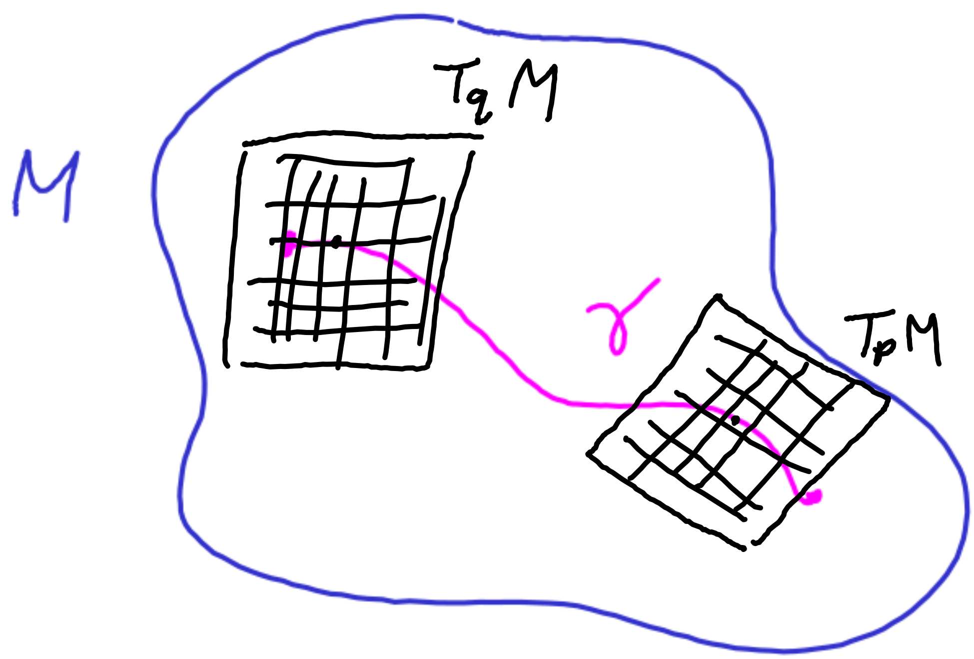

Figure 2.2.1: A visualization of a path traced out in a manifold, and the tangent spaces associated with different points.

Vectors and Vectorfields

In fact, we can do even a bit better than to think about tangents to specific paths. The tangent space is tangents to *all* paths, so if a vector can be defined by a given path: $v_p = \frac{\mathrm{d}}{\mathrm{d}s}|_{s_0}\gamma(s) =$

$\frac{x_\gamma(s)^\mu}{\mathrm{d}s}|_{s_0}\hat{e}_\mu(s_0)$, then surely the same vector can just as well be defined by another path (see figure 2.2.2 that just *happens* to have the same tangent at p: $v_p = \frac{\mathrm{d}}{\mathrm{d}r}|_{r_0}\Gamma(r) = \frac{x_\Gamma(r)^\mu}{\mathrm{d}r}|_{r_0}\hat{e}_\mu(r_0)$ (where $\gamma(s_0) = \Gamma(r_0) = p$), which suggests that the important thing isn't the derivative of a specific path, but rather the derivative itself. That is, we can equally well define the tangent space as the set of all $\frac{\mathrm{d}}{\mathrm{d}s}|_{s_0} = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}|_{s_0}\partial_\mu|_{s_0}$. This even lets us pick out the handy dandy basis vectors $\{\hat{e}_\mu(s_0)\} = \{\partial_\mu|_{s_0}\}$. This identification is so important it warrants its own definition.

Definition 3: Tangent Vectors

An element (a **tangent vector**) of the tangent space T_pM at point $p \in M$ is an object $\frac{\mathrm{d}}{\mathrm{d}s}|_{s_0}$ that generates tangents to curves (where the parameterization is chosen such that $\gamma(s_0) = p$ for all curves). If p lives in a coordinate chart $x^\mu : M \to R^n$, then $\frac{\mathrm{d}}{\mathrm{d}s}|_{s_0} = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}|_{s_0}\partial_\mu|_{s_0}$, and the coordinate partial derivatives $\{\partial_\mu|_{s_0}\}$ form a basis for the tangent space.

Moreover, the expression $\partial_\mu|_{s_0}$ is begging to have that evaluation symbol plucked right off, and in fact we can extend the definition of a vector to a *vectorfield*:

Definition 4: Tangent Vectorfields

A **tangent vectorfield** is an object that returns a tangent vector at every point in space: $v = v^\mu\partial_\mu$ and $v(p) = v^\mu(p)\partial_\mu|_p$. Mathematically, we say that vectorfields live in the **tangent bundle** of the manifold, $TM := \cup_{p\in M}T_pM$.

Aside 10: Evaluating Functions on Manifolds

We've now come across terms like $v^\mu(p)$ and it is worth taking a second to be clear about what we mean. A function on a manifold is formally just some map $f : M \to \mathbb{R}$, but to actually *calculate* anything, we need a representation of that function in terms of some parameters, and that is what the coordinate maps do, they are are nice mappings from the manifold to $\mathbb{R}^n$ that mean we can identify any function on the manifold with an equivalent function $F : \mathbb{R}^n \to \mathbb{R}$ given by composition of f with the coordinate map x^μ: $f(p) = F(x^\mu(p))$, and of course, normally we are sloppy with notation and just write f for both. This is what we mean when we say we write or evaluate things in "local coordinates." Note that these are "local" under the first sense of manifolds being "locally" Minkowski, it just uses the fact that we can parameters patches of points on the manifold with patches of points in $\mathbb{R}^n$.

Aside 11: Differential Geometry Vector Fields

The identification of vectors with derivatives is even deeper than this. In modern differential geometry, vectors are *defined* as derivatives, so that vectorfields on manifolds are defined as linear operators on smooth functions that satisfy the Leibniz law (i.e., the product rule). This definition coincides with the definition in terms of partial derivatives when coordinates are cho-

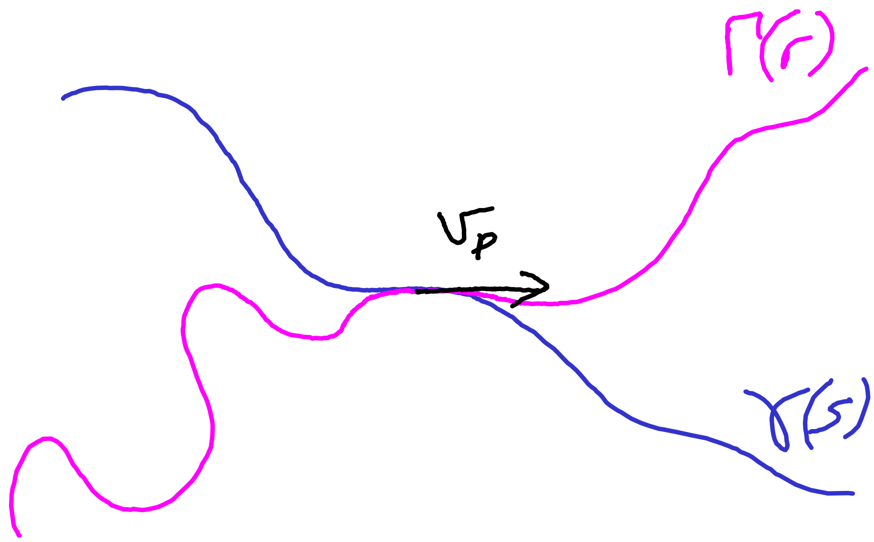

Figure 2.2.2: A visualization of how two curves can have the same tangent at a point p.

sen, but coordinates aren't necessary, the definition is geometric and coordinate-independent in nature.

Tensors

One quick diversion: now that we know the vectors and vectorspaces involved, we can immediately port over our whole discussion of tensors from 1.5, we define tensors at a point p on a manifold to be elements of the tensor product space $T_pM \otimes T_pM \otimes \ldots T_p^*M \otimes T_p^*M \otimes \ldots$, where T_pM is the tangent space as above, and T_p^*M is the dual of the tangent space at a point (we haven't discussed the Minkowski metric yet but the dual space can be defined without a bilinear form, the form is only necessary to provide a *natural* mapping between the two). More generally, tensor fields on manifolds are elements of the tensor product space $TM \otimes TM \otimes \ldots T^*M \otimes T^*M \otimes \ldots$, where TM is the tangent bundle (the collection of all tangent spaces on the manifold) and T^*M is the cotangent bundle (the collection of all dual tangent spaces on the manifold).

Notice that this justifies the effort we put in defining things in terms of displacements! The tangent space is just the space of infinitessimal displacements, so while this might look big and new, it's really the same stuff we've been looking at, just done on a point-by-point basis.

The Metric Tensor

At this point, we have one very important similarity to the content of chapter 1, and one very important divergence. The divergence is that the coordinates used for the basis vectors are *no longer necessarily Cartesian*. Coordinate charts on manifolds can be all sorts of things, there's definitely no restriction to make them be Cartesian (otherwise why bother curving space?), so for example we may have $\partial_1 = \partial_r$ and $\partial_2 = \partial_\theta$—i.e., polar coordinates. The similarity though, is that the Minkowski space our little observer sees has *all* of the structure we discussed in chapter 1, in particular the Minkowski metric tensor may be used to measure lengths of vectors, and may be used to define covectors.

However, there is a giant catch to this similarity! As the gods of this spacetime manifold, we may be kind to little Bob at point p and choose our coordinates such that the coordinate vectors at Bob's location $\partial_\mu|_p$ coincide with Bob's notion of a Cartesian coordinate system: e.g., $\partial_r|_p = \partial_y$ and $\partial_\theta|_p = \partial_x$. In this way, Bob understands how to measure lengths of vectors and define covectors, he just uses the Minkowski metric. However, in those *same* coordinates, little Alice, observing her little world from point q is not so lucky! In her world, we have $\partial_r|_q \neq \partial_y$ and $\partial_\theta|_q \neq \partial_x$. Rather, it is generally true that $\partial_r|_q = a\partial_x + b\partial_y$ and so on. (And of course, Bob is not necessarily special, we could have chosen to bless Alice with orthogonal bases instead, or in a more traditional deity fashion, we may have chosen to smite them both). Since it is impossible to change coordinates every time we want to look at a different point[2], we instead have to live in a world where our basis vectors are in general *not* orthogonal. Still, even if our coordinate basis vectors are not orthogonal, we can always establish some notion of lengths of vectors at a point, we just have to take into account non-trivial overlaps of the basis vectors, where in general $\hat{e}_\mu \cdot \hat{e}_\nu \neq \delta_{\mu\nu}$ and instead varies from point to point. Then just as we always do with point-wise defined objects, we may collect together all of these independent pseudo-inner products into a pseudo-inner product *field*, the metric tensor:

[2] Well, with the help of some auxiliary fields (vielbeins) we technically can, but it's a nuisance

Definition 5: The metric tensor

Collecting together the pseudo-inner products on all tangent spaces in the manifold, we define the **metric tensor** $g : TM \times TM \to \mathbb{R}$ as a tensor field that at each point p in M defines the notion of lengths of vectors. As a pseudo-inner product it is symmetric, and as a physical thing it is continuous and infinitely differentiable. In distinct contrast to the Minkowski metric η, the metric tensor g generally has off-diagonal elements. Notice that according to our definition, $g \in T^*M \otimes T^*M$.

