$$= \frac{1}{2}\|\vec{v}\|^2 + \frac{1}{2N}\sum_{i\neq j}\frac{(v(i)-v(j))^2}{(\lambda_i-\lambda_j)^2} \geq \frac{1}{2}\|\vec{v}\|^2 .$$

Thus, $\nabla^2(\beta N \mathcal{H}) \geq \frac{1}{2}\beta N$. Here, we use a different normalization for the Dirichlet form: $D_\mu(f) = \frac{1}{\beta N}\int \|\nabla f\|^2 d\mu$. Then, the Bakry – Emery estimate gives

$$S_\mu(f) \leq 4 D_\mu(\sqrt{f}) .$$

This shows that the relaxation time is of order 1. (This is the global dynamics.) We will show that the local dynamics converges to equilibrium within time of order $N^{-1+\varepsilon}$.

<u>Thm 6.6</u>. Assume the following conditions:

(i) There exists a constant $\xi > 0$ such that

$$Q := \sup_{0\leq t\leq N} \frac{1}{N}\int \sum_{j=1}^{N}(\lambda_j-\gamma_j)^2 f_t(\vec{\lambda})\,\mu_G(d\vec{\lambda}) \leq CN^{-2+2\xi} .$$

(ii) After time $1/N$, the solution to the equation $\partial_t f_t = \mathcal{L} f_t$, $t\geq 0$, satisfies $S_\mu(f_{1/N}) \leq CN^m$ for some fixed m.

Fix any $\beta\geq 1$ and $n\in\mathbb{N}$. Let $F:\mathbb{R}^n\to\mathbb{R}$ be a smooth function with compact support. Denote:

$$\mathcal{F}_{i,n}(\vec{x}) = F(N(x_{i+1}-x_i), N(x_{i+2}-x_i), \dots, N(x_{i+n}-x_i)) .$$

Then, for any $\xi\in(0,\frac{1}{2})$ and any sufficiently small constant $\varepsilon>0$, there ~~exists a~~ exist constants $C, c>0$ depending only on ε and F, such that for any $J\subset\{1,2,\dots,N-n\}$ we have

$$\sup_{t\geq N^{-1+2\xi+\varepsilon}}\left|\int \frac{1}{|J|}\sum_{i\in J}\mathcal{F}_{i,n}(\vec{x})(f_t d\mu - d\mu)\right| \leq CN^{\varepsilon}\sqrt{\frac{N^2Q}{|J|t}} + Ce^{-cN^{\varepsilon}} .$$

In particular, if $t\geq N^{-1+2\xi+2\varepsilon+\delta}$, we have

$$\left|\int \frac{1}{|J|}\sum_{i\in J}\mathcal{F}_{i,n}(\vec{x})(f_t d\mu - d\mu)\right| \leq \frac{C}{\sqrt{JN^{\delta-1}}} + Ce^{-cN^{\varepsilon}} .$$

Hence, the gap distribution, averaged over J indices, coincides for $f_t d\mu$ and $d\mu$ if $|J|N^{\delta-1}\to+\infty$.

Averaged gap distribution on scale J corresponds to averaged energy distribution on scale $J\sim Nb$.

<u>Lemma</u>. Under the setting of Thm 6.6, suppose $\left|\int \frac{1}{|J|}\sum_{i\in J}\mathcal{F}_{i,n}(\vec{x})(f d\mu - d\mu)\right| \leq \frac{C}{\sqrt{|J|N^{\delta-1}}}$ for some constant $\delta>0$.

Suppose the eigenvalue rigidity holds: $\forall\ \kappa>0,\ \sup_{\kappa N\le j\le(1-\kappa)N} |\lambda_j-\gamma_j| < N^{-1+\xi}$.

Then, for any constant $\varepsilon>0$ and $N^{-1} \ll b_N \ll 1$, we have that

$$\left| \int_{E-b}^{E+b} \frac{dE'}{2b} \int_{\mathbb{R}^n} d\vec{\alpha}\ O(\vec{\alpha}) \left(p^{(n)}_{f\mu,N} - p^{(n)}_{\mu,N} \right) \left(E' + \frac{\vec{\alpha}}{N\rho_{sc}(E)} \right) \right|$$

$$\lesssim N^{2\varepsilon} \left[\frac{N^{-1+\xi}}{b} + \sqrt{\frac{N^{-\delta}}{b}} \right].$$

Hence, the averaged energy distribution over $b \gg N^{-1+\xi+2\varepsilon} \vee N^{4\varepsilon-\delta}$ coincides for $f\mu$ and μ.

The above two results establish the averaged gap/energy universality for $t \gtrsim N^{-1+2\xi+2\varepsilon+\delta}$. Under the optimal rigidity, ξ can be chosen as small as possible. Furthermore, ε and δ are arbitrary. This establishes the Dyson's conjecture: the local dynamics relaxes to equilibrium for $t \gtrsim N^{-1+c}$ $\forall$ constant $c>0$.

<u>Main ideas for the proof:</u> Recall that

$$\langle \vec{v}, \nabla^2\mathcal{H}\, \vec{v} \rangle = \frac{1}{2}\sum_i v(i)^2 + \frac{1}{N}\sum_{i<j} \frac{(v(i)-v(j))^2}{(\lambda_i-\lambda_j)^2} \ge \frac{1}{2}\|\vec{v}\|^2.$$

The key intuition is that the relaxation time is in fact much shorter than order 1 for local variables that depend only on the eigenvalue differences. In particular, with the intuition that $\lambda_i-\lambda_j$ are close, the relaxation time in the direction $v(i)-v(j)$ should be much smaller than 1. However, this effect is hard to use directly in the real proof. An idea is to add an "auxiliary strongly convex potential" to $\mathcal{H}$ to "speed up" the relaxation to local equilibrium. On the other hand, we will show that the effect of this speeding up on the local statistics can be controlled provided with the estimate

$$(+)\quad \sup_{0\le t\le N} \frac{1}{N}\int \sum_{j=1}^N (\lambda_j-\gamma_j)^2 f_t(\vec{\lambda})\, \mu_G(d\vec{\lambda}) \lesssim N^{-2+2\xi}.$$

We introduce the following auxiliary potential $\hat{\mathcal{H}}(\vec{\lambda}) := \frac{1}{2\tau}\sum_{j=1}^N (\lambda_j-\gamma_j)^2$, $0<\tau<1$.
(We will choose $\tau = tN^{-\varepsilon}$ in the proof.)
Then, the new Hamiltonian is $\tilde{\mathcal{H}} = \mathcal{H} + \hat{\mathcal{H}}$, and the corresponding measure is

$d\omega = \omega(\vec{x})\,d\vec{x}$, $\quad \omega(\vec{x}) := e^{-\beta N\tilde{\mathcal{H}}}/\tilde{Z}$, called the "local relaxation measure".

The "local relaxation flow" with the generator characterized by the natural Dirichlet form w.r.t. ω: $\tilde{\mathcal{L}} = \mathcal{L} - \sum_j b_j\partial_j$, $\quad b_j(x_j) = \frac{x_j-\gamma_j}{\tau}$.

