[13] D. M. Kodek, "A lower bound for the increase of finite wordlength minimax approximation error," in *Proc. Electr. Comput. Sci. Conf. (ERK)*, Portorož, Slovenia, Sep. 28–30, 1992, vol. B, pp. 3–6.
[14] D. M. Kodek, "Limits of finite wordlength FIR digital filter design," in *Proc. Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Munich, Germany, Apr. 21–24, 1997, vol. III, pp. 2149–2152.
[15] D. M. Kodek, "Design of optimal finite wordlength FIR digital filters," in *Proc. Eur. Conf. Circuit Theory Des. (ECCTD)*, Stresa, Italy, Aug. 29–31, 1999, vol. I, pp. 401–404.
[16] D. M. Kodek, "Performance limit of finite word-length FIR digital filters," *IEEE Trans. Signal Process.*, vol. 53, pp. 2462–2469, Jul. 2005.
[17] L. J. Karam and J. H. McClellan, "Complex Chebyshev approximation for FIR filter design," *IEEE Trans. Circuits Syst. II*, vol. 42, pp. 207–216, Mar. 1995.
[18] T. W. Parks and J. H. McClellan, "A program for the design of linear phase finite impulse response filters," *IEEE Trans. Audio Electroacoust.*, vol. 20, pp. 195–199, Aug. 1972.
[19] M. J. D. Powell, *Approximation Theory and Methods*. Cambridge, U.K.: Cambridge Univ. Press, 1981, pp. 97–99.
[20] A. K. Lenstra, H. W. Lenstra, and L. Lovász, "Factoring polynomials with rational coefficients," *Math. Ann.*, vol. 261, pp. 515–534, 1982.
[21] D. Micciancio and S. Goldwasser, *Complexity of Lattice Problems: A Cryptographic Perspective*. Boston, MA: Kluwer Academic, 2002, pp. 45–68.
[22] N. P. Smart, *The Algorithmic Resolution of Diophantine Equations*. Cambridge, U.K.: Cambridge Univ. Press, 1998, pp. 59–76.
[23] C. H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization*. Englewood Cliffs, NJ: Prentice-Hall, 1982, pp. 433–444.
[24] D. M. Kodek and M. Krisper, "Telescopic rounding for suboptimal finite wordlength FIR digital filter design," *Digit. Signal Process.*, vol. 15, pp. 522–535, Nov. 2005.

Reconstruction of Uniformly Sampled Sequence From Nonuniformly Sampled Transient Sequence Using Symmetric Extension

Sung-Won Park, Wei-Da Hao, and Chung S. Leung

***Abstract*—In this correspondence, reconstruction of a uniformly sampled sequence from a nonuniformly sampled transient sequence using symmetric extension is described. First, a relationship between the discrete Fourier transform (DFT) of a uniformly sampled sequence and the DFT of a nonuniformly sampled sequence is obtained. From the relationship, the formula to reconstruct the DFT of a uniformly sampled sequence from the DFT of a nonuniformly sampled sequence is derived when the nonuniform sampling ratios are known. Second, a symmetric extension of the nonuniformly sampled sequence is described to avoid discontinuity that adds high-frequency content in the DFT. Finally, experimental results are presented.**

***Index Terms*—Discrete Fourier transform (DFT), nonuniform sampling, symmetric extension, uniform sampling.**

I. Introduction

In many applications, it is desired to reconstruct a uniformly sampled sequence from a nonuniformly sampled sequence. Early works on recurrent nonuniform sampling are available in the literature [1]–[7]. Recurrent nonuniform sampling means that a continuous-time signal is sampled nonuniformly with a periodic pattern. Recurrent nonuniform sampling problems occur in sampling of a high frequency signal using a very high-speed waveform digitizing system with interleaved A/D converters [2], [3]. Early works on recurrent nonuniform sampling considered sampling of relatively long signals. For example, the number of samples considered in [4] was 512 and the number of samples considered in [5] was 2048. Recurrent nonuniform sampling has also been applied to image enhancement where resolution of an image is increased beyond the number of pixels available in the camera by using multiple aliased copies of unknown relative sampling offsets [6]. On the other hand, recurrent nonuniform sampling described in [2] and [4] has been extended to two-dimensional signals [7]. There are two main differences between [6] and [7]. There is a constraint in the nonuniform sampling ratios in [7] and there is no such a constraint in [6]. Nonuniform sampling ratios are estimated from multiple copies in [6] by iteration, but they are estimated by using a prescribed sinusoidal signal in [7] without iteration.

