Section 3 Invariant Ensembles

Consider the density function of an invariant ensemble as

$$\mathbb{P}(H)\,dH = \frac{1}{Z}\exp\left(-\frac{\beta}{2}N\,\mathrm{Tr}\,V(H)\right)dH$$

For GOE / GEU, $V(H) = \frac{1}{2}H^2$. In this case, this model is called a β-ensemble

$\beta=1$ for GOE ; $\beta=2$ for GUE ; $\beta=4$ for GSE.
symplectic
(Gaussian quaternion ensemble)

For general β, it does not correspond to any Wigner ~~sym~~ ensembles.

① Probability density for β-ensemble

For invariant ensembles, their eigenvectors are uniformly distributed on the unit sphere. So people mainly focus on their eigenvalues. We will integrate out the eigenvectors in $\mathbb{P}(H)\,dH$ and show the following result.

Theorem 3.1 The joint probability density of the eigenvalues of H is given by

$$(*)\quad p_N(\lambda_1, \lambda_2, \cdots, \lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{2}N\sum_{i=1}^N V(\lambda_i)}.$$

Rmk 1: Recall that $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant.

Rmk 2: In the above formula, we can neglect the ordering of the eigenvalues. This can be done by multiplying a $\frac{1}{n!}$ factor when calculating various probabilities.

~~Rmk3: We can rewrite (*) as:~~ $\cancel{p_N(\lambda_1, \lambda_2, \cdots, \lambda_N) = \frac{1}{Z_N}\exp\left[\beta\log|\lambda_i - \lambda_j| - \frac{\beta}{2}N\right.}$

Rmk 3: When $|\lambda_i - \lambda_j| = 0$, we have $p_N(\lambda_1, \cdots, \lambda_N) = 0$. This indicates a repulsion between different eigenvalues.

Rmk 4: The eigenvalues are strongly correlated. It is useful to think of p_N as a Gibbs measure on N "particles" $\vec{\lambda} = (\lambda_1, \cdots, \lambda_N)$:

$$p_N(\vec{\lambda})\,d\vec{\lambda} = \exp\left[-\beta N \mathcal{H}(\vec{\lambda})\right]\frac{1}{Z_N},$$

$$\mathcal{H}(\vec{\lambda}) := \underbrace{\frac{1}{2}\sum_{i=1}^N V(\lambda_i)}_{\text{A confining potential.}} - \underbrace{\frac{1}{N}\sum_{i<j}\log|\lambda_i - \lambda_j|}_{\text{A logarithmic potential giving repulsions between particles.}}$$

[The two terms have the same order "N".]

This gives an important statistical mechanical system, called the "log-Coulomb gas".

To prove ~~the~~ Thm 3.1, we need to show that integrating out the eigenvectors gives the Vandermonde determinant.

Let $H = \mathcal{U} D \mathcal{U}^*$ be an eigenvalue decomposition of H, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and $\mathcal{U}$ is an orthogonal / ~~unitary~~ unitary matrix.
We consider the unitary case. Then $\mathcal{U}$ is Haar distributed on $U(N)$.

From the unitary invariance of $\mathbb{P}(H)dH$, we see that after conditioning on D, the eigenmatrix $\mathcal{U}$ is drawn from the Haar measure on $U(n)$. In particular, $\mathcal{U}$ and D can be taken to be independent:

$$\cancel{\mathbb{P}(H = \mathcal{U}_0 D_0 \mathcal{U}_0^* + dH) = \mathbb{P}(H = (V\mathcal{U}_0) D_0 (\mathcal{U}_0^* V^*) + dH)}$$

~~for any $V \in U(N)$~~

$$\mathbb{P}(\mathcal{U} \in B,\ D \in A) = \mathbb{P}(\mathcal{U} \in B \mid D \in A)\, \mathbb{P}(D \in A)$$
$$= \mathbb{P}(\mathcal{U} \in B)\, \mathbb{P}(D \in A).$$

<u>Proof of Theorem 3.1</u>: Fix a diagonal matrix $D_0 = \mathrm{diag}(\lambda_1^0, \cdots, \lambda_N^0)$, $\lambda_1^0 \le \cdots \le \lambda_N^0$.
Let $\varepsilon > 0$ be arbitrarily small. We now compute the probability that H_N lies in an ε-ball around D_0 in the H-S norm (Frobenius norm):

$$\mathbb{P}(\| H_N - D_0 \|_F \le \varepsilon)$$

(The Euclidean norm on the space of $n \times n$ matrices.)

