2

THE SMOOTH, EXPANDING UNIVERSE

Just as the early navigators of the great oceans required sophisticated tools to help them find their way, we will need modern technology to help us work through the ramifications of an expanding universe. In this chapter I introduce two of the necessary tools, general relativity and statistical mechanics. We will use them to derive some of the basic results laid down in Chapter 1: the expansion law of Eq. (1.2), the dependence of different components of energy density on the scale factor which governs expansion, the epoch of equality $a_{\rm eq}$ shown in Figure 1.3, and the luminosity distance needed to understand the implications of the supernovae diagram in Figure 1.7. Indeed, with general relativity and statistical mechanics, we can go a long way toward performing a cosmic inventory, identifying those components of the universe that dominate the energy budget at various epochs.

Implicit in this discussion will be the notion that the universe is smooth (none of the densities vary in space) and in equilibrium (the consequences of which will be explored in Section 2.3). In succeeding chapters, we will see that the deviations from equilibrium and smoothness are the source of much of the richness in the universe. Nonetheless, if only in order to understand the framework in which these deviations occur, a basic knowledge of the "zero order" universe is a must for any cosmologist.

In this chapter, I begin using units in which

$$\hbar = c = k_B = 1. \tag{2.1}$$

Many papers employ these units, so it is important to get accustomed to them. Please work through Exercise 1 if you are uncomfortable with the idea of setting the speed of light to 1.

2.1 GENERAL RELATIVITY

Most of cosmology can be learned with only a passing knowledge of general relativity. One must be familiar with the concept of a metric, understand geodesics, and be able to apply the Einstein equations to the Friedmann-Robertson-Walker (FRW) metric thereby relating the parameters in the metric to the density in the universe. Eq. (1.2) is the result of applying the Einstein equations to the zero order universe. We will derive it in this section. Chapters 4 and 5 apply them to the perturbed universe. With the experience we gain in this section, there will be nothing difficult about these subsequent chapters. The principles are identical; only the algebra will be a touch harder.

2.1.1 The Metric

Figure 1.1 from Chapter 1 highlights the fact that even if one knows the components of a vector, say the difference between two grid points there, the physical distance associated with this vector requires additional information. In the case of a smooth expanding universe, the scale factor connects the coordinate distance with the physical distance. More generally, the *metric* turns coordinate distance into physical distance and so will be an essential tool in our quest to make quantitative predictions in an expanding universe.

We are familiar with the metric for the Cartesian coordinate system which says that the square of the physical distance between two points separated by dx and dy in a 2D plane is $(dx)^2 + (dy)^2$. However, were we to use polar coordinates instead, the square of the physical distance would no longer be the sum of the square of the two coordinate differences. Rather, if the differences dr and $d\theta$ are small, the square of the distance between two points is $(dr)^2 + r^2(d\theta)^2 \neq (dr)^2 + (d\theta)^2$. This distance is *invariant*: an observer using Cartesian coordinates to find it would get the same result as one using polar coordinates. Thus another way of stating what a metric does is this: it turns observer-dependent coordinates into invariants. Mathematically, in the 2D plane, the invariant distance squared $dl^2 = \sum_{i,j=1,2} g_{ij} dx^i dx^j$. The metric g_{ij} in this 2D example is a 2×2 symmetric matrix. In Cartesian coordinates the metric is diagonal with each element equal to 1. In polar coordinates (taking $x^1 = r$ and $x^2 = \theta$) it is also diagonal with $g_{11} = 1$, but g_{22} which multiplies $(d\theta)^2$ is equal to r^2.

There is yet another way of thinking about a metric, using pictures. When handed a vector, we immediately think of a line with an arrow attached, the length of the line corresponding to the length of the vector and the arrow to its direction. In fact, this notion is rooted too firmly in Euclidean space. In actuality, the length of the vector depends on the metric. An intuitive way of understanding this is to consider the contour map in Figure 2.1. The number of lines crossed by a vector is a measure of the vertical distance traveled by a hiker. Vectors of the same apparent 2D length — corresponding to identical coordinate distances — can correspond to significantly different physical distances. Mathematically the surface of the Earth can be parametrized by two coordinates, say θ and ϕ. Then the metric is a very nontrivial function of θ and ϕ which accounts for all the elevation changes on the surface.

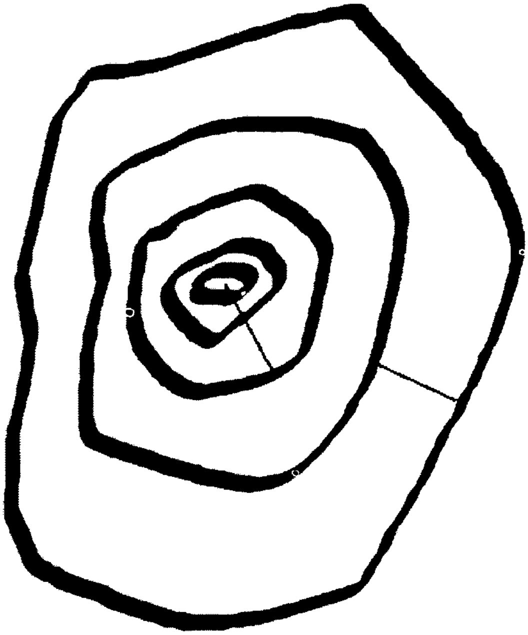

Figure 2.1. Contour map of a mountain. The closely spaced contours near the center correspond to rapid elevation gain. The two thin lines correspond to hikes of significantly different difficulty even though they appear to be of the same length. Similarly, the true length of a vector requires knowledge of the metric.

The great advantage of the metric is that it incorporates gravity. Instead of thinking of gravity as an external force and talking of particles moving in a gravitational field, we can include gravity in the metric and talk of particles moving freely in a distorted or curved space-time, one in which the metric cannot be converted everywhere into Euclidean form.

In four space-time dimensions the invariant includes time intervals as well, so

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu \tag{2.2}$$

where the indices μ and ν range from 0 to 3 (see the box on page 27), with the first one reserved for the time-like coordinate ($dx^0 = dt$) and the last three for spatial coordinates. Here I have explicitly written down the summation sign, but from now on we will use the convention that repeated indices are summed over. The metric $g_{\mu\nu}$ is necessarily symmetric, so in principle has four diagonal and six off-diagonal components. It provides the connection between values of the coordinates and the more physical measure of the interval ds^2 (sometimes called *proper time*). Special relativity is described by Minkowski space-time with the metric: $g_{\mu\nu} = \eta_{\mu\nu}$, with

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

What is the metric which describes the expanding universe? Let us return to the grid depicted in Figure 1.1. We said earlier that two grid points move away from each other, so that the distance between the two points is always proportional to the scale factor. If the comoving distance today is x_0, the physical distance between the two points at some earlier time t was $a(t)x_0$. At least in a flat (as opposed to open or closed) universe, the metric then is almost identical to the Minkowski metric, except that distance must be multiplied by the scale factor. This suggests that the metric in an expanding, flat universe is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}. \tag{2.4}$$

This is called the Friedmann–Robertson–Walker (FRW) metric.

As noted in Eq. (1.2), which we will shortly derive, the evolution of the scale factor depends on the density in the universe. When perturbations are introduced, the metric will become more complicated, and the perturbed part of the metric will be determined by the inhomogeneities in the matter and radiation.

Indices

In three dimensions, a vector $\vec{A}$ has three components, which we refer to as A^i, superscript i taking the values 1,2, or 3. The dot product of two vectors is then

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A^i B^i \equiv A^i B^i \tag{2.5}$$

where I have introduced the Einstein summation convention of not explicitly writing the $\sum$ sign when an index (in this case i) appears twice. Similarly, matrices can be written in component notation. For example, the product of two matrices $\mathbf{M}$ and $\mathbf{N}$ is

$$(\mathbf{MN})_{ij} = M_{ik} N_{kj} \tag{2.6}$$

again with implicit summation over k.

In relativity, two generalizations must be made. First, in relativity a vector has a fourth component, the time component. Since the spatial indices run from 1 to 3, it is conventional to use 0 for the time component. Greek letters are used to represent all four components, so $A^\mu = (A^0, A^i)$. The second, more subtle, feature of relativity is the distinction between upper and lower indices, the former associated with vectors and the latter with 1-forms. One goes back and forth with the metric tensor, so

$$A_\mu = g_{\mu\nu} A^\nu \qquad ; \qquad A^\mu = g^{\mu\nu} A_\nu . \tag{2.7}$$

A vector and a 1-form can be contracted to produce an invariant, a scalar. For example, the statement that the four-momentum squared of a massless particle must vanish is

$$P^2 \equiv P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = 0. \tag{2.8}$$

This contraction is the equivalent of counting the contours crossed by a vector, as depicted in Figure 2.1.

Just as the metric can turn an upper index on a vector into a lower index, the metric can be used to raise and lower indices on tensors with an arbitrary number of indices. For example, raising the indices on the metric tensor itself leads to

$$g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta} . \tag{2.9}$$

If the index $\alpha = \nu$, then the first term on the right is equal to the term on the left, so if the combination of the last two terms on the right force α to be equal to ν, then the equation is satisfied. Therefore,

$$g^{\nu\beta} g_{\alpha\beta} = \delta^\nu{}_\alpha , \tag{2.10}$$

where $\delta^\nu{}_\alpha$ is the Kronecker delta equal to zero unless $\nu = \alpha$ in which case it is 1.

2.1.2 The Geodesic Equation

In Minkowski space, particles travel in straight lines unless they are acted on by a force. Not surprisingly, the paths of particles in more general space-times are more complicated. The notion of a straight line gets generalized to a *geodesic*, the path followed by a particle in the absence of any forces. To express this in equations, we must generalize Newton's law with no forces, $d^2\vec{x}/dt^2 = 0$, to the expanding universe.

The machinery necessary to generalize $d^2\vec{x}/dt^2 = 0$ is perhaps best introduced by starting with a simple example: particle motion in a Euclidean 2D plane. In that case, the equations of motion in Cartesian coordinates $x^i = (x, y)$ are

$$\frac{d^2x^i}{dt^2} = 0. \tag{2.11}$$

However, if we use polar coordinates $x'^i = (r, \theta)$ instead, the equations for a free particle look significantly different. The fundamental difference between the two coordinate systems is that the basis vectors for polar coordinates $\hat{r}, \hat{\theta}$ vary in the plane. Therefore, $d^2\vec{x}'/dt^2 = 0$ does *not* imply that each coordinate r and θ satisfies $d^2x'^i/dt^2 = 0$.

