APPENDIX A

SOLUTIONS TO SELECTED PROBLEMS

The problems at the end of each chapter have a broad range of difficulty. Some are simply repeating calculations in the text in a slightly different context; others are fairly elementary applications of basic formulae; while some are quite difficult, culled from recent papers. Here are some selected solutions. The solutions are heavily weighted to the first several chapters, especially Chapter 2, because it is important to be comfortable with the background cosmology before proceeding to tackle perturbations.

CHAPTER 1

Exercise 1 The ratio

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = (\rho_\Lambda/\rho_c)_0 \left(\frac{H_0}{H}\right)^2 \tag{A.1}$$

where subscript 0 means evaluate today, where it is assumed to be 0.7. Again, by assumption, the universe is forever radiation dominated (clearly not true today, but a good approximation early on), so $H/H_0 = a^{-2}$. The temperature also scales as a^{-1}, so $H/H_0 = (T/T_0)^2$ with $T_0 = 2.7\text{K} = 2.3 \times 10^{-4}$ eV. So,

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = 0.7 \left(\frac{T_0}{T}\right)^4. \tag{A.2}$$

At the Planck scale, $T_0/T = 2.3 \times 10^{-4}/1.22 \times 10^{28}$, so

$$\frac{\rho_\Lambda}{3H^2/(8\pi G)} = 9 \times 10^{-128}. \tag{A.3}$$

This is the so-called fine-tuning problem: for the cosmological constant to be important today, it had to have been fine-tuned to an absurdly small value at early times. It's a deep problem.

Exercise 2 We need to do the integral

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a} \left[\Omega_\Lambda + \frac{1 - \Omega_\Lambda}{a^3} \right]^{-1/2} \tag{A.4}$$

for $\Omega_\Lambda = 0.7$ and 0. The latter case can be done analytically:

$$\int_0^1 \frac{da}{a} a^{3/2} = \frac{2}{3}. \tag{A.5}$$

So $t_0 = 2/3H_0 = 0.67 \times 10^{10} h^{-1}$ yrs. When Ω_Λ is not zero, the integral needs to be done numerically. I find

$$\int_0^1 \frac{da}{a} \left[0.7 + \frac{0.3}{a^3} \right]^{-1/2} = 0.96. \tag{A.6}$$

So for fixed Hubble constant, a cosmological constant universe is older than a matter-dominated one, older by a factor of $0.96/0.67 = 1.43$. For $h = 0.7$, a cosmological constant universe has an age of 14 billion years, in accord with other observations of the age of the universe.

Exercise 4 An inverse wavelength is ν/c, so replacing ν everywhere in Eq. (1.8) by c/λ leads to

$$I_\nu = \frac{4\pi\hbar c}{\lambda^3} \frac{1}{\exp\{2\pi\hbar c/\lambda k_B T\} - 1}. \tag{A.7}$$

This is energy per Hz; we want energy per cm^{-1}, so we need to multiply by c, leaving

$$I_{1/\lambda} = \frac{4\pi\hbar c^2}{\lambda^3} \frac{1}{\exp\{2\pi\hbar c/\lambda k_B T\} - 1}. \tag{A.8}$$

Plugging in numbers leads to

$$I_{1/\lambda} = 1.2 \times 10^{-5} \text{erg sec}^{-1} \text{ cm}^{-1} \text{sr}^{-1} \left(\frac{\text{cm}}{\lambda} \right)^3 \frac{1}{\exp\{0.53\text{cm}/\lambda\} - 1}. \tag{A.9}$$

A quick check verifies that this agrees with Figure 1.10.

To find the peak, differentiate I with respect to $1/\lambda$ and set equal to zero. This leaves

$$\lambda = \frac{1}{3} \frac{(2\pi\hbar c/k_B T)}{1 - \exp\{-2\pi\hbar c/\lambda k_B T\}}. \tag{A.10}$$

So $1/\lambda_{\text{peak}}$ is $3/.53\text{cm}^{-1}$. The exact coefficient, accounting for the exponential is 2.82, so $1/\lambda_{\text{peak}} = 5.3\text{cm}^{-1}$, exactly where it occurs in Figure 1.10.

CHAPTER 2

Exercise 1

(a) To get from kelvin to eV, use $k_B = \text{eV}/(11605K)$. So $2.725\text{K} \to k_B 2.725\text{K} = (2.725/11605)$ eV. Or 2.348×10^{-4} eV.

(b) Since $T_0 = 2.348 \times 10^{-4}$ eV,

$$\rho_\gamma = \frac{\pi^2 T_0^4}{15} = 2.000 \times 10^{-15} \mathrm{eV}^4. \tag{A.11}$$

To get this in g cm^{-3}, first divide by $(\hbar c)^3 = (1.973 \times 10^{-5}\,\mathrm{eV\ cm})^3$ to get $0.2604\,\mathrm{eV\ cm}^{-3}$. Then to change from eV to grams, remember that the mass of the proton is either 1.673×10^{-24} g or 0.9383×10^9 eV, so 1 eV $= 1.783 \times 10^{-33}$ g. Therefore, $\rho_\gamma = 4.643 \times 10^{-34}$ g cm^{-3}.

(c) We have parametrized $H_0 = 100\,h\,\mathrm{km\ sec}^{-1}\ \mathrm{Mpc}^{-1}$, or using the fact that one Mpc is equal to 3.1×10^{19} km, $H_0 = 3.23\,h \times 10^{-18}\,\mathrm{sec}^{-1}$. To get this into inverse cm, divide by the speed of light, $c = 3 \times 10^{10}$ cm sec $^{-1}$; then $H_0 = 1.1\,h \times 10^{-28}$ cm. Or $H_0^{-1} = 9.3\,h^{-1} \times 10^{27}$ cm.

(d) To get the Planck mass (1.2×10^{28} eV) into kelvins, multiply by $k_B^{-1} =$ 11605K/eV; then $m_{\mathrm{Pl}} = 1.4 \times 10^{32}\,\mathrm{K}$. To get it into inverse cm, divide by $\hbar c =$ 1.97×10^{-5} eV cm to get $m_{\mathrm{Pl}} = 6.1 \times 10^{32}\ \mathrm{cm}^{-1}$. To get this is units of time, multiply by the speed of light to get $m_{\mathrm{Pl}} = 6.1 \times 10^{32} \times 3 \times 10^{10}$ cm sec^{-1}, or $m_{\mathrm{Pl}} = 1.8 \times 10^{43}\ \mathrm{sec}^{-1}$.

Exercise 7

Start with

$$\Gamma^0_{\mu\nu} = \frac{g^{0\alpha}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right] \tag{A.12}$$

where μ, ν range from 0 t0 2, 0 being the time index, 1 corresponding to θ, and 2 to ϕ. Since the metric is diagonal, $g^{0\alpha}$ is nonzero only when $\alpha = 0$ in which case it is -1. So

$$\Gamma^0_{\mu\nu} = \frac{-1}{2}\left[\frac{\partial g_{0\mu}}{\partial x^\nu} + \frac{\partial g_{0\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial t}\right]. \tag{A.13}$$

All of these terms vanish: the first two since g_{00} is a constant, and the last because none of the metric elements depend on $x^0 = t$. So $\Gamma^0_{\mu\nu} = 0$ for all μ, ν.

