4

THE BOLTZMANN EQUATIONS

We are interested in the anisotropies in the cosmic distribution of photons and inhomogeneities in the matter. Figure 4.1 shows why these are complicated to calculate. The photons are affected by gravity and by Compton scattering with free electrons. The electrons are tightly coupled to the protons. Both of these, of course, are also affected by gravity. The metric which determines the gravitational forces is influenced by all these components plus the neutrinos and the dark matter. Thus to solve for the photon and dark matter distributions, we need to simultaneously solve for all the other components.

There is a systematic way to account for all of these couplings. We write down a Boltzmann equation for each species in the universe. We have already encountered the Boltzmann equation in its integrated form in Chapter 3. There we were interested solely in the number density of the dark matter, the neutrons, and the free electrons. The number density is the integral over all momenta of the distribution function. Here we will be interested in more detailed information, not just the integrated number density, but the full distribution of photons, say, as a function of momentum. We then need a more primitive version of Eq. (3.1). Schematically, the unintegrated Boltzmann equation is

$$\frac{df}{dt} = C[f]. \tag{4.1}$$

The right-hand side of the Boltzmann equation contains all possible collision terms. These terms in general are complicated functionals of the distribution functions of the various components. In the absence of collisions, the distribution function obeys $df/dt = 0$. This seemingly innocent equation says that the number of particles in a given element of phase space does not change with time. The catch is that the phase space elements themselves are moving in time in complicated ways due to the nontrivial metric. This catch makes the problem more difficult than it seems from Eq. (4.1). Nonetheless, we can still progress systematically by reexpressing the full derivative in terms of partial derivatives.

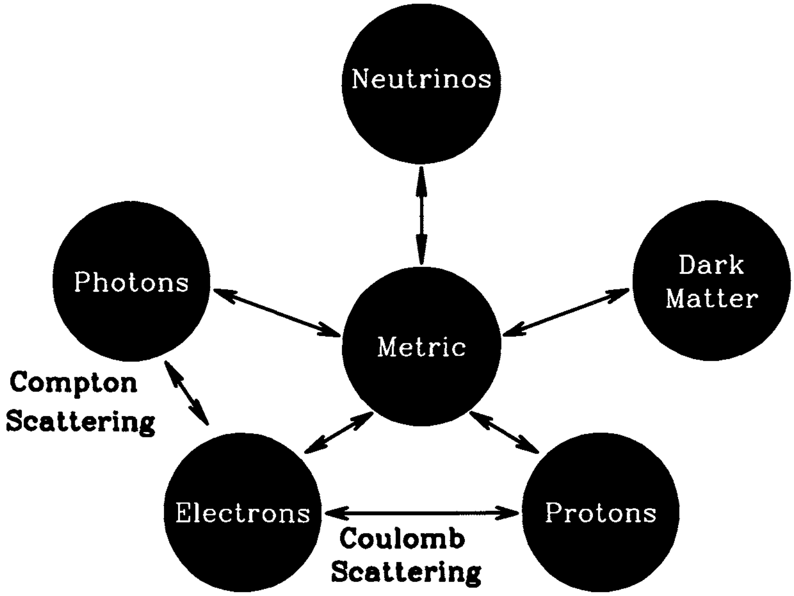

Figure 4.1. The ways in which the different components of the universe interact with each other. These connections are encoded in the coupled Boltzmann–Einstein equations.

In this chapter, we derive the Boltzmann equations for photons, eletrons, protons, dark matter, and massless neutrinos. This set of equations governs the evolution of perturbations in the universe.

4.1 THE BOLTZMANN EQUATION FOR THE HARMONIC OSCILLATOR

Before tackling the problem of interest — the Boltzmann equation for all species in an expanding universe — let us treat a much simpler example of the Boltzmann equation: the nonrelativistic harmonic oscillator. This simple example is very similar to the full general relativistic version we will encounter in the next section, but the algebra is much less cumbersome. So here the physics will be quite transparent. It will be useful to keep this example in mind when the algebra threatens to obscure the physics in the next section.

Consider a one-dimensional harmonic oscillator with energy

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{4.2}$$

The distribution function of the harmonic oscillator depends on time t, position x, and momentum p. Thus, the full time derivative in Eq. (4.1) can be rewritten as

$$\frac{df(t,x,p)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt}. \tag{4.3}$$

Figure 4.2 illustrates the movement through phase space of a distribution of collisionless ($C = 0$) oscillators. The full time derivative df/dt vanishes since the number of particles in the bunch at t_1 equals that at t_2. What has changed is the location of the phase space elements $x(t)$ and $p(t)$ themselves. Alternatively, we can think of x and p as independent variables (not dependent on t) and take partial derivatives of f with respect to t, x, and p. All of these partial derivatives are nonzero, but the appropriate weighted sum of the three vanishes.

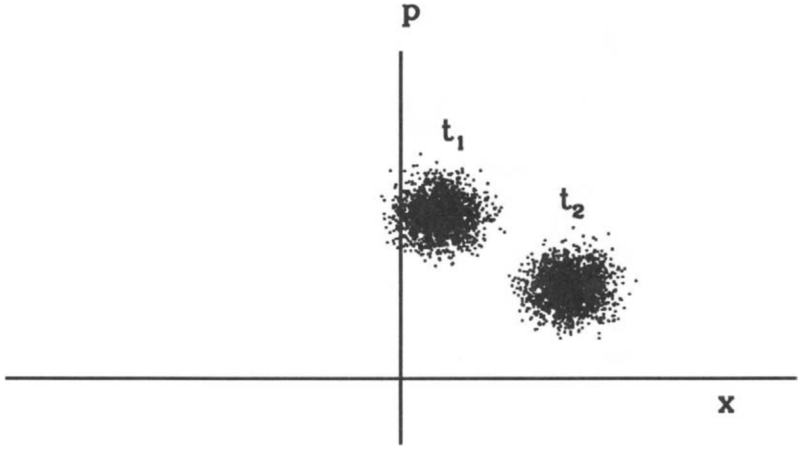

Figure 4.2. Distribution function for a set of collisionless harmonic oscillators. The initial distribution at t_1 moves in phase space by time t_2. The distribution function $f(t, x, p)$ remains constant as long as the evolution of $x(t)$ and $p(t)$ is accounted for.

To determine the coefficients dx/dt and dp/dt we must use the equations of motion. By the definition of momentum,

$$\frac{dx}{dt} \equiv \frac{p}{m}. \tag{4.4}$$

This equation will be generalized to a fully relativistic, three-dimensional version in the next section. Indeed we already got a preview of this when we defined $P^\mu \equiv dx^\mu/d\lambda$ in Chapter 2. Newton's equation governing the motion of the oscillator is

$$\frac{dp}{dt} = -kx. \tag{4.5}$$

The analogue of this familiar equation in the next section will be the geodesic equation of general relativity.

The collisionless Boltzmann equation for the harmonic oscillator is thus

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} - kx\frac{\partial f}{\partial p} = 0. \tag{4.6}$$

The second term here governs how rapidly the oscillator moves in real space; the coefficient in front is just the velocity, p/m. The last term governs how quickly particles lose momentum.

In order to solve the Boltzmann equation, we need to know the initial conditions on the distribution function. Even without these, though, the Boltzmann equation offers some useful physics. Consider the equilibrium distribution, wherein $\partial f/\partial t = 0$. A general solution for the equilibrium distribution is

$$f(p,x) = f_{\rm EQ}(E); \tag{4.7}$$

that is, f is a function only of energy E. To see that this is indeed a solution, consider

$$\begin{aligned}\frac{p}{m}\frac{\partial f(E)}{\partial x} - kx\frac{\partial f(E)}{\partial p} &= \frac{df}{dE}\left[\frac{p}{m}\frac{\partial E}{\partial x} - kx\frac{\partial E}{\partial p}\right] \\ &= 0.\end{aligned} \tag{4.8}$$

So any function of the energy alone is an equilibrium distribution. Of course, in general, there will be interactions, or collisions. The only way for the full Boltzmann equation to be satisfied is if the collision terms also vanish. This will in general drive f to one of the familiar equilibrium distributions, e.g., $e^{-E/T}$ for the classical Maxwell–Boltzmann distribution.

4.2 THE COLLISIONLESS BOLTZMANN EQUATION FOR PHOTONS

Let us begin then by considering the left hand side of Eq. (4.1) for massless photons. First we must specify the form of the metric, accounting for perturbations around the smooth universe described by Eq. (2.4). Whereas the smooth universe is characterized by a single function, $a(t)$, which depends only on time and not on space, the perturbed universe requires two more functions, Ψ and Φ, both of which depend on both space and time. In terms of them, the metric can be written as

$$\begin{aligned}g_{00}(\vec{x},t) &= -1 - 2\Psi(\vec{x},t) \\ g_{0i}(\vec{x},t) &= 0 \\ g_{ij}(\vec{x},t) &= a^2\delta_{ij}\left(1 + 2\Phi(\vec{x},t)\right).\end{aligned} \tag{4.9}$$

In the absence of Ψ and Φ, Eq. (4.9) is simply the FRW metric of the zero-order homogeneous, flat cosmology. Similarly, in the absence of expansion ($a = 1$) this metric describes a weak gravitational field (Exercise 2.3). The perturbations to the metric are Ψ, which corresponds to the Newtonian potential, and Φ, the perturbation to the spatial curvature. Since the perturbations in the universe are small at the times and scales of interest, we will treat these Ψ and Φ as small quantities, dropping all terms quadratic in them.

There are two technical points about the metric in Eq. (4.9) which you don't need to worry about for most of this book, but which nonetheless are important to be aware of, if only to better understand the literature. First, one can break up perturbations into those behaving as scalars, vectors, and tensors under a transformation from one 3D coordinate system to another. Equation (4.9) contains only scalar perturbations. In principle, it is possible that the metric of our universe also has vector or tensor perturbations. If so, $g_{\mu\nu}$ would require other functions besides Ψ and Φ to fully describe all perturbations. For example, the off-diagonal elements become nonzero if there are vector perturbations. Indeed, there are many cosmological theories wherein there are both tensor and vector perturbations. For example, inflation tends to predict that there will be tensor perturbations, while models based on topological defects tend to produce large vector perturbations. For now we focus solely on the scalar perturbations; these are the only ones that couple to matter perturbations and are the most important that couple to photon perturbations as well.