A manifold equipped with a metric tensor is called a **Riemannian manifold** if g is positive-definite, and a **pseudo-Riemannian manifold** otherwise (so in GR we will be concerned with the latter).

Cotangent vectors

Now that we have a metric tensor, we have a natural relationship between vectors (elements of T_pM) and covectors (elements of T_p^*M), induced in just the same way as in 1.3. That is, for every tangent vector $v_p \in T_pM$, there is a natural dual vector given by $v_p^* = g(v, \cdot)|_p \in T_p^*M$. Recall that even in Minkowski space, the *basis* dual vectors satisfied a smoother relationship with the basis displacement vectors, $\hat{e}^\mu(\hat{e}_\nu) = \delta^\mu_\nu$. In fact, the same thing applies here (as it should), we can find a lovely basis given by the gradient of the coordinate functions (and here we use d to represent the gradient instead of ∂ or ∇ both because it is the standard notation in the literature, and to avoid confusion with the tangent vector bases, and later the covariant derivative): $\mathrm{d}(x^\mu)\,(p) = \frac{\mathrm{d}x^\mu}{\mathrm{d}x^\nu}|_p(\hat{e}_p)^\nu = \delta^\mu_\nu|_p(\hat{e}_p)^\nu \implies dx^\mu|_p = (\hat{e}_p)^\mu$. We then have a wonderful shorthand for the action of a dual basis vector on a tangent basis vector: $dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu_\nu$.

Aside 12: Exterior derivative

In fact, this shorthand can be generalized: the gradient of any scalar field $\phi : M \to \mathbb{R}$ is a covector $d\phi_p = \frac{\partial\phi}{\partial x^\mu}|_p dx^\mu|_p$, and the action of this dual vector on any tangent vector is $d\phi_p(v_p) = v_p(\phi) = v_p^\mu \partial_\mu|_p(\phi)$. This is the most basic example of the exterior derivative d, which in general maps $\mathrm{d} : \bigwedge^n T^*M \to \bigwedge^{n+1} T^*M$.

Equivalent vectors

In chapter 1, we made a big deal about the notion of "equivalent" vectors and covectors. With the loss of the Minkowski metric, we lose the special status of Lorentz transformations (except in a few places). This is okay though, because we have a new notion of equivalence, a bigger, bolder notion: general covariance[3]. Geometric structures on manifolds will be said to be generally covariant if the following holds: given two overlapping coordinate patches U, V, $U \cap V \neq \emptyset$, defined respectively by the coordinate functions x^μ and y^μ, the coordinate transformation $x \circ y^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given by

[3] Here is a point of great division in the community. You will hear this type of thing variously referred to as "general coordinate invariance," "general covariance," "diffeomorphism invariance," and others. There are subtle differences between them all, but the truth is that the matter is simply not settled. In these notes I aim for the simplest take, which I understand to be general covariance. A good discussion of the matter can be found in [8].

$x^\mu(y^{-1})$ induces a transformation on vectorfields

$$\partial_{y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\partial_{x^\nu}. \tag{2.2.1}$$

(By the way, we might remember in previous courses defining $\frac{\partial x^\nu}{\partial y^\mu}$ as the "Jacobian" of the coordinate transformation. That name still applies). The components of geometric objects—vectors, covectors, and tensors—then transform in the opposite way, just as we saw with Lorentz transformations:

$$v^\mu \to \frac{\partial y^\mu}{\partial x^\nu}v^\nu, \tag{2.2.2}$$

$$v_\mu \to \frac{\partial x^\nu}{\partial y^\mu}v_\nu, \tag{2.2.3}$$

and tensors follow trivially by sticking one copy of the appropriate vector/covector law onto each contravariant/covariant index.

An especially useful example of this general covariance is expressing the Minkowski metric tensor in polar coordinates (in 2+1 dimensions, for simplicity). Simply turning the crank, we write:

$$(\eta_{\text{Polar}})_{\mu\nu} = \frac{\partial(x_{\text{Cart}})^\rho}{\partial(x_{\text{Polar}})^\mu}\frac{\partial(x_{\text{Cart}})^\sigma}{\partial(x_{\text{Polar}})^\nu}(\eta_{\text{Cart}})_{\rho\sigma}. \tag{2.2.4}$$

Now obviously the switch from Cartesian to polar coordinates does not mess with the time component, so we can pretty easily read off that $(\eta_{\text{Polar}})_{0\nu} = (\eta_{\text{Cart}})_{0\nu} = -\delta_{0\nu}$. For the rest, we have to use the inverse of the coordinate transformation (so the first line of (2.1.1)). We have:

$$\begin{aligned} \frac{\partial(x_{\text{Cart}})_1}{\partial(x_{\text{Polar}})_1} &= \cos\theta, & \frac{\partial(x_{\text{Cart}})_1}{\partial(x_{\text{Polar}})_2} &= -r\sin\theta, \\ \frac{\partial(x_{\text{Cart}})_2}{\partial(x_{\text{Polar}})_1} &= \sin\theta, & \frac{\partial(x_{\text{Cart}})_2}{\partial(x_{\text{Polar}})_2} &= r\cos\theta \end{aligned}. \tag{2.2.5}$$

Then to chug this into (2.2.4), use that $(\eta_{\text{Cart}})_{ij} = \delta_{ij}$, so we find:

$$\begin{aligned} (\eta_{\text{Polar}})_{11} &= \cos^2\theta + \sin^2\theta, & (\eta_{\text{Polar}})_{12} &= -r\sin\theta\cos\theta + r\sin\theta\cos\theta, \\ (\eta_{\text{Polar}})_{22} &= r^2\sin^2\theta + r^2\cos^2\theta, & (\eta_{\text{Polar}})_{21} &= r\sin\theta\cos\theta - r\sin\theta\cos\theta \end{aligned}. \tag{2.2.6}$$

Or in a more conventional form:

$$\eta_{\text{Polar}} = -\mathrm{d}t\otimes\mathrm{d}t + \mathrm{d}r\otimes\mathrm{d}r + r^2\mathrm{d}\theta\otimes\mathrm{d}\theta. \tag{2.2.7}$$

Next up

And there we have it, we now have generalized everything we had cared about in Minkowski space to general manifolds. Well. . . almost. We haven't talked about differentiation of tensors yet, and for good reason. So far, we've resigned ourselves to living in a world of a bajillion tangent spaces and cotangent spaces, none of which can talk to each other. But physics takes place *across* tangent spaces, so next up, we need to learn how to get tangent spaces to talk to each other, how to take physical things along paths through different tangent spaces, and ultimately tease out measures of the underlying curvature of the manifold, and connect that in a physical way to the matter content of spacetime.

2.3 Recap

- The equivalence principle $m_I = m_G$ destroys the idea that there can exist a global inertial reference frame for any physical observer $\longrightarrow$ everything is affected by gravity in the same way, so treat think of gravitational motion as a *property* of spacetime, not something that happens on top of it.
- No global Minkowski space leads to general manifolds instead
- No global Minkowski space means no displacement vectors
- *Local* observers still exist though, so *infinitessimal* displacement vectors still exist $\longrightarrow$ tangent vectors (and cotangent vectors)
- Basis for tangent vectors is ∂_μ
- Basis for cotangent vectors is dx^μ
- At most one point in general can have $\partial_\mu|_p \cdot \partial \nu|_p = \eta_{\mu\nu}|_p$, everywhere else will be generally non-orthogonal: $\partial_\mu|_q \cdot \partial_\nu|_q = g_{\mu\nu}|_q \longrightarrow$ metric tensor field $g_{\mu\nu}$ (symmetric but not diagonal).

2A Appendix

2A.1 Examples of Coordinate Functions and Transformations

To really drive home how to think about coordinates and transformations, it helps to do some examples. First, let's treat some patches on the 1-sphere, the circle. For our maps, let's take the usual angle $\theta \in (0, 2\pi)$, and let's take projection from the left hemisphere onto the y-axis (see figure 2A.1.1, and note that throughout I will engage in a systematic abuse of notation, using θ and y as both the coordinate maps and the points in the images of those maps, and I will use them in both different senses in the same expression. Madness). The polar angle induces a parameterization on

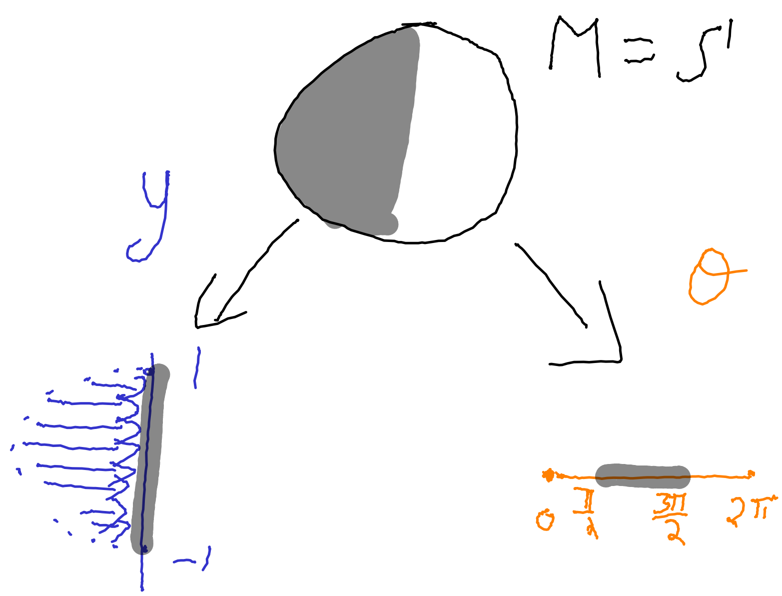

Figure 2A.1.1: The coordinate maps of S^1 under study. The shaded region indicates the overlap in domain between the polar coordinate and the projection coordinate.

the manifold $\theta^{-1}(\theta) = (\cos\theta, \sin\theta)$, while the projection induces the parameterization $y^{-1}(y) =$

$(\sqrt{1-y^2}, y)$. A change of coordinates is then effected by composition, either from polar coordinates to the projection: $y(\theta^{-1})$ or the reverse, $\theta(y^{-1})$. Explicitly, we have $y(\theta^{-1}(\theta)) = y(\cos\theta, \sin\theta) = \sin\theta$, and $\theta(y^{-1}(y)) = \theta(\sqrt{1-y^2}, y) = \operatorname{asin}(y)$.

