

$$= \frac{1}{2} \sum_{i,j} \| \vec{v}_i \|_j^2 + \frac{1}{2N} \sum_{i \neq j} \frac{(v_i(i) - v_j(j))^2}{(\lambda_i - \lambda_j)^2} \geq \frac{1}{2} \| \vec{v} \|_j^2.$$

Thus, $\nabla^2(\beta N f)$ $\geq \frac{1}{2} \beta N$. Here, we use a different normalization for the Dirichlet form: $D_\mu(f) = \frac{1}{\beta N} \int \| \nabla f \|^2 d\mu$. Then, the Bakry-Emery estimate gives $S_\mu(f) \leq 4 D_\mu(\sqrt{f})$.

This shows that the relaxation time is of order 1. (This is the global dynamics.)

We will show that the local dynamics converges to equilibrium ^{within} ~~at~~ time of order $N^{-1+\delta}$.

Thm 6.6. Assume the following conditions:

(i) There exists a constant $\beta > 0$ such that

$$Q := \sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\gamma_j - \bar{\gamma}_j)^2 f_t(d\bar{x}) \mu_G(d\bar{x}) \leq C N^{-2+2\beta}.$$

(ii) After time $1/N$, the solution to the equation $\partial_t f_t = L f_t$, $t \geq 0$, satisfies $S_\mu(f_{1/N}) \leq C N^m$ for some fixed m .

Fix any $\beta \geq 1$ and $n \in \mathbb{N}$. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function with compact support.

Denote:

$$f_{i,n}(\bar{x}) = F(N(x_{i+1} - \bar{x}), N(x_{i+2} - \bar{x}), \dots, N(x_{i+n} - \bar{x})).$$

Then, for any $\beta \in (0, \frac{1}{2})$ and any sufficiently small constant $\epsilon > 0$, there exist constants $C, c > 0$ depending only on ϵ and F , such that for any $J \in \{1, 2, \dots, N-n\}$ we have

$$\sup_{t \geq N^{-1+2\beta+\epsilon}} \left| \int \frac{1}{|J|} \sum_{i \in J} f_{i,n}(\bar{x}) (f_t d\mu - d\mu) \right| \leq C N^\epsilon \sqrt{\frac{N^2 Q}{|J| t}} + C e^{-c N^\epsilon}.$$

In particular, if $t \geq N^{-1+2\beta+2\epsilon+\delta}$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} f_{i,n}(\bar{x}) (f_t d\mu - d\mu) \right| \leq \frac{C}{\sqrt{|J| N^{\delta-1}}} + C e^{-c N^\epsilon}.$$

Hence, the gap distribution, averaged over J indices, coincides for $f_t d\mu$ and $d\mu$ if $|J| N^{\delta-1} \rightarrow \infty$.

Averaged gap distribution on scale J corresponds to averaged energy distribution on scale $J \sim N^\delta$.

Lemma. Under the setting of Thm 6.6, suppose $\left| \int \frac{1}{|J|} \sum_{i \in J} f_{i,n}(\bar{x}) (f_t d\mu - d\mu) \right| \leq \frac{C}{\sqrt{|J| N^{\delta-1}}}$ for some constant $\delta > 0$. (62)

Suppose the eigenvalue rigidity holds: $\forall \kappa > 0$, $\sup_{\kappa N \leq j \leq (1-\kappa)N} |\lambda_j - \bar{\lambda}_j| < N^{-1+\delta}$.

Then, for any constant $\varepsilon > 0$ and $N^{-1} \ll b_N \ll 1$, we have that

$$\left| \int_{E-b}^{E+b} \frac{dE'}{2b} \int_{\mathbb{R}^n} d\vec{\alpha} O(\vec{\alpha}) (P_{f\mu, N}^{(n)} - P_{\mu, N}^{(n)}) (E' + \frac{\vec{\alpha}}{N p_{sc}(E)}) \right| \\ \leq N^{2\varepsilon} \left[\frac{N^{-1+\delta}}{b} + \sqrt{\frac{N^{-\delta}}{b}} \right].$$

Hence, the averaged energy distribution over $b > N^{-1+2\varepsilon+2\delta} \sqrt{N^{-\delta}} \approx N^{-1+2\varepsilon+2\delta}$ coincides for $f\mu$ and μ .

The above two results establish the averaged gap/energy universality for $t \geq N^{-1+2\delta+2\varepsilon+\delta}$. Under the optimal rigidity, δ can be chosen as small as possible. Furthermore, ε and δ are arbitrary. This establishes the Dyson's conjecture: the local dynamics relaxes to equilibrium for $t \geq N^{-1+c}$ \forall constant $c > 0$.

Main ideas for the proof: Recall that

$$\langle \vec{v}, \nabla^2 f \vec{v} \rangle = \frac{1}{2} \sum_i v(i)^2 + \frac{1}{N} \sum_{i,j} \frac{(v(i) - v(j))^2}{(\lambda_i - \lambda_j)^2} \geq \frac{1}{2} \| \vec{v} \|^2.$$

The key intuition is that the relaxation time is in fact much shorter than order 1 for local variables that depend only on the eigenvalue differences. In particular, with the intuition that $\lambda_i - \lambda_j$ are close, the relaxation time in the direction $v(i) - v(j)$ should be much smaller than 1. However, this effect is hard to use directly in the real proof.

An idea is to add an "auxiliary strongly convex" potential to $f\mu$ to "speed up" the relaxation to local equilibrium. On the other hand, we will show that the effect of this speeding up on the local statistics can be controlled provided with the estimate

$$(+) \sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \bar{\lambda}_j)^2 f_t(\vec{\lambda}) \mu_G(d\vec{\lambda}) \lesssim N^{-2+2\delta}.$$

We introduce the following auxiliary potential $\tilde{f}(\vec{\lambda}) := \frac{1}{2\tau} \sum_{j=1}^N (\lambda_j - \bar{\lambda}_j)^2$, $0 < \tau < 1$. (We will choose $\tau = t N^{-\varepsilon}$ in the proof.)

Then, the new Hamiltonian is $\tilde{H} = H + \tilde{f}\mu$, and the corresponding measure is $d\omega = w(\vec{\lambda}) d\vec{\lambda}$, $w(\vec{\lambda}) := e^{-\beta N \tilde{f}(\vec{\lambda})} / \tilde{Z}$, called the "local relaxation measure". The "local relaxation flow" with the generator characterized by the natural Dirichlet form w.r.t. w : $\tilde{L} = L - \sum_j b_j \partial_j$, $b_j |x_j| = \frac{x_j - \bar{x}_j}{\tau}$.

For small τ , $\tilde{f}\mu$ will substantially increase the lower bound on the Hessian, hence speeding up the dynamics so that the relaxation time is at most τ . In addition, we will compare the local statistics of the original system with those of the modified one.

It turns out that the difference is governed by $\|\nabla f_t\|^2$, which can be controlled with the estimate (†).

Proof of Theorem 6.6

Proposition I. Fix any $\beta \geq 1$. Consider the equation $\partial_t g_t = \tilde{L} g_t$, $t \geq 0$, with the reversible measure $w = e^{-\beta N \tilde{L}} / Z_N$. Suppose the initial condition g_0 satisfies $\int g_0 dw = 1$. Then, we have that

$$(1) \quad \partial_t D_w(\sqrt{g_t}) \leq -\frac{2}{t} D_w(\sqrt{g_t}) - \frac{1}{\beta N^2} \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{g_t} - \partial_j \sqrt{g_t})^2}{(x_i - x_j)^2} dw,$$

$$(2) \quad \frac{1}{\beta N^2} \int_0^\infty ds \int \sum_{i,j=1}^N \frac{(\partial_i \sqrt{g_s} - \partial_j \sqrt{g_s})^2}{(x_i - x_j)^2} dw \leq D_w(\sqrt{g_0}),$$

and the LSI $S_w(g_0) \leq C \tau D_w(\sqrt{g_0})$. Hence, the relaxation time to equilibrium is of order τ :

$$(3) \quad S_w(g_t) \leq e^{-Ct/\tau} S_w(g_0), \quad D_w(\sqrt{g_t}) \leq e^{-Ct/\tau} D_w(\sqrt{g_0}).$$

Proof: Performing the same calculation as in the proof of the Bakry - Emery estimate, we obtain that for $h = \sqrt{g_t}$,

$$\partial_t D_w(h_t) = \partial_t \frac{1}{\beta N} \int (\nabla h_t)^2 e^{-\beta N \tilde{L}} dx \stackrel{-\frac{2}{\beta N}}{\leq} \int (\nabla h_t) \cdot (\nabla^2 \tilde{L}) \cdot (\nabla h_t) e^{-\beta N \tilde{L}} dx.$$

For \tilde{L} , we can calculate its Hessian as

$$\nabla h \cdot \nabla^2 \tilde{L} \cdot \nabla h \geq \frac{1}{t} \sum_j (\partial_j h)^2 + \frac{1}{2N} \sum_{i,j} \frac{1}{(x_i - x_j)^2} (\partial_i h - \partial_j h)^2.$$

Plugging it into the above inequality, we conclude (1). Integrating over t , we conclude (2). The LSI and (3) can be derived in the same way as in the proof of Bakry - Emery. \square

The extra term on the RHS of (1) plays a key role in the proof of the following prop.

Proposition II. Let g be a probability density w.r.t. the local relaxation measure w , i.e., $\int g dw = 1$. Fix any $n \geq 1$ and a smooth function F with compact support. Recall that $F_{i,n}(\vec{x}) := F(N(x_{i+1} - x_i), \dots, N(x_{i+n} - x_i))$.