Next consider

$$\Gamma^\theta_{\mu\nu} = \frac{g^{\theta\alpha}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right]. \tag{A.14}$$

Again since the metric is diagonal, and $g^{\theta\theta} = 1/r^2$, this reduces to

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2r^2}\left[\frac{\partial g_{\theta\mu}}{\partial x^\nu} + \frac{\partial g_{\theta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial \theta}\right]. \tag{A.15}$$

Only the $g_{\phi\phi}$ component depends on one of our variables, so only it is nonzero when differentiated. Therefore, the first two terms vanish and the last is nonzero only when $\mu = \nu = \phi$, in which case it is

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2r^2}\left[-r^2\frac{\partial \sin^2\theta}{\partial\theta}\right] = -\sin\theta\cos\theta. \tag{A.16}$$

Finally, when the upper index is ϕ, we have

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2r^2 \sin^{\theta}} \left[\frac{\partial g_{\phi\mu}}{\partial x^{\nu}} + \frac{\partial g_{\phi\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial \phi} \right]. \tag{A.17}$$

The last term vanishes since none of the metric elements depend on ϕ; the first two are nonzero only if one of the indices μ, ν is equal to ϕ and the other is θ, so

$$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}. \tag{A.18}$$

The geodesic equation is

$$\frac{d^2 x^{\mu}}{d\lambda^2} = -\Gamma^{\mu}_{\alpha\beta} P^{\alpha} P^{\beta} \tag{A.19}$$

with

$$P^{\mu} \equiv \frac{dx^{\mu}}{d\lambda}. \tag{A.20}$$

Let's apply this to the $\mu = \theta$ component. The left-hand side is

$$\frac{d^2\theta}{d\lambda^2} = \frac{d}{\lambda} \frac{dt}{d\lambda} \dot{\theta} = E^2 \ddot{\theta} \tag{A.21}$$

since $E = dt/d\lambda$ is constant. The Christoffel symbol on the right-hand side $\Gamma^{\theta}_{\alpha\beta}$ is nonzero only when $\alpha = \beta = \phi$ in which case it is $-\sin\theta\cos\theta$. So,

$$\ddot{\theta} - \sin\theta\cos\theta\,(\dot{\phi})^2 = 0. \tag{A.22}$$

For the second equation, consider the ϕ component of the geodesic equation,

$$\frac{d^2\phi}{d\lambda^2} = -\Gamma^{\phi}_{\alpha\beta} P^{\alpha} P^{\beta}. \tag{A.23}$$

Again the left-hand side is simply $E^2\ddot{\phi}$. The right-hand side gets nonzero contributions when $\alpha = \theta, \beta = \phi$ or an identical term when $\alpha = \phi, \beta = \theta$. Therefore,

$$\ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} = 0. \tag{A.24}$$

Incidentally this is equivalent to

$$\frac{d}{dt}\left(\dot{\phi}\sin^2\theta\right) = 0 \tag{A.25}$$

and the conserved quantity in parentheses is the angular momentum.

The Ricci scalar is

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = -R_{00} + \frac{1}{r^2} R_{\theta\theta} + \frac{1}{r^2 \sin^2\theta} R_{\phi\phi}. \tag{A.26}$$

The time-time component vanishes since all Γ's with time components are zero. We need to compute the two spatial components. First, consider

$$R_{\theta\theta} = \frac{\partial \Gamma^{\alpha}{}_{\theta\theta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}{}_{\theta\alpha}}{\partial \theta} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\theta\theta} - \Gamma^{\alpha}{}_{\beta\theta}\Gamma^{\beta}{}_{\theta\alpha}. \tag{A.27}$$

The first and third terms vanish since the Christoffel symbol with two lower θ's vanishes. For the same reason, the index α in the second term must be equal to ϕ, and both β and α in the last term must equal ϕ:

$$R_{\theta\theta} = -\frac{\partial(\cos\theta/\sin\theta)}{\partial\theta} - \left(\frac{\cos\theta}{\sin\theta}\right)^2. \tag{A.28}$$

Carrying out the derivative then gives

$$R_{\theta\theta} = \left[1 + \frac{\cos^2\theta}{\sin^2\theta}\right] - \left(\frac{\cos\theta}{\sin\theta}\right)^2 = 1. \tag{A.29}$$

The other spatial component is

$$R_{\phi\phi} = \frac{\partial \Gamma^{\alpha}{}_{\phi\phi}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}{}_{\phi\alpha}}{\partial \phi} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\phi\phi} - \Gamma^{\alpha}{}_{\beta\phi}\Gamma^{\beta}{}_{\phi\alpha}. \tag{A.30}$$

The Christoffel symbol in the first term is nonzero only if $\alpha = \theta$, while the one in the second term is always zero. In the third term β must be equal to θ to make the second Christoffel symbol be nonzero, and then $\alpha = \phi$. In the last term β can be θ and $\alpha = \phi$ or vice versa, so

$$R_{\phi\phi} = \frac{\partial \Gamma^{\theta}{}_{\phi\phi}}{\partial \theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\theta}{}_{\phi\phi} - \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\theta}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\beta}{}_{\phi\theta}. \tag{A.31}$$

The middle two terms cancel leaving

$$R_{\phi\phi} = -\frac{\partial(\sin\theta\cos\theta)}{\partial\theta} + \sin\theta\cos\theta\frac{\cos\theta}{\sin\theta}. \tag{A.32}$$

Carrying out the derivative gives

$$R_{\phi\phi} = -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta. \tag{A.33}$$

Summing up, we get

$$\mathcal{R} = \frac{1}{2r^2}. \tag{A.34}$$

The Ricci scalar is therefore a measure of the curvature of the space.

Exercise 9 Accumulating the various Γ's leads to

$$\frac{d^2x^i}{d\lambda^2} = -2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dx^i}{d\lambda}. \tag{A.35}$$

Change to differentiation with respect to η using the facts that $dt/d\lambda = E$ and $d\eta/d\lambda = E/a$. Then the geodesic equation becomes

$$\frac{E}{a}\frac{d}{d\eta}\left(\frac{E}{a}\frac{dx^i}{d\eta}\right) = -2\frac{\dot{a}}{a}\frac{E^2}{a}\frac{dx^i}{d\eta}. \tag{A.36}$$

Since $E/a \propto a^{-2}$, when the derivative on the left acts on E/a, the resulting term (proportional to $dx^i/d\eta$) exactly cancels the term on the right, leaving the result of Eq. (2.99).

Exercise 10 The age integral is

$$t(a) = \int_0^a \frac{da'}{a'H(a')}. \tag{A.37}$$

Since we are assuming only matter and radiation, we can take

$$H(a) = H_0\sqrt{\rho/\rho_{\rm cr}} = \sqrt{\frac{1}{a^3} + \frac{\Omega_r}{a^4}} \tag{A.38}$$

where the $1/a^3$ term is from matter with density equal to the critical density. When the density in matter is equal to critical, $\Omega_r = a_{\rm eq} = 4.15 \times 10^{-5} h^{-2}$. Therefore, the age integral is

$$t = \frac{1}{H_0}\int_0^a \frac{da'a'}{\sqrt{a' + a_{\rm eq}}}. \tag{A.39}$$

Integrate by parts to get

$$H_0 t = 2a\sqrt{a + a_{\rm eq}} - 2\int_0^a da'\sqrt{a' + a_{\rm eq}}. \tag{A.40}$$

Carrying out the last integral leads to

$$H_0 t = 2a\sqrt{a + a_{\rm eq}} - \frac{4}{3}\left\{[a + a_{\rm eq}]^{3/2} - a_{\rm eq}^{3/2}\right\}. \tag{A.41}$$

At very early times, such as when the temperature was 0.1 MeV, a is much smaller than $a_{\rm eq}$, so

$$t \to \frac{a^2}{2H_0\sqrt{a_{\rm eq}}} \qquad ; \qquad a \ll a_{\rm eq}. \tag{A.42}$$

This limit is easiest to see directly in the integral of Eq. (A.39), but you can also get it by Taylor expanding Eq. (A.41). When the temperature is 0.1 MeV, the scale factor is $2.35 \times 10^{-4}\,{\rm eV}/0.1\,{\rm MeV} = 2.35 \times 10^{-9}$, the temperature today divided by 0.1 MeV. Plugging in numbers leads to

$$t\,(0.1\,{\rm MeV}) = 4.28 \times 10^{-16} \times 9.78 \times 10^9\,{\rm yr} = 130\,{\rm sec}. \tag{A.43}$$

At $T = 1/4$ eV, $a = 9.4 \times 10^{-4}$, significantly larger than $a_{\rm eq} = 8.5 \times 10^{-5}$ with $h = 0.7$, so $H_0 t \to (2/3)a^{3/2}$. So,

$$t\,(1/4\,{\rm eV}) = 270,000\,{\rm yr}. \tag{A.44}$$

Exercise 12 The angle subtended is the physical distance divided by the angular diameter distance

$$\theta(z) = \frac{5\text{ kpc}(1+z)}{\chi(z)}. \tag{A.45}$$

In a flat, matter-dominated universe, χ is given by Eq. (2.43). When $z = 0.1$ (1), the term in brackets in Eq. (2.43) is equal to 0.0465 (0.293). The comoving distance out to z is, therefore,

$$\chi = \begin{cases} 280h^{-1}\,\text{Mpc} & z = 0.1 \\ 1760h^{-1}\,\text{Mpc} & z = 1 \end{cases}. \tag{A.46}$$

Carrying out the division and converting radians to arcsec (1 radian equals 2.06×10^5 arcsec) leads to

$$\theta = \begin{cases} 4.0h'' & z = 0.1 \\ 1.2h'' & z = 1 \end{cases}. \tag{A.47}$$

In a universe with $\Omega_\Lambda = 0.7, \Omega_m = 0.3$, χ must be computed numerically. At $z = 1$, I find χ to be larger than in the flat, matter-dominated case by a factor of 1.3, so the angular size will be smaller by this factor, down to $0.9h''$. At $z = 0.1$ the difference in comoving distances is only 5%, so the angular size goes down to $3.8h''$ in the cosmological constant case.

