5

EINSTEIN EQUATIONS

The previous chapter set up the formalism to describe how perturbations in the gravitational field affect particle distributions. This formalism led to the set of equations (4.100) -(4.107). We need to supplement these equations with an account of how the perturbations to the particle distributions affect the gravitational field. For this, we need the Einstein equations of general relativity. The calculation detailed in this chapter expands the Einstein equations perturbatively around the zero-order homogeneous solution. Far from being subtle or complex as one might expect from general relativity's reputation, this calculation is completely straightforward, although a bit long. Still, working through it is a "must-do-once" exercise, so the steps are presented in some detail.

5.1 THE PERTURBED RICCI TENSOR AND SCALAR

The fundamental equation of general relativity (2.30) is a 4D tensor equation, so in principle it represents 16 separate equations. However, since both sides of the equation are symmetric tensors, only 10 of these are distinct. We are interested though in only two of the 10, since the metric we are focusing on has only two independent functions, Φ and Ψ.

Evaluating the left-hand side of the Einstein equation requires three pre-steps:

- Compute the Christoffel symbols, $\Gamma^{\mu}{}_{\alpha\beta}$, for the perturbed metric of Eq. (4.9).
- From these, form the Ricci tensor, $R_{\mu\nu}$, using Eq. (2.31).
- Contract the Ricci tensor to form the Ricci scalar, $\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu}$.

Note that, unfortunately, even if we are interested in only several components of the Einstein equations, we need to compute all the elements of the Ricci tensor. For all the components of the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}\mathcal{R}/2$ depend on the Ricci scalar, which depends on all elements of $R_{\mu\nu}$.

5.1.1 Christoffel Symbols

We have already computed the zero-order Christoffel symbols in Eqs. (2.22) and (2.23). Now we need to look at the first-order terms, those that are linear in Φ and/or Ψ. First let us consider $\Gamma^0{}_{\mu\nu}$, which by definition is

$$\Gamma^0{}_{\mu\nu} = \frac{1}{2} g^{0\alpha} \left[g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right] \tag{5.1}$$

where again $_{,\alpha}$ means the derivative with respect to x^α. The only nonzero component of $g^{0\alpha}$ is the time component,[1] which is the inverse of $g_{00} = -1 - 2\Psi$. So, to first-order in the perturbations, $g^{00} = -1 + 2\Psi$, and

$$\Gamma^0{}_{\mu\nu} = \frac{-1 + 2\Psi}{2} \left[g_{0\mu,\nu} + g_{0\nu,\mu} - g_{\mu\nu,0} \right]. \tag{5.2}$$

Take each component in turn: first the one with $\mu = \nu = 0$. Each of the terms in square brackets is identical, so the brackets give $g_{00,0} = -2\Psi_{,0}$. Since we are interested only in first-order terms the factor of 2Ψ out in front can be dropped and we are left with

$$\Gamma^0{}_{00} = \Psi_{,0}. \tag{5.3}$$

The next possibility is that one of the indices μ or ν is spatial and the other time. It doesn't matter which, since the Christoffel symbol is symmetric in its lower indices. In this case, only one of terms in brackets in Eq. (5.2) is nonzero, $g_{00,i} = -2\Psi_{,i}$. Once again since this is first-order, we can drop the factor of 2Ψ in front, leading to

$$\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Psi_{,i} = ik_i\Psi. \tag{5.4}$$

The final equality here moves to Fourier space, where we will stay from now on. Recall our convention of not using ~s for Fourier transformed variables: Ψ on the far right is really $\tilde{\Psi}$.

Finally, if both lower indices in Eq. (5.2) are spatial, the first two terms in brackets vanish since $g_{0i} = 0$ and only the last term survives, leaving

$$\Gamma^0{}_{ij} = \frac{1 - 2\Psi}{2} \frac{\partial}{\partial t} \left[\delta_{ij} a^2 (1 + 2\Phi) \right]. \tag{5.5}$$

There is a zero-order term here, the one we computed in Eq. (2.22), and three first-order terms:

$$\Gamma^0{}_{ij} = \delta_{ij} a^2 \left[H + 2H(\Phi - \Psi) + \Phi_{,0} \right] \tag{5.6}$$

with $H = (da/dt)/a$.

Computing the Christoffel symbols, $\Gamma^i{}_{\mu\nu}$, will be left as an exercise. They are

$$\Gamma^i{}_{00} = \frac{ik^i}{a^2}\Psi$$

[1] We will do the calculation with $x^0 = t$, not conformal time. Therefore, $\Psi_{,0}$ for example means derivative with respect to time. Since our convention is now $\dot{\Psi} \equiv \partial\Psi/\partial\eta$, $\Psi_{,0} = \dot{\Psi}/a$.

$$\Gamma^i{}_{j0} = \Gamma^i{}_{0j} = \delta_{ij}\left(H + \Phi_{,0}\right)$$
$$\Gamma^i{}_{jk} = i\Phi\left[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i\right]. \tag{5.7}$$

Note that the only nonvanishing zero-order component is $\Gamma^i{}_{j0}$, in agreement with Eq. (2.23). Also remember that both δ_{ij} and the 3-vector k_i live in Euclidean space, so we can freely interchange their upper and lower indices.

5.1.2 Ricci Tensor

The Ricci tensor is most easily expressed in terms of the Christoffel symbols, as in Eq. (2.31). First, consider the time-time component:

$$R_{00} = \Gamma^\alpha{}_{00,\alpha} - \Gamma^\alpha{}_{0\alpha,0} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{00} - \Gamma^\alpha{}_{\beta 0}\Gamma^\beta{}_{0\alpha}. \tag{5.8}$$

All of these terms contribute at first-order. One simplification comes from considering the $\alpha = 0$ part of all these terms. The first and second terms are equal and opposite to each other as are the last two. So the sum over the index α contributes only when α is spatial. Let's consider each of the terms one by one.

- The first is
$$\Gamma^i{}_{00,i} = \frac{-k^2}{a^2}\Psi, \tag{5.9}$$
using the first of equations (5.7).
- The second term in Eq. (5.8) is
$$-\Gamma^i{}_{0i,0} = -3\left(\frac{d^2a/dt^2}{a} - H^2 + \Phi_{,00}\right) \tag{5.10}$$
using the second of equations (5.7). The factor of 3 in front comes from the implicit sum in δ_{ii}.
- The next term is $\Gamma^i{}_{i\beta}\Gamma^\beta{}_{00}$. Note that $\Gamma^\beta{}_{00}$ is first order no matter what β is, so we need keep only the zero-order part of $\Gamma^i{}_{i\beta}$. However, the last of equations (5.7) shows that $\Gamma^i{}_{i\beta}$ is first-order unless $\beta = 0$. So to first-order,
$$\Gamma^i{}_{i\beta}\Gamma^\beta{}_{00} = \Gamma^i{}_{i0}\Gamma^0{}_{00}$$
$$= 3H\Psi_{,0}. \tag{5.11}$$
- Finally the last term is $-\Gamma^i{}_{\beta 0}\Gamma^\beta{}_{0i}$. In this case, if $\beta = 0$ both Γ's are first-order, so their product is second-order and can be neglected. Therefore, only spatial β need be considered, leading to
$$-\Gamma^i{}_{\beta 0}\Gamma^\beta{}_{0i} = -\Gamma^i{}_{j0}\Gamma^j{}_{0i}$$
$$= -3\left(H^2 + 2H\Phi_{,0}\right). \tag{5.12}$$

Collecting these four sets of terms gives

$$R_{00} = -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\Phi_{,00} + 3H(\Psi_{,0} - 2\Phi_{,0}). \tag{5.13}$$

Note that the zero-order term agrees with Eq. (2.34).