Manuscript received August 18, 2011; revised October 26, 2011; accepted November 17, 2011. Date of publication December 14, 2011; date of current version February 10, 2012. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. XiangGen Xia.

The authors are with the Department of Electrical Engineering and Computer Science, Texas A&M University-Kingsville, Kingsville, TX 78363 USA (e-mail: park@tamuk.edu; wei-da.hao@tamuk.edu; chung.leung@tamuk.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.
Digital Object Identifier 10.1109/TSP.2011.2177834

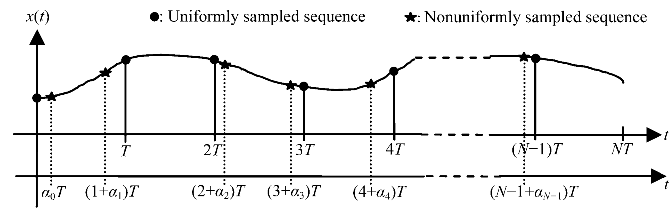

Fig. 1. Uniform and nonuniform sampling. T is the sampling interval for uniform sampling, α_n is the nonuniform sampling ratio, and N is the number of samples.

Reconstruction of uniformly sampled sequence from nonuniformly sampled sequence without any periodic pattern is described in this correspondence. This reconstruction technique is useful when one is interested in reconstructing a short transient sequence. A short sequence, whose length is only 20, is considered in this correspondence.

The correspondence is organized as follows. In Section II, a relationship between the DFT of a uniformly sampled sequence and the DFT of a nonuniformly sampled sequence is obtained. From the relationship, the formula to construct the DFT of a uniformly sampled sequence from the DFT of a nonuniformly sampled sequence is derived when the nonuniform sampling ratios are known. In Section III, symmetric extension of a sequence to avoid a discontinuity that unduly adds high frequency content in the DFT is explained. In addition, results of reconstruction experiments using no extension and symmetric extension are presented. Finally, a conclusion is made in Section IV.

II. Relationship Between Uniform Sampling and Nonuniform Sampling

Suppose a continuous-time signal, $x(t)$, is sampled uniformly at $t = 0, T, 2T, \ldots, (N-1)T$ where T is the sampling interval (see Fig. 1).

The DFT of the uniformly sampled sequence, $x(n)$, for $n = 0, 1, 2, \ldots, N-1$, is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \text{ for } k = 0, 1, 2, \ldots, N-1 \tag{1}$$

where $x(n) = x(nT)$ for all n. The IDFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \text{ for } n = 0, 1, 2, \ldots, N-1. \tag{2}$$

By plugging $x(n)$ of (2) into (1), one obtains

$$X(k) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{j\frac{2\pi}{N}mn} \right] e^{-j\frac{2\pi}{N}kn}. \quad (3)$$

Suppose the signal is nonuniformly sampled such that the nonuniformly sampled sequence is given by (see Fig. 1)

$$\tilde{x}(n) = x((n+\alpha_n)T) \text{ for } n = 0, 1, 2, \ldots, N-1 \quad (4)$$

where α_n is the nonuniform sampling ratio.

By replacing n inside the brackets in (3) with $n+\alpha_n$, the DFT of the nonuniformly sampled sequence is expressed as follows:

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{j\frac{2\pi}{N}m(n+\alpha_n)} \right] e^{-j\frac{2\pi}{N}kn}. \quad (5)$$

The order of the summations in (5) can be reversed such that

$$\tilde{X}(k) = \sum_{m=0}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}m\alpha_n} e^{-j\frac{2\pi}{N}(k-m)n} \right] X(m). \quad (6)$$

Let us define the following:

$$B(m,k) = \sum_{n=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi}{N}m\alpha_n} e^{-j\frac{2\pi}{N}kn}. \quad (7)$$

In other words, $B(m,k)$ for $k = 0, 1, \ldots, N-1$ is the DFT of the sequence given by

$$\left\{ \frac{1}{N} e^{j\frac{2\pi}{N}m\alpha_0}, \frac{1}{N} e^{j\frac{2\pi}{N}m\alpha_1}, \frac{1}{N} e^{j\frac{2\pi}{N}m\alpha_2}, \ldots, \frac{1}{N} e^{j\frac{2\pi}{N}m\alpha_{N-1}} \right\} \quad (8)$$

where $m = 0, 1, \ldots, N-1$.