Exercise 13 Rewriting Eq. (1.8) in terms of momentum $p = h\nu/c = 2\pi\hbar\nu/c$ and recognizing the denominator there as $1/f$ leads to

$$I_\nu = f\frac{4\pi p^3}{(2\pi)^3} \tag{A.48}$$

with $\hbar = c = 1$. So the energy density is the integral of this over all frequencies, with a factor of 4π to count photons from all directions (i.e., I_ν is per steradian):

$$\rho_\gamma = 4\pi\int_0^\infty d\nu I_\nu. \tag{A.49}$$

This can be converted into an integral over momentum, with $d\nu = dp/(2\pi)$:

$$\rho_\gamma = 2\int_0^\infty dp I_\nu. \tag{A.50}$$

Exercise 15 We want to compute $\rho = -T_0^0$. Setting $\mu = \nu = 0$ leads to

$$T^0{}_0 = -g_i\int\frac{dP_1dP_2dP_3}{(2\pi)^3}(-\det[g_{\mu\nu}])^{-1/2}P^0 f_i. \tag{A.51}$$

The matrix $g_{\mu\nu}$ is diagonal, so the determinant is simply the product of the diagonal elements, $-a^6$. By definition, $p^2 = g^{ij}P_iP_j = a^{-2}\delta_{ij}P_iP_j$. So $p_i \equiv \hat{p}_i p = P_i/a$ with $\hat{p}_i$ a unit vector pointing in the direction of the momentum. Therefore, $d^3P = a^3d^3p$ and the factors of a precisely cancel those coming from the determinant. We're left with

$$T_0^0 = -g_i\int\frac{d^3p}{(2\pi)^3}P^0 f_i. \tag{A.52}$$

The four vector P_μ squared is equal to $-m^2$, the mass of the particle, so $g_{00}(P^0)^2 = -m^2 - g_{ij}P^iP^j = -m^2 - p^2$. Since $g_{00} = -1$, $P^0 = \sqrt{p^2+m^2}$, in accord with Eq. (2.59).

Exercise 17 The energy density of a massless boson is $g\pi^2T^4/30$, while that of a fermion is 7/8 times this. So,

$$s = \frac{2\pi^2}{45}\left[\sum_{i=\text{bosons}} g_iT_i^3 + \frac{7}{8}\sum_{i=\text{fermions}} g_iT_i^3\right] \tag{A.53}$$

accounting for the possibility that different species have different temperatures.

CHAPTER 3

Exercise 1 The number density of a species with degeneracy $g = 2$ is

$$n = 2\int \frac{d^3p}{(2\pi)^3} f(p). \tag{A.54}$$

For the distributions we will consider, the phase space density f depends only on the magnitude of the momentum, so the angular part of the integral can be performed leading to the a factor of 4π; therefore,

$$n = \frac{1}{\pi^2}\int_0^\infty dp\; p^2 f(p). \tag{A.55}$$

First let's consider the high m/T limit. In this case, the limit of the Boltzmann distribution is $\exp[-(m + p^2/2m)/T]$. I claim, though, that this is precisely the limit of both the Fermi–Dirac and Bose–Einstein distributions:

$$\frac{1}{e^{E/T} \pm 1} \to e^{-E/T} \tag{A.56}$$

since $E \simeq m \gg T$ so that the exponential in the denominator dwarfs the 1. Therefore the low-temperature limit of all three distributions is

$$n^{\text{low T}} = \frac{e^{-m/T}}{\pi^2}\int_0^\infty dp\; p^2 e^{-p^2/2mT}. \tag{A.57}$$

To do the integral, define a dimensionless parameter $x \equiv p/\sqrt{2mT}$. In terms of the variable $dpp^2 = [2mT]^{3/2} dxx^2$, so

$$n^{\text{low T}} = \frac{e^{-m/T}}{\pi^2}[2mT]^{3/2}\int_0^\infty dx\; x^2 e^{-x^2}. \tag{A.58}$$

But the integral is equal to $\sqrt{\pi}/2$, so we have

$$n^{\text{low T}} = 2e^{-m/T}\left(\frac{mT}{2\pi}\right)^{3/2}. \tag{A.59}$$

The high-temperature Boltzmann limit is

$$n^{\text{Hi T, Boltz}} = \frac{1}{\pi^2}\int_0^\infty dp\; p^2 e^{-p/T}. \tag{A.60}$$

Defining the dummy variable $x \equiv p/T$ leads to

$$n^{\text{Hi T, Boltz}} = \frac{1}{\pi^2}T^3\int_0^\infty dx\; x^2 e^{-x}. \tag{A.61}$$

The x integral is equal to 2. So,

$$n^{\text{Hi T. Boltz}} = \frac{2T^3}{\pi^2}. \tag{A.62}$$

The Bose–Einstein and Fermi–Dirac integrals similarly are

$$n^{\text{Hi T, BE/FD}} = \frac{T^3}{\pi^2} \int_0^\infty \frac{dx\, x^2}{e^x \mp 1}. \tag{A.63}$$

The integrals can be written in terms of the Riemann zeta function, via Eq. (C.27). So the integral in Eq. (A.63) with the minus sign — the Bose–Einstein distribution — is $\zeta(3)\Gamma(3) = 2\zeta(3)$. The integral with the plus sign — the Fermi–Dirac distribution — is $3\zeta(3)\Gamma(3)/4 = 3\zeta(3)/2$, so

$$n^{\text{Hi T}} = \frac{\zeta(3)T^3}{\pi^2} \begin{cases} 2 & \text{Bose–Einstein} \\ 3/2 & \text{Fermi–Dirac} \end{cases}.$$

By the way, $\zeta(3) \simeq 1.202$, so there are more bosons than fermions for the same temperature, and these bracket the Boltzman amount. All of course are proportional to T^3.

Exercise 6 The photon number density is $411\,\text{cm}^{-3}$, while the baryon number density is $n_b = \rho_b/m_p = \rho_{\text{cr}}\Omega_b/m_p$. Plugging in numbers gives

$$n_b = \Omega_b \frac{1.879h^2 \times 10^{-29}\text{g cm}^{-3}}{1.673 \times 10^{-24}\text{g}} = 1.12 \times 10^{-5}\Omega_b h^2\,\text{cm}^{-3}. \tag{A.64}$$

So η_b, the ratio of the baryon to the photon number density, is indeed given by Eq. (3.11).

Exercise 11 To find this ratio, we compute the entropy density $(\mathcal{P} + \rho)/T$ at the two times. In both cases, only relativistic particles contribute to the entropy density significantly so that Eq. (A.53) holds. At high temperatures, the following particles contribute to the energy density: quarks ($g_* = 5 \times 3 \times 2$ for the five least massive types — up, down, strange, charm, bottom — each with three colors and two spin states); anti-quarks ($g_* = 30$ again); leptons ($g_* = 6 \times 2$ for the six types — $e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau$ — each with two spin states); anti-leptons ($g_* = 12$ again); photons (2); and gluons ($g_* = 8 \times 2$ for eight possible colors each with two spin states). This totals up to

$$g_* = 2 + 16 + \frac{7}{8}(30 + 30 + 12 + 12) = 91.5. \tag{A.65}$$

The sixth quark, the top quark, does not contribute because it is too heavy to be around at these temperatures $m_t \simeq 175$ GeV. Today entropy comes only from photons and neutrinos. The former contribute 2 to g_*; the latter contribute $(7/8) \times 3 \times 2 \times (4/11)^{4/3} = 1.36$, so today $g_* = 3.36$. Since the product sa^3 remains constant, we have

$$\left[g_*(aT)^3\right]\Big|_{T=10\text{ Gev}} = \left[g_*(aT)^3\right]\Big|_{T_0}. \tag{A.66}$$