The space-space part of the Ricci tensor is left as an exercise. It is

$$R_{ij} = \delta_{ij}\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1 + 2\Phi - 2\Psi)\right.$$
$$\left. + a^2H\,(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi\right] + k_ik_j(\Phi + \Psi). \tag{5.14}$$

We can now contract the indices on the Ricci tensor and find the Ricci scalar:

$$\begin{aligned}\mathcal{R} \equiv g^{\mu\nu}R_{\mu\nu} &= g^{00}R_{00} + g^{ij}R_{ij} \\ &= [-1 + 2\Psi]\left[-3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\Phi_{,00} + 3H(\Psi_{,0} - 2\Phi_{,0})\right] \\ &+ \left[\frac{1 - 2\Phi}{a^2}\right]\left[3\left\{\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)\Big(1 + 2\Phi - 2\Psi\Big)\right.\right. \\ &\left.\left. + a^2H\Big(6\Phi_{,0} - \Psi_{,0}\Big) + a^2\Phi_{,00} + k^2\Phi\right\} + k^2(\Phi + \Psi)\right].\end{aligned} \tag{5.15}$$

First let us check the zero-order part of $\mathcal{R}$. Combining terms, we find that it is $6(H^2 + \frac{d^2a/dt^2}{a})$, in agreement with Eq. (2.37). To get the first-order part, $\delta\mathcal{R}$, we go through the by-now-familiar routine of multiplying terms, keeping only those first-order in Φ and Ψ. This gives

$$\begin{aligned}\delta\mathcal{R} = &-6\Psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\Psi + 3\Phi_{,00} - 3H(\Psi_{,0} - 2\Phi_{,0}) \\ &- 6\Psi\left(2H^2 + \frac{d^2a/dt^2}{a}\right) + 3H\,(6\Phi_{,0} - \Psi_{,0}) \\ &+ 3\Phi_{,00} + 4\frac{k^2\Phi}{a^2} + \frac{k^2\Psi}{a^2},\end{aligned} \tag{5.16}$$

where the first line contains the terms from R_{00} (the second line in Eq. (5.15)) and the last two from R_{ij} (the last two lines in Eq. (5.15)). Combining these leads to

$$\begin{aligned}\delta\mathcal{R} = &-12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00} \\ &-6H(\Psi_{,0} - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}.\end{aligned} \tag{5.17}$$

5.2 TWO COMPONENTS OF THE EINSTEIN EQUATIONS

We can now derive the evolution equations for Φ and Ψ, the perturbations to the Friedmann–Robertson–Walker metric. There is some freedom here because the Einstein equations

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu} \tag{5.18}$$

have 10 components and we need only two. All of the other eight components will either be zero at first-order or be redundant.[2]

The first component we will use is the time-time component. Thus we need to evaluate

$$\begin{aligned} G^{0}{}_{0} &= g^{00}\left[R_{00} - \frac{1}{2}g_{00}\mathcal{R}\right] \\ &= (-1+2\Psi)R_{00} - \frac{\mathcal{R}}{2}. \end{aligned} \tag{5.19}$$

Here one of the indices has been raised by multiplying G_{00} by g^{00} (recall that g^{0i} vanish). This turns out to simplify the energy–momentum tensor (see Exercise 3) which sources the Einstein tensor. Also note that the second line follows from the first since $g^{00}g_{00} = 1$. We have computed the time-time component of the Ricci tensor (Eq. (5.13)) and the perturbed Ricci scalar (Eq. (5.17)), so the first-order part of the time-time component of the Einstein tensor is

$$\begin{aligned} \delta G^{0}{}_{0} = &-6\Psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\Psi + 3\Phi_{,00} - 3H(\Psi_{,0} - 2\Phi_{,0}) \\ &+ 6\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) - \frac{k^2}{a^2}\Psi - 3\Phi_{,00} \\ &+ 3H(\Psi_{,0} - 4\Phi_{,0}) - 2\frac{k^2\Phi}{a^2}. \end{aligned} \tag{5.20}$$

Combining terms leads to

$$\delta G^{0}{}_{0} = -6H\Phi_{,0} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}. \tag{5.21}$$

Einstein's equation equates $G^0{}_0$ with $8\pi G T^0{}_0$ where $T_{\mu\nu}$ is the energy–momentum tensor. To complete our derivation of the first evolution equation for Φ and Ψ, therefore, we need to compute the first-order part of the source term, $T^0{}_0$. Recall from Section 2.3 that $-T^0{}_0$ is the energy density of all the particles

[2]This is true for scalar perturbations. When we come to conside. tensor perturbations, some of the other components will be useful.

in the universe, and that the contribution from each species is an integral over the distribution function (Eq. (2.59)),

$$T^0{}_0(\vec{x},t) = -\sum_{\text{all species } i} g_i \int \frac{d^3p}{(2\pi)^3}\, E_i(p) f_i(\vec{p},\vec{x},t). \tag{5.22}$$

Recall also that g_i is the spin degeneracy of the species (has nothing to do with the metric); $E_i = \sqrt{p^2+m_i^2}$ is the energy of a particle with momentum p and mass m_i; and f_i is the distribution function. In Section 2.3, we considered the zero-order distributions of the smooth universe. To get the first-order part of the energy–momentum tensor, we must naturally consider the first-order part of the distribution functions, i.e. the perturbation variables we defined in Chapter 4 for the photons, neutrinos, dark matter, and baryons. This is easiest for the dark matter and baryons. For we defined the right-hand side as $-\rho_i(1+\delta_i)$ where i labels either dark matter or baryons. For photons, a little more care is required. Using the definition of Θ in Eq. (4.35), we have

$$T^0{}_0 = -2\int \frac{d^3p}{(2\pi)^3}\, p\left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right] \qquad \text{(photons)}. \tag{5.23}$$

The first term here is just the zero-order photon energy density, ρ_γ. To reduce the second term, we first do the angular integral, which picks out the monopole Θ_0 from Θ. Then, we do the radial integral by parts. This changes the sign and introduces a factor of 4 since $\partial p^4/\partial p = 4p^3$, leading to

$$T^0{}_0 = -\rho_\gamma\left[1+4\Theta_0\right] \qquad \text{(photons)}. \tag{5.24}$$

The factor of 4 here is obvious in retrospect. The perturbation variable Θ is the fractional temperature change, while the energy momentum tensor is interested in the perturbed energy density, $\delta\rho$. We should have expected that since $\rho \propto T^4$, $\delta\rho/\rho = 4\delta T/T$. In any event, it falls out of the algebra. I harp on it only to warn those who turn to the literature that authors are virtually split between those who define Θ as $\delta\rho/\rho$ and those who opt for the convention we use here. Finally, note that the first-order contribution from massless neutrinos is identical in form,

$$T^0{}_0 = -\rho_\nu\left[1+4\mathcal{N}_0\right] \qquad \text{(neutrinos)}. \tag{5.25}$$

In principle, we should also include a term for the perturbation to the dark energy. In practice, though, most models of the dark energy predict that (i) it should be smooth and (ii) it should be important only very recently. Both of these features are inherent in the cosmological constant model for example. There are some models which deviate from one or both of these conditions, but for the most part we are justified in neglecting the dark energy as a source of perturbations to the metric.

