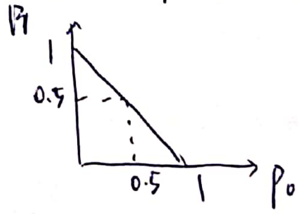
Classical Probability Theory

Probability — coin flipping

$P(X=H)=p_0$

$P(X=T)=p_1$

$p_i \geq 0,\ p_i \in \mathbb{R},\ \sum_i p_i = 1 \rightarrow p_1 = 1-p_0$

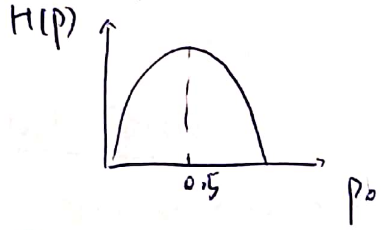


Entropy

$\Rightarrow$ characterise the unpredictability.

$H(p) = -\sum_i p_i \log_2 p_i$

When all outcome with the same probability, this is the most unpredictable case.



Geometry

$\vec{p} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \quad p_i \geq 0$

stochastic vector

normalize to 1 with $\begin{cases} \text{norm } 1: a_0 + a_1 = 1 \\ \text{norm } 2: \sqrt{|a_0|^2 + |a_1|^2} = 1 \end{cases}$

$\sum_i p_i = \sum_i |p_i| = \|\vec{p}\|_1 = 1$

We normalise the P vector in 1 norm

[In quantum states, the normalization will be in a different norm]

Transform probability distributions

transform stochastic vectors

$M\vec{p} = \vec{p}',\ p_i \geq 0$

stochastic matrix

$\|\vec{p}'\|_1 = 1$

In quantum ~~states~~ calculations, the stochastic matrix should be unitary, and ~~they~~ it can transform quantum states to other quantum states.

Quantum States

Quantum state

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ $a_i \in \mathbb{C}$ (complex values)

$\| |\psi\rangle \|_2 = 1$ → the normalization of this vector happens in norm 2.

$\sqrt{|a_0|^2 + |a_1|^2} = 1$

Two-level quantum states: qubits.

Superposition:

A quantum state is also called a wave function.

$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_0|0\rangle + a_1|1\rangle$

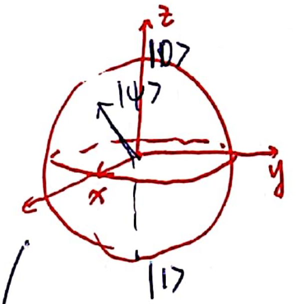
↓ collapse of the wave function

random to deterministic ⟸ out come 0 with prob. $|a_0|^2$

state after the measurement: $|0\rangle$

Bloch Sphere:

Now we have a 2-dimensional complex space, which will take 4-dimensions to visualize, with the restriction $\left(\sqrt{|a_0|^2 + |a_1|^2} = 1\right)$ on the degree of freedom. Thus, we'll ~~have~~ use a 3-dimentional object to visualize ~~quan~~ qubit states.



classical:

Attention: the orthognality is a little ~~bit~~ different in this s~~k~~phere.

[the $|0\rangle$ and $|1\rangle$ are on the same line in this sphere].

Every single point on this sphere is a qubit state.

Every single probability distribution lies on a straight line.

Interference: (干涉)

Can do on quantum computers, but can't on classical ones.

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |1\rangle$

Multiple qubits.

Tensor product

$$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$|\psi'\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$|\psi\rangle \otimes |\psi'\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

$$|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |1\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

· Convention: rightmost qubit is qubit 0.

Beyond product states

$$|\psi\rangle \otimes |\psi'\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

it's in the same space as a product vector, but can't be written in a product vector.

$$|\psi\rangle \otimes |\psi'\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix} = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle$$

Associate $|\Phi^+\rangle = |\psi\rangle \otimes |\psi'\rangle$

$$\therefore \begin{cases} a_0 b_0 = \frac{1}{\sqrt{2}} \\ a_0 b_1 = 0 \longrightarrow a_0 \text{ or } b_1 \text{ has to be } 0, \\ a_1 b_0 = 0 \\ a_1 b_1 = \frac{1}{\sqrt{2}} \end{cases}$$

but if $a_0 = 0$, $a_0 b_0 \neq \frac{1}{\sqrt{2}}$,
if $b_1 = 0$, $a_1 b_1 \neq \frac{1}{\sqrt{2}}$

Thus $|\Phi^+\rangle$ can't be writen as product state.

Such states are called entangled states.