For small τ, $\hat{\mathcal{H}}$ will substantially increase the lower bound on the Hessian, hence speeding up the dynamics so that the relaxation time is at most τ. In addition, we will compare the local statistics of the original system with those of the modified one.

It turns out that the difference is governed by $(\nabla \mathscr{H})^2$, which can be controlled with the estimate (†).

$\boxed{\text{Proof of Theorem 6.6}}$

Proposition I. Fix any $\beta \geq 1$. Consider the equation $\partial_t q_t = \tilde{\mathcal{L}} q_t$, $t \geq 0$, with the reversible measure $\omega = e^{-\beta N \tilde{\mathscr{H}}}/Z_N$. Suppose the initial condition q_0 satisfies $\int q_0 \, d\omega = 1$. Then, we have that

(1) $$\partial_t D_\omega(\sqrt{q_t}) \leq -\frac{2}{\tau} D_\omega(\sqrt{q_t}) - \frac{1}{\beta N^2} \int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_t} - \partial_j \sqrt{q_t})^2}{(x_i - x_j)^2} \, d\omega,$$

(2) $$\frac{1}{\beta N^2} \int_0^\infty ds \int \sum_{i,j=1}^{N} \frac{(\partial_i \sqrt{q_s} - \partial_j \sqrt{q_s})^2}{(x_i - x_j)^2} \, d\omega \leq D_\omega(\sqrt{q_0}),$$

and the LSI $S_\omega(q_0) \leq C\tau D_\omega(\sqrt{q_0})$. Hence, the relaxation time to equilibrium is of order τ:

(3) $$S_\omega(q_t) \leq e^{-Ct/\tau} S_\omega(q_0), \quad D_\omega(\sqrt{q_t}) \leq e^{-Ct/\tau} D_\omega(\sqrt{q_0}).$$

Proof: Performing the same calculation as in the proof of the Bakry-Emery estimate, we obtain that for $h_t = \sqrt{q_t}$,

$$\partial_t D_\omega(h_t) = \partial_t \frac{1}{\beta N} \int (\nabla h_t)^2 e^{-\beta N \tilde{\mathscr{H}}} d\vec{x} \leq -\frac{2}{\beta N} \int (\nabla h_t) \cdot (\nabla^2 \tilde{\mathscr{H}}) \cdot (\nabla h_t) e^{-\beta N \tilde{\mathscr{H}}} d\vec{x}.$$

For $\tilde{\mathscr{H}}$, we can calculate its Hessian as

$$\nabla h_t \cdot \nabla^2 \tilde{\mathscr{H}} \cdot \nabla h \geq \frac{1}{\tau} \sum_j (\partial_j h)^2 + \frac{1}{2N} \sum_{i,j} \frac{1}{(x_i - x_j)^2} (\partial_i h - \partial_j h)^2.$$

Plugging it into the above inequality, we conclude (1). Integrating over t, we conclude (2). The LSI and (3) can be derived in the same way as in the proof of Bakry-Emery. □

The extra term on the RHS of (1) plays a key role in the proof of the following prop.

Proposition II. Let q be a probability density with respect to the local relaxation measure ω, i.e., $\int q \, d\omega = 1$. Fix any $n \geq 1$ and a smooth function F with compact support. Recall that

$$\mathcal{F}_{i,n}(\vec{x}) := F(N(x_{i+1} - x_i), \cdots, N(x_{i+n} - x_i)).$$

Then, for any $J \subseteq \{1, 2, \cdots, N-n\}$ and $t > 0$, we have

$$\left| \int \frac{1}{|J|} \sum_{i \in J} \mathcal{F}_{i,n}(\vec{x}) (q \, d\omega - d\omega) \right| \leq C \left(t \frac{D_\omega(\sqrt{q})}{|J|} \right)^{1/2} + C \sqrt{S_\omega(q)} \, e^{-ct/\tau}.$$

Proof: For simplicity of notation, we take $n = 1$, so $\mathcal{F}_{i,n}(\vec{x}) = F(N(x_{i+1} - x_i))$. Let q_t be a solution to $\partial_t q_t = \tilde{\mathcal{L}} q_t$, $t \geq 0$, with initial condition $q_0 = q$. Then,

$$\int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - 1) \, d\omega = \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q - q_t) \, d\omega + \int \frac{1}{|J|} \sum_{i \in J} G(N(x_{i+1} - x_i)) (q_t - 1) \, d\omega.$$

The second term can be bounded by

Exp. decay of entropy given by the last prop

$$\int |q_t - 1| d\omega \underset{\substack{\uparrow \\ \text{Pinsker inequality}}}{\leq} C\sqrt{S_\omega(q_t)} \overset{\downarrow}{\leq} Ce^{-Ct/\tau}\sqrt{S_\omega(q_0)}\ .$$

To estimate the first term, using $\partial_t q_t = \tilde{\mathcal{L}} q_t$, we get that

$$\int \frac{1}{|J|}\sum_{i\in J} G(N(x_{i+1}-x_i))\, q_t\, d\omega - \int \frac{1}{|J|}\sum_{i\in J} G(N(x_{i+1}-x_i))\, q_0\, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|}\sum_{i\in J} G(N(x_{i+1}-x_i))\, \partial_s q_s\, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|}\sum_{i\in J} G(N(x_{i+1}-x_i))\, \tilde{\mathcal{L}} q_s\, d\omega$$

$$= \int_0^t ds \int \frac{1}{|J|}\sum_{i\in J} G'(N(x_{i+1}-x_i))\,(\partial_{i+1} q_s - \partial_i q_s)\, d\omega. \quad (*)$$

Using $\partial_i q = 2\sqrt{q}\,\partial_i\sqrt{q}$ and the Cauchy-Schwarz inequality, we obtain that

$$|(*)| \leq 2\left[\int_0^t ds \int \frac{N^2}{|J|^2}\sum_{i\in J} |G'(N(x_{i+1}-x_i))|^2 (x_{i+1}-x_i)^2 q_s\, d\omega\right]^{1/2} \times$$

$$\underbrace{\left[\int_0^t ds \int \frac{1}{N^2}\sum_i \frac{1}{(x_i - x_{i+1})^2}\left(\partial_{i+1}\sqrt{q_s} - \partial_i\sqrt{q_s}\right)^2 d\omega\right]^{1/2}}_{\leq C D_\omega(\sqrt{q_0})}$$

Moreover, $|G'(N(x_{i+1}-x_i))|^2 |x_{i+1}-x_i|^2 \leq CN^{-2}$ because G is smooth and compactly supported. So

$$|(*)| \leq C\sqrt{D_\omega(q_0)}\cdot\left(\frac{t}{|J|}\right)^{1/2}. \qquad \square$$

Proposition III. Consider the local relaxation measure ω for a given $\tau > 0$. Set $\psi = \omega/\mu$ and $g_t = f_t/\psi$ with f_t solving $\partial_t f_t = \mathcal{L} f_t$. Suppose there is a const. $C > 0$ such that

$$S(f_t\mu|\omega) \leq N^C.$$

Fix any small $\varepsilon > 0$. For any $t \in [\tau N^\varepsilon, N]$, the entropy and Dirichlet form satisfy

$$S(g_t\omega|\omega) \leq CN^2 Q\tau^{-1}, \quad D_\omega(\sqrt{g_t}) \leq CN^2 Q\tau^{-2},$$

where recall that $Q = \sup_{0\leq t\leq N} \frac{1}{N}\int \sum_{j=1}^N (\lambda_j - \gamma_j)^2 f_t(\vec{\lambda})\,\mu(d\vec{\lambda})\ \ (\leq CN^{-2+2\xi})$.