Then, for any $J \subseteq \{1, 2, \dots, N-n\}$ and $t \geq 0$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} F_{i,n}(\vec{x}) (g dw - dw) \right| \leq C \left(t \frac{D_w(\sqrt{g})}{|J|} \right)^{1/2} + C \sqrt{S_w(g)} e^{-Ct/\tau}.$$

Proof: For simplicity of notation, we take $n=1$, so $F_{i,1}(\vec{x}) = F(N(x_{i+1} - x_i))$. Let g_t be a solution to $\partial_t g_t = \tilde{L} g_t$, $t \geq 0$, with initial condition $g_0 = g$. Then,

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (g - g_t) dw = \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (g - g_t) dw +$$

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (g_t - 1) dw.$$

The second term can be bounded by

$$\int |g_t - 1| d\omega \stackrel{\uparrow}{\leq} C \sqrt{S_\omega(g_t)} \stackrel{\downarrow}{\leq} C e^{-ct/c} \sqrt{S_\omega(g_0)}.$$

Pinsker inequality

To estimate the first term, using $\partial_t g_t = \tilde{L} g_t$, we get that

$$\begin{aligned} & \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) g_t d\omega - \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) g_0 d\omega \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \frac{g_t - g_0}{ds} d\omega \approx g_s d\omega \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \tilde{L} g_s d\omega \\ &= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G'(N(x_{i+1} - x_i)) (\partial_{i+1} g_s - \partial_i g_s) d\omega. \quad (*) \end{aligned}$$

Using ~~$\partial_i g = 2\sqrt{g} \partial_i \sqrt{g}$~~ and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} |(*)| &\leq 2 \left[\int_0^t ds \int \frac{N^2}{|J|^2} \sum_{i \in J} |G'(N(x_{i+1} - x_i))|^2 (x_{i+1} - x_i)^2 g_s d\omega \right]^{1/2} \times \\ &\quad \underbrace{\left[\int_0^t ds \int \frac{1}{N^2} \sum_i \frac{1}{(x_i - x_{i+1})^2} (\partial_{i+1} \sqrt{g_s} - \partial_i \sqrt{g_s})^2 d\omega \right]^{1/2}} \\ &\leq C D_\omega(\sqrt{g_0}) \end{aligned}$$

Moreover, $|G'(N(x_{i+1} - x_i))|^2 |x_{i+1} - x_i|^2 \leq CN^{-2}$ because G is smooth and compactly supported. So

$$|(*)| \leq C \sqrt{D_\omega(g_0)} \cdot \left(\frac{t}{|J|} \right)^{1/2}. \quad \square$$

Proposition III. Consider the local relaxation measure ω for a given $\tau > 0$. Set $\gamma = \omega/\mu$ and $g_t = f_t/\gamma$ with f_t solving $\partial_t f_t = L f_t$. Suppose there is a const. $C > 0$ such that $S(f_t \mu | \omega) \leq N^C$.

Fix any small $\epsilon > 0$. For any $t \in [\tau N^\epsilon, N]$, the entropy and Dirichlet form satisfy

$$S(g_t \omega | \omega) \leq CN^2 Q \tau^{-1}, \quad D_\omega(\sqrt{g_t}) \leq CN^2 Q \tau^{-2},$$

where recall that $Q = \sup_{0 \leq t \leq N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \gamma_j)^2 f_t(\vec{\lambda}) \mu(d\vec{\lambda}) \quad (\leq CN^{-2+2\beta})$.

Proof: We calculate $\partial_t S(f_t \mu | \omega) = \partial_t S(f_t \mu | \gamma \mu)$

$$= \int (L f_t)(\log g_t) d\mu + \int f_t \frac{L f_t}{f_t} d\mu = \int (L f_t)(\log g_t) d\mu$$

$$= \int f_t L(\log g_t) d\mu = \int g_t + 4 L(\log g_t) d\mu = \int \gamma_t + 4 [g_t L(\log g_t) - g_t \frac{L g_t}{g_t}] d\mu$$

$$+ \int 4 L g_t d\mu.$$

We can check directly that $\ell(\log g_t) - \frac{\log g_t}{g_t} = -\frac{1}{\beta N} \frac{|\nabla g_t|^2}{g_t^2} = -\frac{4}{\beta N} |\nabla \sqrt{g_t}|^2 \cdot \frac{1}{g_t}$.

$$\begin{aligned} \text{Hence, we have } \partial_t S(f_t, \mu | w) &= -\frac{4}{\beta N} \int 4 |\nabla \sqrt{g_t}|^2 d\mu + \int 4 \log g_t d\mu, \quad (b_j = \frac{x_j - \bar{x}_j}{\tau}) \\ &= -\frac{4}{\beta N} \int |\nabla \sqrt{g_t}|^2 dw + \int \cancel{4 \log g_t}^0 d\mu + \sum_j \int b_j \partial_j g_t dw. \end{aligned}$$

Using $\partial_j g_t = 2 \sqrt{g_t} \partial_t \sqrt{g_t}$ and C-S, we get

$$\begin{aligned} \partial_t S(f_t, \mu | w) &\leq -4 D_w(\sqrt{g_t}) + 2 \sum_j \int b_j \sqrt{g_t} \partial_j \sqrt{g_t} dw \\ &\leq -2 D_w(\sqrt{g_t}) + C N \sum_j \int b_j^2 g_t dw \\ &= -2 D_w(\sqrt{g_t}) + C \frac{N^2}{\tau^2} \frac{1}{N} \int (x_j - \bar{x}_j)^2 g_t dw \\ &\leq -2 D_w(\sqrt{g_t}) + C N^2 Q \tau^{-2}. \quad (*) \end{aligned}$$

Applying the LSI w.r.t. dw , we get that

$$\partial_t S(f_t, \mu | w) \leq -C \tau^{-1} S(f_t, \mu | w) + C N^2 Q \tau^{-2}.$$

Applying the Gronwall's inequality, integrating this inequality from τ to t and using the assumption $S(f_\tau, \mu | w) \leq N^C$ and $t \geq \tau N^\varepsilon$, we obtain that

$$S(g_t, \mu | w) = S(f_t, \mu | w) \leq C N^2 Q \tau^{-1}. \quad (**) \quad \text{[Wavy line]} \quad \text{[Wavy line]}$$

To prove the second estimate, we notice

$$\begin{aligned} D_\mu(\sqrt{f_s}) &= \frac{1}{\beta N} \int \frac{|\nabla(g_s \psi)|^2}{g_s \psi} \frac{dw}{\psi} \leq \frac{C}{\beta N} \int \left[\frac{|\nabla g_s|^2}{g_s} + |\nabla \log \psi|^2 g_s \right] dw \\ &\leq C D_w(\sqrt{g_s}) + \frac{C}{\beta N} \int \frac{|\nabla(x_j - \bar{x}_j)|^2}{\tau^2} dt \leq C \frac{(\beta N)^2}{\beta N} \int \frac{|\nabla(x_j - \bar{x}_j)|^2}{\tau^2} dt \\ &\leq C D_w(\sqrt{g_s}) + C N^2 Q \tau^{-2}. \quad (*) \end{aligned}$$

Taking integral over $[t-\tau, t]$ ~~over w~~ , we get

$$\int_{t-\tau}^t D_\mu(\sqrt{f_s}) ds \geq \int_{t-\tau}^t \left[\frac{1}{C} D_\mu(\sqrt{f_t}) - C N^2 Q \tau^{-2} \right] ds$$

$$(D_\mu(\sqrt{f_t}) \text{ is decreasing in } t) \Leftrightarrow \frac{1}{C} D_\mu(\sqrt{f_t}) - C N^2 Q \tau^{-1}.$$

On the other hand, with $(*)$ and $(**)$, we get that

$$\int_{t-\tau}^t D_\mu(\sqrt{g_s}) ds \leq C N^2 Q \tau^{-1} + [S(f_t, \mu | w) - S(f_{t-\tau}, \mu | w)] \leq C N^2 Q \tau^{-1}.$$

From the above two inequalities, we obtain that

$$D_\mu(\sqrt{f_t}) \leq C N^2 Q \tau^{-2}.$$

With a similar argument as in $(*)$, we get that

$$D_W(\sqrt{g_t}) \leq C D_W(\sqrt{f_t}) + CN^2 Q t^{-2} \leq CN^2 Q t^{-2}. \quad \square$$

Theorem 6.6 is a consequence of the above three propositions. We choose $\tau = tN^{-\epsilon}$ (recall that $t \geq N^{-1+2\beta+2\epsilon+\delta}$), $\gamma = w/\mu$ and $g_t = g_t/\gamma$. To apply Prop. III, we still need to verify the assumption $S(f_{\tau} \mu | w) \leq N^C$. By definition,

$$\begin{aligned} S(f_{\tau} \mu | w) &= \int f_{\tau} \log f_{\tau} d\mu - \int f_{\tau} \log \gamma d\mu \\ &= S_{\mu}(f_{\tau}) - \int f_{\tau} \log \gamma d\mu, \end{aligned}$$

where $-\log \gamma = \frac{\beta N}{2\tau} \sum_{j=1}^N (x_j - \bar{x}_j)^2 + \log \frac{\tilde{Z}}{Z}$. We know $\tilde{Z} \leq Z$ since $\tilde{f}_{\tau} \geq f_{\tau}$. Hence,

$$-\int f_{\tau} \log \gamma d\mu \leq CN\tau^{-1} \int \sum_{j=1}^N (x_j - \bar{x}_j)^2 f_{\tau}(\bar{x}_j) d\mu(\bar{x}_j) \leq CN^2 \tau^{-1} Q \leq N^C.$$

It remains to bound $S_{\mu}(f_{\tau})$ for a "good initial condition" as a Wigner matrix.