Returning to Einstein's equation, we equate Eq. (5.21) with $8\pi G$ times the first-order part of the time-time component of the energy–momentum tensor. Dividing both sides by 2 leads to

$$-3H\Phi_{,0} + 3\Psi H^2 - \frac{k^2\Phi}{a^2} = -4\pi G\left[\rho_{\rm dm}\delta + \rho_b\delta_{\rm b} + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]. \quad (5.26)$$

It is again useful to write the equation in terms of conformal time. This introduces an extra factor of $1/a$ every time a time derivative appears, so

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left[\rho_{\rm dm}\delta + \rho_b\delta_{\rm b} + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]. \quad (5.27)$$

This is our first evolution equation for Φ and Ψ. In the limit of no expansion (a = constant), Eq. (5.27) reduces to the ordinary Poisson equation for gravity (in Fourier space). The left-hand side is $-\nabla^2\Phi$ while the right-hand side is $4\pi G\delta\rho$. The terms proportional to $\dot{a}$ account for expansion and are typically important for modes with wavelengths ($\sim a/k$) comparable to, or larger than, the Hubble radius, H^{-1}. We need this general relativistic expression when we consider the evolution of parturbations, because almost all modes of interest today used to have wavelengths larger than the Hubble radius. More on this in Chapter 6.

We now obtain a second evolution equation for Φ and Ψ. Since we have already dealt with the time-time component of the Einstein tensor, let's focus on the spatial part of $G^\mu{}_\nu$,

$$G^i{}_j = g^{ik}\left[R_{kj} - \frac{g_{kj}}{2}\mathcal{R}\right] = \frac{\delta^{ik}(1-2\Phi)}{a^2}R_{kj} - \frac{\delta_{ij}}{2}\mathcal{R}. \quad (5.28)$$

From Eq. (5.14), we see that most of the terms in R_{kj} are proportional to δ_{kj}. When contracted with δ^{ik} this will lead to a host of terms proportional to δ_{ij}, in addition to the last term here, the one proportional to $\mathcal{R}$. Therefore,

$$G^i{}_j = A\delta_{ij} + \frac{k_ik_j(\Phi+\Psi)}{a^2} \quad (5.29)$$

where A has close to a dozen terms which we would rather not write down. Since all of these terms are proportional to δ_{ij} they all contribute to the trace of $G^i{}_j$. To avoid dealing with these terms, consider the *longitudinal, traceless* part of $G^i{}_j$, which can be extracted by contracting $G^i{}_j$ with $\hat{k}_i\hat{k}^j - (1/3)\delta_i^j$, a *projection* operator. That is, it picks out the piece which is longitudinal, traceless and only that part (Exercise 4). This projection operator kills all terms proportional to δ_{ij}, leaving only

$$\left(\hat{k}_i\hat{k}^j - (1/3)\delta_i^j\right)G^i{}_j = \left(\hat{k}_i\hat{k}^j - (1/3)\delta_i^j\right)\left(\frac{k^ik_j(\Phi+\Psi)}{a^2}\right) = \frac{2}{3a^2}k^2(\Phi+\Psi). \quad (5.30)$$

This is to be equayed with the longitudinal, traceless part of the energy–momentum tensor, extracted in the same fashion:

$$\left(\hat{k}_i\hat{k}^j - (1/3)\delta_i^j\right) T^i{}_j = \sum_{\text{all species i}} g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2\mu^2 - (1/3)p^2}{E_i(p)} f_i(\vec{p}). \tag{5.31}$$

We can immediately recognize the combination $\mu^2 - 1/3$ as proportional to the second Legendre polynomial, more precisely equal to $(2/3)\mathcal{P}_2(\mu)$. Therefore, the integral picks out the quadrupole part of the distribution. Of course the zero-order part of the distribution function has no quadrupole, so the source term is first order, proportional to Θ_2, which is nonzero only for neutrinos and photons. The integral in Eq. (5.31) for photons is

$$\begin{aligned} -2\int \frac{dpp^2}{2\pi^2} p^2 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^{1} \frac{d\mu}{2} \frac{2\mathcal{P}_2(\mu)}{3} \Theta(\mu) &= 2\frac{2\Theta_2}{3} \int \frac{dpp^2}{2\pi^2} p^2 \frac{\partial f^{(0)}}{\partial p} \\ &= -\frac{8\rho^{(0)}\Theta_2}{3} \end{aligned} \tag{5.32}$$

where the first equality follows from the definition of the quadrupole and the second from an integration by parts. This component of the energy–momentum tensor is called the *anisotropic stress*. Nonrelativistic particles, such as baryons and dark matter, do not contribute anisotropic stress.

For the second Einstein equation, we therefore equate Eq. (5.30) with $8\pi G$ times the photon and neutrino anisotropic stresses:

$$k^2(\Phi + \Psi) = -32\pi G a^2 \left[\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2\right]. \tag{5.33}$$

That is, the two gravitational potentials are equal and opposite unless the photons or neutrinos have appreciable quadrupole moments. In practice, the photons' quadrupole contributes little to this sum, because it is very small during the time when it has appreciable energy density. [Recall the argument after Eq. (4.54).] Only the collisionless neutrino has an appreciable quadrupole moment early on when radiation dominates the universe.

5.3 TENSOR PERTURBATIONS

Until now, we have focused almost exclusively on *scalar* perturbations to the homogeneous FRW universe. Formally, this means that the perturbations $\Phi(\vec{x}, t)$ and $\Psi(\vec{x}, t)$ transform as scalars as $\vec{x} \to \vec{x}'$; i.e., they remain unchanged under a spatial coordinate transformation. This focus is reasonable: as we have seen, scalar perturbations to the metric are sourced by density fluctuations and vice versa. For the most part, the density fluctuations that led to the structure of the universe are our primary interest.

Nonetheless, many theories of structure formation produce, in addition to scalar fluctuations, *tensor* perturbations to the metric. These are potentially detectable because they produce observable distortions in the CMB, especially on large scales. Sprinkled throughout the book, therefore, are exercises (with hints) relating to tensor perturbations. The tools needed to study these are precisely those we crafted when studying scalar perturbations. For the most part, therefore, I regard the evolution of tensor perturbations as one rather large homework problem, one which introduces no new physics.

One question which naturally arises when working out these exercises, though, is why consider scalar and tensor perturbations separately? To answer this question (and to alleviate the homework load) this section derives Einstein's equations for tensor perturbations. We will see that scalar and tensor perturbations *decouple*; that is, they evolve completely independently. So the presence of tensor perturbations does not affect the scalars and vise versa. Contrast this with Φ and Ψ. We have just shown that they are quite tightly coupled to each other. It is impossible to learn about Φ without also solving for Ψ. The decoupling of scalars and tensors is a manifestation of the *decomposition theorem.* Needless to say, it is much more instructive to work out an example of this theorem than to prove it abstractly. Incidentally, as you would expect, the same theorem can be applied to *vector* perturbations. These too are produced by some early-universe models (but not as ubiquitously as tensors) and can be treated completely independently.