The other feature of Eq. (4.9) worth noting is that its form corresponds to a choice of *gauge*. The simplest way to understand this gauge freedom is to think back to electricity and magnetism. There, the vector potential A_μ and its derivatives contain all possible information about the electric and magnetic fields. Since the physical $\vec{E}$ and $\vec{B}$ fields remain unchanged if a constant is added to A_μ, there is some residual freedom in choosing the potential. (For example, one often chooses $A_0 = 0$ or $\partial_\mu A^\mu = 0$.) In our case of perturbations to the metric, a similar freedom exists. Even if only scalar perturbations are considered, there is still considerable freedom in the variables one chooses to describe the fluctuations. Although any physical results must be insensitive to the gauge choice, it is possible to use a gauge which looks quite different from Eq. (4.9) and still describes the same physics. For the record, the gauge in Eq. (4.9) is called the *conformal Newtonian* gauge.[1]

We want to reexpress the total derivative in Eq. (4.1) as a sum of partial derivatives. The distribution function depends on the space-time point $x^\mu = (t, \vec{x})$ and also on the momentum vector defined as

$$P^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{4.10}$$

where λ again parametrizes the particle's path, as in Eq. (2.18) (and again we will not need to specify λ explicitly). Thus, in principle, f is a function defined in an 8-dimensional space. However, not all the components of the momentum vector are independent since the masslessness of the photon implies that

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0. \tag{4.11}$$

[1]Historically, the initial ground-breaking work on the evolution of fluctuations was carried out in synchronous gauge (Peebles and Yu, 1970; Wilson and Silk, 1981; Peebles 1982; Bond and Szalay, 1983; Bond and Efstathiou, 1984). Recently, the physics of the anisotropies has been elucidated most clearly by using conformal Newtonian gauge (e.g., Hu and Sugiyama, 1995). Exercise 2 works out some of the relevant equations in synchronous gauge.

So there are only three independent components of the momentum vector. Before we choose which three we will use, let us enforce the constraint of Eq. (4.11), using the metric of Eq. (4.9).

$$P^2 = 0 = -(1+2\Psi)(P^0)^2 + p^2 = 0 \tag{4.12}$$

where I have defined

$$p^2 \equiv g_{ij}P^iP^j. \tag{4.13}$$

We can use the constraint equation then to eliminate the time component of P^μ:

$$P^0 = \frac{p}{\sqrt{1+2\Psi}} = p\,(1-\Psi)\,. \tag{4.14}$$

This last equality holds since we are doing first-order perturbation theory in the small quantity Ψ. With our sign convention, an overdense region has $\Psi < 0$. Therefore, in an overdense region, the term in parentheses on the right-hand side here is greater than one. Thus, Eq. (4.14) tells us photons lose energy—redshift—as they move out of a potential well.

Equation (4.14) is the generalization of the relativistic expression $E = pc$ to a perturbed Friedmann–Robertson–Walker metric. It allows us to eliminate P^0 whenever it occurs in favor of p, the generalized magnitude of the momentum. Recall that in the harmonic oscillator case, we did not include a term proportional to $\partial f/\partial E$ in Eq. (4.3). Here, too, we do not need to include a term proportional to $\partial f/\partial P^0$ when expanding the total time derivative. We need include only the dependence of f on the momentum: both the magnitude p and the angular direction. For the direction vector, we'll use the unit vector $\hat{p}^i = \hat{p}_i$, which by definition satisfies $\delta_{ij}\hat{p}^i\hat{p}^j = 1$.

We can now write Eq. (4.1) as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\cdot\frac{d\hat{p}^i}{dt}. \tag{4.15}$$

The easiest term in Eq. (4.15) is the last one since it does not contribute at first order in perturbation theory. To see this, first recall that the zero-order distribution function is simply the Bose–Einstein function which depends only on p, not on the direction $\hat{p}^i$. Therefore, $\partial f/\partial \hat{p}^i$ is nonzero only if we consider the perturbation to the zero-order f; i.e., it is a first-order term. But so is the term which multiplies it, $d\hat{p}^i/dt$, for the direction of a photon changes only in the presence of potentials Φ and Ψ. In the absence of these potentials, a photon moves in a straight line. Thus the last term is the product of two first-order terms, rendering it a second-order term. We can neglect it.

Next let us reexpress the second term on the right-hand side of Eq. (4.15) by recalling that (Eq. (4.10)) $P^i \equiv dx^i/d\lambda$ and $P^0 \equiv dt/d\lambda$. Therefore,

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt}$$

$$= \frac{P^i}{P^0}.. \tag{4.16}$$

We want to reexpress this ratio in terms of our favored variables p and $\hat{p}^i$. Equation (4.14) does this for P^0; let's do the same for the numerator P^i. The comoving momentum P^i is proportional to $\hat{p}^i$; call the proportionality constant C:

$$P^i \equiv C\hat{p}^i. \tag{4.17}$$

To determine the coefficient C, we can use Eq. (4.13):

$$\begin{aligned} p^2 &= g_{ij}\hat{p}^i\hat{p}^j C^2 \\ &= a^2(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^j C^2 \\ &= a^2(1+2\Phi)C^2 \end{aligned} \tag{4.18}$$

where the last equality holds because the direction vector is a unit vector. Equation (4.18) tells us that $C = p(1-\Phi)/a$ so whenever we encounter P^i, we can always eliminate it in terms of $p, \hat{p}^i$ via

$$P^i = p\hat{p}^i \frac{1-\Phi}{a}. \tag{4.19}$$

From Eqs. (4.16) and (4.19), we see that

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}(1+\Psi-\Phi). \tag{4.20}$$

An overdense region has $\Psi < 0$ and $\Phi > 0$, rendering the term in parentheses less than one. So, Eq. (4.20) says that a photon slows down (dx/dt becomes smaller) when traveling through an overdense region. This makes perfect sense: we expect the gravitational force of an overdense region to slow down even photons. Having said that, I now claim that we can neglect the potentials in Eq. (4.20). For, in the Boltzmann equation they multiply $\partial f/\partial x^i$ which is a first-order term. (Again, the zero-order distribution function does not depend on position.) So collecting terms up to this point, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt}. \tag{4.21}$$

The remaining term to be calculated is dp/dt. Alas, unlike the harmonic oscillator, here $dp/dt \neq -kx$. Rather we will need the geodesic equation from general relativity and more fortitude to compute dp/dt for photons in a perturbed FRW metric.

To begin, let us recall that the time component of the geodesic equation (2.18) can be written as

$$\frac{dP^0}{d\lambda} = -\Gamma^0{}_{\alpha\beta}P^\alpha P^\beta. \tag{4.22}$$

We can rewrite the derivative with respect to λ as a derivative with respect to time multiplied by $dt/d\lambda = P^0$. Also, we can use Eq. (4.14) to eliminate P^0 in terms of our favored variable p. Then the geodesic equation reduces to

$$\frac{d}{dt}\left[p(1-\Psi)\right] = -\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi). \tag{4.23}$$

Expand out the time derivative to get

$$\frac{dp}{dt}(1-\Psi) = p\frac{d\Psi}{dt} - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi). \tag{4.24}$$

Now we multiply both sides by $(1+\Psi)$; drop all terms quadratic in Ψ; and reexpress the total time derivative of Ψ in terms of partial derivatives so that

$$\frac{dp}{dt} = p\left\{\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - \Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi). \tag{4.25}$$

In order to evaluate dp/dt then we need to evaluate the product $\Gamma^0{}_{\alpha\beta}P^\alpha P^\beta/p$. Recall that the Christoffel symbol is best written as a sum of derivatives of the metric (Eq. (2.19)). Here we are interested only in the $\Gamma^0{}_{\alpha\beta}$ component. It multiplies $P^\alpha P^\beta$, which is symmetric in α, β. Thus, the first two metric derivatives contribute equally, and we have

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{g^{0\nu}}{2}\left[2\frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\frac{P^\alpha P^\beta}{p}. \tag{4.26}$$

Now $g^{0\nu}$ is nonzero only when $\nu = 0$, in which case it is simply the inverse of g_{00}, so

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{P^\alpha P^\beta}{p}. \tag{4.27}$$

Once again, $g_{0\alpha}$ in the first term in brackets is nonzero only when $\alpha = 0$, in which case its derivative is $-2\partial\Psi/\partial x^\beta$. The second term in brackets multiplied by the product of momenta is

$$\begin{aligned} -\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} &= -\frac{\partial g_{00}}{\partial t}\frac{P^0 P^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^i P^j}{p} \\ &= 2\frac{\partial\Psi}{\partial t}p - a^2\delta_{ij}\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]\frac{P^i P^j}{p}. \end{aligned} \tag{4.28}$$

But, via Eq. (4.19), $\delta_{ij}P^iP^j = p^2(1-2\Phi)/a^2$, so we have

$$\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\frac{\partial\Psi}{\partial x^\beta}P^\beta + 2p\frac{\partial\Psi}{\partial t}\right.$$

$$-p\left\{2\frac{\partial\Phi}{\partial t}+2H(1+2\Phi)\right\}(1-2\Phi)\Big]. \tag{4.29}$$

The last line here simplifies since $(1+2\Phi)(1-2\Phi)\to 1$ at first order, and $1-2\Phi$ can be set to 1 when multiplying $\partial\Phi/\partial t$. Summing over the index β in the first term then leads to

$$\begin{aligned}\Gamma^0{}_{\alpha\beta}\frac{P^\alpha P^\beta}{p} &= \frac{-1+2\Psi}{2}\left[-4\Big(\frac{\partial\Psi}{\partial t}p+\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}\Big)+2p\frac{\partial\Psi}{\partial t}-p\Big\{2\frac{\partial\Phi}{\partial t}+2H\Big\}\right]\\ &= \{-1+2\Psi\}\left[-\frac{\partial\Psi}{\partial t}p-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\Big\{\frac{\partial\Phi}{\partial t}+H\Big\}\right].\end{aligned} \tag{4.30}$$

We can insert this into Eq. (4.25) to get

$$\frac{dp}{dt}=p\left\{\frac{\partial\Psi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\}-\frac{\partial\Psi}{\partial t}p-2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}-p\Big\{\frac{\partial\Phi}{\partial t}+H\Big\}. \tag{4.31}$$

Collecting terms, we finally have

$$\frac{1}{p}\frac{dp}{dt}=-H-\frac{\partial\Phi}{\partial t}-\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}. \tag{4.32}$$

Equation (4.32) is what we were after. It describes the change in the photon momentum as it moves through a perturbed FRW universe. The first term accounts for the loss of momentum due to the Hubble expansion. To understand the significance of the next two terms in Eq. (4.32), we first need to remember that an overdense region has $\Phi>0$ and $\Psi<0$ with our sign conventions. Therefore, the second term says that a photon in a deepening gravitational well ($\partial\Phi/\partial t>0$) loses energy. This is understandable: the deepening well makes it more difficult for the photon to emerge, thereby increasing the magnitude of the redshift. Finally, a photon traveling into a well ($\hat{p}^i\partial\Psi/\partial x^i<0$) gains energy because it is being pulled toward the center. Conversely, as it leaves the well, it gets redshifted.

