Section 3 Invariant Ensembles

Consider the density function of an invariant ensemble as

$$\mathbb{P}(H)\,dH = \frac{1}{Z}\exp\left(-\frac{\beta}{2}N\,\mathrm{Tr}\,V(H)\right)dH$$

For GOE / GEU, $V(H) = \frac{1}{2}H^2$. In this case, this model is called a β-ensemble

$\beta=1$ for GOE ; $\beta=2$ for GUE ; $\beta=4$ for GSE.
symplectic
(Gaussian quaternion ensemble)

For general β, it does not correspond to any Wigner ~~sym~~ ensembles.

① Probability density for β-ensemble

For invariant ensembles, their eigenvectors are uniformly distributed on the unit sphere. So people mainly focus on their eigenvalues. We will integrate out the eigenvectors in $\mathbb{P}(H)\,dH$ and show the following result.

Theorem 3.1 The joint probability density of the eigenvalues of H is given by

$$(*)\quad p_N(\lambda_1,\lambda_2,\cdots,\lambda_N) = \frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^{\beta}\, e^{-\frac{\beta}{2}N\sum_{i=1}^{N}V(\lambda_i)}.$$

Rmk1: Recall that $\Delta(\lambda) = \prod_{i<j}(\lambda_i-\lambda_j)$ is the Vandermonde determinant.

Rmk2: In the above formula, we can neglect the ordering of the eigenvalues. This can be done by multiplying a $\frac{1}{n!}$ factor when calculating various probabilities.

~~Rmk3: We can rewrite (*) as:~~ $\cancel{p_N(\lambda_1,\lambda_2,\cdots,\lambda_N) = \frac{1}{Z_N}\exp[\beta\log|\lambda_i-\lambda_j| - \frac{\beta}{2}N}$

Rmk3: When $|\lambda_i - \lambda_j| = 0$, we have $p_N(\lambda_1,\cdots,\lambda_N)=0$. This indicates a repulsion between different eigenvalues.

Rmk 4: The eigenvalues are strongly correlated. It is useful to think of p_N as a Gibbs measure on N "particles" $\vec{\lambda} = (\lambda_1,\cdots,\lambda_N)$:

$$p_N(\vec{\lambda})\,d\vec{\lambda} = \exp[-\beta N\mathcal{H}(\vec{\lambda})]\frac{1}{Z_N},$$

$$\mathcal{H}(\vec{\lambda}) := \underbrace{\frac{1}{2}\sum_{i=1}^{N}V(\lambda_i)}_{\text{A confining potential.}} - \underbrace{\frac{1}{N}\sum_{i<j}\log|\lambda_i-\lambda_j|}_{\text{A logarithmic potential giving repulsions between particles.}}$$

[The two terms have the same order "N".]

This gives an important statistical mechanical system, called the "log-Coulomb gas".

To prove ~~the~~ Thm 3.1, we need to show that integrating out the eigenvectors gives the Vandermonde determinant.

Let $H = UDU^*$ be an eigenvalue decomposition of H, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$ with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and U is an orthogonal / ~~unitary~~ unitary matrix.
We consider the unitary case. Then U is Haar distributed on $U(N)$.

From the unitary invariance of $\mathbb{P}(H)dH$, we see that after conditioning on D, the eigenmatrix U is drawn from the Haar measure on $U(n)$. In particular, U and D can be taken to be independent:

$$\cancel{\mathbb{P}(H = U_0 D_0 U_0^* + dH) = \mathbb{P}(H = (VU_0) D_0 (U_0^* V^*) + dH)}$$

~~for any $V \in U(N)$~~

$$\mathbb{P}(U \in B,\ D \in A) = \mathbb{P}(U \in B \mid D \in A)\,\mathbb{P}(D \in A) = \mathbb{P}(U \in B)\,\mathbb{P}(D \in A).$$

<u>Proof of Theorem 3.1</u>: Fix a diagonal matrix $D_0 = \mathrm{diag}(\lambda_1^0, \cdots, \lambda_N^0)$, $\lambda_1^0 \le \cdots \le \lambda_N^0$.
Let $\varepsilon > 0$ be arbitrarily small. We now compute the probability that H_N lies in an ε-ball around D_0 in the H-S norm (Frobenius norm):

$$\mathbb{P}(\|H_N - D_0\|_F \le \varepsilon)$$

(The Euclidean norm on the space of $n \times n$ matrices.)