Therefore,

$$\frac{(aT)^3\big|_{T=10\text{ Gev}}}{(a_0T_0)^3} = \frac{3.36}{91.5} = \frac{1}{27}. \tag{A.67}$$

CHAPTER 4

Exercise 1 First integrate Eq. (4.6) over all momentum. This gives

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0, \tag{A.68}$$

the $\partial f/\partial p$ term vanishing after integrating by parts and noticing that $f = 0$ at $p = \pm\infty$ (there are no particles with infinite momentum). This is the continuity equation. To get the Euler equation, first multiply by p/m and then integrate over all momentum. This gives

$$\frac{\partial (nv)}{\partial t} + \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\frac{p^2}{m^2} + \frac{kx}{m}n = 0 \tag{A.69}$$

where the last term follows from an integration by parts. The integral over p^2 yields two terms, one a *bulk velocity* term, v^2, and the second a pressure term, P. Using the continuity equation reduces this to

$$\dot{v} + v\frac{\partial v}{\partial x} + \frac{1}{n}\frac{\partial P}{\partial x} + \frac{kx}{m} = 0. \tag{A.70}$$

Exercise 4 From Eq. (3.3), the electron distribution function peaks at zero momentum, with a maximum value of $e^{(\mu - m_e)/T}$. To relate the chemical potential to the density, recall that $n = e^{\mu/T} n^{(0)}$, so in the low-temperature limit (Eq. (3.6)):

$$e^{\mu/T} = \frac{n_e}{2}\left(\frac{2\pi}{m_e T}\right)^{3/2} e^{m_e/T}. \tag{A.71}$$

So the maximum value of f_e is $(n_e/2)(2\pi/m_e T)^{3/2}$. Divide Eq. (3.44) by the Thomson cross-section to get $n_e = 1.12 \times 10^{-5}\Omega_B h^2 \mathrm{cm}^{-3}$ today including both ionized and captured electrons. Taking the electron temperature to be equal to the photon temperature today gives $2\pi/m_e T = 2.04 \times 10^{-11}\mathrm{cm}^2$. Putting back in the factors of a leads to

$$f_e^{\mathrm{MAX}} = 10^{-21}\Omega_B h^2 a^{-3/2}. \tag{A.72}$$

This expression holds only up to $T \leq m_e$, corresponding to $a \simeq 4.6 \times 10^{-10}$. So, as long as the temperature is well below the electron mass, f_e is very small.

Exercise 7 The difference between the amplitude we used in the derivation in Section 4.3 and the more accurate one given in the problem is $2\pi\sigma_T m_e^2[3\cos(\hat{p}\cdot\hat{p}') - 1]$. The combination is square brackets is twice the second Legendre polynomial. Rewrite using the addition formula of spherical harmonics; then the difference becomes

$$2\pi\sigma_T m_e^2 \frac{8\pi}{5}\sum_{m=-2}^{2} Y_{2m}(\hat{p})Y^*_{2m}(\hat{p}'). \tag{A.73}$$

This is the quantity we need to insert into the multiple integral in Eq. (4.49) in place of $\mathcal{M}^2$. When we do this, only the $m = 0$ term will contribute since all other

$Y_{2m}(\hat{p}')$ have an azimuthal dependence which integrates to zero. Therefore, the new collision term due to anisotropic Compton scattering is

$$\delta C[f(\vec{p})] = \frac{\pi^2 n_e \sigma_T}{p} \mathcal{P}_2(\mu) \int \frac{d^3p'}{(2\pi)^3 p'} \mathcal{P}_2(\hat{\gamma}' \cdot \hat{k})$$
$$\times \left\{ \delta(p - p') + (\vec{p} - \vec{p}\,') \cdot \vec{v}\, \frac{\partial \delta(p - p')}{\partial p'} \right\} \{ f(\vec{p}\,') - f(\vec{p}) \}, \quad (A.74)$$

where I have used the fact that $Y_{20} = -\sqrt{5}\mathcal{P}_2/\sqrt{4\pi}$. The only term which survives the angular integral is the one proportional to $\delta(p - p')f(\vec{p}\,')$, leaving

$$\delta C[f(\vec{p})] = -\frac{n_e \sigma_T}{2p} \mathcal{P}_2(\mu) \int_0^\infty dp' p' \delta(p - p') p' \frac{\partial f^{(0)}}{\partial p'}$$
$$\times \int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_2(\mu) \Theta(\mu). \quad (A.75)$$

The angular integral gives $-\Theta_2$. Then integrating over the Dirac δ-function yields

$$\delta C[f(\vec{p})] = +p \frac{\partial f^{(0)}}{\partial p} \frac{n_e \sigma_T}{2p} \mathcal{P}_2(\mu) \Theta_2. \quad (A.76)$$

This adds a factor of $-\mathcal{P}_2\Theta_2/2$ inside the square brackets of Eq. (4.54) and explains the corresponding factor in Eq. (4.100).

CHAPTER 5

Exercise 4 In Fourier space,

$$\epsilon_{ijk}(\hat{k}_k \hat{k}_l - \delta_{kl}/3) G^L_{,jl} = -k^2 \epsilon_{ijk}(\hat{k}_k \hat{k}_j - \hat{k}_j \hat{k}_k/3) G^L$$
$$= -2k^2/3 \epsilon_{ijk} \hat{k}_j \hat{k}_k G^L = 0 \quad (A.77)$$

since ϵ_{ijk} is antisymmetric under interchange of j and k while $\hat{k}_j \hat{k}_k$ is symmetric. The combination is also traceless since $\delta_{ij}(\hat{k}_i \hat{k}_j - \delta_{ij}/3) = 0$.

Exercise 7 (a) By definition,

$$\Gamma^i_{jk} = \frac{g^{ii'}}{2} \left[g_{i'j,k} + g_{i'k,j} - g_{jk,i'} \right]. \quad (A.78)$$

All derivatives here are spatial, and the only spatially varying part of the metric is the first-order piece $\mathcal{H}$. Therefore, we can again use the zero-order $g^{ii'} = \delta_{ii'}/a^2$, leaving Eq. (5.43).

(b) The product $\Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha}$ vanishes when both indices α and β are zero (because $\Gamma^0_{0i} = 0$) and when both indices are spatial (because then each Christoffel symbol is first order). Therefore, this product is

$$\Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} = \Gamma^0_{kj}\Gamma^k_{i0} + \Gamma^k_{0j}\Gamma^0_{ik}$$

$$= \Gamma^0_{kj}\Gamma^k_{i0} + (i \leftrightarrow j). \tag{A.79}$$

But

$$\Gamma^0_{kj}\Gamma^k_{i0} = \frac{1}{2}\left(2Hg_{jk} + a^2\mathcal{H}_{jk,0}\right)\left(H\delta_{ik} + \frac{1}{2}\mathcal{H}_{ik,0}\right)$$

$$= (H)^2\, g_{ij} + a\frac{da}{dt}\mathcal{H}_{ij,0}. \tag{A.80}$$

We must remember to add back in the same set of terms with i and j interchanged. This just introduces a factor of 2, so

$$\Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} = 2\,(H)^2\, g_{ij} + 2a\frac{da}{dt}\mathcal{H}_{ij,0}. \tag{A.81}$$

CHAPTER 6

Exercise 5 There are 411 photons per cm^{-3} today; the Hubble volume is $(4\pi/3)[3000\,h^{-1}\,\mathrm{Mpc}]^3 = 3.3 \times 10^{84}\,h^{-3}\,\mathrm{cm}^3$. So the total number of photons is $1.4 \times 10^{87}\,h^{-3}$. This number remains roughly constant throughout the matter and radiation eras since the number density scales as T^3, the physical volume as a^3, and the temperature as a^{-1}. So another problem of the classical cosmology is: Why is the entropy of the universe so large?

Inflation solves this problem. At first the solution seems obvious: inflation makes the scale factor grow exponentially fast, thereby increasing the product aT and hence the entropy. In fact, the solution is not quite that simple because during inflation, the exponential expansion is adiabatic: the temperature still falls as a^{-1}. So near the end of inflation the temperature has dropped rapidly enough so that if the entropy was initially of order unity, it remained of order unity.