Proof: We calculate $\partial_t S(f_t\mu|\omega) = \partial_t S(f_t\mu|\psi\mu)$

$$= \int (\mathcal{L} f_t)(\log g_t)\,d\mu + \int f_t \frac{\mathcal{L} f_t}{f_t}\,d\mu = \int (\mathcal{L} f_t)(\log g_t)\,d\mu$$

$$= \int f_t\, \mathcal{L}(\log g_t)\,d\mu = \int g_t\psi\, \mathcal{L}(\log g_t)\,d\mu = \int \psi\left[g_t\mathcal{L}(\log g_t) - g_t\frac{\mathcal{L} g_t}{g_t}\right]d\mu$$

$$+ \int \psi\, \mathcal{L} g_t\, d\mu.$$

We can check directly that $\mathcal{L}(\log g_t) - \frac{\mathcal{L}g_t}{g_t} = -\frac{1}{\beta N}\frac{|\nabla g|^2}{g_t^2} = -\frac{4}{\beta N}|\nabla\sqrt{g_t}|^2\cdot\frac{1}{g_t}$.

Hence, we have $\partial_t S(f_t\mu|w) = -\frac{4}{\beta N}\int \psi|\nabla\sqrt{g_t}|^2 d\mu + \int \psi\mathcal{L}g_t d\mu$, $\left(b_j = \frac{x_j - \gamma_j}{\tau}\right)$

$$= -\frac{4}{\beta N}\int |\nabla\sqrt{g_t}|^2 dw + \int \overset{0}{\cancel{\tilde{\mathcal{L}}g_t d\mu}} + \sum_j \int b_j \partial_j g_t \, dw.$$

Using $\partial_j g_t = 2\sqrt{g_t}\,\partial_t\sqrt{g_t}$ and C-S, we get

$$\partial_t S(f_t\mu|w) = -4D_w(\sqrt{g_t}) + 2\sum_j\int b_j\sqrt{g_t}\,\partial_j\sqrt{g_t}\,dw$$

$$\leq -2D_w(\sqrt{g_t}) + CN\sum_j\int b_j^2 g_t\,dw$$

$$= -2D_w(\sqrt{g_t}) + C\frac{N^2}{\tau^2}\frac{1}{N}\int (x_j-\gamma_j)^2 g_t\,dw$$

$$\leq -2D_w(\sqrt{g_t}) + CN^2Q\tau^{-2}. \qquad (+)$$

Applying the LSI w.r.t. dw, we get that

$$\partial_t S(f_t\mu|w) \leq -C\tau^{-1}S(f_t\mu|w) + CN^2Q\tau^{-2}.$$

Applying the Gronwall's inequality, integrating this inequality from τ to t and using the assumption $S(f_t\mu|w) \leq N^C$ and $t \geq \tau N^\varepsilon$, we obtain that

$$S(g_t w|w) = S(f_t\mu|w) \leq CN^2Q\tau^{-1}. \qquad (++)$$

To prove the second estimate, we notice

$$D_\mu(\sqrt{f_s}) = \frac{1}{\beta N}\int \frac{|\nabla(g_s\psi)|^2}{g_s\psi}\frac{dw}{\psi} \leq \frac{C}{\beta N}\int\left[\frac{|\nabla g_s|^2}{g_s} + |\nabla\log\psi|^2 g_s\right]dw$$

$$\leq C D_w(\sqrt{g_s}) + \cancel{\frac{C}{\beta N}\int\frac{\sum_j|x_j-\gamma_j|^2}{\tau^2}f_t\,d\mu}\; C\frac{(\beta N)^2}{\beta N}\int\frac{\sum_j|x_j-\gamma_j|^2}{\tau^2}f_t\,d\mu$$

$$\leq C D_w(\sqrt{g_s}) + CN^2Q\tau^{-2}. \qquad (*)$$

Taking integral over $[t-\tau, t]$ ~~we get and using~~, we get

$$\int_{t-\tau}^t D_w(\sqrt{g_s})\,ds \geq \int_{t-\tau}^t\left[\frac{1}{C}D_\mu(\sqrt{f_s}) - CN^2Q\tau^{-2}\right]ds$$

($D_\mu(\sqrt{f_t})$ is decreasing in t) $\Leftarrow \geq \frac{\tau}{C}D_\mu(\sqrt{f_t}) - CN^2Q\tau^{-1}$.

On the other hand, with $(+)$ and $(++)$, we get that

$$\int_{t-\tau}^t D_w(\sqrt{g_s})\,ds \leq CN^2Q\tau^{-1} + C[S(f_t\mu|w) - S(f_{t-\tau}\mu|w)] \leq CN^2Q\tau^{-1}.$$

From the above two inequalities, we obtain that

$$D_\mu(\sqrt{f_t}) \leq CN^2Q\tau^{-2}.$$

With a similar argument as in $(*)$, we get that

$$D_\omega(\sqrt{g_t}) \leq C D_\mu(\sqrt{f_t}) + CN^2Q\tau^{-2} \leq CN^2Q\tau^{-2}. \qquad \square$$

Theorem 6.6 is a consequence of the above three propositions. We choose $\tau = tN^{-\varepsilon}$ (recall that $t \geq N^{-1+2\zeta+2\varepsilon+\delta}$), $\psi = \omega/\mu$ and $q := g_t = f_t/\psi$. To apply Prop. III, we still need to verify the assumption $S(f_t\mu|\omega) \leq N^C$. By definition,

$$S(f_\tau\mu|\omega) = \int f_\tau \log f_\tau \, d\mu - \int f_\tau \log\psi \, d\mu$$

$$= S_\mu(f_\tau) - \int f_\tau \log\psi \, d\mu,$$

where $-\log\psi = \frac{\beta N}{2\tau}\sum_{j=1}^{N}(x_j-\gamma_j)^2 + \log\frac{\tilde{Z}}{Z}$. We know $\tilde{Z} \leq Z$ since $\tilde{\mathcal{H}} \geq \mathcal{H}$. Hence,

$$-\int f_\tau \log\psi \, d\mu \leq CN\tau^{-1}\int \sum_{j=1}^{N}(x_j-\gamma_j)^2 f_\tau(\vec{x})\,d\mu(\vec{x}) \leq CN^2\tau^{-1}Q \leq N^C.$$

It remains to bound $S_\mu(f_\tau)$ for a "good initial condition" as a Wigner matrix.

