

$$= \frac{1}{2} \|\vec{v}\|^2 + \frac{1}{2N} \sum_{i \neq j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} \geq \frac{1}{2} \|\vec{v}\|^2.$$

Thus, $\nabla^2(\beta N \mathcal{H}) \geq \frac{1}{2} \beta N$. Here, we use a different normalization for the Dirichlet form: $D_u(f) = \frac{1}{\beta N} \int \|\nabla f\|^2 d\mu$. Then, the Bakry - Emery estimate gives $S_u(f) \leq 4 D_u(f)$.

This shows that the relaxation time is of order 1. (This is the global dynamics.)

We will show that the local dynamics converges to equilibrium ^{within} ~~at~~ time of order $N^{-1+\epsilon}$.

Thm 6.6. Assume the following conditions:

(i) There exists a constant $\xi > 0$ such that

$$Q := \sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \sigma_j)^2 f_t(\vec{x}) M_G(d\vec{x}) \leq CN^{-2+2\xi}.$$

(ii) After time $1/N$, the solution to the equation $\partial_t f_t = \mathcal{L} f_t$, $t \geq 0$, satisfies $S_u(f_{1/N}) \leq CN^m$ for some fixed m .

Fix any $\beta \geq 1$ and $n \in \mathbb{N}$. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with compact support. Denote:

$$F_{i,n}(\vec{x}) = F(N(x_{i+1} - x_i), N(x_{i+2} - x_i), \dots, N(x_{i+n} - x_i)).$$

Then, for any $\xi \in (0, \frac{1}{2})$ and any sufficiently small constant $\epsilon > 0$, there ~~exist~~ ~~a~~ exist constants $C, c > 0$ depending only on ϵ and F , such that for any $J \subset \{1, 2, \dots, N-n\}$ we have

$$\sup_{t \geq N^{-1+2\xi+\epsilon}} \left| \int \frac{1}{|J|} \sum_{i \in J} F_{i,n}(\vec{x}) (f_t d\mu - d\mu) \right| \leq C N^\epsilon \sqrt{\frac{N^2 Q}{|J| t}} + C e^{-c N^\epsilon}.$$

In particular, if $t \geq N^{-1+2\xi+2\theta\epsilon+\delta}$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} F_{i,n}(\vec{x}) (f_t d\mu - d\mu) \right| \leq \frac{C}{\sqrt{|J| N^{\delta-1}}} + C e^{-c N^\epsilon}.$$

Hence, the gap distribution, averaged over J indices, coincides for $f_t d\mu$ and $d\mu$ if $|J| N^{\delta-1} \rightarrow +\infty$.

Averaged gap distribution on scale J corresponds to averaged energy distribution on scale $J \sim Nb$.

Lemma. Under the setting of Thm 6.6, suppose $\left| \int \frac{1}{|J|} \sum_{i \in J} F_{i,n}(\vec{x}) (f_t d\mu - d\mu) \right| \leq \frac{C}{\sqrt{|J| N^{\delta-1}}}$ for some constant $\delta > 0$.

Suppose the eigenvalue rigidity holds: $\forall K > 0, \sup_{KN \leq j \leq (1-K)N} |\lambda_j - \bar{\lambda}_j| < N^{-1+\frac{1}{K}}$.

Then, for any constant $\varepsilon > 0$ and $N^{-1} \ll b_N \ll 1$, we have that

$$\left| \int_{E-b}^{E+b} \frac{dE'}{2b} \int_{\mathbb{R}^n} d\vec{\alpha} O(\vec{\alpha}) \left(p_{f_{\mu,N}}^{(n)} - p_{\mu,N}^{(n)} \right) \left(E' + \frac{\vec{\alpha}}{N f_{sc}(E)} \right) \right|$$

$$\leq N^{2\varepsilon} \left[\frac{N^{-1+\frac{1}{K}}}{b} + \sqrt{\frac{N^{-\delta}}{b}} \right].$$

Hence, the averaged energy distribution over $b \gg N^{-1+\frac{1}{K}+2\varepsilon} \vee N^{4\varepsilon-\delta}$ coincides for $f_{\mu,N}$ and f_{μ} .

The above two results establish the averaged gap/energy universality for $t \geq N^{-1+2\frac{1}{K}+2\varepsilon+\delta}$. Under the optimal rigidity, $\frac{1}{K}$ can be chosen as small as possible. Furthermore, ε and δ are arbitrary. This establishes the Dyson's conjecture: the local dynamics relaxes to equilibrium for $t \geq N^{-1+c}$ \forall constant $c > 0$.