Measurement

Bra-ket notation:

Ket: $|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ $\rightarrow$ $a_0 = x_0 + iy_0$

$\bar{a}_0 = x_0 - iy_0$

Bra: $\langle\psi| = |\psi\rangle^{\dagger} = [\bar{a}_0 \ \bar{a}_1]$

Conjugate transpose

Dot Product

$\langle\psi|\psi\rangle = |a_0|^2 + |a_1|^2 = 1$ $= \| |\psi\rangle \|_2^2$ $\Rightarrow$ bra-ket

$\langle 0|1\rangle = [1 \ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ $\Rightarrow$ scala value

ket-bra

$|0\rangle\langle 0| = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ matrix

Measurements

Intuition: measurement is very similar to random variable.

measurement outcome ~ value a random variable

Born Rule

Outcome 0 with prob. $|a_0|^2$

State afterwards: $|0\rangle$ "collapse of the wavefunction"

The measurement outcome is actually a projection.

e.g. $|0\rangle\underbrace{\boxed{\langle 0|\psi\rangle}}_{\text{scala}} = \cancel{|0\rangle} \ |0\rangle \cdot a_0$

$\langle\psi|0\rangle\langle 0|\psi\rangle = \| a_0 |0\rangle \|_2^2 = |a_0|^2$

State afterwards : $\dfrac{|0\rangle\langle 0|\psi\rangle}{\sqrt{\langle\psi|0\rangle\langle 0|\psi\rangle}}$ if we observe the output i.

MIXED STATES

Mixed states

$|\psi\rangle$: pure quantum state

A pure quantum state is a state which can be described by a single ket vector. While a mixed quantum state is a statistical ensemble of pure states. A mixed state is described by its density matrix (density matrix can also work for pure state

$\rho = |\psi\rangle\langle\psi|$ density matrix

Operations on kets can be rewritten as operations on density matrices.

e.g. Born rule : $\underset{\substack{\uparrow \\ \text{trace}}}{Tr}[|0\rangle\langle 0|\rho] \Rightarrow$ the probability of seeing 0.

trace, which is the sum of diagnal elements.

Why density matrix?

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum p_i = 1$ → classical ignorance: this is sth. that we don't know about the underlying quantum system.

for mixed states

classical probability distributions over pure states

e.g. ① $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \rightarrow \rho = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ → density matrix for an equal superposition

② $\rho' = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ → desity matrix for the equally mixed state of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$

diagnal

off-diagnal also called coherence

In ② ~~exp~~ example, the off-diagnal elements are gone. they're also called coherence, and their presence indicates that the state is quantum. The smaller these values are, the closer the quantum state is to a classical probability distribution.

not just a mixed state, but also a maximally mixed state,

A maximally mixed state is the equivalent of a uniform distribution the entropy is maximum, and we have no predictive power of what's going to happen ~~next~~.

Measuring multi-qubit systems

$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ → entangled

if we measure the ~~oth~~ 0th qubit (the right most one) and get the outcome 0:

$(\mathbb{1} \otimes |0\rangle\langle 0|)|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle$

$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then if we measure the other qubit, we must get 0 deterministicly.

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Partial trace: ~~matrix~~ marginal probability

$$S = |\phi^+\rangle\langle\phi^+| = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

$$Tr_1\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} = \begin{bmatrix} a+f & c+h \\ i+n & k+p \end{bmatrix}$$

$$\therefore Tr_1[S] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$ $\Rightarrow$ this is the same result with the maximally mixed state that ~~we int~~ introduced in previous page.

This means that if we ~~may~~ marginalize out one of the qubits in this system, then we end up with a uniform distribution.

$\downarrow$ after measuring 1 qubit, the other is deterministic

the entropy

we have no predictive power over what's going to happen in that remaining quantum system.

Evolution in Closed Systems → Ideal

Classically:

Stocastic vector → $\boxed{\text{stocastic matrix}}$ → stochastic vector'

$M\vec{p} = \vec{p}'$, $\vec{p} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$ where $p_i \geq 0$, $\sum_i p_i = 1$

colums add up to 1

Quantum:

$U|\psi\rangle = |\psi'\rangle$

U: unitary → $UU^{\dagger} = \mathbb{I} = U^{\dagger}U$

$\dagger$ → complex conjugate transpose

properties: l_2-norm, linear, reversible

The cojugate transpose of U is also its inverse.

Example:

① $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$X^{\dagger} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$

$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

$X|1\rangle = |0\rangle$

② $X|\psi\rangle = X(a_0|0\rangle + a_1|1\rangle) =$

$= a_0 X|0\rangle + a_1 X|1\rangle = a_0|1\rangle + a_1|0\rangle$

$X^{\dagger}(a_0 X|0\rangle + a_1 X|1\rangle) = |\psi\rangle$ ⇒ reversible

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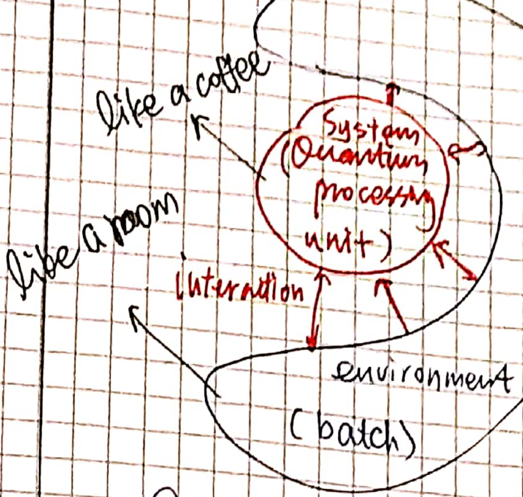
① Open Quantum Systems

real quantum systems are not closed, but open system.



