$\boxed{\text{Section 6}}$ Dyson Brownian Motion and Bulk Universality

I. Various notions of Bulk universality

The local eigenvalue statistics can either be expressed in terms of "local correlation functions" rescaled around some energy E or the "gap statistics" for a gap $\lambda_{j+1}-\lambda_j$ with a given label j. They are called "fixed energy" and "fixed gap" universalites, and they do not coincide. In fact, eigenvalues fluctuate on a scale much larger than the typical eigenvalue spacing, the label j of the eigenvalue λ_j closest to a fixed energy E is not a deterministic function of E. Moreover, the two concepts both have natural averaged versions, which are generall easier to establish.

Rmk: Recall that $\int_{\gamma_j}^{\gamma_{j+1}} \rho_{sc}(x)\,dx = \frac{1}{N} \Rightarrow \gamma_{j+1}-\gamma_j \sim \frac{1}{N\rho_{sc}(\gamma_j)}$. Hence, the ~~fluctuation~~ gaps and correlation functions need to be rescaled by the local density ρ_{sc} to get an universal limit. This holds in more general setups, such as sample covariance matrices.

① Fixed energy universality: $\forall n \in \mathbb{N}$, $F: \mathbb{R}^n \to \mathbb{R}$ is C_c^∞. For any const $\kappa>0$, we have that uniformly in $E \in [-2+\kappa, 2-\kappa]$,

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^n} \int_{\mathbb{R}^n} d\vec{\alpha}\, F(\vec{\alpha})\, p_N^{(n)}\Big(E+\frac{\vec{\alpha}}{N\rho_{sc}(E)}\Big) = \int_{\mathbb{R}^n} d\vec{\alpha}\, F(\vec{\alpha})\, q^{(n)}_{GOE/GUE}(\vec{\alpha}),$$

where $\vec{\alpha}=(\alpha_1,\cdots,\alpha_n)$, $p_N^{(n)}$ is the n-point correlation function, and $q^{(n)}_{GOE/GUE}(\vec{\alpha}) = \det\big(S(\alpha_i-\alpha_j)\big)_{i,j=1}^n$ is the determinant of sine-kenel we derived before.

② Averaged bulk universality (on scale $N^{-1+\varepsilon}$):

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^n} \int_{E-b}^{E+b} \frac{dx}{2b} \int_{\mathbb{R}^n} d\vec{\alpha}\, F(\vec{\alpha})\, p_N^{(n)}\Big(x+\frac{\vec{\alpha}}{N\rho_{sc}(E)}\Big) = \int_{\mathbb{R}^n} d\vec{\alpha}\, F(\vec{\alpha})\, q^{(n)}_{GOE/GUE}(\vec{\alpha})$$

where $b=N^{-1+\varepsilon}$ $\forall$ const $\varepsilon>0$.

③ Fixed gap universality: Fixed any small constant $\delta>0$ and $n\in\mathbb{N}$. For any $F:\mathbb{R}^n\to\mathbb{R}$, $F\in C_c^\infty$ and any $k,m\in[\delta N,(1-\delta)N]$, we have that

$$\lim_{N\to\infty}\Big|\mathbb{E}_{H_N} F\big(N\rho_{sc}(\gamma_k)(\lambda_{k+1}-\lambda_k),\cdots,N\rho_{sc}(\gamma_k)(\lambda_{k+n}-\lambda_k)\big) - \mathbb{E}_{GOE/GUE} F\big(N\rho_{sc}(\gamma_m)(\lambda_{m+1}-\lambda_m),\cdots,N\rho_{sc}(\gamma_m)(\lambda_{m+n}-\lambda_m)\big)\Big| = 0.$$