Main ideas for the proof: Recall that

$$\langle \vec{v}, \nabla^2 \mathcal{H} \vec{v} \rangle = \frac{1}{2} \sum_i v(i)^2 + \frac{1}{N} \sum_{i < j} \frac{(v(i) - v(j))^2}{(\lambda_i - \lambda_j)^2} \geq \frac{1}{2} \|\vec{v}\|^2.$$

The key intuition is that the relaxation time is in fact much shorter than order 1 for local variables that depend only on the eigenvalue differences. In particular, with the intuition that $\lambda_i - \lambda_j$ are close, the relaxation time in the direction $v(i) - v(j)$ should be much smaller than 1. However, this effect is hard to use directly in the real proof.

An idea is to add an "auxiliary strongly convex" potential to \mathcal{H} to "speed up" the relaxation to local equilibrium. On the other hand, we will show that the effect of this speeding up on the local statistics can be controlled provided with the estimate

$$(+)\sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \bar{\lambda}_j)^2 f_t(\vec{\lambda}) \mu_0(d\vec{\lambda}) \lesssim N^{-2+2\frac{1}{K}}.$$

We introduce the following auxiliary potential $\hat{\mathcal{H}}(\vec{\lambda}) := \frac{1}{2\tau} \sum_{j=1}^N (\lambda_j - \bar{\lambda}_j)^2$, $0 < \tau < 1$. (We will choose $\tau = tN^{-\varepsilon}$ in the proof.)

Then, the new Hamiltonian is $\tilde{\mathcal{H}} = \mathcal{H} + \hat{\mathcal{H}}$, and the corresponding measure is $d\tilde{w} = w(\vec{\lambda}) d\vec{\lambda}$, $w(\vec{\lambda}) := e^{-\beta N \tilde{\mathcal{H}}} / \tilde{Z}$, called the "local relaxation measure".

The "local relaxation flow" with the generator characterized by the natural Dirichlet form w.r.t. w : $\tilde{\mathcal{L}} = \mathcal{L} - \sum_j b_j \partial_j$, $b_j(x_j) = \frac{\lambda_j - \bar{\lambda}_j}{\tau}$.

For small τ , $\hat{\mathcal{H}}$ will substantially increase the lower bound on the Hessian, hence speeding up the dynamics so that the relaxation time is at most τ . In addition, we will compare the local statistics of the original system with those of the modified one.

It turns out that the difference is governed by $(\nabla \tilde{h})^2$, which can be controlled with the estimate (1).

Proof of Theorem 6.6

Proposition I. Fix any $\beta \geq 1$. Consider the equation $\partial_t q_t = \tilde{L} q_t$, $t \geq 0$, with the reversible measure $\omega = e^{-\beta N \tilde{H}} / Z_N$. Suppose the initial condition q_0 satisfies $\int q_0 d\omega = 1$. Then, we have that

$$(1) \quad \partial_t D\omega(\sqrt{q_t}) \leq -\frac{2}{\tau} D\omega(\sqrt{q_t}) - \frac{1}{\beta N^2} \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} d\omega,$$

$$(2) \quad \frac{1}{\beta N^2} \int_0^\infty ds \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} d\omega \leq D\omega(\sqrt{q_0}),$$

and the LSI $S\omega(q_0) \leq C\tau D\omega(\sqrt{q_0})$. Hence, the relaxation time to equilibrium is of order τ :

$$(3) \quad S\omega(q_t) \leq e^{-Ct/\tau} S\omega(q_0), \quad D\omega(\sqrt{q_t}) \leq e^{-Ct/\tau} D\omega(\sqrt{q_0}).$$

Proof: Performing the same calculation as in the proof of the Bakry - Emery estimate, we obtain that for $h_t = \sqrt{q_t}$,

$$\partial_t D\omega(h_t) = \partial_t \frac{1}{\beta N} \int (\nabla h_t)^2 e^{-\beta N \tilde{H}} d\vec{x} \leq \frac{2}{\beta N} \int (\nabla h_t) \cdot (\nabla^2 \tilde{H}) \cdot (\nabla h_t) e^{-\beta N \tilde{H}} d\vec{x}.$$

For \tilde{H} , we can calculate its Hessian as

$$\nabla h_t \cdot \nabla^2 \tilde{H} \cdot \nabla h_t \geq \frac{1}{\tau} \sum_i (\partial_i h_t)^2 + \frac{1}{2N} \sum_{i,j} \frac{1}{(x_i - x_j)^2} (\partial_i h_t - \partial_j h_t)^2.$$

Plugging it into the above inequality, we conclude (1). Integrating over t , we conclude (2). The LSI and (3) can be derived in the same way as in the proof of Bakry - Emery. \square

The extra term on the RHS of (1) plays a key role in the proof of the following prop.

Proposition II. Let q be a probability density ω with respect to the local relaxation measure ω , i.e., $\int q d\omega = 1$. Fix any $n \geq 1$ and a smooth function F with compact support. Recall that

$$f_{i,n}(\vec{x}) := F(N(x_{i+1} - x_i), \dots, N(x_{i+n} - x_i)).$$

Then, for any $J \subseteq \{1, 2, \dots, N-n\}$ and $t > 0$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} f_{i,n}(\vec{x}) (q d\omega - d\omega) \right| \leq C \left(t \frac{D\omega(\sqrt{q})}{|J|} \right)^{1/2} + C \sqrt{S\omega(q)} e^{-Ct/\tau}.$$

Proof: For simplicity of notation, we take $n=1$, so $f_{i,n}(\vec{x}) = F(N(x_{i+1} - x_i))$. Let q_t be a solution to $\partial_t q_t = \tilde{L} q_t$, $t \geq 0$, with initial condition $q_0 = q$. Then,

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - 1) d\omega = \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - q_t) d\omega +$$

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q_t - 1) d\omega.$$

The second term can be bounded by $\int |g_t - 1| dw \leq C \sqrt{S_w(g_t)} \leq C e^{-ct/t} \sqrt{S_w(g_0)}$.
 Exp. decay of entropy given by the last prop
 Pinsker inequality

To estimate the first term, using $\partial_t g_t = \tilde{L} g_t$, we get that

$$\begin{aligned} & \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) g_t dw - \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) g_0 dw \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \frac{d}{ds} g_s dw \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \tilde{L} g_s dw \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G'(N(x_{i+1} - x_i)) (\partial_{i+1} g_s - \partial_i g_s) dw. \quad (*) \end{aligned}$$

Using $\partial_i g = 2\sqrt{g} \partial_i \sqrt{g}$ and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} |(*)| &\leq 2 \left[\int_0^t ds \int \frac{N^2}{|J|^2} \sum_{i \in J} |G'(N(x_{i+1} - x_i))|^2 (x_{i+1} - x_i)^2 g_s dw \right]^{1/2} \times \\ &\quad \left[\int_0^t ds \int \frac{1}{N^2} \sum_{i \in J} \frac{1}{(x_i - x_{i+1})^2} (\partial_{i+1} \sqrt{g_s} - \partial_i \sqrt{g_s})^2 dw \right]^{1/2} \\ &\leq C D_w(\sqrt{g_0}) \end{aligned}$$

Moreover, $|G'(N(x_{i+1} - x_i))|^2 |x_{i+1} - x_i|^2 \leq CN^{-2}$ because G is smooth and compactly supported. So

$$|(*)| \leq C \sqrt{D_w(g_0)} \cdot \left(\frac{t}{|J|} \right)^{1/2}.$$

Proposition III. Consider the local relaxation measure w for a given $\tau > 0$. Set $\mu = w/\mu$ and $g_t = f_t/\mu$ with f_t solving $\partial_t f_t = \tilde{L} f_t$. Suppose there is a const. $C > 0$ such that $S(f_t \mu | w) \leq N^C$.

Fix any small $\varepsilon > 0$. For any $t \in [\tau N^\varepsilon, N]$, the entropy and Dirichlet form satisfy

$$S(g_t w | w) \leq CN^2 Q \tau^{-1}, \quad D_w(\sqrt{g_t}) \leq CN^2 Q \tau^{-2},$$

where recall that $Q = \sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - x_j)^2 f_t(\vec{x}) \mu(d\vec{x}) (\leq CN^{-2+2\varepsilon})$.

Proof: We calculate $\partial_t S(f_t \mu | w) = \partial_t S(f_t \mu | \mu \mu)$

$$\begin{aligned} &= \int (\tilde{L} f_t) (\log g_t) d\mu + \int f_t \frac{\tilde{L} f_t}{f_t} d\mu = \int (\tilde{L} f_t) (\log g_t) d\mu \\ &= \int f_t \tilde{L} (\log g_t) d\mu = \int g_t \mu \tilde{L} (\log g_t) d\mu = \int g_t \mu [\tilde{L} (\log g_t) - g_t \frac{\tilde{L} g_t}{g_t}] d\mu \\ &\quad + \int \mu \tilde{L} g_t d\mu. \end{aligned} \quad (65)$$

We can check directly that $\mathcal{L}(\log g_t) - \frac{\mathcal{L}g_t}{g_t} = -\frac{1}{\beta N} \frac{|\nabla g_t|^2}{g_t^2} = -\frac{4}{\beta N} |\nabla \sqrt{g_t}|^2 \cdot \frac{1}{g_t}$.

Hence, we have $\partial_t S(f_t | \mu | w) = -\frac{4}{\beta N} \int 4 |\nabla \sqrt{g_t}|^2 d\mu + \int 4 \mathcal{L}g_t d\mu + \sum_j (b_j = \frac{x_j - \bar{x}_j}{\tau})$
 $= -\frac{4}{\beta N} \int |\nabla \sqrt{g_t}|^2 d\mu + \int \tilde{\mathcal{L}}g_t d\mu + \sum_j \int b_j \partial_j g_t d\mu.$

Using $\partial_j g_t = 2\sqrt{g_t} \partial_j \sqrt{g_t}$ and C-S, we get

$$\begin{aligned} \partial_t S(f_t | \mu | w) &\leq -4D_w(\sqrt{g_t}) + 2 \sum_j \int b_j \sqrt{g_t} \partial_j \sqrt{g_t} d\mu \\ &\leq -2D_w(\sqrt{g_t}) + CN \sum_j \int b_j^2 g_t d\mu \\ &= -2D_w(\sqrt{g_t}) + C \frac{N^2}{\tau^2} \frac{1}{N} \int (x_j - \bar{x}_j)^2 g_t d\mu \\ &\leq -2D_w(\sqrt{g_t}) + CN^2 Q \tau^{-2}. \quad (+) \end{aligned}$$

Applying the LSI w.r.t. $d\mu$, we get that

$$\partial_t S(f_t | \mu | w) \leq -C\tau^{-1} S(f_t | \mu | w) + CN^2 Q \tau^{-2}.$$

Applying the Gronwall's inequality, integrating this inequality from τ to t and using the assumption $S(f_t | \mu | w) \leq NC$ and $t \geq \tau N^E$, we obtain that

$$S(g_t | w | w) = S(f_t | \mu | w) \leq CN^2 Q \tau^{-1}. \quad (H)$$

To prove the second estimate, we notice

$$\begin{aligned} D_\mu(\sqrt{f_s}) &= \frac{1}{\beta N} \int \frac{|\nabla(g_s \psi)|^2}{g_s \psi} \frac{d\mu}{\psi} \leq \frac{C}{\beta N} \int \left[\frac{|\nabla g_s|^2}{g_s} + |\nabla \log \psi|^2 g_s \right] d\mu \\ &\leq CD_w(\sqrt{g_s}) + \frac{C}{\beta N} \int \frac{(\sum_j |x_j - \bar{x}_j|^2)}{\tau^2} f_t d\mu \leq C \frac{(\beta N)^2}{\beta N} \int \frac{\sum_j |x_j - \bar{x}_j|^2}{\tau^2} f_t d\mu \\ &\leq CD_w(\sqrt{g_s}) + CN^2 Q \tau^{-2}. \quad (*) \end{aligned}$$

Taking integral over $[t-\tau, t]$ ~~one gets and using~~, we get

$$\int_{t-\tau}^t D_w(\sqrt{g_s}) ds \geq \int_{t-\tau}^t \left[\frac{1}{C} D_\mu(\sqrt{f_s}) - CN^2 Q \tau^{-2} \right] ds$$

$$(D_\mu(\sqrt{f_t}) \text{ is decreasing in } t) \Leftrightarrow \geq \frac{\tau}{C} D_\mu(\sqrt{f_t}) - CN^2 Q \tau^{-1}.$$

On the other hand, with (+) and (H), we get that

$$\int_{t-\tau}^t D_w(\sqrt{g_s}) ds \leq CN^2 Q \tau^{-1} + [S(f_t | \mu | w) - S(f_{t-\tau} | \mu | w)] \leq CN^2 Q \tau^{-1}.$$

From the above two inequalities, we obtain that

$$D_\mu(\sqrt{f_t}) \leq CN^2 Q \tau^{-2}.$$

With a similar argument as in (*), we get that