Now, (6) becomes

$$\tilde{X}(k) = \sum_{m=0}^{N-1} B(m, (k-m) \bmod N) X(m). \quad (9)$$

In matrix form, (9) becomes

$$\tilde{\mathbf{X}} = \mathbf{B}\mathbf{X} \quad (10)$$

where [see the equation at the bottom of the page]. By performing the following matrix computation, the DFT of the uniformly sampled sequence can be reconstructed from the DFT of the nonuniformly sampled sequence:

$$\mathbf{X} = \mathbf{B}^{-1}\tilde{\mathbf{X}}. \quad (11)$$

This is related to the theory described in [8]. If the signal $x(t)$ is real and the number of samples N is odd, then the term inside the brackets in (3) becomes

$$x(n) = \frac{1}{N} \sum_{m'=-\frac{N-1}{2}}^{\frac{N-1}{2}} X(m') e^{j\frac{2\pi}{N}nm'} \quad (12)$$

where $X(m') = X^*(-m')$ and the superscript * indicates complex conjugate. For every integer index m in $B(m,k)$ in (7), corresponding integer m' as shown in the following should be used for computation of $B(m,k)$:

$$m' = \begin{cases} m, & \text{for } \; 0 \le m \le \frac{N-1}{2} \\ m-N, & \text{for } \; \frac{N-1}{2}+1 \le m \le N-1. \end{cases} \quad (13)$$

If $x(t)$ is real and N is even, then the term inside the brackets in (3) becomes

$$x(n) = \frac{1}{N} \sum_{m'=-(\frac{N}{2}-1)}^{\frac{N}{2}} X(m') e^{j\frac{2\pi}{N}nm'} \quad (14)$$

where $X(m') = X^*(-m')$ and $X(N/2)$ is a real number. For every integer index m in $B(m,k)$ in (7), corresponding integer m' as shown in the following should be used for computation of $B(m,k)$:

$$m' = \begin{cases} m & \text{for } \; 0 \le m \le \frac{N}{2} \\ m-N & \text{for } \; \frac{N}{2}+1 \le m \le N-1. \end{cases} \quad (15)$$

Suppose that $x_p(t)$ is a real periodic signal that is obtained by periodically extending $x(t)$ of Fig. 1 with the period of *NT* [s], where N is an odd integer. The periodic signal $x_p(t)$ will have its own Fourier series with the fundamental radian frequency of $2\pi/NT$ [rad/s] as follows:

$$x_p(t) = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_m e^{j\frac{2\pi}{NT}mt} \quad (16)$$

where F_m are the Fourier series coefficients of $x_p(t)$. Assume that we sample $x_p(t)$ uniformly with the sampling period of T [sec] over the signal period [0, NT]. This results in a sequence as follows:

$$x_p(n) = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} F_m e^{j\frac{2\pi}{N}nm} \text{ for } n = 0, 1, 2, \ldots, N-1. \quad (17)$$

Equation (17) is identical to (12), which, in turn, is identical to (2) when N is odd by equating

$$F_m = \frac{1}{N} X(m). \quad (18)$$

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}, \quad \tilde{\mathbf{X}} = \begin{bmatrix} \tilde{X}(0) \\ \tilde{X}(1) \\ \vdots \\ \tilde{X}(N-1) \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} B(0,0) & B(1,N-1) & \cdots & B(N-2,2) & B(N-1,1) \\ B(0,1) & B(1,0) & \cdots & B(N-2,3) & B(N-1,2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B(0,N-2) & B(1,N-3) & \cdots & B(N-2,0) & B(N-1,N-1) \\ B(0,N-1) & B(1,N-2) & \cdots & B(N-2,1) & B(N-1,0) \end{bmatrix}$$