The production of entropy actually takes place at the end of inflation during the reheating process. Even though the temperature at the end of inflation is extremely small, the energy density (which is almost completely in the scalar field) is not. When the energy in the scalar field transforms into radiation, the temperature of the radiation shoots up from its very low value of T to $\rho^{1/4} \gg T$. Thus, the reheating process is responsible for the large entropy we see today. Another way to say this is to point out that inflation is a very ordered state: the universe supercools while the field is trapped in a false vacuum. The transition to the true vacuum is a transition to the very disordered state of equilibrium.

Exercise 11 (a) With this substitution, the equation becomes

$$\frac{d^2\tilde{v}}{d\eta^2} + \frac{2}{\eta}\frac{d\tilde{v}}{d\eta} + \left(k^2 - \frac{2}{\eta^2}\right)\tilde{v}. \tag{A.82}$$

Defining $x \equiv k\eta$, we see that $\tilde{v}$ satisfies the spherical Bessel equation of order 1 (Eq. (C.13)).

(b) The two general solutions are $j_1(x)$ and $y_1(x)$. The general solution is therefore $Aj_1 + By_1$. Writing these out explicitly leads to

$$
\begin{aligned}
v = \eta\tilde{v} &= \eta\left(A\frac{\sin x - x\cos x}{x^2} - B\frac{\cos x + x\sin x}{x^2}\right) \\
&= \frac{1}{2k^2\eta}\Big(e^{ik\eta}\left[-iA - Ak\eta - B + iBk\eta\right] \\
&\qquad + e^{-ik\eta}\left[iA - Ak\eta - B - iBk\eta\right]\Big). \qquad \text{(A.83)}
\end{aligned}
$$

When $k\eta$ is very large and negative, we want $v \to e^{-ik\eta}/\sqrt{2k}$, so the coefficient of $e^{+ik\eta}$ in this limit, proportional to $-A + iB$, must vanish. Thus, $A = iB$. The coefficient of $e^{-ik\eta}$ is

$$
\frac{1}{2k^2\eta}\Big[-2Ak\eta\Big] = \frac{-A}{k}. \qquad \text{(A.84)}
$$

This must equal $(2k)^{-1/2}$, so $A = -(k/2)^{1/2}$. Therefore the correct solution is

$$
v = \frac{-1}{\sqrt{2k}k\eta}\left(e^{-ik\eta}\Big[i - k\eta\Big]\right) \qquad \text{(A.85)}
$$

in agreement with Eq. (6.57).

Exercise 13 The two components of Einstein's equations are

$$
\begin{aligned}
k^2\Psi + 3aH\left(\dot{\Psi} + aH\Psi\right) &= 4\pi Ga^2\delta T^0_0 \\
ik_i(\dot{\Psi} + aH\Psi) &= -4\pi Ga\delta T^0_i. \qquad \text{(A.86)}
\end{aligned}
$$

Here I have simply copied the results from Chapter 5, replacing Φ with $-\Psi$. Multiply the second of these by $3iaHk_i/k^2$, and then add the two equations to get

$$
k^2\Psi = 4\pi Ga^2\left[\delta T^0_0 - \frac{3Hk_i\delta T^0_i}{k^2}\right]. \qquad \text{(A.87)}
$$

On large scales, the left-hand side is negligible, so the terms in brackets on the right must sum to zero, giving Eq. (6.85).

CHAPTER 7

Exercise 4

(c) To do the integral, introduce a new dummy variable $x \equiv \sqrt{1+y}$. Then Eq. (7.31) becomes

$$\Phi = \frac{3\Phi(0)}{2}\frac{\sqrt{1+y}}{y^3}\int_1^{\sqrt{1+y}} dx \frac{(x^2-1)^2(3x^2+1)}{x^2}. \tag{A.88}$$

Now integrate by parts using the fact that the integral of $1/x^2$ is equal to $-1/x$. The surface term is proportional to the numerator and so vanishes at the lower limit, when $x=1$. Therefore,

$$\begin{aligned}\Phi &= \frac{3\Phi(0)}{2}\frac{\sqrt{1+y}}{y^3}\left[-\frac{y^2(4+3y)}{\sqrt{1+y}} + \int_1^{\sqrt{1+y}} dx\left(18x^4-20x^2+2\right)\right] \\ &= \frac{3\Phi(0)}{2}\frac{\sqrt{1+y}}{y^3}\left[-\frac{y^2(4+3y)}{\sqrt{1+y}} + \left(\frac{18}{5}x^5 - \frac{20}{3}x^3 + 2x\right)\Big|_1^{\sqrt{1+y}}\right]. \end{aligned} \tag{A.89}$$

Evaluating the terms in parentheses at the upper and lower limits leads to Eq. (7.32).

Exercise 9

$$\begin{aligned}\sigma_R^2 &= \langle\left[\int d^3x\delta(x)W_R(x)\right]^2\rangle \\ &= \langle\left[\frac{d^3k}{(2\pi)^3}\tilde{\delta}(\vec{k})\tilde{W}_R^*(\vec{k})\right]^2\rangle \end{aligned} \tag{A.90}$$

where ˜ denotes Fourier transform, and I have used the fact that since $W_R(x)$ is real, $\tilde{W}_R(\vec{k}) = \tilde{W}_R^*(-\vec{k})$. Also I have evaluated δ_R at the origin; and the angular brackets denote the average, now over all realizations of $\tilde{\delta}(\vec{k})$. Squaring and using the fact that

$$\langle\delta(\vec{k})\delta(\vec{k}')\rangle = (2\pi)^3\delta^3(\vec{k}+\vec{k}')P(k) \tag{A.91}$$

leads to

$$\sigma_R^2 = \int\frac{d^3k}{(2\pi)^3}P(k)\left|\tilde{W}_R(\vec{k})\right|^2. \tag{A.92}$$

It remains only to compute the Fourier transform of the top-hat window function,

$$\begin{aligned}\tilde{W}_R(\vec{k}) &= \int d^3xW_R(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \\ &= \frac{2\pi}{V_R}\int_0^R dx x^2\int_{-1}^1 d\mu e^{ikx\mu}. \end{aligned} \tag{A.93}$$

Note that I have normalized the window function so that the integral over it is unity; hence the factor of $V_R = 4\pi R^3/3$. Carrying out the remaining angular and radial integrals leads to

$$\tilde{W}_R(k) = \frac{3}{kR^3}\int_0^R dx x\sin(kx)$$

$$= \frac{3}{k^3R^3}\left[-kR\cos(kR) + \sin(kR)\right]. \tag{A.94}$$

By way of solving Exercise 10, note that

$$\Delta^2(k) = \frac{4\pi}{(2\pi)^3}k^3P(k). \tag{A.95}$$

CHAPTER 8

Exercise 2 Assume a solution of the form $x = e^{i\omega t}$. The damping equation then becomes a quadratic equation for ω:

$$\omega^2 - \frac{ib}{m}\omega - \frac{k}{m} = 0. \tag{A.96}$$

Solving with $k/m > \gamma^2 \equiv (b/2m)^2$ leads to

$$\omega = i\gamma \pm \omega_1. \tag{A.97}$$

The frequency is now $\omega_1 \equiv [k/m - \gamma^2]^{1/2}$, smaller than in the undamped case. The amplitude is also damped by $e^{-\gamma t}$.

Exercise 9 Use the addition theorem of spherical harmonics (C.12) to write

$$\mathcal{P}_{l'}(\hat{\gamma}\cdot\hat{k}) = \frac{4\pi}{2l+1}\sum_{m'} Y^*_{l'm'}(\hat{\gamma})Y_{l'm'}(\hat{k}). \tag{A.98}$$

Then the angular integral becomes an integral over the product of two spherical harmonics, which — because of orthogonality — is equal to 1 if $l' = l$ and $m' = m$ and zero otherwise. This leads directly to the desired reult.

Exercise 12 I get the result show in Figure A.1. The integral of the cross-term is significantly smaller than that of either of the squares, so there is no interefence between the monopole and dipole.

Exercise 17 The generalization of Eq. (8.67) to tensors gives

$$C_l^T = \sum_{l'l''}(-i)^{l'+l''}(2l'+1)(2l''+1)\int \frac{d^3k}{(2\pi)^3}\Theta^T_{l'}(k)\Theta^{T,*}_{l''}(k)I_{lml'}(k)I^*_{lml''}(k) \tag{A.99}$$

where I have defined

$$I_{lml'}(k) \equiv \sqrt{\frac{8\pi}{15}}\int d\Omega\mathcal{P}_{l'}(\hat{k}\cdot\hat{\gamma})Y_{lm}(\Omega)\left[Y_{22}(\Omega) + Y_{2-2}(\Omega)\right]. \tag{A.100}$$

The factor of $[8\pi/15]^{1/2}[Y_{22}+Y_{2-2}]$ is the combination $\sin^2\theta\cos(2\phi)$ which appears in Eq. (4.115), so this expression is valid only for the + mode. However, the $\times$ mode gives exactly the same result.