~~Then $f_N(\lambda^0)$~~

I. ~~The~~ The simpler direction: The probability density of M_n is $\propto \exp(-\frac{\beta}{2} N \sum_i V(\lambda_i^0))$
The ~~volume~~ volume of the ε-ball is $C_N \varepsilon^{N^2}$. So

$$(1) \quad \mathbb{P}(\| H_N - D_0 \|_F \le \varepsilon) = (C_N + \underbrace{o(1)}_{o_\varepsilon(1) \to 0 \text{ as } \varepsilon \to 0 \text{ when } N \text{ is fixed.}})\, \varepsilon^{N^2} \exp(-\frac{\beta}{2} N \sum_i V(\lambda_i))$$

II. The harder direction: Let's calculate $\mathbb{P}(\| H_N - D_0 \|_F \le \varepsilon)$ in a different way.

Weyl's inequality: Given two hermitian matrices A and B with eigenvalues

$$\lambda_1(A) \le \cdots \le \lambda_N(A), \quad \lambda_1(B) \le \cdots \le \lambda_N(B).$$

Then, $\forall i$, $\quad \lambda_i(A) + \lambda_1(B) \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_N(B)$.
In particular, if B is a small perturbation with $\|B\| \le \varepsilon$, then

$$\lambda_i(A) - \varepsilon \le \lambda_i(A+B) \le \lambda_i(A) + \varepsilon.$$

Thus, for $H_N = \mathcal{U} D \mathcal{U}^*$, we have $\|D - D_0\| \le \varepsilon$ when $\| H_N - D_0 \|_F \le \varepsilon$.

~~Moreover,~~ $\cancel{\mathcal{U} D \mathcal{U}^* = D_0 + O(\varepsilon) = D + O(\varepsilon) \Rightarrow \mathcal{U} D - D \mathcal{U} = O(\varepsilon)}$

~~Now, we make the ansatz~~

$$\cancel{D = D_0 + \varepsilon E, \quad \mathcal{U} = \exp(\varepsilon S)},$$

~~[... is diagonal & unitary, i.e. ...]~~

~~where E is a diagonal matrix and S is a skew-adjoint matrix. with zero diagonals~~

~~[Note that $\mathcal{U}\mathcal{U}^* = \exp(\varepsilon S)\exp(\varepsilon S^*) = \exp(\varepsilon S)\exp(-\varepsilon S) = I$]~~

~~Note that $S \mapsto \exp(\varepsilon S)$ has a non-degenerate Jacobian in the small ε-ball, so the inverse function thm applies to specify S from $\mathcal{U}$. (Formally, $S = \frac{1}{\varepsilon} \log \mathcal{U}$.)~~

~~WLOG, suppose all λ_i^0 are different (oth this holds a.s.).~~

Thus, we make the ansatz: $D = D_0 + \varepsilon E$, E is diagonal bounded.

Note, the eigenvalues of D_0 are non-zero and non-degenerate almost surely. Moreover, we have $\quad \mathcal{U} D \mathcal{U}^* = D_0 + O(\varepsilon) \Rightarrow \mathcal{U} D_0 \mathcal{U}^* = D_0 + O(\varepsilon) \Rightarrow \mathcal{U} D_0 - D_0 \mathcal{U} = O(\varepsilon)$. $(*)$

The only unitary matrices that commute with D_0 are diagonal unitary matrices

$$R = (r_1, r_2, \cdots, r_N), \quad |r_i|^2 = 1.$$

From $(*)$, we can make the ansatz: $\mathcal{U} = \exp(\varepsilon S) R$,

Where S is a bounded skew-adjoint ($S^* = -S$) matrix with zero diagonal. (The diagonal parts can be included into R.)

We can check that $\quad \mathcal{U}\mathcal{U}^* = \exp(\varepsilon S) \exp(\varepsilon S^*) = 1$,

$\mathcal{U} D_0 - D_0 \mathcal{U} = \exp(\varepsilon S) R D_0 - D_0 \exp(\varepsilon S) R = R D_0 - D_0 R + O(\varepsilon) = O(\varepsilon)$.

It is easy to check that $(R, S) \mapsto \exp(\varepsilon S) R$ has a non-degenerate Jacobian, so with the inverse function thm, we can uniquely determine R and S from $\mathcal{U}$ in the ~~sam~~ small ε-ball around D_0.

