Section 6 Dyson Brownian Motion and Bulk Universality

I. Various notions of Bulk universality

The local eigenvalue statistics can either be expressed in terms of "local correlation functions" rescaled around some energy E or the "gap statistics" for a gap $\lambda_{j+1}-\lambda_j$ with a given label j. They are called "fixed energy" and "fixed gap" universalites, and they do not coincide. In fact, eigenvalues fluctuate on a scale much larger than the typical eigenvalue spacing, the label j of the eigenvalue λ_j closest to a fixed energy E is not a deterministic function of E. Moreover, the two concepts both have natural averaged versions, which are generall easier to establish.

Rmk: Recall that $\int_{\gamma_j}^{\gamma_{j+1}} \rho_{sc}(x)\,dx = \frac{1}{N} \Rightarrow \gamma_{j+1}-\gamma_j \sim \frac{1}{N\rho_{sc}(\gamma_j)}$. Hence, the ~~fluctuation~~ gaps and correlation functions need to be rescaled by the local density ρ_{sc} to get an universal limit. This holds in more general setups, such as sample covariance matrices.

① Fixed energy universality: $\forall n \in \mathbb{N}$, $F: \mathbb{R}^n \to \mathbb{R}$ is C_c^∞. For any const $\kappa>0$, we have that uniformly in $E \in [-2+\kappa, 2-\kappa]$,

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^n} \int_{\mathbb{R}^n} d\vec{\alpha}\; F(\vec{\alpha})\, p_N^{(n)}\Big(E + \frac{\vec{\alpha}}{N\rho_{sc}(E)}\Big) = \int_{\mathbb{R}^n} d\vec{\alpha}\; F(\vec{\alpha})\, q^{(n)}_{GOE/GUE}(\vec{\alpha}),$$

where $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $p_N^{(n)}$ is the n-point correlation function, and $q^{(n)}_{GOE/GUE}(\vec{\alpha}) = \det\big(S(\alpha_i-\alpha_j)\big)_{i,j=1}^n$ is the determinant of sine-kenel we derived before.

② Averaged bulk universality (on scale $N^{-1+\varepsilon}$):

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^n} \int_{E-b}^{E+b} \frac{dx}{2b} \int_{\mathbb{R}^n} d\vec{\alpha}\; F(\vec{\alpha})\, p_N^{(n)}\Big(x + \frac{\vec{\alpha}}{N\rho_{sc}(E)}\Big)$$
$$= \int_{\mathbb{R}^n} d\vec{\alpha}\; F(\vec{\alpha})\, q^{(n)}_{GOE/GUE}(\vec{\alpha})$$

where $b = N^{-1+\varepsilon}$ $\forall$ const $\varepsilon>0$.

③ Fixed gap universality: Fixed any small constant $\delta>0$ and $n\in\mathbb{N}$. For any $F: \mathbb{R}^n \to \mathbb{R}$, $F\in C_c^\infty$ and any $k, m \in [\delta N, (1-\delta)N]$, we have that

$$\lim_{N\to\infty} \Big| \mathbb{E}_{H_N} F\big(N\rho_{sc}(\gamma_k)(\lambda_{k+1}-\lambda_k), \cdots, N\rho_{sc}(\gamma_k)(\lambda_{k+n}-\lambda_k)\big)$$
$$- \mathbb{E}_{GOE/GUE} F\big(N\rho_{sc}(\gamma_m)(\lambda_{m+1}-\lambda_m), \cdots, N\rho_{sc}(\gamma_m)(\lambda_{m+n}-\lambda_m)\big)\Big| = 0.$$

④ Averaged gap universality: For $\ell = N^\varepsilon$, $\forall$ const $\varepsilon>0$,

$$\lim_{N\to+\infty} \Big| \frac{1}{2\ell+1} \sum_{j=k-\ell}^{k+\ell} \mathbb{E}_{H_N} F\big(N\rho_{sc}(\gamma_k)(\lambda_{j+1}-\lambda_j), \cdots, N\rho_{sc}(\gamma_k)(\lambda_{j+n}-\lambda_j)\big)$$
$$- \mathbb{E}_{GOE/GUE} F\big(N\rho_{sc}(\gamma_m)(\lambda_{m+1}-\lambda_m), \cdots, N\rho_{sc}(\gamma_m)(\lambda_{m+n}-\lambda_m)\big)\Big| = 0.$$