I. The simpler direction: The probability density of M_n is $\propto \exp(-\frac{\beta}{2} N \sum_i V(\lambda_i^0))$
The volume of the ε-ball is $C_N \varepsilon^{N^2}$. So

$$(1) \quad \mathbb{P}(\|H_N - D_0\|_F \le \varepsilon) = (C_N + \underbrace{o(1)}_{o_\varepsilon(1) \to 0 \text{ as } \varepsilon \to 0 \text{ when } N \text{ is fixed.}})\,\varepsilon^{N^2} \exp(-\tfrac{\beta}{2} N \textstyle\sum_i V(\lambda_i))$$

II. The harder direction: Let's calculate $\mathbb{P}(\|H_N - D_0\|_F \le \varepsilon)$ in a different way.

Weyl's inequality: Given two hermitian matrices A and B with eigenvalues

$$\lambda_1(A) \le \cdots \le \lambda_N(A), \quad \lambda_1(B) \le \cdots \le \lambda_N(B).$$

Then, $\forall i$, $\lambda_i(A) + \lambda_1(B) \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_N(B)$.
In particular, if B is a small perturbation with $\|B\| \le \varepsilon$, then

$$\lambda_i(A) - \varepsilon \le \lambda_i(A+B) \le \lambda_i(A) + \varepsilon.$$

Thus, for $H_N = UDU^*$, we have $\|D - D_0\| \le \varepsilon$ when $\|H_N - D_0\|_F \le \varepsilon$.

~~Moreover,~~ $\cancel{UDU^* = D_0 + O(\varepsilon) = D + O(\varepsilon) \Rightarrow UD - DU = O(\varepsilon)}$.

~~Now, we make the ansatz~~

$$\cancel{D = D_0 + \varepsilon E, \quad U = \exp(\varepsilon S)}$$

~~where E is a diagonal matrix and S is a skew-adjoint matrix. with zero diagonals~~

~~[Note that $UU^* = \exp(\varepsilon S)\exp(\varepsilon S^*) = \exp(\varepsilon S)\exp(-\varepsilon S) = I$]~~

~~Note that $S \mapsto \exp(\varepsilon S)$ has a non-degenerate Jacobian in the small ε-ball, so the inverse function thm applies to specify S from U. (Formally, $S = \frac{1}{\varepsilon}\log U$.)~~

~~WLOG, suppose all λ_i^0 are different (othe this holds a.s.).~~

Thus, we make the ansatz: $D = D_0 + \varepsilon E$, E is diagonal bounded.

Note, the eigenvalues of D_0 are non-zero and non-degenerate almost surely. Moreover, we have $\quad UDU^* = D_0 + O(\varepsilon) \Rightarrow UD_0U^* = D_0 + O(\varepsilon) \Rightarrow UD_0 - D_0U = O(\varepsilon)$. $(*)$

The only unitary matrices that commute with D_0 are diagonal unitary matrices

$$R = (r_1, r_2, \cdots, r_N), \quad |r_i|^2 = 1.$$

From $(*)$, we can make the ansatz: $U = \exp(\varepsilon S) R$,

Where S is a bounded skew-adjoint ($S^* = -S$) matrix with zero diagonal. (The diagonal parts can be included into R.)

We can check that $\quad UU^* = \exp(\varepsilon S)\exp(\varepsilon S^*) = 1$,

$UD_0 - D_0U = \exp(\varepsilon S) R D_0 - D_0 \exp(\varepsilon S) R = RD_0 - D_0R + O(\varepsilon) = O(\varepsilon)$.

It is easy to check that $(R, S) \mapsto \exp(\varepsilon S) R$ has a non-degenerate Jacobian, so with the inverse function thm, we can uniquely determine R and S from U in the ~~sam~~ small ε-ball around D_0.

