Section 9.2: The Two-sample t Test and Confidence Interval

Assume

$$X_1, \cdots, X_m \sim^{iid} N(\mu_1, \sigma_1^2)$$

and

$$Y_1, \cdots, Y_n \sim^{iid} N(\mu_2, \sigma_2^2),$$

where either m or n is small and both σ_1^2 and σ_2^2 are unknown.

Case 1: Two Sample t Procedure.

Let

$$\nu = \frac{(\frac{s_1^2}{m} + \frac{s_2^2}{n})^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}.$$

If ν is not an integer, we usually take the greatest integer $\leq \nu$. For example, if $\nu = 10.9$, we choose $\nu = 10$.

Then, the two sample t confidence interval for $\mu_1 - \mu_2$ is

$$\bar{x} - \bar{y} \pm t_{\alpha/2,\nu}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}.$$

Let

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}.$$

Let α be the significance level. Then,

(a) we reject H_0 and conclude H_a when $t > t_{\alpha,\nu}$ if we test

$$H_0 : \mu_1 - \mu_2 = \Delta_0 (\text{or } \leq \Delta_0) \leftrightarrow H_a : \mu_1 - \mu_2 > \Delta_0;$$

(b) we reject H_0 and conclude H_a when $t < -t_{\alpha,\nu}$ if we test

$$H_0 : \mu_1 - \mu_2 = \Delta_0 (\text{or } \geq \Delta_0) \leftrightarrow H_a : \mu_1 - \mu_2 < \Delta_0;$$

(c) we reject H_0 and conclude H_a when $|t| > t_{\alpha/2,\nu}$ if we test

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \leftrightarrow H_a : \mu_1 - \mu_2 \neq \Delta_0.$$

Case 2: Pooled t procedure.

Assume $\sigma_1^2 = \sigma_2^2$. Let

$$S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$$

and write s_p^2 as the observed value of S_p^2. Then,

$$T = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{S_p^2(1/m + 1/n)}} \sim t_{m+n-2}$$

when $\mu_1 - \mu_2 = \Delta_0$. Thus, the $(1-\alpha)100\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{x} - \bar{y} \pm t_{\frac{\alpha}{2},m+n-2} s_p \sqrt{\frac{1}{m} + \frac{1}{n}}.$$

Let

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{s_p\sqrt{1/m + 1/n}}.$$

We reject H_0 and conclude H_a if

$$\begin{cases} t > t_{\alpha,m+n-2} & \text{in the testing problem (a)} \\ t < -t_{\alpha,m+n-2} & \text{in the testing problem (b)} \\ |z| > z_{\frac{\alpha}{2},m+n-2} & \text{in the testing problem (c)} \end{cases}$$

First example of Section 9.2: example 9.6 on textbook.

Type	Sample Size	Sample Mean	Sample SD
Cotton	10	51.71	0.79
Triacetate	10	136.14	3.59

$$\nu = \frac{(\frac{0.79^2}{10} + \frac{3.59^2}{10})^2}{\frac{(0.79^2/10)^2}{9} + \frac{(3.59^2/10)^2}{9}} = 9.87.$$

Thus, we choose $\nu = 9$. Then,, the 95% confidence interval for $\mu_1 - \mu_2$ is

$$51.71 - 136.14 \pm t_{0.025,9}\sqrt{\frac{0.79^2}{10} + \frac{3.59^2}{10}}$$
$$=[-87.06, -81.80].$$

Note: Since s_1^2 and s_2^2 are not close, we cannot use pooled t procedure. Section 9.5 will give the detailed method for that.

Second example of Section 9.2: example 9.7 on textbook.

Type	Sample Size	Sample Mean	Sample SD
No fusion	10	2902.8	277.3
Fused	8	3108.1	205.9

- In the two-sample t method, we have

$$\nu = \frac{(\frac{277.3^2}{10} + \frac{205.9^2}{8})^2}{\frac{(277.3^2/10)^2}{9} + \frac{(205.9^2/8)^2}{7}} = 15.94.$$

Thus, we choose $\nu = 15$. The 95% confidence interval is

$$\begin{aligned} & 2902.8 - 3108.1 \pm t_{0.025,15} \\ & \qquad \times \sqrt{\frac{277.3^2}{10} + \frac{205.9^2}{8}} \\ = & [-448.22, 37.62]. \end{aligned}$$

For $H_0 : \mu_1 \geq \mu_2$ and $H_a : \mu_1 < \mu_2$,

$$t = \frac{2902.8 - 3108.1}{\sqrt{\frac{277.3^2}{10} + \frac{205.9^2}{8}}} = -1.8$$

which is between $-t_{0.05,15} = -1.753$ and $-t_{0.01,15} = -2.602$. Thus, H_0 is rejected at $\alpha = 0.05$ and not rejected at $\alpha = 0.01$.

- Pooled t-procedure.

$$s_p^2 = \frac{9 \times 277.3^2 + 7 \times 205.9^2}{10 + 8 - 2} = 248.6^2.$$

The 95% confidence interval is

$$\begin{aligned} & 2902.8 - 3108.1 \pm t_{0.025,16} \\ & \quad \times 248.6\sqrt{\frac{1}{10} + \frac{1}{8}} \\ = & [-455.29, 44.68]. \end{aligned}$$

For $H_0 : \mu_1 \geq \mu_2$ and $H_a : \mu_1 < \mu_2$,

$$t = \frac{2902.8 - 3108.1}{248.6\sqrt{\frac{1}{10} + \frac{1}{8}}} = -1.74.$$

Since $-t_{0.05,16} = -1.746$, we fail to reject H_0 at $\alpha = 0.05$.

Note: since s_1^2 is close to s_2^2 here, we can use either the two-sample t procedure or pooled t procedure.