④ Averaged gap universality: For $l=N^\varepsilon$, $\forall$ const $\varepsilon>0$,

$$\lim_{N\to+\infty}\Big|\frac{1}{2l+1}\sum_{j=k-l}^{k+l}\mathbb{E}_{H_N} F\big(N\rho_{sc}(\gamma_k)(\lambda_{j+1}-\lambda_j),\cdots,N\rho_{sc}(\gamma_k)(\lambda_{j+n}-\lambda_j)\big) - \mathbb{E}_{GOE/GUE} F\big(N\rho_{sc}(\gamma_m)(\lambda_{m+1}-\lambda_m),\cdots,N\rho_{sc}(\gamma_m)(\lambda_{m+n}-\lambda_m)\big)\Big| = 0.$$

Rmk: Fixed energy $\overset{\Longrightarrow}{\not\Longleftarrow}$ Averaged energy, Fixed gap $\overset{\Longrightarrow}{\not\Longleftarrow}$ Averaged gap,

Fixed energy $\overset{\not\Longrightarrow}{\not\Longleftarrow}$ Fixed gap, Averaged energy $\Longleftrightarrow$ Averaged gap.

We will focus on proving the averaged energy universality.

Theorem 6.1 The averaged energy universality holds on scales $N^{-1+\varepsilon}$ $\forall\ 0<\varepsilon<1$.

This is a version of the famous "Wigner-Dyson-Mehta conjecture".

II. The three-step strategy

Step 1: Local semicircle law. At this step, we get precise estimates on the matrix elements of the resolvent, the rigidity of eigenvalues, and delocalization of eigenvectors.

Step 2: Universality for Gaussian divisible ~~ensembles~~ ensembles.
Gaussian divisible ensembles are random matrices that can be written as $H_0 + \sqrt{t}\, H^G$, where H_0 is Wigner, H^G is GOE / GUE and independent of H_0, and $t>0$ is a parameter.

A convenient way to generate H_t is the "matrix Ornstein-Uhlenbeck (OU) Process":

$$dH_t = \frac{1}{\sqrt{N}}\, dB_t - \frac{1}{2} H_t\, dt, \qquad H_t = H_0,$$

Where B_t is a matrix Brownian motion whose entries are independent BMs up to symmetry $B_t^* = B_t$, and $\frac{1}{\sqrt{N}} B_t \overset{\text{law}}{=} \sqrt{t}$ GOE / GUE. For each entry,

$$dh_{ij}(t) = \frac{1}{\sqrt{N}}\, db_{ij}(t) - \frac{1}{2} h_{ij}(t)\, dt.$$

It has a unique strong solution:

$$h_{ij}(t) = h_{ij}(0)\, e^{-t/2} + \int_0^t \frac{e^{-(t-t')/2}}{\sqrt{N}}\, db_{ij}(t').$$

Note that $\int_0^t \frac{e^{-(t-t')/2}}{\sqrt{N}}\, db_{ij}(t')$ is centered Gaussian of variance $\frac{1}{N}\int_0^t e^{-(t-t')}\, dt' = \frac{1}{N}(1-e^{-t})$.
Hence, with a slight abuse of notation, we write it as $h_{ij}^G \cdot \sqrt{1-e^{-t}}$.
This gives a solution

$$H_t \overset{\text{law}}{=} e^{-t/2} H_0 + \sqrt{1-e^{-t}}\, H^G.$$

A big advantage of this form is that variances are preserved throughout the process:

$$\mathbb{E}|h_{ij}(t)|^2 = e^{-t}\, \mathbb{E}|h_{ij}(0)|^2 + (1-e^{-t})\, \mathbb{E}|h_{ij}^G|^2 = \mathbb{E}|h_{ij}(0)|^2,$$

for $\mathbb{E}|h_{ij}(0)|^2 = 1+\delta_{ij}$ in the real case, and $\mathbb{E}|h_{ij}(0)|^2 = 1$ in the complex case.

The purpose of Step 2 is to show ~~that~~ the bulk universality of H_t for $t = N^{-1+\varepsilon}$ for any $0<\varepsilon<1$.