To determine the equation satisfied by the polar coordinates, we can start from the Cartesian equation and then transform. In particular,

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}. \tag{2.12}$$

$\partial x^i/\partial x'^j$ is called the *transformation matrix* going from one basis to another. In the case of Cartesian to polar coordinates in 2D, $x^1 = x'^1 \cos x'^2$ and $x^2 = x'^1 \sin x'^2$, so the transformation matrix is

$$\frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \cos x'^2 & -x'^1 \sin x'^2 \\ \sin x'^2 & x'^1 \cos x'^2 \end{pmatrix}. \tag{2.13}$$

Therefore, the geodesic equation becomes

$$\frac{d}{dt}\left[\frac{dx^i}{dt}\right] = \frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = 0. \tag{2.14}$$

The derivative with respect to time acts on both terms inside the brackets. If the transformation from the Cartesian basis to the new basis was linear, then the derivative acting on the transformation matrix would vanish, and the geodesic equation in the new basis would still be $d^2x'^i/dt^2 = 0$. In the case of polar coordinates, though, the transformation is not linear, and we need the fact that

$$\frac{d}{dt}\left(\frac{\partial x^i}{\partial x'^j}\right) = \frac{\partial}{\partial x'^j}\left(\frac{dx^i}{dt}\right)$$

$$= \frac{\partial^2 x^i}{\partial x'^j \partial x'^k} \frac{dx'^k}{dt} \tag{2.15}$$

where the first equality holds since derivatives commute and the second comes from inserting dx^i/dt from Eq. (2.12), changing dummy indices from $j \to k$. The geodesic equation in the new coordinates therefore becomes

$$\frac{d}{dt}\left[\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right] = \frac{\partial x^i}{\partial x'^j}\frac{d^2 x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0. \tag{2.16}$$

To get this in a more recognizable form, note that the term multiplying the second time derivative is the transformation matrix. If we multiply the equation by the inverse of this transformation matrix, then the second time derivative will stand alone, leaving

$$\frac{d^2 x'^l}{dt^2} + \left[\left(\left\{\frac{\partial x}{\partial x'}\right\}^{-1}\right)^l_{\ i} \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\right]\frac{dx'^k}{dt}\frac{dx'^j}{dt} = 0. \tag{2.17}$$

You can check that this rather cumbersome expression does indeed give the correct equations of motion in polar coordinates. More importantly, by keeping things general, we have derived the geodesic equation in a non-Cartesian basis.

It is convenient to define the *Christoffel symbol*, $\Gamma^l{}_{jk}$, to be the coefficient of the $(dx'^k/dt)(dx'^j/dt)$ term in Eq. (2.17). Note that by definition it is symmetric in its lower indices j and k. In a Cartesian coordinate system, the Christoffel symbol vanishes and the geodesic equation is simply $d^2x^i/dt^2{=}0$. But in general, the Christoffel symbol does not vanish; its presence describes geodesics in nontrivial coordinate systems. The reason why this generalized geodesic equation is so powerful is that in a nontrivial space-time such as the expanding universe it is not *possible* to find a fixed Cartesian coordinate system, so we need to know how particles travel in the more general case.

There are two small changes we need to make when importing the geodesic equation (2.17) into relativity. The first is trivial: allow the indices to range from 0 to 3 to include time and the three spatial dimensions. The second is also not surprising: since time is now one of our coordinates, it will not do to use it as the evolution parameter. Instead introduce a parameter λ which monotonically increases along the particle's path as in Figure 2.2. The geodesic equation then

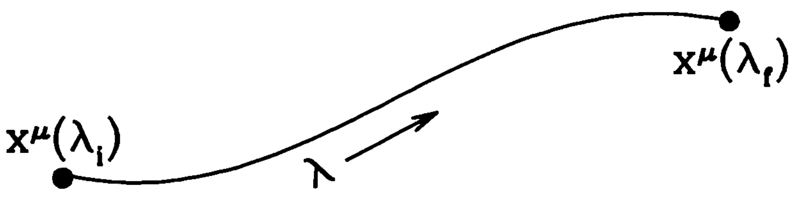

Figure 2.2. A particle's path is parametrized by λ, which monotonically increases from its initial value λ_i to its final value λ_f.

becomes

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}. \tag{2.18}$$

We derived this equation transforming from a Cartesian basis, so that the Christoffel symbol is given by the term in square brackets in Eq. (2.17). It is almost always more convenient, however, to obtain the Christoffel symbol from the metric directly. A convenient formula expressing this dependence is

$$\Gamma^\mu{}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]. \tag{2.19}$$

Note that the raised indices on $g^{\mu\nu}$ are important: $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ (see the box on page 27). So $g^{\mu\nu}$ in the flat, FRW metric is identical to $g_{\mu\nu}$ except that its spatial elements are $1/a^2$ instead of a^2.

Using the general expression in Eq. (2.19) and the FRW metric in Eq. (2.4), we can derive the Christoffel symbol in an expanding, homogeneous universe. First we compute the components with upper index equal to zero, $\Gamma^0{}_{\alpha\beta}$. Since the metric is diagonal, the factor of $g^{0\nu}$ vanishes unless $\nu = 0$ in which case it is -1. Therefore,

$$\Gamma^0{}_{\alpha\beta} = \frac{-1}{2}\left[\frac{\partial g_{\alpha 0}}{\partial x^\beta} + \frac{\partial g_{\beta 0}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right]. \tag{2.20}$$

The first two terms here reduce to derivatives of g_{00}. Since the FRW metric has constant g_{00}, these terms vanish, and we are left with

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^0}. \tag{2.21}$$

The derivative is nonzero only if α and β are spatial indices, which will be identified with Roman letters i, j running from 1 to 3. Since $x^0 = t$, we have

$$\begin{aligned} \Gamma^0{}_{00} &= 0 \\ \Gamma^0{}_{0i} &= \Gamma^0_{i0} = 0 \\ \Gamma^0{}_{ij} &= \delta_{ij}\dot{a}a \end{aligned} \tag{2.22}$$

where overdots indicate derivatives with respect to time.[1] It is a straightforward, but useful, exercise to show that $\Gamma^i{}_{\alpha\beta}$ is nonzero only when one of its lower indices is zero and one is spatial, so that

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij}\frac{\dot{a}}{a} \tag{2.23}$$

with all other $\Gamma^i{}_{\alpha\beta}$ zero.

[1] I will use this convention until Chapter 4. After that, overdots will denote derivatives with respect to conformal time.

This has been a long, rather formal subsection, opening with the generalization of the geodesic equation to curved space-time and then proceeding with a calculation of the Christoffel symbol in the expanding universe described by the FRW metric. Before completing our main task and using the Einstein equations to derive Eq. (1.2), let's take a break and apply the geodesic equation to a single particle. In particular let's see how a particle's energy changes as the universe expands. We'll do the calculation here for a massless particle; an almost identical problem for a massive particle is relegated to Exercise 4.

Start with the four-dimensional energy–momentum vector $P^\alpha = (E, \vec{P})$, whose time component is the energy. We use this four-vector to define the parameter λ in Eq. (2.18):

$$P^\alpha = \frac{dx^\alpha}{d\lambda}. \tag{2.24}$$

This is an implicit definition of λ. Fortunately, one never needs to find λ explicitly, for it can be directly eliminated by noting that

$$\begin{aligned} \frac{d}{d\lambda} &= \frac{dx^0}{d\lambda}\frac{d}{dx^0} \\ &= E\frac{d}{dt}. \end{aligned} \tag{2.25}$$

The zeroth component of the geodesic equation (2.18) then becomes

$$E\frac{dE}{dt} = -\Gamma^0{}_{ij}P^iP^j \tag{2.26}$$

where the equality holds since only the spatial components of $\Gamma^0{}_{\alpha\beta}$ are nonzero. Inserting these components leads to a right-hand side equal to $-\delta_{ij}a\dot{a}P^iP^j$. A massless particle has energy–momentum[2] vector $(E, \vec{P})$ with zero magnitude:

$$g_{\mu\nu}P^\mu P^\nu = -E^2 + \delta_{ij}a^2P^iP^j = 0 \tag{2.27}$$

which enables us to write the right hand side of Eq. (2.26) as $-(\dot{a}/a)E^2$. Therefore, the geodesic equation yields

$$\frac{dE}{dt} + \frac{\dot{a}}{a}E = 0, \tag{2.28}$$

the solution to which is

$$E \propto \frac{1}{a}. \tag{2.29}$$

This confirms our hand-waving argument in Chapter 1 that the energy of a massless particle should decrease as the universe expands since it is inversely proportional to its wavelength, which is being stretched along with the expansion. In Chapter 4 we will rederive this result in yet another way using the Boltzmann equation.

[2]Note that $\vec{P}$ measures motion on the comoving (nonexpanding) grid. The physical momentum which measures changes in physical distance is related to $\vec{P}$ by a factor of a. Hence the factor of a^2 in Eq. (2.27).

2.1.3 Einstein Equations

If you did a word search on the previous two subsections, you might be surprised to discover that the words "general relativity" never appeared. The concept of a metric and the realization that nontrivial metrics affect geodesics both exist completely independently of general relativity. The part of general relativity that is hidden above is that gravitation can be described by a metric, in our case by Eq. (2.4). There is a second aspect of general relativity, though: one which relates the metric to the matter and energy in the universe. This second part is contained in the Einstein equations, which relate the components of the Einstein tensor describing the geometry to the energy–momentum tensor describing the energy:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu}. \tag{2.30}$$

Here $G_{\mu\nu}$ is the Einstein tensor; $R_{\mu\nu}$ is the *Ricci tensor*, which depends on the metric and its derivatives; $\mathcal{R}$, the *Ricci scalar*, is the contraction of the Ricci tensor ($\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu}$); G is Newton's constant; and $T_{\mu\nu}$ is the energy–momentum tensor. We will spend some time on the energy–momentum tensor in Section 2.3. For now, all we need to know is that it's a symmetric tensor describing the constituents of the universe. The left-hand side of Eq. (2.30) is a function of the metric, the right a function of the energy: the Einstein equations relate the two.

The Ricci tensor is most conveniently expressed in terms of the Christoffel symbol,

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha} \Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu} \Gamma^\beta{}_{\mu\alpha}. \tag{2.31}$$

Here commas denote derivatives with respect to x. So, for example, $\Gamma^\alpha{}_{\mu\nu,\alpha} \equiv \partial \Gamma^\alpha{}_{\mu\nu} / \partial x^\alpha$. Although this expression looks formidable, we have already done the hard work by computing the Christoffel symbol in an FRW universe. It turns out that there are only two sets of nonvanishing components of the Ricci tensor: one with $\mu = \nu = 0$ and the other with $\mu = \nu = i$.