Lemma: Let $\beta=1,2$. Suppose the initial data f_0 of the DBM is given by the eigenvalue distribution of a Wigner matrix. Then, $\forall \tau>0$, we have

$$S_\mu(f_\tau) \leq CN^2[1-\log(1-e^{-\tau})].$$

Proof: We only consider the real case with $\beta=1$. Recall that $f_\tau\mu$ is the eigenvalue distribution of $H_\tau = e^{-\tau/2}H_0 + (1-e^{-\tau})^{1/2}\,\mathrm{GOE}$. Notice that $f_\tau\mu$ is the marginal of μ_{H_τ} by integrating out the eigenvector distribution, and μ is the marginal of GOE. Then, using the fact that the entropy is decreasing w.r.t. taking a marginal, we get

$$S_\mu(f_\tau) \leq S(\mu_{H_\tau}|\mu_{\mathrm{GOE}}).$$

Notice that both μ_{H_τ} and μ_{GOE} are product measures of the laws of matrix elements. Then, using the additivity of entropy, $S(\mu_{H_\tau}|\mu_{\mathrm{GOE}})$ is equal to the sum of the relative entropies of the matrix elements. Let's consider the off-diagonal elements: for $\gamma = 1-e^{-\tau}$, let ρ_γ be the probability density of $(1-\gamma)^{1/2}\zeta = e^{-\tau/2}\zeta$, where ζ is the random variable for an off-diagonal matrix element. Then, the prob. density of an off-diagonal element of H_τ is given as: $\zeta_\tau = \rho_\gamma * g_{\gamma/N}$, where g_α denotes the Gaussian distribution

$$g_\alpha(x) = \frac{1}{\sqrt{2\pi\alpha}}\exp\left(-\frac{x^2}{2\alpha}\right).$$

Then, we calculate that

$$S(\zeta_\tau|g_{1/N}) = S\left(\int dy\, \rho_\gamma(y)\, g_{\gamma/N}(\cdot - y) \,\Big|\, g_{1/N}\right)$$

$$(\text{Jensen}) \leftarrow \leq \int dy\, \rho_\gamma(y)\, S(g_{\gamma/N}(\cdot - y)|g_{1/N}).$$

A direct calculation yields that

$$S(g_{\sigma^2}(\cdot - y)|g_{s^2}) = \log\frac{s}{\sigma} + \frac{y^2}{2s^2} + \frac{\sigma^2}{2s^2} - \frac{1}{2}.$$

Applying it to our case, we get

$$S\left(g_{\gamma/N}(\cdot - y) \,|\, g_{1/N}\right) = \frac{1}{2}\left(Ny^2 - \log\gamma + \gamma - 1\right).$$

Thus, $S(f_t \,|\, g_{1/N}) \le \int dy\, \rho_\gamma(y)\, \frac{1}{2}\left(Ny^2 - \log\gamma + \gamma - 1\right)$. Using $\int y^2 \rho_\gamma(y)\, dy = \frac{1}{N}$, we conclude the proof. □

With the above lemma, we can apply Prop. II & III to get that (with ~~$q = f_t$~~ $q\,dw = f_t\,d\mu$, $\tau = tN^{-\varepsilon}$)

$$\left| \int \frac{1}{|J|} \sum_{i \in J} J_{i,n}(\vec{x}) \left(f_t\, d\mu - q\, dw\right) \right| \le C \left(t \, \frac{D_w(\sqrt{q})}{|J|}\right)^{1/2} + C\sqrt{S_w(q)}\, e^{-cN^\varepsilon}$$

$$\left[D_w(\sqrt{q}) = D_w(\sqrt{g_t}) \right] \to \le C \left(t\, \frac{N^2 Q}{|J|\tau^2}\right)^{1/2} + N^C e^{-cN^\varepsilon} \le C N^\varepsilon \left(\frac{N^2 Q}{|J| t}\right)^{1/2} + Ce^{-cN^\varepsilon}.$$

Hence, $f_t\,d\mu$ and dw are close for any initial data. Applying this estimate to the Gaussian initial data, we can also compare $d\mu$ and dw. This concludes the proof.

$\boxed{\text{Section 7}}$ Green function comparison

We now finish Step 3 of the three step strategy. That is, given a Wigner H, we can find a Wigner H_0 and $t \geq N^{-1+c}$ such that the local eigenvalue statistics of H match those of $H_t = e^{-t/2} H_0 + (1-e^{-t})^{1/2} H_G$, $H_G \stackrel{d}{=} \text{GOE}/\text{GUE}$. We have shown the bulk universality of H_t. Then, Step 3 implies the bulk universality of H.

To compare local statistics of two Wigner ensembles, we use the following "correlation function comparison theorem". For the detailed setting, suppose we have two Wigner ensembles, labelled by v and w, respectively: H^v and H^w. Denote their eigenvalues by λ_i^v and λ_i^w, and the corresponding Stieltjes transforms by m^v and m^w, i.e.,

$$m^v(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j^v - z}.$$

Let $p_{v,N}^{(n)}$ and $p_{w,N}^{(n)}$ be the n-point correlation functions of $\{\lambda_i^v\}$ and $\{\lambda_i^w\}$.

Theorem 7.1 Let $\kappa^{>0}$ be a small const. Suppose for some small constants $\sigma, \delta > 0$ such that the following two conditions hold.

(i) $\forall \varepsilon > 0$ and fixed $k \in \mathbb{N}$,

$$\mathbb{E}[\operatorname{Im} m^v(E+iN^{-1+\varepsilon})]^k + \mathbb{E}[\operatorname{Im} m^w(E+iN^{-1+\varepsilon})]^k \leq C$$

for $|E| \leq 2-\kappa$.

(ii) For any sequence $z_j = E_j \pm i\eta_j$, $j=1,\cdots,n$, with $|E_j| \leq 2-\kappa$ and $\eta_j = N^{-1-\sigma_j}$ for some $\sigma_j \leq \sigma$, we have

$$\left| \mathbb{E}[\operatorname{Im} m^v(z_1) \cdots \operatorname{Im} m^v(z_n)] - \mathbb{E}[\operatorname{Im} m^w(z_1) \cdots \operatorname{Im} m^w(z_n)] \right| \leq N^{-\delta}. \quad (*)$$

Then, for any $n \in \mathbb{N}$, there are constants $C_n = C_n(\sigma, \delta)$ s.t. for any $|E| \leq 2-2\kappa$ and C^1 function $O: \mathbb{R}^n \to \mathbb{R}$ with compact support,

$$\left| \int_{\mathbb{R}^n} O(\vec{\alpha}) \left(p_{v,N}^{(n)} - p_{w,N}^{(n)} \right) \left(E + \frac{\vec{\alpha}}{N} \right) d\vec{\alpha} \right| \leq C N^{-C_n}.$$

Roughly speaking, this theorem says that if the finite dimensional marginals of the Stieltjes transforms are close, then the local correlation functions are close. It is crucial that in $(*)$, z_i has imaginary part $N^{-1-\sigma_j}$ below the scale of bulk eigenvalue gaps N^{-1}. Controlling the Stieltjes transforms on this very short scale is necessary to identify and compare local correlation functions at the scale N^{-1}.