Approximation by a Gaussian ~~d~~ divisible ensemble

Step 3 : Given a Wigner matrix, H, there exists a Wigner H_0 such that H_t has asymptotically identical local eigenvalue statistics as H. This is usually done through a Green's function comparison argument by using certain moment matching conditions. Alternatively, one can also use a continuity estimate of the matrix OU process.

The "three-step strategy" is now (one of) the most standard in proving the bulk universality of random matrices (for edge universality, the Step 2 sometimes is not necessary).
Here, Step 1 is model-specific and generally is the "hardest" step. The Steps 2 and 3 are more standard, where general methods / proofs / arguments are known and work for "most" models. In particular, the strongest result for Step 2 has been established for very general initial conditions H_0 (not necessarily a random matrix).

III. Dyson Brownian Motion

The matrix Brownian motion introduces SPDE tools to study the evolution of the eigenvalues of H_t: $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_N(t)$. A classical theorem below will guarantee that the eigenvalues are simple and continuous functions of t. So the labelling is preserved along the evolution.

In principle, the eigenvalues $\{\lambda_i(t)\}$ and eigenvectors $\{\vec{u}_i(t)\}$ of H_t are correlated strongly, and we expect a couple system of stochastic differential equations for them (which is indeed the case if B_t is not chosen to have the law of GOE / GUE).
But, Dyson observe that the eigenvalues themselves satisfy an autonomous system of SDEs that does not involve eigenvectors, which is called the Dyson Brownian Motion (DBM).

Theorem 6.2 The eigenvalues $\{\lambda_i(t)\}$ of H_t satisfy the following system of SDEs:

$$d\lambda_i = \frac{\sqrt{2}}{\sqrt{\beta N}} dB_i(t) + \left(-\frac{\lambda_i}{2} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)dt, \quad 1\le i\le N, \quad \begin{cases}\beta=1 & \text{for real,}\\ \beta=2 & \text{for complex.}\end{cases}$$

$\{B_i(t) : 1\le i\le N\}$ is a collection of independent BMs. The solution to the above equation is called a DBM (where there is not necessarily an underlying matrix model).

Proof: Let $\lambda_\alpha^{(t)}$ be an eigenvalue of $H(t) = (h_{ij}(t))$ with eigenvector $\vec{u}_\alpha(t)$. Almost surely, all eigenvalues are simple. We apply Ito's formula to $\lambda_\alpha(t)$ to derive the DBM.
We only consider the real case with $\beta=1$.
Differentiating : $H\vec{u}_\alpha = \lambda_\alpha \vec{u}_\alpha$, $\vec{u}_\alpha^* \vec{u}_\beta = \delta_{\alpha\beta}$, we obtain that

$$(1) \quad \frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha + H\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \frac{\partial \lambda_\alpha}{\partial h_{ij}}\vec{u}_\alpha + \lambda_\alpha \frac{\partial \vec{u}_\alpha}{\partial h_{ij}},$$

$$(2)\quad \frac{\partial \vec{u}_\alpha^{\,*}}{\partial h_{ij}}\vec{u}_\beta + \vec{u}_\alpha^{\,*}\frac{\partial \vec{u}_\beta}{\partial h_{ij}} = 0, \qquad \vec{u}_\alpha^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = 0.$$

Taking inner product $\vec{u}_\alpha$ with (1) and using (2), we get

$$\vec{u}_\alpha^{\,*}\frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha + \lambda_\alpha \vec{u}_\alpha^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \frac{\partial \lambda_\alpha}{\partial h_{ij}} + \lambda_\alpha \vec{u}_\alpha^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}}.$$

$$\Rightarrow \frac{\partial \lambda_\alpha}{\partial h_{ij}} = \vec{u}_\alpha^{\,*}\frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha. \qquad (*)$$

Taking inner product $\vec{u}_\beta$ with (1) and using (2), we get (for $\beta\neq\alpha$)

$$\vec{u}_\beta^{\,*}\frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha + \lambda_\beta \vec{u}_\beta^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \lambda_\alpha \vec{u}_\beta^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}}.$$