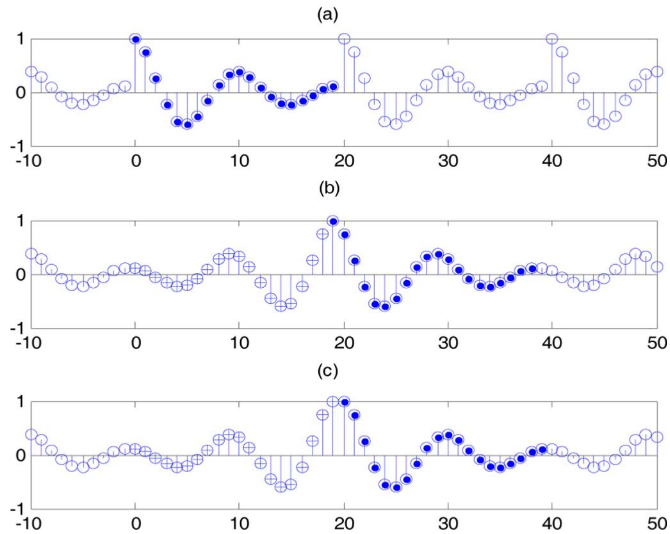

Fig. 2. Extension of $x(n) = e^{-0.1n}\cos(0.2\pi n)$. (a) No extension followed by periodic extension; (b) whole-sample symmetric extension followed by periodic extension; and (c) half-sample symmetric extension followed by periodic extension.

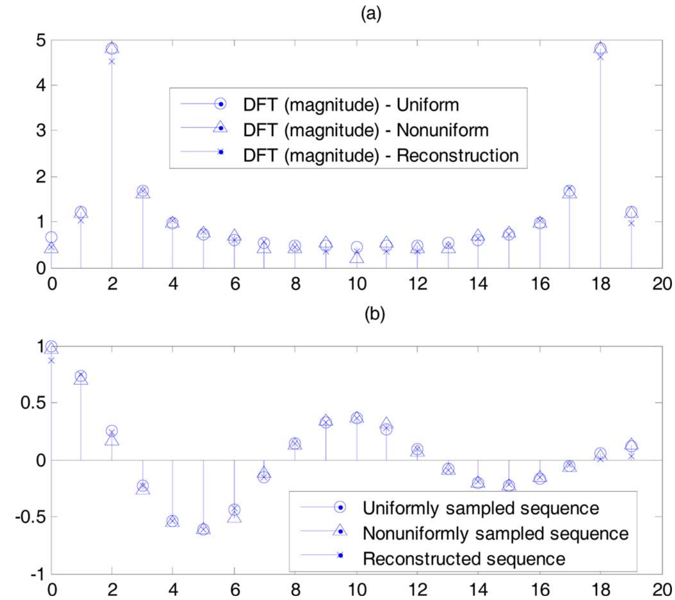

Fig. 3. Reconstruction without symmetric extension. The standard deviation of the nonuniform sampling ratio was 0.16. (a) Reconstruction of DFT without symmetric extension. (b) Reconstruction of sequence without symmetric extension.

When N is odd, for perfect reconstruction the highest harmonic of $x_p(t)$ should not be greater than $(N-1)/2$. When N is even, for perfect reconstruction the highest harmonic of $x_p(t)$ should not be greater than $N/2$. In other words, for perfect reconstruction the bandwidth of $x_p(t)$ should be less than $1/2T$ [Hz] (which is $N/2$ multiplied by $1/NT$).

By taking the IDFT of the reconstructed DFT, one can reconstruct the uniformly sampled sequence.

III. Symmetric Extension and Experimental Results

The following continuous-time signal was used for our experiment in this section:

$$x(t) = e^{-0.1t}\cos(0.2\pi t)u(t) \tag{19}$$

where $u(t)$ is a unit step function. The sampling interval $T = 1$ [sec] and the number of samples $N = 20$. For our simulation, statistically independent zero-mean Gaussian random numbers were used for nonuniform sampling ratios, α_n.