2A.2 Tensor Densities

While we have made a big point about defining everything in terms of invariant geometric (tensorial) objects, there are times when we will really want to use objects that we just made up and definitely aren't tensors[4]. The trick, as ever, is to finagle these objects into actual tensor quantities so that we can work with, in a sense, the "next best thing." The structures that result from this procedure are referred to as **tensor densities** because they carry their ad hoc history in the form of their components.

Maybe the best example of a tensor density is the Levi-Civita tensor density. The point is that we would really like to be able to work with the Levi-Civita symbol in our equations, the object (in n spacetime dimensions):

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} +1, & \text{if even permutation of indices,} \\ -1, & \text{if odd permutation of indices,} \\ 0, & \text{else} \end{cases}, \tag{2A.2.1}$$

where these components are the same *everywhere and in all coordinate systems.* Clearly something like that does *not* represent the components of a tensor, so we need to find something *like* it that does.

Argument about matrix determinant leads to tensor density, metric determinant, and general densities.

[4]This section follows closely the treatment in section 2.8 of [2].

Chapter 3

Covariant Differentiation and Curvature

Finally we get to get on to the task of *quantifying* curvature. Our heuristic argument about the equivalence principle and the curvature of spacetime is all well and good, but it doesn't make for a mathematical theory until we can find a mathematical expression to represent some notion of the curvature of a manifold, and a way to connect it to the matter content of the universe. Ultimately, curvature is a property of the manifold as a whole, so quantifying it necessarily means drawing information from multiple tangent spaces.

The first thing we need to do then, is get tangent spaces to talk to each other. Normally this is first handled by looking at parallel transport (a consistent idea of "moving" a vector along some path in the manifold). While that's a great tool and the reader is encouraged to look it up themselves (see e.g. [2, §3.3], or in the future 3A.1), in the interests of time we'll skip ahead a little and go straight to the idea of defining a derivative operator on vectorfields.

3.1 The Covariant Derivative

Our goal is to study physics in curved space, so at some point we need to set up physical equations of motion, which invariably involve derivatives. At first glance, this seems to be straightforward, we defined a derivative of tensors in 1.6, when discussing the conservation of the stress-energy tensor. Looking closer though, we'll see that the definition (1.6.8) applies *only* to Minkowski space (note: not just flat space, but specifically *Cartesian* flat space). At the time, we noted that the derivative acting trivially on the basis vectors was something to keep an eye on, and indeed, we'll get to that shortly, but an easy way to see why the Minkowski tensor (1.6.8) does not remain a tensor on a general manifold is to see how it transforms under a general coordinate transformation.

The failure of the partial derivative to turn vectors into tensors

For a simple vectorfield, the components of the derivative as given by (1.6.8) have the form $\partial_\mu V^\nu$ and should transform as a type-$\begin{pmatrix}1\\1\end{pmatrix}$ tensor under a general coordinate transformation:

$$\partial_\mu V^\nu \to \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial y^\nu}{\partial x^\sigma}\partial_\rho V^\sigma \qquad \text{(If a tensor).} \tag{3.1.1}$$

However, since V^ν definitely are the components of a tensor, and we know how partial derivatives transform under a coordinate transformation (the chain rule, i.e., covariantly), we can perform the computation explicitly:

$$\partial_\mu V^\nu \to \frac{\partial x^\rho}{\partial y^\mu}\partial_\rho\left(\frac{\partial y^\nu}{\partial x^\sigma}V^\sigma\right) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial^2 y^\nu}{\partial x^\rho \partial x^\sigma}V^\sigma + \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial y^\nu}{\partial x^\sigma}\partial_\rho V^\sigma, \tag{3.1.2}$$

so for a *general* coordinate transformation, where the Jacobian of the transformation is *not* a constant matrix (like the Lorentz transformations), the naive derivative of a vectorfield does *not* transform as a tensor.

So what's really the problem here? We've suggested the derivatives of the basis vectors are throwing a wrench in the works, and while that's true, it's instructive to see why. Recall that formally, a derivative is just a difference, $\frac{\mathrm{d}f}{\mathrm{d}x} := \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. For scalar functions on a manifold, this leads to the definition of tangent spaces; we assert that over infinitessimally small distances, life is Minkowski enough and we can construct an entire vectorspace out of the derivatives of fields and paths at a point. But when it comes time to take the derivative *of those derivatives*, we can no longer pretend we're in a flat Cartesian world, now we have to compute $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \lim_{s\to 0}\frac{f'(x+s)-f'(x)}{s} = \lim_s \lim_{h\to 0}\frac{f(x+h+s)-f(x+s)-f(x+h)+f(x)}{sh}$, which stretches across too much of the manifold, it necessarily involves two distinct tangent spaces (see figure 3.1.1 for a visualization in terms of numerical derivatives). Fundamentally, that's why the derivative of the basis vectors can't be neglected, $\partial_\mu(\hat{e}_\nu)$ is an object in a new tangent space, and in general cannot always be taken to vanish (note that we temporarily restore the notation $\hat{e}_\mu$ for the tangent space basis vectors in order to avoid confusion with the ordinary partial derivative, which is an operator in this context, not a basis vector).

Defining a new derivative

So if we can't just use the partial derivative willy nilly to construct tensorial derivatives of tensors, what can we do? In the long-standing tradition of mathematics, the way out of this jam is to define a new thing to do what we want, and see if we can represent it in terms of things we know. So let's say there exists something like a derivative that really does turn tensors into tensors—we'll get ahead of ourselves and call it a **covariant derivative**[1] ∇.

Well, it better be linear, that's the first thing, but linearity's not so hard, we're dealing with linear things left right and centre. More informatively, if it's going to be some sort of a derivative, it also needs to satisfy the Leibnitz law (product rule), so we need:

$$\nabla(T\otimes S) = \nabla(T)\otimes S + T\otimes\nabla(S). \tag{3.1.3}$$

We also really want this thing to represent what we think of as a gradient operator, so we will require that it *is* the gradient when it acts on scalar fields (since the gradient lives in the cotangent space, and that's well-defined already):

$$\nabla(\phi) = \partial\phi = \partial_\mu\phi\,\mathrm{d}x^\mu, \tag{3.1.4}$$

where the second equality specializes to a coordinate system. Notice that just like the gradient and our naive derivative of vectors in Minkowski space, it must be the case that the covariant derivative turns tensors of type $\begin{pmatrix}M\\N\end{pmatrix}$ into tensors of type $\begin{pmatrix}M\\N+1\end{pmatrix}$.

[1]In some mathy circles, this is called a "connection," since it connects different tangent spaces.

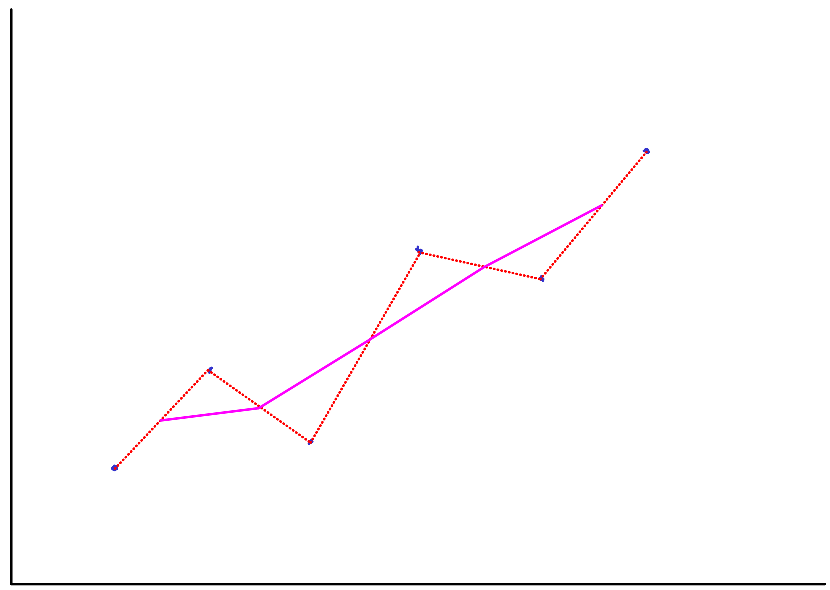

Figure 3.1.1: A numerical example of how second derivatives sample more of a function than first derivatives. Here, the data are blue points, first derivatives are calculated as simple slopes between points and represented by red dotted lines, and second derivatives are calculated as slopes of the first derivative slopes and are represented as solid purple lines. Notice how each second derivative needs information from *three* points.

Consider the action of this new derivative on a vector, $V^\mu \partial_\mu$. We can think of the coordinate representation of a vector as the tensor product between a scalar field (V^μ) and a contravariant vector (∂_μ), so the action of the covariant derivative must follow the product rule and read:

$$\nabla V = (\nabla_\nu V^\mu)\partial_\mu \otimes \mathrm{d}x^\nu + V^\mu \nabla_\nu(\partial_\mu) \otimes \mathrm{d}x^\nu = (\partial_\nu V^\mu)\partial_\mu \otimes \mathrm{d}x^\nu + V^\mu \nabla_\nu(\partial_\mu) \otimes \mathrm{d}x^\nu. \tag{3.1.5}$$

In the second equality, we use that the covariant derivative of a scalar field is just the ordinary gradient. The question remains about what to do with the second term. In general, there is no way to derive what the result of $\nabla_\nu(\partial_\mu)$ must be, but we can at least parameterize our ignorance. For instance, it must be *a vector*, since the equation is tensorial, so we know we can write it in a basis, and we might as well just give it a name,

$$\nabla V = (\partial_\nu V^\mu)\partial_\mu \otimes \mathrm{d}x^\nu + V^\mu (\Gamma_{\nu\mu})^\lambda \partial_\lambda \otimes \mathrm{d}x^\nu. \tag{3.1.6}$$

Note that the indices on Γ are *not* tensorial, they are a *naming convention*. The components Γ, often called the **connection coefficients**, are explicitly *not* the components of a rank-3 tensor, so we do not space the indices, and typically we write them without the brackets as $\Gamma^\lambda_{\nu\mu}$. In components then, we say the covariant derivative takes the form:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda =: V^\mu{}_{;\nu} \,. \tag{3.1.7}$$

(Notice the re-labelling of dummy indices compared to equation (3.1.6)). In the second equality, we define a sometimes handy shorthand for the covariant derivative. Similar to how we defined the ordinary partial derivative shorthand notation with a comma, here we use a semi-colon to denote a covariant derivative.