Lemma: Let $\beta=1, 2$. Suppose the initial data f_0 of the DBM is given by the eigenvalue distribution of a Wigner matrix. Then, $\forall \tau \geq 0$, we have

$$S_{\mu}(f_{\tau}) \leq CN^2 [1 - \log(1 - e^{-\tau})].$$

Proof: We only consider the real case with $\beta=1$. Recall that $f_{\tau} \mu$ is the eigenvalue distribution of $H_{\tau} = e^{-\tau/2} H_0 + (1 - e^{-\tau})^{1/2}$ GOE. Notice that $f_{\tau} \mu$ is the marginal of $\mu_{H_{\tau}}$ by integrating out the eigenvector distribution, and μ is the marginal of GOE. Then, using the fact that the entropy is decreasing w.r.t. taking a marginal, we get

$$S_{\mu}(f_{\tau}) \leq S(\mu_{H_{\tau}} \otimes \mu_{\text{GOE}}).$$

Notice that both $\mu_{H_{\tau}}$ and μ_{GOE} are product measures of the laws of matrix elements. Then, using the additivity of entropy, $S(\mu_{H_{\tau}} \mid \mu_{\text{GOE}})$ is equal to the sum of the relative entropies of the matrix elements. Let's consider the off-diagonal elements:

for $\gamma = 1 - e^{-\tau}$, let p_{γ} be the probability density of $(1-\gamma)^{1/2} g = e^{-\tau/2} g$, where g is the random variable for an off-diagonal matrix element. Then, the prob. density of an off-diagonal element of H_{τ} is given as: $g_{\tau} = p_{\gamma} * g_{\tau/2} \otimes g_{1/N}$, where $g_{\tau/2}$ denotes the Gaussian distribution

$$g_{\tau/2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}).$$

Then, we calculate that

$$S(g_{\tau/2} \otimes g_{1/N}) = S\left(\int dy p_{\gamma}(y) g_{\tau/2}(y) \mid g_{1/N}\right)$$

$$(\text{Jensen}) \leq \int dy p_{\gamma}(y) S(g_{\tau/2}(y) \mid g_{1/N}).$$

A direct calculation yields that

$$S(g_{\tau/2}(y) \mid g_{\tau/2}) = \log \frac{g_{\tau/2}}{g_{\tau/2}} + \frac{g_{\tau/2}^2}{2g_{\tau/2}^2} + \frac{g_{\tau/2}^2}{2g_{\tau/2}^2} - \frac{1}{2}.$$

Applying it to our case, we get

$$S(g_{\gamma/N}(\cdot - y) | g_{1/N}) = \frac{1}{2}(Ny^2 - \log r + r - 1).$$

Thus, $S(g_t | g_{1/N}) \leq \int dy p_\gamma(y) \frac{1}{2}(Ny^2 - \log r + r - 1)$. Using $\int y^2 p_\gamma(y) dy = \frac{1}{N}$, we conclude the proof. \square

With the above lemma, we can apply Prop. II & III to get that (with $g = f_t \otimes g$, $g dw = f_t d\mu$, $t = \tau N^{-\varepsilon}$)

$$\left| \int \frac{1}{|J|} \sum_{i \in J} f_{i,n}(\bar{x}) (f_t d\mu - g dw) \right| \leq C \left(t \frac{D_w(\sqrt{g})}{|J|} \right)^{1/2} + C \sqrt{S_w(g)} e^{-cN^\varepsilon}$$

$$[D_w(\sqrt{g}) = D_w(\sqrt{g_t})] \rightarrow \leq C \left(t \frac{N^2 Q}{|J| \tau^2} \right)^{1/2} + N^\varepsilon e^{-cN^\varepsilon} \leq C N^\varepsilon \left(\frac{N^2 Q}{|J| t} \right)^{1/2} + C e^{-cN^\varepsilon}.$$

Hence, $f_t d\mu$ and dw are close for any initial data. Applying this estimate to the Gaussian initial data, we can also compare $d\mu$ and dw . This concludes the proof.

Section 7 Green function comparison

We now finish Step 3 of the three step strategy. That is, given a Wigner H , we can find a Wigner H_0 and $t \geq N^{-1+\epsilon}$ such that the local eigenvalue statistics of H_t match those of $H_t = e^{-t/2} H_0 + (-e^{-t})^{1/2} H_G$, $H_G \stackrel{d}{=} \text{GOE/GUE}$. We have shown the bulk universality of H_t . Then, Step 3 implies the bulk universality of H .

To compare local statistics of two Wigner ensembles, we use the following "correlation function comparison theorem". For the detailed setting, suppose we have two Wigner ensembles, labelled by v and w , respectively: H^v and H^w . Denote their eigenvalues by λ_i^v and λ_i^w , and the corresponding Stieltjes transforms by m^v and m^w , i.e.,

$$m^v(z) = \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j^v - z}.$$

Let $P_{v,N}^{(n)}$ and $P_{w,N}^{(n)}$ be the n -point correlation functions of $\{\lambda_i^v\}$ and $\{\lambda_i^w\}$.

Theorem 7.1 Let $\kappa' > 0$ be a small const. Suppose for some small constants $\delta, \theta > 0$ such that the following two conditions hold.

(i) $\forall \epsilon > 0$ and fixed $k \in \mathbb{N}$,

$$|\mathbb{E} [\operatorname{Im} m^v(E + iN^{-1+\epsilon})]^k + \mathbb{E} [\operatorname{Im} m^w(E + iN^{-1+\epsilon})]^k| \leq C$$

for $|E| \leq 2 - \kappa$.

(ii) For any sequence $\beta_j = E_j \pm i\eta_j$, $j = 1, \dots, n$, with $|E_j| \leq 2 - \kappa$ and $\eta_j = N^{-1-\theta}$ for some $\theta \leq \delta$, we have

$$|\mathbb{E} [\operatorname{Im} m^v(\beta_1) \cdots \operatorname{Im} m^v(\beta_n)] - \mathbb{E} [\operatorname{Im} m^w(\beta_1) \cdots \operatorname{Im} m^w(\beta_n)]| \leq N^{-\delta}. \quad (*)$$

Then, for any $n \in \mathbb{N}$, there are constants $C_n = C_n(\delta, \theta)$ s.t. for any $|E| \leq 2 - 2\kappa$ and C^1 function $O: \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support,

$$\left| \int_{\mathbb{R}^n} O(\vec{a}) (P_{v,N}^{(n)} - P_{w,N}^{(n)}) (E + \frac{\vec{a}}{N}) d\vec{a} \right| \leq C_n N^{-C_n}.$$

Roughly speaking, this theorem says that if the finite dimensional marginals of the Stieltjes transforms are close, then the local correlation functions are close. It is crucial that in $(*)$, β_i has imaginary part $N^{-1-\theta}$ below the scale of bulk eigenvalue gaps N^{-1} . Controlling the Stieltjes transforms on this very short scale is necessary to identify and compare local correlation functions at the scale N^{-1} .

The condition $(*)$ can be replaced by a condition that is easier to check: let $F: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function such that for any multi-index $a = (a_1, \dots, a_n)$ with $1 \leq |a| = |a_1| + \dots + |a_n| \leq 5$ and sufficiently small constant $\epsilon > 0$,

$$\max \{ |\partial^2 F(x_1, \dots, x_n)| : \max_j |x_j| \leq N^\varepsilon \} \leq N^{C_0 \varepsilon},$$

$$\max \{ |\partial^3 F(x_1, \dots, x_n)| : \max_j |x_j| \leq N^2 \} \leq N^{C_0},$$

for some constant C_0 . For any such F , we have

$$(+) \quad |IE F(Im m^V(z_1) \dots Im m^V(z_n)) - IE F(Im m^W(z_1), \dots, Im m^W(z_n))| \leq N^{-\delta}.$$

Given (+), we can approximate $IE [Im m^W(z_1) \dots Im m^W(z_n)]$ by $IE F(Im m^W(z_1), \dots, Im m^W(z_n))$ with $F(x_1, \dots, x_n) = x_1 x_2 \dots x_n$ if $\max_j |x_j| \leq N^c$ and it is smoothly cutoff to 0 in the regime $\max_j |x_j| \geq N^c$ for some small constant $c > 0$.

To see this, recall that $\eta Im m(E+i\eta) \leq \eta' Im m(E+i\eta')$ for $0 < \eta \leq \eta'$. Then, (i) of Theorem 7.1 implies that for any $\eta \leq N^{-1+\varepsilon}$,

$$IE [\eta Im m(E+i\eta)]^k \leq IE [N^{-1+\varepsilon} Im m(E+iN^{-1+\varepsilon})]^k \\ \leq (CN^{-1+\varepsilon})^k$$

$$\Rightarrow IE [Im m(E+i\eta)]^k \leq (C \frac{N^\varepsilon}{N\eta})^k \quad \text{for any fixed } k \in \mathbb{N}.$$

Let $\eta = N^{-1-\varepsilon}$ and $z = E+i\eta$. Consider $n=1$ for simplicity. By choosing $c = \frac{26}{22}$,

$$|IE F(Im m^V(z)) - IE Im m^V(z)| \leq IE Im m^V(z) \cdot \mathbf{1}(Im m^V(z) \geq N^c) \\ \leq N^{-C(k-1)} IE [Im m^V(z)]^k \leq CN^{k(6+\varepsilon)-C(k-1)} \leq N^{-\delta}$$

as long as ε is small enough and k is large enough. Similar arguments apply to general n .

The conditions on F and $\partial^3 F$ will be used in the Green's function comparison argument. The proof of Theorem 7.1 can be found in Theorem 10.15.3 of Erdős-Yau.

I. The Lindeberg replacement trick

The Green's function comparison uses a classical "Lindeberg replacement trick", which was developed in Lindeberg's own proof of CLT. To explain the basic idea of this argument and the importance of moment matching conditions, we use it to prove the following version of the Berry-Esseen theorem.

Theorem 7.2 (Berry-Esseen theorem) Let X be a random variable with mean 0, variance 1, and finite 3rd moment. Let φ be smooth with uniformly bounded third-order derivatives up to third order. Let X_1, \dots, X_n be i.i.d. copies of X and

$$Z_n := \frac{X_1 + \dots + X_n}{\sqrt{n}}.$$

Then, we have that

$$\mathbb{E} \varphi(Z_n) = \mathbb{E} \varphi(G) + O\left(\frac{1}{\sqrt{n}} \mathbb{E}|x|^3 \cdot \|\varphi''\|_{\infty}\right),$$

where $G \sim N(0, 1)$ is standard Gaussian.

Proof: Note that G has the same distribution as $\frac{Y_1 + \dots + Y_n}{\sqrt{n}}$, where Y_i 's are independent $N(0, 1)$ random variables. Hence, we only need to bound

$$\mathbb{E} \varphi\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) - \mathbb{E}\left(\frac{Y_1 + \dots + Y_n}{\sqrt{n}}\right) = \sum_{i=0}^{n-1} [\mathbb{E} \varphi(Z_{n,i+1}) - \mathbb{E} \varphi(Z_{n,i})],$$

where $Z_{n,i} = \frac{x_1 + \dots + x_i + Y_{i+1} + \dots + Y_n}{\sqrt{n}}$, i.e., we replace ~~the first~~ y_1, \dots, y_i with x_1, \dots, x_i in $Z_{n,i}$.

