$$= \frac{1}{2}\|\vec{v}\|^2 + \frac{1}{2N}\sum_{i\neq j}\frac{(v(i)-v(j))^2}{(\lambda_i-\lambda_j)^2} \geq \frac{1}{2}\|\vec{v}\|^2 .$$

Thus, $\nabla^2(\beta N \mathcal{H}) \geq \frac{1}{2}\beta N$. Here, we use a different normalization for the Dirichlet form: $D_\mu(f) = \frac{1}{\beta N}\int \|\nabla f\|^2 d\mu$. Then, the Bakry – Emery estimate gives

$$S_\mu(f) \leq 4\, D_\mu(\sqrt{f}) .$$

This shows that the relaxation time is of order 1. (This is the global dynamics.) We will show that the local dynamics converges to equilibrium within time of order $N^{-1+\varepsilon}$.

<u>Thm 6.6</u>. Assume the following conditions:

(i) There exists a constant $\xi > 0$ such that

$$Q := \sup_{0\leq t\leq N} \frac{1}{N}\int \sum_{j=1}^{N}(\lambda_j-\gamma_j)^2 f_t(\vec{\lambda})\,\mu_G(d\vec{\lambda}) \leq CN^{-2+2\xi}.$$

(ii) After time $1/N$, the solution to the equation $\partial_t f_t = \mathcal{L} f_t$, $t\geq 0$, satisfies $S_\mu(f_{1/N}) \leq CN^m$ for some fixed m.

Fix any $\beta \geq 1$ and $n \in \mathbb{N}$. Let $F: \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support. Denote:

$$\mathcal{F}_{i,n}(\vec{x}) = F(N(x_{i+1}-x_i), N(x_{i+2}-x_i), \cdots, N(x_{i+n}-x_i)).$$

Then, for any $\xi \in (0,\frac{1}{2})$ and any sufficiently small constant $\varepsilon > 0$, there exist constants $C, c > 0$ depending only on ε and F, such that for any $J \subset \{1, 2, \cdots, N-n\}$ we have

$$\sup_{t\geq N^{-1+2\xi+\varepsilon}} \left| \int \frac{1}{|J|}\sum_{i\in J} \mathcal{F}_{i,n}(\vec{x})(f_t\,d\mu - d\mu)\right| \leq CN^{\varepsilon}\sqrt{\frac{N^2 Q}{|J|t}} + Ce^{-cN^{\varepsilon}} .$$

In particular, if $t \geq N^{-1+2\xi+2\varepsilon+\delta}$, we have

$$\left|\int \frac{1}{|J|}\sum_{i\in J}\mathcal{F}_{i,n}(\vec{x})(f_t\,d\mu - d\mu)\right| \leq \frac{C}{\sqrt{JN^{\delta-1}}} + Ce^{-cN^{\varepsilon}} .$$

Hence, the gap distribution, averaged over J indices, coincides for $f_t\,d\mu$ and $d\mu$ if $|J|N^{\delta-1} \to +\infty$.

Averaged gap distribution on scale J corresponds to averaged energy distribution on scale $J \sim Nb$.

<u>Lemma</u>. Under the setting of Thm 6.6, suppose $\left|\int \frac{1}{|J|}\sum_{i\in J}\mathcal{F}_{i,n}(\vec{x})(f\,d\mu - d\mu)\right| \leq \frac{C}{\sqrt{|J|N^{\delta-1}}}$ for some constant $\delta > 0$.

Suppose the eigenvalue rigidity holds: $\forall\, \kappa>0,\ \sup_{\kappa N \le j \le (1-\kappa)N} |\lambda_j - \gamma_j| \prec N^{-1+\xi}$.

Then, for any constant $\varepsilon>0$ and $N^{-1} \ll b_N \ll 1$, we have that

$$\left| \int_{E-b}^{E+b} \frac{dE'}{2b} \int_{\mathbb{R}^n} d\vec{\alpha}\ O(\vec{\alpha}) \left(p^{(n)}_{f\mu,N} - p^{(n)}_{\mu,N} \right) \left(E' + \frac{\vec{\alpha}}{N\rho_{sc}(E)} \right) \right|$$

$$\lesssim N^{2\varepsilon} \left[\frac{N^{-1+\xi}}{b} + \sqrt{\frac{N^{-\delta}}{b}} \right].$$

Hence, the averaged energy distribution over $b \gg N^{-1+\xi+2\varepsilon} \vee N^{4\varepsilon-\delta}$ coincides for $f\mu$ and μ.