Now if ∇V really is a tensor, it must be the case that the coefficients (3.1.7) transform as a type-$\binom{1}{1}$ tensor. However, we already know what happens to the first term, it's (3.1.2)—the derivative product rules up an extra term out of the derivative of the Jacobian. In order to keep things tensorial then, we have to define that the components of $\nabla_n(\partial_\mu)$ transform in just the right way to cancel off that extra term. That is, we need the connection coefficients to correct for the over-eager ordinary derivative. Some simple algebra shows the connection coefficients must therefore satisfy:

$$\Gamma^\mu_{\nu\lambda} \to \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\beta}{\partial y^\lambda}\frac{\partial y^\mu}{\partial x^\gamma}\Gamma^\gamma_{\alpha\beta} - \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\beta}{\partial y^\lambda}\frac{\partial^2 y^\mu}{\partial x^\alpha \partial x^\beta}. \tag{3.1.8}$$

(Ex. show that this transformation cancels the extra term from (3.1.2)).

One quick follow-up: by repeatedly applying the product rule, what we've derived here for a rank-1 vector field generalizes immediately to completely contravariant tensors of any rank. Each contravariant index comes with a basis vector, $T = T^{\mu\nu\rho\cdots}\partial_\mu \otimes \partial_\nu \otimes \partial_\rho \otimes \cdots$, so when we take the covariant derivative of T, the product rule gives us a different factor of Γ for each index. E.g.:

$$\nabla_\nu T^{\rho\sigma\cdots} = \partial_\nu T^{\rho\sigma\cdots} + \Gamma^\rho_{\nu\alpha}T^{\alpha\sigma\cdots} + \Gamma^\sigma_{\nu\beta}T^{\rho\beta\cdots} + \dots. \tag{3.1.9}$$

So far so good, and in general any choice of Γ that satisfies the properties above is a good covariant derivative, or connection (actually, you can even squeak by without the reduction to the partial derivative, if you're a masochist). This means that in *addition* to the metric being an external object we slap on to a manifold to give it structure, we *also* have to adorn our manifolds with a *choice* of connection. With so much flexibility, it helps to narrow down to a useful choice of connection by asking for a wishlist of nice properties. First up, compatibility with contractions.

It may seem obvious that the covariant derivative should be compatible with contractions—i.e. $\nabla_\nu T^\lambda{}_{\lambda\beta} = (\nabla T)^\lambda{}_{\lambda\beta\nu}$—but we're kind of winging the transformation properties of the connection coefficients here, so there are no guarantees a priori. If we assume this holds though, we get a great relation between the covariant derivatives of vectors and covectors. Generically, there would be no relation between the vector object $\nabla_\nu(\partial_\mu) =: \Gamma^\lambda_{\nu\mu}\partial_\nu$ and the covector object $\nabla_\nu(\mathrm{d}x^\mu) =: \widetilde{\Gamma}^\mu_{\nu\lambda}\mathrm{d}x^\lambda$. However, if we can freely commute contractions with the covariant derivative, something fun happens. Consider the scalar $V^\mu W_\mu$. It's a scalar field, so if we take its covariant derivative we should get $\nabla_\rho(V^\mu W_\mu) = \partial_\rho(V^\mu W_\mu) = \partial_\rho(V^\mu)W_\mu + V^\mu\partial_\rho(W_\mu)$. But if we contract after taking the covariant derivative, we need to bring in the connection coefficients:

$$\begin{aligned}\nabla_\rho\left(V^\mu W_\mu\right) &= (\nabla V)^\mu{}_\rho W_\mu + V^\mu(\nabla W)_{\mu\rho},\\ &= (\partial_\rho V^\mu)W_\mu + \partial_\rho(W_\mu)V^\mu + \Gamma^\mu_{\rho\lambda}V^\lambda W_\mu + \widetilde{\Gamma}^\lambda_{\rho\mu}V^\mu W_\lambda.\end{aligned} \tag{3.1.10}$$

For the last line to equal $\partial_\rho(V^\mu W_\mu)$, it must be the case that (re-labelling some dummy indices):

$$\Gamma^\mu_{\rho\lambda} = -\widetilde{\Gamma}^\mu_{\rho\lambda}. \tag{3.1.11}$$

And with this (and the product rule), we arrive at a way to take the covariant derivative of any rank of mixed tensor: one copy of $+\Gamma$ for each contravariant index, and one copy of $-\Gamma$ for each covariant index.

$$\nabla_\rho T^{\mu\cdots}{}_{\nu\ldots} = \partial_\rho T^{\mu\cdots}{}_{\nu\ldots} + \Gamma^\mu_{\rho\lambda} T^{\lambda\cdots}{}_{\nu\ldots} + \ldots - \Gamma^\lambda_{\rho\nu} T^{\mu\cdots}{}_{\lambda\ldots} - \ldots. \tag{3.1.12}$$

One last quick observation before we move on to the specific choice of connection coefficients we'll be using for the rest of the course. Although the connection coefficients Γ are *not* the coefficients of tensors (they transform funny), it is easy to show that the *difference* between any two connection coefficients *is* a tensor. That is, define two different covariant derivatives ∇ and $\widehat{\nabla}$ out of two different choices for connection coefficients Γ and $\widehat{\Gamma}$. The difference of their actions on a vector must be a tensor (of course, they're proper tensor operators after all), so

$$\begin{aligned}\nabla_\nu V^\mu - \widehat{\nabla}_\nu V^\mu &= \partial_\nu V^\mu - \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda - \widehat{\Gamma}^\mu_{\nu\lambda} V^\lambda, \\ &= \Gamma^\mu_{\nu\lambda} V^\lambda - \widehat{\Gamma}^\mu_{\nu\lambda} V^\lambda\end{aligned} \tag{3.1.13}$$

is a proper tensor. This is particularly useful in one specific case. When working out the components for ∇_ν above, we somewhat arbitrarily decided the order of the indices on Γ. Had we chosen another convention, for example $\widehat{\Gamma}^\rho_{\mu\nu} := \Gamma^\rho_{\nu\mu}$, we would have found another perfectly valid connection, so we can use this to construct an important tensor by taking the difference:

$$T^\rho_{\mu\nu} := \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}, \tag{3.1.14}$$

named the **torsion** tensor for a particular connection.

The Levi-Civita Connection

Finally, we'll impose a couple more handy properties on our covariant derivative to really nail it down to a particular form. First, let's get rid of torsion, it's really ugly. From now on, we'll only work with a connection that is so-called "torsion-free," so satisfies $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. Next, we'll impose what will turn out to be the last condition we need to uniquely define a connection: metric compatibility.

Having attached a metric to our manifold, it is extremely useful to know how the covariant derivative we are trying to add relates to it (if at all). As it happens, we generally have the freedom to choose the best possible relation, called **metric compatibility**, the property of a covariant derivative acting trivially on the metric. That is, we choose ∇ to satisfy:

$$\nabla_\rho g_{\mu\nu} = 0, \tag{3.1.15}$$

for all indices. It turns out that all these conditions are finally sufficient to actually once-and-for-all uniquely define the connection and its coefficients, which in this case gets the name the **Levi-Civita connection**, or the **Christoffel connection**.

To see how we get an expression for the connection coefficients, consider the permutations of (3.1.15)[2]:

$$\begin{aligned}\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \\ \nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \\ \nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0.\end{aligned}$$

[2]The following derivation is taken almost verbatim from Carroll [2, §3.2].

Using the symmetry of the metric, and the newly minted symmetry of the connection coefficients, we can subtract the second and third equations to find:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0, \tag{3.1.16}$$

which can be solved for the coefficients to find the unique expression

$$\Gamma^\lambda_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho} \left(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu}\right). \tag{3.1.17}$$

This is an incredibly useful relation because (apart from all of the lovely conditions we've already imposed) we no longer have to worry about the connection coefficients as independent quantities, and can instead think of them as just complicated functions of the metric, which is enough to worry about on its own. In the form (3.1.17), the connection coefficients are often called the **Christoffel symbols** ("symbols" to reflect that they are *not* tensors), or the **affine connection** (so called because it generalizes the affine structure of Minkowski space that let us play with vectors at different places on the same footing), and in older texts have been given the slightly odd notation $\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}$ (we will not employ this notation).

Finally, another handy formula that falls out of the definition of the Christoffel connection is the expression

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|}\right), \tag{3.1.18}$$

where $|g|$ is the determinant of the metric. We will not derive this here, and will have little use for it, but it is good to know, particularly as a convenient way to write the divergence of a vectorfield $\nabla_\mu V^\mu = (|g|)^{-1/2} \partial_\mu \left(\sqrt{|g|} V^\mu\right)$.

3.2 Geodesics

Now that we have a consistent notion of a derivative on tensors, we can start to think about things to do with it. As physicists, the first thing we might want to do is use this tool to put physics on manifolds. For example, if we look at (two of) the Maxwell equations in their covariant special relativistic form:

$$\partial_\nu F^{\mu\nu} = J^\mu \qquad \text{(Minkowski Space)}, \tag{3.2.1}$$

where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, and J^μ is a four-current, containing both electric and magnetic currents. Don't worry too much about the form here, the important part is just that since the ordinary partial derivative is *not* a tensor operator, this physical law in the form (3.2.1) is *not* a tensor equation on a general manifold. However, we can easily see what *would* be a tensor equation, and indeed we make the bold assumption that the laws of electromagnetism in curved space actually are governed by the re-covariantized equation of motion:

$$\nabla_\nu F^{\mu\nu} = J^\mu \qquad \text{(General Space)}. \tag{3.2.2}$$

(Interestingly, the electromagnetic field strength tensor needs no re-definition, since the difference of partial derivatives cancels out their non-conforming transformation properties. This is an example of the "exterior derivative" acting on one-forms).