Then, it suffices to show that

$$\mathbb{E} \varphi(Z_{n,i+1}) - \mathbb{E} \varphi(Z_{n,i}) = O\left(\mathbb{E}|x|^3 \frac{\|\varphi''\|_{\infty}}{n^{3/2}}\right).$$

Note that $Z_{n,i+1}$ is different from $Z_{n,i}$ only by one entry, i.e., a $\frac{Y_{i+1}}{\sqrt{n}}$ is replaced by $\frac{x_{i+1}}{\sqrt{n}}$.

Define

$$S_{n,i} = \frac{x_1 + \dots + x_i + Y_{i+2} + \dots + Y_n}{\sqrt{n}}, \quad Z_{n,i+1} = S_{n,i} + \frac{x_{i+1}}{\sqrt{n}}, \quad \begin{cases} S_{n,i} \text{ is independent} \\ \text{of } x_{i+1} \text{ and } Y_{i+1}. \end{cases}$$

Using Taylor expansion, $\varphi(Z_{n,i}) = \varphi(S_{n,i}) + \varphi'(S_{n,i}) \frac{Y_{i+1}}{\sqrt{n}} + \frac{1}{2} \varphi''(S_{n,i}) \frac{Y_{i+1}^2}{n} + O\left(\frac{1}{n^{3/2}} \|\varphi''\|_{\infty}\right)$,

$$\varphi(Z_{n,i+1}) = \varphi(S_{n,i}) + \varphi'(S_{n,i}) \frac{x_{i+1}}{\sqrt{n}} + \frac{1}{2} \varphi''(S_{n,i}) \frac{x_{i+1}^2}{n} + O\left(\frac{1}{n^{3/2}} \|\varphi''\|_{\infty}\right).$$

Since the moments of x_{i+1} and Y_{i+1} match to second order. We get

$$\mathbb{E} \varphi(Z_{n,i+1}) - \mathbb{E} \varphi(Z_{n,i}) = O\left(\|\varphi''\|_{\infty} \frac{\mathbb{E}|Y_{i+1}|^3 + \mathbb{E}|x_{i+1}|^3}{n^{3/2}}\right) = O\left(\mathbb{E}|x|^3 \frac{\|\varphi''\|_{\infty}}{n^{3/2}}\right). \quad \square$$

II. Green's function comparison theorem (four-moment theorem)

In Theorem 7.1, (i) follows from the local semicircle law. The condition (ii) follows from the next theorem, which, roughly speaking, shows that two Wigner ensembles have almost identical local statistics provided that their matrix elements have matching "first four moments".

Theorem 7.3 Suppose we have two Wigner matrices H^V and H^W , with matrix elements

$v_{ij}^V = \frac{1}{\sqrt{n}} v_{ij}$, $w_{ij}^W = \frac{1}{\sqrt{n}} w_{ij}$. Suppose v_{ij} and w_{ij} have finite moments up to arbitrary order, i.e., $\forall k \in \mathbb{N}$,

$$\mathbb{E}|v_{ij}|^k + \mathbb{E}|w_{ij}|^k \leq c_k \text{ for a constant } c_k > 0.$$

Suppose v_{ij} and w_{ij} have matching first three moments, i.e.,

$$\mathbb{E} \bar{v}_{ij}^s v_{ij}^{k-s} = \mathbb{E} \bar{w}_{ij}^s w_{ij}^{k-s}, \quad k=1, 2, 3, \quad s=0, 1, \dots, k,$$

and almost identical fourth moments:

$$|\mathbb{E} \bar{v}_{ij}^s v_{ij}^{4-s} - \mathbb{E} \bar{w}_{ij}^s w_{ij}^{4-s}| \leq N^{-\delta}, \quad s=0, 1, 2, 3, 4,$$

for some constant $\delta > 0$. Suppose the local semicircle laws hold for H^V and H^W . (71)

Let F be a function as given below Theorem 7.1. Consider any sequence $\beta_j = E_j \pm i\eta_j$, $j=1, \dots, n$, with $|E_j| \leq 2 - \kappa$ and $\eta_j = N^{-1-\delta}$ for $\delta_j \leq 6$ and an arbitrary choice of the \pm signs. Then, there exists a constant $C_1 > 0$ that does not depend on δ such that

$$|\mathrm{IEF}(m^V(\beta_1), \dots, m^V(\beta_n)) - \mathrm{IEF}(m^W(\beta_1), \dots, m^W(\beta_n))| \leq N^{-\frac{1}{2} + C_1\delta} + N^{-\delta + C_1\delta}.$$

Combining this theorem with Theorem 7.1, we get that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^n} O(\vec{\alpha}) (P_{V,N}^{(n)} - P_{W,N}^{(n)}) (E + \frac{\vec{\alpha}}{N}) d\vec{\alpha} = 0, \quad \forall O \in C_c^\infty(\mathbb{R}^n) : \mathbb{R}^n \rightarrow \mathbb{R}.$$

Proof of Theorem 7.3: We use the Lindeberg replacement trick, i.e., we replace the matrix elements of H^V one by one and estimate the error of each step using a resolvent expansion.

Note that each matrix element has a typical size $N^{-1/2}$ and all resolvents can be controlled using the local semicircle law. Hence, a resolvent expansion up to the fourth order will identify the change of each resolvent element by ~~an error $O(N^{-5/2})$~~ with precision $O(N^{-5/2})$. The expectation values of the terms up to 4th order involve only the first four moments of v_{ij} 's and w_{ij} 's, which can be compared using the four-moment matching conditions. This leads to ~~an~~ errors $O(N^{5/2} + N^{-2-\delta})$, which are negligible after summing over N^2 times.

For simplicity of presentation ~~an~~, to illustrate the main idea, we consider the real case ~~and the case~~ with $n=1$. The general case follows analogously. We need to control

$$(*) \quad |\mathrm{IEF}(m^V(\beta)) - \mathrm{IEF}(m^W(\beta))|, \quad \beta = E \pm i\eta, \quad \eta = N^{-1-\delta}.$$

We fix a ~~bijective~~ ordering map on the index set of independent entries:

$$\phi: \{(i,j) : 1 \leq i \leq j \leq N\} \rightarrow \{1, \dots, \frac{N(N+1)}{2}\}, \quad \frac{N(N+1)}{2} = \frac{N(N+1)}{2}.$$

Fix any $1 \leq \gamma \leq \frac{N(N+1)}{2}$. Let H_γ be the matrix whose entries are h_{ij}^V if $\phi(i,j) \leq \gamma$ and h_{ij}^W if $\phi(i,j) > \gamma$. In particular, we have

$$H_{1:N} = H^V \quad \text{and} \quad H_0 = H^W.$$

Thus, we can write $(*)$ as a telescoping sum

$$\mathrm{IEF}\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H^V - \beta}\right) - \mathrm{IEF}\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H^W - \beta}\right) = \sum_{\gamma=1}^{\frac{N(N+1)}{2}} \left[\mathrm{IEF}\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H_\gamma - \beta}\right) - \mathrm{IEF}\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H_{\gamma-1} - \beta}\right) \right]$$

We need to estimate each summand. Suppose $(i,j) = \phi^{-1}(\gamma)$. Let $E^{(i,j)}$ be the matrix

$E^{(i,j)} = e_i e_j^T + e_j e_i^T$, i.e., $E^{(i,j)}$ is the matrix whose elements are zero everywhere except at (i,j) and (j,i) positions, where it is 1. Then, we can write

$$H_{\gamma-1} = Q + \frac{1}{\sqrt{N}} W_{ij} E^{(i,j)}, \quad H_\gamma = Q + \frac{1}{\sqrt{N}} V_{ij} E^{(i,j)},$$

where Q is the matrix obtained by setting the (i,j) and (j,i) -th entries of H_γ and $H_{\gamma-1}$ to 0.

Note that Q is independent of w_{ij} and v_{ij} .

Define the resolvents $R := \frac{1}{\lambda - z}$. We abbreviate: $S := \frac{1}{H_\lambda - z}$ and $T := \frac{1}{H_{\lambda-1} - z}$.

(1) Resolvent estimates for R, S, T

By local semicircle law, we have $\max_{0 \leq \eta \leq a(N)} \max_{1 \leq k \leq N} \sup_{|\zeta| \leq 2-k} \sup_{\eta \geq N^{-1+\epsilon}} \left| \left(\frac{1}{H_\lambda - E - i\eta} \right)_{kk} \right| < 1,$

for any small constants $\kappa > 0$ and $\epsilon > 0$. Again, using

$$\eta \operatorname{Im} \left(\frac{1}{H_\lambda - E - i\eta} \right) \leq \eta' \operatorname{Im} \left(\frac{1}{H_\lambda - E - i\eta'} \right), \quad 0 < \eta \leq \eta',$$

We obtain that

$$\max_{1 \leq k \leq N} \sup_{|\zeta| \leq 2-k} \sup_{\eta \geq N^{-1-\epsilon}} \left| \operatorname{Im} \left(\frac{1}{H_\lambda - E - i\eta} \right)_{kk} \right| \ll N^{6+\epsilon}.$$

From this estimate, we derive the following result.