Tensor perturbations can be characterized by a metric with $g_{00} = -1$, zero space-time components $g_{0i} = 0$, and spatial elements

$$g_{ij} = a^2 \begin{pmatrix} 1+h_+ & h_\times & 0 \\ h_\times & 1-h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.34}$$

That is, the perturbations to the metric are described by two functions, h_+ and $h_\times$, assumed small. For definiteness, I have chosen the perturbations to be in the x-y plane. This corresponds to an implicit choice of axes; in particular, it corresponds to choosing the z-axis to be in the direction of the wavevector, $\vec{k}$. More generally, h_+ and $h_\times$ are two components of a *divergenceless, traceless, symmetric tensor.* If this perturbation tensor is written as $\mathcal{H}_{ij}$, *divergenceless* means that $k^i\mathcal{H}_{ij} = k^j\mathcal{H}_{ij} = 0$. This is clearly satisfied by Eq. (5.34) since there are no components in the $\hat{k} = \hat{z}$ direction. Tracelessness is also satisfied since the sum of the perturbations along the diagonal vanishes.

Once the metric in Eq. (5.34) has been written down, we can blast away and derive the Einstein equations. Once again the derivation proceeds in three steps: (i) Christoffel symbols, (ii) Ricci tensor, and (iii) Ricci scalar.

5.3.1 Christoffel Symbols for Tensor Perturbations

First consider $\Gamma^0{}_{\alpha\beta}$. The metric we are considering in Eq. (5.34) has constant g_{00} and vanishing g_{0i}. Recall that the Christoffel symbol is a sum of derivatives of the metric. The only terms that will be nonzero are those which involve derivatives of the spatial part of the metric, $g_{ij,\alpha}$. Therefore, we can immediately argue that

$$\Gamma^0{}_{00} = \Gamma^0{}_{i0} = 0. \tag{5.35}$$

The term with two lower spatial indices is

$$\Gamma^0{}_{ij} = -\frac{g^{00}}{2} g_{ij,0}$$
$$= \frac{1}{2} g_{ij,0}. \tag{5.36}$$

Let's use the notation mentioned earlier: the 3D matrix $\mathcal{H}_{ij}$ contains the perturbations, which in this basis (with $\hat{k}$ in the $\hat{z}$ direction) is equal to

$$\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.37}$$

so that $g_{ij} = a^2(\delta_{ij} + \mathcal{H}_{ij})$. Therefore,

$$g_{ij,0} = 2Hg_{ij} + a^2\mathcal{H}_{ij,0} \tag{5.38}$$

where the Hubble rate H is not to be confused with tensor perturbations $\mathcal{H}$. The first nonzero Christoffel symbol is therefore

$$\Gamma^0{}_{ij} = Hg_{ij} + \frac{a^2\mathcal{H}_{ij,0}}{2}. \tag{5.39}$$

When both lower indices on Γ are 0, the Christoffel symbol vanishes. The two remaining components are $\Gamma^i{}_{0j}$ and $\Gamma^i{}_{jk}$. The former is

$$\Gamma^i{}_{0j} = \frac{g^{ik}}{2} g_{jk,0}. \tag{5.40}$$

The time derivative of g_{jk} acts on both the scale factor and on the perturbations $h_{+,\times}$, as in Eq. (5.38), so

$$\Gamma^i{}_{0j} = \frac{g^{ik}}{2}\left[2Hg_{jk} + a^2\mathcal{H}_{jk,0}\right]. \tag{5.41}$$

But $g^{ik}g_{jk} = \delta_{ij}$, so the first term here is simply $\delta_{ij}H$. To get the second, we can set $g^{jk} = \delta_{jk}/a^2$ (i.e., neglect first-order terms) since it multiplies the first-order $\mathcal{H}$. So,

$$\Gamma^i{}_{0j} = H\delta_{ij} + \frac{1}{2}\mathcal{H}_{ij,0}, \tag{5.42}$$

where I have used the fact that $\mathcal{H}_{ij}$ is symmetric.

The last Christoffel symbol we need is $\Gamma^i{}_{jk}$. In Exercise 7 you will show that

$$\Gamma^i{}_{jk} = \frac{i}{2}\left[k_k\mathcal{H}_{ij} + k_j\mathcal{H}_{ik} - k_i\mathcal{H}_{jk}\right]. \tag{5.43}$$

5.3.2 Ricci Tensor for Tensor Perturbations

Following the same steps as in the scalar perturbation case, we now combine these Christoffel symbols to form the Ricci tensor. First we compute the time-time component of the Ricci tensor:

$$R_{00} = \Gamma^{\alpha}{}_{00,\alpha} - \Gamma^{\alpha}{}_{0\alpha,0} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{00} - \Gamma^{\alpha}{}_{\beta 0}\Gamma^{\beta}{}_{0\alpha}. \tag{5.44}$$

We have shown that the Christoffel symbol vanishes for tensor perturbations when the two lower indices are time-time. Therefore, the first and third terms here are zero. Using the same argument, the indices α and β in the second and fourth terms must be spatial, so

$$R_{00} = -\Gamma^{i}{}_{0i,0} - \Gamma^{i}{}_{j0}\Gamma^{j}{}_{0i}. \tag{5.45}$$

Using Eq. (5.42) for $\Gamma^{i}{}_{j0}$ which is the only element appearing, we find that

$$\begin{aligned} R_{00} &= -3\frac{dH}{dt} - \frac{1}{2}\mathcal{H}_{ii,00} \\ &\quad - \left(H\delta_{ij} + \frac{1}{2}\mathcal{H}_{ij,0}\right)\left(H\delta_{ij} + \frac{1}{2}\mathcal{H}_{ij,0}\right). \end{aligned} \tag{5.46}$$

On the first line, the trace $\mathcal{H}_{ii}$ vanishes since h_+ appears in the metric with opposite signs along the diagonal. Expanding the second line out to first-order leads to a similar cancellation: $\mathcal{H}_{ij}$ is multiplied by δ_{ij}, so there are no first-order terms. The zero-order terms combine to form

$$R_{00} = -3\frac{d^2a/dt^2}{a}, \tag{5.47}$$

an equation in which we are by now quite confident since this is the third time we have derived it (see equations (2.34) and (5.13)). Of course the big news here is not that we have correctly derived the zero-order term, but rather that tensor perturbations do not appear at first-order in R_{00}. Looking ahead, we will soon see that the Ricci scalar also has no tensor contribution (even though R_{ij} does). Therefore, we can anticipate that the time-time component of Einstein's equations contains no tensor perturbations. This is important for it tells us that density perturbations — which form the right hand side of the time-time component as shown in Eq. (5.26) — do *not* induce any tensor perturbations. We are beginning therefore to get a glimmer of the decomposition theorem. Density perturbations and scalar perturbations to the metric are coupled; indeed their names are often used as synonyms. Tensor perturbations, however, are decoupled from these and evolve on their own.