The condition $(*)$ can be replaced by a condition that is easier to check: let $F: \mathbb{R}^n \to \mathbb{R}$ be a function such that for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $1 \leq |\alpha| = |\alpha_1| + \cdots + |\alpha_n| \leq 5$ and sufficiently small constant $\varepsilon > 0$,

$$\max\{\, |\partial^\alpha F(x_1,\cdots,x_n)| : \max_j |x_j| \le N^{\varepsilon} \} \le N^{C_0 \varepsilon},$$

$$\max\{\, |\partial^\alpha F(x_1,\cdots,x_n)| : \max_j |x_j| \le N^{2} \} \le N^{C_0},$$

for some constant C_0. For any such F, we have

$$(\dagger) \quad |\mathbb{E} F(\operatorname{Im} m^v(z_1) \cdots \operatorname{Im} m^v(z_n)) - \mathbb{E} F(\operatorname{Im} m^w(z_1), \cdots, \operatorname{Im} m^w(z_n))| \le N^{-\delta}.$$

Given $(\dagger)$, we can approximate $\mathbb{E}[\operatorname{Im} m^w(z_1) \cdots \operatorname{Im} m^w(z_n)]$ by $\mathbb{E} F(\operatorname{Im} m^w(z_1), \cdots, \operatorname{Im} m^w(z_n))$ with $F(x_1,\cdots,x_n) = x_1 x_2 \cdots x_n$ if $\max_j |x_j| \le N^c$ and it is smoothly cutoff to 0 in the regime $\max_j |x_j| \ge N^c$ for some small constant $c>0$.

To see this, recall that $\eta \operatorname{Im} m(E+i\eta) \le \eta' \operatorname{Im} m(E+i\eta')$ for $0<\eta\le\eta'$. Then, (i) of Theorem 7.1 implies that for any $\eta \le N^{-1+\varepsilon}$,

$$\mathbb{E}[\eta \operatorname{Im} m(E+i\eta)]^k \le \mathbb{E}[N^{-1+\varepsilon} \operatorname{Im} m(E+iN^{-1+\varepsilon})]^k$$
$$\le (CN^{-1+\varepsilon})^k$$

$$\Rightarrow \quad \mathbb{E}[\operatorname{Im} m(E+i\eta)]^k \le \left(C\frac{N^{\varepsilon}}{N\eta}\right)^k \quad \text{for any fixed } k \in \mathbb{N}.$$

Let $\eta = N^{-1-\delta}$ and $z = E+i\eta$. Consider $n=1$ for simplicity. By choosing $c = 2\delta$,

$$|\mathbb{E} F(\operatorname{Im} m^v(z)) - \mathbb{E} \operatorname{Im} m^v(z)| \le \mathbb{E} \operatorname{Im} m^v(z) \cdot \mathbb{1}(\operatorname{Im} m^v(z) \ge N^c)$$

$$\le N^{-c(k-1)} \mathbb{E}[\operatorname{Im} m^v(z)]^k \le C N^{k(\delta+\varepsilon) - c(k-1)} \le N^{-\delta}$$

as long as ε is small enough and k is large enough. Similar arguments apply to general n.

The conditions on F and $\partial^\alpha F$ will be used in the Green's function comparison argument. The proof of Theorem 7.1 can be found in Theorem 15.3 of Erdős – Yau.

I. The Lindeberg replacement trick

The Green's function comparison uses a classical "Lindeberg replacement trick", which was developed in Lindeberg's own proof of CLT. To explain the basic idea of this argument and the importance of moment matching conditions, we use it to prove the following version of the Berry – Esseen theorem.

<u>Theorem 7.2</u> (Berry – Esséen theorem) Let X be a random variable with mean 0, variance 1, and finite 3rd moment. Let φ be smooth with uniformly bounded ~~3rd order~~ derivatives up to third order. Let $X_1, \cdots, X_n$ be i.i.d. copies of X and

$$Z_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Then, we have that

$$\mathbb{E}\,\varphi(Z_n) = \mathbb{E}\,\varphi(G) + O\left(\frac{1}{\sqrt{n}}\,\mathbb{E}|X|^3 \cdot \|\varphi'''\|_\infty\right),$$

where $G \sim \mathcal{N}(0,1)$ is standard Gaussian.

Proof: Note that G has the same distribution as $\frac{Y_1 + \cdots + Y_n}{\sqrt{n}}$, where Y_i's are independent $\mathcal{N}(0,1)$ random variables. Hence, we only need to bound

$$\mathbb{E}\,\varphi\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right) - \mathbb{E}\,\varphi\left(\frac{Y_1 + \cdots + Y_n}{\sqrt{n}}\right) = \sum_{i=0}^{n-1}\left[\mathbb{E}\,\varphi(Z_{n,i+1}) - \mathbb{E}\,\varphi(Z_{n,i})\right],$$

where $Z_{n,i} = \frac{X_1 + \cdots + X_i + Y_{i+1} + \cdots + Y_n}{\sqrt{n}}$, i.e., we replace $Y_1, \cdots, Y_i$ with $X_1, \cdots, X_i$ in $Z_{n,i}$.

Then, it suffices to show that

$$\mathbb{E}\,\varphi(Z_{n,i+1}) - \mathbb{E}\,\varphi(Z_{n,i}) = O\left(\mathbb{E}|X|^3\,\frac{\|\varphi'''\|_\infty}{n^{3/2}}\right).$$

Note that $Z_{n,i+1}$ is different from $Z_{n,i}$ only by one entry, i.e., a $\frac{Y_{i+1}}{\sqrt{n}}$ is replaced by $\frac{X_{i+1}}{\sqrt{n}}$.

Define

$$S_{n,i} = \frac{X_1 + \cdots + X_i + Y_{i+2} + \cdots + Y_n}{\sqrt{n}}, \qquad \left.\begin{aligned} Z_{n,i+1} &= S_{n,i} + \frac{X_{i+1}}{\sqrt{n}}, \\ Z_{n,i} &= S_{n,i} + \frac{Y_{i+1}}{\sqrt{n}}. \end{aligned}\right\}$$

$S_{n,i}$ is independent of X_{i+1} and Y_{i+1}.

Using Taylor expansion,

$$\varphi(Z_{n,i}) = \varphi(S_{n,i}) + \varphi'(S_{n,i})\frac{Y_{i+1}}{\sqrt{n}} + \frac{1}{2}\varphi''(S_{n,i})\frac{Y_{i+1}^2}{n} + O\left(\frac{|Y_{i+1}|^3}{n^{3/2}}\|\varphi'''\|_\infty\right),$$

$$\varphi(Z_{n,i+1}) = \varphi(S_{n,i}) + \varphi'(S_{n,i})\frac{X_{i+1}}{\sqrt{n}} + \frac{1}{2}\varphi''(S_{n,i})\frac{X_{i+1}^2}{n} + O\left(\frac{|X_{i+1}|^3}{n^{3/2}}\|\varphi'''\|_\infty\right).$$

Since the moments of X_{i+1} and Y_{i+1} match to second order. We get

$$\mathbb{E}\,\varphi(Z_{n,i+1}) - \mathbb{E}\,\varphi(Z_{n,i}) = O\left(\|\varphi'''\|_\infty\,\frac{\mathbb{E}|Y_{i+1}|^3 + \mathbb{E}|X_{i+1}|^3}{n^{3/2}}\right) = O\left(\mathbb{E}|X|^3 \cdot \frac{\|\varphi'''\|_\infty}{n^{3/2}}\right).$$

□

II. Green's function comparison theorem (four-moment theorem)

In Theorem 7.1, (i) follows from the local semicircle law. The condition (ii) follows from the next theorem, which, roughly speaking, shows that two Wigner ensembles have almost identical local statistics provided that their matrix elements have matching "first four moments".