A much less obvious generalization can be carried out for another crucial physical law: Newton's second law in the absence of an external force:

$$\vec{F} = m\vec{a} = 0 \qquad \text{(Flat space, no external force).} \tag{3.2.3}$$

Here the statement is that unaccelerated particles follow straight lines. The generalization of this notion is the idea of **geodesics**, the "straightest" paths on a manifold. We'll get into more details, but the simplest way to carry out this generalization is by following your nose. Newtonian acceleration is a 3-vector, $\vec{a} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}$, so the first thing we do is swap out the three-vector for a four-vector $\vec{x} \to x^\mu$. Next, recall we identify the formerly universal parameter of time t with an external physical variable, usually a particle's proper time τ, so ideally we would like to write Newton's second law as (dropping the mass because it falls out for unaccelerated trajectories) $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = 0$. Unfortunately, this doesn't quite work! Of course $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} =: U^\mu$ is a good tensor, it's the particle's four-velocity. However, the second derivative, $\frac{\mathrm{d}}{\mathrm{d}\tau} = U^\mu\partial_\mu$ turns the putative equation into $U^\mu\partial_\mu U^\nu = 0$, which is *not* tensorial because of the presence of the ordinary partial derivative. There is an obvious way to remedy this deficiency though, we simply swap out the partial derivative for a covariant one! Then by a simple algorithm, we find the unaccelerated version of Newton's second law becomes

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\nabla_\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \tag{3.2.4}$$

on a general manifold!

This is great to have arrived at this incredibly important equation from a simple algorithm and an old physical law. For later use though, it is beneficial to think about what this equation means geometrically, and how to derive it just from considerations of the properties of straight lines. Here's the basic premise. When we take the covariant derivative of a tensor, we get an object ∇T that will tell you how that tensor changes in the tangent space next door in any given direction. This is the reason the covariant derivative adds a covariant index, the extra covector is waiting for you to give it a vector to tell it which direction you want to compute the change in the tensor. Mathematically, this follows very similarly to the notion of the directional derivative from 1.6, the object $V^\mu\nabla_\mu T^{\rho\cdots}{}_{\sigma\ldots}$ is the (infinitessimal) change in the tensor T along the direction of V.

Now, thinking about paths—especially paths of things, physical things—we recall there is a natural vector associated with any path $\gamma(s)$ along a manifold, which is its tangent vector, $v_s(\gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\partial_\mu|_s\gamma$. Newton's second law states that an unaccelerated particle's trajectory is a straight line, never deviating from the direction of the line. Following the manifold philosophy, we can zoom in on our path γ and ask is it basically a straight line? That is, does the tangent vector *covariantly* stay the same in the direction it was headed? If it does stay the same, we say that the path is a *geodesic*, an unaccelerated path on the manifold. Mathematically, we can write this exactly as (3.2.4). Try to read that equation as: "the change in the tangent vector vanishes in the direction it's pointing." ($\nabla\dot{x}$ is the gradient of the vector $\dot{x}$, so the inner product $\dot{x}\cdot\nabla\dot{x}$ vanishes along $\dot{x}$ if there is no change in $\dot{x}$ in that direction).

Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\partial_\mu$, the geodesic equation (3.2.4) can neatly and conveniently be written:

$$\boxed{\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = 0.} \tag{3.2.5}$$

This is the equation that generalizes the notion of straight lines in curved spaces. To see this explicitly, notice that the components of Γ are all derivatives of the metric, and in Minkowski space (in Cartesian coordinates), the components of the metric are all constants, so the Christoffel symbols all vanish, and (3.2.5) reduces to $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} = 0$, which for $s = \tau$ (proper time) yields $a = 0$, or $x^\mu \propto \tau$. Geodesics are extremely important in GR (so important I've boxed the equation) as the fact that some unaccelerated trajectories are *not* straight lines is the mathematical realization of the statement that gravity is geometry. That is, an unaccelerated particle moving along a curved trajectory is said to be moving in a gravitational field. More on that in chapter 4.

Aside 13: Affine parameters

Here again we run into the word "affine." Strictly speaking, the geodesic equation (3.2.5) only defines geodesics with a specific parameterization, called an affine parameterization. To see this, take a curve that solves the geodesic equation, and change the parameter $s \to s(t)$, where $s(t)$ is some wild, crazy function of some other number t. If you stick this change of parameter into the geodesic equation, the derivatives change by the chain rule, $\mathrm{d}/\mathrm{d}s \to (\mathrm{d}t/\mathrm{d}s)\mathrm{d}/\mathrm{d}t$, but the first term is a *second* derivative, so it picks up an extra term, and in general you would have to solve:

$$\frac{\mathrm{d}^2 x^\mu(t)}{\mathrm{d}t^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu(t)}{\mathrm{d}t}\frac{\mathrm{d}x^\rho(t)}{\mathrm{d}t} = -\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^{-2}\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}. \tag{3.2.6}$$

Fortunately though, we have a great deal of freedom in how we choose to parameterize curves, so to make our lives easier we always choose to parameterize geodesics in a way that the right hand side of the geodesic equation vanishes. This still leaves us with a certain latitude in choosing parameters, and you can verify easily that if s parameterizes a geodesic (with the RHS vanishing), then so does $t = as + b$, where a and b are constants. This relation defines s and t as elements of a one-dimensional affine space (points can be identified with displacement vectors).

3.3 Curvature

At long last, we can finally start to talk about curvature—what it means, how to describe it, and soon how it relates to the matter content of space. There are a number of different ways of thinking about curvature. Intuitively when we think about curved geometries, we usually think about two-dimensional surfaces embedded in three spatial dimensions, like spheres. Here, the embedding carries with it a natural notion of how something like a sphere deviates from three-dimensional Cartesian space in terms of confining 3D objects to the 2D surface (think of stretching out a triangle to fit it onto a sphere, or trying to flatten out the peel of an orange). These intuitive notions of curvature coming from the external embedding are called *extrinsic curvature*, and while that can have its use, it's much more helpful to have an internal, *intrinsic* notion of curvature of a manifold. So far, we've done very well constructing spacetime as a single manifold with the additional structure of a metric and a connection, it would be a real shame if we could only understand the curvature of our spacetime manifold by *also* having to carry around the entire structure of a larger Cartesian manifold as well. Phrased another way, we live in spacetime, so we'd better be able to characterize the curvature of our world while living in it!

Fortunately, there are such intrinsic ways of quantifying curvature. A great overview can be found in [3, §1], but the basic gist is that Euclidean space defines what we mean by "flat" space. Euclidean space is based on a set of axioms, and if you can violate any one of them (or any of the myriad equivalent formulations of them), then you have yourself a curved, non-Euclidean space. One example of (a phrasing of) these axioms is the statement that the internal angles of triangles add up to π, a statement it is easy to verify is contradicted on the sphere: consider the triangle formed by connecting the North pole to the equator, running along the equator over an angle of $\pi/2$ longitude, then running back up to the North pole. Each corner of the triangle is at an angle of $\pi/2$, adding up to a total of $3\pi/2$. Notice that the way we described this triangle did *not* require us to picture it embedded in three dimensions. All we have to do is imagine we're a tiny person living on the sphere who starts walking in a straight line from a particular point, turns to their left when they feel like it, does the same after walking the other direction, and eventually arrives back home. However long this tiny person walks in either direction[3], they will form a triangle with at least two right angles, so a sum of internal angles that is always grater than π.

More convenient for calculations is the requirement that parallel lines remain parallel. One of the ways to define Euclidean space is to include in it the notion of infinitely parallel lines, lines that start parallel and never meet in either direction. A space where lines that were initially parallel ever deviate (either meet or shoot off from each other) is inherently a non-Euclidean, or **curved** space.

3.3.1 Geodesic Deviation

To find a way of mathematically describing curvature by means of the deviation of parallel lines, we start with the generalization of lines, geodesics. This time we'll need more than one of them, so let's go wild and define a whole *field* of geodesics, $\gamma_t(s)$, where t is a continuous parameter (see figure 3.3.1) and all of the geodesics are parameterized by the same s.

Each of the $\gamma_t(s)$ is the generalization of a straight line, so if we zoom in really close on one, we can find a neighbour that is essentially parallel. Mathematically, "looking really close" means looking at tangent vectors, so consider the tangent to a single one of these curves, call it $S^\mu := \frac{\mathrm{d}x^\mu(\gamma_{t_0}(s))}{\mathrm{d}s}|_{s_0}$. Each γ_t is a geodesic, so we must have:

$$S^\nu \nabla_\nu S^\mu = 0. \tag{3.3.1}$$

Next, since our field of geodesics is continuous, we can define a direction in the manifold given by the parameter t, call it $T^\mu := \frac{\mathrm{d}x^\mu(\gamma_t(s_0))}{\mathrm{d}t}|_{t_0}$. The vector T represents the direction toward the nearest geodesic from the one at s_0, t_0, so it gives us a way to define a sort of "neighbour" to the curve we're studying—graphically, picture this as the perpendicular connecting two parallel lines. In fact, we have a great deal of latitude here, so we can go ahead an parameterize things just right so that T really is the perpendicular to S, so that $T^\mu S_\mu = 0$. Note that this is a condition that eliminates a free parameter, so where we had $2n$ for the two vectors, now we have $2n - 1$ free numbers to work with. Remember that, it'll come in handy shortly.

So S^μ is the tangent to a geodesic, and T^μ is the perpendicular infinitessimal distance to a neighbouring geodesics. The next thing we can ask is: does T^μ change if we move along the geodesic that defines S^μ? This can be computed the same way we defined geodesics, take the gradient of T^μ and project out the S^μ component:

$$v^\mu := S^\nu \nabla_\nu T^\mu. \tag{3.3.2}$$

[3]Barring circumnavigations, of course.

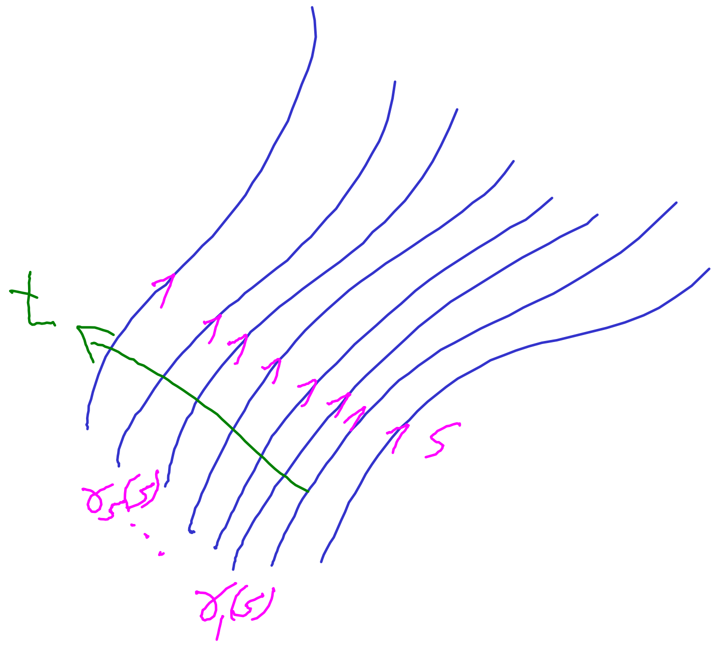

Figure 3.3.1: A smooth field of geodesics. The blue lines are geodesics on the manifold, all conveniently parameterized by the same s, and there is a smooth field of them, so in between all of the blue lines are more geodesics. The parameter t is a continuous variable that labels each of the geodesics. Together, s and t map out a two-dimensional surface in the manifold, each point on the surface corresponding to a point on single geodesic curve. This time it's not just my bad drawing, the lines really can be non-parallel, and even cross at some points if they want to.

