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Topics in Random Matrix Theory

* A dynamical approach to random matrix theory. by L. Erdős and H.-T. Yau.
* Topics in random matrix theory. by Terence Tao.

Random matrix theory aims to study "properties of large random matrices," such as: the operator norm, eigenvalue / eigenvector distributions, condition number, the singular probability, characteristic ~~polynomials~~ polynomials, ... Many of these properties reduce to studying the asymptotic behaviors of the "eigenvalues and eigenvectors" as the matrix dimension tends to ∞.

The grand principle

The key concept of RMT is the "~~unitar~~ universality phenomenon": the "asymptotic eigenvalue & eigenvector statistics" are independent of the law of matrix elements, but only depend on the symmetry class (i.e., symmetric / hermitian). (Same spirit as LLN and CLT.)

We will illustrate this principle with ~~theree~~ three standard examples.

1. Winger ensemble

Wigner's pioneering work in 1955 marks the birth of RMT. He proposed to use a large real symmetric / complex Hermitian random matrix with independent entries to model the ~~Hatm~~ Hamiltonian of large nuclei. This simple-minded model surprisingly produce the correct gap statistics between energy levels of large nuclei, indicating the "universality principle" behind the model.

Wigner matices: $H = (h_{ij})_{1\le i,j\le N}$ is an $N\times N$ self-adjoint matrix with matrix elements having mean 0, variance 1 and independent up to symmetry: $h_{ij} = \bar{h}_{ji}$.

Gaussian orthogonal ensemble (GOE): The entries h_{ij}, $1\le i\le j\le N$, are real Gaussian random variables, and

$$\mathbb{E} h_{ij} = 0, \quad \mathbb{E} h_{ij}^2 = 1 + \delta_{ij}.$$

Gaussian unitary ensemble (GUE): The upper-triangular entries are i.i.d. $N(0,1)_{\mathbb{C}}$ random variables with $\mathbb{E} h_{ij} = 0$, $\mathbb{E}|h_{ij}|^2 = 1$, $\mathbb{E} h_{ij}^2 = 0$ $(1\le i<j\le N)$. The diagonal entries are $N(0,1)$ random variables.

The GOE / GUE is ~~orthogonal/idx~~ invariant under orthogonal / unitary transformations.

Prop:

Let H be a GOE, and O be an ~~orto~~ orthogonal matrix. Then, $\underbrace{O^T H O}^{H'} \overset{d}{=} H$.

Pf: We only need to check that $\mathbb{E} H'_{ij} H'_{i'j'} = \cancel{(1+\delta_{ij})\,\delta_{ii'}\delta_{jj'}}$ ~~for $1 \le i, j \le N$, $1 \le i', j' \le N$,~~

$\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'}$

$$\mathbb{E} \sum_{\substack{k,l,\\k',l'}} H_{kl} O_{ki} O_{lj} H_{k'l'} O_{k'i'} O_{l'j'}$$

$$= \sum_k 2 O_{ki} O_{kj} O_{ki'} O_{kj'} + \sum_{k \ne l} (\delta_{kk'}\delta_{ll'} + \delta_{kl'}\delta_{lk'}) O_{ki} O_{lj} O_{k'i'} O_{l'j'}$$

$$= 2\sum_k O_{ki} O_{kj} O_{ki'} O_{kj'} + \sum_{k \ne l} (O_{ki} O_{ki'} O_{lj} O_{lj'} + O_{ki} O_{kj'} O_{lj} O_{li'})$$

$$= \sum_{k,l} (O_{ki} O_{ki'} O_{lj} O_{lj'} + O_{ki} O_{kj'} O_{lj} O_{li'}) = \delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'} \qquad \square$$

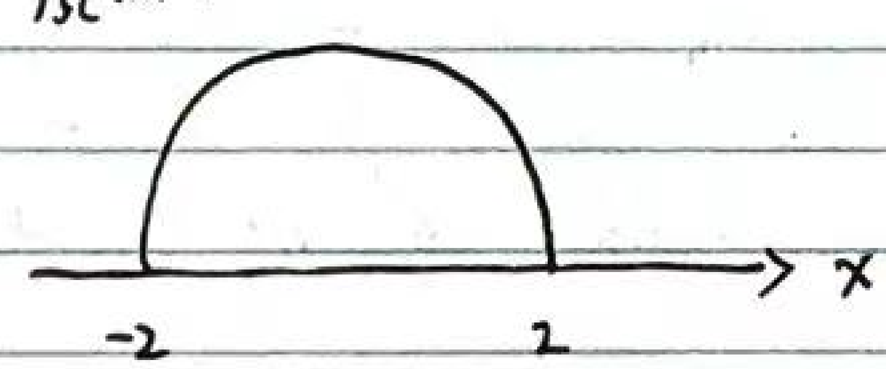
Wigner proved an "LLN" for the empirical spectral density (ESD) of $\frac{1}{\sqrt{N}} H_N$:

$$\mu_{\frac{1}{\sqrt{N}} H_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\lambda_i}{\sqrt{N}}}, \quad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N \text{ are the eigenvalues of } H_N.$$

$\mu_{\frac{1}{\sqrt{N}} H_N} \to \mu_{sc}$ weakly, where μ_{sc} has density

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}, \quad -2 \le x \le 2.$$

$\rho_{sc}(x)$:



Rmk: The above result implies that for any small constant $\varepsilon > 0$, interval $I \subseteq \mathbb{R}$ with $|I| = \varepsilon$,

$$\frac{1}{N}\#\{i : \tfrac{1}{\sqrt{N}}\lambda_i \in I\} = \int_I d\mu_{\frac{1}{\sqrt{N}} H_N}(x) \xrightarrow[N\to+\infty]{} \int_I \rho_{sc}(x)\,dx$$

$\boxed{\text{Q1:}}$ Does the SC law holds in a stronger sense, i.e., $\cancel{\frac{1}{|I|}\int_I d\mu_{\frac{1}{\sqrt{N}}H_N}(x) \to \frac{1}{|I|}\int_I \rho_{sc}(x)\,dx}$

for $\frac{1}{N} \ll a_N \ll 1$, $\displaystyle\frac{1}{a_N}\int_{[E-a_N,\,E+a_N]} d\mu_{\frac{1}{\sqrt{N}} H_N}(x) \xrightarrow[N\to+\infty]{} \rho_{sc}(E)$?