Theorem 7.3 Suppose we have two Wigner matrices H^v and H^w, with matrix elements $h^v_{ij} = \frac{1}{\sqrt{n}} v_{ij}$, $h^w_{ij} = \frac{1}{\sqrt{n}} w_{ij}$. Suppose v_{ij} and w_{ij} have finite moments up to arbitrary order, i.e., $\forall\, k \in \mathbb{N}$,

$$\mathbb{E}|v_{ij}|^k + \mathbb{E}|w_{ij}|^k \le C_k \quad \text{for a constant } C_k > 0.$$

Suppose v_{ij} and w_{ij} have matching first three moments, i.e.,

$$\mathbb{E}\,\bar{v}_{ij}^{\,s}\, v_{ij}^{k-s} = \mathbb{E}\,\bar{w}_{ij}^{\,s}\, w_{ij}^{k-s}, \quad k = 1, 2, 3, \quad s = 0, 1, \cdots, k,$$

and almost identical fourth moments:

$$\left|\mathbb{E}\,\bar{v}_{ij}^{\,s}\, v_{ij}^{4-s} - \mathbb{E}\,\bar{w}_{ij}^{\,s}\, w_{ij}^{4-s}\right| \lesssim N^{-\delta}, \quad s = 0, 1, 2, 3, 4,$$

for some constant $\delta > 0$. Suppose the local semicircle laws hold for H^v and H^w.

Let F be a function as given below Theorem 7.1. Consider any sequence $z_j = E_j \pm i\eta_j$, $j=1,\cdots,n$, with $|E_j| \le 2-\kappa$, $\eta_j = N^{-1-\sigma_j}$ for $\sigma_j \le \sigma$ and an arbitrary choice of the $\pm$ signs. Then, there exists a constant $C_1>0$ that does not depend on σ such that

$$\left| \mathbb{E} F(m^v(z_1), \cdots, m^v(z_n)) - \mathbb{E} F(m^w(z_1), \cdots, m^w(z_n)) \right| \le N^{-\frac{1}{2}+C_1\sigma} + N^{-\delta+C_1\sigma}.$$

Combining this theorem with Theorem 7.1, we get that

$$\lim_{N\to+\infty} \int_{\mathbb{R}^n} O(\vec{\alpha}) \left(p^{(n)}_{v,N} - p^{(n)}_{w,N} \right) \left(E + \frac{\vec{\alpha}}{N} \right) d\vec{\alpha} = 0, \quad \forall\, O \in C_c^\infty(\mathbb{R}^n) : \mathbb{R}^n \to \mathbb{R}.$$

Proof of Theorem 7.3: We use the Lindeberg replacement trick, i.e., we replace the matrix elements of H^v one by one and estimate the error of each step using a resolvent expansion.
Note the each matrix element has a typical size $N^{-1/2}$ and ~~all~~ resolvents can be controlled using the local semicircle law. Hence, a resolvent expansion up to the fourth order will identify the change of each resolvent element ~~by an error $O(N^{-5/2}+N^{-2-\delta})$~~ with precision $O(N^{-5/2})$.
The expectation values of the terms up to 4th order involve only the first four moments of v_{ij}'s and w_{ij}'s, which can be compared using the four-moment matching conditions.
This leads to errors $O(N^{-5/2} + N^{-2-\delta})$, which are negligible after summing over N^2 times.

For simplicity of presentation, to illustrate the main idea, we consider the real case ~~and the case~~ with $n=1$. The general case follows analogously. We need to control

$$(*) \quad \mathbb{E} F(m^v(z)) - \mathbb{E} F(m^w(z)), \quad z = E \pm i\eta, \quad \eta = N^{-1-\sigma}.$$

We fix a bijective ordering map on the index set of independent entries:

$$\phi : \{(i,j) : 1 \le i \le j \le N\} \to \{1, \cdots, a(N)\}, \quad a(N) = \frac{N(N+1)}{2}.$$

Fix any $1 \le \gamma \le a(N)$. Let H_γ be the matrix whose entries are h^v_{ij} if $\phi(i,j) \le \gamma$ and h^w_{ij} if $\phi(i,j) > \gamma$. In particular, we have

$$H_{a(N)} = H^v \quad \text{and} \quad H_0 = H^w.$$

Thus, we can write $(*)$ as a telescoping sum

$$\mathbb{E} F\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H^v - z} \right) - \mathbb{E} F\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H^w - z} \right) = \sum_{\gamma=1}^{a(N)} \left[\mathbb{E} F\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H_\gamma - z} \right) - \mathbb{E} F\left(\frac{1}{N} \mathrm{Tr} \frac{1}{H_{\gamma-1} - z} \right) \right]$$

We need to estimate each summand. Suppose $(i,j) = \phi^{-1}(\gamma)$. Let $E^{(ij)}$ be the matrix

$$E^{(ij)} = e_i e_j^T + e_j e_i^T,$$

i.e., $E^{(ij)}$ is the matrix whose elements are zero everywhere except at the (i,j) and (j,i) positions, where it is 1. Then, we can write

$$H_{\gamma-1} = Q + \frac{1}{\sqrt{N}} w_{ij} E^{(ij)}, \quad H_\gamma = Q + \frac{1}{\sqrt{N}} v_{ij} E^{(ij)},$$

where is the matrix obtained by setting the (i,j) and (j,i)-th entries of H_γ and $H_{\gamma-1}$ to 0.

Note that Q is independent of w_{ij} and v_{ij}.

Define the resolvents $R := \frac{1}{Q - z}$. We abbreviate: $S := \frac{1}{H_\gamma - z}$ and $T := \frac{1}{H_{\gamma-1} - z}$.

(1) Resolvent estimates for R, S, T

By local semicircle law, we have $\max_{0 \le \gamma \le a(N)} \max_{1 \le k \le N} \sup_{|E| \le 2-\kappa} \sup_{\eta \ge N^{-1+\varepsilon}} \left| \left(\frac{1}{H_\gamma - E - i\eta} \right)_{kk} \right| \prec 1,$

for any small constants $\kappa > 0$ and $\varepsilon > 0$. Again, using

$$\eta \, \mathrm{Im} \left(\frac{1}{H_\gamma - E - i\eta} \right) \le \eta' \, \mathrm{Im} \left(\frac{1}{H_\gamma - E - i\eta'} \right), \quad 0 < \eta \le \eta',$$

we obtain that

$$\max_{1 \le k \le N} \sup_{|E| \le 2-\kappa} \sup_{\eta \ge N^{-1-\delta}} \left| \mathrm{Im} \left(\frac{1}{H_\gamma - E \pm i\eta} \right)_{kk} \right| \prec N^{\delta + \varepsilon}.$$

From this estimate, we derive the following result.