We are now in a position to write down the Boltzmann equation for photons. Using Eq. (4.32) in Eq. (4.21) leads to

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}-p\frac{\partial f}{\partial p}\left[H+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]. \tag{4.33}$$

This equation incorporates much of the physics with which we are already familiar, such as the fact that photons redshift in an expanding universe. It also leads directly to the equations governing anisotropies. Working through the terms on the right, the first two are familiar from standard hydrodynamics; when integrated, they lead to the continuity and Euler equations (Exercise 1). The third term dictates that photons lose energy in an expanding universe. We saw some of this in Chapter 2 when considering geodesics. Shortly, we will see how the Boltzmann formalism enforces this result. Finally, the last two encode the effects of under-/overdense regions on the photon distribution function.

To go further we must now expand the photon distribution function f about its zero-order Bose–Einstein value. I will do this in a way that may seem odd at first. Let us write

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1 + \Theta(\vec{x}, \hat{p}, t)]}\right\} - 1\right]^{-1}. \tag{4.34}$$

Here the zero-order temperature T is a function of time only (i.e., scales as a^{-1}), not space. The perturbation to the distribution function is characterized by Θ, which could also be called $\delta T/T$. In the smooth zero-order universe, photons are distributed homogeneously, so T is independent of $\vec{x}$, and isotropically, so T is independent of the direction of propagation $\hat{p}$. Now that we want to describe perturbations about this smooth universe, we need to allow for inhomogeneities in the photon distribution (so Θ depends on $\vec{x}$) and anisotropies (so Θ depends on $\hat{p}$). There is one assumption built into Eq. (4.34). I have explicitly written down that Θ depends on $\vec{x}, \hat{p}$, and t. This assumes that it does *not* depend on the magnitude of the momentum p. We will soon see that this is a valid assumption, following directly from that fact that the magnitude of the photon momentum is virtually unchanged during a Compton scatter. The perturbation Θ is small, so we can expand (again keeping only terms up to first order)

$$\begin{aligned} f &\simeq \frac{1}{e^{p/T} - 1} + \left(\frac{\partial}{\partial T}\left[\exp\left\{\frac{p}{T}\right\} - 1\right]^{-1}\right) T\Theta \\ &= f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta. \end{aligned} \tag{4.35}$$

In the last line I have identified the zero-order distribution function as the Bose–Einstein distribution with zero chemical potential,

$$f^{(0)} \equiv \left[\exp\left\{\frac{p}{T}\right\} - 1\right]^{-1}, \tag{4.36}$$

and made use of the fact that for this function $T\partial f^{(0)}/\partial T = -p\partial f^{(0)}/\partial p$.

4.2.1 Zero-Order Equation

We can now set about systematically collecting the terms of similar order in Eq. (4.33). Let us start with the zero-order terms, those with no Φ, Ψ, or Θ. These lead immediately to

$$\left.\frac{df}{dt}\right|_{\text{zero order}} = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0. \tag{4.37}$$

I have set df/dt here equal to zero, i.e., set the collision term on the right of Eq. (4.1) to zero. I could justify this by claiming that we are now looking only at the collisionless Boltzmann equation. But there is a much deeper justification. In fact, even when we come around to including collisions, we will see that there is no zero-order collision term. That is, the collision terms will be proportional to Θ and other perturbatively small quantities. There is a profound reason for this: the zero-order distribution function is set precisely by the requirement that the collision term vanishes. Another, perhaps more familiar, way of saying this is to point out that any collision term includes the rate for the given reaction and for its inverse. If the distribution functions are set to their equilibrium values, the rate for the reaction precisely cancels the rate for its inverse. If a given component is out of equilibrium, collisions will drive it toward its equilibrium distribution. This is the reason we expected a Bose–Einstein distribution in the first place. Its observation is convincing evidence that photons were at one point in the early universe tightly coupled to the electrons.

Returning to Eq. (4.37), we can rewrite the time derivative as

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{dT/dt}{T}p\frac{\partial f^{(0)}}{\partial p}$$

so that the zero-order equation becomes

$$\left[-\frac{dT/dt}{T} - \frac{da/dt}{a}\right]\frac{\partial f^{(0)}}{\partial p} = 0. \tag{4.38}$$

Thus $dT/T = -da/a$ or

$$T \propto \frac{1}{a}. \tag{4.39}$$

This is precisely what we expected from the heuristic argument about the photon's wavelength getting stretched as the universe expands (Section 1.1) and the more concrete argument of Section 2.1. It is reassuring to see this result emerge from the Boltzmann treatment.

4.2.2 First-Order Equation

We now return to Eq. (4.33) and extract the equation for the deviation of the photon temperature from its zero-order value, i.e., an equation for Θ. To do this, everywhere we encounter f in Eq. (4.33), we insert the expansion of Eq. (4.35):

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]$$
$$-p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]. \tag{4.40}$$

Consider the first term on the right-hand side here. The time derivative can be rewritten as a temperature derivative so

$$-p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p}$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} + p\Theta\frac{dT/dt}{T}\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]. \tag{4.41}$$

The second line follows here since $\partial f^{(0)}/\partial T = -(p/T)\partial f^{(0)}/\partial p$. The second term on this second line cancels the third term on the right in Eq. (4.40), so we can finally write down the equation governing the perturbation Θ:

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]. \tag{4.42}$$

The first two terms here account for "free streaming," which translates into anisotropies on increasingly small scales as the universe evolves. The last two account for the effect of gravity. Note that every time x appears it is multiplied by a, the scale factor. This must happen, for physical distances are ax.

4.3 COLLISION TERMS: COMPTON SCATTERING

Our task in this section is to determine the influence Compton scattering has on the photon distribution function. The scattering process of interest is

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,'), \tag{4.43}$$

where I have explicitly indicated the momentum of each particle.

We are interested in the change of distribution of photons with momentum $\vec{p}$ (with magnitude p and direction $\hat{p}$). Therefore we must sum over all other momenta $(\vec{q}, \vec{q}\,', \vec{p}\,')$ which affect $f(\vec{p})$. Schematically, then, the collision term is

$$C[f(\vec{p})] = \sum_{\vec{q},\vec{q}\,',\vec{p}\,'} |\text{Amplitude}|^2 \left\{f_e(\vec{q}\,')f(\vec{p}\,') - f_e(\vec{q})f(\vec{p})\right\}. \tag{4.44}$$

The amplitude is reversible so it multiplies both the reaction and its inverse. The products of the electron distribution function f_e and the photon distribution function simply count the number of particles with the given momenta. I have neglected stimulated emission and Pauli blocking, which would lead to factors of $1 + f$ and $1 - f_e$ with the appropriate momenta. At first order this turns out to be a valid assumption. If one were to go to second order, though, stimulated emission would have to be included. Pauli blocking is never important after electron–positron annihilation because the occupation numbers f_e are very small (Exercise 4).

Unfortunately, the collision term becomes a bit messier than the schematic version when we put in all the factors of 2π to properly account for the sums over phase space. Explicitly, the collision term is[2]

$$C[f(\vec{p})] = \frac{1}{p}\int \frac{d^3q}{(2\pi)^3 2E_e(q)}\int \frac{d^3q'}{(2\pi)^3 2E_e(q')}\int \frac{d^3p'}{(2\pi)^3 2E(p')}|\mathcal{M}|^2(2\pi)^4$$

[2]Most of the phase space factors here follow from our discussion in Section 3.1, the exception being the factor of $1/p$ in front. You may have wondered about one other feature of the Boltzmann equation presented in this chapter: I started at the outset taking df/dt; does not general relativity require us to take the derivative with respect to the affine parameter λ? Exercise 5 illustrates how these problems solve each other.

$$\times\ \delta^3\left[\vec{p}+\vec{q}-\vec{p}\,'-\vec{q}\,'\right]\delta\left[E(p)+E_e(q)-E(p')-E_e(q')\right]$$
$$\times\ \left\{f_e(\vec{q}\,')f(\vec{p}\,')-f_e(\vec{q})f(\vec{p})\right\}. \tag{4.45}$$

Here the delta functions enforce energy momentum conservation. The energies at this order are the relativistic limit for photons and non-relativistic limit for electrons: $E(p) = p$ and $E_e(q) = m_e + q^2/(2m_e)$. Note the similarity between this collision term and the general one (Eq. (3.1)) we considered in Chapter 3. The only difference is that I have not integrated this collision term over all photon momenta $\vec{p}$, so there are only three momentum integrals. Again, this reflects our need to understand how photons traveling in different directions interact: we will see that the collision term depends on $\hat{p}$.

Since the kinetic energy of the electrons is very small at the epochs of interest compared with their rest energy, the factors of E_e in the denominator of Eq. (4.45) may be replaced with m_e. Then using the three-dimensional momentum delta function to do the $\vec{q}\,'$ integral, we have

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3 p'}\,\delta\left[p+\frac{q^2}{2m_e}-p'-\frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}\right]$$
$$\times\ |\mathcal{M}|^2\left\{f_e(\vec{q}+\vec{p}-\vec{p}\,')f(\vec{p}\,')-f_e(\vec{q})f(\vec{p})\right\}. \tag{4.46}$$