The above two results establish the averaged gap/energy universality for $t \geq N^{-1+2\xi+2\varepsilon+\delta}$. Under the optimal rigidity, ξ can be chosen as small as possible. Furthermore, ε and δ are arbitrary. This establishes the Dyson's conjecture: the local dynamics relaxes to equilibrium for $t \geq N^{-1+c}$ $\forall$ constant $c>0$.

<u>Main ideas for the proof:</u> Recall that

$$\langle \vec{v}, \nabla^2 \mathcal{H} \vec{v} \rangle = \frac{1}{2}\sum_i v(i)^2 + \frac{1}{N}\sum_{i<j} \frac{(v(i)-v(j))^2}{(\lambda_i-\lambda_j)^2} \geq \frac{1}{2}\|\vec{v}\|^2.$$

The key intuition is that the relaxation time is in fact much shorter than order 1 for local variables that depend only on the eigenvalue differences. In particular, with the intuition that $\lambda_i - \lambda_j$ are close, the relaxation time in the direction $v(i)-v(j)$ should be much smaller than 1. However, this effect is hard to use directly in the real proof. An idea is to add an "auxiliary strongly convex potential" to $\mathcal{H}$ to "speed up" the relaxation to local equilibrium. On the other hand, we will show that the effect of this speeding up on the local statistics can be controlled provided with the estimate

$$(+)\quad \sup_{0\le t\le N} \frac{1}{N}\int \sum_{j=1}^{N} (\lambda_j-\gamma_j)^2 f_t(\vec{\lambda})\, \mu_G(d\vec{\lambda}) \lesssim N^{-2+2\xi}.$$

We introduce the following auxiliary potential $\hat{\mathcal{H}}(\vec{\lambda}) := \frac{1}{2\tau}\sum_{j=1}^{N}(\lambda_j-\gamma_j)^2$, $0<\tau<1$.
(We will choose $\tau = tN^{-\varepsilon}$ in the proof.)
Then, the new Hamiltonian is $\tilde{\mathcal{H}} = \mathcal{H} + \hat{\mathcal{H}}$, and the corresponding measure is

$$d\omega = \omega(\vec{x})\, d\vec{x}, \quad \omega(\vec{x}) := e^{-\beta N \tilde{\mathcal{H}}}/\tilde{Z},$$ called the "local relaxation measure".

The "local relaxation flow" with the generator characterized by the natural Dirichlet form w.r.t. ω: $\tilde{\mathcal{L}} = \mathcal{L} - \sum_j b_j \partial_j$, $b_j(x_j) = \frac{x_j - \gamma_j}{\tau}$.

For small τ, $\hat{\mathcal{H}}$ will substantially increase the lower bound on the Hessian, hence speeding up the dynamics so that the relaxation time is at most τ. In addition, we will compare the local statistics of the original system with those of the modified one.

It turns out that the difference is governed by $(\nabla \mathcal{H})^2$, which can be controlled with the estimate $(\dagger)$.

$\boxed{\text{Proof of Theorem 6.6}}$

Proposition I. Fix any $\beta \geq 1$. Consider the equation $\partial_t q_t = \tilde{\mathcal{L}} q_t$, $t \geq 0$, with the reversible measure $\omega = e^{-\beta N \tilde{\mathcal{H}}}/Z_N$. Suppose the initial condition q_0 satisfies $\int q_0 \, d\omega = 1$. Then, we have that

$$(1) \quad \partial_t D_\omega(\sqrt{q_t}) \leq -\frac{2}{\tau} D_\omega(\sqrt{q_t}) - \frac{1}{\beta N^2} \int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} \, d\omega,$$

$$(2) \quad \frac{1}{\beta N^2} \int_0^\infty ds \int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} \, d\omega \leq D_\omega(\sqrt{q_0}),$$

and the LSI $S_\omega(q_0) \leq C\tau D_\omega(\sqrt{q_0})$. Hence, the relaxation time to equilibrium is of order τ:

$$(3) \quad S_\omega(q_t) \leq e^{-Ct/\tau} S_\omega(q_0), \qquad D_\omega(\sqrt{q_t}) \leq e^{-Ct/\tau} D_\omega(\sqrt{q_0}).$$