Now with the ansatz: $D = D_0 + \varepsilon E$, $\quad U = \exp(\varepsilon S) R$, we get

$$\begin{aligned} H_N = UDU^* &= \exp(\varepsilon S) R (D_0 + \varepsilon E) R^* \exp(-\varepsilon S) \\ &= D_0 + \varepsilon E + \varepsilon S R D_0 R^* - \varepsilon R D_0 R^* S + O(\varepsilon^2) \\ &= D_0 + \varepsilon E + \varepsilon(SD_0 - D_0 S) + O(\varepsilon^2). \end{aligned}$$

Thus, $\quad \mathbb{P}(\|H_N - D_0\|_F \le \varepsilon) = \mathbb{P}(\|E + (SD_0 - D_0S)\| \le 1 + O(\varepsilon))$. $(+)$

Since U is Haar distributed on $U(N)$, S is locally distributed as $C_N' \varepsilon^{N^2 - N}$ times the Lebesgue measure on the space of skew-adjoint matrices with 0 diagonal. On the other hand, E is distributed as $(\rho_N(\vec{\lambda}^0) + o(1))\varepsilon^N$ times the Lebesgue measure on the space of diagonal matrices. Thus we can calculate $(+)$ as:

$$C_N' \varepsilon^{N^2} (\rho_N(\vec{\lambda}^0) + o(1)) \iint_{\|E + SD_0 - D_0S\|_F \le 1 + O(\varepsilon)} dE\, dS$$

Consider the map $S \mapsto SD_0 - D_0S$, $\quad$ where $(SD_0 - D_0S)_{ij} = (\lambda_j^0 - \lambda_i^0) S_{ij}$.

In other words, the map dilates the (i,j)th entry of S by $\lambda_j^0 - \lambda_i^0$. Hence, the Jacobian of this map is $\quad \prod_{i \ne j} |\lambda_j^0 - \lambda_i^0| = \prod_{1 \le i < j \le N} |\lambda_j^0 - \lambda_i^0|^2$.

Applying the change of variable, we get

$$\begin{aligned} \iint_{\|E + SD_0 - D_0S\|_F \le 1 + O(\varepsilon)} dE\, dS &= \frac{1}{\prod_{1 \le i < j \le N} |\lambda_j^0 - \lambda_i^0|^2} \iint_{\|E + S\|_F \le 1 + O(\varepsilon)} dE\, dS \\ &= \frac{C_N''}{\prod_{1 \le i < j \le N} |\lambda_j^0 - \lambda_i^0|^2}. \end{aligned}$$

In sum, we have found that

$$\mathbb{P}(\|H_N - D_0\|_F \le \varepsilon) = (C_N + o(1))\,\varepsilon^{N^2} \exp\left[-\frac{\beta}{2} N \sum_i V(\lambda_i^0)\right]$$

$$= \frac{C_N'''}{\prod_{1\le i<j\le N} |\lambda_j^0 - \lambda_i^0|^2} \left[\rho_N(\vec{\lambda}^0) + o(1)\right] \varepsilon^{N^2}$$

$$\Rightarrow \rho_N(\vec{\lambda}^0) = \frac{1}{Z_N} \prod_{1\le i<j\le N} |\lambda_j^0 - \lambda_i^0|^2 \exp\left[-\frac{\beta}{2} N \sum_{i=1}^N V(\lambda_i^0)\right]. \qquad \square$$

② Mean field approximation of semicircle law

$$\rho_N(\vec{\lambda}) = \frac{1}{Z_N} \exp\left[-\beta N \mathcal{H}(\vec{\lambda})\right], \quad \mathcal{H}(\vec{\lambda}) = \frac{1}{2}\sum_{i=1}^N V(\lambda_i) - \frac{1}{N}\sum_{i<j} \log|\lambda_i - \lambda_j|.$$

Intuitively, it is plausible that the spectum should concentrate around the maximum of $\rho_N(\vec{\lambda})$ as $N \to +\infty$. It is equivalent to study the minimum of $\mathcal{H}(\vec{\lambda})$.

Heuristically, we make a mean-field approximation, i.e., the ESD $\frac{1}{N}\sum_i \delta_{\lambda_i}$ can be approximated by a continuous probability measure $\rho(x)\,dx$ ($\to$ this is reasonable for GOE/GUE at least with $\rho = \rho_{sc}$.)

Then, we expect that $\mathcal{H}(\vec{\lambda})$ is approximately given by

$$\cancel{\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\,dx}$$

$$\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\rho(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)\rho(y) \log\frac{1}{|x-y|}\,dx\,dy.$$

ρ should minimize this functional. Now, we derive the Euler-Lagrange equation for $\mathcal{H}(\rho)$: consider $\rho + \delta\rho$, where ρ is the minimizer. We should have

$$\frac{\delta \mathcal{H}(\rho)}{\delta\rho} = 0, \quad \text{because } \mathcal{H}(\rho+\delta\rho) \ge \mathcal{H}(\rho) \ \forall\, \delta\rho.$$