Consider

$$R_{00} = \Gamma^\alpha{}_{00,\alpha} - \Gamma^\alpha{}_{0\alpha,0} + \Gamma^\alpha{}_{\beta\alpha} \Gamma^\beta{}_{00} - \Gamma^\alpha{}_{\beta 0} \Gamma^\beta{}_{0\alpha}. \tag{2.32}$$

Recall that the Christoffel symbol vanishes if its two lower indices are zero, so the first and third terms on the right vanish. Similarly, the indices α and β in the second and fourth terms must be spatial. We are left with

$$R_{00} = -\Gamma^i{}_{0i,0} - \Gamma^i{}_{j0} \Gamma^j{}_{0i}. \tag{2.33}$$

Using Eq. (2.23) leads directly to

$$\begin{aligned} R_{00} &= -\delta_{ii} \frac{\partial}{\partial t} \left(\frac{\dot{a}}{a} \right) - \left(\frac{\dot{a}}{a} \right)^2 \delta_{ij} \delta_{ij} \\ &= -3 \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] - 3 \left(\frac{\dot{a}}{a} \right)^2 \end{aligned}$$

$$= -3\frac{\ddot{a}}{a}. \tag{2.34}$$

The factors of 3 on the second line arise since δ_{ii} means sum over all three spatial indices, counting one for each. I will leave the space-space component as an exercise; it is

$$R_{ij} = \delta_{ij}\left[2\dot{a}^2 + a\ddot{a}\right]. \tag{2.35}$$

The next ingredient in the Einstein equations is the Ricci scalar, which we can now compute since

$$\begin{aligned}\mathcal{R} &\equiv g^{\mu\nu}R_{\mu\nu} \\ &= -R_{00} + \frac{1}{a^2}R_{ii}.\end{aligned} \tag{2.36}$$

Again the sum over i leads to a factor of 3, so

$$\mathcal{R} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{2.37}$$

To understand the evolution of the scale factor in a homogeneous universe, we need consider only the time-time component of the Einstein equations:

$$R_{00} - \frac{1}{2}g_{00}\mathcal{R} = 8\pi G T_{00}. \tag{2.38}$$

The terms on the left sum to $3\dot{a}^2/a^2$, and the time-time component of the energy–momentum tensor is simply the energy density ρ. So we finally have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{2.39}$$

To get this into the form of Eq. (1.2), recall that the left-hand side here is the square of the Hubble rate and that the critical density was defined as $\rho_{\rm cr} \equiv 3H_0^2/8\pi G$. So, dividing both sides by H_0^2 leads to

$$\frac{H^2(t)}{H_0^2} = \frac{\rho}{\rho_{\rm cr}}. \tag{2.40}$$

Here the energy density ρ counts the energy density from all species: matter, radiation, and the dark energy. In our derivation, we have assumed the universe is flat, so Eq. (2.40) does not contain a term corresponding to the curvature of the universe. I leave it as an exercise to derive the Einstein equation in an open universe.

2.2 DISTANCES

We can anticipate that measuring distance in an expanding universe will be a tricky business. Referring back to the expanding grid of Figure 1.1, we immediately see two possible ways to measure distance, the comoving distance which remains fixed as the universe expands or the physical distance which grows simply because of the expansion. Frequently, neither of these two measures accurately describes the process of interest. For example light leaving a distant QSO at redshift 3 starts its journey towards us when the scale factor was only a quarter of its present value and ends it today when the universe has expanded by a factor of 4. Which distance do we use in that case to relate, say, the luminosity of the QSO to the flux we see?

The fundamental distance measure, from which all others may be calculated, is the distance on the comoving grid. If the universe is flat, as we will assume through most of this book, then computing distances on the comoving grid is easy: the distance between two points $\vec{x}_1$ and $\vec{x}_2$ is equal to $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}$.

One very important comoving distance is the distance light could have traveled (in the absence of interactions) since $t = 0$. In a time dt, light travels a comoving distance $dx = dt/a$ (recall that we are setting $c = 1$), so the total comoving distance light could have traveled is

$$\eta \equiv \int_0^t \frac{dt'}{a(t')}. \tag{2.41}$$

The reason this distance is so important is that no information could have propagated further (again on the comoving grid) than η since the beginning of time. Therefore, regions separated by distance greater than η are not causally connected. If they appear similar, we should be suspicious! We can think of η then as the *comoving horizon*. We can also think of η, which is monotonically increasing, as a time variable and call it the *conformal time*. Just like the time t, the temperature T, the redshift z, and the scale factor a, η can be used to discuss the evolution of the universe. In fact, for most purposes η is the most convenient time variable, so when we begin to study the evolution of perturbations, we will use it instead of t. In some simple cases, η can be expressed analytically in terms of a (Exercise 11). For example, in a matter-dominated universe, $\eta \propto a^{1/2}$, while $\eta \propto a$ in a radiation-dominated universe.

Another important comoving distance is that between a distant emitter and us. In that case, the comoving distance out to an object at scale factor a (or redshift $z = 1/a - 1$) is

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')}. \tag{2.42}$$

Here I have changed the integration over t' to one over a', which brings in the additional factor of $da/dt = aH$ in the denominator. Typically we can see objects out to $z \lesssim 6$; at these late times radiation can be ignored (recall Figure 1.3). If the universe is purely matter dominated at such times, then $H \propto a^{-3/2}$ and we can do the integral in Eq. (2.42) analytically,

$$\chi^{\text{Flat.MD}}(a) = \frac{2}{H_0}\left[1 - a^{1/2}\right]$$

$$= \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]. \tag{2.43}$$

This comoving distance goes as z/H_0 for small z (verifying our hand-waving discussion of the small-z Hubble diagram in Section 1.2) and then asymptotes to $2/H_0$ as z gets very large.

A classic way to determine distances in astronomy is to measure the angle θ subtended by an object of known physical size l. The distance to that object (assuming the angle subtended is small) is then

$$d_A = \frac{l}{\theta}. \tag{2.44}$$

The subscript $_A$ here denotes *angular diameter distance.* To compute the angular diameter distance in an expanding universe, we first note that the comoving size of the object is l/a. The comoving distance out to the object is given by Eq. (2.42), so the angle subtended is $\theta = (l/a)/\chi(a)$. Comparing with Eq. (2.44), we see that the angular diameter distance is

$$d_A^{\text{flat}} = a\chi = \frac{\chi}{1+z}. \tag{2.45}$$

Note that the angular diameter distance is equal to the comoving distance at low redshift, but actually decreases at very large redshift. At least in a flat universe, objects at large redshift appear larger than they would at intermediate redshift! The superscript here is a warning that this result holds only in a flat universe. In an open or closed universe, the curvature density is defined as $\Omega_k = 1 - \Omega_0$ where Ω_0 is the ratio of total to critical density today, including contributions from matter, radiation, and any other form of energy such as a cosmological constant. If the curvature is nonzero, the angular diameter distance generalizes to

$$d_A = \frac{a}{H_0\sqrt{|\Omega_k|}} \begin{cases} \sinh\left[\sqrt{\Omega_k}H_0\chi\right] & \Omega_k > 0 \\ \sin\left[\sqrt{-\Omega_k}H_0\chi\right] & \Omega_k < 0 \end{cases}. \tag{2.46}$$

Note that both of these expressions reduce to the flat case in the limit that the curvature density Ω_k goes to zero. Figure 2.3 shows the angular diameter distance in a flat universe, both with and without a cosmological constant.

Another way of inferring distances in astronomy is to measure the flux from an object of known luminosity. Recall that (forgetting about expansion for the moment) the observed flux F a distance d from a source of known luminosity L is

$$F = \frac{L}{4\pi d^2} \tag{2.47}$$

since the total luminosity through a spherical shell with area $4\pi d^2$ is constant. How does this result generalize to an expanding universe? Again it is simplest to work

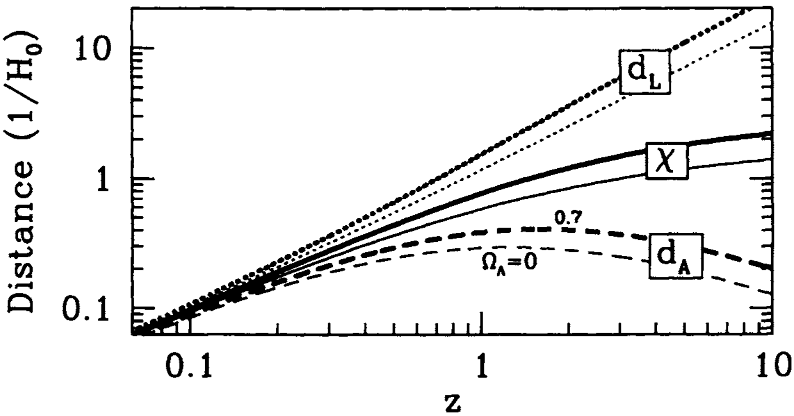

Figure 2.3. Three distance measures in a flat expanding universe. From top to bottom, the luminosity distance, the comoving distance, and the angular diameter distance. The pair of lines in each case is for a flat universe with matter only (light curves) and 70% cosmological constant Λ (heavy curves). In a Λ-dominated universe, distances out to fixed redshift are larger than in a matter-dominated universe.

on the comoving grid, this time with the source centered at the origin. The flux we observe is

$$F = \frac{L(\chi)}{4\pi\chi^2(a)} \tag{2.48}$$

where $L(\chi)$ is the luminosity through a (comoving) spherical shell with radius $\chi(a)$. To further simplify, let's assume that the photons are all emitted with the same energy. Then $L(\chi)$ is this energy multiplied by the number of photons passing through a (comoving) spherical shell per unit time. In a fixed time interval, photons travel farther on the comoving grid at early times than at late times since the associated physical distance at early times is smaller. Therefore, the number of photons crossing a shell in the fixed time interval will be smaller today than at emission, smaller by a factor of a. Similarly, the energy of the photons will be smaller today than at emission, because of expansion. Therefore, the energy per unit time passing through a comoving shell a distance $\chi(a)$ (i.e., our distance) from the source will be a factor of a^2 smaller than the luminosity at the source. The flux we observe therefore will be

$$F = \frac{La^2}{4\pi\chi^2(a)} \tag{2.49}$$

where L is the luminosity at the source. We can keep[3] Eq. (2.47) in an expanding universe as long as we define the *luminosity distance*

$$d_L \equiv \frac{\chi}{a}. \tag{2.50}$$

[3]Actually there is one more difference that needs to be accounted for: the observed luminosity is related to the emitted luminosity at a different wavelength. Here we have assumed a detector which counts all the photons.

The luminosity distance is shown in Figure 2.3.

