# 6

# INITIAL CONDITIONS

In order to understand structure in the universe, we have derived the equations governing perturbations around a smooth background. Before we start solving these equations, we need to know the initial conditions. This quest for initial conditions will lead to an entirely new realm of physics, the theory of inflation. Inflation was introduced (Guth, 1981; Linde 1982; Albrecht and Steinhardt, 1982) partly to explain how regions which could not have been in causal contact with each other have the same temperature. It was soon realized (Starobinsky, 1982; Guth and Pi, 1982; Hawking, 1982; Bardeen, Steinhardt, and Turner, 1983; Brandenberger, Kahn, and Press, 1983; Guth and Pi, 1985) that the very mechanism that explains the uniformity of the temperature in the universe can also account for the origin of perturbations in the universe. Therefore, in order to produce a set of initial conditions, we will need to detour into the world of inflation. One warning: we are not sure that inflation is the mechanism that generated the initial perturbations. It is very difficult to test a theory based on energy scales well beyond the reach of accelerators. Nonetheless, it is by far the most plausible explanation. Indeed, one of the current problems in cosmology is that there is really no viable alternative to inflation. Also, the next generation of CMB and large-scale structure observations will put inflation to some stringent tests.

## 6.1 THE EINSTEIN–BOLTZMANN EQUATIONS AT EARLY TIMES

Chapters 4 and 5 contain nine first-order differential equations for the nine perturbation variables we need to track. In principle, we need initial conditions for all of these variables. In practice, though, a combination of arguments will relate many of these variables to each other, and we need only determine the initial conditions for one of these. This section determines the way all variables depend on the gravitational potential $\Phi$ at early times; the remaining sections work out the initial conditions for $\Phi$.

Let us consider first the Boltzmann equations (4.100)–(4.107) at very early times. In particular, we want to consider times so early that for any $k$-mode of interest, $k\eta \ll 1$. This inequality immediately leads to several important simplifications. Consider the terms $\dot{\Theta}$ and $ik\mu\Theta$ in Eq. (4.100). The first term is of order $\Theta/\eta$, while the second is of order $k\Theta$. Therefore, the first is larger than the second by a factor of order $1/(k\eta)$, which, by assumption, is much greater than 1. In a similar way, we can argue that all terms in the Boltzmann equations multiplied by $k$ can be neglected at early times. Physically, this means that, at early times, all perturbations of interest have wavelengths ($\sim k^{-1}$) much larger than the distance over which causal physics operates. A hypothetical observer then who sees only photons from within his causal horizon will see a uniform sky. Thus higher multipoles ($\Theta_1, \Theta_2, \ldots$) are much smaller than the monopole, $\Theta_0$. Therefore, the perturbations to the photon and neutrino temperatures evolve according to

$$\dot{\Theta}_0 + \dot{\Phi} = 0$$
$$\dot{\mathcal{N}}_0 + \dot{\Phi} = 0. \tag{6.1}$$

The same principles can be applied to the matter distributions. The overdensity equations reduce to

$$\dot{\delta} = -3\dot{\Phi}$$
$$\dot{\delta}_{\rm b} = -3\dot{\Phi}. \tag{6.2}$$

The velocities are comparable to the first moments of the radiation distributions, so they are smaller than the overdensities by a factor of order $k\eta$ and may be set to zero initially. In fact, the baryon velocity is not only comparable to the photon first moment, $\Theta_1$: it is equal to it by virtue of the strength of Compton scattering. That is, the largeness of $\dot{\tau}$ in Eq. (4.106) ensures that $v_{\rm b} = -3i\Theta_1$. We will use this later when reexamining the Boltzmann equations closer to decoupling. For now, we are interested in times so early that the only relevant fact is that higher moments are all negligibly small.

Now let us turn to the Einstein equations at early times. First consider Eq. (5.27). The first term there contains a factor of $k^2$ so may be neglected. Also the two matter terms on the right are negligible at early times since radiation dominates. Therefore, we have

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\right) = 16\pi G a^2 \left(\rho_\gamma \Theta_0 + \rho_\nu \mathcal{N}_0\right). \tag{6.3}$$

But since radiation dominates, $a \propto \eta$ (recall Eq. (2.100) and the discussion immediately afterward) so $\dot{a}/a = 1/\eta$. Therefore,

$$\frac{\dot{\Phi}}{\eta} - \frac{\Psi}{\eta^2} = \frac{16\pi G\rho a^2}{3}\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0\right)$$
$$= \frac{2}{\eta^2}\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0\right) \tag{6.4}$$

where the last equality follows by virtue of the zero-order Einstein equation.

To simplify further, we can define the ratio of neutrino energy density to the total radiation density as

$$f_\nu \equiv \frac{\rho_\nu}{\rho_\gamma + \rho_\nu}. \tag{6.5}$$

Then multiplying Eq. (6.4) by $\eta^2$ leads to

$$\dot{\Phi}\eta - \Psi = 2\left(\left[1 - f_\nu\right]\Theta_0 + f_\nu \mathcal{N}_0\right). \tag{6.6}$$

Recall that Eq. (6.1) relates the derivative of the monopoles to the derivative of the potential. We can therefore eliminate both monopoles from Eq. (6.6) by differentiating both right- and left-hand sides. Then,

$$\ddot{\Phi}\eta + \dot{\Phi} - \dot{\Psi} = -2\dot{\Phi} \tag{6.7}$$

where the right-hand side follows since both $\dot{\Theta}_0$ and $\dot{\mathcal{N}}_0$ are equal to $-\dot{\Phi}$ for these large scale modes.

So far we have used only one Einstein equation. The second, Eq. (5.33), describes how the higher moments of the photon and neutrino distributions cause $\Psi + \Phi$ to be nonzero. Let us here neglect these higher order moments, which cause the sum of the gravitational potentials to be slightly nonzero.[1] Under this approximation, we can eliminate $\Psi$ everywhere by simply setting it to $-\Phi$. Then,

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0. \tag{6.8}$$

Setting $\Phi = \eta^p$ leads to the algebraic equation

$$p(p-1) + 4p = 0 \tag{6.9}$$

which allows two solutions: $p = 0, -3$. The $p = -3$ mode is the decaying mode. If it is excited very early on, it will quickly die out and have no impact on the universe. The $p = 0$ mode, on the other hand, does not decay if excited. It is the mode we are interested in. If some mechanism can be found which excites this mode, this mechanism may well be responsible for the perturbations in the universe.

Focusing therefore on only the $p = 0$ mode, we see that Eq. (6.6) relates the gravitational potential to the neutrino and photon overdensities:

$$\Phi = 2\left(\left[1 - f_\nu\right]\Theta_0 + f_\nu \mathcal{N}_0\right). \tag{6.10}$$

Both $\Theta_0$ and $\mathcal{N}_0$ are also constant in time. In most models of structure formation, they are equal since whatever causes the perturbations tends not to distinguish between photons and neutrinos. Therefore, we will set

$$\Theta_0(k, \eta_i) = \mathcal{N}_0(k, \eta_i) \tag{6.11}$$

which leads to

$$\Phi(k, \eta_i) = 2\Theta_0(k, \eta_i) \tag{6.12}$$

where I have explicitly written the $k$-dependence of all these variables and the fact that we are setting up the initial conditions at some early time $\eta_i$.

[1] See Exercise 2 for a careful accounting of the effect of the neutrino quadrupole; the photon quadrupole is kept minuscule by Compton scattering, so it really does not contribute to Eq. (5.33).

The initial conditions for matter, both $\delta$ and $\delta_b$, depend upon the nature of the primordial perturbations. Combining the first of equations (6.1) and (6.2) leads to

$$\delta = 3\Theta_0 + \text{constant} \tag{6.13}$$

for the dark matter overdensity, with an identical equation for the baryon overdensity. Primordial perturbations are often divided into those for which the constant in Eq. (6.13) is zero (*adiabatic* perturbations) and those for which the constant is nonzero (*isocurvature* perturbations). Adiabatic perturbations have a constant matter-to-radiation ratio everywhere since

$$\frac{n_{\rm dm}}{n_\gamma} = \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}} \left[\frac{1+\delta}{1+3\Theta_0}\right]. \tag{6.14}$$

The prefactor, the ratio of zero-order number densities, is a constant in both space and time. For the ratio of matter to radiation number density to be uniform, therefore, the combination inside the brackets which linearizes to $1 + \delta - 3\Theta_0$ must be independent of space. So the perturbations must sum to zero,

$$\delta = 3\Theta_0, \tag{6.15}$$

for adiabatic perturbations. By similar arguments for the baryons, $\delta_b = 3\Theta_0$. There are models based on isocurvature perturbations, but these have not been very successful to date; we will focus on adiabatic initial conditions.

For the most part, velocities and dipole moments are negligibly small in the very early universe. However, we will encounter situations where we need to know the initial conditions for these as well. You will show in Exercise 3 that the appropriate initial conditions are

$$\begin{aligned} \Theta_1 = \mathcal{N}_1 &= \frac{iv_b}{3} = \frac{iv}{3} \\ &= -\frac{k\Phi}{6aH}. \end{aligned} \tag{6.16}$$

## 6.2 THE HORIZON

If this book were a novel or a biography, a better title for this section might be *Midlife Crisis*. The main character would have attended a good high school, studied hard, and gone on to a solid university. There he fell in love with an exciting, but sensible, woman; upon graduating, he set up some interviews, and got a good job downtown. He married his college girlfriend, and after several years in the city, they moved to the suburbs and had three kids. Our hero contributed to the community and was recognized all over town as a solid citizen. He was moving up fast in his company and there was talk about a political position. Just when he was about to declare his candidacy, he began to have doubts. "What have I been doing with my life? What is really important? Were all those years of study and work simply a 'track'? Did I take this path just because everyone else was moving in the same direction? Where is the innovation and the signature that my life is mine?" And worse, he has a secret, an underlying feeling that everything he has built is based on a fallacy.

OK, maybe it wouldn't be a bestseller, but it does serve as a useful metaphor for our study of perturbations in the universe. Until now, we have done everything in a systematic, proper way. We reviewed the standard Big Bang cosmology. We expanded about this zero-order smooth universe, getting evolution equations for the perturbations to the particle distributions and to the gravitational fields. We realized that these coupled differential equations needed initial conditions so in the last section we set those up. However, now we must ask, What caused those initial perturbations? It is one thing to say that $\Phi = 2\Theta_0$ initially. It is quite another to explain what caused $\Phi$ to be nonzero in the first place.

And it is worse than that. To understand why let us recall the physical meaning of the conformal time $\eta$: it is the maximum comoving distance traveled by light since the beginning of the universe. Equivalently, objects separated by comoving distances larger than $\eta$ today were not ever in causal contact: there is simply no way information could have propagated over distances larger than $\eta$. For this reason, $\eta$ is called the comoving horizon.

With this in mind, we can now revisit the condition used in the previous section that $k\eta \ll 1$. The wavenumber $k$ is roughly equal to the inverse of the wavelength of the mode in question (give or take a factor of $2\pi$). Therefore $k\eta$ is the ratio of the comoving horizon to the comoving wavelength of the perturbation. If this ratio is much smaller than 1, then the mode in question has a wavelength so large that no causal physics could possibly have affected it. A picture worth remembering is shown in Figure 6.1. The horizon grows as the scale factor increases. On the other hand, comoving wavelengths remain constant. All modes of cosmological interest therefore had wavelengths much larger than the horizon early on. Eventually these cosmological modes *enter* the horizon; after that, causal physics begins to operate on them.

The truly disturbing feature of this realization is most apparent when considering the microwave background today. On all scales observed the CMB is very close to isotropic. How can this be? The largest scales observed have entered the horizon just recently, long after decoupling. (An example is the scale corresponding to the quadrupole moment of the CMB, shown in Figure 6.1.) Before decoupling, the wavelengths of these modes are so large that no causal physics could force deviations from smooothness to go away. After decoupling, the photons do not interact at all; they simply freestream. So even though it is technically possible that photons reaching us today from opposite directions had a chance to communicate with each other and equilibrate to the same temperature, practically this could not have happened. Why then is the CMB temperature so uniform? This is a profound problem that we have glossed over by simply assuming that the temperature is uniform and that perturbations about the zero-order temperature are small.

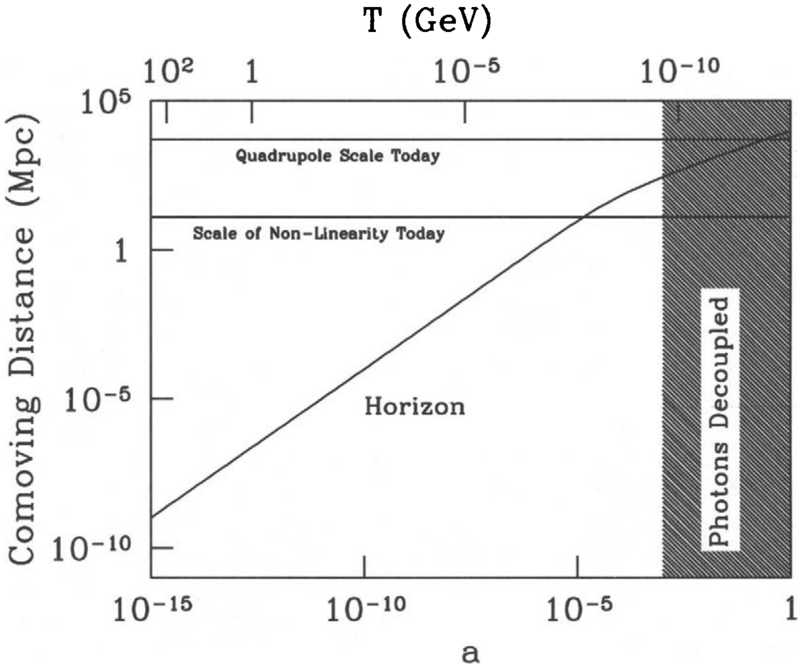


**Figure 6.1.** The comoving horizon as a function of the scale factor. Also shown are two comoving wavelengths, which remain constant with time. Early in the history of the universe, both of these modes — as well as all other modes of cosmological interest — had wavelengths much larger than the horizon. The CMB comes from the last scattering surface at $a \simeq 10^{-3}$. At that time, the largest scales (e.g., the one labeled "quadrupole") were still outside the horizon. The horizon problem asks how regions separated by distances larger than the horizon at the last scattering surface can have the same temperature.