The spatial components of the Ricci tensor do depend on the tensor perturbation variables. We now turn to

$$R_{ij} = \Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} + \Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{ij} - \Gamma^{\alpha}{}_{\beta j}\Gamma^{\beta}{}_{i\alpha}. \tag{5.48}$$

Let's consider the first two terms together. Expanding out leads to

$$\Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} = \Gamma^{0}{}_{ij,0} + \Gamma^{k}{}_{ij,k} - \Gamma^{k}{}_{ik,j} \tag{5.49}$$

since $\alpha = 0$ does not contribute in $\Gamma^{\alpha}{}_{i\alpha,j}$ because of Eq. (5.35). The hardest (i.e., longest) term here is the first, which involves multiple time derivatives. Let's postpone its calculation by recalling that $\Gamma^{0}{}_{ij} = g_{ij,0}/2$ so that the first term can be written in shorthand as $g_{ij,00}/2$. The last term in Eq. (5.49) vanishes since $\Gamma^{k}{}_{ik} = 0$ for tensor perturbations. Combining the other terms then leads to

$$\Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} = \frac{g_{ij,00}}{2} + \frac{1}{2}\left[-k_i k_k \mathcal{H}_{jk} - k_j k_k \mathcal{H}_{ik} + k^2 \mathcal{H}_{ji}\right]. \tag{5.50}$$

Recall that we chose $\vec{k}$ to be along the z-axis. Therefore, the indices i and k in the first term in brackets must be equal to 3. But these multiply $\mathcal{H}_{jk} = \mathcal{H}_{j3} = 0$ so this term and its cousin $k_j k_k \mathcal{H}_{ik}$ must both vanish. Therefore,

$$\Gamma^{\alpha}{}_{ij,\alpha} - \Gamma^{\alpha}{}_{i\alpha,j} = \frac{g_{ij,00}}{2} + \frac{k^2}{2}\mathcal{H}_{ji}. \tag{5.51}$$

The third term in Eq. (5.48), $\Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{ij}$, is nonzero only when the index α is spatial, so

$$\Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{ij} = \Gamma^{k}{}_{k0}\Gamma^{0}{}_{ij} + \Gamma^{k}{}_{kl}\Gamma^{l}{}_{ij}. \tag{5.52}$$

But each of the Christoffel symbols in the second term here are first-order, so their product vanishes. In the first term, the sum over k makes the first-order terms go away, so $\Gamma^{k}{}_{k0}$ is purely zero-order, $3H$. Therefore,

$$\Gamma^{\alpha}{}_{\alpha\beta}\Gamma^{\beta}{}_{ij} = \frac{3}{2}H g_{ij,0}. \tag{5.53}$$

The final term in Eq. (5.48) will be left as an exercise; it is

$$\Gamma^{\alpha}{}_{\beta j}\Gamma^{\beta}{}_{i\alpha} = 2H^2 g_{ij} + 2a^2 H \mathcal{H}_{ij,0}. \tag{5.54}$$

We can now combine all four terms in Eq. (5.48) to get

$$\begin{aligned} R_{ij} = {} & \frac{g_{ij,00}}{2} + \frac{k^2}{2}\mathcal{H}_{ji} + \frac{3}{2}H g_{ij,0} \\ & - 2H^2 g_{ij} - 2a^2 H \mathcal{H}_{ij,0}. \end{aligned} \tag{5.55}$$

We now need to expand out the time derivatives of the metric. Using Eq. (5.38), one finds

$$g_{ij,00} = 2g_{ij}\left(\frac{d^2a/dt^2}{a} + H^2\right) + 4a^2 H \mathcal{H}_{ij,0} + a^2 \mathcal{H}_{ij,00}. \tag{5.56}$$

Therefore the Ricci tensor is

$$R_{ij} = g_{ij}\left(\frac{d^2a/dt^2}{a} + 2H^2\right) + \frac{3}{2}a^2 H \mathcal{H}_{ij,0}$$

$$+ a^2 \frac{\mathcal{H}_{ij,00}}{2} + \frac{k^2}{2}\mathcal{H}_{ij}. \tag{5.57}$$

Again we see that we have successfully recaptured the zero-order part of the Ricci tensor. Remarkably, we will see that the first-order parts — when used in Einstein's equations — do not couple to the scalar perturbations.

First, though, we must compute the Ricci scalar:

$$\mathcal{R} = g^{00}R_{00} + g^{ij}R_{ij}. \tag{5.58}$$

The time-time product is all zero-order, so we can neglect it when considering the first-order piece $\delta\mathcal{R}$. The space-space contraction has two types of terms. First, there are the terms in Eq. (5.57) proportional to the metric g_{ij}. But $g^{ij}g_{ij} = 3$, so there are no first-order terms here. All the other terms in Eq. (5.57) are proportional to $\mathcal{H}_{ij}$, so when contracting them we can set g^{ij} to its zero-order value, δ_{ij}/a^2. This corresponds to taking the trace of the first-order terms in Eq. (5.57). Since all first-order terms are proportional to $\mathcal{H}_{ij}$, the trace vanishes. Therefore, tensor perturbations do not affect (at first order) the Ricci scalar.

5.3.3 Einstein Equations for Tensor Perturbations

Now let's read off the perturbations to the Einstein tensor induced by tensor modes. Since the Ricci scalar is unperturbed by tensor perturbations, the first-order Einstein tensor is simply

$$\delta G^i{}_j = \delta R^i{}_j. \tag{5.59}$$

To get $R^i{}_j$, we contract $g^{ik}R_{kj}$, using the Ricci tensor we computed in Eq. (5.57). The first term, proportional to the contraction of $g^{ik}g_{kj} = \delta^i_j$, has no first-order piece; the remaining terms are explicitly first-order in $\mathcal{H}$, so we can set $g^{ik} = \delta^{ik}/a^2$, leading to

$$\delta G^i{}_j = \delta^{ik}\left[\frac{3}{2}H\mathcal{H}_{kj,0} + \frac{\mathcal{H}_{kj,00}}{2} + \frac{k^2}{2a^2}\mathcal{H}_{kj}\right]. \tag{5.60}$$

We can now derive a set of equations governing the evolution of the tensor variables, h_+ and $h_\times$.

To derive an equation for h_+, let us consider the difference between the ${}^1{}_1$ and ${}^2{}_2$ components of the Einstein tensor. The Einstein tensor in Eq. (5.60) is proportional to $\mathcal{H}_{ij}$ and its derivatives. Since $\mathcal{H}_{11} = -\mathcal{H}_{22} = h_+$, $\delta G^1{}_1$ is equal and opposite to $\delta G^2{}_2$. Therefore,

$$\delta G^1{}_1 - \delta G^2{}_2 = 3Hh_{+,0} + h_{+,00} + \frac{k^2 h_+}{a^2}. \tag{5.61}$$

Change to conformal time so that $h_{+,0} = \dot{h}_+/a$ and $h_{+,00} = \ddot{h}_+/a^2 - (\dot{a}/a^3)\dot{h}_+$. Then,

$$a^2\left[\delta G^1{}_1 - \delta G^2{}_2\right] = \ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+. \tag{5.62}$$

The right-hand side of this component of Einstein's equations is zero (Exercise 8), and $h_\times$ obeys the same equation (Exercise 9), so the tensor modes are governed by

$$\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0 \tag{5.63}$$

where $\alpha = +, \times$. Equation (5.63) is a wave equation, and the corresponding solutions are called *gravity waves.* For example, if we neglect the expansion of the universe so that the damping term in Eq. (5.63) vanishes, we immediately see that the two solutions are $h_\alpha \propto e^{\pm ik\eta}$. In real space, then the perturbation to the metric is of the form

$$h_\alpha(\vec{x},\eta) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \left[Ae^{ik\eta} + Be^{-ik\eta}\right] \qquad \text{(no expansion).} \tag{5.64}$$

The two modes here corresponds to waves traveling in the $\pm\hat{z}$ direction at the speed of light.