Rmk: Fixed energy $\overset{\Longrightarrow}{\not\Longleftarrow}$ Averaged energy, Fixed gap $\overset{\Longrightarrow}{\not\Longleftarrow}$ Averaged gap,

Fixed energy $\overset{\not\Rightarrow}{\not\Leftarrow}$ Fixed gap, Averaged energy $\Leftrightarrow$ Averaged gap.

We will focus on proving the averaged energy universality.

Theorem 6.1 The averaged energy universality holds on scales $N^{-1+\varepsilon}$ $\forall\ 0<\varepsilon<1$.

This is a version of the famous "Wigner-Dyson-Mehta conjecture".

II. The three-step strategy

Step 1: Local semicircle law. At this step, we get precise estimates on the matrix elements of the resolvent, the rigidity of eigenvalues, and delocalization of eigenvectors.

Step 2: Universality for Gaussian divisible ~~ensembles~~ ensembles.
Gaussian divisible ensembles are random matrices that can be written as $H_0 + \sqrt{t}\, H^G$, where H_0 is Wigner, H^G is GOE / GUE and independent of H_0, and $t>0$ is a parameter.

A convenient way to generate H_t is the "matrix Ornstein-Uhlenbeck (OU) Process":

$$dH_t = \frac{1}{\sqrt{N}}\, dB_t - \frac{1}{2} H_t\, dt, \qquad H_t = H_0,$$

Where B_t is a matrix Brownian motion whose entries are independent BMs up to symmetry $B_t^* = B_t$, and $\frac{1}{\sqrt{N}} B_t \overset{\text{law}}{=} \sqrt{t}$ GOE / GUE. For each entry,

$$dh_{ij}(t) = \frac{1}{\sqrt{N}}\, db_{ij}(t) - \frac{1}{2} h_{ij}(t)\, dt.$$

It has a unique strong solution:

$$h_{ij}(t) = h_{ij}(0)\, e^{-t/2} + \int_0^t \frac{e^{-(t-t')/2}}{\sqrt{N}}\, db_{ij}(t').$$

Note that $\int_0^t \frac{e^{-(t-t')/2}}{\sqrt{N}}\, db_{ij}(t')$ is centered Gaussian of variance $\frac{1}{N}\int_0^t e^{-(t-t')}\, dt' = \frac{1}{N}(1-e^{-t})$.
Hence, with a slight abuse of notation, we write it as $h_{ij}^G \cdot \sqrt{1-e^{-t}}$.
This gives a solution

$$H_t \overset{\text{law}}{=} e^{-t/2} H_0 + \sqrt{1-e^{-t}}\, H^G.$$

A big advantage of this form is that variances are preserved throughout the process:

$$\mathbb{E}|h_{ij}(t)|^2 = e^{-t}\, \mathbb{E}|h_{ij}(0)|^2 + (1-e^{-t})\, \mathbb{E}|h_{ij}^G|^2 = \mathbb{E}|h_{ij}(0)|^2,$$

for $\mathbb{E}|h_{ij}(0)|^2 = 1+\delta_{ij}$ in the real case, and $\mathbb{E}|h_{ij}(0)|^2 = 1$ in the complex case.

The purpose of Step 2 is to show ~~that~~ the bulk universality of H_t for $t = N^{-1+\varepsilon}$ for any $0<\varepsilon<1$.

Approximation by a Gaussian ~~div~~ divisible ensemble

Step 3 : Given a Wigner matrix, H, there exists a Wigner H_0 such that H_t has asymptotically identical local eigenvalue statistics as H. This is usually done through a Green's function comparison argument by using certain moment matching conditions. Alternatively, one can also use a continuity estimate of the matrix OU process.