A more intuitive picture of the horizon problem is shown in Figure 6.2. At any given time, the region within the cone is causally connected to us (at the center). Photons that we observe today from the last scattering surface were well outside our horizon when they were first emitted. The most disturbing aspect of this is the observation of large-angle isotropy, an indication that photons apparently separated by many horizons at the last scattering surface nonetheless shared the same temperature (to a part in $10^5$).

## 6.3 INFLATION

This section describes a beautiful solution to the horizon problem outlined in the previous section. First, we explore a logical way out of the previous argument by realizing that an early epoch of rapid expansion solves the horizon problem. Then we consider the Einstein equations to tell us what type of energy is needed in order to produce this rapid expansion, showing that negative pressure is required.

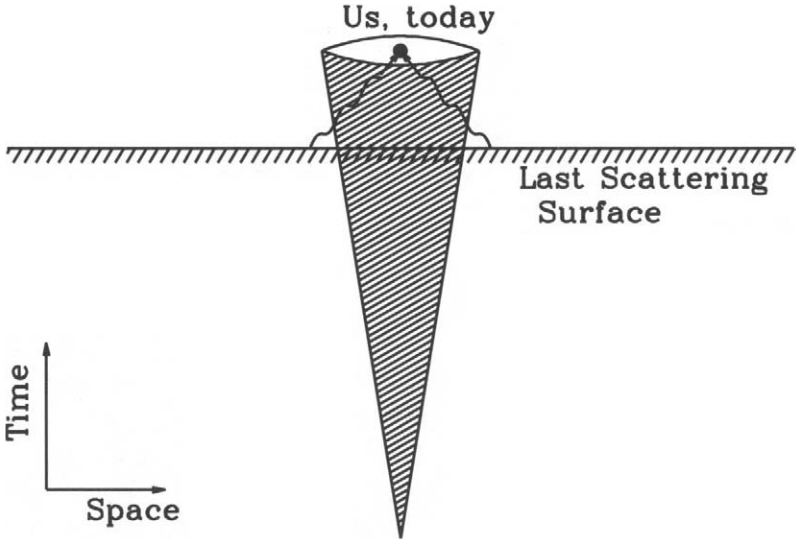


**Figure 6.2.** The horizon problem. The region inside the cone at any time is causally connected to us (at the center). Photons emitted from the last scattering surface (at redshift $\sim 1000$) started outside of this region. Therefore, at the last scattering surface, they were not in causal contact with us and certainly not with each other. Yet their temperatures are almost identical.

Finally, we consider a scalar field theory and show that negative pressure is easy to accommodate in such a theory.

Two comments about the field theory implementation. First, field theory has a reputation as a difficult subject. It is, but the part we will need for inflation is decidedly simple. Indeed, almost all we will need to know about field theory we've already used in the previous chapter on general relativity. The second point is that there is no known scalar field which can drive inflation. (A skeptic might point out that there is no known fundamental scalar field at all!) Therefore, it may well be true that the idea of inflation is correct but it is driven by something other than a scalar field. Having said that, there are a number of reasons to work with scalar fields, as we will do whenever we need to specify the source of inflation. Almost all fundamental particle physics theories contain scalar fields. In fact, historically it was particle physicists studying high-energy extensions of the Standard Model (in particular Grand Unified Theories) who proposed the idea of inflation driven by a scalar field as a natural byproduct of some of these extensions. Indeed, almost all current work on inflation is based on a scalar field (or sometimes two). The alternative from a particle physics point of view is to use a vector field (such as the electromagnetic potential) or a set of fermions (similar to the way condensates induce superconductivity) to drive inflation. Neither of these choices works very well, but they both complicate things severely, so we will stick to a scalar field.

### 6.3.1 A Solution to the Horizon Problem

To motivate a solution to the horizon problem, let me rewrite the comoving horizon as

$$\eta = \int_0^a \frac{da'}{a'} \frac{1}{a'H(a')}. \tag{6.17}$$

The comoving horizon then is the logarithmic integral of the *comoving Hubble radius*, $1/aH$. The Hubble radius is the distance over which particles can travel in the course of one expansion time, i.e., roughly the time in which the scale factor doubles. So the Hubble radius is another way of measuring whether particles are causally connected with each other: if they are separated by distances larger than the Hubble radius, then they cannot currently communicate. There is a subtle distinction between the comoving horizon $\eta$ and the comoving Hubble radius $(aH)^{-1}$. If particles are separated by distances greater than $\eta$, they *never* could have communicated with one another; if they are separated by distances greater than $(aH)^{-1}$, they cannot talk to each other *now*. It is therefore possible that $\eta$ could be much larger than $(aH)^{-1}$ now, so that particles cannot communicate today but were in causal contact early on. This might happen if the comoving Hubble radius early on was much larger than it is now so that $\eta$ got most of its contribution from early times. This could happen, but it does not happen during matter- or radiation-dominated epochs. In those cases, the comoving Hubble radius increases with time, so typically we expect the largest contribution to $\eta$ to come from the most recent times. Indeed, this is precisely what Figure 6.1 indicates.

Look again at Figure 6.1. On top of the figure I have drawn an axis which depicts the temperature of the cosmic plasma for the given value of the scale factor. We know quite a bit about physics going up to the limits on the plot, several hundred GeV. Beneath these energies, the standard model of particle physics works very well. Beyond those energies, although we have ideas, there is no experimental reason to prefer one theory over another. Since the energy content of the universe determines $a(t)$, when you mentally extrapolate the horizon in Figure 6.1 back to $a = 0$, or equivalently to infinitely high temperatures, you are really making an assumption. You are assuming that nothing strange happened early on, in particular that the universe was always radiation dominated at early times. If this were true, then it does indeed follow that the comoving horizon received a negligible contribution from the very early universe, that photons can travel only very small distances in the first fraction of a second after the Big Bang.

This suggests a solution to the horizon problem: perhaps early on, the universe was not dominated by either matter or radiation. Perhaps, for at least a brief time, the comoving Hubble radius decreased. Then, we would have the situation depicted in Figure 6.3. The comoving Hubble radius would decrease dramatically during this epoch. In that case, the comoving horizon would get most of its contribution not from recent times, but rather from primordial epochs before the rapid expansion of the grid. Particles separated by many Hubble radii today, for example those outside the circle in the bottom panel of the figure, were in causal contact — were inside the Hubble circle in the top panel — before this epoch of rapid expansion.

How must the scale factor evolve in order to solve the horizon problem? We can first answer this question qualitatively. If the comoving Hubble radius is to decrease, then $aH$ must increase. That is,

$$\frac{d}{dt}\left[a\frac{da/dt}{a}\right] = \frac{d^2a}{dt^2} > 0. \qquad (6.18)$$

So to solve the horizon problem, the universe must go through a period in which it is accelerating, expanding ever more rapidly. This is the origin of the term *inflation*. To understand the epoch of inflation more quantitatively, let me give away the punchline that most inflationary models typically operate at energy scales of order $10^{15}$ GeV or larger. How big was the comoving Hubble radius when the temperature was $10^{15}$ GeV? We can get an order of magnitude estimate by ignoring the relatively brief epoch of recent matter domination and assuming that the universe has been radiation dominated since the end of inflation (you can correct this assumption in Exercise 6). Then $H$ scales as $a^{-2}$ so $a_0H_0/a_eH_e = a_e$ where $a_e$ is the scale factor at the end of inflation. If $a_e$ corresponds to a time at which the temperature was $10^{15}$ GeV, then $a_e \simeq T_0/10^{15}$ GeV $\simeq 10^{-28}$. So the comoving Hubble radius at the end of inflation was 28 orders of magnitude smaller than it is today. For inflation to work, the comoving Hubble radius at the onset of inflation had to be larger than the largest scales observable today, i.e., larger than the current comoving Hubble radius. So during inflation, the comoving Hubble radius had to decrease by some 28 orders of magnitude.

The most common way to arrange this is to construct a model wherein $H$ is constant during inflation. In that case, since $da/a = Hdt$, the scale factor evolves as

$$a(t) = a_e e^{H(t-t_e)} \qquad t < t_e \qquad (6.19)$$

where $t_e$ is the time at the end of inflation. The decrease in the comoving Hubble radius $(aH)^{-1}$ is now due solely to the exponential increase in the scale factor. For the scale factor to increase by a factor of $10^{28}$, the argument of the exponential must be of order $\ln(10^{28}) \sim 64$ (but remember the corrections in Exercise 6), so inflation can solve the horizon problem if the universe expands exponentially for more than 60 e-folds.

Thus, consider Figure 6.4, which shows the comoving Hubble radius as a function of the scale factor. The right side of this plot is virtually identical to Figure 6.1, which tells us that the comoving scales of interest to us were much larger than the Hubble radius in the standard cosmology. The left-hand side of the plot though shows that an inflationary epoch reduces the comoving Hubble radius dramatically. This makes sense: since the scale factor is inflating very rapidly, it becomes increasingly difficult for photons to move along the comoving grid (which is itself expanding with $a$). Before inflation started, the comoving Hubble radius was very large, larger than any scale of cosmological interest today, so all such scales were well within the horizon.

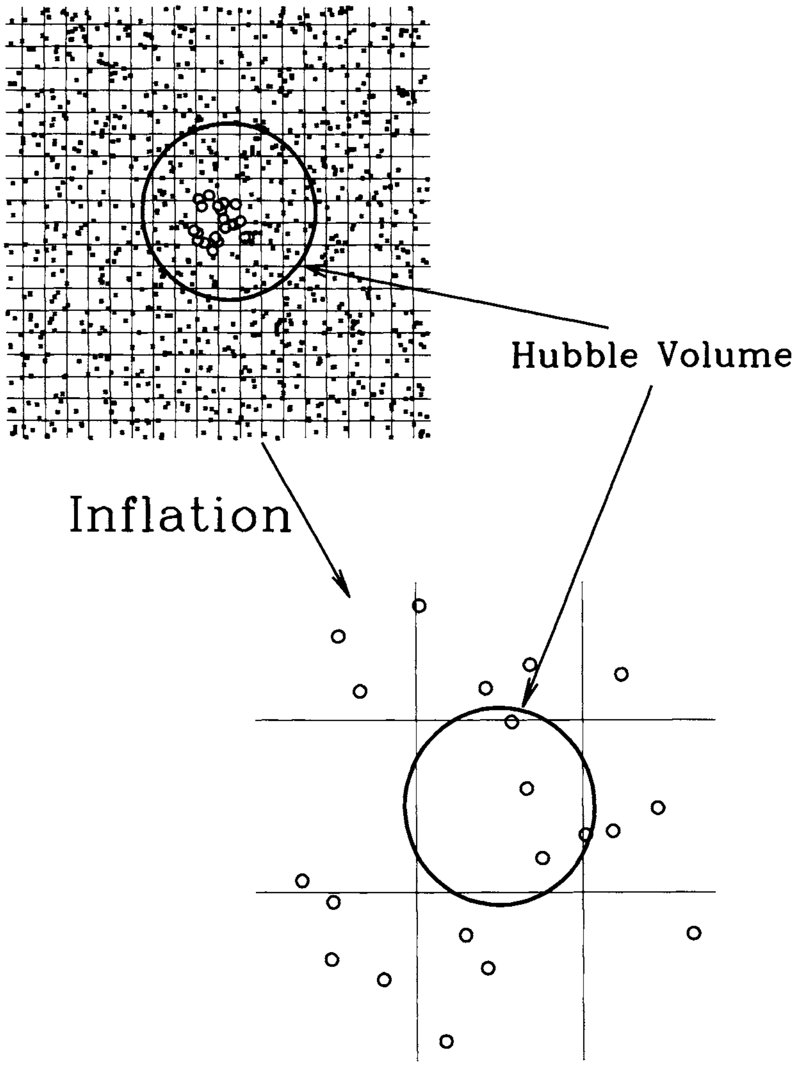


**Figure 6.3.** Particles on the comoving grid before (top) and after (bottom) inflation. Open circles are the same particles on top and bottom. Before inflation, the comoving Hubble radius was quite large, encompassing dozens of cells on the grid. After inflation, the comoving Hubble radius has shrunk to just one cell. (In this caricature, the scale factor has grown by a factor of order 7; during inflation the scale factor increases by greater than $e^{60}$.) The shrinkage of the comoving Hubble radius means that particles which were initially in causal contact with one another (within the large circle at top) can now no longer communicate. Note that the physical Hubble radius, depicted by large circles on the top and bottom grids, remains roughly constant during inflation.