The integral $I_{lml'}$ is not trivial. By rewriting the Legendre polynomial as $[4\pi/(2l'+1)]^{1/2}Y_{l'0}/i^{l'}$, we can turn $I_{ll'}$ into an integral over the product of three

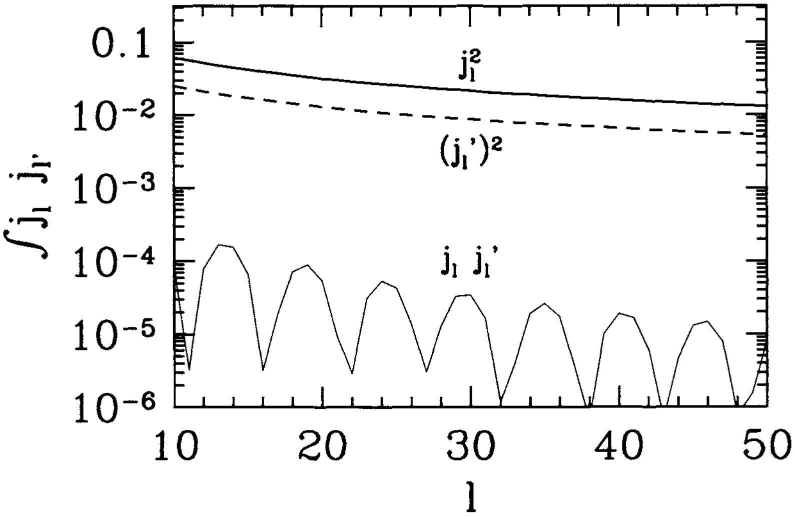

Figure A.1. The integrals of products of spherical Bessel functions.

spherical harmonics. Such integrals are intensively studied in quantum mechanics and can be expressed in terms of the Wigner 3-j symbols. By the way, my favorite reference for these things — especially useful for this integral — is *Quantum Mechanics* (Landau and Lifshitz), like all the other texts in their Course of Theoretical Physics a wonderful investment. The integral is then

$$I_{lml'} = \sqrt{\frac{32\pi^2}{15(2l'+1)}}\frac{1}{i^{l'}}\langle lm|Y_{22}+Y_{2-2}|l'0\rangle \tag{A.101}$$

which vanishes unless $m=2$ or $m=-2$. When m takes on one of these two values, the matrix element is

$$\langle l2|Y_{22}+Y_{2-2}|l'0\rangle = i^{l'-l}\begin{pmatrix} l & 2 & l' \\ 0 & 0 & 0\end{pmatrix}\left[\frac{5(2l'+1)(2l+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & 2 & l' \\ -2 & 2 & 0\end{pmatrix}. \tag{A.102}$$

The first 3-j symbol here, the one with the bottom row all zero, vanishes unless the sum of the elements in the top row $l+l'+2$ is even. And of course l' cannot differ from l by more than 2 since the combination of $Y_{22}\, Y_{l'0}$ leads to angular momenta ranging from $l'-2$ to $l'+2$. So the only time the matrix element is nonzero is when $l'=l-2, l, l+2$. Using Table 9 in Section 106 of *Quantum Mechanics* leads to the final result:

$$I_{lml'} = \sqrt{\frac{8\pi}{3}}\,\sqrt{2l+1}\; i^{-l}\;\left(\delta_{m,2}+\delta_{m,-2}\right)\left[c_{-2}\delta_{l',l-2}+c_0\delta_{l',l}+c_2\delta_{l',l+2}\right] \tag{A.103}$$

where here $\delta_{m,2}$ (and all other δ's) is the Kronecker delta, equal to 1 if $m=2$ and zero otherwise. The coefficients are

$$c_{-2} = \frac{\sqrt{6}}{4} \frac{[(l-1)l(l+1)(l+2)]^{1/2}}{(2l-3)(2l-1)(2l+1)}$$

$$c_0 = \frac{-2\sqrt{6}}{4} \frac{[(l-1)l(l+1)(l+2)]^{1/2}}{(2l-1)(2l+1)(2l+3)}$$

$$c_2 = \frac{\sqrt{6}}{4} \frac{[(l-1)l(l+1)(l+2)]^{1/2}}{(2l+1)(2l+3)(2l+5)}. \tag{A.104}$$

The result in Eq. (8.93) then follows.

Exercise 18

(a) On large scales, we can take the matter-dominated solution for h, so

$$\Theta_{l,i} = \frac{-1}{2} \int_{\eta_*}^{\eta_0} d\eta \, j_l[k(\eta_0 - \eta)] \frac{d}{d\eta} \left[\frac{3j_1(k\eta)}{k\eta} \right] P_h^{1/2}. \tag{A.105}$$

Here I have used the fact that the initial amplitude of the gravity waves is $P_h^{1/2}$ with the time dependence given in the square brackets. Plug this into Eq. (8.93) to get

$$C_l^T = 2\frac{9(l-1)l(l+1)(l+2)}{4\pi} \int_0^\infty dk \, k^2 P_h(k) \left| \int_0^{\eta_0} d(k\eta) \frac{j_2(k\eta)}{k\eta} \right.$$
$$\left. \times \left[\frac{j_{l-2}(k[\eta_0 - \eta])}{(2l-1)(2l+1)} + 2\frac{j_l(k[\eta_0 - \eta])}{(2l-1)(2l+3)} + \frac{j_{l+2}(k[\eta_0 - \eta])}{(2l+1)(2l+3)} \right] \right|^2, \tag{A.106}$$

where I have set the lower limit on the time integral to zero since $\eta_* \ll \eta_0$. Also, I have used the identity $(j_1/x)' = -j_2/x$. The factor of 2 out in front comes from the sum over the $+$ and $\times$ components. Using Eq. (6.100) for P_h (in the slow-roll approximation $\epsilon = 0$ and $\nu = 3/2$) and defining new integration variables $y \equiv k\eta_0$ and $x \equiv k\eta$ leads to

$$C_l^T = 36 \left(\frac{H_{\text{inf}}}{m_{\text{Pl}}} \right)^2 (l-1)l(l+1)(l+2) \int_0^\infty \frac{dy}{y} \left| \int_0^y dx \frac{j_2(x)}{x} \right.$$
$$\left. \times \left[\frac{j_{l-2}(y-x)}{(2l-1)(2l+1)} + 2\frac{j_l(y-x)}{(2l-1)(2l+3)} + \frac{j_{l+2}(y-x)}{(2l+1)(2l+3)} \right] \right|^2. \tag{A.107}$$

Here H_{inf} denotes the Hubble rate during inflation, or more precisely the Hubble rate when the modes in question crossed the horizon (when $k\eta = -1$ early on). This expression does well on the low multipoles. To get even better results stick in the transfer function of Eq. (5.88).

(b) For the $l = 2$ mode, the double integral in Eq. (A.107) is equal to 2.139×10^{-4}, so $C_2^T = 0.185(H/m_{\text{Pl}})^2$. The scalar C_2 is equal to $\pi\delta_H^2/12$. Using Eq. (6.100) for δ_H leads to

$$r = 13.86\epsilon. \tag{A.108}$$

(c) Combining with Eq. (6.104), we expect

$$r = -6.93 n_T. \tag{A.109}$$

For many models, the inflationary parameter $\delta = -\epsilon$, so

$$n - 1 = -n_T. \tag{A.110}$$

CHAPTER 9

Exercise 2 Expand the power spectrum about $k_3 = 0$:

$$P\left(\sqrt{k_3^2 + (H_0\kappa/\chi)^2}\right) = P(H_0\kappa/\chi) + \frac{1}{2k}\frac{dP}{dk}|_{k_3=0} k_3^2 + \ldots. \tag{A.111}$$

For a smooth power spectrum dP/dk is of order P/k, so the coefficient of k_3^2 is of order P/k^2. For us, $k = H_0\kappa/\chi$, so this coefficient is of order $P\chi^2/(H_0\kappa)^2$. We can write k_3^2 as $-H_0^2\partial^2/\partial\chi^2$ acting on the exponential of Eq. (9.9). Assuming the selection function is relatively smooth, this is of order H_0^2/χ^2. So the first correction to the leading term is of order $1/\kappa^2$, which is small as long as the angular scales probed are not too large.

