APPENDIX C

SPECIAL FUNCTIONS

Here is a very brief summary of special functions, focusing primarily on properties relevant to the calcuations in the text. For a more complete treatment, see, e.g., *Handbook of Mathematical Functions* (Abramowitz and Stegun).

C.1 LEGENDRE POLYNOMIALS

The Legendre polynomial $\mathcal{P}_l(\mu)$ is an lth-order polynomial of μ. For $-1 \le \mu \le 1$, $\mathcal{P}_l$ has l zeroes in this interval. Some special values are

$$\mathcal{P}_0(\mu) = 1 \qquad ; \qquad \mathcal{P}_1(\mu) = \mu \qquad ; \qquad \mathcal{P}_2(\mu) = \frac{3\mu^2 - 1}{2}. \tag{C.1}$$

The property observed in these first few, that $\mathcal{P}_l$ is an even function of μ for l even and an odd function for l odd, holds for all l. They are orthogonal so that

$$\int_{-1}^{1} d\mu \mathcal{P}_l(\mu)\mathcal{P}_{l'}(\mu) = \delta_{ll'}\frac{2}{2l+1}. \tag{C.2}$$

To generate the higher order ones starting from the low ones, one can use the recurrence relation

$$(l+1)\mathcal{P}_{l+1}(\mu) = (2l+1)\mu\mathcal{P}_l(\mu) - l\mathcal{P}_{l-1}(\mu). \tag{C.3}$$

This relation is useful for expressing the Boltzmann equations in terms of moments.

C.2 SPHERICAL HARMONICS

Spherical harmonics are eigenfunctions of the angular part of the Laplacian,

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi). \tag{C.4}$$

In the text, we decomposed the CMB temperature into spherical harmonics (Eq. (8.60)); this decomposition is the analogue of a Fourier decomposition in flat space. The CMB temperature is defined on the sphere, i.e., is a function of θ, ϕ, while the 3D galaxy density, for example, is a function of all three spatial coordinates so is expanded in Fourier coefficients. Some special values are

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \tag{C.5}$$

$$Y_{10}(\theta,\phi) = i\sqrt{\frac{3}{4\pi}}\cos(\theta) \tag{C.6}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp i\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi} \tag{C.7}$$

$$Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(1-3\cos^2\theta\right) \tag{C.8}$$

$$Y_{2,\pm1}(\theta,\phi) = \pm i\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\phi} \tag{C.9}$$

$$Y_{2,\pm2}(\theta,\phi) = -\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}. \tag{C.10}$$

These functions are orthogonal, with normalization

$$\int d\Omega Y^*_{lm}(\Omega)Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}. \tag{C.11}$$

Another useful expression is the Legendre polynomial in terms of a sum of products of the spherical harmonics:

$$\mathcal{P}_l(\hat{x}\cdot\hat{x}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{x})Y^*_{lm}(\hat{x}'). \tag{C.12}$$

C.3 SPHERICAL BESSEL FUNCTIONS

Spherical Bessel functions are crucial in the study of the CMB and large-scale structure in part because they project the inhomogeneities at last scattering onto the sky today. They satisfy the differential equation

$$\frac{d^2 j_l}{dx^2} + \frac{2}{x}\frac{dj_l}{dx} + \left[1 - \frac{l(l+1)}{x^2}\right] j_l = 0. \tag{C.13}$$

The lowest several are

$$j_0(x) = \frac{\sin(x)}{x} \quad ; \quad j_1(x) = \frac{\sin x - x\cos x}{x^2}. \tag{C.14}$$

The key integral relating Legendre polynomials to spherical Bessel functions is

$$\frac{1}{2}\int_{-1}^{1} d\mu P_l(\mu)e^{iz\mu} = \frac{j_l(z)}{(-i)^l}. \tag{C.15}$$

The inverted version of this leads to a useful expansion for Fourier basis functions:

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kx)P_l(\hat{k}\cdot\hat{x}). \tag{C.16}$$

Another important integral for the Sachs-Wolfe effect is

$$\int_0^\infty dx\ x^{n-2}j_l^2(x) = 2^{n-4}\pi\frac{\Gamma\left(l+\frac{n}{2}-\frac{1}{2}\right)\Gamma\left(3-n\right)}{\Gamma\left(l+\frac{5}{2}-\frac{n}{2}\right)\Gamma^2\left(2-\frac{n}{2}\right)}. \tag{C.17}$$

Another important relation which eliminates derivatives is

$$\frac{dj_l}{dx} = j_{l-1} - \frac{l+1}{x}j_l. \tag{C.18}$$

C.4 FOURIER TRANSFORMS

Our Fourier convention is

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\ \tilde{f}(\vec{k})$$

$$\tilde{f}(\vec{k}) = \int d^3xe^{-i\vec{k}\cdot\vec{x}}\ f(\vec{x}). \tag{C.19}$$

The power spectrum is then the Fourier transform of the correlation function, with

$$\langle\tilde{\delta}(\vec{k})\tilde{\delta}(\vec{k}')\rangle = (2\pi)^3\delta^3(\vec{k}-\vec{k}')P(k). \tag{C.20}$$

C.5 MISCELLANEOUS

We just need a couple of relations involving ordinary Bessel functions,

$$J_n(x) = \frac{i^{-n}}{\pi}\int_0^\pi d\theta\ e^{ix\cos\theta}\cos(n\theta) \tag{C.21}$$

and

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x). \tag{C.22}$$

The Γ function for integers is simply related to factorials:

$$\Gamma(n+1) = n!. \tag{C.23}$$

More generally

$$\Gamma(x+1) = x\Gamma(x) \tag{C.24}$$

even if x is not an integer. The Sachs–Wolfe integral (Eq. (C.17)) for a Harrison–Zel'dovich–Peebles spectrum ($n = 1$) depends on

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}. \tag{C.25}$$

The Riemann zeta function is useful for evaluating integrals in statistical mechanics. In particular,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x - 1} = \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_0^\infty dx \frac{x^{s-1}}{e^x + 1}. \tag{C.26}$$

Some low integer Riemann zeta functions are

$$\zeta(2) = \frac{\pi^2}{6} \quad ; \quad \zeta(3) = 1.202 \quad ; \quad \zeta(4) = \frac{\pi^4}{90}. \tag{C.27}$$