Proof: Performing the same calculation as in the proof of the Bakry – Emery estimate, we obtain that for $h_t = \sqrt{q_t}$,

$$\partial_t D_\omega(h_t) = \partial_t \frac{1}{\beta N} \int (\nabla h_t)^2 e^{-\beta N \tilde{\mathcal{H}}} \, d\vec{x} \leq -\frac{2}{\beta N} \int (\nabla h_t) \cdot (\nabla^2 \tilde{\mathcal{H}}) \cdot (\nabla h_t) \, e^{-\beta N \tilde{\mathcal{H}}} \, d\vec{x}.$$

For $\tilde{\mathcal{H}}$, we can calculate its Hessian as

$$\nabla h_t \cdot \nabla^2 \tilde{\mathcal{H}} \cdot \nabla h \geq \frac{1}{\tau} \sum_j (\partial_j h)^2 + \frac{1}{2N} \sum_{i,j} \frac{1}{(x_i - x_j)^2} (\partial_i h - \partial_j h)^2.$$

Plugging it into the above inequality, we conclude (1). Integrating over t, we conclude (2). The LSI and (3) can be derived in the same way as in the proof of Bakry – Emery. $\square$

The extra term on the RHS of (1) plays a key role in the proof of the following prop.

Proposition II. Let q be a probability density with respect to the local relaxation measure ω, i.e., $\int q \, d\omega = 1$. Fix any $n \geq 1$ and a smooth function F with compact support. Recall that

$$\mathcal{F}_{i,n}(\vec{x}) := F(N(x_{i+1} - x_i), \cdots, N(x_{i+n} - x_i)).$$

Then, for any $J \subseteq \{1, 2, \cdots, N-n\}$ and $t > 0$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} \mathcal{F}_{i,n}(\vec{x}) (q \, d\omega - d\omega) \right| \leq C \left(t \frac{D_\omega(\sqrt{q})}{|J|} \right)^{1/2} + C \sqrt{S_\omega(q)} \, e^{-ct/\tau}.$$

Proof: For simplicity of notation, we take $n = 1$, so $\mathcal{F}_{i,n}(\vec{x}) = F(N(x_{i+1} - x_i))$. Let q_t be a solution to $\partial_t q_t = \tilde{\mathcal{L}} q_t$, $t \geq 0$, with initial condition $q_0 = q$. Then,

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - 1) \, d\omega = \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - q_t) \, d\omega + \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q_t - 1) \, d\omega.$$

The second term can be bounded by

Exp. decay of entropy given by the last prop

$$\int |q_t - 1| \, d\omega \underset{\text{Pinsker inequality}}{\le} C\sqrt{S_\omega(q_t)} \le C e^{-Ct/\tau} \sqrt{S_\omega(q_0)} .$$

To estimate the first term, using $\partial_t q_t = \tilde{\mathcal{L}} q_t$, we get that

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \, q_t \, d\omega - \int \frac{1}{|J|} \sum_{i\in J} G(N(x_{i+1} - x_i)) \, q_0 \, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) \, \cancel{q_t \, d\omega} \, \partial_s q_s \, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|} \sum_{i\in J} G(N(x_{i+1} - x_i)) \, \tilde{\mathcal{L}} q_s \, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|} \sum_{i \in J} G'(N(x_{i+1} - x_i)) (\partial_{i+1} q_s - \partial_i q_s) d\omega. \quad (*)$$