The quantity v^μ can be thought of as the *geodesic separation velocity*, it tells us whether the geodesic and its neighbour are moving towards each other, away from each other, or parallel to each other. Our interest is in checking Euclid's Parallel Postulate for the manifold, so we'd like to consider a setup where $v^\mu = 0$, which we have more than enough degrees of freedom to do. Setting the n components of the geodesic separation velocity to zero cuts our free parameters down to $n-1$.

Now we have S^μ the tangent to a geodesic, T^μ the perpendicular to this tangent connecting it to another geodesics, and we have chosen initial conditions and coordinates such that the separation T^μ is initially stationary in the direction of the geodesics. Finally we can ask, will that separation *always* be stationary? To find out, just take the derivative again to find the *geodesic separation acceleration*:

$$a^\mu := S^\nu \nabla_\nu v^\mu = S^\nu \nabla_\nu \left(S^\rho \nabla_\rho T^\mu \right). \tag{3.3.3}$$

Now remember, we're only working with $n-1$ free parameters at this point, and the geodesic separation acceleration is an n-component vector, so we no longer have enough degrees of freedom to *force* the acceleration to vanish, so if it is *non-zero*, then we know the manifold did it and we're living in a non-Euclidean world! The only trouble is, this expression at the moment is highly dependent on our particular choices of S and T. What we need is to massage it into a form where we can yank out the components of the tangent vector and its perpendicular, and have something that depends only on the properties of the manifold, its metric and connection (and for the Levi-Civita connection, *just* the metric).

At this point, it helps to temporarily choose our coordinates wisely. Let's pick the x-direction to be along S and the y-direction to be along T, so $S = \delta^\mu_1 \partial_\mu$ and $T = \delta^\mu_2 \partial_\mu$. Then the covariant derivative is fairly easy to evaluate: the components of S and T are constant so we have $\nabla_\nu S^\mu = \Gamma^\mu_{\nu\rho} S^\rho$ and $\nabla_\nu T^\mu = \Gamma^\mu_{\nu\rho} T^\rho$, so in particular, $T^\nu \nabla_\nu S^\mu = T^\nu \Gamma^\mu_{\nu\rho} S^\rho = S^\rho \Gamma^\mu_{\rho\nu} T^\nu = S^\rho \nabla_\rho T^\mu$. In other words, in these coordinates we can swap S and T in the directional derivative, but since that is a tensor equation it is valid in *any* coordinate system, so we have in general:

$$\begin{aligned} a^\mu &= S^\nu \nabla_\nu \left(S^\rho \nabla_\rho T^\mu \right), \\ &= S^\nu \nabla_\nu \left(T^\rho \nabla_\rho S^\mu \right), \\ &= S^\nu \left(\nabla_\nu T^\rho \right) \nabla_\rho S^\mu + S^\nu T^\rho \nabla_\nu \nabla_\rho S^\mu, \\ &= T^\rho \left(\nabla_\rho S^\nu \right) \nabla_\nu S^\mu + S^\nu T^\rho \nabla_\rho \nabla_\nu S^\mu + S^\nu T^\rho [\nabla_\nu, \nabla_\rho] S^\mu, \\ &= T^\rho \nabla_\rho \left(S^\nu \nabla_\nu S^\mu \right) + S^\nu T^\rho [\nabla_\nu, \nabla_\rho] S^\mu, \end{aligned} \tag{3.3.4}$$

but since S is the tangent to a geodesic, the first term vanishes by the geodesic equation. (Note: there was a lot of re-labelling dummy indices and swapping S and T in there, read it carefully). We are therefore left with the following expression for the geodesic separation acceleration:

$$a^\mu = S^\nu T^\rho [\nabla_\nu, \nabla_\rho] S^\mu. \tag{3.3.5}$$

This may not look like much of an improvement, but it's actually great! The commutator of the covariant derivatives is a tensor operator, and astoundingly its components do *not* depend on S! The easy way to see why is to note that the partial derivatives commute, so the only non-commuting parts of the covariant derivative operators comes from the Christoffel symbols and their derivatives, all depending only on the metric. In components, we have:

$$\boxed{[\nabla_\nu, \nabla_\rho] S^\mu =: R^\mu{}_{\lambda\nu\rho} S^\lambda = \left(\partial_\nu \Gamma^\mu_{\rho\lambda} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\nu\sigma} \Gamma^\sigma_{\rho\lambda} - \Gamma^\mu_{\rho\sigma} \Gamma^\sigma_{\nu\lambda} \right) S^\lambda.} \tag{3.3.6}$$

This defines the incredibly important **Riemann Curvature Tensor** $R^\mu{}_{\lambda\nu\rho}$, which encodes the intrinsic notion of curvature for Riemannian manifolds equipped with the Levi-Civita connection (the definition can be extended with little difficulty to more general connections). In terms of the curvature tensor, the geodesic separation acceleration is

$$a^\mu = R^\mu{}_{\lambda\nu\rho} S^\nu T^\rho S^\lambda, \tag{3.3.7}$$

and now we know what it means: if initially parallel lines remain parallel, then $R^\mu{}_{\lambda\nu\rho} = 0$ and the manifold is Euclidean, otherwise the spacetime is *curved*.

In its final form, (3.3.7) is known as the *geodesic deviation equation*, and represents a very important physical effect. What are two parallel lines you made extensive use of eons ago when deriving special relativity? How about the worldlines of the ends of some extended object, like a rod? The ultimate divergence or convergence of parallel lines is the same thing as a force that twists, stretches, and contracts macroscopic objects—a **tidal** force. Tidal forces arise from *non-homogeneous* gravitational fields, and in general relativity, that manifests as intrinsically *curved* spacetimes. It is also very interesting to note that this is a *local* expression, so tidal forces are observable locally. This does not contradict the equivalence principle though, the elevator thought-experiment still works if you give it a non-uniform acceleration (but in general, you know, don't be a jerk to people in elevators).

3.3.2 Properties of the Curvature Tensor

The Riemann tensor is such an important object, it will be very helpful later for us to evaluate some of its key properties now. For listing symmetries, it's easiest if all the indices are on the same level, so to start, we'll consider the purely covariant form, $R_{\mu\lambda\nu\rho} := g_{\mu\sigma}R^{\sigma}{}_{\lambda\nu\rho}$, which satisfies the following symmetries:

1. Anti-symmetry in pairs:

$$R_{\mu\lambda\nu\rho} = -R_{\lambda\mu\nu\rho}, \quad \text{and} \tag{3.3.8}$$

$$R_{\mu\lambda\nu\rho} = -R_{\mu\lambda\rho\nu}. \tag{3.3.9}$$

2. Symmetry of pairs:

$$R_{\mu\lambda\nu\rho} = R_{\nu\rho\mu\lambda}. \tag{3.3.10}$$

3. Cyclic permutations:

$$R_{\mu[\lambda\nu\rho]} = R_{\mu\lambda\nu\rho} + R_{\mu\nu\rho\lambda} + R_{\mu\rho\lambda\nu} = 0. \tag{3.3.11}$$

4. The Bianchi identity:

$$\nabla_{[\sigma}R_{\mu\lambda]\nu\rho} = \nabla_{\sigma}R_{\mu\lambda\nu\rho} + \nabla_{\mu}R_{\lambda\sigma\nu\rho} + \nabla_{\lambda}R_{\sigma\mu\nu\rho} = 0. \tag{3.3.12}$$

With these symmetries, we may also identify the *unique* non-vanishing contraction of the curvature tensor:

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}. \tag{3.3.13}$$

(Note: The choice of contraction is a *convention*, and it is equally possible to define this with a relative minus sign as well). This is known as the **Ricci Tensor**. It is also helpful to notice that the symmetries above imply the Ricci tensor is symmetric: $R_{\mu\nu} = R_{\nu\mu}$. Finally, we may also identify the one and only non-vanishing scalar that can be constructed from (one copy of) the curvature tensor and metric alone, and that is the **Ricci Scalar**:

$$R := R^{\mu}{}_{\mu}. \tag{3.3.14}$$

For the reader in a hurry, this is sufficient to move on to the next chapter. For those with a little more time on their hands, it is a good exercise to prove the symmetries above, particularly because the easiest way to prove them involves a very helpful tool, the *locally inertial* or *Riemann normal* coordinates.

Locally Inertial (aka Riemann Normal) Coordinates

Locally inertial coordinates are the formal notion of us as rulers of our manifold being kind to Tiny Alice and choosing our coordinates such that they align with her local frame. For any single point on the manifold, we are allowed to establish a system of coordinates such that our cardinal directions align with the coordinate system established by the tiny person that lives there, and the tangent space to the manifold at that point has an inner product given by the Minkowski metric $(g_{\mu\nu})|_p = \eta_{\mu\nu}$.

What's surprising is that this notion actually *extends* just a little ways away from Tiny Alice's room. Not only can we establish coordinates such that Tiny Alice's metric looks like η, but we can also arrange for the *first derivatives* of the metric to vanish at her location. One way to see why this should be the case is that as far as Tiny Alice can tell, inertial objects follow straight lines, so the geodesic equation from her perspective reads:

$$\frac{\mathrm{d}^2 x^\mu(\tau)}{\mathrm{d}\tau^2} + \Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu(\tau)}{\mathrm{d}\tau^2} = 0, \qquad \text{(at } p\text{)} \tag{3.3.15}$$

so we also have $\Gamma|_p = 0$ which is certainly evidence for the derivative of the metric vanishing at that point. Notice however that the vanishing of a derivative at a point *does not imply the vanishing of a second derivative* at that point. This goes back to the idea of second derivatives sampling more of an underlying distribution—just because things look flat at a single point, and even a shmidge around that point, that doesn't mean you're actually looking at something that's flat overall (think about a local minimum of some curvy function).