The "three - step strategy" is now (one of) the most standard in proving the bulk universality of random matrices (for edge universality, the Step 2 sometimes is not necessary). Here, Step 1 is model-specific and generally is the "hardest" step. The Steps 2 and 3 are more standard, where general methods / proofs / arguments are known and work for "most" models. In particular, the strongest result for Step 2 has been established for very general initial conditions H_0 (not necessarily a random matrix).

III. Dyson Brownian Motion

The matrix Brownian motion introduces SPDE tools to study the evolution of the eigenvalues of H_t: $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_N(t)$. A classical theorem below will guarantee that the eigenvalues are simple and continuous functions of t. So the labelling is preserved along the evolution.

In principle, the eigenvalues $\{\lambda_i(t)\}$ and eigenvectors $\{\vec{u}_i(t)\}$ of H_t are correlated strongly, and we expect a couple system of stochastic differential equations for them (which is indeed the case if B_t is not chosen to have the law of GOE / GUE).
But, Dyson observe that the eigenvalues themselves satisfy an autonomous system of SDEs that does not involve eigenvectors, which is called the Dyson Brownian Motion (DBM).

Theorem 6.2 The eigenvalues $\{\lambda_i(t)\}$ of H_t satisfy the following system of SDEs:

$$d\lambda_i = \frac{\sqrt{2}}{\sqrt{\beta N}} dB_i(t) + \left(-\frac{\lambda_i}{2} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)dt, \quad 1\le i\le N, \quad \begin{cases}\beta=1 & \text{for real,}\\ \beta=2 & \text{for complex.}\end{cases}$$

$\{B_i(t): 1\le i\le N\}$ is a collection of independent BMs. The solution to the above equation is called a DBM (where there is not necessarily an underlying matrix model).

Proof: Let $\lambda_\alpha^{(t)}$ be an eigenvalue of $H(t) = (h_{ij}(t))$ with eigenvector $\vec{u}_\alpha(t)$. Almost surely, all eigenvalues are simple. We apply Ito's formula to $\lambda_\alpha(t)$ to derive the DBM.
We only consider the real case with $\beta=1$.
Differentiating: $H\vec{u}_\alpha = \lambda_\alpha \vec{u}_\alpha$, $\vec{u}_\alpha^* \vec{u}_\beta = \delta_{\alpha\beta}$, we obtain that

$$(1) \quad \frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha + H\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \frac{\partial \lambda_\alpha}{\partial h_{ij}}\vec{u}_\alpha + \lambda_\alpha \frac{\partial \vec{u}_\alpha}{\partial h_{ij}},$$

$$(2)\quad \frac{\partial \vec{u}_\alpha^*}{\partial h_{ij}} \vec{u}_\beta + \vec{u}_\alpha^* \frac{\partial \vec{u}_\beta}{\partial h_{ij}} = 0, \qquad \vec{u}_\alpha^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = 0.$$

Taking inner product $\vec{u}_\alpha$ with (1) and using (2), we get

$$\vec{u}_\alpha^* \frac{\partial H}{\partial h_{ij}} \vec{u}_\alpha + \lambda_\alpha \vec{u}_\alpha^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \frac{\partial \lambda_\alpha}{\partial h_{ij}} + \lambda_\alpha \vec{u}_\alpha^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}}.$$

$$\Rightarrow \frac{\partial \lambda_\alpha}{\partial h_{ij}} = \vec{u}_\alpha^* \frac{\partial H}{\partial h_{ij}} \vec{u}_\alpha. \qquad (*)$$

Taking inner product $\vec{u}_\beta$ with (1) and using (2), we get (for $\beta \neq \alpha$)

$$\vec{u}_\beta^* \frac{\partial H}{\partial h_{ij}} \vec{u}_\alpha + \lambda_\beta \vec{u}_\beta^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \lambda_\alpha \vec{u}_\beta^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}}.$$