Using $\partial_i q = 2\sqrt{q}\, \partial_i \sqrt{q}$ and the Cauchy-Schwarz inequality, we obtain that

$$|(*)| \le 2\Big[\int_0^t ds \int \frac{N^2}{|J|^2} \sum_{i\in J} |G'(N(x_{i+1} - x_i))|^2 (x_{i+1} - x_i)^2 q_s \, d\omega \Big]^{1/2} \times$$

$$\underbrace{\Big[\int_0^t ds \int \frac{1}{N^2} \sum_i \frac{1}{(x_i - x_{i+1})^2} (\partial_{i+1}\sqrt{q_s} - \partial_i \sqrt{q_s})^2 d\omega \Big]^{1/2}}_{\le C D_\omega(\sqrt{q_0})}$$

Moreover, $|G'(N(x_{i+1} - x_i))|^2 |x_{i+1} - x_i|^2 \le C N^{-2}$ because G is smooth and compactly supported. So

$$|(*)| \le C\sqrt{D_\omega(q_0)} \cdot \left(\frac{t}{|J|}\right)^{1/2} . \qquad \square$$

Proposition III. Consider the local relaxation measure ω for a given $\tau > 0$. Set $\psi = \omega/\mu$ and $g_t = f_t/\psi$ with f_t solving $\partial_t f_t = \mathcal{L} f_t$. Suppose there is a const. $C > 0$ such that

$$S(f_t \mu | \omega) \le N^C .$$

Fix any small $\varepsilon > 0$. For any $t \in [\tau N^\varepsilon, N]$, the entropy and Dirichlet form satisfy

$$S(g_t \omega | \omega) \le C N^2 Q \tau^{-1}, \quad D_\omega(\sqrt{g_t}) \le C N^2 Q \tau^{-2},$$

where recall that $Q = \sup_{0 \le t \le N} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \gamma_j)^2 f_t(\vec{\lambda}) \, \mu(d\vec{\lambda}) \ (\le C N^{-2+2\xi})$.

Proof: We calculate $\partial_t S(f_t \mu | \omega) = \partial_t S(f_t \mu | \psi \mu)$

$$= \int (\mathcal{L} f_t)(\log g_t) d\mu + \int f_t \frac{\mathcal{L} f_t}{f_t} d\mu = \int (\mathcal{L} f_t)(\log g_t) d\mu$$

$$= \int f_t \, \mathcal{L}(\log g_t) d\mu = \int g_t \psi \, \mathcal{L}(\log g_t) d\mu = \int \psi \Big[g_t \mathcal{L}(\log g_t) - g_t \frac{\mathcal{L} g_t}{g_t} \Big] d\mu$$

$$+ \int \psi \, \mathcal{L} g_t \, d\mu .$$

We can check directly that $\mathcal{L}(\log g_t) - \frac{\mathcal{L} g_t}{g_t} = -\frac{1}{\beta N}\frac{|\nabla g|^2}{g_t^2} = -\frac{4}{\beta N}|\nabla\sqrt{g_t}|^2 \cdot \frac{1}{g_t}$.

Hence, we have $\partial_t S(f_t\mu|w) = -\frac{4}{\beta N}\int \psi|\nabla\sqrt{g_t}|^2 d\mu + \int \psi \mathcal{L} g_t d\mu$, $\left(b_j = \frac{x_j-\gamma_j}{\tau}\right)$

$$= -\frac{4}{\beta N}\int |\nabla\sqrt{g_t}|^2 dw + \int \cancel{\tilde{\mathcal{L}} g_t\, d\mu}^{\,0} + \sum_j \int b_j \partial_j g_t \, dw.$$

Using $\partial_j g_t = 2\sqrt{g_t}\,\partial_t\sqrt{g_t}$ and C-S, we get

$$\partial_t S(f_t\mu|w) = -4D_w(\sqrt{g_t}) + 2\sum_j \int b_j \sqrt{g_t}\,\partial_j\sqrt{g_t}\, dw$$

$$\le -2D_w(\sqrt{g_t}) + CN\sum_j \int b_j^2 g_t\, dw$$

$$= -2D_w(\sqrt{g_t}) + C\frac{N^2}{\tau^2}\frac{1}{N}\int (x_j-\gamma_j)^2 g_t\, dw$$