Lemma: For any $\eta \geq N^{-1-\epsilon}$, we have

$$(*) \quad \max_{\lambda} \max_{1 \leq k, l \leq N} \sup_{|\zeta| \leq 2-k} |G_{kk}^{\lambda}(E \pm i\eta)| \ll N^{6+\epsilon}, \quad G^{\lambda}(z) := \frac{1}{H_\lambda - z}.$$

Proof: With the eigen decomposition of G_{kk}^{λ} , for $\zeta = E + i\eta$,

$$|G_{kk}^{\lambda}| \leq \sum_{j=1}^N \frac{|U_j(k)| |U_j(l)|}{|\lambda_j - \zeta|} \leq \left[\sum_{j=1}^N \frac{|U_j(k)|^2}{|\lambda_j - \zeta|} \right]^{1/2} \left[\sum_{j=1}^N \frac{|U_j(l)|^2}{|\lambda_j - \zeta|} \right]^{1/2}.$$

Consider a dyadic decomposition $U_n = \{ j : 2^{n-1} \eta \leq |\lambda_j - E| \leq 2^n \eta \}, n=1, 2, \dots, C \log N$,

$$U_0 = \{ j : |\lambda_j - E| \leq \eta \},$$

$$U_{\infty} = \{ j : |\lambda_j - E| \geq 2^{C \log N} \eta \}. \quad (\text{Take } C \text{ large s.t. } 2^{C \log N} \eta \geq 1.)$$

Then, we divide the summation over j into U_n, U_0 :

$$\begin{aligned} \sum_{j=1}^N \frac{|U_j(k)|^2}{|\lambda_j - \zeta|} &= \sum_n \sum_{j \in U_n} \frac{|U_j(k)|^2}{\sqrt{|\lambda_j - E|^2 + \eta^2}} \leq C \sum_{n \in \infty} \sum_{j \in U_n} \frac{2^n \eta |U_j(k)|^2}{|\lambda_j - E|^2 + (2^n \eta)^2} \\ &\quad + C \sum_{j \in U_0} |U_j(k)|^2 \\ &\leq C \sum_{n \in \infty} \sum_{j \in U_n} \operatorname{Im} \left(\frac{|U_j(k)|^2}{\lambda_j - E - i2^n \eta} \right) + C \leq C \sum_{n=0}^{C \log N} \operatorname{Im} G_{kk}^{\lambda}(E + i2^n \eta) + C \\ &\ll N^{6+\epsilon}. \end{aligned}$$

Next, we claim that the above estimate also holds for $R(z)$:

$$(+) \quad \max_{1 \leq k, l \leq N} \sup_{|\zeta| \leq 2-k} |R_{kk}(E \pm i\eta)| \ll N^{6+\epsilon}.$$

Let $V = V_{ij} E^{(ij)}$. Then we have the resolvent expansion: $\forall k \in N$,

$$R = (\lambda - z)^{-1} = (H_\lambda - \frac{1}{\sqrt{n}} V^\dagger - z)^{-1} = (S^{-1} - \frac{1}{\sqrt{n}} V)^{-1}$$

$$= S + \frac{1}{\sqrt{N}} SVS + \frac{1}{N} (SV)^2 S + \cdots + \frac{1}{N^{k/2}} (SV)^k S + \frac{1}{N^{(k+1)/2}} (SV)^{k+1} R.$$

Using the above estimate (*) for S and the trivial bound $\|R\| \leq \eta^{-1}$, taking $k=5$ in the above expansion, we can obtain (+) for R .

(2) The Green's function comparison

We now expand $S = (H_\gamma - z)^{-1}$ using R :

$$S(z) = (R^{-1} + \frac{1}{\sqrt{N}} V)^{-1} = R - N^{-\frac{1}{2}} RVR + N^{-1} (RV)^2 R - N^{-\frac{3}{2}} (RV)^3 R + N^{-2} (RV)^4 R - N^{-\frac{5}{2}} (RV)^5 S.$$

Taking trace, we get

$$\frac{1}{N} \text{Tr } S(z) = \underline{R} + \sum_{k=1}^{\lfloor \frac{4}{N} \rfloor} N^{-\frac{k}{2}} \underline{(RV)^k R} + N^{-\frac{5}{2}} \underline{D_V}, \quad \text{where } M := \frac{1}{N} \text{Tr } M \text{ for any } N \times N \text{ matrix}$$

and $D_V = -\underline{(RV)^5 S}$.

$$=: \underline{R} + \underline{s}.$$

Then, performing Taylor expansion, we get that

$$\mathbb{E} F(\frac{1}{N} \text{Tr } S(z)) = \mathbb{E} F(\underline{R} + \underline{s}) = \mathbb{E}[F(\underline{R}) + F'(\underline{R}) \underline{s} + \frac{1}{2} F''(\underline{R}) \underline{s}^2 + \cdots + \frac{1}{5!} F^{(5)}(\underline{R} + \underline{s}') \underline{s}^5]$$

where \underline{s}' is a value between 0 and \underline{s} , and it depends on F , R and \underline{s} . We organize these terms according to the number of $N^{-\frac{1}{2}}$ factors:

$$\mathbb{E} F(\frac{1}{N} \text{Tr } S(z)) = \sum_{m=0}^5 N^{-\frac{m}{2}} \mathbb{E} A_V^{(m)},$$

where $A_V^{(0)} = F(\underline{R})$, $A_V^{(1)} = F'(\underline{R})(-\underline{RVR})$, $A_V^{(2)} = \frac{1}{2} F''(\underline{R})(\underline{RVR})^2 + F'(\underline{R})(\underline{RV})^2 R$,

~~A_V⁽³⁾~~ and ~~A_V⁽⁴⁾~~ are third and fourth order terms that can be defined in a similar way.

Finally, $A_V^{(5)}$ contains terms of order at least 5:

$$A_V^{(5)} = F'(\underline{R}) D_V + \frac{1}{5!} F^{(5)}(\underline{R} + \underline{s}') (\underline{RV})^5 + \cdots.$$

Similarly, we can get that for $T(z) = (H_{\gamma-1} - z)^{-1}$,

$$\mathbb{E} F(\frac{1}{N} \text{Tr } T(z)) = \sum_{m=0}^5 N^{-\frac{m}{2}} \mathbb{E} A_W^{(m)}, \quad \text{where } A_W^{(m)} \text{ are obtained by}$$

replacing all V in $A_V^{(m)}$ by $W = w_{ij} E^{(ij)}$.

Since \underline{R} is independent of V and W , we notice that $\mathbb{E} A_V^{(m)}$, $m=0,1,2,3,4$, are determined by the first m moments of v_{ij} , and similarly for $\mathbb{E} A_W^{(m)}$. Using the moment matching condition, we get that

$$\mathbb{E} A_V^{(m)} = \mathbb{E} A_W^{(m)}, \quad \text{for } m=0,1,2,3.$$

For $m=4$, using the estimate $|R_{kl}| < N^{6+\epsilon}$ and the moment matching condition, we get

$$|\mathbb{E} A_V^{(4)} - \mathbb{E} A_W^{(4)}| < N^{C(6+\epsilon)-\delta}, \quad \text{for an absolute constant } C>0.$$

(We can choose 6 and ϵ small enough such that $C(6+\epsilon) < \delta$.)

Finally, for $A_V^{(5)}$, all terms without D_V can be handled in a similar way.

The terms containing D_V can be bounded using the high moment condition on v_{ij} , the estimates on S and R , and the condition on the derivative of F : for example, (74)

$$|IE[F'(R)S]| \leq \frac{1}{N} |IE[F'(R) Tr(RV)^5 S]| \leq N^{C(6+\epsilon)} \max_i |IE[(RV)^5 S]_{ii}| \leq N^{C(6+\epsilon)}.$$

In sum, we get that

$$\begin{aligned} |IEF\left(\frac{1}{N} Tr \frac{1}{H_{R-3}}\right) - IEF\left(\frac{1}{N} Tr \frac{1}{H_{R-1-3}}\right)| &\leq N^{C(6+\epsilon)-\delta-2} + N^{-\frac{5}{2}} (|IE A_D^{15}| + |IE A_W^{15}|) \\ &\leq N^{-\delta+C(6+\epsilon)} + N^{-\frac{5}{2}+C(6+\epsilon)}. \end{aligned}$$

Taking $\delta=\epsilon$ and summing over R , we get

$$|IEF\left(\frac{1}{N} Tr \frac{1}{H_{R-3}}\right) - IEF\left(\frac{1}{N} Tr \frac{1}{H_{W-3}}\right)| \leq N^{-\delta+C(6+\epsilon)} + N^{-\frac{1}{2}+C(6+\epsilon)}.$$

The general case can be proved in a similar way. \square

III. Conclusion of the three step strategy.

To show the bulk universality of a Wigner matrix H , we want to find a Gaussian divisible ensemble with a sufficiently large Gaussian component that approximates H in the sense of four-moment matching.

Given a real random variable ζ , denote its moments by $m_k = IE\zeta^k$. Suppose $m_1=0$ and $m_2=1$.

$$\text{Then: } m_3^2 = (IE\zeta^3)^2 = [IE\zeta(\zeta^2-1)]^2 \leq (IE\zeta^2)[IE(\zeta^2-1)^2] = m_2(m_4 - 2m_2 + 1) = m_4 - 2m_2 + 1 = m_4 - 1.$$

In particular, the " $=$ " holds only if the support of ζ consists of only two points.

Hence, given a random variable (e.g., Bernoulli), we may not find a Gaussian divisible ensemble random variable with matching first 4 moments. But, this can be done approximately if we allow for some room in the fourth moment.

Lemma. Let m_3 and m_4 be two real numbers s.t.

$$m_4 - m_3^2 - 1 \geq 0 \quad \text{and} \quad m_4 \leq C_1.$$

Let $g \sim N(0,1)$ be a standard Gaussian. For any small $\gamma > 0$, there exists a real random variable ζ_γ with finite moments up to arbitrary high order and independent of g , such that $\zeta = (1-\gamma)^{1/2} \zeta_\gamma + \gamma^{1/2} g$ has first four moments $IE\zeta=0$, $IE\zeta^2=1$, $IE\zeta^3=m_3$ and $IE\zeta^4=m_4'$ s.t. $|m_4' - m_4| \leq C\gamma$ for some constant $C > 0$.

Pf: First, it is easy to construct ~~a~~ a random variable X with first four moments given by $0, 1, m_3$ and m_4 . We take the law of X as

$$P(X=a)=p, \quad P(X=-b)=q, \quad P(X=0)=1-p-q,$$

where $p = \frac{1}{a(a+b)}$, $q = \frac{1}{b(a+b)}$. (Note the condition $p+q \leq 1$ implies $ab \geq 1$.)