We have
$$\delta\mathcal{H}(\rho) = \frac{1}{2}\int_{\mathbb{R}} V(x)\,\delta\rho(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} (\delta\rho(x))\rho(y) \log\frac{1}{|x-y|}\,dx\,dy$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x)(\delta\rho(y)) \log\frac{1}{|x-y|}\,dx\,dy$$

$$\Rightarrow \frac{1}{2}\int_{\mathbb{R}} V(x)\,\delta\rho(x)\,dx + 2\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \rho(y)\log\frac{1}{|x-y|}\,dy\right)\delta\rho(x)\,dx = 0$$

Since $\delta\rho$ is arbitrary, we ought to have $\frac{1}{2}V(x) + 2\int_{\mathbb{R}} \rho(y)\log\frac{1}{|x-y|}\,dy = C$.
with $\int_{\mathbb{R}} \delta\rho(x)\,dx = 0$ (since $\rho + \delta\rho$ should be a prob. density).

This is the E-L equation for ρ.

Solving the E-L equation, we get a density $\rho(x)$, which should be the limiting ESD of $\mathbb{P}(H)\,dH = \frac{1}{Z_N}\exp[-\frac{\beta}{2}N\,\mathrm{Tr}\,V(H)]\,dH$.

Let's solve it for GUE with $V(x)=x^2$. We need to solve

$$\frac{1}{2}x^2 + 2\int_{\mathbb{R}} \rho(y)\log\frac{1}{|x-y|}\,dy = C \Rightarrow x + 2\int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy = 0 \quad (*)$$

Here, $\int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy$ ~~should~~ should be formally understood as a principal value

$$\text{p.v.}\int_{\mathbb{R}} \frac{\rho(y)}{y-x}\,dy := \lim_{\varepsilon\to 0}\int_{|y-x|\geq\varepsilon}\frac{\rho(y)}{x-y}\,dy = \lim_{\eta\to 0^+} \mathrm{Re}\int_{\mathbb{R}}\frac{\rho(y)}{y-(x+i\eta)}\,dy$$

$$= \lim_{\eta\to 0^+}\mathrm{Re}\, S_\rho(x+i\eta) =: \mathrm{Re}\,S_\rho(x+i0^+)$$

On the other hand, recall $\rho(x) = \frac{1}{\pi}\,\mathrm{Im}\,S_\rho(x+i0^+)$.
Then the equation $(*)$ reads: $\quad x + 2\,\mathrm{Re}\,S_\rho(x+i0^+) = 0$

$$\Rightarrow x\,\mathrm{Im}\,S_\rho(x+i0^+) + 2[\mathrm{Re}\,S_\rho(x+i0^+)][\mathrm{Im}\,S_\rho(x+i0^+)] = 0$$

$$\Rightarrow \mathrm{Im}[x S_\rho(x+i0^+) + S_\rho^2(x+i0^+)] = 0$$

Define function $f(z) = zS_\rho(z) + S_\rho^2(z)$. The above function means we can make an analytic continuation from $z\in\mathbb{C}_+ \to \mathbb{C}$ by making a reflection along $\mathbb{R}$:

$$f(z) = \overline{f(\bar z)} \quad \text{for } z\in\mathbb{C}_-.$$

On the other hand, since $S(z) = \frac{-(1+o(1))}{z}$ as $z\to\infty$, $f\to -1$ as $z\to+\infty$.
By Liouville's theorem, f is a constant and

$$f(z) \equiv \lim_{z\to+\infty}[zS_\rho(z) + S_\rho^2(z)] = -1$$

This gives the familiar self-consistent equation:

$$S_\rho^2(z) + zS_\rho(z) + 1 = 0 \Rightarrow S_\rho(z) = m_{sc}(z) \Rightarrow \rho = \rho_{sc}.$$

The above arguments can be made rigorous for general β ensembles as long as we impose some conditions on V (in particular, need certain condition on its growth at ∞).

③ Bulk eigenvalue distribution of GUE and orthogonal polynomials

Recall that the Vandermont determinant $\Delta(\vec\lambda)$ can be expressed (up to a $\pm$ sign) as a determinant of an $n\times n$ matrix

$$M = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

[We take $n\equiv N$ in this section.]