Lemma: For any $\eta \ge N^{-1-\varepsilon}$, we have

$$(*) \quad \max_{\gamma} \max_{1 \le k, l \le N} \sup_{|E| \le 2-\kappa} | G^\gamma_{kl} (E \pm i\eta) | \prec N^{\delta + \varepsilon}, \quad G^\gamma(z) := \frac{1}{H_\gamma - z}.$$

Proof: With the eigendecomposition of G^γ, for $z = E + i\eta$,

$$|G^\gamma_{kl}| \le \sum_{j=1}^{N} \frac{|u_j(k)| \, |u_j(l)|}{|\lambda_j - z|} \le \left[\sum_{j=1}^{N} \frac{|u_j(k)|^2}{|\lambda_j - z|} \right]^{1/2} \left[\sum_{j=1}^{N} \frac{|u_j(l)|^2}{|\lambda_j - z|} \right]^{1/2}.$$

Consider a dyadic decomposition
$$U_n = \{ j : 2^{n-1} \eta \le |\lambda_j - E| \le 2^n \eta \}, \quad n = 1, 2, \cdots, C \log N,$$
$$U_0 = \{ j : |\lambda_j - E| \le \eta \},$$
$$U_\infty = \{ j : |\lambda_j - E| \ge 2^{C \log N} \eta \}. \quad (\text{Take } C \text{ large s.t. } 2^{C \log N} \eta \ge 1.)$$

Then, we divide the summation over j into $\bigcup_n U_n$:

$$\sum_{j=1}^{N} \frac{|u_j(k)|^2}{|\lambda_j - z|} = \sum_n \sum_{j \in U_n} \frac{|u_j(k)|^2}{[|\lambda_j - E|^2 + \eta^2]^{1/2}} \le C \sum_{n < +\infty} \sum_{j \in U_n} \frac{2^n \eta \, |u_j(k)|^2}{|\lambda_j - E|^2 + (2^n \eta)^2}$$
$$+ C \sum_{j \in U_\infty} |u_j(k)|^2$$

$$\le C \sum_{n < +\infty} \sum_{j \in U_n} \mathrm{Im} \left(\frac{|u_j(k)|^2}{\lambda_j - E - i 2^n \eta} \right) + C \le C \sum_{n=0}^{C \log N} \mathrm{Im} \, G^\gamma_{kk} (E + i 2^n \eta) + C$$
$$\prec N^{\delta + \varepsilon}.$$

□

Next, we claim that the above estimate also holds for $R(z)$:

$$(+) \quad \max_{1 \le k, l \le N} \sup_{|E| \le 2-\kappa} | R_{kl} (E \pm i\eta) | \prec N^{\delta + \varepsilon}.$$

Let $V = v_{ij} E^{(ij)}$. Then we have the resolvent expansion: $\forall k \in \mathbb{N}$,

$$R = (Q - z)^{-1} = \left(H_\gamma - \tfrac{1}{\sqrt{N}} V - z \right)^{-1} = \left(S^{-1} - \tfrac{1}{\sqrt{N}} V \right)^{-1}$$

$$= S + \frac{1}{\sqrt{N}} SVS + \frac{1}{N}(SV)^2 S + \cdots + \frac{1}{N^{k/2}}(SV)^k S + \frac{1}{N^{(k+1)/2}}(SV)^k R.$$

Using the above estimate $(*)$ for S and the trivial bound $\|R\| \le \eta^{-1}$, taking $k=5$ in the above expansion, we can obtain $(+)$ for R.

(2) The Green's function comparison

We now expand $S = (H_\gamma - z)^{-1}$ using R:

$$S(z) = (R^{-1} + \frac{1}{\sqrt{N}} V)^{-1} = R - N^{-\frac{1}{2}} RVR + N^{-1}(RV)^2 R - N^{-3/2}(RV)^3 R + N^{-2}(RV)^4 R - N^{-\frac{5}{2}}(RV)^5 S.$$

Taking trace, we get

$$\frac{1}{N}\mathrm{Tr}\, S(z) = \underline{R} + \sum_{k=1}^{4} (-1)^k N^{-\frac{k}{2}} \underline{(RV)^k R} + N^{-\frac{5}{2}} \Omega_V, \quad \text{where } \underline{M} := \frac{1}{N}\mathrm{Tr}\, M \text{ for any } N\times N \text{ matrix}$$

and $\Omega_V = -\underline{(RV)^5 S}$.

$$=: \underline{R} + \zeta.$$

Then, performing Taylor expansion, we get that

$$\mathbb{E} F(\frac{1}{N}\mathrm{Tr}\, S(z)) = \mathbb{E} F(\underline{R} + \zeta) = \mathbb{E}[F(\underline{R}) + F'(\underline{R})\zeta + \frac{1}{2} F''(\underline{R})\zeta^2 + \cdots + \frac{1}{5!} F^{(5)}(\underline{R} + \zeta')\zeta^5]$$

where ζ' is a value between 0 and ζ, and it ~~depedi~~ depends on F, $\underline{R}$ and ζ. We organize these terms according to the number of $N^{-\frac{1}{2}}$ factors:

$$\mathbb{E}F(\frac{1}{N}\mathrm{Tr}\, S(z)) = \sum_{m=0}^{5} N^{-\frac{m}{2}} \mathbb{E} A_V^{(m)},$$

where $A_V^{(0)} = F(\underline{R})$, $A_V^{(1)} = F'(\underline{R})(-\underline{RVR})$, $A_V^{(2)} = \frac{1}{2} F''(\underline{R})(\underline{RVR})^2 + F'(\underline{R})\underline{(RV)^2 R}$, ~~and~~ $A_V^{(3)}$ and $A_V^{(4)}$ are third and fourth order terms that can be defined in a similar way. Finally, $A_V^{(5)}$ ~~are terms~~ contains terms of order at least 5:

$$A_V^{(5)} = F'(\underline{R})\Omega_V + \frac{1}{5!} F^{(5)}(\underline{R}+\zeta')(R_V^{(1)})^5 + \cdots.$$

Similarly, we can get that for $T(z) = (H_{\gamma-1} - z)^{-1}$,

$$\mathbb{E} F(\frac{1}{N}\mathrm{Tr}\, T(z)) = \sum_{m=0}^{5} N^{-\frac{m}{2}} \mathbb{E} A_W^{(m)},$$ where $A_W^{(m)}$ are obtained by replacing all V in $A_V^{(m)}$ by $W = w_{ij} E^{(ij)}$.