To go further, we need to understand the kinematics of nonrelativistic Compton scattering. The most important feature of this process for our purposes is that very little energy is transferred. In particular,

$$E_e(q)-E_e(\vec{q}+\vec{p}-\vec{p}\,') = \frac{q^2}{2m_e}-\frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}$$
$$\simeq \frac{(\vec{p}\,'-\vec{p})\cdot\vec{q}}{m_e}, \tag{4.47}$$

where the last approximate equality holds since q is much larger than $\vec{p}$ and $\vec{p}\,'$. In nonrelativistic Compton scattering, $p' \simeq p$, scattering is nearly elastic. Therefore, $\vec{p}\,'-\vec{p}$ is of order p, of order the ambient temperature T. So the right-hand side of Eq. (4.47) is of order $Tq/m_e \sim Tv_b$ where the baryonic velocity v_b is very small. The change in the electron energy due to Compton scattering is therefore of order Tv_b. Since the typical kinetic energy of the electrons is also of order T, the fractional energy change in a single Compton collision is very small, of order v_b. It makes sense, therefore, to expand the final electron kinetic energy $(\vec{q}+\vec{p}-\vec{p}\,')^2/(2m_e)$ around its zero-order value of $q^2/(2m_e)$. The delta function can be expanded as

$$\delta\left[p+\frac{q^2}{2m_e}-p'-\frac{(\vec{q}+\vec{p}-\vec{p}\,')^2}{2m_e}\right] \simeq \delta(p-p')$$
$$+\left(E_e(q')-E_e(q)\right)\quad \left.\frac{\partial\delta\left(p+E_e(q)-p'-E_e(q')\right)}{\partial E_e(q')}\right|_{E_e(q)=E_e(q')}$$

$$= \delta(p - p') + \frac{(\vec{p} - \vec{p}\,') \cdot \vec{q}}{m_e} \frac{\partial \delta(p - p')}{\partial p'} \tag{4.48}$$

where the second equality makes use of the fact that for a general function f of the sum of two variables, $\partial f(x-y)/\partial x = -\partial f(x-y)/\partial y$. This formal expansion appears ill-defined at present, but when integrating over momenta, the derivatives of delta functions will be handled by integrating by parts. With this expansion, and using the fact that $f_e(\vec{q} + \vec{p} - \vec{p}\,') \simeq f_e(\vec{q})$, the collision term becomes

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p} \int d^3q \frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3 p'} |\mathcal{M}|^2$$
$$\times \left\{ \delta(p - p') + \frac{(\vec{p} - \vec{p}\,') \cdot \vec{q}}{m_e} \frac{\partial \delta(p - p')}{\partial p'} \right\} \{ f(\vec{p}\,') - f(\vec{p}) \} . \tag{4.49}$$

To proceed further, we need the amplitude for Compton scattering. This can be computed using Feynman rules as explicated for example in Bjorken and Drell (1965). We will take it to be constant:

$$|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2 \tag{4.50}$$

where σ_T is the Thomson cross-section. This is wrong, and it is wrong for two reasons. First of all, the amplitude squared has an angular dependence $\propto (1 + \cos^2[\hat{p}\cdot\hat{p}\,'])$. Ignoring this angular dependence, as I now propose to do, makes a small difference in the final collision term. It needs to be included in calculations which aspire to 1% accuracy. But it would simply distract us here, so let us ignore it for the present. The second reason a constant amplitude is wrong is a little more subtle and, when properly accounted for, opens up a whole new branch of CMB study. In particular, the amplitude squared has a polarization dependence ($\propto |\hat{\epsilon} \cdot \hat{\epsilon}'|^2$, where $\hat{\epsilon}$ and $\hat{\epsilon}'$ are the polarizations of the incoming and outgoing photons) which I have implicitly summed over here. The dependence on polarization means that at a small level the CMB will be polarized due to Compton scattering (Bond and Efstathiou, 1984; Polnarev, 1985). It turns out that the information carried by the polarization spectrum is as valuable as that carried by the temperature spectrum (Seljak, 1997; Seljak and Zaldarriaga, 1997; Kamionkowski, Kosowsky, and Stebbins, 1997a,b). We will devote considerable time in Chapter 10 to understanding polarization. Even if we were not concerned with polarization, the temperature anisotropies are coupled to the polarization field, so an accurate determination of the former requires a treatment of the latter. Again, though, I will neglect this small effect here in the derivation of the collision term. It is straightforward to include both the effects of polarization and the angular dependence of Compton scattering using the same formalism we are now in the midst of. The algebra is simply a bit more tedious.

Once we have assumed that $|\mathcal{M}|^2$ is constant, we can multiply out the terms in brackets in Eq. (4.49) keeping only terms first order in energy transfer. Also, the $\vec{q}$ integral simply gives a factor of n_e (or $n_e \vec{v}_b$ for the term which has a factor of $\vec{q}/m_e$). So,

$$C[f(\vec{p})] = \frac{2\pi^2 n_e \sigma_T}{p} \int \frac{d^3p'}{(2\pi)^3 p'} \left\{ \delta(p-p') + (\vec{p}-\vec{p}\,') \cdot \vec{v}_{\rm b} \frac{\partial \delta(p-p')}{\partial p'} \right\}$$
$$\times \left\{ f(\vec{p}\,') - f(\vec{p}) - p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}\,') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right\}$$
$$= \frac{n_e \sigma_T}{4\pi p} \int_0^\infty dp' p' \int d\Omega' \left[\delta(p-p') \left(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}\,') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right) \right.$$
$$\left. + (\vec{p}-\vec{p}\,') \cdot \vec{v}_{\rm b} \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p)) \right] \tag{4.51}$$

where Ω' is the solid angle spanned by the unit vector $\hat{p}\,'$. On the first line, I have broken up the difference $f(\vec{p}\,') - f(\vec{p})$ into a zero-order piece,[3] which doesn't contribute when multiplying $\delta(p-p')$, and a first-order part which can be neglected when multiplying the velocity term.

There are only two terms in Eq. (4.51) which depend on $\hat{p}\,'$ and therefore which must be accounted for when integrating over solid angle Ω'. First, there is the perturbation to the distribution function, $\Theta(\hat{p}\,')$. It is convenient at this stage to introduce the notation

$$\Theta_0(\vec{x}, t) \equiv \frac{1}{4\pi} \int d\Omega' \, \Theta(\hat{p}\,', \vec{x}, t). \tag{4.52}$$

So Θ_0 does not depend on the direction vector; it is an integral of the perturbation over all directions. In other words, it is the *monopole* part of the perturbation. Note that we *cannot* absorb this monopole into the definition of the zero order temperature since the latter is constant over all space. The perturbation Θ_0 therefore represents the deviation of the monopole at a given point in space from its average in all space. Later on we will generalize Eq. (4.52) to all other multipoles.

The second term in Eq. (4.51) which depends on $\hat{p}\,'$ is the explicit factor $\hat{p}\,' \cdot \vec{v}_{\rm b}$. This term integrates to zero since $\vec{v}_{\rm b}$ is a fixed vector. Thus, the integration over solid angle leaves

$$C[f(\vec{p})] = \frac{n_e \sigma_T}{p} \int_0^\infty dp' p' \left[\delta(p-p') \left(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta_0 + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right) \right.$$
$$\left. + \vec{p} \cdot \vec{v}_{\rm b} \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p)) \right]. \tag{4.53}$$

Now the p' integral can be done: in the first line by trivially integrating over the delta function and in the second by integrating by parts. We are left with

[3] Note that we are expanding in two small quantities simultaneously, the small perturbations and the small energy transfer. Here, we are breaking up $f(\vec{p}\,') - f(\vec{p})$ into terms zero and first order in the small perturbations.

$$C[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]. \tag{4.54}$$

Already, we can anticipate the effect of Compton scattering on the photon distribution. In the absence of a bulk velocity for the electrons ($v_b = 0$), the collision terms serve to drive Θ to Θ_0. That is, when Compton scattering is very efficient, only the monopole perturbation survives; all other moments are washed out (Figure 4.3). Intuitively, strong scattering means that the mean free path of a photon is very small. Therefore, photons arriving at a given point in space last scattered off very nearby electrons if Compton scattering is efficient. These nearby electrons most likely had a temperature very similar to the point of observation. Therefore, photons from all directions have the same temperature. This is the characteristic signature of a monopole distribution: the temperature on the sky is uniform.

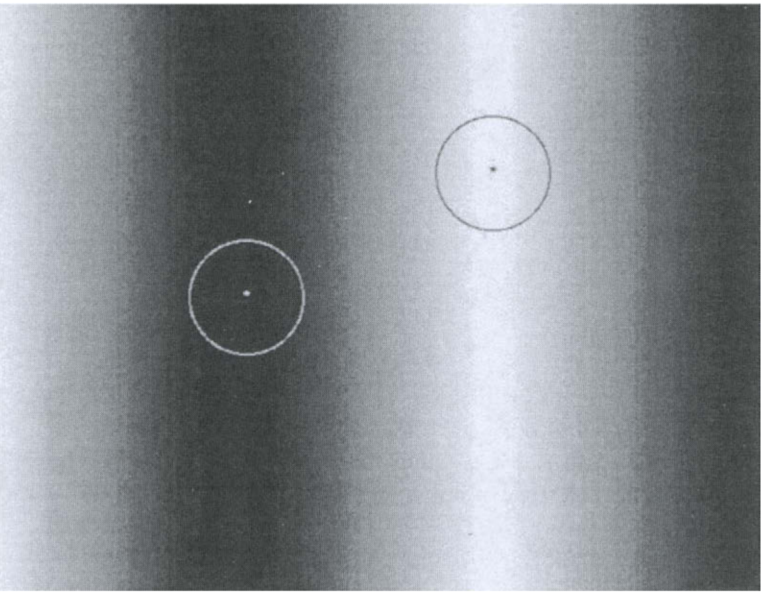

Figure 4.3. A plane wave perturbation in the matter and its effect on tightly coupled photons. Dark (white) regions represent hot (cold) spots in the electron temperature. If Compton scattering is very efficient then photons last scattered very near the point of observation. Circles denote last scattering surfaces for observation points indicated by stars. The temperature on these surfaces is very close to uniform, so the distribution is almost all monopole. Note though that different circles (corresponding to different observers) have different temperatures due to the perturbation. So the monopole varies in space.

The situation changes slightly if the electrons carry a bulk velocity. In that case, the photons will also have a dipole moment, fixed by the amplitude and direction of the electron velocity. Even in this case, though, all higher moments will vanish. Thus Compton scattering produces a photon distribution which is extremely simple to categorize: it has only a nonvanishing monopole and dipole. This is equivalent to saying that the photons behave like a fluid. Indeed, strong scattering, or tight coupling, produces a situation wherein the photons and electrons behave as a single fluid.

4.4 THE BOLTZMANN EQUATION FOR PHOTONS

We can now collect the left- and right-hand sides of the Boltzmann equations from the previous two sections. A few more definitions will complete the first goal of this chapter, a linear equation for the perturbation to the photon distribution. Equating Eqs. (4.42) and (4.54) leads to

$$\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}=n_e\sigma_T\left[\Theta_0-\Theta+\hat{p}\cdot\vec{v}_{\rm b}\right]. \tag{4.55}$$

At this point, it is convenient to reintroduce the conformal time η, defined in Eq. (2.41), as our time variable. In terms of the conformal time, the Boltzmann equation becomes

$$\dot{\Theta}+\hat{p}^i\frac{\partial\Theta}{\partial x^i}+\dot{\Phi}+\hat{p}^i\frac{\partial\Psi}{\partial x^i}=n_e\sigma_T a\left[\Theta_0-\Theta+\hat{p}\cdot\vec{v}_{\rm b}\right]. \tag{4.56}$$

Here, and from now on, overdots represent derivatives with respect to conformal time.