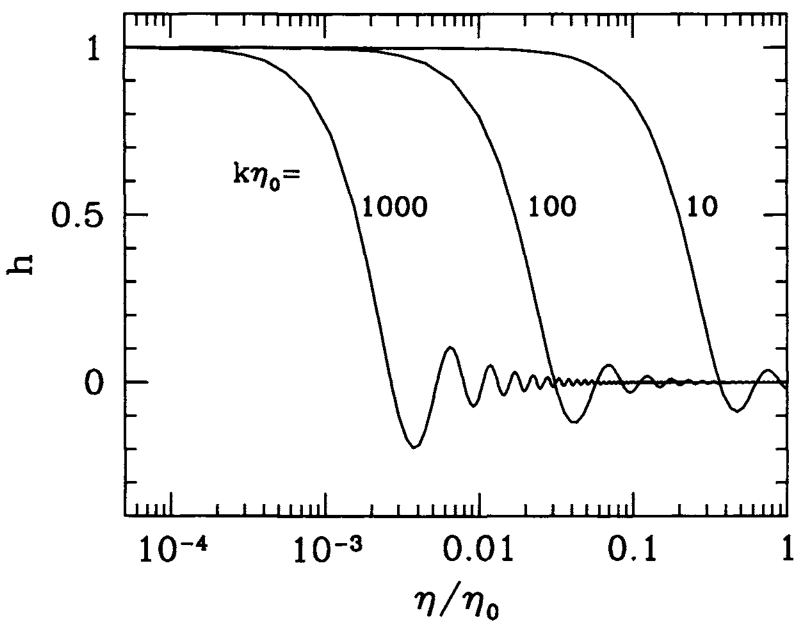

Figure 5.1. Evolution of gravity waves as a function of conformal time. Three different modes are shown, labeled by their wave numbers. Smaller scale modes decay earlier.

Equation (5.63) is a generalization of the wave equation to an expanding universe. Exercise 12 illustrates that if the universe is purely radiation or matter dominated, exact analytic solutions can be obtained. These are oscillatory, like the simple ones in Eq. (5.64), but also damp out. Figure 5.1 shows the evolution of h_α for three different wavelength modes. The large-scale mode (with $k\eta_0 = 10$) remains constant at early times when its wavelength is larger than the horizon $k\eta < 1$. Once its wavelength becomes comparable to the horizon, the amplitude begins to die off, oscillating several times until the present. The small-scale mode $k\eta_0 = 1000$ shown in Figure 5.1 also begins to decay when its wavelength becomes comparable to the horizon. Its entry into the horizon occurs much earlier, though, so the decay is much more efficient. By today, the amplitude is extremely small.

An important point about the effect of gravity waves on the CMB anisotropy spectrum can be gleaned from Figure 5.1. Because small-scale modes decay earlier than large-scale modes, at decoupling (at $\eta/\eta_0 \simeq 0.02$) only modes with $k\eta_0$ less than about 100 persist. All smaller scale modes can be neglected. Therefore, anisotropies on small angular scales will not be affected by gravity waves. Only the large-scale anisotropies are impacted by gravity waves.

5.4 THE DECOMPOSITION THEOREM

The decomposition theorem states that perturbations to the metric can be divided up into three types: *scalar, vector,* and *tensor.* Each of these types of perturbations evolves independently. That is, if some physical process in the early universe sets up tensor perturbations, these do not induce scalar perturbations. Conversely, to determine the evolution of scalar perturbations, we will not have to worry about possible vector or tensor perturbations.

Now that we have computed the contributions to the Einstein tensor $G_{\mu\nu}$ from scalars and tensors, we can demonstrate the decomposition of these two types of perturbations. To do this, remember that we obtained the scalar equations by considering the two components of Einstein's tensor:

$$G^0{}_0 \qquad ; \qquad \left(\hat{k}_i\hat{k}_j - (1/3)\delta_{ij}\right) G^i{}_j. \tag{5.65}$$

Inserting these components into Einstein's equations led to equations (5.27) and (5.33). If we can show that tensor perturbations do not contribute to these two components, then we will have convinced ourselves of at least part of the decomposition theorem, namely that the equations governing scalar equations are not affected by tensors.

It is easy to see that tensor perturbations do not contribute to $G^0{}_0$. For $G^0{}_0$ depends on R_{00} and $\mathcal{R}$. But we have seen that both of these do not depend on h_+ or $h_\times$.

Now let's show that $(\hat{k}_i\hat{k}_j - \delta_{ij}/3)G^i{}_j$ also does not pick up a contribution from tensor perturbations. Multiply Eq. (5.60) by the projection operator:

$$\begin{aligned}\left(\hat{k}_i\hat{k}_j - (1/3)\delta_{ij}\right) \delta G^i{}_j &= \left(\delta_{i3}\delta_{j3} - (1/3)\delta_{ij}\right) \\ &\quad \times \left[\frac{3}{2}H\mathcal{H}_{ij,0} + \frac{\mathcal{H}_{ij,00}}{2} + \frac{k^2}{2a^2}\mathcal{H}_{ij}\right]\end{aligned} \tag{5.66}$$

where the equality holds since we have chosen $\hat{k}$ to lie in the $\hat{z}$ direction. The terms in which indices i and j are set to 3 vanish since $\mathcal{H}_{33} = 0$. The only remaining terms are those proportional to δ_{ij}. But the Kronecker delta instructs us to take the trace of $\mathcal{H}$. This too vanishes. The scalar equations we derived in the previous section are therefore unchanged by the presence of tensor modes. This is a manifestation of the decomposition theorem.

5.5 FROM GAUGE TO GAUGE

Let's go back to scalar perturbations. Until now, we have characterized these with Ψ and Φ in the form of Eq. (4.9). This corresponds to a choice of *gauge* or a choice of a coordinate system with which to study the space-time. If we changed the coordinate system we use, we would get a metric of a different form, i.e., a different gauge. Although we will work almost exclusively in the gauge corresponding to Eq. (4.9), the conformal Newtonian gauge, historically many other gauges have been used, and for different parts of the "cosmological perturbation" problem, different gauges have their advantages. Indeed, we will see in Section 6.5.3 that people who work on the theory of inflation sometimes prefer a gauge with *spatially flat slicing* (g_{ij} unperturbed), since the equations for the perturbations generated by inflation simplify considerably. Also, the code currently used by most people to compute anisotropies and inhomogeneities in the universe uses *synchronous gauge*, partly because the equations are better behaved numerically in that gauge. So the ability to move back and forth between different gauges is useful, and I want to spend a few pages describing how to do this.

Most generally, scalar perturbations to the metric can be written down as

$$\begin{aligned} g_{00} &= -(1+2A) \\ g_{0i} &= -aB_{,i} \\ g_{ij} &= a^2\left(\delta_{ij}\left[1+2\psi\right] - 2E_{,ij}\right). \end{aligned} \tag{5.67}$$

There are four functions which characterize scalar perturbations to the metric: A, B, ψ, and E. They all depend on space and time, and they are all scalars. For example, the g_{0i} components are the derivatives of a scalar function, not an independent vector function with its own orientation. In conformal Newtonian gauge, $A = \Psi$ and $\psi = \Phi$, while $B = E = 0$.