There are more thorough, formal ways of showing the existence of this coordinate system, but the easiest way to convince yourself they exist is through a parameter-counting argument. If these coordinates exist on the manifold, then they can be related to any other coordinates on the manifold by a coordinate transformation: $x(\hat{x})$, where $\hat{x}$ are our putative locally inertial coordinates. In the vicinity of p, we may write this as a Taylor series: $x(\hat{x}) = \hat{x}(x(p)) + \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\big|_p (\hat{x} - \hat{x}(p)) + \ldots$. More importantly, the Jacobian of the transformation may similarly be expanded in a Taylor series:

$$\frac{\partial x^\mu}{\partial \hat{x}^\nu} = \frac{\partial x^\mu}{\partial \hat{x}^\nu}\Big|_p + \frac{\partial^2 x^\mu}{\partial \hat{x}^\alpha \partial \hat{x}^\nu}\Big|_p (\hat{x}^\alpha - \hat{x}^\alpha(p)) + \ldots, \tag{3.3.16}$$

and so may be the components of the metric (individually, as functions, not components of a tensor):

$$g_{\mu\nu}(\hat{x}) = g_{\mu\nu}(\hat{x}(p)) + \frac{\partial g_{\mu\nu}}{\partial \hat{x}^\alpha}\Big|_p (\hat{x}^\alpha - \hat{x}^\alpha(p)) + \ldots \qquad \text{(as functions).} \tag{3.3.17}$$

In this way, we can perturbatively change coordinates for the metric near our point of interest and see how far our freedom to choose the components of the coordinate transformation will take us.

So crunch away, we have (now as components of a tensor):

$$\begin{aligned} g_{\mu\nu}(\hat{x}) &= \frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} g_{\alpha\beta}(\hat{x}), \\ &= \frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu}\Big|_p g_{\alpha\beta}(\hat{x}(p)) \\ &\quad + (\hat{x}^\gamma - \hat{x}^\gamma(p))\left\{\frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu}\frac{g_{\alpha\beta}(\hat{x})}{\partial \hat{x}^\gamma} + \frac{\partial^2 x^\alpha}{\partial \hat{x}^\gamma \partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} g_{\alpha\beta}(\hat{x}) + \frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial^2 x^\beta}{\partial \hat{x}^\gamma \partial \hat{x}^\nu} g_{\alpha\beta}(\hat{x})\right\}\Big|_p \\ &\quad + \ldots \\ &\stackrel{?}{=} \eta_{\mu\nu}. \end{aligned} \tag{3.3.18}$$

Now, obviously we have enough degrees of freedom to rotate the metric at a single point into the Minkowski form, so it's not hard to see there's a solution to $\frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu}\Big|_p g_{\alpha\beta}(\hat{x}(p)) = \eta_{\mu\nu}$. In fact, it is more than possible, the system is overdetermined. In 3+1 dimensions, we have an excess freedom of 6 parameters, which corresponds to the 6 generators of the Lorentz group, so not only can we choose the components of the Jacobian to take us to the Minkowski metric at a point, but we have enough degrees of freedom left over to also rotate it by any Lorentz transformation. The question then is, is it technically possible to solve the linear system of equations $\{\ldots\} = 0$ for the terms in braces in (3.3.18)? Without working too hard, what we can do is count parameters and see if the system of equations is under, over, or identically determined.

The idea here is that the original metric and its derivatives are generic, so we want to pick the components of our coordinate transformation (and its derivatives) so as to eliminate the original metric and its derivatives. The Hessian matrix (the "Jacobian of the Jacobian," if you will) looks at first like it has n^3 components, but it's actually smaller than that. Since partial derivatives commute, the Hessian has only $\frac{1}{2}n(n+1) \times n$ degrees of freedom (a symmetric $n \times n$ matrix has $\frac{1}{2}n(n+1)$ components, so think of the gradient of the Jacobian as n symmetric $n \times n$ matrices). Meanwhile, the metric is a symmetric $n \times n$ matrix, so it has $\frac{1}{2}n(n+1)$ degrees of freedom, and so its gradient has n times that many components (one for each partial derivative). Altogether then, we want to kill off $\frac{1}{2}n^2(n+1)$ components, and we have $\frac{1}{2}n^2(n+1)$ degrees of freedom to do it with, so the system is *exactly* determined, and we can indeed choose coordinates such that the first derivative of the metric at vanishes at a point.

Finally, we have to ask if we can go further, can we set the second derivative to vanish? Here again we count. The next order up involves a gradient of the gradient of the metric, so $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1) = \frac{1}{4}n^2(n+1)^2$ components (again, partial derivatives commute, cutting the degrees of freedom down a bit), and the Jacobian of the Jacobian of the Jacobian, with $\frac{1}{3!}n^2(n+1)(n+2)$ degrees of freedom (symmetry of all 3 partial derivatives leads to n copies of $\binom{n+3-1}{3}$ independent terms). In 3+1 dimensions, these are 100 and 80 degrees of freedom respectively, which means we're 20 degrees of freedom shy of being able to use a coordinate transformation to flatten a metric beyond its first derivative. Those 20 degrees of freedom are important, by the way, they are exactly the 20 degrees of freedom of the Riemann curvature tensor! With that out of the way, we can move on to proving the symmetries of $R^{\mu}{}_{\nu\rho\sigma}$.

Proofs of Riemann Tensor Symmetries

The symmetries above are most easily seen by direct computation in the Riemann normal coordinates. In those coordinates, at the point p where the metric is Minkowski, we have $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$, $\partial_\rho \hat{g}_{\mu\nu} = 0$ and so $\widehat{\Gamma}^{\mu}_{\nu\lambda} = 0$. The Riemann tensor then boils down to:

$$\begin{aligned}
\hat{R}_{\mu\lambda\nu\rho} &= \hat{g}_{\mu\sigma}\hat{R}^{\sigma}{}_{\lambda\nu\rho}, \\
&= \hat{g}_{\mu\sigma}\left(\widehat{\Gamma}^{\sigma}_{\rho\lambda,\nu} - \widehat{\Gamma}^{\sigma}_{\nu\lambda,\rho}\right), \\
&= -\frac{1}{2}\hat{g}_{\mu\sigma}\hat{g}^{\sigma\alpha}\left(\hat{g}_{\rho\lambda,\alpha\nu} - \hat{g}_{\lambda\alpha,\rho\nu} - \hat{g}_{\alpha\rho,\lambda\nu}\right. \\
&\qquad \left. - \hat{g}_{\nu\lambda,\alpha\rho} + \hat{g}_{\lambda\alpha,\nu\rho} + \hat{g}_{\alpha\nu,\lambda\rho}\right), \\
&= -\frac{1}{2}\left(\hat{g}_{\rho\lambda,\mu\nu} - \hat{g}_{\lambda\mu,\rho\nu} - \hat{g}_{\mu\rho,\lambda\nu}\right. \\
&\qquad \left. - \hat{g}_{\nu\lambda,\mu\rho} + \hat{g}_{\lambda\mu,\nu\rho} + \hat{g}_{\mu\nu,\lambda\rho}\right), \\
&= -\frac{1}{2}\left(\hat{g}_{\rho\lambda,\mu\nu} - \hat{g}_{\mu\rho,\lambda\nu} - \hat{g}_{\nu\lambda,\mu\rho} + \hat{g}_{\mu\nu,\lambda\rho}\right).
\end{aligned} \tag{3.3.19}$$

(Here we used that $\partial_\rho \hat{g}_{\mu\nu} = 0$ to pull the overall factor of $\hat{g}^{\sigma\alpha}$ out front, and we have used that partial derivatives commute). Now normally if we were trying to derive a tensor relation from this, it would be fairly tedious to try and find another tensor that is equal to the RHS in this particular coordinate system at that particular point, but since our aim is only to permute indices, the job is already done! We know the tensor it will be equal to, it's just the (covariant) Riemann tensor with its indices permuted. So for instance, it is easy to see that if we swap μ and λ, the first pair and second pair of terms on the RHS swap themselves around inducing an overall minus sign, which immediately yields $\hat{R}_{\mu\lambda\nu\rho} = -\hat{R}_{\lambda\mu\nu\rho}$, but since this is a *tensor* equation, we can drop the hats and observe that it holds in any coordinate system (and at any point). In this way, (3.3.19) contains all the information needed to directly compute all of the symmetries listed above. It is a good exercise to pick one or two of them and follow the calculation through.

3.4 Recap

- Physics needs derivatives, but the ordinary partial derivative is *not* a good operator on tensors (it does not take tensors to tensors).
- Define new kind of derivative, the *Covariant Derivative* (or *connection*) ∇ to carry out the duties of a derivative in a manner consistent with tensor equations. It has the form of the ordinary derivative plus a correction: $\nabla = \partial + \Gamma$, the correction being called the "connection coefficients."
- The choice of a covariant derivative is an *additional structure* on a manifold, so points+coordinates $\rightarrow$ (points+coordinates)+metric $\rightarrow$ (points+coordinates+metric)+connection.
- For simplicity (among other reasons), we choose the *Levi-Civita* connection, defined by having zero torsion ($\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$) and being metric-compatible ($\nabla g = 0$). For the Levi-Civita connection, the connection coefficients are called *Christoffel symbols* and take on the simple form (3.1.17).
- Generalizing equations from flat to curved space typically amounts to replacing $\partial \rightarrow \nabla$.
- Straight lines are generalized on manifolds to *geodesics*, given by curves that solve the geodesic equation (3.2.5).
- Intrinsic curvature of manifolds deduced by geodesic deviation, the failure of initially parallel lines to remain parallel. This failure is the geometric manifestation of tidal forces, and is entirely encoded in the *Riemann Curvature tensor*, given in components by (3.3.6).
- A very useful tool in calculating tensor relations is the *locally inertial coordinate system*, where the metric at a single point takes the form of the Minkowski metric, and at which point the first derivative of the metric (and hence the Christoffel symbols) vanishes. Note that the *second* derivative of the metric (and hence the derivative of the Christoffel symbols) does *not* usually vanish there, which is thanks to the curvature of the manifold.

3A Appendix

3A.1 Parallel Transport of Vectorfields

Although we don't have time to go through this in the course, it is well worth exploring the notion of parallel transport on manifolds.

For an initial tangent vector v_p and a curve $\gamma(s)$ on a manifold, we can construct a vectorfield along γ by "dragging" the initial vector along the direction of the tangent vector $\dot{\gamma}$. The equation that defines the vectorfield is the equation of **parallel transport**:

$$\nabla_{\dot{\gamma}(s)} v(s) = \dot{\gamma}^{\mu}(s) \nabla_{\mu} v^{\nu}(s) = 0, \tag{3A.1.1}$$

which, together with the initial condition $v(s_0) = v_p$ is a set of ordinary differential equations for the components $v^{\nu}(s)$.