This implies that $\frac{\partial \vec{u}_\alpha}{\partial h_{ij}} = \sum_{\beta\neq\alpha}\vec{u}_\beta\left(\vec{u}_\beta^{\,*}\frac{\partial \vec{u}_\alpha}{\partial h_{ij}}\right) = \sum_{\beta\neq\alpha}\vec{u}_\beta\frac{1}{\lambda_\alpha-\lambda_\beta}\left(\vec{u}_\beta^{\,*}\frac{\partial H}{\partial h_{ij}}\vec{u}_\alpha\right)$ $\quad(+)$

For $(*)$, it writes: $\frac{\partial\lambda_\alpha}{\partial h_{ij}} = (2-\delta_{ij})u_\alpha(i)u_\alpha(j)$. For $(+)$, it writes

$$\frac{\partial u_\alpha(k)}{\partial h_{ij}} = \sum_{\beta\neq\alpha}\frac{u_\beta(i)u_\alpha(j)+u_\beta(j)u_\alpha(i)(1-\delta_{ij})}{\lambda_\alpha-\lambda_\beta}u_\beta(k).$$

With these two formulas, we can also compute the second order partial derivatives:

$$\frac{\partial^2\lambda_\alpha}{\partial h_{ik}\partial h_{\ell j}} = (2-\delta_{ik})\left[\frac{\partial u_\alpha(i)}{\partial h_{\ell j}}u_\alpha(k) + u_\alpha(i)\frac{\partial u_\alpha(k)}{\partial h_{\ell j}}\right]$$

$$= (2-\delta_{ik})\sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha-\lambda_\beta}\Big[\big(u_\beta(j)u_\alpha(\ell)+u_\beta(\ell)u_\alpha(j)(1-\delta_{\ell j})\big)u_\beta(i)u_\alpha(k)$$
$$+\big(u_\beta(j)u_\alpha(\ell)+u_\beta(\ell)u_\alpha(j)(1-\delta_{\ell j})\big)u_\beta(k)u_\alpha(i)\Big]$$

$$= (2-\delta_{ik})\sum_{\beta\neq\alpha}\frac{1}{\lambda_\alpha-\lambda_\beta}\big(u_\beta(j)u_\alpha(\ell)+u_\beta(\ell)u_\alpha(j)(1-\delta_{\ell j})\big)\big(u_\beta(i)u_\alpha(k)+u_\beta(k)u_\alpha(i)\big).$$

Now, using Ito's formula, we obtain that

$$d\lambda_\alpha = \sum_{i\le k}\frac{\partial\lambda_\alpha}{\partial h_{ik}}dh_{ik} + \frac{1}{2}\sum_{i\le k}\sum_{j\le \ell}\frac{\partial^2\lambda_\alpha}{\partial h_{ik}\partial h_{\ell j}}[dh_{ik}, dh_{\ell j}]$$

$$= \sum_{i,k}u_\alpha(i)u_\alpha(k)\left[\frac{db_{ik}}{\sqrt{N}} - \frac{1}{2}h_{ik}dt\right] + \sum_{i,k}\sum_{\alpha\neq\beta}\frac{1}{2N}\frac{1}{\lambda_\alpha-\lambda_\beta}\left[u_\beta(i)u_\alpha(k)+u_\beta(k)u_\alpha(i)\right]^2dt$$

$$= \frac{1}{\sqrt{N}}\sum_{i,k}u_\alpha(i)u_\alpha(k)db_{ik} - \frac{1}{2}\lambda_\alpha dt + \frac{1}{N}\sum_{\alpha\neq\beta}\frac{1}{\lambda_\alpha-\lambda_\beta}dt,$$

where we used $[dh_{ik}, dh_{\ell j}] = \frac{1}{N}\delta_{i\ell}\delta_{kj}(1+\delta_{ik})dt$, and $\sum_k h_{ik}u_\alpha(k) = \lambda_\alpha u_\alpha(i)$.