↑

("Local" semicircle law)

In the bulk, ~~around $E \in (-2, 2)$,~~ what is the typical gap between $\frac{\lambda_i}{\sqrt{N}}$ & $\frac{\lambda_{i+1}}{\sqrt{N}}$ for $\varepsilon N \le i \le (1-\varepsilon) N$?

$$\int_{\frac{\lambda_i}{\sqrt{N}}}^{\frac{\lambda_{i+1}}{\sqrt{N}}} \rho_{sc}(x)\,dx = \frac{1}{N} \Rightarrow \frac{\lambda_{i+1}}{\sqrt{N}} - \frac{\lambda_i}{\sqrt{N}} \sim \frac{1}{N}$$

$\boxed{Q2:}$ Does $\sqrt{N}(\lambda_{i+1}-\lambda_i)$ has a limiting distribution in the bulk? Does this distribution depends on the distribution of h_{ij}? (Does bulk universality holds?)

Near the edge: $\displaystyle\int_{-2}^{\frac{\lambda_1}{\sqrt{N}}} \rho_{sc}(x)\,dx = \frac{1}{N} \Rightarrow \int_{-2}^{\frac{\lambda_1}{\sqrt{N}}} \sqrt{2+x}\,dx \sim \frac{1}{N}$

$$\Rightarrow \left(\frac{\lambda_1}{\sqrt{N}}+2\right)^{3/2} \sim \frac{1}{N} \Rightarrow N^{1/6}(\lambda_1+2\sqrt{N}) \sim 1$$

$\boxed{Q3:}$ Does $N^{1/6}(\lambda_1+2\sqrt{N})$ has a limiting distribution? Is this distribution universal?

Every eigenvector of H (GOE) is uniformly distributed on the unit sphere $S(N-1)$.
(Think about why?)

Rmk: A uniformly distributed unit vector can be generated as $\frac{\vec{g}}{\|\vec{g}\|}$, where $\vec{g}=(g_1,\cdots,g_N)$ is a Gaussian vector with i.i.d. $N(0,1)$ entries.

$\boxed{Q4:}$ What is the asymptotic behavior of the eigenvectors of a (non-invariant) Wigner matrix?

We expect that an eigenvector $\vec{u}_k$ is "asyptotically uniform" on $S(N-1)$. But defining this concept is already very non-trivial.

2. Sample covariance matrices

$X=(x_{ij})_{1\le i\le M,\ 1\le j\le N}$, the entries of X are independent random variables of mean 0, variance 1.

① To study the SVD of $X=UDV^*$, $\left.\begin{matrix}U: M\times M\\ V: N\times N\end{matrix}\right\}$ orthogonal/unitary, $D=\begin{pmatrix} d_1 & & & \\ & \ddots & & 0 \\ & & d_M & \end{pmatrix}$ rectangular diagonal,

it reduces to studying the eigendecomposition of XX^* and X^*X.

② Let $\vec{x}=\begin{pmatrix}x_1\\ \vdots \\ x_M\end{pmatrix}\in\mathbb{R}^M$ be a random vector with independent entries of mean 0, variance 1.

The covariance matrix of $\vec{x}$ is given by $\mathbb{E}(\vec{x}\vec{x}^*)=I_{M\times M}$.
Suppose we draw N i.i.d. copies of $\vec{x}$: $\vec{x}_1,\cdots,\vec{x}_N$. Then we form the sample covariance matrix

$$Q_N=\frac{1}{N}\sum_{i=1}^N \vec{x}_i\vec{x}_i^* = \frac{1}{N}XX^*, \quad X=(\vec{x}_1,\cdots,\vec{x}_N).$$

Rmk: By LLN, if M is fixed, letting $N\to\infty$ we have: Q_N converges a.s. to the true covariance I_M. This is called the "low-dimensional" setting.

~~Q1: Does the ESD of Q_N also converge? What is its limit?~~
~~A: The limit~~

Rmk: The "high-dimension" setting considers $c_N = \frac{M}{N} \to c \in (0, +\infty)$, where M and N are of the same order. Then LLN fails, and the behavior of Q_N is very different from that in the low-d setting. This is related to the so-called "curse of dimensionality" in statistics.

When the entries of X are i.i.d. Gaussian, then Q_N is called the Wishart ensemble (1928):

$$Q_N \sim W_M(I, N)$$

M: data dimension; I: covariance; N: degrees of freedom

> **Exercise:** Check that UXV^* ~~is~~ has the same distribution as X in the Wishart case for any unitary U & V.

A useful extension is: $\vec{x} \sim \mathcal{N}_M(0, \Sigma)$, p-variate normal with covariance Σ.
Then $Q_N \sim W_M(\Sigma, N)$. We can write that $X = \Sigma^{1/2} Y$, where the entries of Y are i.i.d. $N(0,1)$. Then $Q_N = \frac{1}{N} \Sigma^{1/2} Y Y^* \Sigma^{1/2}$.

As a further extension, the entries of Y are not necessarily Gaussian. We only require that they are independent, of mean 0 & variance 1.

Q1: Does the ESD of Q_N also converge? What is the limit?
(We will see that the limit is called the Marchenko – Pastur law.)

Q2: Bulk universality?

Q3: Edge universality?

Q4: Eigenvectors?

3. non-Hermitian random matrices

$$X = (x_{ij})_{1 \le i,j \le N}, \quad \text{the entries of } X \text{ are i.i.d., mean 0, variance 1.}$$

We want to study the asymptotic behavior of the eigenvalues & eigenvectors of X.

Note that: almost surely, X has N different eigenvalues. For general X, $\mathbb{P}(X \text{ is singular}) \to$
as $N \to$

Unlike Hermitian matrices, the eigenvalues of a non-Hermitian can be complex.
People find that the ESD of X satisfies a circular law:

$$\mu_{\frac{1}{\sqrt{N}} X_N}(dx\,dy) \to \frac{1}{\pi} \mathbb{1}_{\{z \in \mathbb{C}:\, |z| \le 1\}}\, dx\,dy \quad \text{weakly.}$$

The bulk universality & edge universality are still open. The study of eigenvectors is even harder.

4. Invariant ensembles

For GOE, $$\mu_{H_N} = C_N \prod_{1\le i\le N} e^{-h_{ii}^2/4} \prod_{1\le i<j\le N} e^{-h_{ij}^2/2}\, dH_N$$
$$= C_N\, e^{-\sum_i h_{ii}^2/4 - \sum_{1\le i<j\le N} h_{ij}^2/2}\, dH_N$$
$$= C_N\, e^{-tr(H_N^2)/4}\, dH_N .$$

For GUE, $$\mu_{H_N} = C_N\, e^{-tr(H_N^2)/2}\, dH_N .$$

Under the conjugation by any unitary matrix $\mathcal{U}$, $H_N \to \mathcal{U} H_N \mathcal{U}^{-1}$, we have that $tr(H_N^2)$ is invariant.

In general, we can define a density function on the set of random matrices as $\mathbb{P}(H_N)\, dH_N = \frac{1}{Z_N} \exp(-\mathrm{Tr}\, V(H_N))\, dH_N$, where $dH = \prod_{1\le i\le j\le N} dH_{ij}$ is the Lebesgue measure, V is a "potential function" that grows mildly at ∞ (to guarantee integrability), Z_N is the normalization factor (partition function).