Then, we can calculate that

$$m_3 = \mathbb{E} X^3 = a-b, \quad m_4 = \mathbb{E} X^4 = a^2 + ab + b^2 = m_3^2 + ab.$$

It is easy to check that $m_4 - m_3^2 - 1 \geq 0$ implies that the above system of equations has a solution.
 with first four moments 0, 1, $m_3(\xi_\gamma)$ and $m_4(\xi_\gamma)$,

For any random variable $\xi_\gamma \perp\!\!\!\perp g$, the first four moments of ξ is given by 0, 1, and

$$m_3(\xi) = (1-\gamma)^{3/2} m_3(\xi_\gamma), \quad m_4(\xi) = (1-\gamma)^2 m_4(\xi_\gamma) + 6\gamma - 3\gamma^2.$$

By the previous result, we can choose ξ_γ s.t. $m_3(\xi_\gamma) = (1-\gamma)^{-\frac{3}{2}} m_3$ and $m_4(\xi_\gamma) = m_3(\xi_\gamma)^2 + (m_4 - m_3^2)$.

Then, we can check that $m_4(\xi) = m_4$ and

$$m_4(\xi) = (1-\gamma)^2 [(1-\gamma)^{-3} m_3^2 + m_4 - m_3^2] + 6\gamma - 3\gamma^2 = m_4 + O(\gamma). \quad \square$$

Given H , by the above lemma, we can construct a Gaussian divisible ensemble $H_t = e^{-tH} H_0 + (1-e^{-t})^{1/2} \text{GOE}$
 such that $\mathbb{E}(H_t)_{ij} = 0$, $\mathbb{E}(H_t)_{ij}^2 = 1$, $\mathbb{E}(H_t)_{ij}^3 = \mathbb{E} H_{ij}^3$, and $|\mathbb{E}(H_t)_{ij}^4 - \mathbb{E}(H_{ij})^4| \leq C t N^{-2}$.

In fact, given the first four moments 0, 1, m_3 and m_4 of the entries $\sqrt{N} h_{ij}$, we
 choose ξ_γ as the law of $\sqrt{N} (H_t)_{ij}$ as ξ_γ for $\gamma = 1 - e^{-t}$ as constructed in the above lemma.

Then, from the bulk universality of H_t , we get that for $t \geq N^{-\epsilon}$ and $b = N^{-1+10\epsilon}$,

$$\frac{1}{2b} \int_{E-b}^{E+b} dE' \int_{\mathbb{R}^n} d\vec{\alpha} O(\vec{\alpha}) (P_{t,N}^{(n)} - P_{G,N}^{(n)}) (E' + \frac{\vec{\alpha}}{N}) \rightarrow 0.$$

Finally, using the Green function comparison result, we get the bulk universality for H .

Section 8 Proof of Local Semicircle Law

In this section, we complete the proof of the local semicircle law. We consider real symmetric Wigner matrices only.

Theorem 8.1 For any small constant $\varepsilon > 0$, uniformly in $\lambda \in S(\varepsilon) := \{\lambda \in \mathbb{C}^+ : |\lambda| \leq \varepsilon^{-1}, \eta \geq N^{-1+\varepsilon}\}$, the following local laws hold:

$$(1) \text{ Entrywise local law: } \max_{i,j} |G_{ij}(\lambda) - m_{sc}(\lambda) \delta_{ij}| \leq \bar{\Psi}(\lambda), \quad \bar{\Psi}(\lambda) = \sqrt{\frac{\operatorname{Im} m_{sc}}{N\eta}} + \frac{1}{N\eta};$$

$$(2) \text{ Averaged local law: } |\operatorname{Im} m(\lambda) - m_{sc}(\lambda)| \leq \frac{1}{N\eta};$$

$$(3) \text{ Averaged local law outside } [-2, 2]: |\operatorname{Im} m(\lambda) - m_{sc}(\lambda)| \leq \frac{1}{N(K_E + \eta)} + \frac{1}{(N\eta)^2 \sqrt{K_E + \eta}} \quad \text{for } \lambda \in S(\varepsilon) \text{ with } 2 \leq |\lambda| \leq \varepsilon^{-1}, \quad N^{-1+\varepsilon} \leq \eta \leq 10, \quad N\eta \sqrt{K_E + \eta} \geq N^\varepsilon. \quad (\text{Recall } K_E := |\lambda| - 2.)$$

For simplicity of presentation, we only prove (1) and (2) for λ "in the bulk", i.e., $\lambda \in [-2+\kappa, 2-\kappa]$ for a small constant $\kappa > 0$: $\lambda \in S_{in}(\varepsilon) := \{\lambda \in \mathbb{C}^+ : |\lambda| \leq 2-\kappa, \quad N^{-1+\varepsilon} \leq \eta \leq 1\}$.

Recall the notations of resolvent minors $G^{(i)}, G^{(ii)}$; partial expectation $E_i := \operatorname{E}[L \cdot H^{(i)}]$, we abbreviate $P_i := E_i$ and $Q_i := 1 - P_i$; and the resolvent identities:

$$G_{ii}^{-1} = h_{ii} - \lambda - \sum_{k \neq i}^{(i)} h_{ik} G_{kk} h_{ki}; \quad \text{for } i \neq j, \quad G_{ij} = -G_{ii} \sum_k^{(i)} h_{ik} G_{kj}^{(i)} = -G_{jj} \sum_k^{(j)} G_{ik} h_{kj}; \quad G_{ij} = G_{ij}^{(k)} + \frac{G_{ik} G_{kj}}{G_{kk}};$$

$$(\text{Ward's identity}) \quad \sum_j |G_{ij}|^2 = \sum_j |G_{ji}|^2 = \frac{\operatorname{Im} G_{ii}}{\eta}; \quad |\operatorname{Im} G_{ii} - m| \leq \frac{1}{N\eta}.$$

Furthermore, for $\lambda \in S_{in}(\varepsilon)$, $\operatorname{Im} m_{sc}(\lambda) \asymp 1$ and $|1 - m_{sc}(\lambda)| \asymp 1$.

Again, we start with the formula $\frac{1}{G_{ii}} = h_{ii} - \lambda - \sum_{k \neq i}^{(i)} h_{ik} G_{kk} h_{ki} = h_{ii} - \lambda - \frac{1}{N} \sum_k^{(i)} G_{kk} - z_i$,

$$\begin{aligned} z_i &= Q_i \left(\sum_{k \neq i}^{(i)} G_{kk} h_{ik} h_{ki} \right) \\ &= \sum_{k \neq i}^{(i)} h_{ik} G_{kk} h_{ki} + \sum_k^{(i)} \left(h_{ik}^2 - \frac{1}{N} G_{kk} \right) G_{kk}^{(i)} \end{aligned} = -\lambda - m(\lambda) + \underbrace{\left(h_{ii} + \frac{1}{N} \sum_k^{(i)} \frac{G_{ki} G_{ik}}{G_{kk}} - z_i \right)}_{Y_i}$$

We will show that Y_i is a small error. We abbreviate $v_i = G_{ii} - m_{sc}$, $[v] = m - m_{sc} = \frac{1}{N} \sum_i v_i$.

Then, the above equation implies

$$\begin{cases} v_i = \frac{1}{-\lambda - m_{sc} - [v] + Y_i} - m_{sc} \quad (*) \\ [v] = \frac{1}{N} \sum_i \frac{1}{-\lambda - m_{sc} - [v] + Y_i} - m_{sc} \quad (+) \end{cases}$$

Lemma We denote $\Lambda_0 := \max_{i \neq j} |G_{ij}|$, $\Lambda_d := \max_i |G_{ii} - m_{sc}|$, $\Lambda := \Lambda_0 \vee \Lambda_d$. Define the event

$$E(\lambda) := \{|\lambda(\lambda)| \leq (\log N)^{-1}\}. \quad \text{Then, we have the following large deviation estimates for } Y_i.$$

Lemma: For any spectral domain D of \mathbb{E} , if $\mathbb{E}(\mathbb{E}) \cap D$ holds with high probability, then

Lemma 1: For any $\delta \in S(\mathbb{E})$ we have

$$\mathbb{E}(\mathbb{E}(\delta)) (\Lambda_0 + |\mathbb{E}| + |\mathbb{Y}_i|) \leq \sqrt{\frac{2m_{sc} + \Lambda}{N\eta}} \text{ uniformly in } \delta \in S(\mathbb{E}).$$

Moreover, for any constant $c > 0$, we have

$$\Lambda_0 + |\mathbb{E}| + |\mathbb{Y}_i| \leq N^{-\frac{1}{2}} \text{ uniformly in } \delta \in S(\mathbb{E}) \cap \{ \delta : \operatorname{Im} \delta \geq c \}.$$

Proof: Note that on $\mathbb{E}(\mathbb{E})$, we have

$$1 \leq |\operatorname{Im}_{sc}(\delta)| - \Lambda \leq |G_{ii}(\delta)| \leq |\operatorname{Im}_{sc}(\delta)| + \Lambda \leq 1, \quad \mathbb{E}_{i \neq j} |G_{ij}(\delta)| \leq \Lambda \leq (\log N)^{-1},$$

and $|G_{ii}^{(k)}| \asymp 1$, $|G_{ij}^{(k)}| \asymp 1$, $\forall k, l \neq i, j$, $\mathbb{E}_{i \neq j} |G_{ij}^{(k)}| \lesssim (\log N)^{-1}$, $\mathbb{E}_{i \neq j} |G_{ij}^{(k)}| \lesssim (\log N)^{-1}$.

(Here, we can apply $G_{ij}^{(k)} = G_{ij} - \frac{G_{ik}G_{kj}}{G_{kk}}$ repeatedly to obtain these estimates.) $\forall k, l \neq i, j$.
(large deviation estimates we proved before)

① We first estimate Λ_0 : on $\mathbb{E}(\mathbb{E})$,

$$|G_{ij}| = |G_{ii} \sum_k^{(i)} h_{ik} G_{kj}^{(i)}| \leq \left| \sum_k^{(i)} h_{ik} G_{kj}^{(i)} \right| \leq \left(\frac{1}{N} \sum_k^{(i)} |G_{kj}^{(i)}|^2 \right)^{1/2} = \left(\frac{\operatorname{Im} G_{ii}^{(i)}}{N\eta} \right)^{1/2}.$$

We can estimate that $\mathbb{E}(\mathbb{E}(\delta)) G_{ij}^{(i)} = \mathbb{E}(\mathbb{E}(\delta)) (G_{ii} - \frac{G_{ij}G_{ji}}{G_{ii}})$ Ward's identity

$$= \mathbb{E}(\mathbb{E}(\delta)) (G_{jj} - m_{sc} + m_{sc} + O(\Lambda^2)) = \mathbb{E}(\mathbb{E}(\delta)) (m_{sc} + O(\Lambda)).$$

Hence, we get: $\mathbb{E}(\mathbb{E}(\delta)) |G_{ij}| \leq \sqrt{\frac{2m_{sc} + \Lambda}{N\eta}}$. Taking union bound over $i \neq j$, we get $\mathbb{E}(\mathbb{E}(\delta)) \Lambda_0 \leq \sqrt{\frac{2m_{sc} + \Lambda}{N\eta}}$.