Now, we define a new Gaussian process: $\tilde{B}_\alpha := \sum_{i,k}u_\alpha(i)u_\alpha(k)db_{ik}$. Clearly, $\mathbb{E}\tilde{B}_\alpha = 0$.
We now calculate its covariance:

$$\mathbb{E}[(d\tilde{B}_\alpha),(d\tilde{B}_\beta)] = \sum_{i,k} \sum_{l,j} u_\alpha(i)\, u_\alpha(k)\, u_\beta(l)\, u_\beta(j) [db_{ik}, db_{lj}]$$

$$= \sum_{i,k} \sum_{l,j} u_\alpha(i)\, u_\alpha(k)\, u_\beta(l)\, u_\beta(j)\, (\delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl})\, dt$$

$$= 2 \sum_{i,k} u_\alpha(i)\, u_\beta(i)\, u_\alpha(k)\, u_\beta(k)\, dt = 2\,\delta_{\alpha\beta}\, dt.$$

Thus, $\tilde{B}_\alpha = \sqrt{2} B_\alpha$, where $B_\alpha(t)$ is a standard real BM and B_α's are independent of each other. This gives the DBM with $\beta = 1$. □

A standard SPDE argument shows that there is a strong solution to the DBM:

$$(\ddagger) \quad d\lambda_i = \frac{\sqrt{2}}{\sqrt{\beta N}} dB_i(t) + \Big(-\frac{\lambda_i}{2} + \frac{1}{N}\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}\Big) dt, \quad 1 \le i \le N, \quad \beta \ge 1.$$

Note: if $\lambda_j < \lambda_i$, then the "interaction force" $\frac{1}{\lambda_i - \lambda_j}$ is >0, while if $\lambda_j > \lambda_i$, $\frac{1}{\lambda_i - \lambda_j} < 0$. This gives a repulsion between particles $\{\lambda_i\}$.

<u>Theorem 6.3</u>. Let $\bar{\Delta}_N := \{\vec{\lambda} : \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N\}$. Let $\beta \ge 1$ and suppose that the initial cond. $\vec{\lambda}(0) \in \bar{\Delta}_N$. Then, there exists a unique strong solution to $(\ddagger)$ in the space of continuous functions $(\vec{\lambda}(t))_{t\ge 0} \in C(\mathbb{R}_+, \bar{\Delta}_N)$. Moreover, $\forall\, t > 0$, we have $\vec{\lambda}(t) \in \Delta_N$ and $\vec{\lambda}(t)$ depends continuously on $\vec{\lambda}(0)$. In particular, if $\vec{\lambda}(0) \in \Delta_N$, then $(\vec{\lambda}(t))_{t\ge 0} \in C(\mathbb{R}_+, \Delta_N)$, i.e., the particles are separated ~~for all~~ for all times along the evolution.

<u>Rmk</u>: The DBM can be regarded as a Itô drift-diffusion process. Hence, we can mimic the proof of the existence and uniqueness of the strong solution there. But, one needs to deal with the sigularities $(\lambda_i - \lambda_j)^{-1}$. The "level repulsion mechanism" will play a significant role in the proof.

<u>IV.</u> Strong local ergodicity of DBM

~~DBM~~ The Gaussian measure (GOE / GUE) is the only stationary measure of DBM and the DBM dynamics converges to this equilibrium from any initial condition.

Recall the invariant β-ensemble: $\vec{\lambda} = (\lambda_1, \cdots, \lambda_N)$,

$$\mu_N(d\lambda) = \frac{1}{Z_N} \exp(-\beta N \mathcal{H}_N(\vec{\lambda}))\, d\vec{\lambda},$$

$\mathcal{H}_N(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^N V(\lambda_i) - \frac{1}{N}\sum_{i<j} \log|\lambda_i - \lambda_j|$. For us, we are interested in the GOE / GUE case with $V(\lambda) = \frac{1}{2}\lambda^2$.