All three distances are larger in a universe with a cosmological constant than in one without. This follows from the fact that the energy density, and therefore the expansion rate, is smaller in a Λ-dominated universe. The universe was therefore expanding more slowly early on, and light had more time to travel from distant objects to us. These distant objects will therefore appear fainter to us than if the universe was dominated by matter only.

2.3 EVOLUTION OF ENERGY

Let us return to the energy-momentum tensor on the right-hand side of the Einstein equations. We will eventually include perturbations to $T^\mu{}_\nu$, but in the spirit of this chapter, first consider the case of a perfect isotropic fluid. Then,

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix} \tag{2.51}$$

where $\mathcal{P}$ is the pressure of the fluid.

How do the components of the energy-momentum tensor evolve with time? Consider first the case where there is no gravity and velocities are negligible. The pressure and energy density in that case evolve according to the continuity equation, $\partial\rho/\partial t = 0$, and the Euler equation, $\partial\mathcal{P}/\partial x^i = 0$. This can be promoted to a 4 component conservation equation for the energy-momentum tensor: $\partial T^\mu{}_\nu/\partial x^\mu = 0$. In an expanding universe, however, the conservation criterion must be modified. Instead, conservation implies the vanishing of the *covariant* derivative:

$$T^\mu{}_{\nu;\mu} \equiv \frac{\partial T^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu}T^\mu{}_\alpha. \tag{2.52}$$

The vanishing of $T^\mu{}_{\nu;\mu}$ is four separate equations; let's consider the $\nu = 0$ component. This is

$$\frac{\partial T^\mu{}_0}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_0 - \Gamma^\alpha{}_{0\mu}T^\mu{}_\alpha = 0 \tag{2.53}$$

Since we are assuming isotropy, $T^i{}_0$ vanishes, so the dummy indices μ in the first term and α in the second must be equal to zero:

$$\frac{-\partial\rho}{\partial t} - \Gamma^\mu{}_{0\mu}\rho - \Gamma^\alpha{}_{0\mu}T^\mu{}_\alpha = 0. \tag{2.54}$$

From Eq. (2.23), $\Gamma^\alpha{}_{0\mu}$ vanishes unless α, μ are spatial indices equal to each other, in which case it is $\dot{a}/a$. So, the conservation law in an expanding universe reads

$$\frac{\partial\rho}{\partial t} + \frac{\dot{a}}{a}\left[3\rho + 3\mathcal{P}\right] = 0. \tag{2.55}$$

Rearranging terms, we have

$$a^{-3}\frac{\partial\left[\rho a^3\right]}{\partial t} = -3\frac{\dot{a}}{a}\mathcal{P}. \tag{2.56}$$

The conservation law can be applied immediately to glean information about the scaling of both matter and radiation with the expansion. Matter has effectively zero pressure, so

$$\frac{\partial\left[\rho_{\rm m} a^3\right]}{\partial t} = 0 \tag{2.57}$$

implying that the energy density of matter $\rho_{\rm m} \propto a^{-3}$. We anticipated this result in Chapter 1 based on the simple notion that the mass remains constant, while the number density scales as the inverse volume. The application to radiation also offers no surprises. Radiation has $\mathcal{P} = \rho/3$ (Exercise 14), so working from Eq. (2.55),

$$\begin{aligned}\frac{\partial\rho_r}{\partial t} + \frac{\dot{a}}{a}4\rho_r &= a^{-4}\frac{\partial\left[\rho_r a^4\right]}{\partial t}\\ &= 0.\end{aligned} \tag{2.58}$$

Therefore, the energy density of radiation $\rho_r \propto a^{-4}$, accounting for the decrease in energy per particle as the universe expands.

Through most of the early universe, reactions proceeded rapidly enough to keep particles in equilibrium, different species sharing a common temperature. We will often want to express the energy density and pressure in terms of this temperature. For this reason, and many others which will emerge over the next few chapters, it is convenient to introduce the occupation number, or *distribution function*, of a species. This counts the number of particles in a given region in phase space around position $\vec{x}$ and momentum $\vec{p}$.[4] The energy of a species is then obtained by summing the energy over all of phase space elements: $\sum f(\vec{x},\vec{p})E(p)$ with $E(p) = \sqrt{p^2+m^2}$. How many phase space elements are there in a region of "volume" $d^3x d^3p$? By Heisenberg's principle, no particle can be localized into a region of phase space smaller than $(2\pi\hbar)^3$, so this is the size of a fundamental element. Therefore, the number of phase space elements in d^3xd^3p is $d^3xd^3p/(2\pi\hbar)^3$ (see Figure 2.4), and the energy density is

$$\rho_i = g_i\int\frac{d^3p}{(2\pi)^3}\, f_i(\vec{x},\vec{p})E(p) \tag{2.59}$$

where i labels different species, g_i is the degeneracy of the species (e.g., equal to 2 for the photon for its spin states), and I have gone back to $\hbar = 1$. In equilibrium at temperature T, bosons such as photons have Bose–Einstein distributions,

$$f_{\rm BE} = \frac{1}{e^{(E-\mu)/T}-1}, \tag{2.60}$$

and fermions such as electrons have Fermi-Dirac distributions,

[4] By p here I mean not the comoving momentum defined in Eq. (2.24), but rather the proper momentum which decreases with the expansion. See Exercise 15 for a discussion.

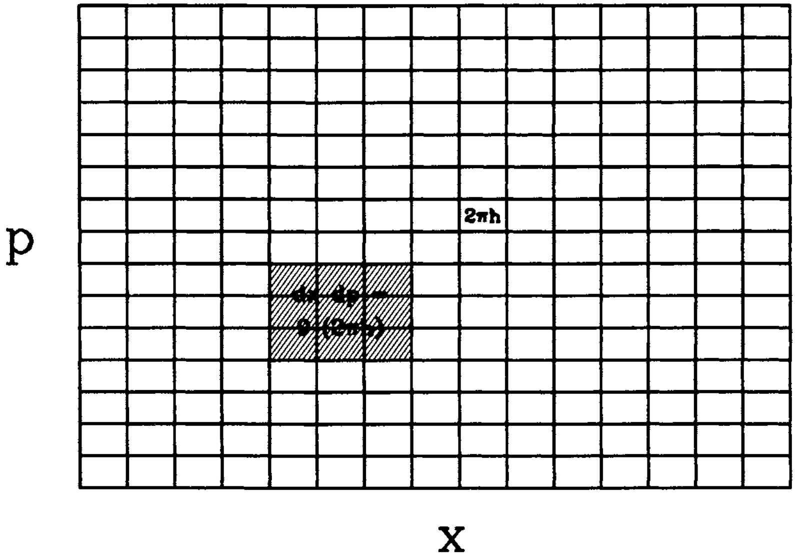

Figure 2.4. Phase space of position and momentum in one dimension. Volume of each cell is $2\pi\hbar$, the smallest region into which a particle can be confined because of Heisenberg's principle. Shaded region has infinitesmal volume $dxdp$. This covers nine cells. To count the appropriate number of cells, therefore, the phase space integral must be $\int dxdp/(2\pi\hbar)$.

$$f_{\rm FD} = \frac{1}{e^{(E-\mu)/T} + 1}, \tag{2.61}$$

with μ the chemical potential. It should be noted that these distributions do not depend on position $\vec{x}$ or on the direction of the momentum $\hat{p}$, simply on the magnitude p. This is a feature of the zero-order, smooth universe. When we come to consider inhomogenities and anisotropies, we will see that the distribution functions have small perturbations around these zero order values, and the perturbations do depend on position and on the direction of propagation.

The pressure can be similarly expressed as an integral over the distribution function,

$$\mathcal{P}_i = g_i \int \frac{d^3p}{(2\pi)^3}\, f_i(\vec{x},\vec{p}) \frac{p^2}{3E(p)}. \tag{2.62}$$

For almost all particles at almost all times in the universe, the chemical potential is much smaller than the temperature. To a good approximation, then, the distribution function depends only on E/T and the pressure satisfies (Exercise 14)

$$\frac{\partial \mathcal{P}_i}{\partial T} = \frac{\rho_i + \mathcal{P}_i}{T}. \tag{2.63}$$

This relation can be used to show that the entropy density in the universe scales as a^{-3}. To see this, let's rewrite Eq. (2.56) as

$$a^{-3}\frac{\partial\left[(\rho+\mathcal{P})a^3\right]}{\partial t}-\frac{\partial\mathcal{P}}{\partial t}=0. \tag{2.64}$$

The derivative of the pressure with respect to time can be written as $(dT/dt)(\partial\mathcal{P}/\partial T)$ so

$$\begin{aligned}a^{-3}\frac{\partial\left[(\rho+\mathcal{P})a^3\right]}{\partial t}-\frac{dT}{dt}\frac{\rho+\mathcal{P}}{T}&=a^{-3}T\frac{\partial}{\partial t}\left[\frac{(\rho+\mathcal{P})a^3}{T}\right]\\&=0.\end{aligned} \tag{2.65}$$

So the entropy density[5]

$$s\equiv\frac{\rho+\mathcal{P}}{T} \tag{2.66}$$

scales as a^{-3}. Although we have framed the argument in terms of a single species, this scaling holds for the total entropy including all species in equilibrium. In fact, even if two species have different temperatures, the sum of their entropy densities still scales as a^{-3}. We will make use of this fact shortly when computing the relative temperatures of neutrinos and photons in the universe.

2.4 COSMIC INVENTORY

Armed with an expression for the energy density of a given species (Eq. (2.59)), and a knowledge of how it evolves in time (Eq. (2.56)), we can now tackle quantitatively the question of how much energy is contributed by the different components of the universe.

2.4.1 Photons

The temperature of the CMB photons has been measured extraordinarily precisely by the FIRAS instrument aboard the COBE satellite, $T=2.725\pm0.002$K (Mather *et al.*, 1999). The energy density associated with this radiation is

$$\rho_\gamma=2\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{e^{p/T}-1}\,p. \tag{2.67}$$

The factor of 2 in front of Eq. (2.67) accounts for the two spin states of the photon. The energy of a given state is simply equal to p since the photon is massless. The chemical potential is zero; we expect this theoretically because early in the universe, photon number is not conserved (e.g., electrons and positrons can annihilate to produce photons). We also know it observationally because the spectrum of the CMB has been measured so accurately. The limits on a chemical potential are $\mu/T<9\times10^{-5}$ (Fixsen *et al.*, 1996), so μ can be safely ignored. Since there is

[5]Technically, there is another term in the entropy density—proportional to the chemical potential—but, as mentioned above, this term is usually irrelevant in cosmology. Even with nonzero chemical potential, though, the entropy density scales as a^{-3}.

no angular dependence in the integrand of Eq. (2.67). the angular integral yields a factor of 4π and we are left with a one-dimensional integral. Define a dummy variable $x \equiv p/T$; then

$$\rho_\gamma = \frac{8\pi T^4}{(2\pi)^3} \int_0^\infty \frac{dx\, x^3}{e^x - 1}. \tag{2.68}$$

The integral can be expressed in terms of the Riemann ζ function; it is $6\zeta(4) = \pi^4/15$, so

$$\rho_\gamma = \frac{\pi^2}{15} T^4. \tag{2.69}$$

Since we derived (Eq. (2.58)) that the energy density of radiation scales as a^{-4}, the temperature of the CMB must scale as a^{-1}.