Then, $|\Delta_n(\vec\lambda)|^2$ can be expressed as $\det(MM^*)$, where

$$(MM^*)_{ij} = \sum_{k=0}^{n-1}\lambda_i^k\lambda_j^k, \quad 1\leq i\leq j\leq n.$$

Through ~~row~~ column operations of M, we observe the following fact: if $P_0(\lambda), P_1(\lambda), \cdots, P_{n-1}(\lambda)$ are any sequence of polynomials, in which $P_i(x)$ has degree i and the degree i term is exactly x^i, then $\Delta(\lambda_1, \cdots, \lambda_n)$ is equal to the det of the matrix $(P_{j-1}(\lambda_i))_{1\le i,j\le n}$.
Hence, $|\Delta(\vec{\lambda})|^2$ is equal to the det of the following matrix

$$\left(\sum_{k=0}^{n-1} P_k(\lambda_i) P_k(\lambda_j)\right)_{1\le i,j\le n}$$

Then, we rewrite the prob. density for the eigenvalues of GUE as

We resort to the original scaling with $\mathbb{E}|h_{ij}|^2=1$

$$p(\vec{\lambda}) \propto \det\left(\sum_{k=0}^{n-1} P_k(\lambda_i) e^{-\frac{\lambda_i^2}{4}} P_k(\lambda_j) e^{-\frac{\lambda_j^2}{4}}\right)_{1\le i,j\le n}.$$

One particular class of polynomials $\{P_i(\lambda)\}$ is of particular importance to us, that is, the "orthogonal polynomials" with respect to the Gaussian measure.
These are the famous "Hermite polynomials". Define the inner product

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x) g(x) e^{-\frac{x^2}{2}} dx.$$

Let $P_0(x) = 1$. Then we define $P_i(x)$ through the Gram-Schmidt process, i.e., given $P_0(x), P_1(x), \cdots, P_{n-1}(x)$ we define $P_n(x)$ (of degree n ~~and has degree n term as x^n~~) such that $\langle P_n, P_i\rangle = 0$ for $0 \le i \le n-1$. Furthermore, we can normalize $P_n(x)$ so that $\langle P_n, P_n\rangle = 1$. Then we get a sequence of orthonormal polynomials with respect to the Gaussian measure: $\langle P_i, P_j\rangle = \delta_{ij}$.
Then, define the kernel

$$K_n(x,y) := \sum_{k=0}^{n-1} P_k(x) e^{-\frac{x^2}{4}} P_k(y) e^{-\frac{y^2}{4}}.$$

This kernel defines an orthogonal projection Π_{V_n} in $L^2(\mathbb{R})$ to the span V_n of $\{1, x, \cdots, x^n\}$. In addition, we have

$$p_n(\vec{\lambda}) \propto \det(K_n(\lambda_i, \lambda_j))_{1\le i,j\le n}.$$

<u>Lemma 3.2.</u> For any $k \ge 0$, we have

$$\int_{\mathbb{R}} \det[K_n(\lambda_i, \lambda_j)]_{1\le i,j\le k+1} \, d\lambda_{k+1} = (n-k)\det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}.$$

<u>Rmk</u>: Here, we have neglected the ordering of eigenvalues.

<u>Pf</u>: We observe that

① $$\int_{\mathbb{R}} K_n(x,x)\,dx = \sum_{k=0}^{n-1} \int P_k(x)^2 e^{-\frac{x^2}{2}} dx = n.$$

② $$\int_{\mathbb{R}} K_n(x,z) K_n(z,y)\,dz = \int_{\mathbb{R}} \sum_{k,k'=0}^{n-1} P_k(x) e^{-\frac{x^2}{4}} \left[P_k(z) e^{-\frac{z^2}{4}} P_{k'}(z) e^{-\frac{z^2}{4}}\right] P_{k'}(y) e^{-\frac{y^2}{4}} dz$$

$$= \sum_{k=0}^{n-1} P_k(x) e^{-\frac{x^2}{4}} P_k(y) e^{-\frac{y^2}{4}} = K_n(x,y).$$

When $k=0$, ① completes the proof. Now suppose the result has been proved for $k \geq 0$. Then, we consider the $k+1$ case. We apply the cofactor expansion to the $(k+1)$-th row of $\det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k+1}$. Then,

$$\det(K(\lambda_i,\lambda_j))_{1\le i,j\le k+1} = \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}\, K_n(\lambda_{k+1},\lambda_{k+1}) + \sum_{l=1}^{k} (-1)^{k+1-l} K_n(\lambda_l,\lambda_{k+1}) \det(K_n(\lambda_i,\lambda_j))_{1\le i\le k;\ 1\le j\le k+1,\ j\ne l}$$

$$\int \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}\, K_n(\lambda_{k+1},\lambda_{k+1})\, d\lambda_{k+1} = n \det(\lambda_i,\lambda_j)_{1\le i,j\le k}.$$

For the sum, we use ② to get that

$$\int K_n(\lambda_l,\lambda_{k+1}) \det(K_n(\lambda_i,\lambda_j))_{1\le i\le k;\ 1\le j\le k+1,\ j\ne l}\, d\lambda_{k+1}$$

$$= \det\Big((K_n(\lambda_i,\lambda_j))_{1\le i\le k;\ 1\le j\le k,\ j\ne l},\ (K_n(\lambda_i,\lambda_l))_{1\le i\le k} \Big)$$

After row exchange, we get $(-1)^{k-l} \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}$. □

Iterating the formula, we get $\displaystyle\int_{\mathbb{R}^n} \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le n}\, d\lambda_1\cdots d\lambda_n = n!$.

On the other hand, neglecting the ordering eigenvalues, if we extend $p_n(\vec{\lambda})$ from the simplex $\{\lambda_1\le\lambda_2\le\cdots\le\lambda_n\}$ to $\mathbb{R}^n$, its integral is $n!$. Since $\det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le n}$ is symmetric in $\lambda_1,\cdots,\lambda_n$, we thus get the Gaudin-Mehta formula

$$p_n(\lambda_1,\cdots,\lambda_n) = \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le n} \quad \text{on } \{\lambda_1\le\lambda_2\le\cdots\le\lambda_n\}.$$

Moreover, the above lemma 3.2 shows that $p_k(\lambda_1,\cdots,\lambda_k) = \det(K_n(\lambda_i,\lambda_j))_{1\le i,j\le k}$ is the k-point correlation function:

$$\int_{\mathbb{R}^k} p_k(\lambda_1,\cdots,\lambda_k)\, F(\lambda_1,\cdots,\lambda_k)\, d\lambda_1\cdots d\lambda_k$$

$$= \mathbb{E} \sum_{\substack{1\le i_1,\cdots,i_k\le n \\ \text{distinct}}} F(\lambda_{i_1},\cdots,\lambda_{i_k}),$$

$F(x_1,\cdots,x_k)$ supported on $\{x_1\le x_2\le\cdots\le x_k\}$

When $k=1$, we get $\mathbb{E}\,\mu_{H_n}(dx) = \frac{1}{n} K_n(x,x)\,dx$, or

$$\mathbb{E}\,\mu_{\frac{1}{\sqrt{n}}H_n} = \frac{1}{\sqrt{n}} K_n(\sqrt{n}x, \sqrt{n}x)\,dx.$$

When $k=2$, we get

$$p_2(\lambda_1,\lambda_2) = \det\begin{pmatrix} K(\lambda_1,\lambda_1) & K(\lambda_1,\lambda_2) \\ K(\lambda_2,\lambda_1) & K(\lambda_2,\lambda_2) \end{pmatrix}.$$

Recall that $P_i(x)$ is orthogonal to x^j for $0 \le j < i$. This implies that $xP_i(x)$ is orthogonal to x^j for $0 \le j < i-1$. On the other hand, $xP_i(x)$ has degree $i+1$, so $xP_i(x)$ must lie in the span of $\{P_{i-1}, P_i, P_{i+1}\}$. This gives that

$$x P_i = a P_{i-1} + b P_i + c P_{i+1}, \text{ where } c \neq 0.$$

Rearrange, we can write that

$$P_{i+1}(x) = (a_i x + b_i) P_i - c_i P_{i-1}, \quad a_i \neq 0.$$

Note $\displaystyle\int_{\mathbb{R}} x P_i(x) P_{i+1}(x) e^{-\frac{x^2}{2}} dx = \frac{1}{a_i}, \quad \int_{\mathbb{R}} x P_i(x) P_{i-1}(x) dx = \frac{c_i}{a_i}.$

$$\Rightarrow \frac{1}{a_{i-1}} = \frac{c_i}{a_i} \Rightarrow c_i = \frac{a_i}{a_{i-1}}, \quad \text{with } a_{-1} = \infty.$$

Consider $P_{i+1}(x) P_i(y) - P_i(x) P_{i+1}(y) = a_i (x-y) P_i(x) P_i(y) - \dfrac{a_i}{a_{i-1}} \left(P_{i-1}(x) P_i(y) - P_{i-1}(y) P_i(x) \right).$

$$\Rightarrow P_i(x) P_i(y) = \frac{P_{i+1}(x) P_i(y) - P_i(x) P_{i+1}(y)}{a_i (x-y)} - \frac{P_i(x) P_{i-1}(y) - P_{i-1}(x) P_i(y)}{a_{i-1}(x-y)}.$$

Summing over them gives the Christoffel – Darboux formula:

$$K_n(x,y) := \sum_{i=0}^{n-1} P_i(x) P_i(y) e^{-\frac{x^2+y^2}{4}} = \frac{P_n(x) P_{n-1}(y) - P_{n-1}(x) P_n(y)}{a_{n-1}(x-y)} e^{-\frac{x^2+y^2}{4}}.$$

To understand the asymptotic behavior of $K_n(x,y)$ as $n \to +\infty$, need to understand the behavior $P_n(x)$ as $n \to +\infty$.

Note that the above discussion is very general and not restricted to Hermite polynomials. It only requires a system of orthogonal polynomials.

Hermite polynomials are defined as

$$H_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2} \quad \text{with leading coefficient } 1.$$

Hermite functions are defined as

$$\psi_k(x) = \underbrace{\frac{H_k(x)}{(2\pi)^{1/4}\sqrt{k!}}}_{P_k(x)} e^{-x^2/4}, \quad \int \psi_k(x) \psi_l(x) dx = \delta_{kl}.$$

Hermite functions are eigenfunctions of the Hamiltonian for the Harmonic oscillator in quantum mechanics. They have the celebrated asymptotics:

$$\psi_{2m}(x) = \frac{(-1)^m}{n^{1/4}\sqrt{\pi}} \cos(\sqrt{n}\, x) + \bar{o}(n^{-\frac{1}{4}}),$$

$$\psi_{2m+1}(x) = \frac{(-1)^m}{n^{1/4}\sqrt{\pi}} \sin(\sqrt{n}\, x) + \bar{o}(n^{-\frac{1}{4}}), \quad \text{as } n \to +\infty \text{ and } |2m-n| = O(1).$$

They are uniform over $|x| \le C n^{-\frac{1}{2}}$. In addition, $a_{n-1} \sim \dfrac{1}{\sqrt{n}}$.

Hence, we get $K_n(x,y) \approx \sqrt{n}\,\dfrac{\psi_n(x)\psi_{n-1}(y) - \psi_{n-1}(x)\psi_n(y)}{x-y}$

$$\approx \frac{1}{\pi}\frac{\sin(\sqrt{n}x)\cos(\sqrt{n}y) - \sin(\sqrt{n}y)\cos(\sqrt{n}x)}{x-y}$$

$$= \frac{1}{\pi}\frac{\sin(\sqrt{n}(x-y))}{x-y}$$

We consider the scaling $\dfrac{y}{\sqrt{n}\rho_{sc}(0)}$, $\rho_{sc}(0) = \frac{1}{\pi}$. Then

$$K_n\left(\frac{y_1}{\sqrt{n}\rho_{sc}(0)}, \frac{y_2}{\sqrt{n}\rho_{sc}(0)}\right) \approx \sqrt{n}\rho_{sc}(0)\,\frac{\sin[\pi(y_1-y_2)]}{\pi(y_1-y_2)}.$$

This gives the Dyson sine kernel around 0. In general, we have

$$\cancel{\frac{1}{\sqrt{n}\rho_{sc}(x)}K_n\left(\frac{y_1}{\sqrt{n}\rho_{sc}(x)}, \frac{y_2}{\sqrt{n}\rho_{sc}(x)}\right)} \to S(y_1-y_2) := \frac{\sin(\pi(y_1-y_2))}{\pi(y_1-y_2)} \quad \text{for } x\in(-2,2).$$

$$\frac{1}{\sqrt{n}\rho_{sc}(x)}K_n\left(\sqrt{n}x + \frac{y_1}{\sqrt{n}\rho_{sc}(x)}, \sqrt{n}x + \frac{y_2}{\sqrt{n}\rho_{sc}(x)}\right)$$

We resort to the scaling $H_n \to \frac{1}{\sqrt{n}}H_n$, where the typical eigengap is of order $\frac{1}{n}$. Then,

$$\frac{1}{[\rho_{sc}(E)]^n}\,p_k\left(E + \frac{x_1}{n\rho_{sc}(E)}, \cdots, E + \frac{x_k}{n\rho_{sc}(E)}\right) \to \det\left(S(x_i - x_j)\right)_{1\le i,j\le n}.$$

This means, for any $E\in(-2,2)$ and bounded F,

$$\mathbb{E}\sum_{1\le i_1<i_2<\cdots<i_k\le n} F\left(n\rho_{sc}(E)(\lambda_{i_1}-E), \cdots, n\rho_{sc}(E)(\lambda_{i_k}-E)\right)$$

$$\to \int_{y_1\le y_2\le\cdots\le y_k} F(y_1,y_2,\cdots,y_k)\det\left(S(y_i-y_j)\right)_{1\le i,j\le k}\,dy_1\cdots dy_k.$$

In particular, using the 2-point correlation function, one can derive that "Wigner surmise": given a bulk energy E, let i s.t. $i = N\left(\int_{-2}^{E}\rho_{sc}(x)dx + \bar{o}(1)\right)$. Then

$N\rho_{sc}(E)(\lambda_{i+1}-\lambda_i)$ converges in law to $p_W(x) = \frac{32x^2}{\pi^2}e^{-4x^2/\pi}$ in the Hermitian case.

Later, we will show the "bulk universality", i.e., the Dyson sine kernel and the Wigner surmice also occur for general Wigner matrices.

At edge, a different scaling has to be used. Using the definition of Hermite functions, we can check the following identity:

$$\psi_N'(x) = -\frac{x}{2}\psi_N(x) + \sqrt{N}\psi_{N-1}(x).$$

With this identity, we can check that

$$K_N(x,y) = \sqrt{N}\,\frac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{x-y} = \frac{\psi_N(x)\psi_N'(y) - \psi_N(y)\psi_N'(x)}{x-y} - \frac{1}{2}\psi_N(x)\psi_N(y).$$

The Plancherel-Rotach edge asymptotics for ψ_N gives that

$$N^{\frac{1}{12}} \psi_N\left(2\sqrt{N} + \frac{u}{N^{1/6}}\right) \to Ai(u), \quad \text{where } Ai(u) \text{ is the Airy function:}$$

$$Ai(u) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + ut\right)dt$$

(It is called Airy function because it is a solution to the Airy equation $y'' - xy = 0$.)

Then, we see that

$$\frac{1}{N^{1/6}} K_N\left(2\sqrt{N} + \frac{u}{N^{1/6}}, 2\sqrt{N} + \frac{v}{N^{1/6}}\right) \to A(u,v),$$

$$A(u,v) := \frac{Ai(u)\,Ai'(v) - Ai'(u)\,Ai(v)}{u-v} \quad \text{the Airy kernel.}$$

Resort to the scaling $H_N \to \frac{1}{\sqrt{N}} H_N$, we have

$$N^{k/3} p_k\left(2 + \frac{\alpha_1}{N^{2/3}}, 2 + \frac{\alpha_2}{N^{2/3}}, \cdots, 2 + \frac{\alpha_k}{N^{2/3}}\right) \to \det\left(A(\alpha_i, \alpha_j)\right)_{1 \le i,j \le k}.$$

When $k=2$, we have

$$\sum_{j \ne k} \mathbb{E}\, F\left(N^{2/3}(\lambda_j - 2), N^{2/3}(\lambda_k - 2)\right) \to \int_{\mathbb{R}^2} d\alpha_1\, d\alpha_2\, F(\alpha_1, \alpha_2) \det\left(A(\alpha_i, \alpha_j)\right)_{1 \le i,j \le 2}.$$

A much harder problem is to derive the limiting distribution of $N^{2/3}(\lambda_N - 2)$.
Tracy and Widom (1993, 1994) show that it converges to a limiting distribution, which is now referred to as the Tracy-Widom law: $F_\beta(x)$, $\beta = 1, 2, 4$.

In particular,

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)\, q^2(x)\, dx\right), \quad \text{where } q(x) \text{ is the solution to}$$

Painlevé equation II: $q''(s) = sq(s) + 2q(s)^3$,

$q(s) \sim Ai(s)$ as $s \to \infty$.

"Edge universality" also holds, i.e., $N^{2/3}(\lambda_N - 2)$ also converges to the TW law as $N \to +\infty$ for general Wigner matrices.