Then, we define the Dirichlet form associated with μ_N:

$$D_\mu(f) := \frac{1}{\beta N}\sum_{i=1}^{N}\int (\partial_i f)^2 d\mu = \frac{1}{\beta N}\|\nabla f\|^2_{L^2(\mu_N)}, \quad \partial_i := \partial_{\lambda_i}.$$

The symmetric operator associated with the Dirichlet form is called generator and denoted by $\mathcal{L}_\mu \equiv \mathcal{L}$. It is defined through ($\langle , \rangle$: inner product)

$$D_\mu(f) = \langle f, (-\mathcal{L})f\rangle_{L^2(\mu)} = -\int f \mathcal{L} f \, d\mu_N . \quad (-\mathcal{L} \text{ is a positive operator})$$

Note that $\mathcal{L}$ can be chosen as $\mathcal{L} = \frac{1}{\beta N}\Delta - (\nabla \mathcal{H})\cdot\nabla$:

$$-\int f \mathcal{L} f \ \frac{1}{Z_N}\exp(-\beta N \mathcal{H}_N(\vec\lambda))\, d\vec\lambda = -\int f \frac{1}{\beta N}\Delta f \ \frac{1}{Z_N}\exp(-\beta N\mathcal{H}_N)\, d\vec\lambda$$

$$+ \int f (\nabla\mathcal{H})\cdot\nabla f \ \frac{1}{Z_N}\exp(-\beta N\mathcal{H}_N)\, d\vec\lambda$$

$$= D_\mu(f) + \frac{1}{\beta N}\int f(\nabla f)\cdot\nabla\left(\frac{1}{Z_N}\exp(-\beta N\mathcal{H}_N)\right) d\vec\lambda + \int f(\nabla\mathcal{H})\cdot\nabla f \, d\mu_N = D_\mu(f).$$

In components: $\mathcal{L} = \sum_{i=1}^{N}\frac{1}{\beta N}\partial_i^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}V'(\lambda_i) + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)\partial_i$.

For $V(\lambda) = \frac{1}{2}\lambda^2$, we have $\mathcal{L}_G = \frac{1}{\beta N}\sum_{i=1}^{N}\partial_i^2 + \sum_{i=1}^{N}\left(-\frac{1}{2}\lambda_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right)\partial_i$.

~~Con~~ With the DBM and Ito's formula, we notice that for any twice differentiable g,

$$\partial_t \mathbb{E}\, g(\vec\lambda(t)) = \mathbb{E}\, \mathcal{L}_G\, g(\vec\lambda(t)) .$$

Write the distribution of $\vec\lambda(t)$ at time t as $f_t(\vec\lambda)\mu_N(d\vec\lambda)$. We have

$$\partial_t \int g(\vec\lambda) f_t(\vec\lambda)\mu_N(d\vec\lambda) = \int (\mathcal{L}_G g(\vec\lambda)) f_t(\vec\lambda)\mu_N(d\vec\lambda)$$

$$= \int g(\vec\lambda)[\mathcal{L}_G f_t(\vec\lambda)]\mu_N(d\vec\lambda).$$

In other words, the density $f_t(\vec\lambda)$ satisfies $\partial_t f_t(\vec\lambda) = \mathcal{L}_G f_t(\vec\lambda)$. (★)
Note that $f(\vec\lambda)\equiv 1$ is a solution to this equation, i.e., $\mu_N(d\vec\lambda)$ is a stationary measure of the DBM. Our goal is to show that for any initial condition f_0, $f_t \to f_\infty \equiv 1$. A much harder and more important question is: how fast the dynamics reach equilibrium?

$\boxed{\text{Dyson's conjecture}}$ The global equilibrium of DBM is reached in time of order 1 and the local equilibrium (in the bulk) is reached in time of order $\frac{1}{N}$.

From $H_t = e^{-t/2}H_0 + \sqrt{1-e^{t}}\,H_G$, we see that the global equilibrium is indeed reached within a time of order 1. The key is that the local equilibrium is achieved much faster if an a priori estimate on the initial locations of the eigenvalues holds, which verifies Dyson's conjecture.