How do we transform from one gauge to another? The invariant distance of Eq. (2.2) does not change if different coordinates $\tilde{x}$ are used instead of x. Therefore,

$$\tilde{g}_{\alpha\beta}(\tilde{x})d\tilde{x}^{\alpha}d\tilde{x}^{\beta} = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}, \tag{5.68}$$

where I have used a different set of dummy indices on both sides to make the upcoming few lines clearer. One of the differentials on the left-hand side can be written as $d\tilde{x}^\alpha = (\partial\tilde{x}^\alpha/\partial x^\mu)dx^\mu$ and similarly with the other differential, so equating coeffients of $dx^\mu dx^\nu$ leads to

$$\tilde{g}_{\alpha\beta}(\tilde{x})\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu} = g_{\mu\nu}(x). \tag{5.69}$$

This equation is what we are after: a prescription for how the metric changes under a coordinate transformation.

The most general coordinate transformation is generated by

$$\begin{aligned} t &\rightarrow \tilde{t} = t + \xi^0(t,\vec{x}) \\ x^i &\rightarrow \tilde{x}^i = x^i + \delta^{ij}\xi_{,j}(t,\vec{x}), \end{aligned} \tag{5.70}$$

where we take ξ^0 and ξ to be small perturbations of the same order as the variables characterizing the perturbations. Let's examine how the metric changes under such a transformation. I'll work out one component explicitly and leave the rest as an exercise. Consider the $_{00}$ component of Eq. (5.69):

$$\tilde{g}_{\alpha\beta}(\tilde{x})\frac{\partial\tilde{x}^\alpha}{\partial t}\frac{\partial\tilde{x}^\beta}{\partial t} = -\left[1+2A\right]. \tag{5.71}$$

I claim that the only term that contributes to the left-hand side is the one with $\alpha = \beta = 0$. Consider for example $\alpha = 0$ and $\beta = i$. The off-diagonal component of the metric $\tilde{g}_{0i}$ is proportional to $\tilde{B}_{,i}$ a first-order perturbation. But $\partial\tilde{x}^i/\partial t$ is proportional to the first-order variable ξ, so the product is second-order and can be neglected. A similar argument holds for the $\alpha = i; \beta = j$ terms. Therefore, the left-hand side is simply

$$\begin{aligned} -\left[1+2\tilde{A}\right]\left(\frac{\partial\tilde{t}}{\partial t}\right)^2 &= -\left[1+2\tilde{A}\right]\left(1+\frac{\partial\xi^0}{\partial t}\right)^2 \\ &\simeq -1-2\tilde{A}-2\,\frac{\partial\xi^0}{\partial t}. \end{aligned} \tag{5.72}$$

Equating this with g_{00} leads to

$$-2\tilde{A}-2\,\frac{\partial\xi^0}{\partial t} = -2A, \tag{5.73}$$

so under the coordinate transformation specified by Eq. (5.70)

$$A \rightarrow \tilde{A} = A - \frac{1}{a}\dot{\xi}^0. \tag{5.74}$$

In a similar vein, the other components of the metric transform into

$$\tilde{\psi} = \psi - H\xi^0$$

$$\tilde{B} = B - \frac{\xi^0}{a} + \dot{\xi}$$

$$\tilde{E} = E + \xi. \tag{5.75}$$

One technical point: Eqs. (5.74) and (5.75) describe how the components of the metric tensor transform under general coordinate changes. These equations, which have become standard, are a bit misleading, though, because each of the individual functions, A for example, transforms as a scalar, i.e., does not change under a spatial coordinate transformation. Here we have transformed the metric and accomodated the resulting changes in new definitions of A, B, ψ, and E. This is not the same thing as seeing how A by itself changes under a transformation.

To sum up, then, there are four functions which characterize scalar perturbations, but these can be manipulated with two other functions which characterize coordinate transformations. For example, starting with a metric in which $E \neq 0$, it is trivial to make a transformation to eliminate E: simply choose $\xi = -E$, and the resulting $\tilde{E} = 0$. Thus, there are really only $4 - 2 = 2$ functions which matter. Indeed, this is the reason that we had only Φ and Ψ in conformal Newtonian gauge. More generally, one might hope to construct two *gauge invariant* variables, those which remain unchanged under a general coordinate transformation. Bardeen (1980) first identified two such variables:

$$\Phi_A \equiv A + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{E} - B)\right]$$

$$\Phi_H \equiv -\psi + aH(B - \dot{E}). \tag{5.76}$$

In conformal Newtonian gauge, in which $E = B = 0$, $\Phi_A = \Psi$ and $\Phi_H = -\Phi$. These invariants are very useful: if equations simplify in a particular gauge, then one can do calculations in that gauge, form the gauge-invariant variables, and then turn these into the perturbations in any other gauge. We will do precisely this in Section 6.5.3. In other words, Φ_A and Φ_H are useful shortcuts or recipes for transforming from one gauge to another.

Under a general coordinate transformation, the components of the energy–momentum tensor $T_{\mu\nu}$ also change. In exact analogy with the metric tensor,

$$\tilde{T}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(x). \tag{5.77}$$

Again, though, Bardeen found combinations of the components of $T_{\mu\nu}$ which remain invariant and therefore facilitate mapping from one gauge to another. In particular, in Fourier space

$$v \equiv ikB + \frac{\hat{k}^i T^0{}_i}{(\rho + \mathcal{P})a} \tag{5.78}$$

remains invariant under a coordinate transformation. In conformal Newtonian gauge, for matter, v is indeed equal to the v we defined in Chapter 4. For radiation, $v = -3i\Theta_{r,1}$, i.e., proportional to the dipole, again in conformal Newtonian gauge. A second invariant is the generalized perturbation to the energy density,

$$\epsilon_m \equiv -1 - \frac{T^0{}_0}{\rho} + \frac{3H}{k^2\rho}k^i T^0{}_i. \tag{5.79}$$

For matter, in conformal Newtonian gauge, $\epsilon_m = \delta + (3aHv/k)$; i.e., it reduces to the ordinary overdensity δ on scales smaller than the horizon. For radiation, $\epsilon_m = 4\Theta_{r,0} - 12i\Theta_{r,1}aH/k$, again reducing to the standard overdensity on small scales.

5.6 SUMMARY

The Einstein equations relate perturbations in the metric to perturbations in the matter and radiation. Taking two components of the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we found equations governing the evolution of the two functions which describe scalar metric perturbations, Φ and Ψ of Eq. (4.9). It is easiest to write these equations in Fourier space. Again recalling our convention of dropping the ˜s on transformed variables, we can write

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left[\rho_{\rm m}\delta_m + 4\rho_r\Theta_{r,0}\right] \tag{5.27}$$

$$k^2(\Phi + \Psi) = -32\pi G a^2 \rho_r \Theta_{r,2}. \tag{5.33}$$

Here subscript m includes all matter such as baryons and dark matter and subscript r all radiation such as neutrinos and photons. More precisely

$$\rho_{\rm m}\delta_m \equiv \rho_{\rm dm}\delta + \rho_b\delta_{\rm b} \quad ; \quad \rho_r\Theta_{r,0} \equiv \rho_\gamma\Theta_0 + \rho_\nu\mathcal{N}_0$$

$$\rho_{\rm m}v_m \equiv \rho_{\rm dm}v + \rho_b v_{\rm b} \quad ; \quad \rho_r\Theta_{r,1} \equiv \rho_\gamma\Theta_1 + \rho_\nu\mathcal{N}_1. \tag{5.80}$$

Some of the other components of Einstein's equation are redundant; they add no new information about the evolution of Φ and Ψ. An example is the time-space component, which you can derive in Exercise 5. At times, though, one form of the evolution equation will be more useful than another. For example, one combination (Exercise 6) of these equations leads to an algebraic equation for the potential,

$$k^2\Phi = 4\pi G a^2\left[\rho_{\rm m}\delta_m + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{\rm m}v_m + 4\rho_r\Theta_{r,1}\right)\right]. \tag{5.81}$$

Other components of Einstein's equation contain information not about the scalar perturbations Φ and Ψ, but about vector and tensor perturbations. Scalar, vector, and tensor perturbations are decoupled: each evolves independently of the others. We will see in Chapter 6 that inflation can produce tensor perturbations, so it is important to know what the Einstein equation says about their evolution. We showed that there are two functions which can characterize tensor perturbations, h_+ and $h_\times$; each of these evolves independently and satisfies

$$\ddot{h}_\alpha + 2\frac{\dot{a}}{a}\dot{h}_\alpha + k^2 h_\alpha = 0 \tag{5.63}$$

where α denotes $+, \times$. In an expanding universe, the amplitude of a gravity wave described by Eq. (5.63) falls off once the mode enters the horizon.