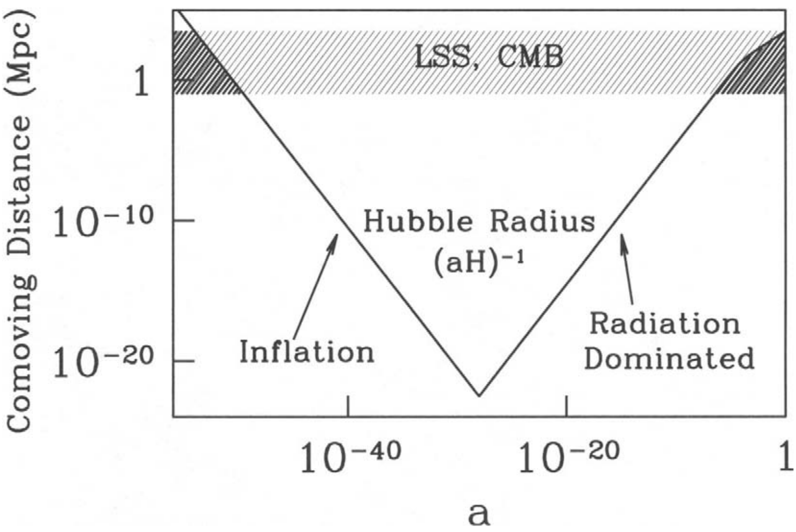


**Figure 6.4.** The comoving Hubble radius as a function of scale factor. Scales of cosmological interest (shaded band) were larger than the Hubble radius until $a \sim 10^{-5}$. Dark shaded regions show when these scales were smaller than the Hubble radius, and therefore susceptible to microphysical processes. Very early on, before inflation operated, all scales of interest were smaller than the Hubble radius and therefore susceptible to microphysical processing. Similarly, at very late times, scales of cosmological interest came back within the Hubble radius.

Note the symmetry in Figure 6.4. Scales just entering the horizon today — roughly 60 e-folds after the end of inflation — left the horizon 60 e-folds before the end of inflation. The amplitude of the perturbations on these scales remained constant as long as they were super-horizon. So, when we measure them today, we are actually seeing them as they were when they first left the horizon during the inflationary era (modulo whatever processing has taken place since they reentered the horizon, processing we will study in great detail in Chapters 7 and 8). To explain the structure in the universe today, then, it is clearly important to understand the generation of perturbations during inflation.

We have until now discussed inflation in comoving coordinates. But it is also profitable to think of the exponential expansion in physical coordinates. The idea that the horizon blows up early on is depicted in Figure 6.5. The physical size ($a$ times the comoving size) of a causally connected region blows up exponentially quickly during inflation. So regions that we observe to be astronomical today were actually microscopically small before inflation, and they were in causal contact with each other.

The total comoving horizon ceases to be an effective time parameter after inflation because it becomes large very early on, and then changes very little as the universe expands during the matter- and radiation-dominated eras. A simple way to rectify this is to subtract off its primordial part $\eta_{\rm prim}$, and redefine $\eta$ as

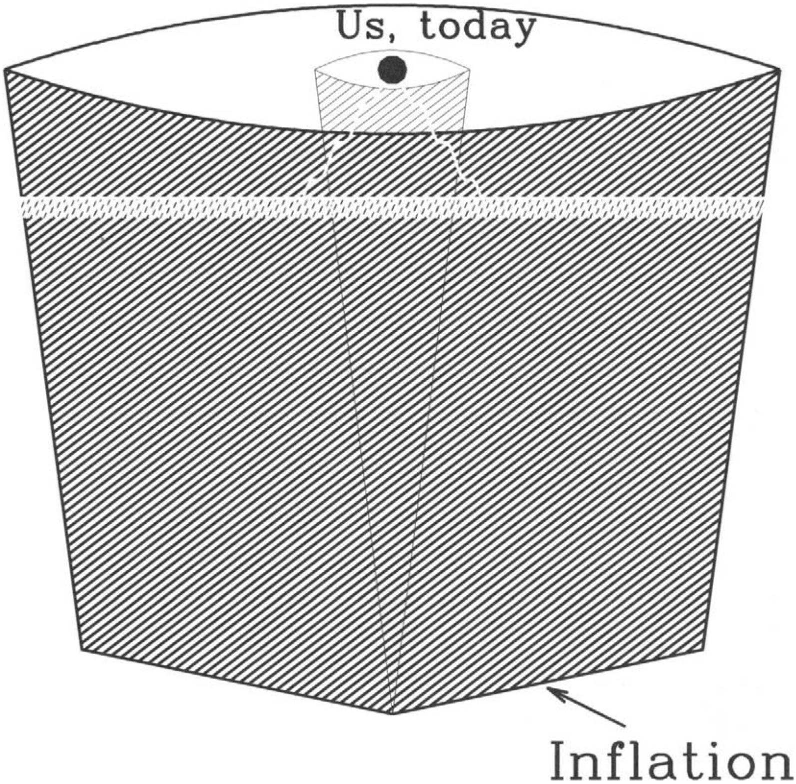


**Figure 6.5.** Inflationary solution to the horizon problem. Larger cone shows the true horizon in an inflationary model; smaller inner cone shows the horizon without inflation. During inflation, the physical horizon blows up very rapidly. All scales in the shaded region were once in causal contact so it is not surprising that the temperature is uniform.

$$\eta = \int_{t_e}^{t} \frac{dt'}{a(t')}, \tag{6.20}$$

so that the total comoving horizon is $\eta_{\rm prim} + \eta$. This is the convention we will follow; note that this means that during inflation, $\eta$ is negative, but always monotonically increasing. A scale leaves the horizon in the sense of Figure 6.4 when $k|\eta|$ becomes less than 1, and returns at late times when $k\eta$ becomes larger than 1.

To sum up, inflation — an epoch in which the universe accelerates — solves the horizon problem. During the accelerated expansion the physical Hubble radius remains fixed, so particles initially in causal contact with one another can no longer communicate. Regions which are separated by vast distances today were actually in causal contact before and during inflation. At that time, these regions were given the necessary initial conditions, the smoothness we observe today, but also, as we will soon see, the small perturbations about smoothness that eventually grew into galaxies and other structure in the universe.

### 6.3.2 Negative Pressure

We have shown that an accelerating universe can solve the horizon problem. Since general relativity ties the expansion of the universe to the energy in it, we now need to ask what type of energy can produce acceleration. We can get an immediate answer if we appeal to the time-time and space-space components of the zero-order Einstein equations. They are (Eqs. (2.39) and (2.93))

$$\left(\frac{da/dt}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{d^2a/dt^2}{a} + \frac{1}{2}\left(\frac{da/dt}{a}\right)^2 = -4\pi G\mathcal{P}. \tag{6.21}$$

Multiplying the first of these by one-half and then subtracting one from the other eliminates the first derivative of $a$, leaving

$$\frac{d^2a/dt^2}{a} = -\frac{4\pi G}{3}\left(\rho + 3\mathcal{P}\right). \tag{6.22}$$

Acceleration is defined to mean that $d^2a/dt^2$ is positive. For this to happen, the terms in parentheses on the right must be negative. So inflation requires

$$\mathcal{P} < -\frac{\rho}{3}. \tag{6.23}$$

Since the energy density is always positive, the pressure must be negative. This result is perhaps not surprising: we saw back in Chapter 2 that the accelerated expansion which cause supernovae to appear very faint can be caused only by dark energy with negative pressure. Inflation was apparently driven by the a similar form of energy, one with $\mathcal{P} < 0$. To reiterate what we emphasized in Chapter 2, negative pressure is not something with which we have any familiarity. Nonrelativistic matter has small positive pressure proportional to temperature divided by mass, while a relativistic gas has $\mathcal{P} = +\rho/3$, again positive. So whatever it is that drives inflation is not ordinary matter or radiation.

### 6.3.3 Implementation with a Scalar Field

We have become familiar with the fields $\Psi(\vec{x}, t)$ and $\Phi(\vec{x}, t)$, deriving equations for them which govern their evolution (the Einstein equations) and the evolution of particles which are affected by them. These two fields are parts of the multicomponent field, the metric $g_{\mu\nu}$. The metric is one of the fundamental fields in physics, but there are others. Every elementary particle — the electron, neutrino, quarks, photon, etc. — is associated with its own field. It would be wonderful if one of these fields, the electromagnetic potential associated with photons say, could serve as the source for an inflationary model. Unfortunately we do not yet have such a concrete model. Instead, I will discuss inflation in terms of a generic scalar field (not a fermion like the quarks and leptons or a vector like the electromagnetic field). The simplest version of the standard model does indeed have within it one such field, the Higgs field. But, again unfortunately, we know too much about the Higgs of the standard model. Its interactions and properties are constrained enough for us to know that it cannot serve as the source for inflation. So we will drop any pretensions of connecting the generic scalar field which drives inflation to known physics. Making this connection is left as a homework problem for a future Nobel laureate.

We want to know if a scalar field — which I will call $\phi(\vec{x},t)$, not to be confused with the metric perturbation $\Phi(\vec{x},t)$ — can have negative $\rho + 3\mathcal{P}$. So our first task is to write down the energy–momentum tensor for $\phi$. This is

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - g^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]. \tag{6.24}$$

Here $V(\phi)$ is the potential for the field. For example a free field with mass $m$ has a potential $V(\phi) = m^2\phi^2/2$. A warning about signs: if you delve into the literature you will invariably find different signs than those in Eq. (6.24). These are dictated by the choice of metric. Although our metric signature $(-,+,+,+)$ is probably most common in the context of cosmology, it is probably not as common in particle physics. Beware. We will assume that $\phi$ is mostly homogeneous, consisting of a zero-order part, $\phi^{(0)}(t)$, and a first-order perturbation, $\delta\phi(\vec{x},t)$. In this section we will derive information about the zero-order homogeneous part, its energy density and pressure, and its time evolution. Later we will consider its perturbations, $\delta\phi$, and how they are generated.

For the homogeneous part of the field, only time derivatives of $\phi$ are relevant so the indices $\alpha$ and $\beta$ in the first term in Eq. (6.24) and $\mu,\nu$ in the second must be equal to zero. The energy–momentum tensor then reduces to

$$T^{(0)\alpha}{}_{\beta} = -g^{\alpha}{}_{0}g^{0}{}_{\beta}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + g^{\alpha}{}_{\beta}\left[\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi^{(0)})\right]. \tag{6.25}$$

The time-time component of $T^{0}{}_{0}$ is equal to $-\rho$, so the energy density is

$$\rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi^{(0)}). \tag{6.26}$$

The first term here is the kinetic energy density of the field, the second its potential energy density. A homogeneous scalar field therefore has much the same dynamics as a single particle moving in a potential [think of $\phi^{(0)}(t)$ as the position of the particle $x(t)$]. In fact this analogy dominates even the language used to describe inflation. The pressure for the homogeneous field is $\mathcal{P} = T^{(0)i}{}_{i}$ (no sum over spatial index $i$), so

$$\mathcal{P} = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi^{(0)}). \tag{6.27}$$

A field configuration with negative pressure is therefore one with more potential energy than kinetic. An example is shown in Figure 6.6, in which a field is trapped in a *false vacuum*, i.e., a local, but not the global, minimum of the potential.

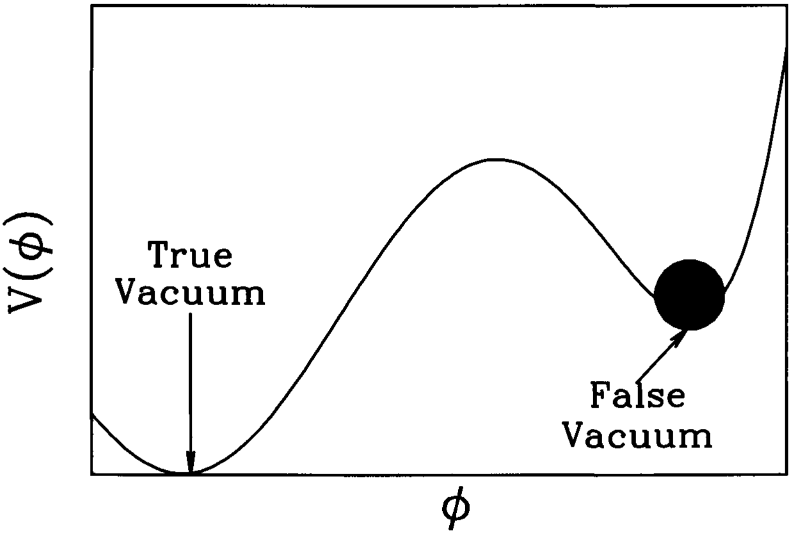


**Figure 6.6.** A scalar field trapped in a false vacuum. Since it is trapped, it has little kinetic energy. The potential energy is nonzero, however, so the pressure is negative. The global minimum of the potential is called the true vacuum, since a homogeneous field sitting at the global minimum of the potential is in the ground state of the system.

There is something important to notice about a field trapped in a false vacuum. Since $\phi^{(0)}$ is constant, its energy density, which is all potential, remains constant with time. Constant energy density is much different than anything with which we are familiar. The densities of both matter and radiation, for example, fall off very rapidly as the universe expands. Therefore, even if the universe initially contains a mixture of matter, radiation, and false vacuum energy, it will quickly become dominated by the vacuum energy. For a trapped field, it is trivial to determine the evolution of the scale factor. Since the energy density is constant, Einstein's equation for the evolution of $a$ is

$$\frac{da/dt}{a} = \sqrt{\frac{8\pi G\rho}{3}} = \text{constant}. \tag{6.28}$$

We immediately see that a field trapped in a false vacuum produces exponential expansion as in Eq. (6.19), with $H \propto \rho^{1/2}$ constant. The primordial comoving horizon, that generated before the end of inflation, is then obtained by integrating the inverse of Eq. (6.19) over time,

$$\eta_{\rm prim} = \frac{1}{H_e a_e}\left(e^{H(t_e - t_b)} - 1\right), \tag{6.29}$$

where $t_b$ is the beginning of inflation. So if the field is trapped for at least 60 e-foldings ($H(t_e - t_b) > 60$), the horizon problem is solved.

Guth's (1981) initial formulation of inflation used a scalar field trapped in a false minimum of the potential, but it was quickly realized (Guth and Weinberg, 1983; Hawking, Moss, and Stewart, 1982) that such a scenario is not viable. The only way for the field to evolve to its true minimum is similar to the way an alpha particle migrates out of the potential barrier in a nucleus: it tunnels quantum mechanically. Thus, initially small localized regions tunnel from the false to the true vacuum. These bubbles of the true vacuum state must coalesce in order for the universe as a whole to move to the true vacuum state. Careful calculations showed that these bubbles would never coalesce, that the regions of false vacuum would expand rapidly and remain, so that the true vacuum state of the universe would never be attained.