Exercise 3 Define the dummy variable $\chi \equiv H_0\kappa/k$. Then

$$F = H_0 \int \frac{d\chi}{2\pi\chi} J_0(k\theta\chi/H_0) W^2(\chi), \tag{A.112}$$

an expression which clearly depends only on the combination $k\theta$.

Exercise 5 To express $C_l^{\rm matter}$ in terms of w, multiply both sides of Eq. (9.66) by $\mathcal{P}_{l'}(\cos\theta)$ and integrate over $\mu \equiv \cos\theta$. This gives

$$C_l^{\rm matter} = 2\pi \int_{-1}^{1} d\cos\theta \mathcal{P}_l(\cos\theta) w(\theta). \tag{A.113}$$

Express w as an integral over the 2D power spectrum as in the first line of Eq. (9.13). Then,

$$C_l^{\rm matter} = \int_0^\infty dl'\ l' P_2(l') \int_{-1}^{1} d\cos\theta \mathcal{P}_l(\cos\theta) J_0(l'\theta). \tag{A.114}$$

Note the difference in P's: the first P_2 here is the 2D power spectrum, the second $\mathcal{P}_l$ is the Legendre polynomial. In the limit that l' is large, the Bessel function becomes

$$J_0(l'\theta) \to \mathcal{P}_{l'}(\cos\theta). \tag{A.115}$$

Therefore, the integral over θ vanishes unless $l = l'$, in which case it is equal to $2/(2l+1)$. The integral over l' is identical to a sum over l' at large l' since $dl' \to 1$. The factor of $2/(2l+1)$ in the denominator cancels the factor of l' in the numerator, leaving the desired equality between the 2D power spectrum and $C_l^{\rm matter}$.

CHAPTER 10

Exercise 2 The total intensity received at the detector is the angular integral of $I_{\rm obs}(\theta)$ over θ. The total intensity emitted is the angular integral of $I_{\rm true}(\theta_S)$ over θ_S. The magnification μ is the ratio of the two:

$$\mu \equiv \frac{\int d^2\theta I_{\rm obs}(\theta)}{\int d^2\theta_S I_{\rm true}(\theta_S)}. \tag{A.116}$$

Change variables in the denominator to θ, leading to a factor of $\det(A)$ where A is defined in Eq. (10.15). Recall now that $I_{\rm true}(\theta_S) = I_{\rm obs}(\theta)$, so except for the determinant, the numerator and denominator cancel. This leaves

$$\mu = \frac{1}{\det(A)} = \frac{1}{(1-\kappa)^2 - (\gamma_1^2 + \gamma_2^2)}. \tag{A.117}$$

If all the perturbations are small, then the magnification depends only on κ:

$$\mu \simeq 1 + 2\kappa. \tag{A.118}$$

Exercise 3 (a) Reading off from Eq. (10.14), we see immediately that

$$\phi = 2\int_0^{\chi_S} d\chi \, \frac{\chi_S - \chi}{\chi_S \chi} \, \Phi(\chi\vec\theta, \chi), \tag{A.119}$$

where I have let $\chi \to \chi_S$ in Eq. (10.14) and replaced the dummy variable χ' there with χ. The only subtlety here is the extra factor of χ in the denominator. This comes from changing the derivative with respect to position (the comma in Eq. (10.14)) to an angular derivative.

Exercise 4 Recall that, in the Newtonian limit, the gravitational potential can be written in terms of the mass density:

$$\Phi(\vec x) = -G\int \frac{d^3x'}{|\vec x - \vec x'|}\rho(\vec x'). \tag{A.120}$$

We will do this integral in cylindrical coordinates, so that $\vec x' = (\vec R, \chi')$. Thus,

$$\phi \simeq -2G\,\frac{\chi_S - \chi_L}{\chi_S\chi_L} \int d^2R \int d\chi' \rho(\vec R, \chi') \int_0^{\chi_S} \frac{d\chi}{\sqrt{(\vec R - \chi_L\vec\theta)^2 + (\chi - \chi')^2}} \tag{A.121}$$

where I have set $\chi = \chi_L$ in the slowly varying factors out front. The innermost integral can be done analytically: it is equal to

$$2\ln\left| x + \sqrt{(\vec R - \chi_L\vec\theta)^2 + x^2)}\right|_0^\infty$$

where I have set the upper limit to infinity because there is no contribution to the relevant part of the projected potential from large x. In fact, the only part which depends on $\vec\theta$ (and hence is relevant when derivatives are taken) comes from the lower limit: $-2\ln|\vec R - \chi_L\vec\theta|$. The integral over χ' then becomes the surface density leaving the desired result.

CHAPTER 11

Exercise 4 The noise matrix is

$$\left(C_N\right)_{ij} = \frac{1}{\bar{n}} \int d^3x \psi_i(\vec{x})\psi_j(\vec{x}) \tag{A.122}$$

in this case of constant $\bar{n}$. Let's consider first the diagonal elements of the matrix. For these, both ψ_i and ψ_j require x to be within a radius R of the center of the ith cell, so

$$\left(C_N\right)_{ii} = \bar{n} \int_{x<R} d^3x = \frac{4\pi\bar{n}R^3}{3} \tag{A.123}$$

(no sum over i intended). For cells separated by more than $2R$, the integral vanishes since $\vec{x}$ cannot be within a distance R of both cell centers. For distances less than $2R$ there is some overlap and the integral becomes

$$\bar{n} \int_{x<R} d^3x \Theta\left(R - \left|\vec{x} - \vec{r}\right|\right) = 2\pi\bar{n} \int_0^R dx x^2 \int_{-1}^{1} d\mu \Theta\left(R - \sqrt{x^2 + r^2 - 2xr\mu}\right) \tag{A.124}$$

where $\vec{r}$ is the difference between the positions of the two cell centers and Θ is the step function equal to 1 if its argument is positive and zero otherwise. The μ integral therefore goes runs from $(x^2 + r^2 - R^2)/2xr$ up to 1. If this lower limit is greater than 1, then the μ integral vanishes; otherwise it is unity. The only contribution then comes when the lower limit is less than 1, which happens when x lies between $r \pm R$. The integral is therefore

$$\int_{r-R}^{R} dx x^2 \left[1 - \frac{x^2 + r^2 - R^2}{2xr}\right] = \frac{1}{2r} \int_{r-R}^{R} dx \left[2xr - (x^2 + r^2 - R^2)\right]. \tag{A.125}$$

The x integral here is then tedious but completely straightforward since the integrand is simply powers of x. I find that

$$\left(C_N\right)_{ij} = \frac{\pi\bar{n}R^3}{12} \left(2 - \frac{r_{ij}}{R}\right)^2 \left(4 + \frac{r_{ij}}{R}\right). \tag{A.126}$$

Exercise 7 We need to compute the integral of Eq. (11.55). Since the window function is sharply peaked for small scale modes, we can set k everywhere to k_i. Then inserting our explicit expression for the window function in a volume-limited survey (Eq. (11.59)), we are left with

$$(C_S)_{ii} \simeq \frac{9P(k_i)}{4\pi^2 R^2} \int_0^\infty dk \int_{|k-k_i|R}^{(k+k_i)R} \frac{dy}{y} \, j_1^2(y). \tag{A.127}$$

The best way to do the integrals here is to switch orders of integration. Consider the Figure A.2, which shows the region of integration. The region below the horizontal