As shown in the previous section, computation of the DFT of a sequence is in fact computing the Fourier series coefficients of the periodically extended sequence. The periodically extended sequence is shown in Fig. 2(a). Note that there is discontinuity at the edges. This discontinuity or sudden jump unduly adds substantial high-frequency content in the DFT.

To prevent such discontinuity, symmetric extension is considered. Two types of symmetric extensions [9] are shown in Fig. 2(b) and (c). The extended sequence is whole-sample symmetric if it is symmetric about one of its samples as shown in Fig. 2(b). The extended sequence is half-sample symmetric if it is symmetric about a point halfway between two samples as shown in Fig. 2(c). Whole-sample symmetric extension results in $2N-1$ points and half-sample symmetric extension results in $2N$ points in the extended sequence. It should be noted that the corresponding nonuniform sampling ratios should be symmetrically extended as well for reconstruction.

Statistically independent zero-mean Gaussian random numbers were used for nonuniform sampling ratios to generate a nonuniformly sampled sequence. The standard deviations were chosen as 0.01, 0.02, 0.04, 0.08, 0.16, and 0.32.

A. Reconstruction Without Symmetric Extension

The DFT of the nonuniformly sampled sequence is computed and the DFT of the uniformly sampled sequence is estimated using (11). The IDFT of the estimated DFT is computed for reconstruction of the uniformly sampled sequence. 5 000 trials were performed at each standard deviation. The average signal to ratio is computed using the following method:

$$\text{average SNR (in dB)} = 10\log_{10}\frac{\text{signal power}}{\frac{1}{5000}\sum_{m=1}^{5000}[\text{noise power in each trial}]} \tag{20}$$

where

$$\text{signal power} = \frac{1}{N}\sum_{n=0}^{N-1}x^2(n) \tag{21}$$

$$\text{noise power in each trial} = \frac{1}{N}\sum_{n=0}^{N-1}[x(n)-\hat{x}(n)]^2 \tag{22}$$

where $\hat{x}(n)$ is the reconstructed uniformly sampled sequence by taking the IDFT of the $\hat{X}(k)$ obtained according to (11) in each trial.

Fig. 3 shows an example of reconstruction without symmetric extension. The standard deviation of the nonuniform sampling ratio was 0.16. Note that both the reconstructed DFT and the reconstructed sequence do not match the uniformly sampled counterparts at several points. Note also that there is substantial high frequency content in the DFT due to discontinuity. The second column of Table I shows that even when the standard deviation is small there is significant error in the reconstruction.

B. Reconstruction With Whole-Sample or Half-Sample Symmetric Extension

The DFT of the symmetrically extended nonuniformly sampled sequence is computed and the DFT of the extended uniformly sam-

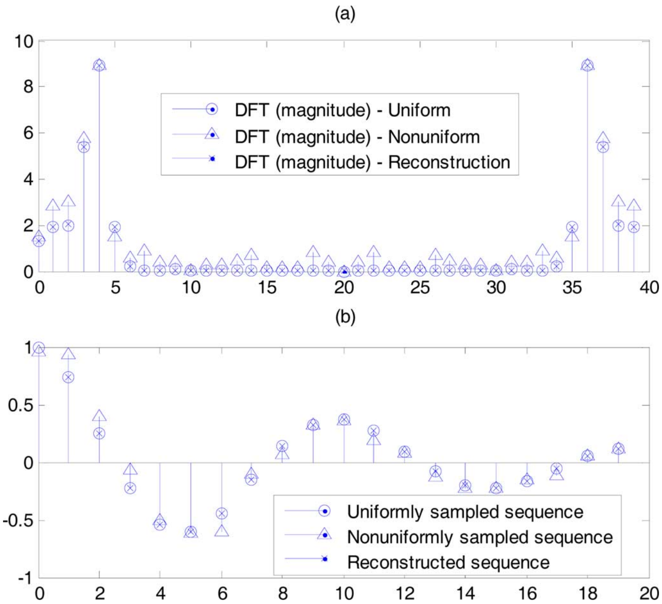

Fig. 4. Reconstruction using half-sample symmetric extension. The standard deviation of the nonuniform sampling ratio was 0.32. (a) Reconstruction of DFT via half-sample symmetric extension. (b) Reconstruction of sequence via half-sample symmetric extension.