That (3A.1.1) makes sense as a notion of "parallel" transport of vectors can be understood by reading the equation a little differently. Normally one thinks of $\nabla_{\dot{\gamma}} v = 0$ as "the covariant derivative of v in the direction $\dot{\gamma}$ vanishes." However, more relevant for this discussion is the component formulation which can be read "the inner product of $\dot{\gamma}$ with the covariant gradient of v vanishes." In other words, the differential equation (3A.1.1) says that at every point, we compute how v changes in any direction (its gradient), and we define $v(s)$ by the condition that it *not* change in the direction of the curve of interest. In this way, $v(s)$ describes a continuous sequence of tangent vectors that can be thought of as following the curve γ. Note also that the same can be done for tensors of any rank just by applying the appropriate calculation of the covariant derivative.

*** Add example here, probably calculate the usual example on the sphere ***

3A.2 More on Torsion

The notions of intrinsic torsion and intrinsic curvature can get very confusing (at least for me), so we elaborate here on the geometric picture of torsion, and how it differs from the closely-related but distinct notion of local curvature.

*** Torsion is "rotation" of basis vectors, has to do with how the covariant derivative messes up the relative orientation of coordinate axes. Curvature is holonomy around a closed loop (defined by external curve), or as above, geodesic deviation. ***

Chapter 4

The Einstein Field Equations and General Relativity

Finally we have all the pieces we need to construct the general theory of relativity! We have a working understanding of tensors on general manifolds, how to take their derivatives in a tensorial way, how to describe matter on a manifold, and a local, intrinsic measure of curvature. The question now is how to piece them together. It turns out there is actually no way to *derive* the fundamental equation that relates the curvature of a manifold to its matter content, that has to be a postulate of the theory, and for general relativity is it given by the **Einstein Field Equations**:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.0.1}$$

where $\kappa = 8\pi G$ is a convenient rescaling of the gravitational constant G. There are two main ways to go about justifying this postulate, a "bottom-up" way and a "top-down" way.

Two quick notes about (4.0.1) just before moving on: 1) the choice of $+\kappa$ on the RHS instead of $-\kappa$ is a *convention*, just like the sign on the Ricci tensor, so be mindful of that when reading the literature, and 2) the LHS is sometimes also written with a so-called "cosmological constant" term $\Lambda g_{\mu\nu}$. While the cosmological constant can be thought of as a geometrical property of spacetime, it is just as easy (and probably better) to think of it as a constant energy density, so it really belongs in the stress-energy tensor, which is the approach we will take here.

4.1 The EFEs By Construction

The first way to come at the Einstein field equations is to think "gravity is geometry, and matter sources gravity, so how do I get matter to source geometry?" Obviously we only have the one way to describe matter, it's the stress-energy tensor $T_{\mu\nu}$, and fundamentally the geometry just boils down to the metric, but there are lots of important constructions that involve the metric and its derivatives, like the connection and the various curvature tensors. What we need is to narrow things down a bit, and to do so, we'll use the correspondence theorem—i.e., the principle that we should be able to recover Newtonian gravity in an appropriate limit. In particular, we recall the Newtonian Poisson equation:

$$\vec{\nabla}^2\phi = 4\pi G\rho, \tag{4.1.1}$$

where $\vec{\nabla}^2$ is the old 3+0-dimensional Laplacian (the notation not being far off, as it can indeed be represented by $\nabla_i\nabla^i$!), ϕ is the gravitational potential, and ρ is the mass density. Well, mass density is just non-relativistic energy density (hence the same variable), which is just T^{00}, and we are hypothesizing that the gravitational potential is basically the metric, so we'll try looking for an equation that has two derivatives of the metric set proportional to the stress-energy tensor. A good try would be

$$R_{\mu\nu} \overset{?}{\propto} T_{\mu\nu}. \tag{4.1.2}$$

This would be nice because we know $T_{\mu\nu}$ is symmetric, and so is the Ricci tensor, so it's a good start but we're missing something important with this equation: conservation of stress-energy. Recall we had imposed $\nabla^\nu T_{\mu\nu} = 0$ (I've flipped the superscripts and subscripts, but that's okay because the connection is metric-compatible). If (4.1.2) is going to be a tensor equation, then the same must be true on the other side, but $\nabla^\nu R_{\mu\nu} \neq 0$ in general.

What we need is something like $R_{\mu\nu}$ but that satisfies its covariant gradient vanishes. As it turns out, the Bianchi identities conveniently furnish exactly the tensor we need. Take (3.3.12) and contract on the first and third indices, as we did to define the Ricci tensor:

$$\begin{aligned} g^{\mu\nu}\left(R_{\nu\rho[\mu\lambda;\sigma]}\right) &= g^{\mu\nu}\left(R_{\nu\rho\mu\lambda;\sigma} + R_{\nu\rho\lambda\sigma;\mu} + R_{\nu\rho\sigma\mu;\lambda}\right), \\ &= R_{\rho\lambda;\sigma} + R^{\mu}{}_{\rho\lambda\sigma;\mu} - R_{\rho\sigma;\lambda}, \\ &= 0. \end{aligned} \tag{4.1.3}$$

Metric compatibility is a beautiful thing, isn't it? Since the metric can pass through the covariant derivative willy-nilly, we can raise and lower indices inside and outside of the derivative with impunity. Equation (4.1.3) is nice and all, but it's not quite what we're looking for yet. To finally get there, contract again, this time on ρ and λ:

$$\begin{aligned} g^{\rho\lambda}g^{\mu\nu}\left(R_{\nu\rho[\mu\lambda;\sigma]}\right) &= g^{\rho\lambda}\left(R_{\rho\lambda;\sigma} + R^{\mu}{}_{\rho\lambda\sigma;\mu} - R_{\rho\sigma;\lambda}\right), \\ &= R_{;\sigma} - R^{\mu}{}_{\sigma;\mu} - R^{\mu}{}_{\sigma;\mu}, \\ &= \left(\delta^{\mu}_{\sigma}R - 2R^{\mu}{}_{\sigma}\right)_{;\mu}, \\ &= 0. \end{aligned} \tag{4.1.4}$$

(The second equality uses $R^{\mu}{}_{\rho\lambda\sigma;\mu} = R_{\mu\rho\lambda\sigma}{}^{;\mu}$ and the symmetries of the curvature tensor). Finally, raising an index, we find:

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} =: G^{\mu\nu}{}_{;\mu} = 0, \tag{4.1.5}$$

which defines the **Einstein tensor** $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The (aptly named) Einstein tensor is exactly the symmetric, conserved tensor composed only of the metric and up to two derivatives that we were looking for. GR is then the theory of physics defined by the postulate

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{4.1.6}$$

for some proportionality constant κ (which we will elaborate more on later).

4.2 The EFEs from an Action Principle

This is a nice enough way of deriving the field equations, but a more modern line of thinking involves the bewilderingly impressive action formalism. In a nutshell, classical mechanics can be formulated in a way such that the physical equations of motion followed by all forms of matter boil down to a single number, the action S. It is written in terms of an integral of a function called the Lagrangian L, which calculates the total kinetic energy T minus the total potential energy V of a system based on the matter content of the system. The action is related to the Lagrangian by $S = \int \mathrm{d}t\, L$, and the simple principle that makes physics tick is the *principle of stationary action*, that the configuration of matter that is actually seen in Nature is the one that extremizes the action. Mathematically, we say the variation of the action $\delta S = S - S_0$ vanishes.

All of the fundamental laws of physics can be arranged in a way that they are the equations of motion of some Lagrangian. The geodesic equation (3.2.5), for example, is the equation of motion associated with the action $S = \int \mathrm{d}\tau \sqrt{-U^\mu U^\mu}$ for a particle's four-velocity. The Maxwell equations of electromagnetism derive from the action $S = -\int \mathrm{d}^x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where the Lagrangian is $L = \int \mathrm{d}^3 \, \mathcal{L} := -\int \mathrm{d}^3 x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, which defines the Lagrangian density $\mathcal{L} := -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. One huge advantage of the Lagrangian formulation is that it lends itself readily to quantization. Through the Feynman path integral, almost any action can become a quantum theory at the drop of a hat. Much of modern particle and condensed matter physics has to do with finding the right Lagrangian to describe the quantum physics of interest.

What does all of this have to do with general relativity? Well one can play the same game and try to find a Lagrangian whose variation yields the Einstein equations as its equation of motion. The usual strategy is to think of the simplest scalar thing you can that is made of the fields you're interested in, and complexify from there, if you have to. So what's the simplest scalar thing we have in gravity? Why the Ricci scalar, of course. So we simply give the following action a go:

$$S_H = \int \mathrm{d}^4 x \, \sqrt{-g} R. \tag{4.2.1}$$

This is known as the **Einstein-Hilbert action**. Here, R is just the Ricci scalar. The factor of $\sqrt{-g}$ (the square root of the absolute value of the metric determinant) is there because otherwise the volume form $\mathrm{d}^4 x$ is not an invariant tensor (don't worry about that part, we didn't have time to cover it because we had to skip over differential forms and integration on manifolds). We could also include an additional term in the Lagrangian, $\mathcal{L}_M$, to represent the matter contributions. As it turns out, variation of this total action with respect to changes in the metric yields *both* the Einstein field equations *and* a general definition of the stress-energy tensor in terms of the Lagrangian density of matter.

We can do the variation in pieces (we'll follow Carroll [2, §4.3] quite closely). First, it's helpful to note that there is an easy relation between the variation of the metric $\delta g_{\mu\nu}$ and the variation of its inverse, $\delta g^{\mu\nu}$. Simply note that the identity matrix is an invariant, and write $\delta(\delta^\mu_\nu) = \delta(g^{\mu\sigma} g_{\sigma\nu}) = \delta g^{\mu\sigma} g_{\sigma\nu} + g^{\mu\sigma} \delta g_{\sigma\nu} = 0 \implies \delta g_{\rho\nu} = -g_{\mu\rho} g_{\sigma\nu} \delta g^{\mu\sigma}$. Then a stationary point of the action with respect to the metric will also be a stationary point with respect to the metric inverse, and it turns out to be a little more convenient to use the latter, so let's compute $\frac{\delta S_H}{\delta g^{\mu\nu}}$ (just the gravitational part first).

($T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$).

4.3 The Newtonian Limit

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