Equation (4.56) is a partial differential linear equation coupling Θ to other variables Φ,Ψ, and $\vec{v}_{\rm b}$ which also evolve linearly. If we Fourier transform all these variables, so that $\partial/\partial x^i\to k_i(\equiv k^i)$, the resulting Fourier amplitudes obey ordinary differential equations, which are much simpler to solve. In the case of small perturbations around a smooth universe, there is an added benefit of Fourier transforming. Since the background is smooth, the only $\vec{x}$ dependence in Eq. (4.56) is hidden in the perturbation variables themselves. In general, an equation of the form

$$aA(\vec{x})=bB(\vec{x}) \tag{4.57}$$

gets transformed into

$$a\tilde{A}(\vec{k})=b\tilde{B}(\vec{k}). \tag{4.58}$$

That is, every Fourier mode evolves independently: $A(\vec{k}_1)$ can be evolved even if we know nothing of $A(\vec{k}_2)$. So the Fourier transform of Eq. (4.56) produces a set of ordinary differential equations for the Fourier modes, and this set of equations is uncoupled. Instead of solving an infinite number of coupled equations, we can solve for one k-mode at a time.

Note that this simplification arises because the perturbations are small (equivalently the equations are linear). In this case, the different Fourier modes all evolve independently. Perturbations to the CMB remain small at all cosmological epochs, so Fourier transforms are very useful. In contrast, perturbations to matter are more complicated. Initially they are small, and they remain small until relatively recently.

The largest scales today are still in the linear regime, so Fourier transforming is certainly useful for the matter perturbations as well. However, to completely characterize the matter field today requires accounting for nonlinearities, and for this purpose, Fourier transforms lose much of their appeal. Different Fourier modes couple when nonlinear behavior becomes important, so the codes which follow matter perturbations all the way until today work in real space. Even these codes, however, start at $z \sim 20$ with the initial conditions set by linear evolution.

Our Fourier convention will be

$$\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\Theta}(\vec{k}). \tag{4.59}$$

We will often characterize a mode by the magnitude of its wavevector[4] : $k = \sqrt{k^i k^i}$.

Before rewriting Eq. (4.56) in terms of Fourier modes, let us make two final definitions. First, define the cosine of the angle between the wavenumber $\vec{k}$ and the photon direction $\hat{p}$ to be

$$\mu \equiv \frac{\vec{k}\cdot\hat{p}}{k}. \tag{4.60}$$

From now on, μ will be the variable describing the direction of photon propagation. A good way to think of μ is to go back to Figure 4.3. The wavevector $\vec{k}$ is pointing in the direction in which the temperature is changing, so it is perpendicular to the gradient ($\vec{k}$ is horizontal in the figure). When $\mu = 1$ then the photon direction is aligned with $\vec{k}$, so the photon is traveling in the direction along which the temperature is changing. A photon traveling in a direction in which the temperature remains the same (vertically in the figure) has $\mu = 0$. We will typically assume that the velocity points in the same direction as $\vec{k}$ (this is equivalent to saying that the velocity is irrotational), so $\tilde{\vec{v}}_{\rm b} \cdot \hat{p} = \tilde{v}_{\rm b}\mu$. Next, we define the optical depth

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta'\, n_e \sigma_T a. \tag{4.61}$$

At late times, the free electron density is small, so $\tau \ll 1$, while at early times, it is very large. Note that I have defined the limits of integration in such a way that

$$\dot{\tau} \equiv \frac{d\tau}{d\eta} = -n_e \sigma_T a. \tag{4.62}$$

With these definitions, we are finally left with

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_{\rm b}\right]. \tag{4.63}$$

[4]Note that k^i is a 3D vector in Eudclidean space so that $k_i = k^i$; you do not need a factor of g_{ij} to go back and forth. The same goes for the velocity $v^i_{\rm b}$.

4.5 THE BOLTZMANN EQUATION FOR COLD DARK MATTER

We can apply the formalism developed in the previous sections to derive the Boltzmann equation for any other constituent in the universe. Of particular importance is the evolution of the dark matter. In almost all currently popular models of structure formation, dark matter plays an important role in structure formation and in determining the gravitational field in the universe.

It is perhaps simplest to derive the evolution equations for dark matter by imposing conservation of the energy–momentum tensor, as we did in Chapter 2 in the homogeneous case. Unlike the photons, the dark matter always behaves like a fluid so can always be described completely by $T_{\mu\nu}$. Nonetheless, here we will sacrifice simplicity and use the Boltzmann formalism to derive the dark matter equations. This will (i) reinforce the calculations of the previous sections and also (ii) pave the way for the electron/proton equations of the next section.

There are several ways in which the dark matter distribution differs from that of the photons. First, by definition, "dark" matter does not interact with any of the other constituents in the universe. Thus we need not deal with any collision terms. Second, cold dark matter, in contrast to the photons, is nonrelativistic. So we need to redo some of the kinematics which led to the left side of the Boltzmann equation. In particular, the constraint Eq. (4.11) now becomes

$$g_{\mu\nu}P^\mu P^\nu = -m^2 \tag{4.64}$$

where m is the mass of the dark matter particle. It is also useful to define the energy as

$$E \equiv \sqrt{p^2 + m^2}, \tag{4.65}$$

where p is defined exactly as in Eq. (4.13): $p^2 = g_{ij}P^iP^j$. In the massless case, of course, Eq. (4.65) says that $E = p$, so E is superfluous. Here it will be convenient to let E replace p as one of the variables on which the distribution function depends (in addition to position $\vec{x}$, time t, and the direction vector $\hat{p}$). We can now derive the equivalent of equations (4.14) and (4.19) for the four-momentum of a massive particle:

$$P^\mu = \left[E(1-\Psi), p\hat{p}^i\frac{1-\Phi}{a}\right]. \tag{4.66}$$

Only the time component is different from that of a massless particle, with E replacing p.

Using E as one of the dependent variables means that the total time derivative of the dark matter distribution function $f_{\rm dm}$ is

$$\frac{df_{\rm dm}}{dt} = \frac{\partial f_{\rm dm}}{\partial t} + \frac{\partial f_{\rm dm}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{\rm dm}}{\partial E}\frac{dE}{dt} + \frac{\partial f_{\rm dm}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}. \tag{4.67}$$

Once again, the last term here vanishes since it is the product of two first-order terms. Because of the change in the constraint equation, the coefficients of the derivatives of the distribution function with respect to x^i and E are slightly different than they were in the massless case. Working through the algebra, which is otherwise identical to the calculation presented in Section 4.2, leads to the collisionless Boltzmann equation for nonrelativistic matter:

$$\frac{\partial f_{\rm dm}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{\rm dm}}{\partial x^i} - \frac{\partial f_{\rm dm}}{\partial E}\left[\frac{da/dt}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\right] = 0. \tag{4.68}$$

Equation (4.68) reduces to Eq. (4.33) in the massless limit as it must. The main difference between the two is the presence of factors of p/E, or velocity. For dark matter particles, these velocity factors suppress any free streaming, as we will shortly see.

In the massless case, to proceed further we used our knowledge of the distribution function. Namely, we knew that the zero-order distribution function was Bose–Einstein, and we perturbed around this zero-order solution. For cold dark matter particles, we do not need such detailed information about the zero-order distribution function. All we need to know is that these particles are very nonrelativistic. So we can neglect the thermal motion of the dark matter (Exercise 9). We cannot however neglect p/m completely, because the density perturbations themselves induce velocity flows in the dark matter via the continuity equation. These coherent flows give rise to $p/m \sim v$ terms, which must be retained. What we can do, however, in our linear treatment, is to neglect terms second-order in p/E.

Instead of assuming a form for $f_{\rm dm}$, we will take moments of Eq. (4.68). First, multiply both sides by the phase space volume $d^3p/(2\pi)^3$ and integrate. This leads to

$$\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_{\rm dm} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^i}{E} - \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right]\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^2}{E}$$
$$-\frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\hat{p}^i p = 0. \tag{4.69}$$

Note that, since they are independent variables, the integral over p passes through the partial derivatives with respect to x^i and t. The last term here can be neglected since the integral over the direction vector is nonzero only for the perturbed part of $f_{\rm dm}$. Thus the integral is first order and it multiplies the first-order term $\partial\Psi/\partial x^i$. The rest of the terms are all relevant, though. To simplify, let us recall that the dark matter density is[5]

$$n_{\rm dm} = \int \frac{d^3p}{(2\pi)^3} f_{\rm dm} \tag{4.70}$$

while the velocity is defined as

$$v^i \equiv \frac{1}{n_{\rm dm}}\int \frac{d^3p}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^i}{E}. \tag{4.71}$$

[5]Here I have incorporated the spin degeneracy $g_{\rm dm}$ into the phase space distribution $f_{\rm dm}$. We implicitly did the same thing in the last section for the electrons.

The first two terms in Eq. (4.69), then, can be simply expressed in terms of the velocity and the density. The third term is a bit more subtle: to relate it to the density, we need to integrate by parts. Since $dE/dp = p/E$, the integrand can be reexpressed as $p\,\partial f_{\rm dm}/\partial p$. Thus, the integral becomes

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3} p \frac{\partial f_{\rm dm}}{\partial p} &= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, p^3 \frac{\partial f_{\rm dm}}{\partial p} \\ &= -3\frac{4\pi}{(2\pi)^3}\int_0^\infty dp p^2 f_{\rm dm} \\ &= -3n_{\rm dm}.\end{aligned} \tag{4.72}$$

So the zeroth moment of the Boltzmann equation leads to the cosmological generalization of the continuity equation:

$$\frac{\partial n_{\rm dm}}{\partial t} + \frac{1}{a}\frac{\partial(n_{\rm dm}v^i)}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right] n_{\rm dm} = 0. \tag{4.73}$$

The first two terms here are the standard continuity equation from fluid mechanics. The last term arises due to the FRW metric and its perturbations.