Now with the ansatz: $D = D_0 + \varepsilon E$, $\quad \mathcal{U} = \exp(\varepsilon S) R$, we get

$$\begin{aligned} H_N = \mathcal{U} D \mathcal{U}^* &= \exp(\varepsilon S) R (D_0 + \varepsilon E) R^* \exp(-\varepsilon S) \\ &= D_0 + \varepsilon E + \varepsilon S R D_0 R^* - \varepsilon R D_0 R^* S + O(\varepsilon^2) \\ &= D_0 + \varepsilon E + \varepsilon (S D_0 - D_0 S) + O(\varepsilon^2). \end{aligned}$$

Thus, $\quad \mathbb{P}(\|H_N - D_0\|_F \leq \varepsilon) = \mathbb{P}(\|E + (S D_0 - D_0 S)\| \leq 1 + O(\varepsilon))$. $(\dagger)$

Since $\mathcal{U}$ is Haar distributed on $\mathcal{U}(N)$, S is locally distributed as ~~$C_N' \varepsilon^{N^2+N}$~~ $C_N' \varepsilon^{N^2-N}$ times the Lebesgue measure on the space of skew-adjoint matrices with 0 diagonal. ~~Bot~~ On the other hand, E is distributed as $(\rho_N(\vec{\lambda}^0) + o(1)) \varepsilon^N$ times the Lebesgue measure on the space of diagonal matrices. Thus we can calculate $(\dagger)$ as:

$$C_N' \varepsilon^{N^2} (\rho_N(\vec{\lambda}^0) + o(1)) \iint_{\|E + S D_0 - D_0 S\|_F \leq 1 + O(\varepsilon)} dE\, dS$$

Consider the map $S \mapsto S D_0 - D_0 S$, where $(S D_0 - D_0 S)_{ij} = (\lambda_j^0 - \lambda_i^0) S_{ij}$.

In other words, the map dilates the (i,j)th entry of S by $\lambda_j^0 - \lambda_i^0$. Hence, the Jacobian of this map is $\quad \prod_{i \neq j} |\lambda_j^0 - \lambda_i^0| = \prod_{1 \leq i < j \leq N} |\lambda_j^0 - \lambda_i^0|^2$.

Applying the change of variable, we get

$$\begin{aligned} \iint_{\|E + S D_0 - D_0 S\|_F \leq 1 + O(\varepsilon)} dE\, dS &= \frac{1}{\prod_{1 \leq i < j \leq N} |\lambda_j^0 - \lambda_i^0|^2} \iint_{\|E + S\|_F \leq 1 + O(\varepsilon)} dE\, dS \\ &= \frac{C_N''}{\prod_{1 \leq i < j \leq N} |\lambda_j^0 - \lambda_i^0|^2}. \end{aligned}$$

In sum, we have found that

$$\mathbb{P}(\|H_N - D_0\|_F \le \varepsilon) = (C_N + o(1))\,\varepsilon^{N^2} \exp\left[-\frac{\beta}{2} N \sum_i V(\lambda_i^\circ)\right]$$

$$= \frac{C_N'''}{\prod_{1\le i<j\le N} |\lambda_j^\circ - \lambda_i^\circ|^2} \left[\rho_N(\vec{\lambda}^\circ) + o(1)\right] \varepsilon^{N^2}$$

$$\Rightarrow \rho_N(\vec{\lambda}^\circ) = \frac{1}{Z_N} \prod_{1\le i<j\le N} |\lambda_j^\circ - \lambda_i^\circ|^2 \exp\left[-\frac{\beta}{2} N \sum_{i=1}^N V(\lambda_i^\circ)\right].$$ □

② Mean field approximation of semicircle law

$$\rho_N(\vec{\lambda}) = \frac{1}{Z_N} \exp\left[-\beta N \mathcal{H}(\vec{\lambda})\right], \quad \mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^N V(\lambda_i) - \frac{1}{N}\sum_{i<j} \log|\lambda_i - \lambda_j|.$$

Intuitively, it is plausible that the spectum should concentrate around the maximum of $\rho_N(\vec{\lambda})$ as $N \to +\infty$. It is equivalent to study the minimum of $\mathcal{H}(\vec{\lambda})$.

Heuristically, we make a mean-field approximation, i.e., the ESD $\frac{1}{N}\sum_i \delta_{\lambda_i}$ can be approximated by a continuous probability measure $\rho(x)\,dx$ (→ this is reasonable for GOE/GUE at least with $\rho = \rho_{sc}$.)