$$\le -2D_w(\sqrt{g_t}) + CN^2Q\tau^{-2}. \qquad (+)$$

Applying the LSI w.r.t. dw, we get that

$$\partial_t S(f_t\mu|w) \le -C\tau^{-1}S(f_t\mu|w) + CN^2Q\tau^{-2}.$$

Applying the Gronwall's inequality, integrating this inequality from τ to t and using the assumption $S(f_t\mu|w) \le N^C$ and $t \ge \tau N^{\varepsilon}$, we obtain that

$$S(g_t w|w) = S(f_t\mu|w) \le CN^2Q\tau^{-1}. \qquad (++)$$

To prove the second estimate, we notice

$$D_\mu(\sqrt{f_s}) = \frac{1}{\beta N}\int \frac{|\nabla(g_s\psi)|^2}{g_s\psi}\frac{dw}{\psi} \le \frac{C}{\beta N}\int \left[\frac{|\nabla g_s|^2}{g_s} + |\nabla\log\psi|^2 g_s\right] dw$$

$$\le C D_w(\sqrt{g_s}) + \cancel{\frac{C}{\beta N}\int\frac{\sum|x_j-\gamma_j|^2}{\tau^2} f_t\, d\mu}\; C\frac{(\beta N)^2}{\beta N}\int \frac{\sum_j |x_j-\gamma_j|^2}{\tau^2} f_t\, d\mu$$

$$\le C D_w(\sqrt{g_s}) + CN^2Q\tau^{-2}. \qquad (*)$$

Taking integral over $[t-\tau, t]$ ~~we get and using~~, we get

$$\int_{t-\tau}^{t} D_w(\sqrt{g_s})\, ds \ge \int_{t-\tau}^{t}\left[\frac{1}{C}D_\mu(\sqrt{f_s}) - CN^2Q\tau^{-2}\right] ds$$

$$(D_\mu(\sqrt{f_t}) \text{ is decreasing in } t) \Leftarrow \ge \frac{\tau}{C}D_\mu(\sqrt{f_t}) - CN^2Q\tau^{-1}.$$

On the other hand, with $(+)$ and $(++)$, we get that

$$\int_{t-\tau}^{t} D_w(\sqrt{g_s})\, ds \le CN^2Q\tau^{-1} + C[S(f_t\mu|w) - S(f_{t-\tau}\mu|w)] \le CN^2Q\tau^{-1}.$$

From the above two inequalities, we obtain that

$$D_\mu(\sqrt{f_t}) \le CN^2Q\tau^{-2}.$$

With a similar argument as in $(*)$, we get that

$$D_\omega(\sqrt{g_t}) \le C D_\mu(\sqrt{f_t}) + CN^2 Q\tau^{-2} \le CN^2 Q \tau^{-2}.$$ □

Theorem 6.6 is a consequence of the above three propositions. We choose $\tau = tN^{-\varepsilon}$ (recall that $t \ge N^{-1+2\zeta+2\varepsilon+\delta}$), $\psi = \omega/\mu$ and $q := g_t = f_t/\psi$. To apply Prop. III, we still need to verify the assumption $S(f_\tau\mu|\omega) \le N^C$. By definition,

$$S(f_\tau\mu|\omega) = \int f_\tau \log f_\tau \, d\mu - \int f_\tau \log\psi \, d\mu$$

$$= S_\mu(f_\tau) - \int f_\tau \log\psi \, d\mu,$$

where $-\log\psi = \frac{\beta N}{2\tau}\sum_{j=1}^{N}(x_j-\gamma_j)^2 + \log\frac{\tilde{Z}}{Z}$. We know $\tilde{Z} \le Z$ since $\tilde{\mathcal{H}} \ge \mathcal{H}$. Hence,

$$-\int f_\tau \log\psi \, d\mu \le CN\tau^{-1}\int \sum_{j=1}^{N}(x_j-\gamma_j)^2 f_\tau(\vec{x})\,d\mu(\vec{x}) \le CN^2\tau^{-1}Q \le N^C.$$

It remains to bound $S_\mu(f_\tau)$ for a "good initial condition" as a Wigner matrix.