Since $\underline{R}$ is independent of V and W, we notice that $\mathbb{E}A_V^{(m)}$, $m=0,1,2,3,4$, are determined by the first m moments of v_{ij}, and similarly for $\mathbb{E}A_W^{(m)}$. Using the moment matching condition, we get that

$$\mathbb{E} A_V^{(m)} = \mathbb{E} A_W^{(m)}, \quad \text{for } m = 0,1,2,3.$$

~~And For~~ For $m=4$, using the estimate $|R_{k\ell}| < N^{\delta+\varepsilon}$ and the moment matching condition, we get

$$|\mathbb{E}A_V^{(4)} - \mathbb{E}A_W^{(4)}| < N^{C(\delta+\varepsilon)-\delta}, \quad \text{for an absolute constant } C>0.$$

(We can choose δ and ε small enough such that $C(\delta+\varepsilon) < \delta$.)

Finally, for $A_V^{(5)}$ ~~and $A_W^{(5)}$~~, all terms without Ω can be handled in a ~~similary~~ similar way. The terms containing Ω can be bounded using the high moment condition on v_{ij}, the estimates on S and R, and the condition on the derivative of F: for example,

$$\left|\mathbb{E}[F'(\underline{R})\Omega]\right| \le \frac{1}{N}\mathbb{E}\left|F'(\underline{R})\,\mathrm{Tr}(RV)^5 S\right| \le N^{C_0(\sigma+\varepsilon)} \max_i \mathbb{E}\left|[(RV)^5 S]_{ii}\right| \le N^{C(\sigma+\varepsilon)}.$$

In sum, we get that

$$\left|\mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_\gamma - z}\right) - \mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H_{\gamma-1}-z}\right)\right| \le N^{C(\sigma+\varepsilon)-\delta-2} + N^{-\frac{5}{2}}\left(|\mathbb{E}A_v^{(5)}| + |\mathbb{E}A_w^{(5)}|\right)$$
$$\le N^{-2-\delta+C(\sigma+\varepsilon)} + N^{-\frac{5}{2}+C(\sigma+\varepsilon)}.$$

Taking $\sigma = \varepsilon$ and summing over γ, we get

$$\left|\mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^v - z}\right) - \mathbb{E}F\left(\frac{1}{N}\mathrm{Tr}\frac{1}{H^w - z}\right)\right| \le N^{-\delta + C\sigma} + N^{-\frac{1}{2}+C\sigma}.$$

The general case can be proved in a similar way. □

III. Conclusion of the three step strategy.

To show the bulk universality of a Wigner matrix H, we want to find a Gaussian divisible ensemble with a sufficiently large Gaussian component that approximates H in the sense of four-moment matching.

Given a real random variable ξ, denote its moments by $m_k = \mathbb{E}\xi^k$. Suppose $m_1 = 0$ and $m_2 = 1$.
Then: $m_3^2 = (\mathbb{E}\xi^3)^2 = [\mathbb{E}\xi(\xi^2-1)]^2 \le (\mathbb{E}\xi^2)[\mathbb{E}(\xi^2-1)^2] = m_2(m_4 - 2m_2 + 1)$
$= m_4 - 1.$

In particular, the "=" holds only if the support of ξ consists of only two points.
Hence, given a random variable (e.g., Bernoulli), we may not find a Gaussian divisible ~~ensemble~~ random variable with matching first 4 moments. But, this can be done approximately if we allow for some room in the fourth moment.

Lemma. Let m_3 and m_4 be two real numbers s.t.
$$m_4 - m_3^2 - 1 \ge 0 \quad \text{and} \quad m_4 \le C_1.$$
Let $g \sim N(0,1)$ be a standard Gaussian. For any small $\gamma > 0$, there exists a real random variable ξ_γ with finite moments up to arbitrary high order and independent of g, such that $\xi = (1-\gamma)^{1/2}\xi_\gamma + \gamma^{1/2} g$ has first four moments $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$, $\mathbb{E}\xi^3 = m_3$ and $\mathbb{E}\xi^4 = m_4'$ s.t.
$$|m_4' - m_4| \le C\gamma \quad \text{for some constant } C > 0.$$

Pf: First, it is easy to construct a random variable X with first four moments given by $0, 1, m_3$ and m_4. We take the law of X as
$$\mathbb{P}(X = a) = p, \quad \mathbb{P}(X = -b) = q, \quad \mathbb{P}(X = 0) = 1 - p - q,$$
where $p = \frac{1}{a(a+b)}$, $q = \frac{1}{b(a+b)}$. (Note the condition $p + q \le 1$ implies $ab \ge 1$.)

Then, we can calculate that

$$m_3 = \mathbb{E}X^3 = a-b, \qquad m_4 = \mathbb{E}X^4 = a^2 - ab + b^2 = m_3^2 + ab.$$

It is easy to check that $m_4 - m_3^2 - 1 \geq 0$ implies that the above system of equations has a solution.

For any random variable $\xi_\gamma \perp\!\!\!\perp g$, with first four moments $0, 1, m_3(\xi_\gamma)$ and $m_4(\xi_\gamma)$, the first four moments of ξ is given by $0, 1,$ and

$$m_3(\xi) = (1-\gamma)^{3/2} m_3(\xi_\gamma), \quad m_4(\xi) = (1-\gamma)^2 m_4(\xi_\gamma) + 6\gamma - 3\gamma^2.$$

By the previous result, we can choose ξ_γ s.t. $m_3(\xi_\gamma) = (1-\gamma)^{-\frac{3}{2}} m_3$ and $m_4(\xi_\gamma) = m_3(\xi_\gamma)^2 + (m_4 - m_3^2)$.
Then, we can check that $m_3(\xi) = m_3$ and

$$m_4(\xi) = (1-\gamma)^2 [(1-\gamma)^{-3} m_3^2 + m_4 - m_3^2] + 6\gamma - 3\gamma^2 = m_4 + O(\gamma).$$

$\square$

Given H, by the above lemma, we can construct a Gaussian divisible encemble $H_t = e^{-t/2} H_0 + (1-e^{-t})^{1/2}\mathrm{GOE}$ such that $\mathbb{E}(H_t)_{ij} = 0$, $\mathbb{E}(H_t)_{ij}^2 = 1$, $\mathbb{E}(H_t)_{ij}^3 = \mathbb{E}H_{ij}^3$, and $|\mathbb{E}(H_t)_{ij}^4 - \mathbb{E}(H_{ij})^4| \leq Ct N^{-2}$.
~~We only~~ In fact, given the first four moments $0, 1, m_3$ and m_4 of the entries $\sqrt{N}h_{ij}$, we choose as the law of $\sqrt{N}(H_0)_{ij}$ as ξ_γ for $\gamma = 1 - e^{-t}$ as constructed in the above lemma.

Then, from the bulk universality of H_t, we get that for $t \geq N^{-\varepsilon}$ and $b = N^{-1+10\varepsilon}$,

$$\frac{1}{2b}\int_{E-b}^{E+b} dE' \int_{\mathbb{R}^n} d\vec{\alpha}\, O(\vec{\alpha})\left(p_{t,N}^{(n)} - p_{G,N}^{(n)}\right)\left(E' + \frac{\vec{\alpha}}{N}\right) \to 0.$$

Finally, using the Green function comparison result, we get the bulk universality for H.