SUGGESTED READING

Most cosmology books offer some treatment of the perturbed Einstein equations in cosmology. Again *The Large Scale Structure of the Universe* (Peebles) is a useful reference, especially for synchronous gauge. *Cosmological Inflation and Large Scale Structure* (Liddle and Lyth) has a very nice treatment which, among other virtues, explains the physics of gauge choices. Probably the two most comprehensive works are the review articles by Mukhanov, Feldman, and Brandenberger (1992) and Kodama and Sasaki (1984), with the former slightly more accessible and the latter more general. These are both based on the seminal Bardeen (1980) article which is remarkable for its clarity and conciseness in its treatment of gauge invariant variables.

The general relativity books mentioned in Chapter 2 all have good discussions of gravity waves. Before turning to any of the technical literature, though, you must read *Black Holes and Time Warps* (Thorne), a wonderful mixture of the history, science, and personalities associated with 20th-century general relativity. It is the best popular science book I have ever read.

EXERCISES

Exercise 1. Derive the Christoffel symbols, $\Gamma^i{}_{\mu\nu}$, given in Eq. (5.7). When doing this, you will need g^{ij}. Show that it is equal to $\delta_{ij}(1-2\Phi)/a^2$.

Exercise 2. Show that R_{ij} is given by Eq. (5.14).

Exercise 3. Use the full general relativistic expression for the energy momentum tensor given in Eq. (2.101), which holds even in the presence of metric perturbations. Show that, with scalar perturbations to the metric, the phase space integral for the time-time component reduces to that in Eq. (5.22). Show that the contribution from species α to $T^0{}_i$ is

$$T^0{}_i = g_\alpha a \int \frac{d^3p}{(2\pi)^3} p_i f_\alpha(\vec{p}, \vec{x}, t). \tag{5.82}$$

Note the extra factor of a.

Exercise 4. Consider a 3D matrix with components $G_{ij} = (\hat{k}_i\hat{k}_j - \delta_{ij}/3)G^L$. Show that this form is traceless and satisfies $\epsilon_{ijk}G_{kl,jl} = 0$ so it is the proper form for the longitudinal component.

Exercise 5. Compute the time-space component of the Einstein tensor. Show that, in Fourier space,

$$G^0{}_i = 2ik_i\left(\frac{\dot{\Phi}}{a} - H\Psi\right). \tag{5.83}$$

Combine with the energy-momentum tensor derived in Exercise 3 to show that

$$\dot{\Phi} - aH\Psi = \frac{4\pi G a^2}{ik}\left[\rho_{\rm dm} v + \rho_b v_{\rm b} - 4i\rho_\gamma \Theta_1 - 4i\rho_\nu \mathcal{N}_1\right]. \tag{5.84}$$

The time-space component of Einstein's equations adds no new information once we already have the two equations derived in the text. Deciding which two to use is a matter of convenience.

Exercise 6. Take the Newtonian limit of Einstein's equations. Combine the time-time equation (5.27) with the time-space equation of Exercise 5 to obtain the algebraic (i.e., no time derivatives) equation for the potential given in Eq. (5.81). Show that this reduces to Poisson's equation (with the appropriate factors of a) when the wavelength is much smaller than the horizon ($k\eta >> 1$).

Exercise 7. Fill in the blanks in the derivation of the tensor equation.
(a) Show that $\Gamma^i{}_{jk}$ is given by Eq. (5.43) in the presence of tensor perturbations.
(b) Show that the last term in Eq. (5.48) is given by Eq. (5.54).

Exercise 8. We defined the perturbations to the photon distribution function via Eq. (4.34). Show that, if Θ depends only on μ, the cosine of the angle between $\hat{k}(\equiv \hat{z}$ here) and $\hat{p}$, then $T^1{}_1 - T^2{}_2$ vanishes. This is indeed the dependence we have been dealing with so far. This is yet another aspect of the decomposition theorem: the terms Θ that source the scalar perturbations (and are sourced by them) do not affect tensor perturbations. Anisotropies induced by tensor perturbations will have Θ of the form

$$\Theta(\mu, \phi) = (1 - \mu^2)\cos(2\phi)\Theta_+(\mu) \tag{5.85}$$

for those perturbations generated by h_+ and a similar expression for $h_\times$ with the cos replaced by a sin. These, however, have a negligible impact on the evolution of the gravity waves, so we are justified in setting the right-hand side of Eq. (5.63) to zero.

Exercise 9. Use the ${}^1{}_2$ component of the Einstein equations to show that $h_\times$ obeys the same equation as does h_+.

Exercise 10. Show that scalar perturbations (Φ and Ψ) do not contribute to either $G^1{}_1 - G^2{}_2$ or to $G^1{}_2$. This completes the demonstration of the decomposition theorem for scalars and tensors.

Exercise 11. Consider vector perturbations to the metric. These can be described by two function h_{xz} and h_{yz} where again only the spatial part of the metric is perturbed. The perturbative part of g_{ij} is

$$h^V_{ij} = \begin{pmatrix} 0 & 0 & h_{xz} \\ 0 & 0 & h_{yz} \\ h_{xz} & h_{yz} & 0 \end{pmatrix}. \tag{5.86}$$

Show that h_{xz} and h_{yz} do not affect any of the equations we have derived so far for scalar or tensor evolution — yet another aspect of the decomposition theorem.

Exercise 12. Solve the wave equation (5.63) if the universe is purely matter dominated. Do the same for the radiation-dominated case.

Exercise 13. Define the *transfer function* for gravity wave evolution as

$$T(k) \equiv \frac{h_\alpha(k,\eta)}{h_\alpha(k,\eta=0)} \left(\frac{k\eta}{3j_1(k\eta)} \right) . \tag{5.87}$$

You should recognize the term in parentheses as the inverse of the matter-dominated solution you derived in Exercise 12. Solve Eq. (5.63) numerically and compute the transfer function. Compare your solution with the fit of Turner, White, and Lidsey (1993),

$$T(y) = \left[1 + 1.34y + 2.5y^2\right]^{1/2} \tag{5.88}$$

where $y \equiv (k\eta_0/370h)$ (with h parametrizing the Hubble constant). Assume the universe today is flat and matter dominated, but account for transition from matter to radiation.

Exercise 14. Derive the transformations in the metric components given by Eq. (5.75). Show that Φ_A and Φ_H do not change under a general coordinate transformation.