To go further, we can collect zero-order and first-order terms in Eq. (4.73). The velocity is first order as is Φ, so the only zero-order terms are

$$\frac{\partial n^{(0)}_{\rm dm}}{\partial t} + 3\frac{da/dt}{a} n^{(0)}_{\rm dm} = 0 \tag{4.74}$$

where $n^{(0)}_{\rm dm}$ is the zero-order, homogeneous part of the density. Equivalently, we have

$$\frac{d(n^{(0)}_{\rm dm}a^3)}{dt} = 0 \Longrightarrow n^{(0)}_{\rm dm} \propto a^{-3}, \tag{4.75}$$

a relation we anticipated back in Chapter 1 as an obvious ramification of the expansion. We also proved this scaling in Chapter 2 by using the conservation of the energy momentum tensor.

Now let us extract the first-order part of Eq. (4.73). All factors of $n_{\rm dm}$ multiplying the first-order quantities v and Φ may be set to $n^{(0)}_{\rm dm}$. Everywhere else, we need to expand $n_{\rm dm}$ out to include a first-order perturbation. In particular, we will set

$$n_{\rm dm} = n^{(0)}_{\rm dm}\left[1 + \delta(\vec{x}, t)\right], \tag{4.76}$$

which defines the first-order piece as $n^{(0)}_{\rm dm}\delta$. Since the energy density of matter is equal to mass times n, δ is also the fractional overdensity, $\delta\rho/\rho$, of the dark matter. After dividing by $n^{(0)}_{\rm dm}$, the first-order equation is therefore

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{4.77}$$

As it stands, we have introduced two new perturbation variables for the dark matter, the density perturbation δ and the velocity $\vec{v}$. Equation (4.77) is only one equation, though, for these two variables. We need another. To get it, we return to the unintegrated Boltzmann equation (4.68). We have just taken its zeroth moment; to extract a second equation, let us take its first moment. In particular, multiply Eq. (4.68) by $d^3p(p/E)\hat{p}^j/(2\pi)^3$ and then integrate. The first moment equation is then

$$0 = \frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^j}{E} + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_{\rm dm}\frac{p^2\hat{p}^i\hat{p}^j}{E^2}$$
$$- \left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^3\hat{p}^j}{E^2} - \frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{\hat{p}^i\hat{p}^j p^2}{E}. \quad (4.78)$$

The first two terms are straightforward: the first is the time derivative of $n_{\rm dm}v^i$ while the second can be safely neglected since it is of order $\langle(p/E)^2\rangle$. The last sets of terms must be handled more carefully, though, because of the partial derivatives. Since $(p/E)\partial/\partial E = \partial/\partial p$ the third term is actually of order p/E while the last is independent of velocity. Let us do the integration by parts explicitly in the third term. The integral is:

$$\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial p}\frac{p^2\hat{p}^j}{E} = \int \frac{d\Omega\,\hat{p}^j}{(2\pi)^3}\int_0^\infty dp\frac{p^4}{E}\frac{\partial f_{\rm dm}}{\partial p}$$
$$= -\int \frac{d\Omega\,\hat{p}^j}{(2\pi)^3}\int_0^\infty dp f_{\rm dm}\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right). \quad (4.79)$$

The p^5/E^3 term is completely negligible, so the only relevant contribution to the integral comes from the $-4p^3/E$ term: its integral is $-4n_{\rm dm}v^j$. The same steps carry through for the last term in Eq. (4.78); the one additional fact we need is that

$$\int d\Omega\hat{p}^i\hat{p}^j = \delta^{ij}\frac{4\pi}{3}. \quad (4.80)$$

So the first moment of the Boltzmann equation is

$$\frac{\partial(n_{\rm dm}v^j)}{\partial t} + 4\frac{da/dt}{a}n_{\rm dm}v^j + \frac{n_{\rm dm}}{a}\frac{\partial\Psi}{\partial x^j} = 0. \quad (4.81)$$

This equation has no zero-order parts, since the velocity is a first-order quantity. Therefore, we need extract only the first-order terms, which allows us to set $n_{\rm dm} \to n^{(0)}_{\rm dm}$ everywhere. Using the time dependence we found in Eq. (4.75) we arrive at

$$\frac{\partial v^j}{\partial t} + \frac{da/dt}{a}v^j + \frac{1}{a}\frac{\partial\Psi}{\partial x^j} = 0. \quad (4.82)$$

Equations (4.77) and (4.82) are the two equations governing the evolution of the density and the velocity of the cold, dark matter. The momentum conservation equation (4.82) does not have the standard $(\vec{v}\cdot\nabla)\vec{v}$ term, since any term with two factors of v is manifestly second order. An interesting feature of the two equations is generic to this process of integrating the Boltzmann equations to get the fluid equations. Note that the equation for the density depends on the next highest moment, the velocity. This is general: the integrated Boltzmann equation for the lth moment depends on the $l+1$ moment. In principle, then, this process of integrating leads to an infinite heirarchy of equations for the moments of the distribution function. Indeed, we will see that this is one way of solving the Boltzmann equation for the photons, Eq. (4.63), which we have not yet integrated over. One might expect, then, that the velocity equation would depend on the next highest moment, the quadrupole, of the dark matter distribution. Why doesn't it? The answer lies in our assumption that the dark matter is *cold*. We have explicitly dropped all terms of order $(p/E)^2$ and higher. These terms correspond to the higher moments of the distribution, but since we are dealing with cold, dark matter they are irrelevant. Thus, the set of two equations, (4.77) and (4.82), are a closed set of equations for the cold, dark matter distribution.[6] If we were interested in dark matter particles with much smaller masses, such as massive neutrinos, we would need to keep these higher moments.

Let us finally rewrite Eqs. (4.77) and (4.82) in terms of conformal time η and the Fourier transforms. The density equation becomes

$$\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0 \tag{4.83}$$

where I have assumed that the velocity is irrotational so $\tilde{v}^i = (k^i/k)\tilde{v}$. The velocity equation is

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0. \tag{4.84}$$

4.6 THE BOLTZMANN EQUATION FOR BARYONS

The final components of the universe which require a set of Boltzmann equations are the electrons and protons. These components are often grouped together and called *baryons*, nomenclature which is obviously ridiculous (electrons are leptons, not baryons) but nonetheless common.

Electrons and protons are coupled by Coulomb scattering ($e + p \to e + p$). The Coulomb scattering rate is much larger than the expansion rate at all epochs of interest (Exercise 12). This tight coupling forces the electron and proton overdensities to a common value:

$$\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b \tag{4.85}$$

where we bow to common usage with the subscript b. Similarly the velocities of the two species are forced to a common value,

[6] Of course, we still need equations for the gravitational potentials Φ and Ψ. These come from Einstein's equations, as does the zero-order equation for a.

$$\vec{v}_e = \vec{v}_p \equiv \vec{v}_{\rm b}. \tag{4.86}$$

We need to derive equations then for δ_b and $v_{\rm b}$. The starting point will be the unintegrated equations for electrons and protons:

$$\frac{df_e(\vec{x},\vec{q},t)}{dt} = \langle c_{ep}\rangle_{QQ'q'} + \langle c_{e\gamma}\rangle_{pp'q'} \tag{4.87}$$

$$\frac{df_p(\vec{x},\vec{Q},t)}{dt} = \langle c_{ep}\rangle_{qq'Q'}. \tag{4.88}$$

The notation here is more compact, and therefore more deceiving, than that in previous sections. We will need this compactness in what follows, so let's walk through it slowly. First, notice that initial and final momenta for the photon are $\vec{p}$ and $\vec{p}\,'$; for electron $\vec{q}$ and $\vec{q}\,'$; and the proton has been assigned $\vec{Q}$ and $\vec{Q}'$. Consider the Compton collision term in the equation for the electron distribution function. I have defined the unintegrated part of the collision term as

$$c_{e\gamma} \equiv (2\pi)^4\delta^4(p+q-p'-q')\frac{|\mathcal{M}|^2}{8E(p)E(p')E_e(q)E_e(q')}\left\{f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\right\} \tag{4.89}$$

and the angular brackets denote integration over all momenta in the subscripts:

$$\langle(\ldots)\rangle_{pp'q'} \equiv \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3q'}{(2\pi)^3}(\ldots). \tag{4.90}$$

The Coulomb collision term is similar, the main difference being the amplitude for the two processes.

In principle, Eq. (4.88) should contain a term accounting for scattering of protons off photons. In practice, though, the cross section for this process is much smaller than for Compton scattering off electrons (in each case the cross section is inversely proportional to the mass squared). So the interactions of the combined electron–proton fluid with the photons is driven by Compton scattering of electrons, and the proton–photon process can be ignored. Also, in principle, we should include ionization and recombination terms in Eqs. (4.87) and (4.88). These however would merely distract us here, so we treat all electrons as ionized.

With this notation defined, we can now proceed and derive equations for δ_b and $v_{\rm b}$. First, multiply both sides of Eq. (4.87) by the phase space volume $d^3q/(2\pi)^3$ and integrate. The left-hand side then becomes identical to the left-hand side we derived for dark matter in Eq. (4.73). So we can immediately write

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial(n_e v^i_{\rm b})}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]n_e = \langle c_{ep}\rangle_{QQ'q'q} + \langle c_{e\gamma}\rangle_{pp'q'q}. \tag{4.91}$$

Both terms on the right vanish. The mathematical way to see this is to realize that the integration measure in the first term on the right, e.g., is completely symmetric under the interchange of $Q \leftrightarrow Q'$ and $q \leftrightarrow q'$. Because of the factors of the distribution function, the integrand — c_{ep} — is antisymmetric under this interchange. So the full integral vanishes. More intuitively, the processes we are considering conserve electron number so they certainly cannot contribute to dn/dt. That is, the integral over $f_e(q')f_p(Q')$ counts the total number of electrons that are produced in Coulomb scattering. But this is obviously equal to the integral over $f_e(q)f_p(Q)$, which counts the number of electrons lost in Coulomb scattering. More generally, any time we multiply an unintegrated collision term by a conserved quantity and then integrate we will get zero.