On the other hand, without assuming $\mathbb{E}(\mathbb{E})$ holds, we let $\eta \geq c$. Then, we have trivial bounds $|G(\delta)| \leq \frac{1}{\eta} \leq 1$, and $|G^{(i)}| \leq \frac{1}{\eta} \leq 1$. Hence, we can repeat

the above arguments to get that
 ~~$\frac{|G_{ij}G_{ji}|}{G_{ii}} = |G_{ij}| + \sum_k h_{ik}h_{kj} G_{kk}^{(i)}$~~
 $|G_{ij}| \leq \sqrt{\frac{\operatorname{Im} G_{ii}^{(i)}}{N\eta}} \leq (\frac{1}{N})^{1/2}$.

② Next, we estimate $\mathbb{E}(\mathbb{E})$: on $\mathbb{E}(\mathbb{E})$, $|\mathbb{E}| \leq \left| \sum_{k \neq l} h_{ik}h_{lk} G_{kk}^{(i)} \right| + \left| \sum_k (h_{ik}^2 - \frac{1}{N}) G_{kk}^{(i)} \right|$

$$\leq \frac{1}{N} \left(\sum_{k \neq l} |G_{kk}^{(i)}|^2 \right)^{1/2} + \frac{1}{N} \left(\sum_k |G_{kk}^{(i)}|^2 \right)^{1/2} \approx \frac{1}{N} \left(\sum_{k \neq l} |G_{kk}^{(i)}|^2 \right)^{1/2} = \frac{1}{N} \left(\sum_k \frac{\operatorname{Im} G_{kk}^{(i)}}{\eta} \right)^{1/2} = \frac{1}{N} \left(\frac{N \operatorname{Im} m^{(i)}}{\eta} \right)^{1/2}$$

$$= \left(\frac{\operatorname{Im} m^{(i)}}{N\eta} \right)^{1/2} \leq \sqrt{\frac{2m_{sc} + \Lambda}{N\eta}}.$$

Again, when $\eta \geq c$, the above estimate gives $|\mathbb{E}| \leq N^{-\frac{1}{2}}$.

③ For \mathbb{Y}_i , we notice that $|\mathbb{Y}_i| \leq N^{-\frac{1}{2}}$ and $\left| \frac{1}{N} \sum_k \frac{G_{ki}G_{ik}}{G_{ii}} \right| \leq \frac{1}{N} \sum_k |G_{ki}|^2 = \frac{1}{N\eta} \frac{\operatorname{Im} G_{ii}}{G_{ii}} \leq \frac{1}{N\eta} \leq \sqrt{\frac{\operatorname{Im} m^{(i)}}{N\eta}}$.

Together with ②, we conclude the proof. \square

Next, we prove an initial bound $\Lambda \leq N^{-\frac{1}{2}}$ for δ with large imaginary part $\eta \geq c$.

Lemma 2: We have $\Lambda \leq N^{-\frac{1}{2}}$ uniformly in $\delta \in S(\mathbb{E})$ on $\{ \delta : \operatorname{Im} \delta \geq c \}$.

Proof: With the previous lemma, we have obtain that $\Lambda_0 \leq N^{-\frac{1}{2}}$. Furthermore, we have

$$\frac{1}{G_{ii}} = -z - m(z) + Y_i = -z - m(z) + O_\epsilon(N^{-\frac{1}{2}}) \Rightarrow G_{ii} = \frac{1}{-z - m(z) + O_\epsilon(N^{-\frac{1}{2}})} \quad (*)$$

For $\eta \geq c$, we have $| -z - m(z) | \geq \operatorname{Im}(z + m(z)) \geq \operatorname{Im} z \geq c$. Thus, performing Taylor expansion,

$$G_{ii} = \frac{1}{-z - m(z)} + O_\epsilon(N^{-\frac{1}{2}}) \Rightarrow m(z) = \frac{1}{N} \sum_i G_{ii} = \frac{1}{-z - m(z)} + O_\epsilon(N^{-\frac{1}{2}})$$

$$\Rightarrow m(z)^2 + z m(z) + 1 = O_\epsilon(N^{-\frac{1}{2}}) \Rightarrow |m - m_{sc}| \leq N^{-\frac{1}{2}}.$$

(stability of the self-consistent equation when $\eta \geq c$)

Plugging this estimate into (*), we get $(\text{union bound over } i)$

$$G_{ii}(z) = \frac{1}{-z - m_{sc} + O_\epsilon(N^{-\frac{1}{2}})} = m_{sc} + O_{\epsilon, N^{-\frac{1}{2}}} \Rightarrow \Lambda_d \leq N^{-\frac{1}{2}}.$$

Hence, we have $\Lambda = \Lambda_0 \vee \Lambda_d \leq N^{-\frac{1}{2}}$. \square

Remark (Uniform estimates over z) In the above proof, we have obtained estimates for each "fixed z ". We can extend them uniformly to all z using a standard "perturbation". More precisely, suppose we have an estimate $|G_{ij}(z)| \leq \phi(z)$ at each $z \in D$ for some spectral domain $D \subseteq S(\epsilon)$ and some deterministic quantity $\phi(z) > 0$. Then, we take an N^{-4} -net Σ of D with cardinality $|\Sigma| = O(N^8)$. With a simple union bound, we get that $\forall \epsilon > 0$, $\bigcap_{z \in \Sigma} \{ |G_{ij}(z)| \leq N^\epsilon \phi(z) \}$ hold with high probability.

On this event, for any $z \in D$,

we can find $z' \in \Sigma$ s.t. $|z - z'| \leq N^{-4}$. Then, using the trivial bound $\|G(z)\| \leq \frac{1}{\operatorname{Im} z} \leq N^2$, we get that $|G_{ij}(z)| \leq |G_{ij}(z')| + O(|z - z'| N^{-4}) \leq N^\epsilon \phi(z') + O(N^{-2})$. In ~~all~~ all our proofs, we have $\phi(z) \asymp \phi(z')$ for $|z - z'| \leq N^{-4}$ and $O(N^{-2})$ is a sufficiently small error. Hence, $\bigcap_{z \in \Sigma} \{ |G_{ij}(z)| \leq N^{2\epsilon} \phi(z) \}$ contains $\bigcap_{z \in \Sigma} \{ |G_{ij}(z)| \leq N^\epsilon \phi(z) \}$ and is a high probability event.

Lemma 3 (A rough bound on Λ) We have $\Lambda \leq N^{-\frac{1}{2}}$ uniformly in $z \in S(\epsilon)$.

Proof: The core of the proof is a continuity argument. We first establish the following dichotomy for Λ . Roughly speaking, it asserts that with high probability, we have either $\Lambda(z) \leq N^{-\epsilon/2}$ or $\Lambda(z) \geq N^{-\epsilon/4}$.

Lemma 4 (Dichotomy estimate) $\mathbb{1}(\Lambda \leq N^{-\epsilon/4}) \Lambda \leq N^{-\epsilon/2}$ uniformly in $z \in S(\epsilon)$.

Proof of the dichotomy: By previous large deviation estimates, we have that

$$\mathbb{1}(\Lambda \leq N^{-\epsilon/4}) (\Lambda_0 + |Y_i|) \leq \sqrt{\frac{\operatorname{Im} m_{sc} + \Lambda}{N}}.$$

Recall the equation (*): $Y_i = \frac{1}{-z - m_{sc} - [v] + Y_i} - m_{sc} + \sqrt{\frac{1}{(-z - m_{sc} - [v])^2} - \Lambda}$

$$= m_{sc} \frac{1}{1 - m_{sc}([v] - Y_i)} - m_{sc} = m_{sc} \left(\frac{m_{sc}^2([v] - Y_i)}{m_{sc}^2([v] - Y_i)^2} + O([v]^2 + Y_i^2) \right).$$

\Rightarrow Average over i : $[v] = m_{sc}^2([v] - \frac{1}{N} \sum_i Y_i) + O([v]^2 + Y_i^2)$

Using $|1 - m_{sc}^2| \asymp 1$, we get from the above equation that $[v] = O\left(\frac{1}{N} \sum_i Y_i + \frac{[v]^2}{N} + Y_i^2\right)$. Since $|[v]| \leq \lambda \leq N^{-\varepsilon/4}$, the above equation gives: $[v] = O\left(\frac{1}{N} \sum_i Y_i + Y_i^2\right) \quad (\neq)$

$$\leftarrow \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}} \text{ on } \{\lambda \leq N^{-\varepsilon/4}\}.$$

Plugging it into $Y_i = \frac{1}{-\beta - m_{sc} - [v] + Y_i} - m_{sc}$ gives $\mathbb{E}[Y_i | \lambda \leq N^{-\varepsilon/4}] |v| < \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}} \Rightarrow \lambda_d < \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}}$.

In ~~some~~ sum, $\mathbb{E}[\lambda | \lambda \leq N^{-\varepsilon/4}] \leq \mathbb{E}[\lambda | \lambda \leq N^{-\varepsilon/4}] \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}} \lesssim N^{-\varepsilon/2}$, since $\text{Im } m_{sc} = o(1)$ and $\eta \geq N^{-1+\varepsilon}$. \square

This result shows that $\lambda(z)$ is excluded from the interval $[N^{-\varepsilon/2}, N^{\varepsilon/4}]$ w.h.p. Furthermore, we know that $\lambda(z)$ is in $[0, N^{-\varepsilon/2}]$ when $\eta \geq c$. Then, it suffices to show that $\lambda(z)$ is continuous in $\text{Im } z$ when we decrease η from c to $N^{-1+\varepsilon}$, which shows that $\lambda(z)$ must be below $N^{-\varepsilon/2}$ for all $z \in \text{Sin}(\varepsilon)$.