It will be useful to have all energy densities in the same units. The simplest way to do this is to divide all energy densities by the critical density today.[6] Thus,

$$\begin{aligned}\frac{\rho_\gamma}{\rho_{\rm cr}} &= \frac{\pi^2}{15}\left(\frac{2.725\text{K}}{a}\right)^4 \frac{1}{8.098 \times 10^{-11} h^2 \text{eV}^4} \\ &= \frac{2.47 \times 10^{-5}}{h^2 a^4}\end{aligned} \tag{2.70}$$

where to get the last line, it is useful to remember the conversion between kelvin and eV: 11605 K = 1 eV. To reiterate an important point, the photon energy density in Eq. (2.70) depends on time via the scale factor, but has no spatial dependence. This is because we have used the zero-order distribution function, the Bose–Einstein function, for the photons. In fact there are small perturbations around this zero-order distribution function. These do have a spatial dependence and correspond to the anisotropies in the CMB.

2.4.2 Baryons

Unlike the CMB, baryons[7] cannot be simply described as a gas with a temperature and zero chemical potential. Therefore, the baryon density must be measured directly, not via a temperature. There are now four established ways of measuring the baryon density, and these all seem to agree reasonably well (Fukugita, Hogan, and Peebles 1998). These are all measurements at different redshifts, and we know that the density scales as a^{-3}, so to facilitate comparison, one defines Ω_b via

$$\frac{\rho_b}{\rho_{\rm cr}} = \Omega_b a^{-3}. \tag{2.71}$$

That is, Ω_b is the ratio of the baryon density to the critical density today.

[6] The critical density — just like the Hubble rate which defines it — changes with time. However, it is common to define $\rho_{\rm cr}$ to be a constant. the critical density today, and I will follow this convention.

[7] I refer to all the nuclei and electrons in the universe as baryons. This is technically incorrect (electrons are *leptons*), but nuclei are so much more massive than electrons that virtually all the mass *is* in the baryons.

The simplest way is to observe the baryons today in galaxies. The greatest contribution to the density, though, comes not from stars in galaxies, but rather from gas in groups of galaxies. In these groups, Ω_b is about 0.02. The second way to count baryons is by looking at the spectra of distant quasars. The amount of light absorbed from these beacons is a measure of the intervening hydrogen, and hence the baryon density. These estimates (Rauch *et al.*, 1997) suggest $\Omega_b h^{1.5} \simeq 0.02$ with a fairly large uncertainty. A third method is to infer the baryon density by careful scrutiny of the anisotropies in the universe. As we will see in Chapter 8, these depend on the baryon density. Preliminary results (Pryke *et al.*, 2001; Netterfield *et al.*, 2001) give $\Omega_b h^2 = 0.024^{+0.004}_{-0.003}$ from the CMB. Finally, we will see in Chapter 3 that the light element abundances are sensitive to the baryon density, and that estimates of these abundances pin down $\Omega_b h^2 = 0.0205 \pm 0.0018$.

Remarkably, then, these estimates of the baryon density with very different techniques all agree.[8] They all place the baryon density at roughly 2–5% of the critical density. The total matter density in the universe is higher than this, so there must be matter in the universe that is nonbaryonic.

2.4.3 Matter

All of the methods of measuring the baryon density mentioned above involve the interaction of matter and radiation. For example, simply counting stars works at some level because we roughly know how much mass is required to output the light from a typical star. There are, however, methods of measuring the mass of matter that do not rely on the way light and matter interact. These classically have involved measuring the gravitational field in a given system, thereby inferring information about the mass responsible for that field. Figure 2.5 shows the inferred mass-to-light ratios of many systems, ranging from galaxies to superclusters. Historically this ratio was measured on small scales first, suggesting that the density in the universe was far less than critical. As more large-scale data were obtained, the steady increase in the mass/light ratio led some cosmologists to speculate that eventually we would find that the density was critical. Bahcall and collaborators (Bahcall, Lubin, and Dorman 1995; Bahcall *et al.*, 2000), however, have argued that mass-to-light ratios do not increase past $R \sim 1$Mpc; a leveling off occurs consistent with a matter density $\Omega_m \simeq 0.3$, where Ω_m is the ratio of the total matter density today to the critical density and

$$\rho_{\rm m} = \Omega_m \rho_{\rm cr} a^{-3}. \tag{2.72}$$

Recently a number of other techniques for inferring the matter density have emerged. We will see in Chapter 7 that the distribution of galaxies in the universe, in particular the power spectrum of this distribution, is very sensitive to $\Omega_m h$;

[8] Whether or not agreement holds is subject to debate. There have been claims (e.g., Persic and Salucci, 1992) that there is a *missing baryon problem* because the present-day abundance appears to be lower than that inferred from the light element abundances.

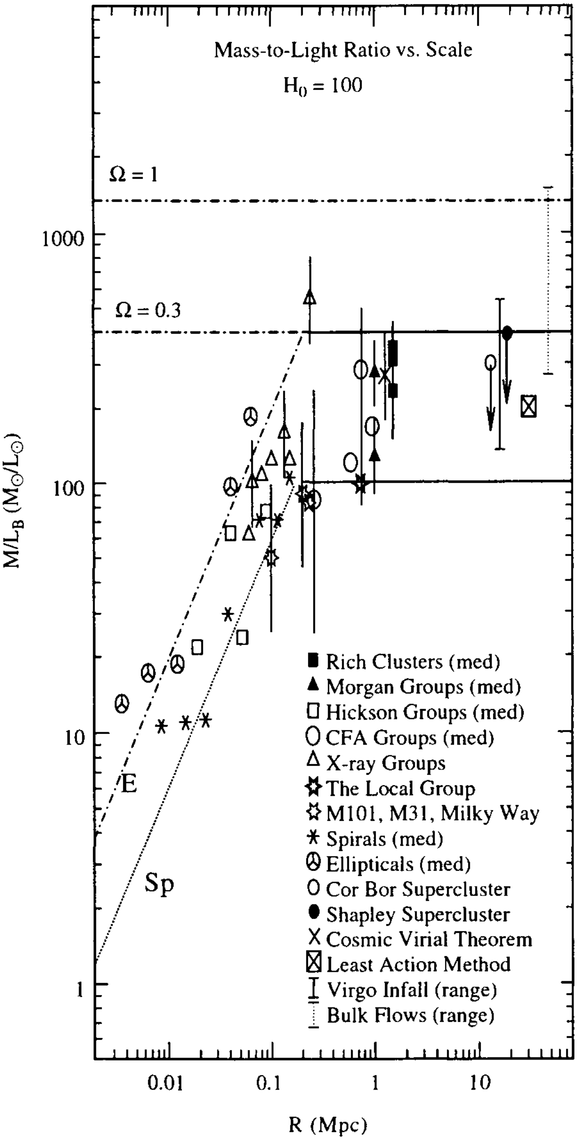

Figure 2.5. Mass vs. light ratio as a function of scale (Bahcall *et al.*, 2000). On the largest scales, the ratio flattens so that $\Omega_m \simeq 0.3$.

virtually all galaxy surveys[9] have inferred $\Omega_m h \simeq 0.2$. Another cosmological probe

[9]To mention three examples, the Automated Plate Measuring (APM) Survey, to be discussed further in Chapter 9, has been analyzed by Efstathiou and Moody (2001); the Two Degree Field (2DF) by Percival *et al.* (2001), and early data from the Sloan Digital Sky Survey by Dodelson *et al.* (2001). These groups found $\Omega_m h = 0.14^{+0.24}_{-0.09}, 0.20 \pm 0.03$, and $0.14^{+0.11}_{-0.06}$, respectively.

we will encounter in Chapter 9 is the cosmic velocity field (Strauss and Willick, 1995) and its relation to the observed galaxy distribution. These are related by the continuity equation, a relation sensitive to Ω_m. Again most of the measurements are clustered around $\Omega_m \simeq 0.3$.

Another way of measuring the total mass density is to pick out observations sensitive to Ω_b/Ω_m and use the apparent value of Ω_b to infer the matter density. For example, most of the baryonic mass in a galaxy cluster is in the form of hot gas. The ratio of the mass of gas in clusters of galaxies to the total mass can be measured either by looking for X-ray emission (White *et al.*, 1993) or by looking at the electron-heated CMB in the direction of the cluster (Grego *et al.*, 2001). If this ratio is characteristic of the universe as a whole — and it probably is, because clusters are so large — then the cosmic baryon to matter ratio is around 20%. Since baryons make up only about 5% of the critical density, the total matter density is inferred to be about 0.25. Another way of inferring the baryon/matter ratio is by looking for features in the power spectrum of galaxies; if the baryon fraction is truly of order 20%, then there will be wiggles in the spectrum (again Chapter 7). There are tentative hints of these wiggles in the early data from the Two Degree Field (2DF) survey (Percival *et al.*, 2001). These pin down $\Omega_b/\Omega_m = 0.15 \pm 0.07$, consistent with the cluster observations. Finally, the anisotropies in the CMB (Chapter 8) are sensitive to the matter density $\Omega_m h^2$. Recent determinations indicate $\Omega_m h^2 = 0.16 \pm 0.04$ (Pryke *et al.*, 2001; Netterfield *et al.*, 2001). Given the fact that current best estimates of the Hubble constant give $h = 0.72$, the CMB observations also are consistent with a matter density equal to 30% of the critical density.

We therefore have an enormous amount of evidence telling us that the baryon density is of order 5% the critical density, while the total matter density is some five times larger. Most of the matter in the universe must not be baryons. It must be some new form of matter: dark matter.

2.4.4 Neutrinos

The next component we need consider is the neutrino. Unlike photons and baryons, cosmic neutrinos have not been observed, so arguments about their contribution to the energy density are necessarily theoretical. However, these theoretical arguments are quite strong, based on very well-understood physics.

A basic understanding of the interaction rates of neutrinos enables us to argue that neutrinos were once kept in equilibrium with the rest of the cosmic plasma. Since they are fermions, their distribution was Fermi–Dirac with zero chemical potential. At late times, they lost contact with the plasma because their interactions are *weak*. Nonetheless, their distribution remained Fermi–Dirac, with their temperature simply falling as a^{-1}. The main task therefore is to relate the neutrino temperature to the photon temperature today. The tricky part of this is the annihilation of electrons and positrons when the cosmic temperature was of order the electron mass. Neutrinos lost contact with the cosmic plasma slightly before this annihilation, so they did not inherit any of the associated energy. The photons, which did, are therefore hotter than the neutrinos.

We can account for the annihilation of electrons and positrons by using the fact that the total entropy density s (Eq. (2.66)) scales as a^{-3}. Massless bosons contribute $2\pi^2 T^3/45$ to the entropy density for each spin state, while massless fermions contribute 7/8 this, and massive particles contribute negligibly (Exercise 17). Before annihilation, the fermions are electrons (2 spin states), positrons (2), neutrinos (3 generations and one spin state) and anti-neutrinos (3). The bosons are photons (2 spin states). So at a_1 before annihilation,

$$s(a_1) = \frac{2\pi^2}{45} T_1^3 \left[2 + (7/8)(2 + 2 + 3 + 3)\right]$$
$$= \frac{43\pi^2}{90} T_1^3 \tag{2.73}$$

where T_1 is the common temperature at a_1. After annihilation, the electrons and positrons have gone away and the photon and neutrino temperatures are no longer identical: we must distinguish between them. Therefore, the entropy density is

$$s(a_2) = \frac{2\pi^2}{45} \left[2T_\gamma^3 + \frac{7}{8} 6T_\nu^3\right]. \tag{2.74}$$

Equating $s(a_1)a_1^3$ with $s(a_2)a_2^3$ leads to

$$\frac{43}{2} (a_1 T_1)^3 = 4 \left[\left(\frac{T_\gamma}{T_\nu}\right)^3 + \frac{21}{8}\right] (T_\nu(a_2) a_2)^3. \tag{2.75}$$

But the neutrino temperature scales throughout as a^{-1}, so $a_1 T_1 = a_2 T_\nu(a_2)$. Therefore, the ratio of the two temperatures is

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}. \tag{2.76}$$

We can now evaluate the energy density of neutrinos in the universe. Let's sum up what we know about the cosmic abundance of neutrinos

- One spin degree of freedom for neutrinos
- Neutrino has antiparticle
- Three generations of neutrinos
- Neutrinos are fermions $\rightarrow$ Fermi–Dirac distribution function
- Neutrino temperature is lower by a factor of $(4/11)^{1/3}$ since photons are heated by e^+e^- annihilation

The first three items on the list then imply that the degeneracy factor of neutrinos is equal to six. The fourth means we need to change the denominator in the integrand in Eq. (2.67) to $e^{p/T} + 1$. The Fermi-Dirac integral is then smaller by a factor of 7/8. Finally, since the energy density of a massless particle scales as T^4, the last item implies that the neutrino energy density is smaller than the photon density by $(4/11)^{4/3}$. Putting all these factors together leads to

$$\rho_\nu = 3\,\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma. \tag{2.77}$$

Equivalently, if there were three species of massless neutrinos today, then their contribution to the energy density would be

$$\Omega_\nu \equiv \left.\frac{\rho_\nu}{\rho_{\rm cr}}\right|_{\rm today} = \frac{1.68\times10^{-5}}{h^2} \qquad m_\nu = 0. \tag{2.78}$$

In reality, all the neutrinos do not appear to be massless. Observations of neutrinos from both the sun (Bahcall, 1989) and from our atmosphere (Fukuda *et al.*, 1998) strongly suggest that neutrinos of different flavors (generations) oscillate into each other. This can happen only if the neutrinos have mass. The atmospheric neutrino observations in particular imply that at least one neutrino has a mass larger than 0.05 eV.[10] The energy density of a massive neutrino is

$$\rho_\nu = 2\int \frac{d^3p}{(2\pi)^3}\,\frac{\sqrt{p^2+m_\nu^2}}{e^{p/T_\nu}+1}. \tag{2.79}$$

At high temperatures, this reduces to Eq. (2.77) (without the 3), so when considering neutrinos in the early universe, it is often sufficient to use Eq. (2.77). Indeed, we will do this shortly when we come to esimtate the epoch at which the energy density of matter equals that of radiation. At late times, the massive neutrino energy density is $m_\nu n_\nu$, with the neutrino number density equal to $3n_\gamma/11$ (Exercise 18). As can be seen from Figure 2.6, the transition takes place when $T_\nu \sim m_\nu$. Therefore,

$$\Omega_\nu = \frac{m_\nu}{94h^2\mathrm{eV}} \qquad m_\nu \neq 0. \tag{2.80}$$

Those who trafficked in both astrophysics and particle physics (Gerstein and Zel'dovich, 1966; Marx and Szalay, 1972; Cowsik and McClelland, 1972) early on noted that the simple observation that the total density was not much greater than the critical density leads to constraints on neutrino mass, constraints much more stringent than those obtainable at accelerators. When the need for nonbaryonic dark matter first became evident, a number of cosmologists (e.g., Gunn *et al.*, 1978) proposed neutrinos as the natural candidate. Subsequent studies (Bond, Efstathiou, and Silk, 1980; White, Frenk, and Davis 1983) of the structure of the universe with neutrinos as the dominant dark matter component looked significantly different from the actual universe. Nonetheless, the possibility that neutrinos might make up a *fraction* of the total density reemerged in the 1990s. We can then hope to detect a trace amount — corresponding to masses smaller than an eV — by observing its effect on large-scale structure.

[10] The oscillation experiments are sensitive to mass *differences*, $m_2^2 - m_1^2$, so the actual constraint is that the mass squared difference is of order 10^{-3}. This could also be accomodated with two nearly degenerate masses with a small splitting.

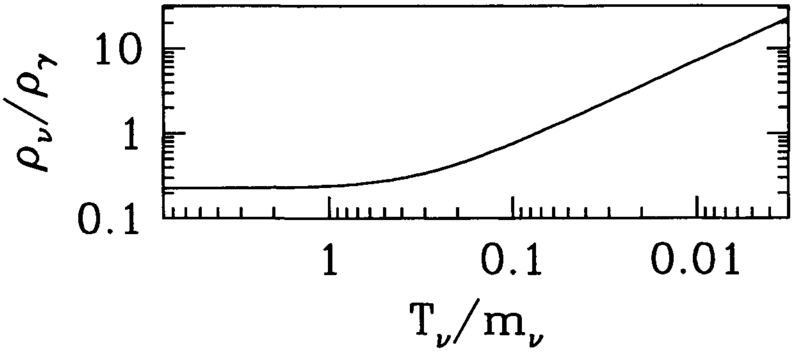

Figure 2.6. Energy density of one generation of massive neutrinos as compared with the density in the CMB. At high temperatures, the ratio is a fixed constant; at low temperatures, the neutrino behaves like nonrelativistic matter (scaling as a^{-3}) and so begins to dominate over the photon density (which scales as a^{-4}).

2.4.5 Dark Energy

There are two sets of evidence pointing toward the existence of something else, something beyond the radiation and matter itemized above. The first comes from a simple budgetary shortfall. The total energy density of the universe is very close to critical. We expect this theoretically (Chapter 6) and we observe it in the anisotropy pattern of the CMB (Chapter 8). Yet, the total matter density inferred from observations is only a third critical. The remaining two-thirds of the density in the universe must be in some smooth, unclustered form, dubbed[11] *dark energy*. The second set of evidence is more direct. Given the energy composition of the universe, one can compute a theoretical distance vs redshift diagram. This relation can then be tested observationally.

In 1998, two groups (Riess *et al.*, 1998, Perlmutter *et al.*, 1999) observing supernovae reported direct evidence for dark energy. The evidence is based on the difference between the luminosity distance in a universe dominated by dark matter and one dominated by dark energy. As Figure 2.3 indicates, the luminosity distance is larger for objects at high redshifts in a dark-energy-dominated universe. Therefore, objects of fixed intrinsic brightness will appear fainter if the universe is composed of dark energy.

[11] After the discovery, quite a bit of attention was focused on choosing an appropriate name. *Cosmological constant*, everyone's initial moniker, is too restrictive in that the energy density is constant at all times, and we do not yet know that this is true of the dark energy. *Variable cosmological constant* fixes that problems but introduces an inherent contradiction ("variable" and "constant"?). *Variable Lambda* using the Greek letter reserved for the cosmological constant is too obscure. *Quintessence* is a good choice: it expresses the fact that, after cosmological photons, baryons, neutrinos, and dark matter, there is a fifth essence in the universe. It seems to me that dark energy has become a bit more popular, with quintessence referring to the subset of models in which the energy density can be associated with a time-dependent scalar field.

More concretely, the luminosity distance of Eq. (2.50) can be used to find the apparent magnitude m of a source with absolute magnitude M. Magnitudes are related to fluxes via $m = -(5/2)\log(F)+$ constant. Since the flux scales as d_L^{-2}, the apparent magnitude $m = M + 5\log(d_L)+$ constant. The convention is that

$$m - M = 5\log\left(\frac{d_L}{10\mathrm{pc}}\right) + K \tag{2.81}$$

where K is a correction for the shifting of the spectrum into or out of the wavelength range measured due to expansion.

The two groups measured the apparent magnitudes of dozens of Type Ia supernovae, which are known to be standard candles, i.e., they have nearly identical absolute magnitudes. Although they were able to place tight constraints on dark energy using the many supernovae that they detected, we can get a feel for the measurement by simply considering two of these. Consider then Supernova 1997ap, found at redshift $z = 0.83$ with apparent magnitude $m = 24.32$, and Supernova 1992P, found at low redshift $z = 0.026$ with apparent magnitude $m = 16.08$. Since the absolute magnitudes of these are the same, the difference in apparent magnitudes is due solely to the difference in luminosity distance:

$$24.32 - 16.08 = 5\log\left(d_L(z = 0.83)\right) - 5\log\Big(d_L(z = 0.026)\Big). \tag{2.82}$$

The nearby luminosity distance is independent of cosmology, simply equal to $z/H_0 = 0.026/H_0$. Therefore, the only unknown remaining in Eq. (2.82) is fixed by the observations to be

$$H_0 d_L(z = 0.83) = 1.16. \tag{2.83}$$

In a flat, matter-dominated universe ($\Omega_m = 1$), the luminosity distance out to $z = 0.83$ is equal to $0.95H_0^{-1}$, whereas a universe with $\Omega_m = 0.3$ and a cosmological constant $\Omega_\Lambda = 0.7$ has a luminosity distance of $1.23H_0^{-1}$. The apparent magnitude of this single distant supernova then suggests that dark energy pervades the universe.

Of course, the discussion of the previous paragraph does not account for uncertainties (typical uncertainties in the magnitudes are of order 0.2), nor does it do a careful fit to all known supernovae, allow for extinction by dust, or allow for the variation of the absolute magnitude correlated with the duration. The supernova groups did all of those things, and emerged with the constraints shown in Figure 2.7. The two free cosmological parameters are the matter density Ω_m and a cosmological constant Ω_Λ, something we now recognize as one possible form of dark energy, one in which the energy density is constant. Note that the "theorists' dream" universe — flat and matter-dominated ($\Omega_m = 1$) — is excluded with high confidence. Indeed, even a pure open universe with $\Omega_m = 0.3, \Omega_\Lambda = 0$ is strongly disfavored by the supernova data.

While highly popular, Figure 2.7 suffers from two drawbacks. It allows for too much freedom in one sector and too little in another. Most of the region in the figure is taken up by a universe with both dark energy and nonzero curvature (not flat).

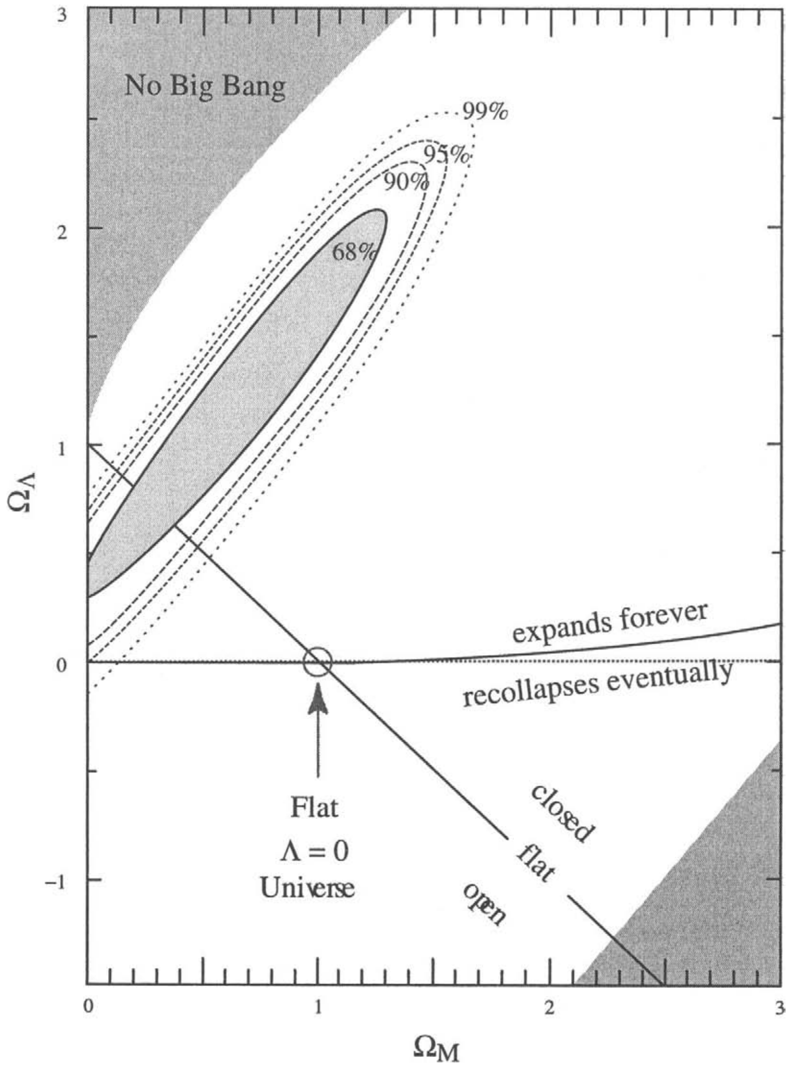

Figure 2.7. Constraints from Type Ia supernovae on the parameters $(\Omega_m, \Omega_\Lambda)$ (Perlmutter *et al.*, 1999). Flat, matter-dominated universe, the dot with $\Omega_m = 1, \Omega_\Lambda = 0$, is ruled out with high confidence. The line extending from upper left to lower right corresponds to a flat universe.

Although one or the other of these has been argued for, seldom have cosmologists suggested that the universe contains both. Thus, except for the "flat" line and the $\Omega_\Lambda = 0$ line, most of the region in Figure 2.7 is, at least aesthetically, unappealing: the figure allows for too much freedom. On the other hand, the only form of dark energy budgeted for is the cosmological constant. To open up other possibilities,

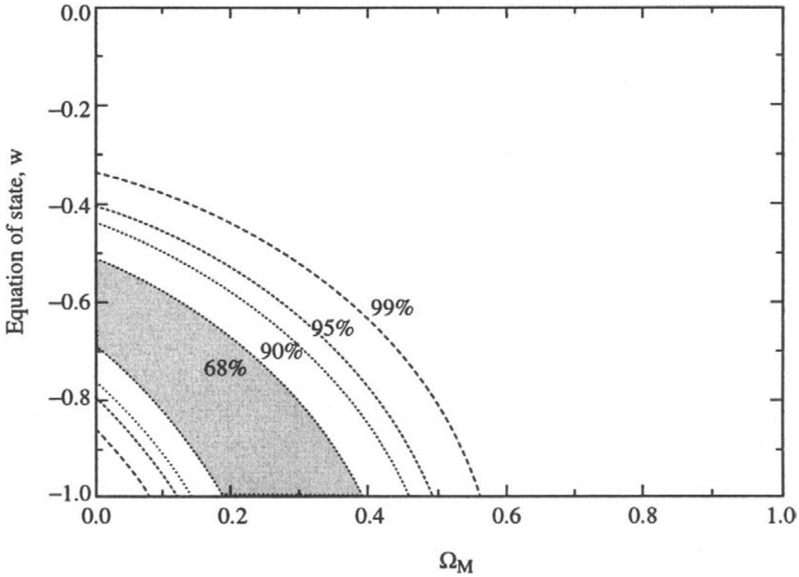

Figure 2.8. Constraints in a flat universe from Type Ia supernovae on the matter density (Ω_m) and equation of state of the dark energy (w) (Perlmutter *et al.*, 1999). Cosmological constant corresponds to $w = -1$, matter to $w = 0$.

consider Eq. (2.55) as applied to the cosmological constant. The only way for this equation to be satisfied with constant energy density is if the pressure is equal to $-\rho$. One might imagine energy with a slightly different pressure and therefore energy evolution. Define

$$w \equiv \frac{\mathcal{P}}{\rho}. \tag{2.84}$$

A cosmological constant corresponds to $w = -1$, matter to $w = 0$, and radiation to $w = 1/3$. With this new freedom, let's see what the supernova data imply for the equation of state of the dark energy if we fix the universe to be flat. Figure 2.8 shows that values of w greater than ~ -0.5 are disfavored; a cosmological constant is consistent with the data, but it is by no means the only possibility. Equation (2.55) can be integrated to find the evolution of the dark energy,

$$\rho_{\rm de} \propto \exp\left\{-3\int^a \frac{da'}{a'}\left[1 + w(a')\right]\right\}. \tag{2.85}$$

Note that—if w is constant—this expression agrees with our knowledge of the cases explicated above $w = 1/3, 0, -1$.

2.4.6 Epoch of Matter–Radiation Equality

The epoch at which the energy density in matter equals that in radiation is called *matter–radiation equality*. It has a special significance for the generation of large-scale structure and for the development of CMB anisotropies, because perturbations grow at different rates in the two different eras. It is therefore a useful exercise to calculate the epoch of matter–radiation equality. To do this, we need to compute the energy density of both matter and radiation, and then find the value of the scale factor at which they were equal.

Using Eqs. (2.70) and (2.78), we see that the total energy density in radiation is

$$\frac{\rho_r}{\rho_{\rm cr}} = \frac{4.15 \times 10^{-5}}{h^2 a^4} \equiv \frac{\Omega_r}{a^4}. \tag{2.86}$$

To calculate the epoch of matter–radiation equality, we equate Equations (2.86) and (2.72) to find

$$a_{\rm eq} = \frac{4.15 \times 10^{-5}}{\Omega_m h^2}. \tag{2.87}$$

A different way to express this epoch is in terms of redshift z; the redshift of equality is

$$1 + z_{\rm eq} = 2.4 \times 10^4 \Omega_m h^2. \tag{2.88}$$

Note that — obviously — as the amount of matter in the universe today, $\Omega_m h^2$, goes up, the redshift of equality also goes up. For our purposes it will be very important that the redshift of equality is at least several times larger than the redshift when photons decouple from matter, $z_* \simeq 10^3$. Thus, we expect photons to decouple when the universe is already well into the matter-dominated era.

2.5 SUMMARY

The smooth universe can be described with the Friedmann–Robertson–Walker metric given in Eq. (2.4), which implies that physical distances are related to coordinate (comoving) distances with the time-dependent scale factor $a(t)$. The time dependence of the metric is determined by the Einstein equations. The time-time component of the Einstein equations reduces to Eq. (2.39) in a flat universe.

Measuring distances in the expanding universe is tricky, but all relevant distances can be obtained from the comoving distance between us and a source at redshift z:

$$\chi(z) = \int_0^z \frac{dz'}{H(z')}. \tag{2.89}$$

Another important distance is that light could have traveled since $t = 0$. This is usually expressed as a time, the conformal time,

$$\eta = \int_0^t \frac{dt'}{a(t')} = \int_z^\infty \frac{dz'}{H(z')}. \tag{2.90}$$

The conformal time will be the natural time variable when we come to consider the evolution of perturbations in the universe.

Photons in the universe have a Bose–Einstein distribution with zero chemical potential, so their energy density can be determined by measuring their temperature. Neutrinos have a Fermi-Dirac distribution, also probably with zero chemical potential, but there is some ambiguity in their energy density because of our ignorance of the neutrino masses. Early on, this ambiguity is irrelevant since the temperatures are so much larger than the masses and neutrinos behave relativistically. Thus, the uncertainty in neutrino mass does not affect Big Bang nucleosynthesis at temperatures of order 1 MeV and probably not even the epoch of matter radiation equality at temperatures of order 1 eV. The neutrino temperature is a factor of $(4/11)^{1/3}$ smaller than the photon temperature. This, and the difference in statistics, implies that a species of massless neutrinos has an energy density equal to 0.23 times that of photons. A single neutrino generation with mass m_ν contributes $\Omega_\nu = 0.01(m_\nu/0.94\,\mathrm{eV}\,h^2)$. In addition to photons and neutrinos, the universe consists of baryons, best determined by nucleosynthesis to have $\Omega_b h^2 = 0.0205 \pm 0.0018$; dark matter ($\Omega_m \simeq 0.3$); and dark energy (with $\Omega_{\mathrm{de}} \simeq 0.7$), a new form of energy with negative pressure.