To avoid the problem of the universe never reaching its true vacuum state, subsequent models of inflation (Linde, 1982; Albrecht and Steinhardt, 1982) made use of a scalar field slowly rolling toward its true ground state. The energy density of such a field is also very close to constant (if the potential is not too steep) so it quickly comes to dominate. To determine the evolution of $\phi^{(0)}$ in general when the field is not trapped, we return to the Einstein equations as given in Eq. (6.21). Consider the first of these. If the dominant component in the universe is $\phi$, then the energy density on the right-hand side becomes $(d\phi^{(0)}/dt)^2/2 + V$. Differentiating this first equation therefore leads to

$$2\frac{da/dt}{a}\left[\frac{d^2a/dt^2}{a} - \left(\frac{da/dt}{a}\right)^2\right] = \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\frac{d\phi^{(0)}}{dt}\right] \tag{6.30}$$

where $V'$ is defined as the derivative of $V$ with respect to the field $\phi^{(0)}$. We can replace the first term in brackets on the left by $-4\pi G(\rho/3+\mathcal{P})$ as in Eq. (6.22). Similarly the second term on the left is $8\pi G\rho/3$. The left-hand side therefore becomes

$$\frac{da/dt}{a}8\pi G\left[-(\rho/3) - \mathcal{P} - 2\rho/3\right] = -8\pi GH\left(\frac{d\phi^{(0)}}{dt}\right)^2. \tag{6.31}$$

Equating this to the right side of Eq. (6.30) leads to the evolution equation for a homogeneous scalar field in an expanding universe,

$$\frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + V' = 0. \tag{6.32}$$

A more useful form for us will be with the conformal time $\eta$ as the time variable; then it is straightforward (Exercise 8) to show that

$$\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2V' = 0 \tag{6.33}$$

where overdots still denote derivatives with respect to conformal time $\eta$.

Most models of inflation are *slow roll* models, in which the zero-order field, and hence the Hubble rate, vary slowly. Therefore, a simple relation between the conformal time $\eta$ and the expansion rate holds. In particular, during inflation

$$\eta \equiv \int_{a_e}^{a}\frac{da}{Ha^2}$$

$$\simeq \frac{1}{H}\int_{a_e}^{a}\frac{da}{a^2}$$

$$\simeq \frac{-1}{aH} \tag{6.34}$$

where the rough equality on the second line holds because $H$ is nearly constant, and the one on the third because the scale factor at the end of inflation is much larger than in the middle ($a_e \gg a$). To quantify slow roll, cosmologists typically define two variables which vanish in the limit that $\phi$ remains constant. First, define

$$\epsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = \frac{-\dot{H}}{aH^2}. \tag{6.35}$$

Since $H$ is always decreasing, $\epsilon$ is always positive. During inflation, it is typically small, whereas it is equal to 2 during a radiation era. In fact, one definition of an inflationary epoch is one in which $\epsilon < 1$. A complementary variable which also quantifies how slowly the field is rolling is:

$$\delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \frac{-1}{aH\dot{\phi}^{(0)}}\left[aH\dot{\phi}^{(0)} - \ddot{\phi}^{(0)}\right]$$

$$= \frac{-1}{aH\dot{\phi}^{(0)}}\left[3aH\dot{\phi}^{(0)} + a^2V'\right]. \tag{6.36}$$

Here the paucity of Greek letters becomes a hindrance. The second slow-roll parameter is more conventionally defined as $\eta$, but we obviously cannot follow that convention as $\eta$ is our conformal time. (Early universe cosmologists use $\tau$ for conformal time, freeing up $\eta$, but we do not have that luxury since we need $\tau$ for optical depth.) My choice of $\delta$ is also fairly common, but we need to bear in mind that this has nothing to do with the overdensities introduced in Chapter 4. The second line here follows from Eq. (6.33). Again, in most models $\delta$ is small. We will see in Section 6.6 that some unique features of inflation, deviations from the simplest possible spectrum and the production of gravity waves, are proportional to $\epsilon$ and $\delta$. If these features are one day measured, they will not only be unique signatures of inflation but also allow us to learn something about the physics driving inflation.

## 6.4 GRAVITY WAVE PRODUCTION

Inflation does more than solve the horizon problem. The power of inflation is its ability to correlate scales that would otherwise be disconnected. The zero-order scheme outlined in the previous section ensures that the universe will be uniform on all scales of interest today. There are perturbations about this zero-order scheme, though, and these perturbations — produced early on when the scales are causally connected — persist long after inflation has terminated.

We are most interested in *scalar* perturbations to the metric since these couple to the density of matter and radiation and ultimately are responsible for most of the inhomogeneities and anisotropies in the universe. In Section 6.5 we will study these in detail. In addition to scalar perturbations, though, inflation also generates *tensor* fluctuations in the gravitational metric, so-called gravity waves. As we saw in Chapter 5, these are not coupled to the density and so are not responsible for the large-scale structure of the universe, but they do induce fluctuations in the CMB. In fact, these fluctuations turn out to be a unique signature of inflation and offer the best window on the physics driving inflation, so they are clearly worthy of our study. I choose to study the production of tensor perturbations first before scalar perturbations for a subtle technical reason. Tensor perturbations to the metric are not coupled to any of the other perturbation variables,[2] so when we consider them, we will be looking at the fluctuations in a single field. Scalar perturbations to the metric couple to energy density fluctutations. The coupled fields fluctuate together and this coupling requires a bit of work to understand. This work, while important, is *not* the main point: the most important idea is that quantum mechanical fluctuations during inflation are responsible for the variations around the smooth background that so fascinate us. This idea is best introduced in the much simpler context of a single field, so we start with tensor perturbations.

During inflation, the universe consists primarily of a uniform scalar field and a uniform background metric. Against this background, the fields fluctuate quantum mechanically. At any given time, the average fluctuation is zero, because there are regions in which the field is slightly larger than its average value and regions in which it is smaller. The average of the square of the fluctuations (the variance), however, is not zero. Our goal is to compute this variance and see how it evolves as inflation progresses. Looking ahead, once we know this variance, we can draw from a distribution with this variance to set the initial[3] conditions.

### 6.4.1 Quantizing the Harmonic Oscillator

In order to compute the quantum fluctuations in the metric, we need to quantize the field. The way to do this, in the case of both tensor and scalar perturbations, is to rewrite the problem so that it looks like a simple harmonic oscillator. Once that is done, we will appeal to our knowledge of this simple system. Therefore, let's first record some basic facts about the quantization of the harmonic oscillator.

- A simple harmonic oscillator with unit mass and frequency $\omega$ is governed by the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0. \tag{6.37}$$

- Upon quantization, $x$ becomes a quantum operator

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger \tag{6.38}$$

[2]This is not quite true. The quadrupole moments act as sources for tensor perturbations, but these vanish if a scalar field drives inflation. See Exercise 10

[3]*Initial* here means those when the modes of interest are still far outside the horizon. This is well before any processing can take place, but well after inflation has generated them.

where $\hat{a}$ is a quantum operator which acts on the state of the system, and $v$ is a solution to Eq. (6.37), $v \propto e^{-i\omega t}$.

- In particular, $\hat{a}$ annihilates the *vacuum* state $|0\rangle$, in which there are no particles. It also satisfies the commutation relation

$$[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{6.39}$$

Other commutators vanish: $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$. It is straightforward to show (Exercise 9) that these commutation relations are equivalent to the (perhaps more familiar) relations between $\hat{x}$ and its momentum $\hat{p}$:

$$[\hat{x}, \hat{p}] = i, \tag{6.40}$$

as long as $v$ is normalized via

$$v(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}. \tag{6.41}$$

These facts enable us to compute the quantum fluctuations of the operator $\hat{x}$ in the ground state $|0\rangle$:

$$\begin{aligned}\langle|\hat{x}|^2\rangle &\equiv \langle 0|\hat{x}^\dagger\hat{x}|0\rangle \\ &= \langle 0|\left(v^*\hat{a}^\dagger + v\hat{a}\right)\left(v\hat{a} + v^*\hat{a}^\dagger\right)|0\rangle.\end{aligned} \tag{6.42}$$

Since $\hat{a}|0\rangle = 0$, the first term in the second set of parentheses vanishes. Similarly, $\langle 0|\hat{a}^\dagger = (a|0\rangle)^\dagger = 0$, so we are left with

$$\begin{aligned}\langle|\hat{x}|^2\rangle &= |v(\omega, t)|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\ &= |v(\omega, t)|^2\langle 0|[\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|0\rangle.\end{aligned} \tag{6.43}$$

The second term again vanishes since $\hat{a}$ annihilates the vacuum, while the first is unity, so the variance in $\hat{x}$ is

$$\langle|\hat{x}|^2\rangle = |v(\omega, t)|^2, \tag{6.44}$$

in this case $1/2\omega$. This is (almost) all we need to know about quantum fluctuations in order to compute the fluctuations in the early universe generated by inflation.

### 6.4.2 Tensor Perturbations

Recall that tensor perturbations to the metric are described by two functions $h_+$ and $h_\times$, each of which obeys Eq. (5.63),

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 h = 0. \tag{6.45}$$

We would like to massage this equation into the form of a harmonic oscillator, so that $h$ can be easily quantized. To do this, define[4]

$$\tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}. \tag{6.46}$$

Derivatives of $h$ with respect to conformal time can be rewritten as

$$\frac{\dot{h}}{\sqrt{16\pi G}} = \frac{\dot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\tilde{h} \tag{6.47}$$

and

$$\frac{\ddot{h}}{\sqrt{16\pi G}} = \frac{\ddot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a^2}\dot{\tilde{h}} - \frac{\ddot{a}}{a^2}\tilde{h} + 2\frac{(\dot{a})^2}{a^3}\tilde{h}. \tag{6.48}$$

Inserting these into Eq. (6.45), and multiplying by $\sqrt{16\pi G}$, gives

$$\begin{aligned} &\frac{\ddot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a^2}\dot{\tilde{h}} - \frac{\ddot{a}}{a^2}\tilde{h} + 2\frac{(\dot{a})^2}{a^3}\tilde{h} + 2\frac{\dot{a}}{a}\left(\frac{\dot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\tilde{h}\right) + k^2\frac{\tilde{h}}{a} \\ &\qquad = \frac{1}{a}\left[\ddot{\tilde{h}} + \left(k^2 - \frac{\ddot{a}}{a}\right)\tilde{h}\right] = 0. \end{aligned} \tag{6.49}$$

This is precisely the form we know how to use. It has no damping term ($\propto \dot{\tilde{h}}$) so we can immediately write down an expression for the quantum operator

$$\hat{\tilde{h}}(\vec{k}, \eta) = v(k, \eta)\hat{a}_{\vec{k}} + v^*(k, \eta)a^\dagger_{\vec{k}}, \tag{6.50}$$

where the coefficients of the creation and annihilation operators satisfy the equation

$$\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right)v = 0. \tag{6.51}$$

We will shortly solve Eq. (6.51), but first let's see how the eventual solution determines the power spectrum of the fluctuations of the tensor perturbations.

[4]Regarding the factor of $\sqrt{16\pi G}$ here, the only way I know of deriving this is to write down the action for the fields $h_{+,\times}$. The kinetic term is then multiplied by a factor of $1/32\pi G$. A canonical scalar field has prefactor equal to a half. So the additional $16\pi G$ must be absorbed into a redefinition of the field. The hard part of this is writing down the action to second order in perturbation variables. We have seen that even first-order perturbations are cumbersome to track. On the other hand, by dimensional analysis — the fact that $h(\vec{x})$ is dimensionless while a canonical scalar field has dimensions equal to mass — we could have guessed that the factor of $m_{\rm Pl} = G^{-1/2}$ is required. Note that this prefactor does not affect the equation for $\tilde{h}$; it simply provides the normalization that becomes important when trying to determine the amplitude of the gravity-wave spectrum.