From the dichotomy estimate, we get that $\forall z \in \text{Sin}(\varepsilon)$, $\mathbb{P}(N^{-\varepsilon/3} \leq \lambda(z) \leq N^{-\varepsilon/4}) \leq N^{-D}$ for large $D > 10$.

Again, we take an N^{-4} -net of $\text{Sin}(\varepsilon)$ with $|\Sigma| = O(N^8)$. With a union bound, we get

$$\mathbb{P}(\exists w \in \Sigma : N^{-\varepsilon/3} \leq \lambda(w) \leq N^{-\varepsilon/4}) \leq N^{-D+9}.$$

With the Lipschitz condition $\|G''(z)\| \leq N^2$, we get $\mathbb{P}(\exists z \in \text{Sin}(\varepsilon) : 2N^{-\varepsilon/3} \leq \lambda(z) \leq \frac{1}{2}N^{-\varepsilon/4}) \leq N^{-D+9}$.

Hence, there is an event E with $\mathbb{P}(E) \geq 1 - N^{-D+9}$ s.t. $\forall z \in \text{Sin}(\varepsilon)$, either $\mathbb{E}[E] \lambda(z) \leq 2N^{-\varepsilon/3}$ or $\mathbb{E}[E] \lambda(z) \geq \frac{1}{2}N^{-\varepsilon/4}$. Since $\lambda(z)$ is continuous in z and $\text{Sin}(\varepsilon)$ is connected, we get that

either $\bigcap_{z \in \text{Sin}(\varepsilon)} \{\lambda(z) \leq 2N^{-\varepsilon/3}\}$ or $\bigcap_{z \in \text{Sin}(\varepsilon)} \{\lambda(z) \geq \frac{1}{2}N^{-\varepsilon/4}\}$ holds w.h.p.

But we already know the latter event does not hold w.h.p. Hence, the former event must hold w.h.p. \square

Given lemma 3, the proof of lemma 4 gives that $[v] \leq |\frac{1}{N} \sum_i Y_i| + Y_i^2$. We have known that

$|Y_i|^2 \leq \frac{\text{Im } m_{sc} + \lambda}{N\eta}$. For $Y_i = h_{ii} + \frac{1}{N} \sum_k \frac{G_{ki} G_{ik}}{G_{kk}} - z_i$, we have $|\frac{1}{N} \sum_i h_{ii}| \leq \frac{1}{N} \left(\frac{1}{N} \sum_i 1\right)^{1/2} = \frac{1}{N}$,

$\left| \frac{1}{N^2} \sum_{i,k} \frac{G_{ki} G_{ik}}{G_{kk}} \right| \leq \frac{1}{N^2} \sum_i \frac{\text{Im } G_{ii}}{|G_{ii}|} \leq \frac{1}{N\eta} \leq \frac{\text{Im } m_{sc}}{N\eta}$. We also know $|z_i| \leq \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}}$. But, we will show that the average $\frac{1}{N} \sum_i z_i$ is one order better.

Theorem 8.2 (Fluctuation averaging) If $\lambda_d < N^{-c}$ for some constant $c > 0$ and $\lambda_0 < \mathbb{E}_0$ for some deterministic parameter $N^{-\frac{1}{2}} \leq \mathbb{E}_0 \leq N^{-c}$, then we have

$$|\frac{1}{N} \sum_i z_i| \leq \mathbb{E}_0^2.$$

Now, we use this result to conclude the proof. ~~we prove the following iterative estimate~~

With Lemma 3, we get that $\lambda < N^{-\varepsilon/3}$ uniformly in $z \in \text{Sin}(\varepsilon)$. Hence, $\mathbb{E}(z) = \{\lambda(z) \leq (\log N)^{-1}\}$ holds w.h.p., and Lemma 1 implies that

$$\lambda_0 + |Y_i| \leq \sqrt{\frac{\text{Im } m_{sc} + \lambda}{N\eta}} \leq \sqrt{\frac{\text{Im } m_{sc}}{N\eta}}.$$

Then, using Theorem 8.2 with $\mathbb{E}_0 = \sqrt{\frac{\text{Im } m_{sc}}{N\eta}}$, we get $|\frac{1}{N} \sum_i z_i| \leq \frac{\text{Im } m_{sc}}{N\eta} \leq \frac{1}{N\eta}$, (80)

which gives $|\frac{1}{N} \sum_i Y_i| \leq \frac{1}{N\eta} \Rightarrow |V| \leq |\frac{1}{N} \sum_i Y_i| + Y_i^2 \leq \frac{1}{N\eta}$. This gives the averaged local law. Plugging this estimate into $v_i = \frac{1}{-\beta - m_{sc} - [V] + Y_i}$, we get

$$|v_i| \leq |[V]| + |Y_i| \leq \sqrt{\frac{Im_{sc}}{N\eta}} + \frac{1}{N\eta} = \mathbb{E}(z) \Rightarrow \Lambda_d \leq \mathbb{E}(z).$$

Hence, $\Lambda = \Lambda_0 v \Lambda_d \leq \mathbb{E}(z)$. \square

[Proof of a second moment bound for the fluctuation averaging estimate:] Instead of proving Thm 8.2, we only prove a weaker 2nd moment bound:

$$\mathbb{E} \left(\frac{1}{N} \sum_i Z_i \right)^2 \leq \mathbb{E}_0^+$$

Recall that $Z_i = Q_i \left(\sum_{k,l} h_{ik} G_{kl}^{(i)} h_{il} \right) = -Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) + h_{ii}$. We have proved the estimate for $\frac{1}{N} \sum_i h_{ii}$.

Hence, it suffices to show that $\mathbb{E} \left(\frac{1}{N} \sum_i Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \right)^2 \leq \mathbb{E}_0^+$.

We will use the resolvent identities

$$G_{ij}^{(k)} = G_{ij}^{(k)} + \frac{G_{ik} G_{kj}^{(i)}}{G_{kk}^{(i)}} \quad , \quad \frac{1}{G_{ii}^{(k)}} = \frac{1}{G_{ii}^{(k)}} - \frac{G_{ik} G_{ii}^{(i)}}{G_{ii}^{(k)} G_{kk}^{(i)}}.$$

$$\begin{aligned} \text{Then, } \mathbb{E} \left(\frac{1}{N} \sum_i Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \right)^2 &= \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \overline{Q_j \left(\frac{1}{G_{jj}^{(j)}} \right)} = \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \overline{Q_j \left(\frac{1}{G_{jj}^{(j)}} \right)} \\ &\quad + \frac{1}{N^2} \sum_i \mathbb{E} \left| Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \right|^2. \end{aligned}$$

Applying the large deviation estimates for Z_i , we get

$$|Z_i| \leq \left(\frac{1}{N^2} \sum_{k,l} |G_{kl}^{(i)}|^2 \right)^{1/2} \leq \left(\Lambda_0^2 + \frac{1}{N^2} \sum_k |G_{kk}^{(i)}|^2 \right)^{1/2} \leq (\Lambda_0^2 + N^{-1})^{1/2} \leq \mathbb{E}_0^+.$$

Hence, we have

$$\begin{aligned} \mathbb{E} \left| \frac{1}{N} \sum_i Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \right|^2 &= \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) Q_j \left(\frac{1}{G_{jj}^{(j)}} \right) + O_N(N^{-1} \mathbb{E}_0^2) \\ &= \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) - \frac{G_{ij} G_{ii}^{(i)}}{G_{ii}^{(i)} G_{jj}^{(j)} G_{jj}^{(i)}} \right] \overline{Q_j \left(\frac{1}{G_{jj}^{(j)}} \right)} - \frac{G_{ij} G_{ij}^{(i)}}{G_{jj}^{(i)} G_{jj}^{(i)} G_{ii}^{(i)}} \Big] \\ &\quad + O_N(\mathbb{E}_0^+). \end{aligned}$$

Since $G_{ii}^{(i)}$ is independent of the j -th row and column of H , we have

$$\mathbb{E}_j G_{ij}^{(i)} = \mathbb{E}(G_{ij}^{(i)} | H^{(j)}) = G_{ij}^{(i)}.$$

Moreover

Thus, for any random variable X ,

$$\mathbb{E}_j \left(\frac{1}{G_{ii}^{(i)}} X \right) = \frac{1}{G_{ii}^{(i)}} \mathbb{E}_j(X) \Rightarrow Q_j \left(\frac{1}{G_{ii}^{(i)}} X \right) = \frac{1}{G_{ii}^{(i)}} Q_j(X).$$

Furthermore, Q_j and Q_i commute. Thus, we have

$$\mathbb{E} Q_i \left(\frac{1}{G_{ii}^{(i)}} \right) \overline{Q_j \left(\frac{1}{G_{jj}^{(j)}} \right)} = \mathbb{E} Q_j \left(\frac{1}{G_{ii}^{(i)}} \right) \overline{Q_i \left(\frac{1}{G_{ii}^{(i)}} \right)} = 0.$$

Similarly, we have $\mathbb{E} Q_i \left(\frac{G_{ij} G_{ii}^{(i)}}{G_{ii}^{(i)} G_{jj}^{(i)} G_{jj}^{(i)}} \right) \overline{Q_j \left(\frac{1}{G_{jj}^{(j)}} \right)} = 0$.

Finally, we are left with the estimate

$$\begin{aligned} \mathbb{E} \left| \frac{1}{N} \sum_i Q_i \left(\frac{1}{G_{ii}} \right) \right|^2 &= \frac{1}{N} \sum_{i \neq j} \mathbb{E} \left[Q_i \left(\frac{G_{ij} G_{ji}}{G_{ii} G_{jj}^{(i)} G_{jj}} \right) \overline{Q_j \left(\frac{G_{ji} G_{ij}}{G_{jj} G_{ii}^{(i)} G_{ii}} \right)} \right] + o_N(\bar{\Phi}_0^4) \\ &= o_N(\bar{\Phi}_0^4). \end{aligned}$$