The perturbed version of Eq. (4.91) equation is therefore identical to Eq. (4.77). Switching to Fourier space and using conformal time leads to

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_{\rm b} + 3\dot{\tilde{\Phi}} = 0. \tag{4.92}$$

The second equation for the baryons is obtained by taking the first moments of both Eqs. (4.87) and (4.88) and adding them together. We did something similar for the dark matter; there we first multiplied by $\vec{p}/E$ and then integrated over all momenta. Here we will take the moments by first multiplying the unintegrated equations by $\vec{q}$ (and $\vec{Q}$ for the protons) instead of by $\vec{q}/E$. Therefore, our results from the dark matter case carry over as long as we multiply them by a factor of m. The left-hand side of the integrated electron equation, for example, will look exactly like the left-hand side of Eq. (4.81) except it will be multiplied by m_e. The proton equation will be multiplied by m_p. Since the proton mass is so much larger than the electron mass, the sum of the two left-hand sides will be dominated by the protons. So, following Eq. (4.81), we have

$$\begin{aligned} m_p\frac{\partial(n_{\rm b}v_{\rm b}^j)}{\partial t} + 4\frac{da/dt}{a}m_p n_{\rm b}v_{\rm b}^j + \frac{m_p n_{\rm b}}{a}\frac{\partial\Psi}{\partial x^i} \\ = \langle c_{ep}(q^j+Q^j)\rangle_{QQ'q'q} + \langle c_{e\gamma}q^j\rangle_{pp'q'q}. \end{aligned} \tag{4.93}$$

The right-hand side here is the sum of that from both the electron and proton equations. Both equations have the Coulomb term, so it is weighted by $\vec{q}$ (by which we multiplied the electron equation) $+\vec{Q}$ (from the proton equation). Only the electron equation has the Compton term, so there is only the factor of $\vec{q}$ there. Once again we can use a conservation law, this time conservation of momentum, to argue that the integral of $c_{ep}(\vec{q}+\vec{Q})$ over all momenta vanishes. So dividing both sides by[7] $\rho_b = m_p n_{\rm b}^{(0)}$, we are left with

$$\frac{\partial v_{\rm b}^j}{\partial t} + \frac{da/dt}{a}v_{\rm b}^j + \frac{1}{a}\frac{\partial\Psi}{\partial x^j} = \frac{1}{\rho_b}\langle c_{e\gamma}q^j\rangle_{pp'q'q}. \tag{4.94}$$

Here I have used the by now familiar $n_{\rm b}^{(0)} \propto a^{-3}$ scaling to eliminate the $n_{\rm b}^{(0)}$ time derivative and three of the four factors of the da/dt term on the left.

[7]Note that convention here, which I will stick with, that ρ_b is the zero order baryon energy density. The total baryon density is therefore $\rho_b(1+\delta_{\rm b})$.

The final step is to evaluate the average momentum $\vec{q}$ in Compton scattering. As before, we can use the conservation of total momentum $\vec{q}+\vec{p}$ to argue that

$$\langle c_{e\gamma}\vec{q}\rangle_{pp'q'q} = -\langle c_{e\gamma}\vec{p}\rangle_{pp'q'q}. \tag{4.95}$$

Now switch to Fourier space and multiply both sides of Eq. (4.94) by $\hat{k}^j$. Since $\hat{k}\cdot\vec{p} = p\mu$, the right-hand side of Eq. (4.94) becomes $-\langle c_{e\gamma}p\mu\rangle_{pp'q'q}/\rho_b$. We have already computed $\langle c_{e\gamma}\rangle_{p'q'q}$ in Eq. (4.54). We need simply multiply this by $p\mu$ and integrate over all $\vec{p}$ to find the right-hand side of Eq. (4.94):

$$\begin{aligned}
-\frac{\langle c_{e\gamma}p\mu\rangle_{pp'q'q}}{\rho_b} &= \frac{n_e\sigma_T}{\rho_b}\int\frac{d^3p}{(2\pi)^3}p^2\frac{\partial f^{(0)}}{\partial p}\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_{\rm b}\mu\right] \\
&= \frac{n_e\sigma_T}{\rho_b}\int_0^\infty\frac{dp}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p}\int_{-1}^1\frac{d\mu}{2}\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_{\rm b}\mu\right].
\end{aligned} \tag{4.96}$$

The integral over p can be done by integrating by parts: it is $-4\rho_\gamma$. The μ-integration over the first and third terms is straightforward (first term vanishes and second gives $v_{\rm b}/3$). The second term is the first moment of the perturbation Θ. Recall that the zeroth moment was defined as Θ_0. It makes sense therefore to define the first moment as

$$\Theta_1 \equiv i\int_{-1}^1\frac{d\mu}{2}\mu\Theta(\mu) \tag{4.97}$$

where the factor of i is a convention and the definition holds in either real or Fourier space.

We now have an expression for the collision term which can be inserted into Eq. (4.94), and after switching to conformal time, we have:

$$\dot{\tilde{v}}_{\rm b} + \frac{\dot{a}}{a}\tilde{v}_{\rm b} + ik\tilde{\Psi} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_{\rm b}\right]. \tag{4.98}$$

Why is there a factor of ρ_b in the denominator? That is, since photons scatter primarily off electrons, why does the total baryon density (which is dominated by protons) appear in this velocity equation? Physically, it arises from the fact that moving electrons is difficult because they are tightly coupled to protons via Coulomb scattering. If the proton was infinitely heavy, so $\rho_b\to\infty$, Compton scattering would not change the electron velocity at all; it would not have any impact on the combined proton–electron fluid. We derived Eq. (4.98) by setting $n_e = n_p = n_b$, but it turns out to be valid even if there is an appreciable amount of neutral hydrogen, so that $n_e \neq n_b$. Indeed after recombination, most protons are bound in neutral hydrogen atoms. And even before recombination, a small fraction are in helium atoms or ions. You might be tempted therefore to replace ρ_b in the denominator of Eq. (4.98) by the density of free protons. In fact, though, even neutral hydrogen and helium are tightly coupled to electrons and protons (see Exercise 12), so all baryons should be included. Equation (4.98) quite generally governs the evolution of the baryon velocity.

4.7 SUMMARY

The constituents of the universe are not distributed completely uniformly in space. For the nonrelativistic components such as the dark matter and the baryons, this means that some regions are more dense than others and that there are small coherent velocities. For the dark matter we denote the fractional overdensity as $\delta(\vec{x}, t)$ and the velocity as $\vec{v}(\vec{x}, t)$. The equivalent perturbations for the baryons are $\delta_b(\vec{x}, t)$ and $\vec{v}_b(\vec{x}, t)$. In solving the linear evolution equations, it is simplest to work with Fourier transforms of all of these. It turns out that the evolution of a mode associated with wavevector $\vec{k}$ depends only on the magnitude of $\vec{k}$, so we have equations for $\tilde{\delta}(k, t)$. We have found it convenient to use conformal time η as the evolution variable. Also, it is conventional[8] in the literature to drop the $\tilde{}$s over Fourier transformed variables, so our equations will be for $\delta(k, \eta), \delta_b(k, \eta), v(k, \eta)$, and $v_b(k, \eta)$. The scalar velocities here are the components parallel to $\vec{k}$; these are the only ones that are cosmologically relevant.

Relativistic particles such as photons and neutrinos require more information to characterize. They have not only a monopole perturbation (the equivalent of δ) and a dipole (the equivalent of a velocity), but also a quadrupole, octopole, and higher moments as well. In other words, the photon distribution depends not only on $\vec{x}$ and time but also on the direction of propagation of the photon, $\hat{p}$. In Fourier space, therefore, the photon perturbations depend not only on k and η but also on $\hat{p} \cdot \hat{k}$, which we defined as μ. Thus, the photon perturbation variable is $\Theta(k, \mu, \eta)$, the Fourier transform of $\delta T/T$, the fractional temperature difference. Neutrino perturbations require a separate variable with the same dependence; let's call it $\mathcal{N}(k, \mu, \eta)$.

We found it useful to define the monopole (Eq. (4.52)) and dipole (Eq. (4.97)) of the photon distribution. These moments, $\Theta_0(k, \eta)$ and $\Theta_1(k, \eta)$, do not completely characterize the photon distribution. More generally, it is useful to define the lth multipole moment of the temperature field as

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_l(\mu) \Theta(\mu), \tag{4.99}$$

where $\mathcal{P}_l$ is the Legendre polynomial of order l. The quadrupole corresponds to $l = 2$, octopole to $l = 3$, etc. The higher Legendre polynomials have structure on smaller scales (see Figure 4.4), so the higher moments capture information about the small scale structure of the temperature field. So the photon perturbations can be described either by $\Theta(k, \mu, \eta)$ or by a whole hierarchy of moments, $\Theta_l(k, \eta)$. And of course similar freedom applies to the neutrino distribution.

I have postponed a discussion of polarization until Chapter 10, but I mentioned in Section 4.3 that a completely accurate treatment of anisotropies in the temperature requires us to incorporate polarization effects. Again, waiting until Chapter 10 for more formal definitions, let's call the strength of the polarization Θ_P. It

[8] Conventional, but "abominable" according to one early reviewer.

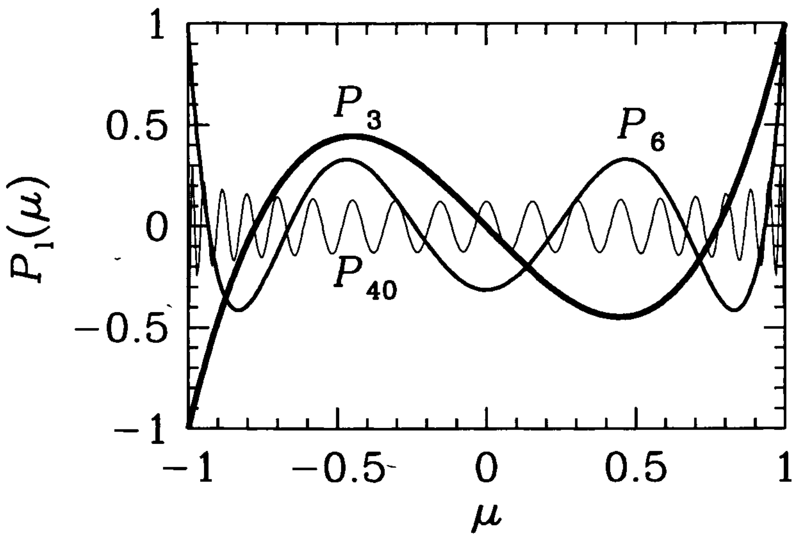

Figure 4.4. Some Legendre polynomials. Note that the higher order ones vary on smaller scales than do the low-order ones. In general $\mathcal{P}_l$ crosses zero l times between -1 and 1.

describes the change in the polarization field in space. Upon Fourier transforming, it too depends on k, μ, and η.

We now collect the equations we have derived for the photons, dark matter, and baryons and supplement them with a trivial extension to massless neutrinos:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_{\rm b} - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\right] \tag{4.100}$$

$$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0} \tag{4.101}$$

$$\dot{\Theta}_P + ik\mu\Theta_P = -\dot{\tau}\left[-\Theta_P + \frac{1}{2}(1 - \mathcal{P}_2(\mu))\Pi\right] \tag{4.102}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{4.103}$$

$$\dot{v} + \frac{\dot{a}}{a}v = -ik\Psi \tag{4.104}$$

$$\dot{\delta}_{\rm b} + ikv_{\rm b} = -3\dot{\Phi} \tag{4.105}$$

$$\dot{v}_{\rm b} + \frac{\dot{a}}{a}v_{\rm b} = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_{\rm b} + 3i\Theta_1\right] \tag{4.106}$$

$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi. \tag{4.107}$$

Equation (4.100) is the Boltzmann equation for photons we have derived. The one change from our derivation is the last term $\mathcal{P}_2\Pi/2$, which requires some explanation. First, note that it is proportional to the second Legendre polynomial, $\mathcal{P}_2(\mu) = (3\mu^2 - 1)/2$. From Eq. (4.101), one of the new terms then is $\mathcal{P}_2\Theta_2/2$; this term accounts for the angular dependence of Compton scattering, which we ignored in Section 4.3. The other parts of Π represent the fact that the temperature field is also coupled to the strength of the polarization field Θ_P which obeys Eq. (4.102). Note that Θ_P is sourced by the quadrupole, Θ_2, and none of the other temperature moments.

In the equation for the baryon velocity (4.106), the ratio of photon to baryon density has been defined as

$$\frac{1}{R} \equiv \frac{4\rho_\gamma^{(0)}}{3\rho_b^{(0)}}. \tag{4.108}$$

Equation (4.107) governs perturbations to the neutrino distribution, $\mathcal{N}$. It is identical to the photon equation except that there is no scattering term since neutrinos interact only very weakly. Here I have assumed that the neutrinos are massless. If any of the neutrinos had appreciable mass, then Eq. (4.107) would have to be amended to account for this. Exercise 11 discusses the question of how large a mass is interesting.

SUGGESTED READING

In the 1960s a national magazine ran a cartoon showing dozens of businessmen and -women walking the streets of Manhattan looking very important and serious. Thought bubbles over each head revealed their true focus: each was imagining a raucus sex scene. In at least some ways, the Boltzmann equation plays a similar role for physicists and astronomers: no one ever talks about it, but everyone is always thinking about it.

Two excellent astronomy textbooks which do make abundant use of the Boltzmann equation — either explicitly or implicitly — are *Radiative Processes in Astrophysics* (Rybicki and Lightman) and *Galactic Dynamics* (Binney and Tremaine). In the context of cosmology, in addition to the books mentioned in Chapter 1, *The Large Scale Structure of the Universe* (Peebles), written by the field's pioneer, uses the Boltzmann equation extensively, working in synchronous gauge. If you struggled through Section 4.3, you will be amused (angered?) to see §92 of Peebles' book, where he takes much less space to derive terms due to Compton scattering.

A number of papers deriving the Boltzmann equation for cosmological perturbations are well worth reading. There is the path-breaking work by Lifshitz (1946), Peebles and Yu (1970), and Bond and Szalay (1983). A nice review was written by Efstathiou (1990). The treatment of Compton scattering presented here is based on Dodelson and Jubas (1995). If you were to read just one paper in this area, I would recommend Ma and Bertschinger (1995), which skips many of the steps presented here but has all the relevant formulae and the added virtue of equations in both conformal Newtonian and synchronous gauges. For derivation of the polarization terms in the Boltzmann equations, see Kosowsky (1996). The first paper to present the Boltzmann equation for tensors was Crittenden *et al.* (1993).

We will not spend too much time in this book on different gauges or on the decomposition of perturbations into scalar, vector, and tensor parts. Two excellent review articles which discuss both of these topics in detail are Mukhanov, Feldman, and Brandenberger (1992) and Kodama and Sasaki (1984). Both of these are also very good on the subjects of the next two chapters, the perturbed Einstein equations and inflation.

EXERCISES

Exercise 1. Derive the fluid equations for the collisionless, one-dimensional harmonic oscillator by taking the moments of Eq. (4.6). The relevant quantities are the number density and the velocity defined as integrals over the distribution function:

$$n \equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi} f \quad ; \quad v \equiv \frac{1}{n} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p}{m} f. \tag{4.109}$$

Exercise 2. The metric in a synchronous gauge is

$$g_{00}(\vec{x}, t) = -1$$

$$g_{0i}(\vec{x},t) = 0$$

$$g_{ij}(\vec{x},t) = a^2 \left[\delta_{ij} + h_{ij}\right], \tag{4.110}$$

with perturbations

$$h_{ij} = \begin{pmatrix} -2\tilde{\eta} & 0 & 0 \\ 0 & -2\tilde{\eta} & 0 \\ 0 & 0 & h + 4\tilde{\eta} \end{pmatrix} \tag{4.111}$$

where $\tilde{\eta}$ has nothing to do with conformal time. Here I have chosen the wavevector $\vec{k}$ to lie in the $\hat{z}$ direction. Derive the equivalent of Eq. (4.63) in synchronous gauge:

$$\dot{\Theta} + ik\mu\Theta - \frac{\mu^2 \dot{h}}{2} - \mathcal{P}_2(\mu)\dot{\tilde{\eta}} = -\dot{\tau}\left[\Theta_0 - \Theta + \mu v\right]. \tag{4.112}$$

Exercise 3. Start from the zero-order unintegrated Boltzmann equation (4.37). Integrate this equation over all momenta to show that the number density falls off as a^3. In the course of this, you will have justified the left-hand side of Eq. (3.1).

Exercise 4. Show that the Pauli blocking factor $1 - f_e$ can be set to 1 for all epochs of interest. First find f_e as a function of temperature and number density using the results/approximations of Section 3.1 (i.e. assume that $T_e \ll m_e$). Then, show that as long as the temperature is much less than m_e, f_e is much less than 1.

Exercise 5. Suppose we started this chapter by writing

$$\frac{df}{d\lambda} = C'. \tag{4.113}$$

Change from this form to the one in Eq. (4.1) (with df/dt) on the left. How is the collision term here, C' related to C in Eq. (4.1)? Argue that the first-order perturbations in the factor relating the two collision terms can be dropped since the collision terms themselves are first-order.

Exercise 6. Derive Eq. (4.68), the unintegrated Boltzmann equation for a massive particle.

Exercise 7. Account for the angular dependence of Compton scattering. Start from Eq. (4.49) but instead of assuming the amplitude is constant, take

$$|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2 (1 + \cos^2[\hat{p}\cdot\hat{p}\,']).$$

Show that correctly accounting for the angular dependence introduces the factor of $(1/2)\mathcal{P}_2(\mu)\Theta_2$ presented in Eq. (4.100).

Exercise 8. Show that the temperature of nonrelativistic matter scales as a^{-2} in the absence of interactions. Start from the zero-order part of Eq. (4.68) and assume a form $f_{\rm dm} \propto e^{-E/T} = e^{-p^2/2mT}$. Note that this argument does *not* apply to electrons and protons: as long as they are coupled to the photons, their temperature scales as a^{-1}.

Exercise 9. In Exercise 8, you showed that a thermal distribution of nonrelativistic particles which do not interact has a temperature which scales as a^{-2} , as opposed to that of relativistic particles which we have seen scales as a^{-1}. So $T_{\rm dm} \propto T^2$. Fix the normalization by requiring $T_{\rm dm} = T$ when each is equal to the dark matter mass. Estimate the typical thermal velocity of a dark matter particle with mass equal to 100 GeV when the photon temperature is 1 eV.

Exercise 10. The purpose of this problem is to derive the results of Section 2.3 using the Boltzmann equation. Multiply the zero-order part of Eq. (4.68) by $d^3pE(p)/(2\pi)^3$ and integrate. Show that the resulting equation is identical to Eq. (2.55).

Exercise 11. Consider the effect of a massive neutrino on the evolution equations.
(a) Start from the Boltzmann equation for a massive particle (4.68). Turn it into an equation for $\mathcal{N}$, the perturbation to the massive neutrino distribution function. Use the fact that to first order the neutrino distribution function is

$$f_\nu = f_\nu^{(0)} + \frac{\partial f_\nu^{(0)}}{\partial T_\nu} T_\nu \mathcal{N} \tag{4.114}$$

where $f_\nu^{(0)} = [e^{p/T_\nu} + 1]^{-1}$. Express the final equation in Fourier space using conformal time as the evolution variable.
(b) Recent experiments measuring the atmospheric neutrino flux suggest that the mass of the tau neutrino is 0.07 eV, far larger than either the electron or muon neutrino. Find the contribution of a 0.07-eV neutrino to the energy density today. You may assume it is nonrelativistic.
(c) Consider the following two scenarios. Each has energy density equal to the critical density divided up between only two components: a cold, dark matter particle and a neutrino. The neutrino in each case has the standard abundance and temperature. The only difference between the two scenarios is in one the neutrino is massless while in the other it has a mass of 0.07 eV. Plot the energy density as a function of scale factor in each of the these scenarios. Note that they should agree very early on (in each case there is only a relativistic neutrino early on) and very late. The only difference comes in the middle.

Exercise 12. Show that ordinary matter is tightly coupled during the relevant epochs in the early universe.
(a) Compute the ratio of the Coulomb scattering rate to the Hubble rate. You may assume that all electrons and protons are ionized.
(b) Show that the rate for neutral hydrogen to scatter off ionized protons is always much larger than the expansion rate even when the ionization fraction is on the

order of 10^{-4}.

Exercise 13. Consider tensor perturbations to the metric. These do not perturb $g_{00}(= -1)$ or $g_{0i}(= 0)$. However, the spatial part of the metric is now

$$g_{ij} = a^2 \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Derive the equation for the photon distribution function in the presence of tensor perturbations. Unlike scalar perturbations, tensor perturbations induce an azimuthal dependence in Θ_l, so decompose the anisotropy due to tensors into

$$\Theta^T(k, \mu, \phi) = \Theta_+^T(k, \mu)(1 - \mu^2)\cos(2\phi) + \Theta_\times^T(k, \mu)(1 - \mu^2)\sin(2\phi). \tag{4.115}$$

Show that both the $+$ and the $\times$ component satisfy

$$\frac{d\Theta_i^T}{d\eta} + ik\mu\Theta_i^T + \frac{1}{2}\frac{dh_i}{d\eta} = \dot{\tau}\left[\Theta_i^T - \frac{1}{10}\Theta_{i,0}^T - \frac{1}{7}\Theta_{i,2}^T - \frac{3}{70}\Theta_{i,4}^T\right] \tag{4.116}$$

where i stands for either $\times$ or $+$, and the moments are defined as were the scalar moments, in Eq. (4.99).