Lemma: Let $\beta = 1, 2$. Suppose the initial data f_0 of the DBM is given by the eigenvalue distribution of a Wigner matrix. Then, $\forall \tau > 0$, we have

$$S_\mu(f_\tau) \le CN^2[1 - \log(1-e^{-\tau})].$$

Proof: We only consider the real case with $\beta = 1$. Recall that $f_\tau\mu$ is the eigenvalue distribution of $H_\tau = e^{-\tau/2}H_0 + (1-e^{-\tau})^{1/2}\,\mathrm{GOE}$. Notice that $f_\tau\mu$ is the marginal of μ_{H_τ} by integrating out the eigenvector distribution, and μ is the marginal of GOE. Then, using the fact that the entropy is decreasing w.r.t. taking a marginal, we get

$$S_\mu(f_\tau) \le S(\mu_{H_\tau} | \mu_{\mathrm{GOE}}).$$

Notice that both μ_{H_τ} and μ_{GOE} are product measures of the laws of matrix elements. Then, using the additivity of entropy, $S(\mu_{H_\tau}|\mu_{\mathrm{GOE}})$ is equal to the sum of the relative entropies of the matrix elements. Let's consider the off-diagonal elements: for $\gamma = 1-e^{-\tau}$, let p_γ be the probability density of $(1-\gamma)^{1/2}\xi = e^{-\tau/2}\xi$, where ξ is the random variable for an off-diagonal matrix element. Then, the prob. density of an off-diagonal element of H_τ is given as: $\varsigma_\tau = p_\gamma * g_{\gamma/N}$, where g_α denotes the Gaussian distribution

$$g_\alpha(x) = \frac{1}{\sqrt{2\pi\alpha}}\exp\left(-\frac{x^2}{2\alpha}\right).$$

Then, we calculate that

$$S(\varsigma_\tau | g_{1/N}) = S\left(\int dy\, p_\gamma(y)\, g_{\gamma/N}(\cdot - y) \,\Big|\, g_{1/N}\right)$$

$$(\text{Jensen}) \leftarrow \le \int dy\, p_\gamma(y)\, S(g_{\gamma/N}(\cdot - y) | g_{1/N}).$$

A direct calculation yields that

$$S(g_{\sigma^2}(\cdot - y) | g_{s^2}) = \log\frac{s}{\sigma} + \frac{y^2}{2s^2} + \frac{\sigma^2}{2s^2} - \frac{1}{2}.$$

Applying it to our case, we get

$$S(g_{\gamma/N}(\cdot - y) \mid g_{1/N}) = \frac{1}{2}\left(Ny^2 - \log\gamma + \gamma - 1\right).$$

Thus, $S(f_t \mid g_{1/N}) \leq \int dy\, p_\gamma(y)\, \frac{1}{2}\left(Ny^2 - \log\gamma + \gamma - 1\right)$. Using $\int y^2 p_\gamma(y)\,dy = \frac{1}{N}$, we conclude the proof. □

With the above lemma, we can apply Prop. II & III to get that (with $q\,d\omega = f_t\,d\mu$, $\tau = tN^{-\varepsilon}$)

$$\left| \int \frac{1}{|J|} \sum_{i\in J} \mathcal{J}_{i,n}(\vec{x})\,(f_t\,d\mu - d\omega) \right| \leq C\left(t\,\frac{D_\omega(\sqrt{q})}{|J|}\right)^{1/2} + C\sqrt{S_\omega(q)}\,e^{-cN^\varepsilon}$$

$$\left[D_\omega(\sqrt{q}) = D_\omega(\sqrt{g_t})\right] \rightarrow \leq C\left(t\,\frac{N^2Q}{|J|\tau^2}\right)^{1/2} + N^C e^{-cN^\varepsilon} \leq C N^\varepsilon \left(\frac{N^2Q}{|J|t}\right)^{1/2} + Ce^{-cN^\varepsilon}.$$

Hence, $f_t\,d\mu$ and $d\omega$ are close for any initial data. Applying this estimate to the Gaussian initial data, we can also compare $d\mu$ and $d\omega$. This concludes the proof.