Note: $\mathrm{Tr}\, V(\mathcal{U} H_N \mathcal{U}^{-1}) = \mathrm{Tr}[\mathcal{U}\, V(H_N)\, \mathcal{U}^{-1}] = \mathrm{Tr}\, V(H_N)$, i.e. orthogonal/unitary conjugation leaves the distribution $\mathbb{P}(H_N)\, dH_N$ invariant. So we call it "invariant ensemble".

Invariant ensembles are very different from Wigner ensembles: Gaussian ensembles are the only invariant Wigner ensembles.

As discussed before, the eigenvectors of invariant ensembles are uniformly distributed on unit sphere.

$\boxed{\text{Q1:}}$ What is the prob. density function for all the N eigenvalues only?

$\boxed{\text{Q2:}}$ Bulk universality? Edge universality?

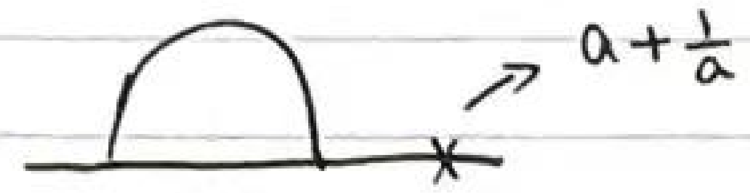
5. Deformed random matrices

Deformed Wigner

$H(a) := \frac{1}{\sqrt{N}} H_N + a \mathcal{u}\mathcal{u}^*$, H_N: Wigner matrix, ~~$a>0$~~ $a\in\mathbb{R}$, $\mathcal{u}$ is an arbitrary unit vector.

WLOG, let $a>0$. A BBP transition as a crosses 1:

* If $a<1$, semicircle law still holds.
* If $a>1$, we have semicircle law + an outlier:



$\lambda_N - (a+\frac{1}{a})\sqrt{N}$ is asymptotically normal as $N \to +\infty$.

<u>Spiked covariance</u>

$$Q_N = \frac{1}{N}\Sigma^{1/2} Y Y^* \Sigma^{1/2}, \quad \Sigma = I + a u u^*, \ a>0, \ u: \text{unit vector}.$$

A similar BBP transition occurs at $a = \sqrt{\frac{M}{N}}$.

Section 1: Why $\frac{1}{\sqrt{N}}$ is the correct saling for Wigner?

Let H_N be a Wigner matrices. We have

$$\mathbb{E}(\sum_i \lambda_i^2) = \mathbb{E}\,\mathrm{Tr}(H^2) = \mathbb{E}\sum_{i,j} H_{ij}H_{ji} = N^2$$

$\Rightarrow \frac{1}{N}\mathbb{E}(\sum_i \lambda_i^2) = N$, i.e. the averaged size of λ_i^2 is of order N.
So the eigenvalues of $\frac{1}{\sqrt{N}}H_N$ are of order 1.

Next, we aim to show the following bound on the operator norm of H_N:
~~there exists a~~ $\|H_N\| := \sup_{x\in\mathbb{C}^n:\,|x|=1} |H_N x|$, $|\cdot|$ means the L^2-norm.

<u>Thm 1.1</u>: Suppose the upper-triangular entries of H_N are independent, have mean zero, and uniformly bounded by 1 (i.e., $|h_{ij}|\le 1$ a.s.). Then, there exists absolute constants $c, C>0$ such that

$$\mathbb{P}(\|H_N\| > A\sqrt{N}) \le C\exp(-cAN) \quad \text{for } A \ge C.$$

(In words, $\|H_N\| = O(\sqrt{N})$ with very high probability.)

<u>Lemma 1.2</u>: Suppose M_N is $\overbrace{\text{a}}^{N\times N}$ random matrix whose entries are independent, have mean zero, and uniformly bounded by 1. Then, there exist absolute constants $c, C>0$ such that

$$\mathbb{P}(\|M_N\| > A\sqrt{N}) \le C\exp(-cAN) \quad \text{for } A\ge C.$$

<u>Pf of Thm 1.1</u>: We write $H_N = U_N + L_N$, U_N consists of the upper-triangular entries, L_N consists of strict lower-triangular entries.

By Lemma 1.2, $\mathbb{P}(\|U_N\| > A\sqrt{N}) \le C\exp(-cAN)$, $\mathbb{P}(\|L_N\| > A\sqrt{N}) \le C\exp(-cAN)$ for $A\ge C$.

Then for $A \ge 2C$,

$$\mathbb{P}(\|H_N\| > A\sqrt{N}) \le \mathbb{P}(\|U_N\| > AN/2) + \mathbb{P}(\|L_N\| > AN/2) \le 2C\exp(-cAN/2). \quad \square$$

The proof of Lemma 1.2 uses some "standard" concentration inequalities & ε-net argument.

Thm 1.3 (Hoeffding's inequality) Let $X_1, \cdots, X_N$ be independent bounded random variables with $X_i \in [a_i, b_i]$ a.s. Let $S_N := X_1 + \cdots + X_N$. Then $\forall \lambda > 0$,

$$\mathbb{P}(|S_N| \geq \lambda\sigma) \leq C\exp(-c\lambda^2), \qquad \sigma^2 := \sum_{i=1}^N |b_i - a_i|^2.$$

Lem 1.4 (Hoeffding's lemma) For $Z \in [a,b]$, $\mathbb{E}e^{\lambda(Z-\mathbb{E}Z)} \leq \exp\left(\frac{\lambda^2(b-a)^2}{2}\right)$.

Rmk: The RHS can be improved to $\exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$.

Pf of Lem 1.4: Let Z' be an independent copy of Z. Then

$$\mathbb{E}_Z \exp(\lambda(Z - \mathbb{E}Z)) = \mathbb{E}_Z \exp(\lambda(Z - \mathbb{E}_{Z'}(Z'))) \underset{\substack{\uparrow \\ \text{Jensen's ineq.}}}{\leq} \mathbb{E}_Z\mathbb{E}_{Z'} \exp(\lambda(Z - Z'))$$

Since $Z - Z'$ is symmetric about 0, for a random sign s, $\mathbb{P}(s=1) = \mathbb{P}(s=-1) = \frac{1}{2}$, $s(Z-Z') \overset{d}{=} Z - Z'$. So

$$\mathbb{E}_Z\mathbb{E}_{Z'} \exp(\lambda(Z-Z')) = \mathbb{E}_{Z,Z'}\mathbb{E}_s \exp(\lambda s(Z-Z')) = \mathbb{E}_{Z,Z'}\left[\tfrac{1}{2}e^{\lambda(Z-Z')} + \tfrac{1}{2}e^{-\lambda(Z-Z')}\right]$$

$$\underset{\substack{\uparrow \\ (\cosh(x) \leq \exp(\frac{x^2}{2}))}}{\leq} \mathbb{E}\exp\left(\frac{\lambda^2}{2}(Z-Z')^2\right) \leq \exp\left(\frac{\lambda^2}{2}(b-a)^2\right). \qquad \square$$

Pf of Thm 1.3: $\forall t > 0$, $\mathbb{E}\exp(tS_N) = \prod_{i=1}^N \mathbb{E}\exp(tX_i) \leq \prod_{i=1}^N \exp\left(\frac{t^2}{2}(b_i - a_i)^2\right) = \exp\left(\frac{t^2}{2}\sigma^2\right)$.

So $\mathbb{P}(S_N > \lambda\sigma) \underset{\text{Markov}}{\leq} \exp(-t\lambda\sigma)\exp\left(\frac{t^2}{2}\sigma^2\right) = \exp\left(\frac{t^2}{2}\sigma^2 - t\lambda\sigma\right)$

~~Taking $t = \lambda$, $\mathbb{P}(S_N > \lambda\sigma) \leq \exp(-\frac{1}{2}\lambda^2\sigma^2)$.~~

$$= \exp\left(\tfrac{1}{2}(t\sigma - \lambda)^2 - \tfrac{\lambda^2}{2}\right).$$

Taking $t = \lambda/\sigma$ gives $\mathbb{P}(S_N > \lambda\sigma) \leq \exp(-\lambda^2/2)$. Can get a similar bound for $\mathbb{P}(S_N < -\lambda\sigma)$. $\square$

Lem 1.5 Under the setting of Lemma 1.2, for any fixed unit vector $x \in \mathbb{R}^N$,

$$\mathbb{P}(|M_N x| \geq A\sqrt{N}) \leq C\exp(-cAN) \quad \text{for } A \geq C.$$

Pf: Let $M_N = \begin{pmatrix} -X_1- \\ -X_2- \\ \vdots \\ -X_N- \end{pmatrix}$, X_i are the row vectors of M_N.

Then, $M_N x = \begin{pmatrix} X_1 \cdot x \\ X_2 \cdot x \\ \vdots \\ X_N \cdot x \end{pmatrix}$. For each $X_i \cdot x = \sum_{j=1}^N X_{ij} x_j$, applying Hoeffding, we get

$$\mathbb{P}(|X_i \cdot x| \geq \lambda\sigma) \leq C\exp(-c\lambda^2), \quad \text{where } \sigma^2 = \sum_{j=1}^N 4x_j^2 = 4.$$

For any $c' < c$, $\mathbb{E}\exp(c'|X_i \cdot x|^2) \leq C'$ for a constant $C' > 0$

[Use the tail-probability formula,

$\mathbb{E}f(X) = \int_0^{+\infty} \mathbb{P}(X \geq t) f'(t)\,dt$, X positive, f increasing on $[0, +\infty)$, & $f(0) = 0$.]

Thus, $\mathbb{E}\exp(c'|Mx|^2) = \prod_{i=1}^{N} \exp(c'|X_i \cdot x|^2) \le (C')^N$.

$$\Rightarrow \mathbb{P}(|Mx| \ge A\sqrt{N}) \le \exp(-c'A^2N)(C')^N \le C\exp(-cAN) \text{ for } A \text{ large enough.} \qquad \square$$

How to extend Lem 1.5 to a bound on

$$\mathbb{P}(\|M_N\| \ge A\sqrt{N})$$
$$= \mathbb{P}\Big(\bigcup_{x \in S^N} \{|M_N x| \ge A\sqrt{N}\}\Big).$$

Of course, we cannot take a union bound over a uncountable set. The idea is to "discretize" S^N.

<u>Def</u> (ε-net) An ε-net of the sphere S^N denotes a set of points in S^N that are separated from each other by a distance of at least ε, and which is <u>maximal</u> with respect to set inclusion.

Let Σ be such an ε-net. By maximality, for any point $x \in S^N$, there exists a point $y \in \Sigma$ such that $|x - y| < \varepsilon$.

<u>Lemma 1.6</u> (Volume packing) Let $0 < \varepsilon < 1$, and Σ be an ε-net. Then $|\Sigma| \le (3/\varepsilon)^N$.

<u>Pf</u>: Consider the collection of balls of radius $\varepsilon/2$ centered around each point in Σ. Then these balls are disjoint. On the other hand, they are also contained in the ball of radius $3/2$ centered at the origin. The volume of the larger ball is $(3/\varepsilon)^N$ times the volume of each small ball. $\square$

<u>Proof of Lemma 1.2</u>: Let Σ be a $\frac{1}{2}$-net of S^N. Then $|\Sigma| \le 6^N$.
Taking a union bound, we get

$$\mathbb{P}\big(\max_{x\in\Sigma} |Mx| > A\sqrt{N}\big) \le \sum_{x \in \Sigma} \mathbb{P}(|Mx| > A\sqrt{N}) \le C\exp(-cAN)\cdot 6^N \le C\exp(-\tfrac{c}{2}AN) \qquad (\dagger)$$

for large enough $A > 0$.

Next, we show that $\mathbb{P}(\|M\| > \lambda) \le \mathbb{P}(\max_{x \in \Sigma} |Mx| > \lambda/2)$ $(*)$ for any $\lambda > 0$.

To show $(*)$, let $x \in S^N$ be such that

$$\|M\| = |Mx|.$$

Then we can find $y \in \Sigma$ so that $|x - y| < \frac{1}{2}$. Then $|M(x-y)| < \frac{1}{2}\|M\|$.

By triangle ineq., $|My| \ge |Mx| - |M(x-y)| > \|M\| - \frac{1}{2}\|M\| = \frac{1}{2}\|M\|$.

Combining $(*)$ and $(\dagger)$ completes the proof. $\square$

Rmk: The above proofs can be extended to Wigner matrices with sub-gaussian entries.
A random variable X is said to be sub-gaussian if there exists an absolute constant $c>0$, so that
$$\mathbb{P}(|X|\geq t) \leq 2\exp(-ct^2) \quad \forall\ t\geq 0.$$
* Gaussian r.v.s are sub-gaussian
* If a random variable is bounded by a const, then it is subgaussian.

The sub-gaussian norm of X, $\|X\|_{\psi_2}$, is defined as
$$\|X\|_{\psi_2} := \inf\{t>0 : \mathbb{E}\exp(X^2/t^2)\leq 2\}.$$

Then we have the general Hoeffding's inequality

Thm 1.7 Let $X_1, \cdots, X_N$ be independent, mean-zero, sub-gaussian r.v.s. Then, $\forall\ t\geq 0$,
$$\mathbb{P}\left\{\left|\sum_{i=1}^N X_i\right|\geq t\right\} \leq 2\exp\left(\frac{-ct^2}{\sum_{i=1}^N \|X_i\|_{\psi_2}^2}\right).$$

Rmk: In accordance with the semicircle law, we should have that $\forall$ constant $\varepsilon>0$,
$\mathbb{P}(\|H_N\|\geq(2+\varepsilon)\sqrt{N})$ with high probability.
One slick way to prove this result is the important "moment method in RMT":
for any $k\in 2\mathbb{N}$, note
$$\mathrm{tr}(H_N^k) = \sum_{i=1}^n \lambda_i^k \geq \max_i |\lambda_i|^k = \|H_N\|^k.$$
Hence,
$$\mathbb{E}\|H_N\|^k \leq \mathbb{E}\,\mathrm{tr}(H_N^k) \Rightarrow \mathbb{P}(\|H_N\|^k\geq(2+\varepsilon)\sqrt{N}) \leq [(2+\varepsilon)\sqrt{N}]^{-k}\,\mathbb{E}\,\mathrm{tr}(H_N^k).$$
The moment method aims to control $\mathbb{E}\,\mathrm{tr}(H_N^k)$. One can show that
$$\mathbb{E}\,\mathrm{tr}(H_N^k) = [2+o_k(1)]^k N^{\frac{k}{2}+1} \quad (*) \quad \text{for } k \text{ as large as } C\log N.$$
Then, we have
$$\mathbb{P}(\|H_N\|^k\geq(2+\varepsilon)\sqrt{N}) \leq \left(1-\frac{\varepsilon}{4}\right)^k N \ll 1 \quad \text{for } k=C\log N \text{ if } C \text{ is large enough.}$$
For details, see Tao, Section 2.3.4.
We will give a proof using a different method. In fact, we will show a much stronger result: $\|H_N\|\leq 2+N^{-\frac{2}{3}+\varepsilon}$ w.h.p. for any const. $\varepsilon>0$.

But, we will use the moment method to prove the first important RMT result, i.e., the Wigner semicircle law. It requires us to calculate $\mathbb{E}\,\mathrm{tr}(H_N^k)$ for large but finite $k\in\mathbb{N}$.

Rmk: Moment method together with a truncation argument gives the operator norm bound for Wigner matrices whose entries have finite fourth moment.

Section 2 Wigner Semicircle Law

For the rest of this course, we rescale H_N to $\frac{1}{\sqrt{N}} H_N$, so that the eigenvalues of H_N are typically of order 1.

~~Thm~~ Thm 2.1 (Semicircle law) Let H be a real Wigner matrix whose entries have finite moments up to any order, i.e., $\forall k \in \mathbb{N}$, $\exists\, C_k > 0$ so that

$$\max_{i,j} \mathbb{E} |\sqrt{N} h_{ij}|^k \le C_k .$$

Then, the ESD μ_{H_N} converges in distribution to μ_{sc} almost surely.

① Moment method

We will prove a weaker convergence in expectation of μ_{H_N} under a stronger sub-gaussian assumption on the entries of H_N.

$$\forall\, \varphi \in C_c(\mathbb{R}), \quad \mathbb{E} \int \varphi(x)\, d\mu_{H_N}(x) \to \int \varphi(x)\, d\mu_{sc}(x) .$$

Define a sequence of measures $\mathbb{E}\mu_{H_N}(A) := \mathbb{E} \int \mathbb{1}(x \in A)\, d\mu_{H_N}(x)$.
The tightness of $\{\mathbb{E}\mu_{H_N}\}$ follows from the operator norm bounds.

To show the convergence, it suffices to show the convergence of moments, i.e., $\forall k \in \mathbb{N}$,

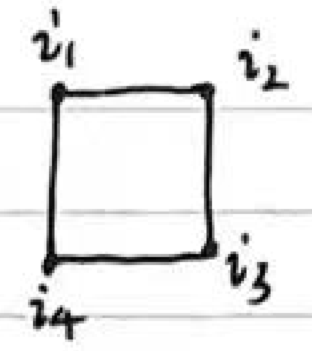
(*) $\mathbb{E} \int x^k\, d\mu_{H_N}(x) \to \int x^k\, d\mu_{sc}(x)$. [This follows from an application of Taylor.]

Rmk: To show convergence in prob. of μ_{H_N}, we need to show concentration of measure, i.e., μ_{H_N} concentrates around $\mathbb{E}\mu_{H_N}$. For that purpose, we need to show that $\mathrm{Var}\left[\int x^k\, d\mu_N(x)\right] \to 0$, $\forall k \in \mathbb{N}$. To show ~~convergence almost surely~~, almost sure convergence, we need to control $\mathbb{P}\left(\left| \int x^k d\mu_N(x) - \mathbb{E}\int x^k\, d\mu_N(x) \right| > \varepsilon \right)$ and use Borel-Cantelli.
While these can be done, we refrain from doing that and will prove the semicircle law using another "~~possibly~~ more powerful" method — the Stieltjes transform method.

By definition, $\displaystyle\int x^k\, d\mu_{H_N}(x) = \int x^k \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}(x)\, dx = \frac{1}{N} \sum_{i=1}^N \lambda_i^k = \frac{1}{N} \mathrm{Tr}(H_N^k)$.

To show (*), we need to show that $\mathbb{E} \frac{1}{N} \mathrm{Tr}(H_N^k) \to \int x^k\, d\mu_{sc}(x)$ for any fixed k.

$\displaystyle\frac{1}{N} \mathbb{E}\, \mathrm{Tr}(H_N^k) = \frac{1}{N} \mathbb{E} \sum_{i_1, \cdots, i_k} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_k i_1}$. For the expectation to be non-zero, every h_{xy} must be paired with anothe $h_{yx} = h_{xy}$.

Let's starts with $k=4$ case: $\frac{1}{N}\mathbb{E}$ 

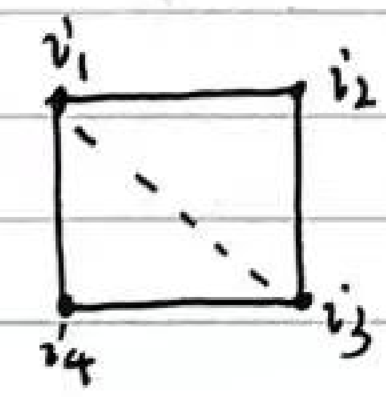
We have the following cases to have non-zero expectation (up to cyclic permutation of vertices)

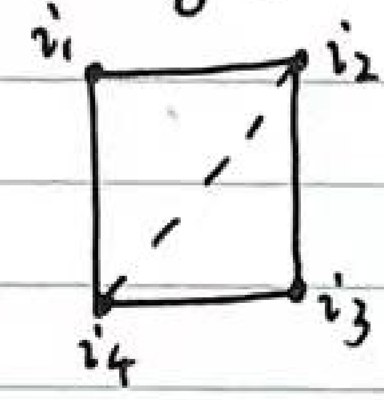
(i) $i_1 = i_3 \neq i_2 \neq i_4$ Typical order $\left(\frac{1}{N} \times N^3 \times \frac{1}{N^2} = 1\right)$

(ii) $i_1 = i_3 \neq i_2 = i_4$ $\left(\frac{1}{N} \times N^2 \times \frac{1}{N^2} = \frac{1}{N}\right)$

(iii) $i_1 = i_2 = i_3 \neq i_4$ $\left(\frac{1}{N} \times N^2 \times \frac{1}{N^2} = \frac{1}{N}\right)$

(iv) $i_1 = i_2 = i_3 = i_4$ $\left(\frac{1}{N} \times N \times \frac{1}{N^2} = \frac{1}{N^2}\right)$

Case (i) is dominating, and there are two such graphs: $i_1 = i_3 \neq i_2 \neq i_4$ 

$i_2 = i_4 \neq i_1 \neq i_3$  . With the fact that $\mathbb{E}\, h_{ij}^2 = \frac{1}{N}$, we get

$$\mathbb{E}\,\frac{1}{N}\mathrm{Tr}(H_N^4) = 2 \times \frac{1}{N} \times N^2 \times \frac{1}{N^2} + O\left(\frac{1}{N}\right) = 2 + O\left(\frac{1}{N}\right).$$

Let's turn to $k=6$. There are four types of graphs to deal with:

(i) There are 3 distinct edges, each occuring twice, and hence 4 distinct vertices. $\left(\frac{1}{N} \times N^4 \times \frac{1}{N^3} = 1\right)$

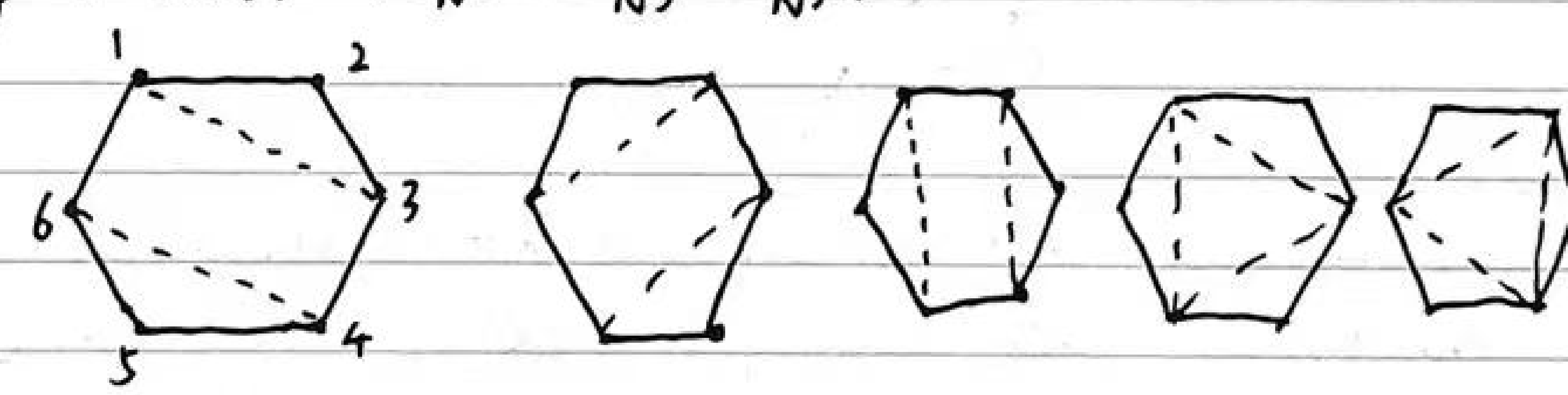
(ii) 2 distinct edges, one occuring twice & one occuring four times, and hence 3 distinct vertices $\left(\frac{1}{N} \times N^3 \times \frac{1}{N^3} = \frac{1}{N}\right)$

(iii) 2 distinct edges, each occuring three times, and hence 3 distinct vertices.

(iv) Only one distinct edge, occuring 6 times. $\left(\frac{1}{N} \times N \times \frac{1}{N^3} = \frac{1}{N^3}\right)$

How many graphs in case (i)?

There are "5" non-crossing planar graphs with 6 edges.



Hence, $\mathbb{E}\,\frac{1}{N}\mathrm{Tr}(H_N^6) = 5 \times \frac{1}{N} \times N^4 \times \frac{1}{N^3} + O\left(\frac{1}{N}\right) = 5 + O\left(\frac{1}{N}\right)$.

In general, we consider $\frac{1}{N}\mathbb{E}\sum_{i_1,\cdots,i_k} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_k i_1}$. The sequence $(i_1, i_2, \cdots, i_k, i_1)$ can be regarded as a cycle with at most k vertices and over all possible labellings of $i_j \in \{1, \cdots, n\}$. Since each ~~edge~~ distinct edge is traversed by two times, there are at most $k/2$ distinct edges and $(\frac{k}{2}+1)$ vertices traversed by the cycle.

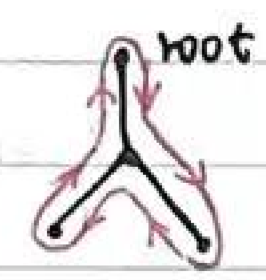
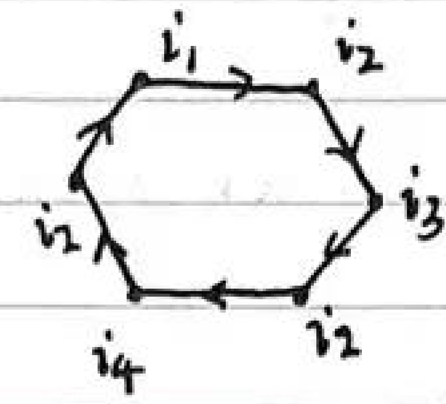
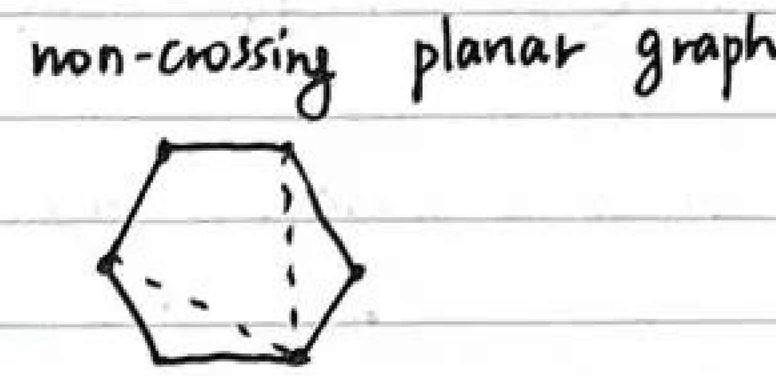
For the cycles with at most $k/2$ distinct vertices, their order is $O\left(\frac{1}{N} \times N^{\frac{k}{2}} \times \frac{1}{N^{k/2}}\right) = O\left(\frac{1}{N}\right)$.

So we only need to consider cycles which traverse exactly $(\frac{k}{2}+1)$ vertices and has $\frac{k}{2}$ distinct edges. We call such cycles <u>non-crossing cycles of length k</u>.

We need to count the number of non-crossing cycles.

Rmk: If k is odd, then ~~∃~~ each cycle has at most $\frac{k+1}{2}$ distinct vertices. Hence its order is $O(\frac{1}{N} \times N^{\frac{k+1}{2}} \times \frac{1}{N^{k/2}}) = O(\frac{1}{\sqrt{N}})$. Hence, $\mathbb{E}\, \frac{1}{N} Tr(H_N^k) \to 0$ for any odd k.

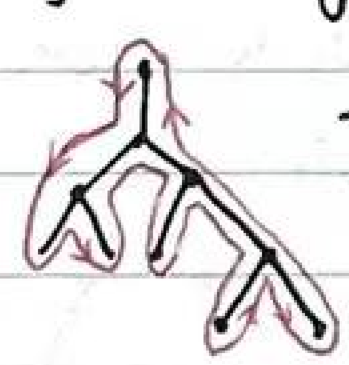
Lemma 2.2 There is a one-to-one correspondance between non-crossing cycles of length k and rooted trees of $k/2$ edges and $(\frac{k}{2}+1)$ edges.
* The cycle lies in the corresponding tree and traverses each edge in the tree exactly twice.

For example,  $\Longleftrightarrow$  $\longleftrightarrow$ 

With lemma 2.2, we can get the following corollary:

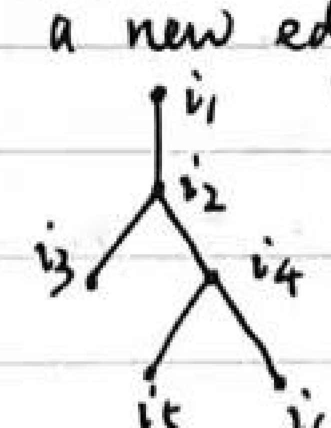
Exercise: Let $i_1, \cdots, i_k$ be a cycle of length k. Arrange the integers $1, 2, \cdots, k$ around a circle. Whenever $1 \le a < b \le k$ s.t. $i_a = i_b$ with no c ~~bett~~ between a, b for which $i_a = i_c = i_b$, draw a dashed line between a & b. ~~∃~~ Then the cycle is non-crossing if ~~⇔~~ and only if the number of dashed lines is exacty $\frac{k}{2} - 1$ and the dashed lines do not cross each other.

Pf of Lemma 2.2: Given an unlabelled rooted tree, starting from the root, traverse the tree from left to right gives a non-crossing cycle. For example:

 $\rightarrow (i_1, i_2, i_3, i_4, i_3, i_5, i_3, i_2, i_6, i_7, i_6, i_8, i_9, i_8, i_{10}, i_8, i_6, i_2, i_1)$

We now show that we can construct a unique tree from a non-crossing cycle: $(i_1, i_2, \cdots, i_k)$. We traverse this cycle from i_1 to i_2, then from i_2 to i_3, and so on. At a step, say, from i_j to i_{j+1}, we either use an edge that we have not seen before, or else we are using an edge for the second time. We call ~~an edge~~ a step of former type an "innovative (I)" ~~edge~~ step, and ~~an edge~~ a step of the later type an "returning (R)" ~~edge~~ step. Then there are $k/2$ (I) ~~edges~~ steps, and $k/2$ (R) ~~edges~~ steps. It is obvious that only the (I) ~~edge~~ step can bring us to new vertices we have not seen before. On the other hand, since we have to visit $(\frac{k}{2}+1)$ vertices starting from i_1, each (I) step must take us to a new vertex.

Then, traversing the cycle $(i_1, i_2, \cdots, i_k, i_1)$, we construct a graph as follows. Let i_1 be the root. For each (I) step, we add a new vertex and a new edge. For example:

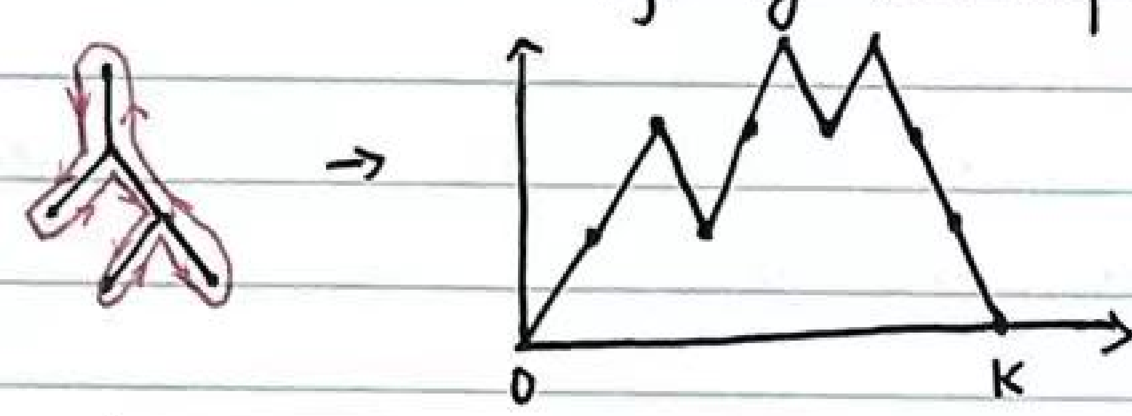
$(i_1, i_2, i_3, i_2, i_4, i_5, i_4, i_6, i_4, i_2, i_1) \rightarrow$ 

This clearly gives a rooted tree. $\square$

<u>Fact:</u> The number of unlabelled rooted trees with $\frac{k}{2}+1$ vertices is the ~~Catalan~~ ~~numb~~ <u>Catalan number $C_{k/2}$</u>.

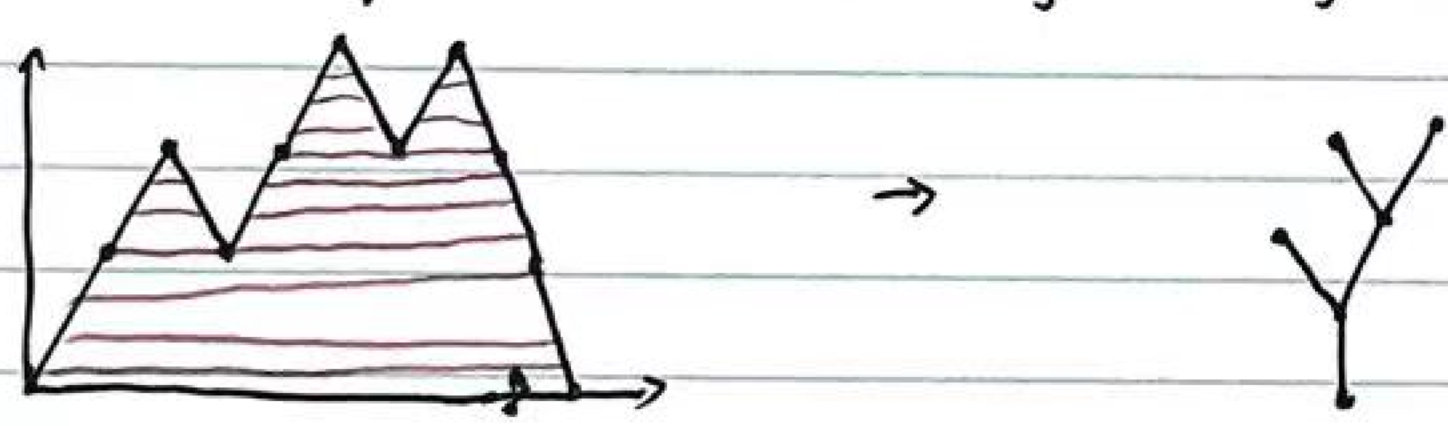
<u>Catalan number</u> $C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$.

<u>One version of poof of the Fact:</u> We further construct a 1-1 correspondance between the rooted trees and random walks on the positive half line.
~~Trans~~ Traversing a rooted tree, if we travese a (I) step, then walk to the right; otherwise, walk to the left by one step. For example:



It gives a RW from 0 to 0 and stays to the right of 0.

From the graph of a random walk, say f, on $[0,k]$. Define a quotient graph under the equivalence relation: $(a, f(a)) \sim (b, f(b))$ if $f(a) = f(b) = \min_{t\in[a,b]} f(t)$.



How many different k-step random walks from 0 to 0 and stays positive?

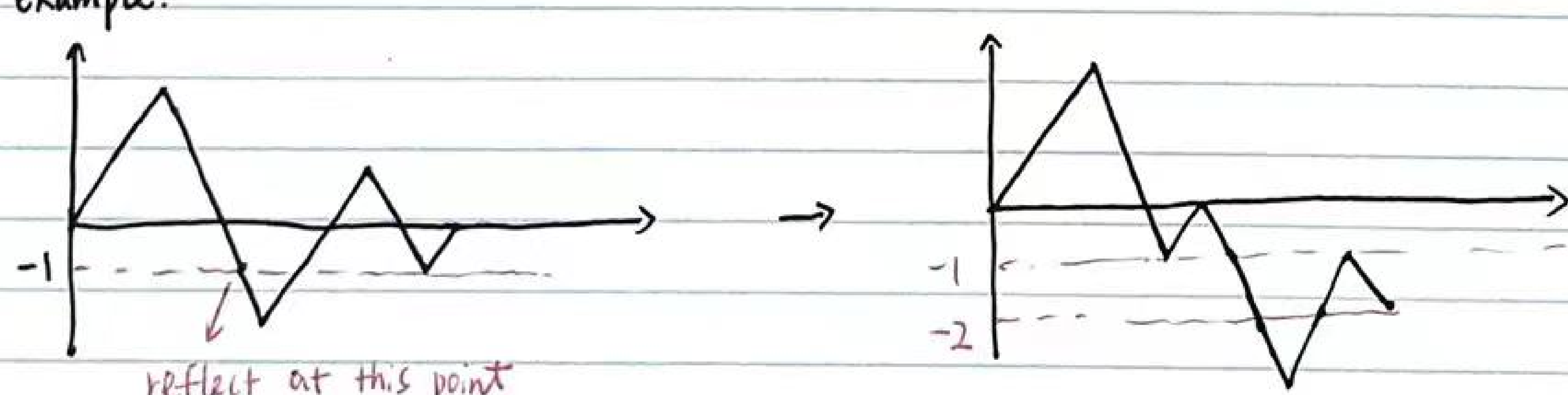
$$\#\{\text{~~Random~~ } k\text{-step ~~walks~~ simple walks from 0 to 0}\} - \#\left\{\begin{array}{l} k\text{-step simple walks from 0 to 0} \\ \text{and hit } -1 \end{array}\right\}$$

$$\| \qquad\qquad\qquad\qquad \uparrow$$

$\cancel{\binom{2n}{n}} \binom{k}{k/2}$ Use reflection principle

$$= \#\{k\text{-step simple walks from 0 to } -2\} = \binom{k}{k/2+1}$$

For example:



$$\binom{k}{k/2} - \binom{k}{k/2+1} = C_{k/2}.$$

$\square$

It remains to show that: $\int x^k \, d\mu_{sc}(x) = C_{k/2} \cdot 1$ (k is even).

This is trivial for $k=$ odd. For k even,

$$I_k = \int_{-2}^{2} x^k \frac{\sqrt{4-x^2}}{2\pi} dx = \int_0^{\pi} (2\cos\theta)^k \frac{2\sin\theta}{2\pi} \cdot 2\sin\theta \, d\theta = \frac{2^{k+1}}{\pi} \int_0^{\pi} (\cos\theta)^k \sin^2\theta \, d\theta$$

$$= \frac{2^{k+1}}{\pi} \int_0^{\pi} (\cos\theta)^{k-1} \sin^2\theta \, d\sin\theta = -\frac{2^{k+1}}{\pi} \int_0^{\pi} \sin\theta \cdot \left[2\sin\theta \cdot (\cos\theta)^k - (k-1)\sin^3\theta \, (\cos\theta)^{k-2}\right] d\theta$$

$$= -2 I_k + \frac{2^{k+1}}{\pi} \int_0^{\pi} (k-1) \sin^2\theta \, (1-\cos^2\theta)(\cos\theta)^{k-2} \, d\theta$$

$$= -(k+1) I_k + 4(k-1) \frac{2^{k-1}}{\pi} \int_0^{\pi} \sin^2\theta \, (\cos\theta)^{k-2} \, d\theta = -(k+1) I_k + 4(k-1) I_{k-2}.$$

$$\Rightarrow \quad I_k = \frac{4(k-1)}{k+2} I_{k-2}.$$

On the other hand, it is simple to check: $C_{k/2} = \frac{4(k-1)}{k+2} C_{(k-2)/2}$.

Sometimes, it is very challenging to recover a measure from its moments. We now use a different method to "derive" the semicircle law directly.

② The Stieltjes transform method