The system interacts with env.

⇓

uncontrolled

The evolution is no longer unitary!

①

e.g. Decoherence → take this as an example.

$$\underbrace{v}_{\text{visibility},\ v\in[0,1]}\, S + (1-v)\frac{\mathbb{I}}{d} \rightarrow \text{random noise} = \frac{\text{identity matrix}}{\text{No. of dimensions}}$$

We can control how much noise we have by tuning v.

When $v=1$ ⇒ pure state

When $v<1$ ⇒ mixed state

The lower the v, the more noise we mixed, the more coherences vanish

$$\underset{v=1}{\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}} \rightarrow \begin{pmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{pmatrix} \rightarrow \underset{v=0}{\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}}$$

$\xrightarrow{\text{v becomes lower}}$

$v=0$ ⇓ maximally mixed state.

⇓

The speed of v becoming lower affects quantum calculations, and the time before it completely decoheres is called T_2 time.

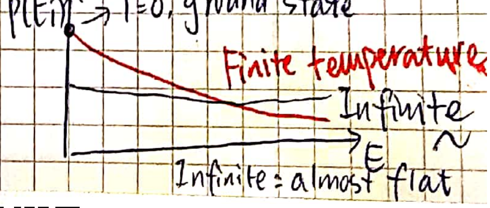
T_2 time: an important property for building quantum computer.

② Equilibration (平衡) ⇒ like a cup of coffee, cools down in a room.

the energy follows this distribution ⇒ $P(E_i) = \dfrac{\exp(-\frac{E_i}{T})}{Z}$

$$Z = \sum_i \exp\left(-\frac{E_i}{T}\right)$$

The higher the temperature (T), the closer to uniform distribution

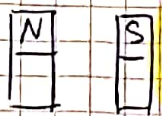


Classical Ising model

Classical and Quantum Many-body Systems

Spin System

magnets {

① 

$\sigma_1 = +1$ $\sigma_2 = -1$

energy: $\sigma_1\sigma_2 = -1$

$\sigma_1, \sigma_2, \sigma_3$: random variables

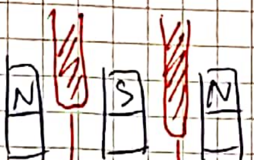
② 

$\sigma_1 = +1$ $\sigma_2 = -1$ $\sigma_3 = +1$

energy: $\sigma_1\sigma_2 + \sigma_2\sigma_3 = -\sum_{\langle i,j \rangle} \sigma_i\sigma_j$

the "−" here: convention

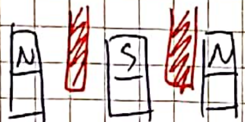
$\langle i,j \rangle$ stands for nearest neighbor

③ 

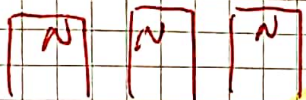
energy: $-\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j$

add some coefficients for the walls

walls, which can change the interaction strength between them

④ 

the operater that describe system energy — Hamiltonian



⇒ external magnetic field.

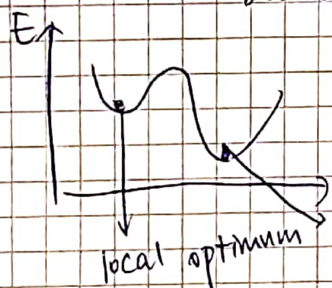
can flip the magnets upside down.

e.g. it can flip the 2nd magnet upside down.

There will be some ~~fur~~ frustration between the 1st and 3rd magnets, but the strength of the field will overide it.

energy: $-\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$

between magnets and magnetic fields



In real cases, because the influence of environment, they can get stuck in the local minimum, but there isn't enough energy to hop out and get to the global optimum point

Very similar to NP-hard Problem in CS.

$$\min \sum_{ij} W_{ij}x_ix_j + \sum_i b_ix_i \qquad x_i, x_j \in \{0,1\}$$

Quadratic unconstrained binary optimization

Transverse Ising model

Classical Ising model rewritten

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^z|0\rangle = |0\rangle \qquad \sigma^z|1\rangle = -|1\rangle$$

$$H = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \Rightarrow H = \sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^z$$

(Classical) (Quantum)

Every single one of the matrices commutes with each other

Commuting Hamiltonian ↔ Classical system

Hamiltonian → total energy of the system, denoted by H.

$$\langle H\rangle = \langle\psi|H|\psi\rangle$$

Transverse Field

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ → add one more type of interaction

$$\sigma^x\sigma^z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^z\sigma^x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

↓

it matters in which order you use the two operators.

$$H = \underbrace{-\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum h_i\sigma_i^z}_{①} \underbrace{- g_i\sigma_i^x}_{②}$$

The eigen value of operator will correspond to the energy:

Eigenvectors of σ^z: $|0\rangle, |1\rangle$ → same with classical Ising model

Eigen vectors of σ^x: $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ → with superposition

∵ The lowest eigenvalue will correspond to the lowest energy for the particular operator in that site.

∴ ② will try to push it into superposition.

① will be deterministically 0 or 1.

②: non-commuting term, Quantum effect.