There is significantly more energy today in nonrelativistic matter than in radiation. However, since the energy density of radiation scales as a^{-4} while that of matter as a^{-3}, the very early universe was radiation dominated. The epoch at which the matter density was equal to the radiation density delineates these two regimes: $a_{\mathrm{eq}} = 4.15 \times 10^{-5}/\Omega_m h^2$.

SUGGESTED READING

My favorite book on general relativity at this level is *A First Course in General Relativity* (Schutz), which gives many simple examples to introduce the seemingly profound ideas of general relativity. Also very good are *Flat and Curved Spacetimes* (Ellis and Williams) and *Essential Relativity* (Rindler). Slightly more advanced is *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Weinberg), which also has a nice discussion of the early universe in Chapter 15. Two more advanced books are *General Relativity* (Wald) and the classic *Gravitation* (Misner, Thorne, and Wheeler). Some of the thermodynamics and statistical mechanics introduced in this chapter is presented in *The Early Universe* (Kolb and Turner). The distance formulae of Section 2.2 are covered in all standard texts. Neutrinos and their relation to cosmology are covered in the standard texts as well, but there are also several other good books focused solely on neutrinos and astrophysics, *Neutrino Astrophysics* by the pioneer of the field, Bahcall, and *Massive Neutrinos in Physics and Astrophysics* by Mohapatra and Pal.

A number of papers treat the topics in this chapter at an accessible level. An especially coherent review of all the different distance measures is given by Hogg (1999). Fukugita, Hogan, and Peebles (1999) do the baryon inventory outlined in Section 2.4.2. Since the supernova discoveries in the late 1990s, many popular articles have appeared attempting to explain the dark energy. The two seminal articles though — Perlmutter *et al.* (1999) and Riess *et al.* (1998) — are extremely clear and well worth reading.

EXERCISES

Exercise 1. Convert the following quantities by inserting the appropriate factors of $c, \hbar$, and k_B:

- $T_0 = 2.725\text{K} \to \text{eV}$
- $\rho_\gamma = \pi^2 T_0^4/15 \to \text{eV}^4$ and g cm^{-3}
- $1/H_0 \to$ cm
- $m_{\text{Pl}} \equiv 1.2 \times 10^{19}$ GeV $\to$K, cm^{-1}, sec^{-1}

Exercise 2. Show that the geodesic equation gets the correct equations of motion for a particle traveling freely in two dimensions using polar coordinates. You can get the Christoffel symbols one of two ways (or both!) and then proceed to **(b)**.
(a) Get the Christoffel symbol either directly from the term in brackets in Eq. (2.17) or from the 2D metric

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{2.91}$$

using Eq. (2.19). Show that the only nonzero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \quad ; \quad \Gamma^1_{22} = -r \tag{2.92}$$

with $1, 2$ corresponding to r, θ.

(b) Write down the two components of the geodesic equation using these Christoffel symbols. Show that these give the proper equations of motion for a particle traveling in a plane.

Exercise 3. The metric for a particle traveling in the presence of gravitational field is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{00} = -2\phi$ where ϕ is the Newtonian gravitational potential; $h_{i0} = 0$; and $h_{ij} = -2\phi\delta_{ij}$. Find the equation of motion for a massive particle traveling in this field.
(a) Show that $\Gamma^0{}_{00} = \partial\phi/\partial t$ and $\Gamma^i{}_{00} = \delta^{ij}\partial\phi/\partial x^j$.
(b) Show that the time component of the geodesic equation implies that energy $p^0 + m\phi$ is conserved.
(c) Show that the space components of the geodesic equation lead to $d^2x^i/dt^2 = -m\delta^{ij}\partial\phi/\partial x^j$ in agreement with Newtonian theory. Use the fact that the particle is nonrelativistic so $p^0 \gg p^i$.

Exercise 4. Find how the energy of a *massive*, nonrelativistic particle changes as the universe expands. Recall that in the massless case we used the fact that $g_{\mu\nu}P^\mu P^\nu = 0$. In this case, it is equal not to zero, but to $-m^2$.

Exercise 5. Fill in some of the blanks left in our derivation of the Einstein equations.
(a) Compute the Christoffel symbol $\Gamma^i{}_{\alpha\beta}$ for a flat FRW metric.
(b) Compute the spatial components of the Ricci tensor in a flat FRW universe, R_{ij}. Show that the space-time component, R_{i0}, vanishes.

Exercise 6. Show that the space-space component of the Einstein equations in a flat universe is

$$\frac{d^2a/dt^2}{a} + \frac{1}{2}\left(\frac{da/dt}{a}\right)^2 = -4\pi G\mathcal{P} \tag{2.93}$$

where $\mathcal{P}$ is the pressure, the $T^i{}_i$ (no sum over i) component of the energy–momentum tensor.

Exercise 7. Find and apply the metric, Christoffel symbols, and Ricci scalar for a particle trapped on the surface of a sphere with radius r.
(a) Using coordinates t, θ, ϕ, the metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{2.94}$$

Show that the only nonvanishing Christoffel symbols are $\Gamma^\theta{}_{\phi\phi}, \Gamma^\phi{}_{\phi\theta}$, and $\Gamma^\phi{}_{\theta\phi}$. Express these in terms of θ.
(b) Use these and the geodesic equation to find the equations of motion for the particle.

(c) Find the Ricci tensor. Show that contraction of this tensor leads to

$$\mathcal{R} \equiv g^{\mu\nu} R_{\mu\nu} = \frac{2}{r^2}. \tag{2.95}$$

Exercise 8. Apply the Einstein equations to the case of an open universe. The interval in an open universe is

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1+\Omega_k H_0^2 r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} \tag{2.96}$$

where r, θ, ϕ are the standard 3D spherical coordinates, and Ω_k is the curvature density.

(a) First compute the Christoffel symbols. Show that the only nonzero ones are equal to

$$\Gamma^i{}_{0j} = H\delta^i{}_j \qquad \Gamma^0{}_{ij} = g_{ij} H$$
$$\Gamma^i{}_{jk} = \frac{g^{il}}{2}\left[g_{lj,k} + g_{lk,j} - g_{jk,l}\right]. \tag{2.97}$$

(b) Show that the components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{ij} = g_{ij}\left[\frac{\ddot{a}}{a} + 2H^2 - \frac{2\Omega_k H_0^2}{a^2}\right]. \tag{2.98}$$

(c) From these, compute the Ricci scalar, and then derive the time-time component of Einstein equations.

Exercise 9. Show that the geodesic equation we derived in a flat universe implies that

$$\frac{d^2\vec{x}}{d\eta^2} = 0 \tag{2.99}$$

where η is the conformal time.

Exercise 10. Assume that there is only matter and radiation in the universe (no cosmological constant) and that the universe is flat ($\rho_0 = \rho_{\rm cr}$). Integrate Eq. (1.2) to determine the times when the cosmic temperature was 0.1 MeV and 1/4 eV.

Exercise 11. Derive some simple expressions for the conformal time η as a function of a.

(a) Show that $\eta \propto a^{1/2}$ in a matter dominated universe and a in one dominated by radiation.

(b) Consider a universe with only matter and radiation, with equality at $a_{\rm eq}$. Show that

$$\eta = \frac{2}{\sqrt{\Omega_m H_0^2}} \left[\sqrt{a + a_{\rm eq}} - \sqrt{a_{\rm eq}}\right] . \tag{2.100}$$

What is the conformal time today? At decoupling?

Exercise 12. Consider a galaxy of physical (visible) size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? Redshift 1? Do the calculation in a flat universe, first matter-dominated and then with 30% matter and 70% cosmological constant.

Exercise 13. How is the energy density of a gas of photons with a blackbody spectrum related to the specific intensity of the radiation? That is, what is the relation between ρ_γ and I_ν defined in Eq. (1.8)?

Exercise 14. (a) Compute the pressure of a relativistic species in equilibrium with temperature T. Show that $\mathcal{P} = \rho/3$ for both Fermi–Dirac and Bose–Einstein statistics.

(b) Suppose the distribution function depends only on E/T as it does in equilibrium. Find $d\mathcal{P}/dT$. A simple way to do this is to rewrite df/dT in the integral as $-(E/T)df/dE$ and then integrate Eq. (2.62) by parts.

Exercise 15. The general relativistic expression for the energy–momentum tensor in terms of the distribution functions is given by

$$T^\mu{}_\nu(\vec{x},t)\Big|_{\text{species i}} = g_i \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3} (-\det[g_{\alpha\beta}])^{-1/2} \frac{P^\mu P_\nu}{P^0} f_i(\vec{x},\vec{p},t) \tag{2.101}$$

where P_μ was defined in Eq. (2.24), g_i is the number of spin states for species i, and $\det[g_{\mu\nu}]$ is the determinant of the 4D matrix $g_{\mu\nu}$. Eliminate the comoving momenta P_μ in favor of the magnitude of the proper momentum defined via

$$p^2 \equiv g^{ij} P_i P_j \tag{2.102}$$

and the direction vector $\hat{p}$. Note that while the comoving momenta P_i remain constant as the universe expands, p falls off as a^{-1}. Show that the time-time component of Eq. (2.101) agrees with the expression for the energy density given in Eq. (2.59). Use the fact that $P^2 \equiv g_{\mu\nu}P^\mu P^\nu = -m^2$ for a particle of mass m.

Exercise 16. Plot $m-M$ as a function of redshift for a flat, matter-dominated universe (this can be done analytically) and for a flat universe with $\Omega_\Lambda = 0.7, \Omega_m = 0.3$ (for this you need to evaluate numerically a 1D integral). Neglect the K correction. Compare with Figure 1.7.

Exercise 17. Consider the entropy density, s, defined in Eq. (2.66). For a massless particle, you showed in Exercise 14 that $\mathcal{P} = \rho/3$, so $s = 4\rho/3T$. Express s as a function of T for both bosons and fermions (assumed massless) in equilibrium with zero chemical potential. Show that the entropy density for a massive particle in equilibrium ($T \ll m; \mu = 0$) is exponentially small.

Exercise 18. Show that the number density of one generation of neutrinos and anti-neutrinos in the universe today is

$$n_\nu = \frac{3}{11} n_\gamma = 112\text{cm}^{-3}.$$

For this calculation, you will also have to compute the photon number density; both can be expressed in terms of Riemann zeta functions (Eq. (C.27)). Using this result, verify Eq. (2.80).

Exercise 19. We computed the epoch of equality in the event that all three neutrinos are massless. Suppose instead that two are massless, but the third has mass $m = 0.1$ eV. What is a_{eq} is this case?