TABLE I
COMPARISON OF PERFORMANCE BETWEEN NO EXTENSION AND SYMMETRIC EXTENSIONS. 5000 TRIALS WERE PERFORMED AT EACH STANDARD DEVIATION

Standard deviation of nonuniform sampling ratio	average SNR [dB]		
	No extension	Whole-sample extension	Half-sample extension
0.01	44.9420 dB	63.7654 dB	74.9079 dB
0.02	38.9975 dB	57.6672 dB	68.8192 dB
0.04	32.8412 dB	51.5969 dB	62.7685 dB
0.08	26.3213 dB	45.3986 dB	56.6499 dB
0.16	18.3775 dB	37.6366 dB	50.2906 dB
0.32	too much error	12.9070 dB	41.8432 dB

pled sequence is estimated using (11). The IDFT of the estimated DFT is computed for reconstruction of the uniformly sampled sequence. Out of $2N-1$ or $2N$ point long sequence, the original N point sequence is extracted for computing the SNR. Fig. 4 shows that reconstruction works well even when the standard deviation of the nonuniform sampling ratio is 0.32. It should be noted that half-sample symmetric extension performed the best as shown in Table I.

There is at least 11-dB advantage in SNR with the half-sample symmetric extension over the whole-sample symmetric extension.

This reconstruction technique can be used for a long signal as long as computation of the inverse of a large matrix does not cause any numerical problems. Recurrent nonuniform sampling for a long signal will still have a discontinuity problem. However, the effect of discontinuity to long signals is not as critical as that to short transient signals.

IV. CONCLUSION

In this correspondence, reconstruction of a uniformly sampled sequence from a nonuniformly sampled transient sequence using symmetric extension is described. It has been shown that perfect reconstruction is possible when the periodically extended signal has no harmonics greater than $(N-1)/2$, where N is the number of samples. It is shown by experiment that reconstruction using half-sample symmetric extension worked fairly well in general case.

REFERENCES

[1] J. L. Yen, "On nonuniform sampling of bandwidth-limited signals," *IRE Trans. Circuit Theory*, vol. 3, pp. 251–257, Dec. 1956.

[2] Y.-C. Jenq, "Digital spectra of nonuniformly sampled signals: Fundamental and high-speed waveform digitizers," *IEEE Trans. Instrum. Meas.*, vol. 37, no. 2, pp. 245–251, Jun. 1988.

[3] D. Wei, Q. Ran, and Y. Li, "Reconstruction of band-limited signals from multichannel and periodic nonuniform samples in the linear canonical transform domain," *Opt. Commun.*, vol. 284, no. 19, pp. 4307–4315, Sep. 2011.

[4] Y.-C. Jenq, "Perfect reconstruction of digital spectrum from nonuniformly sampled signals," *IEEE Trans. Instrum. Meas.*, vol. 46, no. 3, pp. 649–652, Jun. 1997.

[5] P. Sommen and K. Janse, "On the relationship between uniform and recurrent nonuniform discrete-time sampling schemes," *IEEE Trans. Signal Process.*, vol. 56, no. 10, pp. 5147–5156, Oct. 2008.

[6] P. Vandewalle, L. Sbaiz, J. Wandewalle, and M. Vetterli, "Super-resolution from unregistered and totally aliased signals using subspace methods," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3687–3702, Jul. 2007.

[7] Y.-C. Jenq and L. Cheng, "Digital spectrum of a nonuniformly sampled two-dimensional signal and its reconstruction," *IEEE Trans. Instrum. Meas.*, vol. 54, no. 3, pp. 1180–1187, Jun. 2005.

[8] S. Bagchi and S. K. Mitra, *The Nonuniform Discrete Fourier Transform and Its Applications in Signal Processing*. Norwell, MA: Kluwer, 1999.

[9] C. M. Brislawn, "Preservation of subband symmetry in multirate signal coding," *IEEE Trans. Signal Process.*, vol. 43, no. 12, pp. 3046–3050, Dec. 1995.