This implies that $\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \sum_{\beta \neq \alpha} \vec{u}_\beta \left(\vec{u}_\beta^* \frac{\partial \vec{u}_\alpha}{\partial h_{ij}} \right) = \sum_{\beta \neq \alpha} \vec{u}_\beta \frac{1}{\lambda_\alpha - \lambda_\beta} \left(\vec{u}_\beta^* \frac{\partial H}{\partial h_{ij}} \vec{u}_\alpha \right) \qquad (\dagger)$

For $(*)$, it writes: $\frac{\partial \lambda_\alpha}{\partial h_{ij}} = (2-\delta_{ij}) u_\alpha(i) u_\alpha(j)$. For $(\dagger)$, it writes

$$\begin{cases} \frac{\partial \lambda_\alpha}{\partial h_{ij}} = (2-\delta_{ij}) u_\alpha(i) u_\alpha(j). \\ \frac{\partial u_\alpha(k)}{\partial h_{ij}} = \sum_{\beta \neq \alpha} \frac{u_\beta(i) u_\alpha(j) + u_\beta(j) u_\alpha(i)(1-\delta_{ij})}{\lambda_\alpha - \lambda_\beta} u_\beta(k). \end{cases}$$

With these two formulas, we can also compute the second order partial derivatives:

$$\frac{\partial^2 \lambda_\alpha}{\partial h_{ik} \partial h_{\ell j}} = (2-\delta_{ik}) \left[\frac{\partial u_\alpha(i)}{\partial h_{\ell j}} u_\alpha(k) + u_\alpha(i) \frac{\partial u_\alpha(k)}{\partial h_{\ell j}} \right]$$

$$= (2-\delta_{ik}) \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} \Big[\big(u_\beta(j) u_\alpha(\ell) + u_\beta(\ell) u_\alpha(j)(1-\delta_{\ell j}) \big) u_\beta(i) u_\alpha(k)$$

$$+ \big(u_\beta(j) u_\alpha(\ell) + u_\beta(\ell) u_\alpha(j)(1-\delta_{\ell j}) \big) u_\beta(k) u_\alpha(i) \Big]$$

$$= (2-\delta_{ik}) \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} \big(u_\beta(j) u_\alpha(\ell) + u_\beta(\ell) u_\alpha(j)(1-\delta_{\ell j}) \big) \big(u_\beta(i) u_\alpha(k) + u_\beta(k) u_\alpha(i) \big).$$

Now, using Ito's formula, we obtain that

$$d\lambda_\alpha = \sum_{i \le k} \frac{\partial \lambda_\alpha}{\partial h_{ik}} dh_{ik} + \frac{1}{2} \sum_{i \le k} \sum_{j \le \ell} \frac{\partial^2 \lambda_\alpha}{\partial h_{ik} \partial h_{\ell j}} [dh_{ik}, dh_{\ell j}]$$

$$= \sum_{i,k} u_\alpha(i) u_\alpha(k) \left[\frac{db_{ik}}{\sqrt{N}} - \frac{1}{2} h_{ik} dt \right] + \sum_{i,k} \sum_{\alpha \neq \beta} \frac{1}{2N} \frac{1}{\lambda_\alpha - \lambda_\beta} \left[u_\beta(i) u_\alpha(k) + u_\beta(k) u_\alpha(i) \right]^2 dt$$

$$= \frac{1}{\sqrt{N}} \sum_{i,k} u_\alpha(i) u_\alpha(k) db_{ik} - \frac{1}{2} \lambda_\alpha dt + \frac{1}{N} \sum_{\alpha \neq \beta} \frac{1}{\lambda_\alpha - \lambda_\beta} dt,$$

where we used $[dh_{ik}, dh_{\ell j}] = \frac{1}{N} \delta_{i\ell} \delta_{kj} (1+\delta_{ik}) dt$, and $\sum_k h_{ik} u_\alpha(k) = \lambda_\alpha u_\alpha(i)$.

Now, we define a new Gaussian process: $\tilde{B}_\alpha := \sum_{i,k} u_\alpha(i) u_\alpha(k) db_{ik}$. Clearly, $\mathbb{E} \tilde{B}_\alpha = 0$.
We now calculate its covariance: