Considering that the nonlinear factor of PF calculation comes from the quadratic term of voltage, PFEs can be represented by introducing variables $\boldsymbol{V}_{\sin}$, $\boldsymbol{V}_{\mathrm{squ}}$, $\boldsymbol{V}_{\cos}$ to replace the quadratic sine term, the quadratic term and the quadratic cosine term respectively:

$$\begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B^P} & \boldsymbol{G}^{\wedge} & \boldsymbol{G^P} \\ \boldsymbol{G^Q} & \boldsymbol{B}^{\wedge} & \boldsymbol{B^Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{\sin} \\ \boldsymbol{V}_{\mathrm{squ}} \\ \boldsymbol{V}_{\cos} \end{bmatrix} \tag{2}$$

$$\boldsymbol{B^P} = (A_{ik} \cdot B_{ij})_{n\times m}, \boldsymbol{G^P} = (|A_{ik}| \cdot G_{ij})_{n\times m} \ \forall \boldsymbol{e}(i,j) = \boldsymbol{e_k},\ \boldsymbol{e_k} \in \boldsymbol{E} \tag{3}$$

$$\boldsymbol{G^Q} = (A_{ik} \cdot G_{ij})_{n\times m}, \boldsymbol{B^Q} = (-|A_{ik}| \cdot B_{ij})_{\boldsymbol{n\times m}} \forall \boldsymbol{e}(i,j) = \boldsymbol{e_k}, \boldsymbol{e_k} \in \boldsymbol{E} \tag{4}$$

$$\boldsymbol{G}^{\wedge} = \boldsymbol{diag}(\boldsymbol{G}),\ \boldsymbol{B}^{-\wedge} = -\boldsymbol{diag}(\boldsymbol{B}) \tag{5}$$

$$\boldsymbol{V}_{\cos} = (V_i V_j \cos\theta_{ij})_{m\times 1}, \boldsymbol{V}_{\sin} = (V_i V_j \sin\theta_{ij})_{m\times 1} \ \forall \boldsymbol{e}(i,j) \in \boldsymbol{E} \tag{6}$$

$$\boldsymbol{V}_{\mathrm{squ}} = (V_i^2)_{n\times 1} \ \ \forall i \in \boldsymbol{N} \tag{7}$$

where $\boldsymbol{B^P}$, $\boldsymbol{G}^{\wedge}$, $\boldsymbol{G^P}$, $\boldsymbol{G^Q}$, $\boldsymbol{B}^{\wedge}$, $\boldsymbol{B^Q}$ represent the segmentation matrix developed based on the traditional incidence matrix $\boldsymbol{A} = (A_{ik})_{n\times m}$. $\boldsymbol{E}$ and $\boldsymbol{N}$ represent line set and bus set. $\boldsymbol{e}(i,j) = \boldsymbol{e_k}$ indicates that the k-th line take i, j as start bus and end bus. Eq.(2)-(5) indicate the ULE with $[\boldsymbol{V}_{\sin} \quad \boldsymbol{V}_{\mathrm{squ}} \quad \boldsymbol{V}_{\cos}]$ serving as variables. Eq.(6)-(7) represent the newly introduced NNC.

Assuming the rank of the coefficient matrix $\boldsymbol{Rank}(\begin{bmatrix} \boldsymbol{B^P} & \boldsymbol{G}^{\wedge} & \boldsymbol{G^P} \\ \boldsymbol{G^Q} & \boldsymbol{B}^{\wedge} & \boldsymbol{B^Q} \end{bmatrix}) = \boldsymbol{r}$, the underdetermined linear equations of Eq.(2)-Eq.(5) $\begin{bmatrix} \boldsymbol{B^P} & \boldsymbol{G}^{\wedge} & \boldsymbol{G^P} \\ \boldsymbol{G^Q} & \boldsymbol{B}^{\wedge} & \boldsymbol{B^Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{\sin} \\ \boldsymbol{V}_{\mathrm{squ}} \\ \boldsymbol{V}_{\cos} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}$ can determine r pivot variables $V_1, V_2, ..., V_r$ and $2m+n-r$ free variables $V_{r+1}, V_{r+2}, ..., V_{2m+n}$ based on Gaussian Jordan elimination. The pivot variables can be represented by free variables:

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_r \end{bmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,2m+n-r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,2m+n} \end{pmatrix} \begin{bmatrix} V_{r+1} \\ V_{r+2} \\ \vdots \\ V_{2m+n} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2m+n-r} \end{bmatrix} \tag{1*}$$

where $(a_{i,j})_{r\times 2m+n-r}$ and $(b_i)_{2m+n-r\times 1}$ are related to node admittance matrix and node injection power, both of which are known quantities. Combined with the pivot and free variables, the solution of the whole equation can be expressed as:

$$\begin{bmatrix} \hat{\boldsymbol{V}}_{\text{sin}} \\ \hat{\boldsymbol{V}}_{\text{squ}} \\ \hat{\boldsymbol{V}}_{\text{cos}} \end{bmatrix} = \boldsymbol{P}_{\boldsymbol{row}} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_r \\ V_{r+1} \\ \vdots \\ V_{2m+n} \end{bmatrix} = \boldsymbol{P}_{\boldsymbol{row}} \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,2m+n-r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,2m+n} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_{r+1} \\ V_{r+2} \\ \vdots \\ V_{2m+n} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2m+n-r} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \tag{2*}$$

where $\boldsymbol{P}_{\boldsymbol{row}}$ represents row swapping operation that corresponds the pivot variables and free variables with the original variable order. The free variables $\begin{bmatrix} V_{r+1} & V_{r+2} & \cdots & V_{2m+n} \end{bmatrix}^{\mathbf{T}}$ are not restricted by the equations and can be assigned arbitrary values, which are further represented by another group of 2m+n dimensional free variables for format alignment:

$$\begin{bmatrix} V_{r+1} \\ V_{r+2} \\ \vdots \\ V_{2m+n} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_1^{(0)} \\ V_2^{(0)} \\ \vdots \\ V_r^{(0)} \\ V_{r+1}^{(0)} \\ \vdots \\ V_{2m+n}^{(0)} \end{bmatrix} \tag{3*}$$

Combined with Eq. (2*) and (3*), the general solution expression of ULE is further expressed as:

$$\begin{bmatrix} \hat{\boldsymbol{V}}_{\text{sin}} \\ \hat{\boldsymbol{V}}_{\text{squ}} \\ \hat{\boldsymbol{V}}_{\text{cos}} \end{bmatrix} = \boldsymbol{P}_{\boldsymbol{row}} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_r \\ V_{r+1} \\ \vdots \\ V_{2m+n} \end{bmatrix} = \boldsymbol{P}_{\boldsymbol{row}} \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,2m+n-r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,2m+n} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} V_1^{(0)} \\ V_2^{(0)} \\ \vdots \\ V_r^{(0)} \\ V_{r+1}^{(0)} \\ \vdots \\ V_{2m+n}^{(0)} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2m+n-r} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \tag{4*}$$

Assigning $\boldsymbol{w} = \boldsymbol{P}_{\boldsymbol{row}} \begin{bmatrix} a_{1,1} & \cdots & a_{1,2m+n-r} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,2m+n} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$, $\boldsymbol{b} = \boldsymbol{P}_{\boldsymbol{row}} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2m+n-r} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

$[\boldsymbol{V}_{\text{sin}}^{(0)} \quad \boldsymbol{V}_{\text{squ}}^{(0)} \quad \boldsymbol{V}_{\text{cos}}^{(0)}]^{\mathbf{T}} = \boldsymbol{P}_{\boldsymbol{row}} \begin{bmatrix} V_1^{(0)} \\ V_2^{(0)} \\ \vdots \\ V_r^{(0)} \\ V_{r+1}^{(0)} \\ \vdots \\ V_{2m+n}^{(0)} \end{bmatrix}$, the final general solution expression can be described as Eq.(5*), which corresponds to Eq.(8) in the manuscript:

$$\begin{bmatrix} \hat{\boldsymbol{V}}_{\text{sin}} \\ \hat{\boldsymbol{V}}_{\text{squ}} \\ \hat{\boldsymbol{V}}_{\text{cos}} \end{bmatrix} = \begin{pmatrix} w_{1,1} & \cdots & w_{1,2\boldsymbol{m}+\boldsymbol{n}} \\ \vdots & \ddots & \vdots \\ w_{2\boldsymbol{m}+\boldsymbol{n},1} & \cdots & w_{2\boldsymbol{m}+\boldsymbol{n},2\boldsymbol{m}+\boldsymbol{n}} \end{pmatrix} \begin{bmatrix} \boldsymbol{V}_{\text{sin}}^{(0)} \\ \boldsymbol{V}_{\text{squ}}^{(0)} \\ \boldsymbol{V}_{\text{cos}}^{(0)} \end{bmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_{2m+n} \end{pmatrix} \tag{5*}$$