Using our harmonic oscillator analogy, we can write the variance of perturbations in the $\tilde{h}$ field as

$$\langle \hat{\tilde{h}}^{\dagger}(\vec{k},\eta)\hat{\tilde{h}}(\vec{k}',\eta)\rangle = |v(\vec{k},\eta)|^2(2\pi)^3\delta^3(\vec{k}-\vec{k}'). \tag{6.52}$$

There is one difference between this expression and the analogous expression for the one-dimensional harmonic oscillator in Eq. (6.44). A quantum field is defined in all space, so it can be considered as a collection, an infinite collection, of oscillators, each at a different spatial position (or, in Fourier space, at different values of $\vec{k}$). The quantum fluctuations in each of these oscillators are independent (as long as the equations are linear) so $\hat{\tilde{h}}(\vec{k})$ is completely uncorrelated with $\hat{\tilde{h}}(\vec{k}')$ if $\vec{k} \neq \vec{k}'$. The Dirac delta function in Eq. (6.52) enforces this independence; the $(2\pi)^3$ allows for the fact that we have moved to the continuum limit. Recalling that $\tilde{h} = ah/\sqrt{16\pi G}$, we see that

$$\begin{aligned}\langle \hat{h}^{\dagger}(\vec{k},\eta)\hat{h}(\vec{k}',\eta)\rangle &= \frac{16\pi G}{a^2}\ |v(k,\eta)|^2\ (2\pi)^3\delta^3(\vec{k}-\vec{k}') \\ &\equiv (2\pi)^3 P_h(k)\delta^3(\vec{k}-\vec{k}')\end{aligned} \tag{6.53}$$

where the second line defines the *power spectrum* of the primordial perturbations to the metric. Conventions for the power spectrum abound in the literature; the one I've chosen in Eq. (6.53) is not the most popular in the early universe community. Often a factor of $k^{-3}$ is added so that the power spectrum is dimensionless. I prefer to omit this factor to be consistent with the large scale structure community which likes its power spectra to have dimensions of $k^{-3}$. In any event, with this definition,

$$P_h(k) = 16\pi G\ \frac{|v(k,\eta)|^2}{a^2}. \tag{6.54}$$

We have now reduced the problem of determining the spectrum of tensor perturbations produced during inflation to one of solving a second-order differential equation for $v(k,\eta)$, Eq. (6.51). To solve this equation, we first need to evaluate $\ddot{a}/a$ during inflation. Recall that overdots denote derivative with respect to conformal time, so $\dot{a} = a^2 H \simeq -a/\eta$ by virtue of Eq. (6.34). Therefore, the second derivative of $a$ in Eq. (6.51) is

$$\begin{aligned}\frac{\ddot{a}}{a} &\simeq -\frac{1}{a}\frac{d}{d\eta}\left(\frac{a}{\eta}\right) \\ &\simeq \frac{2}{\eta^2}.\end{aligned} \tag{6.55}$$

So the equation for $v$ is

$$\ddot{v} + \left(k^2 - \frac{2}{\eta^2}\right)v = 0. \tag{6.56}$$

The initial conditions necessary to solve this equation come from considering $v$ at very early times before inflation has done most of its work. At that time, $-\eta$ is large, of order $\eta_{\rm prim}$, so the $k^2$ term dominates, and the equation reduces precisely to that of the simple harmonic oscillator. In that case, we know (Eq. (6.41)) that the properly normalized solution is $e^{-ik\eta}/\sqrt{2k}$. This knowledge enables us to choose (Exercise 11) the proper solution to Eq. (6.56),

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right]. \tag{6.57}$$

This obviously goes into the correct solution when the mode is well within the horizon ($k|\eta| \gg 1$). Even if you don't work through Exercise 11 (which arrives at the relatively simple solution of Eq. (6.57) in a rather tortured way), you should at least check that the $v$ given here is indeed a solution to Eq. (6.56).

After inflation has worked for many e-folds $k|\eta|$ becomes very small. Now that $v$ has been normalized, we can determine the amplitude of $v$, and hence the variance of the super-horizon gravitational wave amplitude, by taking the small argument limit of Eq. (6.57):

$$\lim_{-k\eta\to 0} v(k,\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{-i}{k\eta}. \tag{6.58}$$

Figure 6.7 shows the evolution of $h \propto v/a$ during inflation. At early times $h$ falls as $1/a$ as inflation reduces the amplitude of the modes. Once $-k\eta$ becomes smaller than unity, the mode leaves the horizon, after which $h$ remains constant.

The primordial power spectrum for tensor modes, which scales as $|v|^2/a^2$, is therefore constant in time after inflation has stretched the mode to be larger than the horizon. This constant determines the initial conditions for the gravity waves, those with which to start off $h_{+,\times}$ at early times (where in this context "early" means well after inflation has ended but before decoupling). Equations (6.54) and (6.58) show that this constant is

$$\begin{aligned} P_h(k) &= \frac{16\pi G}{a^2}\frac{1}{2k^3\eta^2} \\ &= \frac{8\pi G H^2}{k^3}. \end{aligned} \tag{6.59}$$

The second line here follows from Eq. (6.34). We have assumed that $H$ is constant in deriving this result; more generally, $H$ is to be evaluated at the time when the mode of interest leaves the horizon. This is our final expression for the primordial power spectrum of gravity waves. Detection of these waves would, quite remarkably, measure the Hubble rate during inflation. Since potential energy usually dominates kinetic energy in inflationary models, a measure of $H$ would be tantamount to measuring the potential $V$, again quite remarkable in view of the likelihood that inflation was generated by physics at energy scales above $10^{15}$ GeV, 12 orders of magnitude beyond the capacity of present-day accelerators. There is no guarantee that gravity waves produced during inflation will be detectable. Indeed, since $H^2 \propto \rho/m_{\rm Pl}^2$, the power spectrum is proportional to $\rho/m_{\rm Pl}^4$, the energy density at the time

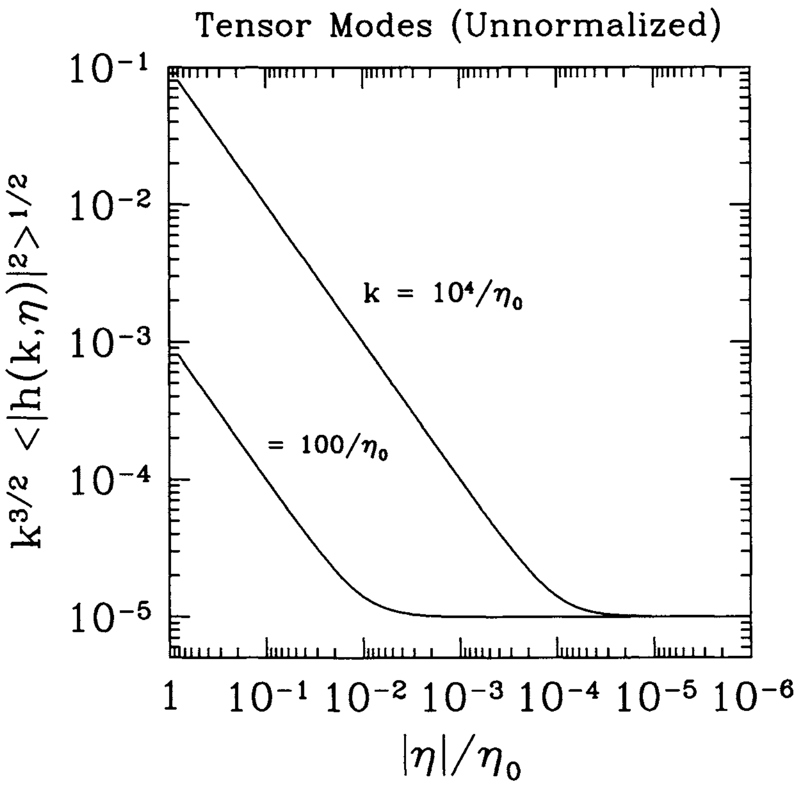


**Figure 6.7.** The root mean square fluctuations in the tensor field during inflation for two different $k$-modes. Time evolves from left to right: $\eta$ is negative but gets closer to zero during inflation. Once a mode "leaves the horizon" ($\eta \sim -1/k$), its RMS amplitude remains constant. Note that after a mode has left the horizon, its RMS amplitude times $k^{3/2}$ is the same for all modes. This is called a scale-free spectrum, strictly true only if the Hubble rate is constant when the scale of interest leaves the horizon (the choice here).

of inflation in units of the Planck mass. If inflation takes place at scales sufficiently smaller than the Planck scale, then primordial gravity waves will not be detected. Later in the book, we will develop the machinery necessary to answer the question, How small can the gravity wave component be and still be detected?

Two final technical points are in order regarding Eq. (6.59). Although I have not emphasized this feature of the spectrum of these primordial perturbations, the fluctuations in $h$ are Gaussian, just as are the quantum-mechanical fluctuations of the simple harmonic oscillator. Gaussianity is a fairly robust prediction of inflation; as such, many studies have been undertaken searching for signs of primordial non-Gaussianity in CMB and large-scale structure data, signs that would jeopardize the inflationary picture. Although there have been some hints, none have held up under greater scrutiny, so this prediction of inflation too appears to be verified. Second, Eq. (6.59) is the power spectrum for $h_+$ and $h_\times$ separately; these are uncorrelated, so the power spectrum for all modes must be multiplied by a factor of 2.

## 6.5 SCALAR PERTURBATIONS

The goal of this chapter is to find the perturbation spectrum of $\Psi$ (or $\Phi$; we assume throughout that they are equal in magnitude) emerging from inflation. With that spectrum, we can use the relations in Section 6.1 to determine the spectrum of the other perturbation variables. Finding the spectrum for $\Psi$ ,however, turns out to be complicated, more so than was the tensor case considered earlier. The primary complication is the presence of perturbations in the scalar field driving inflation, perturbations which are coupled to $\Psi$.

To deal with this problem, we will first ignore it: in Section 6.5.1, we compute the spectrum of perturbations in the scalar field $\phi$ generated during inflation, neglecting $\Psi$. This turns out to be relatively simple to do, since it is virtually identical to the tensor calculation we went through above. Why are we justified in neglecting $\Psi$ and how do the perturbations get transferred from $\phi$ to $\Psi$? The next two subsections take turns answering this question from two different points of view. First, Section 6.5.2 argues that — in a sense to be defined there — until a mode moves far outside the horizon, $\Psi$ is indeed negligibly small. Once it is far outside the horizon, this no longer holds, but we will find that a linear combination of $\Psi$ and $\delta\phi$ (the perturbations to the scalar field driving inflation) is conserved. This will allow us to convert the initial spectrum for $\delta\phi$ into a final spectrum for $\Psi$. The second way of justifying the neglect of perturbations to the metric is to switch gauges and work in a gauge in which the spatial part of the metric is unperturbed, a so-called *spatially flat slicing*. In such a gauge, the calculation of Section 6.5.1 is exact; the only question remaining is how to convert back to conformal Newtonian gauge to move on with the rest of the book. In Section 6.5.3, we identify a *gauge-invariant variable*, one which does not change upon a gauge transformation, which is proportional to $\delta\phi$ in a spatially flat slicing. It is then a simple matter to determine this variable in conformal Newtonian gauge, thereby linking $\Psi$ in conformal Newtonian gauge to $\delta\phi$ in spatially flat slicing. Note that the two solutions to the coupling problem, as worked out in Sections 6.5.2 and 6.5.3, are simply alternative approaches to the same problem. If you are comfortable with gauge transformations, Section 6.5.3 is probably a more elegant approach; the more brute-force approach of Section 6.5.2 gives the same answer though and requires less formalism and background.

### 6.5.1 Scalar Field Perturbations around a Smooth Background

Let's decompose the scalar field into a zero-order homogeneous part and a perturbation,

$$\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t), \tag{6.60}$$

and find an equation governing $\delta\phi$ in the presence of a smoothly expanding universe, i.e., with metric $g_{00} = -1; g_{ij} = \delta_{ij}a^2$. Consider the conservation of the energy–momentum tensor,

$$T^\mu{}_{\nu;\mu} = \frac{\partial T^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu}T^\mu{}_\alpha = 0. \tag{6.61}$$

The $\nu = 0$ component of this equation, expanded out to first order, gives the desired equation for $\delta\phi$. Since we are assuming a smooth metric, the only first-order pieces are perturbations in the energy–momentum tensor. All the $\Gamma$'s are either zero-order ($\Gamma^0{}_{ij} = \delta_{ij}a^2H$ and $\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij}H$ ) or zero (the rest of the components), as we found in Eqs. (2.22) and (2.23). So, writing the perturbed part of the energy–momentum tensor as $\delta T^\mu{}_\nu$ and considering the $\nu = 0$ component of the perturbed conservation equation leads to

$$0 = \frac{\partial \delta T^0{}_0}{\partial t} + ik_i\delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i. \tag{6.62}$$

It remains to determine the perturbations to the energy–momentum tensor in terms of the perturbations to the scalar field.

First let's compute $\delta T^i{}_0$. Since the time-space components of the scalar metric are zero, the second set of terms in Eq. (6.24), those with prefactor $g^\alpha{}_\beta$, vanish. Therefore,

$$T^i{}_0 = g^{i\nu}\phi_{,\nu}\phi_{,0} \tag{6.63}$$

where I have returned to using $_{,\nu}$ to denote the derivative with respect to $x^\nu$. Since $g^{i\nu} = a^{-2}\delta_{i\nu}$, the index $\nu$ must be equal to $i$. Recall that the zero-order field $\phi^{(0)}$ is homogeneous, so $\phi^{(0)}_{,i} = 0$. The space-time component of the energy–momentum tensor therefore has no zero-order piece. To extract the first-order piece, we can set $\phi_{,i}$ to $\delta\phi_{,i} = ik_i\delta\phi$. Then, setting all other factors to their zero-order values leads to

$$\delta T^i{}_0 = \frac{ik_i}{a^3}\dot{\phi}^{(0)}\delta\phi. \tag{6.64}$$

The additional factor of $a$ enters the denominator here because $\phi^{(0)}_{,0} = \dot{\phi}^{(0)}/a$ (recall that $\dot{}$ is derivative with respect to conformal time).

The time-time component of the energy–momentum tensor is a little more difficult:

$$T^0{}_0 = g^{00}(\phi_{,0})^2 - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V. \tag{6.65}$$

Setting $\phi = \phi^{(0)} + \delta\phi$ leads to

$$T^0{}_0 = \frac{-1}{2}\left(\phi^{(0)}_{,0} + \delta\phi_{,0}\right)^2 - \frac{1}{2a^2}\delta\phi_{,i}\delta\phi_{,i} - V(\phi^{(0)} + \delta\phi). \tag{6.66}$$

The spatial derivatives come in pairs, and pairs of first-order variables ($\delta\phi_{,i}$) lead to second-order terms. These may therefore be neglected. The potential may be expanded as a zero-order term, $V(\phi^{(0)})$ plus a first-order correction, $V'\delta\phi$, so the first-order correction to the energy–momentum tensor is

$$\begin{aligned}\delta T^0{}_0 &= -\phi^{(0)}_{,0}\delta\phi_{,0} - V'\delta\phi \\ &= \frac{-\dot{\phi}^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi.\end{aligned} \tag{6.67}$$

Similarly, you can show that the space-space component is

$$\delta T^i{}_j = \delta_{ij}\left(\frac{\dot{\phi}^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right). \tag{6.68}$$

Therefore, the conservation equation (6.62) becomes

$$\left(\frac{1}{a}\frac{\partial}{\partial\eta} + 3H\right)\left(\frac{-\dot{\phi}^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) - \frac{k^2}{a^3}\dot{\phi}^{(0)}\delta\phi - 3H\left(\frac{\dot{\phi}^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) = 0. \tag{6.69}$$

Carrying out the time derivatives (the only subtle one is $\partial V'/\partial\eta = V''\dot{\phi}^{(0)}$), multiplying by $a^3$, and collecting terms leads to

$$-\dot{\phi}^{(0)}\ddot{\delta\phi} + \dot{\delta\phi}\left(-\ddot{\phi}^{(0)} - 4aH\dot{\phi}^{(0)} - a^2V'\right) + \delta\phi\left(-a^2V''\dot{\phi}^{(0)} - k^2\dot{\phi}^{(0)}\right) = 0. \tag{6.70}$$

The $V''$ term here is typically small, proportional to the slow-roll variables $\epsilon$ and $\delta$ (Exercise 14), so it can be neglected. The coefficient of $\dot{\delta\phi}$, the first set of parentheses, is equal to $-2aH\dot{\phi}^{(0)}$ using the zero-order equation (6.33), so after dividing by $-\dot{\phi}^{(0)}$, we are left with

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} + k^2\delta\phi = 0. \tag{6.71}$$

This equation for perturbations to $\delta\phi$ is identical to Eq. (6.45) for tensor perturbations to the metric. Thus we can trivially copy our result from Section 6.4.2 and write immediately that the power spectrum of fluctuations in $\delta\phi$ is equal to

$$P_{\delta\phi} = \frac{H^2}{2k^3}. \tag{6.72}$$

Compare this with Eq. (6.57). It is identical apart from a factor of $16\pi G$. Recall that we inserted this factor (with a bit of hand-waving; see the footnote on page 158) in the tensor case to turn the dimensionless $h$ into a field with dimensions of mass. To get the result for $\delta\phi$ which is already a scalar field with the proper dimensions, we simply remove this factor.

### 6.5.2 Super-Horizon Perturbations

Until now, we have neglected the metric perturbations. When the wavelength of the perturbation is of order the horizon or smaller, this approximation is valid, as we will shortly see. In the process of seeing this, we will also find that, by the end of inflation, the metric perturbation has become important. So, although the inflation-induced perturbations start out all-"$\delta\phi$," they end up as a linear combination of $\Psi$ and $\delta\phi$ or more generally as a linear combination of $\Psi$ and perturbations to the energy–momentum tensor. The trick is to find the linear combination which is conserved outside the horizon. The value of this conserved linear combination is determined by $\delta\phi$ at horizon crossing; we can then evaluate it after inflation solely in terms of $\Psi$. The resulting equation will be of the form $\Psi \propto \delta\phi$ with the left-hand side the post-inflation metric perturbation and the right the scalar field perturbation produced during inflation (the power spectrum for which we have calculated above). We can then finally relate $P_\Psi$ (and the spectra of all other perturbation variables using the results of Section 6.1) to the $P_{\delta\phi}$ of Eq. (6.72).

Let's begin by rewriting the equation for conservation of energy, this time in the presence of the metric perturbation. It is straightforward to show that Eq. (6.62) gets generalized to

$$\frac{\partial \delta T^0{}_0}{\partial t} + ik_i\delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(\mathcal{P}+\rho)\frac{\partial \Psi}{\partial t} \tag{6.73}$$

where $\mathcal{P}$ and $\rho$ are the zero-order pressure and energy density. Were we correct to neglect $\Psi$ in the last section? We were, as long as the right-hand side is significantly smaller than the individual terms on the left. Taking the first term on the left as an example, we require

$$\Psi \ll \frac{\delta T^0{}_0}{\mathcal{P}+\rho}. \tag{6.74}$$

A simple way to see that this inequality holds is to use the Einstein equations we derived in Chapter 5. The most convenient for these purposes is the time-time (Eq. (5.21)) component:

$$k^2\Psi + 3aH(\dot\Psi + aH\Psi) = 4\pi G a^2 \delta T^0{}_0. \tag{6.75}$$

Here I have simply copied the results from Chapter 5, replacing $\Phi$ with $-\Psi$. The left-hand side here is of order $k^2\Psi \sim a^2H^2\Psi$ for modes which are crossing the horizon. Therefore,

$$\begin{aligned}\Psi &\sim G\frac{\delta T^0{}_0}{H^2} \sim \frac{\delta T^0{}_0}{\rho}\\ &= \frac{\mathcal{P}+\rho}{\rho}\left(\frac{\delta T^0{}_0}{\mathcal{P}+\rho}\right).\end{aligned} \tag{6.76}$$

The left-hand side must be much less than the term in parentheses; equivalently, the prefactor $(\mathcal{P}+\rho)/\rho$ must be small. In fact, during inflation, the pressure is almost equal to minus the energy density, so this prefactor is very small. In terms of the slow-roll parameters, it is equal to $2\epsilon/3$. So, at least in slow-roll models of inflation, we are justified in neglecting metric perturbations when computing the spectrum of $\delta\phi$.

The above argument holds only for modes that have not yet passed outside the horizon. Super-horizon modes, on the other hand, require more careful treatment. Indeed, it is inevitable that the inequality of Eq. (6.74) will break down sometime before the end of inflation. To see this, recall that after inflation, when the universe is dominated by radiation, $\delta T^0{}_0 = -4\rho_r\Theta_0$ and $\mathcal{P}+\rho = 4\rho_r/3$. Therefore, after inflation, the right-hand side of Eq. (6.74) is $-3\Theta_0$. According to Eq. (6.12), $\Psi = -2\Theta_0$ right after inflation, so it is certainly not true that the inequality of Eq. (6.74) is satisfied for all times. At some point before inflation ends, perturbations to $\Psi$ must grow in importance relative to those in the energy–momentum tensor.

One way to deal with the coupling between the metric perturbations and those to the energy density is to define

$$\zeta \equiv -\frac{ik_i\delta T^0{}_i H}{k^2(\rho+\mathcal{P})} - \Psi. \tag{6.77}$$

For sub-horizon modes and those which have just left the horizon, $\Psi$ is negligible; $P+\rho = (\dot\phi^{(0)}/a)^2$ from Eqs. (6.26) and (6.27); and Eq. (6.64) fixes the numerator of the first term in $\zeta$. We are left with

$$\zeta = -aH\delta\phi/\dot\phi^{(0)} \tag{6.78}$$

around the time of horizon crossing. After inflation ends, $ik_i\delta T^0{}_i = 4ak\rho_r\Theta_1$, proportional to the dipole of the radiation. Since the pressure of radiation is equal to a third of the energy density,

$$\begin{aligned}\zeta &= -\frac{3aH\Theta_1}{k} - \Psi \\ &= -\frac{3}{2}\Psi \qquad \text{(post inflation)}.\end{aligned} \tag{6.79}$$

The second equality follows from the initial conditions relating the dipole to the potential (Eq. (6.16)).

The variable $\zeta$ is so important because it is conserved when the perturbation moves outside the horizon (Figure 6.8). We will show that $\zeta$ is conserved shortly, but first let's appreciate the importance of this conservation. Since we know that, after inflation, $\zeta = -3\Psi/2$, we can immediately relate $\Psi$ coming out of inflation to the $\delta\phi$ at horizon crossing,

$$\Psi\Big|_{\text{post inflation}} = \frac{2}{3}aH\frac{\delta\phi}{\dot\phi^{(0)}}\Big|_{\text{horizon crossing}}. \tag{6.80}$$

Equivalently, the post-inflation power spectrum of $\Psi$ is simply related to the horizon-crossing spectrum of $\delta\phi$:

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 P_{\delta\phi}\Big|_{aH=k}$$

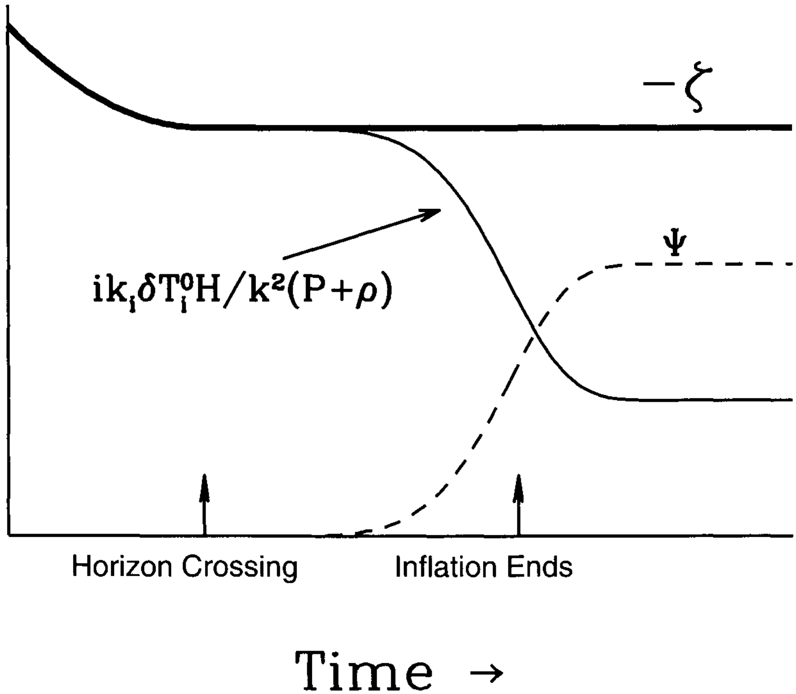


**Figure 6.8.** Cartoon view of the evolution of scalar, adiabatic perturbations during inflation in conformal Newtonian gauge. When a mode is sub-horizon, quantum-mechanical fluctuations are set up in the scalar field driving inflation $(ik_i\delta T^0{}_i H/k^2(\rho+\mathcal{P}) = aH\delta\phi/\dot{\phi}^{(0)})$. Scalar perturbations to the metric are negligible at this time. Once the mode leaves the horizon, the linear combination $\zeta = -ik_i\delta T^0{}_i H/k^2(\rho+\mathcal{P}) - \Psi$ is conserved. Well after inflation has ended, the metric perturbation has grown in importance, but the linear combination $\zeta$ remains unchanged.

$$= \frac{2}{9k^3}\left(\frac{aH^2}{\dot{\phi}^{(0)}}\right)^2\Bigg|_{aH=k}, \tag{6.81}$$

the second line following from Eq. (6.72). Another way to express the power spectrum of scalar perturbations is to eliminate $\dot{\phi}^{(0)}$ in favor of the slow-roll parameter $\epsilon$. You will show (Exercise 12) that $(aH/\dot{\phi}^{(0)})^2 = 4\pi G/\epsilon$, so

$$P_\Psi = P_\Phi(k) = \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon}\Bigg|_{aH=k}. \tag{6.82}$$

The first equality here follows from our ubiquitous assumption that anisotropic strsses are small, so that $\Psi = -\Phi$. Comparing to Eq. (6.59), we see that the ratio of scalar to tensor modes is of order $1/\epsilon$; that is, we expect scalar modes to dominate. Finally, another way of writing the scalar power spectrum is to eliminate $\epsilon$ in favor of the potential and its derivative, using the result of Exercise 14,

$$P_\Phi(k) = \frac{128\pi^2 G^2}{9k^3}\left(\frac{H^2V^2}{V'^2}\right)\Bigg|_{aH=k}. \tag{6.83}$$

It remains to prove that $\zeta$ is conserved on super-horizon scales. To see this, let's turn to the conservation equation, Eq. (6.73). On large scales, $k_i\delta T^i{}_0$ is proportional to $k^2$ and so can be ignored, leaving

$$\frac{\partial \delta T^0{}_0}{\partial t} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(\mathcal{P}+\rho)\frac{\partial \Psi}{\partial t}. \tag{6.84}$$

On large scales, you will show (Exercise 13) that the energy-momentum tensor satisfies

$$\frac{ik_i\delta T^0{}_i H}{k^2} = \frac{\delta T^0{}_0}{3}. \tag{6.85}$$

Therefore, on large scales

$$\zeta = -\Psi - \frac{1}{3}\frac{\delta T^0{}_0}{\rho+\mathcal{P}}. \tag{6.86}$$

Eliminating $\Psi$ in favor of $\zeta$ in the conservation equation leads to

$$\frac{\partial \delta T^0{}_0}{\partial t} + 3H\delta T^0{}_0 - H\delta T^i{}_i = 3(\mathcal{P}+\rho)\frac{\partial \zeta}{\partial t} + (\rho+\mathcal{P})\frac{\partial}{\partial t}\left[\frac{\delta T^0{}_0}{\rho+\mathcal{P}}\right]. \tag{6.87}$$

The partial derivative on the right acting on $\delta T^0{}_0$ cancels the first term on the left, leaving

$$\delta T^0{}_0\left[3H + \frac{1}{\rho+\mathcal{P}}\left(\frac{d\rho}{dt}+\frac{d\mathcal{P}}{dt}\right)\right] - H\delta T^i{}_i = 3(\mathcal{P}+\rho)\frac{\partial\zeta}{\partial t}. \tag{6.88}$$

Recall from Eq. (2.55) that $d\rho/dt = -3H(\rho+\mathcal{P})$, so the first term in brackets cancels the second, and

$$\frac{\partial\zeta}{\partial t} = \frac{-1}{3(\rho+\mathcal{P})^2}\left[H(\rho+\mathcal{P})\delta T^i{}_i - \delta T^0{}_0\frac{d\mathcal{P}}{dt}\right]. \tag{6.89}$$

I claim that the two terms in brackets on the right cancel for the class of perturbations we are considering. To see this, first rewrite $H(\rho+\mathcal{P})$ as $-(1/3)d\rho/dt$. Thus, the terms in brackets are proportional to

$$\frac{\delta T^i{}_i}{3} + \frac{d\mathcal{P}}{d\rho}\delta T^0{}_0 = \delta\mathcal{P} - \frac{d\mathcal{P}}{d\rho}\delta\rho \tag{6.90}$$

since $-\delta T^0{}_0$ is the perturbation to the energy density, while $\delta T^i{}_i/3$ is the perturbation to the pressure. If we know the background pressure–energy density relation $d\mathcal{P}/d\rho$, then given an overdensity $\delta\rho$, we expect the pressure perturbation to be proportional to the overdensity with coefficient $d\mathcal{P}/d\rho$. Indeed, this is the characteristic feature of adiabatic perturbations, precisely those set up during inflation. Thus, $\zeta$ is indeed conserved on large scales.

### 6.5.3 Spatially Flat Slicing

The treatment of the previous subsection is complete, but it is not the most elegant way to understand scalar perturbations in inflation. A much simpler way is to move back and forth between different gauges, making use along the way of the concept of a gauge-invariant variable, one which does not change under these transformations. Here I outline this method, leaving some of the more detailed calculations as problems.

We saw earlier that one of the major complications in conformal Newtonian gauge was that perturbations to the scalar field $\delta\phi$ are coupled to the potential $\Psi$. It would obviously be nice to transform to a gauge in which these perturbations decoupled. Consider a gauge with *spatially flat slicing*, with the spatial part of the metric $g_{ij} = \delta_{ij}a^2$. In this gauge the line element is

$$ds^2 = -(1+2A)dt^2 - 2aB_{,i}dx^i dt + \delta_{ij}a^2 dx^i dx^j, \tag{6.91}$$

i.e., there are two functions $A$ and $B$ characterizing the perturbations. In this case, the equation for $\delta\phi$ is given exactly (Exercise 16) by Eq. (6.71): the perturbations in the scalar field do not couple to those in the gravitational metric. Therefore, without having to neglect any couplings, we can identify the power spectrum for $\delta\phi$ as given by Eq. (6.72).

The next step is to identify a gauge-invariant variable, one which remains the same when transforming from one gauge to the next. Bardeen (1980) identified several such variables, two characterizing scalar perturbations to the metric and two characterizing perturbations to the matter. Of course any linear combination of these is still gauge invariant. We would like to identify the combination that is proportional to $\delta\phi$ in the gauge with spatially flat slicing. In this gauge, Bardeen's velocity (Eq. (5.78)) is

$$v = ikB - \frac{ik\dot{\phi}^{(0)}\delta\phi}{(\rho+\mathcal{P})a^2} \qquad \text{(spatially flat slicing)} \tag{6.92}$$

where I have evaluated $\delta T^0{}_i$ with Eq. (6.64). Thus, we can create a gauge-invariant variable proportional to $\delta\phi$ in a spatially flat slicing if we subtract off the $kB$ term. Bardeen's $\Phi_H$ (Eq. (5.76)) is simply equal to $aHB$, so the combination

$$\zeta \equiv -\Phi_H - \frac{iaH}{k}v \tag{6.93}$$

is gauge invariant and in spatially flat slicing is equal to

$$\zeta = -\frac{aH}{\dot{\phi}^{(0)}}\delta\phi \qquad \text{(spatially flat slicing).} \tag{6.94}$$

We can immediately relate the power in $\zeta$ to the power in $\delta\phi$,

$$P_\zeta = \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi}. \tag{6.95}$$

We know $P_{\delta\phi}$ from Eq. (6.72) and the prefactor is $4\pi G/\epsilon$, so

$$P_\zeta = \left.\frac{2\pi G H^2}{\epsilon k^3}\right|_{aH=k} . \tag{6.96}$$

Equation (6.96) is very useful, for it expresses the power spectrum of a gauge invariant quantity. Although we computed it in a gauge of the form in Eq. (6.91), once we have this answer, we can compute $\zeta$ in any gauge and then relate the power in the perturbation variables of that gauge to $P_\zeta$.

Throughout this book, we have been working in conformal Newtonian gauge. In this gauge, $\Phi_H = -\Phi$, so $\zeta$ as defined in Eq. (6.93) is indeed given by Eq. (6.77). We argued in Section 6.5.2 that in conformal Newtonian gauge, after inflation, $\zeta = 3\Phi/2$, so $P_\Phi = 4P_\zeta/9$, or using Eq. (6.96),

$$P_\Phi = \left.\frac{8\pi G H^2}{9\epsilon k^3}\right|_{aH=k} \tag{6.97}$$

in exact agreement with our earlier calculation finalized in Eq. (6.82).

This is the end of the calculation, but not quite the end of the story. Bardeen and others have argued that $\Phi_H$ has a nice geometrical interpretation, one shared by $\zeta$ in certain gauges. In particular, the curvature of the three-dimensional space at fixed time is equal to $4k^2\Phi_H/a^2$. Therefore, perturbations in $\Phi_H$ represent *curvature perturbations*: even though the zero-order space is flat, perturbations induce a curvature which varies from place to place. In conformal Newtonian gauge or in a spatially flat slicing this interpretation would seem irrelevant to perturbations in $\zeta$, since $\zeta$ is a combination of both $\Phi_H$ and the velocity. However, if one moves to a *comoving gauge*, one in which the velocities vanish, then $\zeta$ is equal to $\Phi_H$. In comoving gauges, then, it is clear that a perturbation to $\zeta$ is a curvature perturbation, and indeed the scalar perturbations generated during inflation are often called curvature perturbations.

## 6.6 SUMMARY AND SPECTRAL INDICES

In order to understand how scales which should be uncorrelated today are observed to have almost identical temperatures, we are virtually forced into the theory of inflation. In addition to explaining away the nagging fine-tuning problems of the standard cosmology, inflation is also a mechanism for generating primordial perturbations over the smooth universe.

Inflation predicts that quantum-mechanical perturbations in the very early universe are first produced when the relevant scales are causally connected. Then these scales are whisked outside the horizon by inflation, only to reenter much later to serve as initial conditions for the growth of structure and anisotropy in the universe. The perturbations are best described in terms of the Fourier modes. The mean of a given Fourier mode, for example for the gravitational potential, is zero:

$$\langle \Phi(\vec{k}) \rangle = 0. \tag{6.98}$$

Further, the perturbations to one Fourier mode are uncorrelated with those to another. However, a given mode has nonzero variance, so

$$\langle \Phi(\vec{k})\Phi^*(\vec{k}') \rangle = (2\pi)^3 P_\Phi(k) \delta^3(\vec{k} - \vec{k}'), \tag{6.99}$$

the Dirac delta function enforcing the independence of the different modes. In the case of scalar perturbations, the ones of most importance for us, the power spectrum is given by Eq. (6.82). Perturbations to the tensor part of the metric are also produced and are also Gaussian with mean zero; the power spectrum of tensor modes is given by Eq. (6.59). The scalar spectrum depends on the slow roll parameter $\epsilon$, defined in Eq. (6.35), which is proportional to the derivative of the Hubble rate. Since the Hubble rate is close to constant during inflation — because of the dominance of potential energy — $\epsilon$ is typically small.

A spectrum in which $k^3 P_\Phi(k)$ is constant (i.e., does not depend on $k$) is called a *scale-invariant* or *scale-free* spectrum. Apart from small deviations encoded in the slow-roll parameters, both the scalar and the tensor perturbations are scale free. This is both a blessing and a curse. It is good because it is a fairly definite prediction, easy to test. It is unfortunate because a scale-free spectrum is what one might have expected even without the complex machinery of inflation. Indeed, a scale-free spectrum is also referred to as a Harrison–Zel'dovich–Peebles spectrum, crediting the smart people who first proposed it as the appropriate distribution for the initial conditions, a proposal that predates inflation by many years. This really is too bad, because if we observe a scale-free spectrum, and most present observations are consistent with this, then inflation cannot fairly claim all the credit. However, if we observe a small mixture of tensor modes and/or a small deviation from a scale-free spectrum, then this will go a long way toward convincing skeptics that inflation is responsible for the primordial perturbations.

To quantify the deviations from scale invariance, it is conventional to write the primordial power spectra as

$$\begin{aligned} P_\Phi(k) &= \frac{8\pi}{9k^3} \frac{H^2}{\epsilon m_{\rm Pl}^2}\bigg|_{aH=k} \equiv \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0}\right)^{n-1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2 \\ P_h(k) &= \frac{8\pi}{k^3} \frac{H^2}{m_{\rm Pl}^2}\bigg|_{aH=k} \equiv A_T k^{n_T - 3}. \end{aligned} \tag{6.100}$$

These equations serve to define the scalar and tensor amplitudes, $\delta_H$ (subscript $_H$ for amplitude at horizon crossing) and $A_T$, and the spectral indices, $n$ and $n_T$. Note that this convention — which has become common — says that a scale-free scalar spectrum corresponds to $n = 1$, while $n_T = 0$ for a scale-free tensor spectrum. The factor of $\Omega_m / D_1(a = 1)$, where $\Omega_m$ is the fraction of the critical density in matter today and $D_1$ is the *growth function* which will be defined in Chapter 7 (Eqs. (7.4) and (7.77)), is part of this convention. It is inconvenient at this stage because we have not even encountered the growth function yet, but it has become standard to include in the definition of $\delta_H$ (Liddle and Lyth, 1993; Bunn and White, 1997). The resulting expression for the matter power spectrum today looks much simpler when these factors are included here. We pay the price of complexity now for the benefit of simplicity later.

We can relate the primordial spectral indices $n$ and $n_T$ to the slow-roll parameters $\epsilon$ and $\delta$. Consider first the tensor spectrum. By virtue of the definition in Eq. (6.100),

$$\frac{d\ln[P_h]}{d\ln k} = n_T - 3. \tag{6.101}$$

The logarithmic derivative has two terms, first the trivial one $d\ln(k^{-3})/d\ln(k)$ which cancels the $-3$ here, leaving $n_T = 2d\ln H/d\ln(k)$. The logarithmic derivative of the Hubble rate at horizon crossing is a bit subtle:

$$\left.\frac{d\ln H}{d\ln k}\right|_{aH=k} = \left.\frac{k}{H}\frac{dH}{d\eta} \times \frac{d\eta}{dk}\right|_{aH=k}. \tag{6.102}$$

By definition (Eq. (6.35)), $\dot{H} = -aH^2\epsilon$, and $d\eta|_{aH=k}/dk = -d(aH)^{-1}|_{aH=k}/dk = 1/k^2$, so

$$\left.\frac{d\ln H}{d\ln k}\right|_{aH=k} = -\left.\frac{k}{H}\frac{aH^2\epsilon}{k^2}\right|_{aH=k} = -\epsilon. \tag{6.103}$$

Therefore, the primordial spectral index of tensor perturbations produced by inflation is

$$n_T = -2\epsilon. \tag{6.104}$$

The scalar spectral index follows from a similar argument. Taking the logarithmic derivative of $P_\Phi$ leads to

$$n - 1 = \frac{d}{d\ln(k)}\left[\ln(H^2) - \ln(\epsilon)\right]. \tag{6.105}$$

The derivative of $H$ again gives $-2\epsilon$ while the logarithmic derivative of $\epsilon$ is $-2(\epsilon+\delta)$ (Exercise 12). So,

$$n = 1 - 4\epsilon - 2\delta. \tag{6.106}$$

The fact that the tensor index $n_T$ is proportional to $\epsilon$ leads to one of the robust predictions of inflation. Many inflationary models have been proposed which offer different predictions for $\epsilon$ and $\delta$. Almost all of these, however, maintain the feature that the ratio of tensor to scalar modes (which we saw earlier was proportional to $\epsilon$) is directly related to the tensor spectral index (here also seen to be directly proportional to $\epsilon$). As you progress through this book, moving from the evolution of anisotropies to their analyses, try to bear in mind the crucial question of whether this prediction can be put to the observational test.

The slow-roll parameters are a convenient way to summarize the predictions of an inflationary model. However, ultimately we are interested in the physics, so we are interested in how these parameters relate back to the fundamental entity, the potential $V$ of the scalar field responsible for inflation. You will show in Exercise 14 that these parameters can be expressed in terms of the potential and its derivatives. Therefore, extracting the values of $\epsilon$ and $\delta$ from the data is tantamount to probing the potential of the field driving inflation. Given that the expected scale of this potential is on the order of $10^{15}$ GeV (Exercise 18), this is quite an impressive probe!

## SUGGESTED READING

The 30 or so pages on inflation in this chapter, which were heavily slanted toward production of perturbations, offer but a glimpse into the many facets of this remarkable theory. Recently, Guth wrote a popular account of his discovery of inflation, *The Inflationary Universe*. One of the other originators of the theory, Linde, has a more technical book, *Inflation and Quantum Cosmology*, which emphasizes model building much more than I have here. As I mentioned earlier, *The Early Universe* (Kolb and Turner) has an excellent chapter on inflation. The recent *Cosmological Inflation and Large Scale Structure* (Liddle and Lyth) is most similar in spirit to this book, with a heavy emphasis on perturbations. The discussion there of the perturbation spectrum is laden with less algebra than the one in Section 6.5 so is worth reading. (Beware that their Planck mass is our $m_{\rm Pl}/\sqrt{8\pi}$.)

An extremely clear and deep look into inflation is given in *300 Years of Gravitation* (ed. Hawking and Israel) in the article by Blau and Guth. Many other articles in that thick compilation volume are also fascinating. The initial article by Guth (1981) is completely accessible and as clear a statement possible of the problems that led to inflation and the initial attempt (old inflation) to solve them. Indeed, I would recommend reading Guth's initial article because this chapter motivates inflation with the horizon problem, while Guth had several different problems in mind, including the monopole problem and the flatness problem (Exercise 4).

There have been many papers reviewing the production of perturbations during inflation. Two clear reviews are Lidsey *et al.* (1997) and Lyth and Riotto (1999). The former focuses on methods for going beyond the predictions elucidated here, which are accurate only to first order in the slow-roll parameters $\epsilon$ and $\delta$, and on extracting the potential $V$ from observations. The latter summarizes efforts to tie inflation to realistic particle physics models. The eight-page paper of Stewart and Lyth (1993) is a remarkably concise treatment of the techniques used to go beyond the first-order slow-roll approximation. Hollands and Wald (2002) have written a thoughtful critique of inflation, which is a refreshing antidote to some of the euphoria emanating from the discoveries of the late 1990s. Besides the importance of this critique in its own right, the paper has one of the clearest qualitative descriptions of perturbation generation during inflation that I have ever read.

The initial conditions relating the various perturbations described in Section 6.1 are perhaps most clearly discussed in the review article by Efstathiou (1990). Isocurvature perturbations, for the most part ignored here, are treated in detail there.

I have ignored the possibility of perturbations produced by topological defects. These theories, while fascinating, have not succeeded in making robust predictions; to the extent that predictions can be extracted from them, they are wrong. Nonetheless the numerics involved in their study is sufficiently complicated that I would not be shocked to see them make a comeback some day. There exist many books with comprehensive discussions of topological defects. Among them are *Cosmic Strings and Other Topological Defects* (Vilenkin and Shellard) and *The Formation and Evolution of Cosmic Strings* (ed. Gibbons, Hawking, and Vachaspati).

## EXERCISES

**Exercise 1.** Find the ratio of neutrino to radiation energy density, $f_\nu$. Assume that there are three species of massless neutrinos.

**Exercise 2.** Account for the neutrino quadrupole moment when setting up initial conditions.
**(a)** Start with Eq. (4.107). This is an equation for $\mathcal{N}(\mu)$. Turn this into a hierarchy of equations for the neutrino moments:

$$\begin{aligned} \dot{\mathcal{N}}_0 + k\mathcal{N}_1 &= -\dot{\Phi} \\ \dot{\mathcal{N}}_1 - \frac{k}{3}\left(\mathcal{N}_0 - 2\mathcal{N}_2\right) &= \frac{k}{3}\Psi \\ \dot{\mathcal{N}}_2 - \frac{2}{5}k\mathcal{N}_1 &= 0. \end{aligned} \tag{6.107}$$

To do this, you need to recall the definition of these moments, which is equivalent to that for photons, Eq. (4.99). A good way to reduce Eq. (4.107) into this hierarchy is to multiply it first by $\mathcal{P}_0$ and then integrate over $\int_{-1}^{1} d\mu$. This leads to the first equation above. Then multiply Eq. (4.107) by $\mathcal{P}_1$ to get the second and $\mathcal{P}_2$ to get the third. More details are given in Section 8.3, where we go through the same exercise for the photon moments. In the third equation you may neglect $\mathcal{N}_3$ because it is smaller than $\mathcal{N}_2$ by a factor of order $k\eta$ (prove this!).
**(b)** Eliminate $\mathcal{N}_1$ from these equations and show that

$$\ddot{\mathcal{N}}_2 = \frac{2k^2}{15}\left(\Psi + \mathcal{N}_0 - 2\mathcal{N}_2\right). \tag{6.108}$$

Drop $\mathcal{N}_2$ on the right-hand side because it is much smaller than $\Psi + \mathcal{N}_0$.
**(c)** Rewrite Einstein's equation (5.33) as

$$\mathcal{N}_2 = -(k\eta)^2 \frac{\Phi + \Psi}{12 f_\nu}. \tag{6.109}$$

This neglects the photon quadrupole. Argue that Compton scattering sets $\Theta_2 \ll \mathcal{N}_2$ so this is a reasonable assumption.
**(d)** Now differentiate this form of Einstein's equation twice to get an expression for $\ddot{\mathcal{N}}_2$. Equate this to the expression for $\ddot{\mathcal{N}}_2$ derived in part (b). (You may drop all derivatives of $\Phi$ and $\Psi$ when doing this since the mode of interest is the $p = 0$ constant mode.) Use this equation to express $\mathcal{N}_0$ in terms of $\Phi$ and $\Psi$.
**(e)** Finally assume that $\Theta_0 = \mathcal{N}_0$ and use your expression for $\mathcal{N}_0$ to rewrite Eq. (6.12) as a relation between the two gravitational potentials. Show that this relation is

$$\Phi = -\Psi\left(1 + \frac{2f_\nu}{5}\right). \tag{6.110}$$

**Exercise 3.** Show that the initial conditions for the velocities and dipoles of matter and radiation are as given in Eq. (6.16).

**Exercise 4.** Inflation also solves the *flatness* problem. This is the question of why the energy density today is so close to critical.
**(a)** Suppose that

$$\Omega(t) \equiv \frac{8\pi G\rho(t)}{3H^2(t)} \tag{6.111}$$

is equal to 0.3 today, where $\rho$ counts the energy density in matter and radiation (assume zero cosmological constant). From Eq. (1.2), plot $\Omega(t) - 1$ as a function of the scale factor. How close to one would $\Omega(t)$ have been back at the Planck epoch (assuming no inflation took place so that the scale factor at the Planck epoch was of order $10^{-32}$)? This fine-tuning of the initial conditions is the flatness problem. If not for the fine tuning, an open universe would be *obviously* open (i.e., $\Omega$ would be almost exactly zero) today.
**(b)** Now show that inflation solve the flatness problem. Extrapolate $\Omega(t) - 1$ back to the end of inflation, and then through 60 e-folds of inflation. What is $\Omega(t) - 1$ right before these 60 e-folds of inflation?

**Exercise 5.** Another way of looking at the problems that inflation solves is to consider the entropy within our Hubble volume. This is proportional to the total number of particles in the volume, with a proportionality constant of order unity. How many photons are there within our Hubble volume today? Explain how inflation produces entropy this large.

**Exercise 6.** We showed that, if the universe was always dominated by ordinary matter or radiation early on, then the comoving horizon when the scale factor was $a_e$ (very small) was $a_0H_0/a_eH_e$ times the comoving Hubble radius today. Compute this ratio assuming that the temperature was equal to $10^{15}$ GeV at $a_e$. Account for the radiation-to-matter transition at $a \sim 10^{-4}$.

**Exercise 7.** Consider a free, homogeneous scalar field with mass $m$. The potential for this field is $V = m^2\phi^2/2$. Show that, if $m \gg H$, the scalar field oscillates with frequency equal to its mass. Also show that its energy density falls off as $a^{-3}$, so it behaves exactly like ordinary nonrelativistic matter.

**Exercise 8.** Show that Eq. (6.33) follows from Eq. (6.32) by changing variables from $t$ to $\eta$.

**Exercise 9.** Compute some well-known properties of the quantized harmonic oscillator.
**(a)** The momentum of the harmonic oscillator with unit mass is $p = dx/dt$. Compute

$$[\hat{x}, \hat{p}]$$

and show that it is equal to $i$. You can obtain the operator $\hat{p}$ by differentiating $\hat{x}$ (Eq. (6.38)) with respect to time.
**(b)** Compute the zero-point energy of the harmonic oscillator with unit mass. Do this by quantizing the energy

$$E = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}$$

and then computing its expectation value in the ground state: $\langle 0|\hat{E}|0\rangle$.

**Exercise 10.** Show that gravity waves are not sourced by the scalar field during inflation. To do this, recall that the right-hand side of Eq. (6.45) is

$$\delta T^1{}_1 - \delta T^2{}_2$$

where $\delta T$ is the perturbation to the energy–momentum tensor (assumed to be dominated by $\phi$) and, as in the derivation of Eq. (5.63), I have chosen $\vec{k}$ to be in the $\hat{z}$ direction. Show that this right-hand side is indeed zero for the scalar field.

**Exercise 11.** Show that Eq. (6.57) is the appropriate solution to Eq. (6.56).
**(a)** Define $\tilde{v} = v/\eta$ and rewrite Eq. (6.56) in terms of $\tilde{v}$.
**(b)** The resulting equation is the spherical Bessel equation. Write down the general solution to this as a linear combination of two functions of $k\eta$.
**(c)** Use the boundary conditions of Eq. (6.58) to determine the coefficients of part (b). Show that Eq. (6.57) is the correct solution for these boundary conditions.

**Exercise 12.** Derive some useful identities involving the slow-roll parameters during inflation.
**(a)** Show that

$$\frac{d}{d\eta}\left(\frac{1}{aH}\right) = \epsilon - 1.$$

**(b)** Show that

$$4\pi G(\dot{\phi}^{(0)})^2 = \epsilon a^2 H^2. \tag{6.112}$$

**(c)** Using the definitions of $\epsilon$ and $\delta$, show that

$$\frac{d\epsilon}{d\eta} = -2aH\epsilon(\epsilon + \delta). \tag{6.113}$$

Use this to show that $d\ln\epsilon|_{aH=k}/d\ln(k) = -2(\epsilon + \delta)$.

**Exercise 13.** Show that on large scales Eq. (6.85) holds. One way to do this is to combine Einstein's equations, the time-time (5.27) and time-space (Exercise 5 of Chapter 5) components, and take the large-scale limit.

**Exercise 14.** Express the slow-roll parameters $\epsilon$ and $\eta$ in terms of the potential $V$ and its derivatives with respect to $\phi$. Show that, to lowest order,

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$$

and

$$\delta = \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

where primes denote derivatives with respect to $\phi^{(0)}$.

**Exercise 15.** There are a number of ways of describing pressure in the universe and of relating the pressure to the energy density. One was introduced back in Chapter 2, the *equation of state*,

$$w \equiv \frac{\mathcal{P}}{\rho}. \tag{6.114}$$

The second is the *sound speed*,

$$c_s^2 \equiv \frac{d\mathcal{P}}{d\rho}. \tag{6.115}$$

The way to compute $c_s^2$ is to differentiate both $\mathcal{P}$ and $\rho$ with respect to time and take the ratio. Finally, there is the ratio of perturbations in the energy density to those in the pressure,

$$\frac{\delta\mathcal{P}}{\delta\rho} = \frac{-3\delta T^0{}_0}{\delta T^i{}_i}, \tag{6.116}$$

where the minus sign accounts for the fact the the time-time component of the energy–momentum tensor is minus the energy density with our convention, and the factor of 3 negates the sum over the three spatial indices. For adiabatic perturbations, $\delta\mathcal{P}/\delta\rho = c_s^2$. Show that this holds for three separate cases: matter, radiation, and a single scalar field during inflation at the time of horizon crossing. For the last case, it is enough to show that the difference $\delta\mathcal{P}/\delta\rho - c_s^2$ is of order the slow-roll parameters $\epsilon$ and $\delta$.

**Exercise 16.** Show that in a gauge given by Eq. (6.91), the equation governing the perturbations to a scalar field $\delta\phi$ is Eq. (6.71).

**(a)** Bardeen's equation for the gauge-invariant density in the absence of anisotropic stress is

$$\frac{d}{d\eta}\left(a^3\rho\epsilon_m\right) = -(\rho+\mathcal{P})a^3kv \tag{6.117}$$

with gauge-invariant density defined as

$$\rho\epsilon_m = -\rho - \delta T^0{}_0 + \frac{3iH}{k^2}k_i\delta T^0{}_i \tag{6.118}$$

and velocity $v$ via Eq. (6.92). Compute $\rho\epsilon_m$ for a scalar field in a gauge with spatially flat slicing. Show that, to lowest order in slow-roll parameters $\epsilon$ (not $\epsilon_m$) and $\delta$, the equation reduces to

$$\frac{d}{d\eta}\left(a^3\dot{\phi}^{(0)}\delta\phi\right) = -k^2 a\dot{\phi}^{(0)}\delta\phi. \tag{6.119}$$

**(b)** Again using the slow-roll approximation reduce Eq. (6.119) to the form of Eq. (6.45).

**Exercise 17.** Show that the curvature in conformal Newtonian gauge is equal to $4k^2\Phi/a^2$. To do this, compute the three-dimensional Ricci scalar arising from the spatial part of the metric $g_{ij} = \delta_{ij}a^2(1+2\Phi)$.

**Exercise 18.** Determine the predictions of an inflationary model with a quartic potential,

$$V(\phi) = \lambda\phi^4.$$

**(a)** Compute the slow roll parameters $\epsilon$ and $\delta$ in terms of $\phi$.
**(b)** Determine $\phi_e$, the value of the field at which inflation ends, by setting $\epsilon = 1$ at the end of inflation.
**(c)** To determine the spectrum, you will need to evaluate $\epsilon$ and $\delta$ at $-k\eta = 1$. Choose the wavenumber $k$ to be equal to $a_0 H_0$, roughly the horizon today. Show that the requirement $-k\eta = 1$ then corresponds to

$$e^{60} = \int_0^N dN' \frac{e^{N'}}{(H(N')/H_e)}$$

where $H_e$ is the Hubble rate at the end of inflation, and $N$ is defined to be the number of e-folds before the end of inflation:

$$N \equiv \ln\left(\frac{a_e}{a}\right).$$

**(d)** Take the Hubble rate to be a constant in the above with $H/H_e$ equal to 1. This implies that $N \simeq 60$. Turn this into an expression for $\phi$. The simplest way to do this is to note that $N = \int_t^{T_e} dt' H(t')$ and assume that $H$ is dominated by potential energy. Show that this mode leaves the horizon when $\phi^2 = 60m_{\rm Pl}^2/\pi$.
**(e)** Determine the predicted values of $n$ and $n_T$.
**(f)** Estimate the scalar amplitude in terms of $\lambda$. As a rough estimate, assume that $k^3 P_\Phi(k)$ for this mode is equal to $10^{-8}$ (we will find a more precise value when we normalize to large-angle anisotropies in Chapter 8). What value does this imply for $\lambda$?
This model illustrates many of the features of contemporary models. In it, (i) the field is of order — even greater than — the Planck scale, but (ii) the energy scale $V$ is much smaller because of (iii) the very small coupling constant.