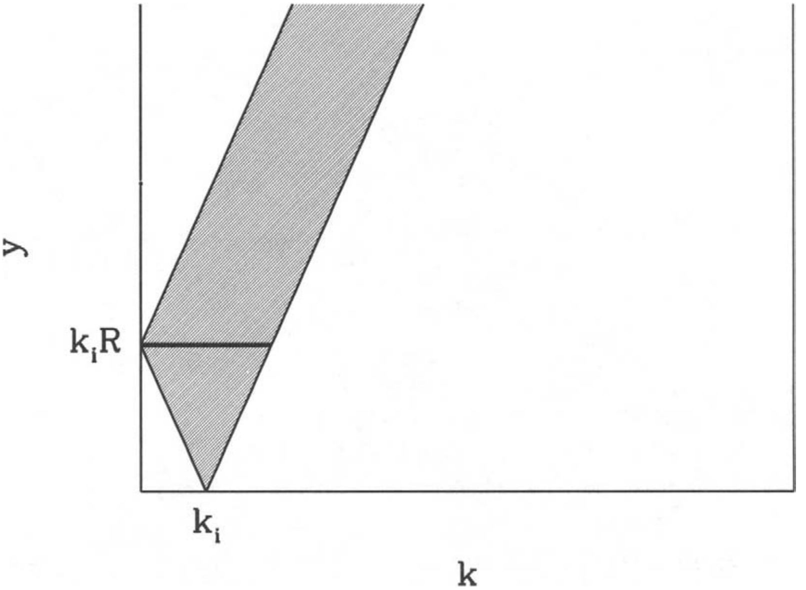

Figure A.2. Region of integration for the double integral in Eq. (A.128). The region below the horizontal line constitutes the first term, above the second term.

line corresponds to $y < k_iR$ and $k_i - y/R < k < k_i + y/R$. In the region above k is bounded by $y/R \pm k_i$. Therefore,

$$\int_0^\infty dk \int_{|k-k_i|R}^{(k+k_i)R} \frac{dy}{y}\, j_1^2(y) = \int_0^{k_iR} \frac{dy}{y}\, j_1^2(y) \int_{k_i-y/R}^{k_i+y/R} dk + \int_{k_iR}^\infty \frac{dy}{y}\, j_1^2(y) \int_{y/R-k_i}^{k_i+y/R} dk$$

$$= \frac{2}{R}\int_0^{k_iR} dy\, j_1^2(y) + 2k_i \int_{k_iR}^\infty \frac{dy}{y}\, j_1^2(y). \tag{A.128}$$

In the limit $k_iR \gg 1$, the first term here is much larger than the second (since $j_1^2(y)$ goes as $1/y^2$ for large y). In the first integral, we may replace the upper limit by infinity, again since k_iR is large. The resulting integral is (Eq. (C.17)) $\pi\Gamma(3/2)/4\Gamma(5/2) = \pi/6$. Multiplying this by $2/R$ and then by $9/4\pi^2R^2$ leads to a factor of $1/V$.

Exercise 9 We want to compute the variance

$$\langle \left(\hat\lambda - \bar\lambda\right)^2 \rangle = \left\langle \left(F^{-1}_{\lambda\lambda} \frac{\Delta C^{-1} C_{,\lambda} C^{-1}\Delta - \mathrm{Tr}[\mathrm{C}^{-1}\mathrm{C}_{,\lambda}]}{2} \right)^2 \right\rangle \tag{A.129}$$

where the estimator $\hat\lambda$ is given by Eq. (11.92), and I have assumed that $\lambda^{(0)} = \bar\lambda$, i.e., we are at the true maximum of the likelihood function. Upon squaring there are terms with no Δ; those with two Δ's; and those with four. The ones with two Δ's can be simply evaluated by using

$$\langle \Delta_i \Delta_j \rangle = C_{ij} \tag{A.130}$$

where the indices label the pixels. Since the distribution is assumed Gaussian, the expectation value of four Δ's is

$$\langle \Delta_i \Delta_j \Delta_k \Delta_l \rangle = C_{ij}C_{kl} + C_{ik}C_{jl} + C_{il}C_{jk}. \tag{A.131}$$

Putting in these expectation values leads to

$$\langle \left(\hat{\lambda} - \bar{\lambda}\right)^2 \rangle = \frac{F_{\lambda\lambda}^{-2}}{4} \Bigg\{ \left[C^{-1}C_{,\lambda}C^{-1}\right]_{ij} \left[C^{-1}C_{,\lambda}C^{-1}\right]_{kl} \left(C_{ij}C_{kl} + C_{ik}C_{jl} + C_{il}C_{jk}\right)$$
$$- \left(\mathrm{Tr}[\mathrm{C}^{-1}\mathrm{C}_{,\lambda}]\right)^2 \Bigg\}. \tag{A.132}$$

The $C_{ij}C_{kl}$ terms lead to $(\mathrm{Tr}[\mathrm{C}^{-1}\mathrm{C}_{,\lambda}])^2$, cancelling the similar term on the last line, so

$$\langle \left(\hat{\lambda} - \bar{\lambda}\right)^2 \rangle = \frac{F_{\lambda\lambda}^{-2}}{4} \left[C^{-1}C_{,\lambda}C^{-1}\right]_{ij} \left[C^{-1}C_{,\lambda}C^{-1}\right]_{kl} \left(C_{ik}C_{jl} + C_{il}C_{jk}\right). \tag{A.133}$$

Both terms here contribute identically (giving one factor of 2). The matrix multiplication simplifies since all matrices are symmetric. For example,

$$\left[C^{-1}C_{,\lambda}C^{-1}\right]_{ij} \left[C^{-1}C_{,\lambda}C^{-1}\right]_{kl} C_{ik}C_{jl} = \mathrm{Tr}\left[\mathrm{C}^{-1}\mathrm{C}_{,\lambda}\mathrm{C}^{-1}\mathrm{C}_{,\lambda}\right] \tag{A.134}$$

and we recognize the right-hand side as $2F_{\lambda\lambda}$ (another factor of 2). Therefore,

$$\langle \left(\hat{\lambda} - \bar{\lambda}\right)^2 \rangle = F_{\lambda\lambda}^{-1}. \tag{A.135}$$

It is important to keep in mind that this equality holds only if the overdensities are distributed as Gaussians and if we truly have reached the point in parameter space which is the true maximum.

Exercise 15 (a) Use a likelihood approach. The likelihood function for the parameters, here the amplitudes of the different components Θ^α, is proportional to $e^{-\chi^2/2}$ with

$$\chi^2 = (d - W\Theta)\, N^{-1}\, (d - W\Theta) + \sum_{\alpha=1}^{\mathrm{N_{foregrounds}}} (\Theta^\alpha)^2 / C^\alpha. \tag{A.136}$$

The first term is identical to that generated by Eq. (11.144). The second accounts for the prior, that Θ^α has mean zero and variance C^α. Maximizing the likelihood corresponds to minimizing the χ^2, with respect to the parameters Θ. Since the χ^2 is quadratic, the minimization leads to a linear equation for Δ^α, the minimum variance estimator of Θ^α:

$$\Delta = \left(WN^{-1}W + C^{-1}\right)^{-1} WN^{-1}d. \tag{A.137}$$

The new covariance matrix is the first term on the right,

$$C_N = \left(WN^{-1}W + C^{-1}\right)^{-1}. \tag{A.138}$$

Here the matrix C is diagonal with $_{00}$ element equal to zero, and the other diagonal elements equal to the assumed power spectra of the foregrounds.

(b) If there is only one foreground with shape vector $W^1 = (1, 1/2)$, and if this foreground has assumed power equal to the noise, then the new inverse covariance matrix goes from that in Eq. (11.151) to the same matrix with $1/\sigma_n^2$ added to the $_{11}$ component. Thus,

$$C_N^{-1} = \frac{1}{\sigma_n^2}\begin{pmatrix} 2 & 3/2 \\ 3/2 & 9/4 \end{pmatrix}. \tag{A.139}$$

Note that the $_{00}$ component of this is unchanged, as it must be since it is the inverse covariance if all foregrounds are known. The inverse of this gives the new covariance matrix,

$$C_N = \frac{4\sigma_n^2}{9}\begin{pmatrix} 9/4 & -3/2 \\ -3/2 & 2 \end{pmatrix}. \tag{A.140}$$

Immediately, we see that the noise in the presence of foregrounds is σ_n. This is a factor of $\sqrt{5}$ smaller than if we had no prior knowledge of the foreground amplitude. It is only a factor of $\sqrt{2}$ larger than the case without foregrounds; thus the new FDF in this case is $\sqrt{2}$. The minimum variance estimator is

$$\begin{aligned}\Delta^0 &= \frac{4\sigma_n^2}{9}(9/4, -3/2)\begin{pmatrix} 1 & 1 \\ 1 & 1/2 \end{pmatrix}\begin{pmatrix} 1/\sigma_n^2 & 0 \\ 0 & 1/\sigma_n^2 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ &= \frac{d_1 + 2d_2}{3}. \end{aligned} \tag{A.141}$$