Then, we expect that $\mathcal{H}(\vec{\lambda})$ is approximately given by

$$\cancel{\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\,dx}$$

$$\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\rho(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)\rho(y) \log\frac{1}{|x-y|}\,dx\,dy.$$

ρ should minimize this functional. Now, we derive the Euler-Lagrange equation for $\mathcal{H}(\rho)$: consider $\rho + \delta\rho$, where ρ is the minimizer. We should have

$$\frac{\delta\mathcal{H}(\rho)}{\delta\rho} = 0, \quad \text{because } \mathcal{H}(\rho+\delta\rho) \ge \mathcal{H}(\rho) \ \forall\, \delta\rho.$$

We have $$\delta\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\,\delta\rho(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} (\delta\rho(x))\rho(y) \log\frac{1}{|x-y|}\,dx\,dy$$

$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)(\delta\rho(y)) \log\frac{1}{|x-y|}\,dx\,dy$$

$$\Rightarrow \frac{1}{2}\int_{\mathbb{R}} V(x)\,\delta\rho(x)\,dx + 2\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \rho(y)\log\frac{1}{|x-y|}\,dy\right)\delta\rho(x)\,dx = 0$$

Since $\delta\rho$ is arbitrary, with $\int_{\mathbb{R}} \delta\rho(x)\,dx = 0$ (since $\rho+\delta\rho$ should be a prob. density), we ought to have $\frac{1}{2}V(x) + 2\int_{\mathbb{R}} \rho(y)\log\frac{1}{|x-y|}\,dy = C$.

This is the E-L equation for ρ.

Solving the E-L equation, we get a density $\rho(x)$, which should be the limiting ESD of $\mathbb{P}(H)\,dH = \frac{1}{Z_N} \exp[-\frac{\beta}{2} N \,\mathrm{Tr}\, V(H)]\,dH$.

Let's solve it for GUE with $V(x) = x^2$. We need to solve

$$\frac{1}{2}x^2 + 2\int_{\mathbb{R}} \rho(y) \log \frac{1}{|x-y|}\,dy = C \Rightarrow x + 2\int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy = 0 \quad (*)$$

Here, $\int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy$ ~~should~~ should be formally understood as a principal value

$$\text{p.v.} \int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy := \lim_{\varepsilon\to 0} \int_{|y-x|\geq\varepsilon} \frac{\rho(y)}{x-y}\,dy = \lim_{\eta\to 0^+} \mathrm{Re} \int_{\mathbb{R}} \frac{\rho(y)}{y-(x+i\eta)}\,dy$$

$$= \lim_{\eta\to 0^+} \mathrm{Re}\, S_\rho(x+i\eta) =: \mathrm{Re}\, S_\rho(x+i0^+)$$

On the other hand, recall $\rho(x) = \frac{1}{\pi}\,\mathrm{Im}\, S_\rho(x+i0^+)$.

Then the equation $(*)$ reads: $\quad x + 2\,\mathrm{Re}\, S_\rho(x+i0^+) = 0$

$$\Rightarrow x\,\mathrm{Im}\, S_\rho(x+i0^+) + 2[\mathrm{Re}\, S_\rho(x+i0^+)][\mathrm{Im}\, S_\rho(x+i0^+)] = 0$$

$$\Rightarrow \mathrm{Im}[x S_\rho(x+i0^+) + S_\rho^2(x+i0^+)] = 0$$

Define function $f(z) = z S_\rho(z) + S_\rho^2(z)$. The above function means we can make an analytic continuation from $z\in\mathbb{C}_+ \to \mathbb{C}$ by making a reflection along $\mathbb{R}$:

$$f(z) = \overline{f(\bar{z})} \quad \text{for } z\in \mathbb{C}_-.$$

On the other hand, since $S(z) = \frac{-(1+o(1))}{z}$ as $z\to\infty$, $f\to -1$ as $z\to+\infty$.

By Liouville's theorem, f is a constant and

$$f(z) \equiv \lim_{z\to+\infty} [z S_\rho(z) + S_\rho^2(z)] = -1$$

This gives the familiar self-consistent equation:

$$S_\rho^2(z) + z S_\rho(z) + 1 = 0 \Rightarrow S_\rho(z) = m_{sc}(z) \Rightarrow \rho = \rho_{sc}.$$

The above arguments can be made rigorous for general β ensembles as long as we impose some conditions on V (in particular, need certain condition on its growth at ∞).

③ Bulk eigenvalue distribution of GUE and orthogonal polynomials