0. A crash Course on Quantum Information

0.1 Classic VS Quantum Bits

Lecture 1. the Qubit

① Mathematical description of qubit

Classical bits

$0 \to |0\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$

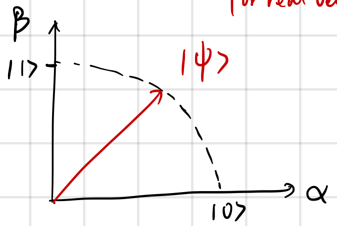
$1 \to |1\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$

Qubits

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Superposition of $|0\rangle$ and $|1\rangle$

For real vectors $\alpha^2 + \beta^2 = 1$



② What do qubits look like

e.g.1 In a maze, classically, a particle will either go left or go right.
quantumly, it can go left and right at the same time

e.g.2 Physical implementation: represent bits by energy levels in an atom

classical bits: in either ground or excited state

Qubits: In two states at the same time

$|1\rangle$ ——— Excited state

$|0\rangle$ ——— Ground state

③ Basis and amplitudes

Standard basis / Computational basis:

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Qubit:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ → A qubit is a element of a complex vector space of dimetion 2

amplitudes

Kets and bras:

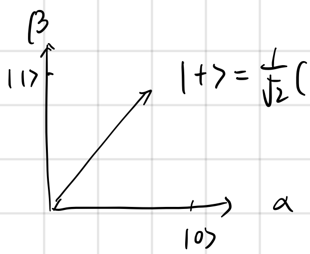
ket ⇒ $|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

bra ⇒ $\langle\psi| = (|\psi\rangle^*)^T = \begin{pmatrix}\alpha^*\\\beta^*\end{pmatrix}^T = (\alpha^* \ \ \beta^*)$

Conjugate, Transpose

Inner product ⇒ $\langle\psi||\psi\rangle = \langle\psi|\psi\rangle = (\alpha^* \ \ \beta^*)\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2 = 1$

e.g.



$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\langle +|+\rangle = \frac{1}{2}(1 \ \ 1)\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}(1+1) = 1$

Lecture 2 More than One Qubit

① Standard / computational basis:

$x = x_1 x_2 \cdots x_n \in \{0,1\}^n$, $d = 2^n$ possible strings

$x \to |x\rangle \to \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ Make the vector zero everywhere except at the position indexed by x.

e.g. $n=0 \Rightarrow |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ⇐ Quantum state of one qubits

② Quantum States of n qubits:

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} a_x |x\rangle, \quad \sum_{x \in \{0,1\}^n} |a_x|^2 = 1 \leftarrow \text{length 1 vector}$$

$\therefore |\psi\rangle \in \mathbb{C}^d$ with $d = 2^n$, $\langle\psi|\psi\rangle = 1$

e.g.1 two qubits in equal superposition

Standard basis for 2 qubis:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Equal superposition:

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

e.g.2. two qubits in an EPR pair

$$|\psi\rangle = |EPR\rangle = \underbrace{\frac{1}{\sqrt{2}}}_{\text{normalization}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$$

$$= \frac{1}{2}\Big(\underbrace{\langle 00|00\rangle}_{=1} + \underbrace{\langle 00|11\rangle}_{=0} + \underbrace{\langle 11|00\rangle}_{=0} + \underbrace{\langle 11|11\rangle}_{=1}\Big)$$

$$= \frac{1}{2}(1+1) = 1$$

e.g.3 Another two qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

0.2 Combining qubits using the tensor product

Lecture 1. Combining qubits using the tensor product

① How do we combine two qubits?

Two seperate qubits:

Ⓐ $|\psi\rangle_A = \alpha_A|0\rangle_A + \beta_A|1\rangle_A$

Ⓑ $|\psi\rangle_B = \alpha_B|0\rangle_B + \beta_B|1\rangle_B$

What is $|\psi\rangle_{AB}$?

Tensor product:

$$|\psi\rangle_A \otimes |\psi\rangle_B = \begin{pmatrix} \alpha_A \\ \beta_A \end{pmatrix} \otimes |\psi\rangle_B = \begin{pmatrix} \alpha_A|\psi\rangle_B \\ \beta_A|\psi\rangle_B \end{pmatrix} = \begin{pmatrix} \alpha_A\alpha_B \\ \alpha_A\beta_B \\ \beta_A\alpha_B \\ \beta_A\beta_B \end{pmatrix}$$

e.g. Constructing the standard basis for two qubits:

Standard basis for 2 qubis:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Another way to get ⇑, is to form the standard basis by combining the basis of two single qubits.

$\{|0\rangle_A, |1\rangle_A\}$, $\{|0\rangle_B, |1\rangle_B\}$

$$|0\rangle_A \otimes |0\rangle_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

② Lazy notation

$$|\psi\rangle_A \otimes |\phi\rangle_B = |\psi\rangle_A |\phi\rangle_B$$

e.g. $|0\rangle_A \otimes |0\rangle_B = |0\rangle_A|0\rangle_B = |00\rangle_{AB}$

$$|\psi\rangle_1 \otimes \cdots \otimes |\psi\rangle_n = |\psi\rangle^{\otimes n}$$

③ Properties of tensor product

- △ distributive: $|\psi\rangle \otimes (|V_1\rangle + |V_2\rangle) = |\psi\rangle \otimes |V_1\rangle + |\psi\rangle \otimes |V_2\rangle$
- △ associative: $|\psi\rangle \otimes (|\Phi\rangle \otimes |\Gamma\rangle) = (|\psi\rangle \otimes |\Phi\rangle) \otimes |\Gamma\rangle$
- △ not commutative: $|\psi\rangle \otimes |\Phi\rangle \neq |\Phi\rangle \otimes (|\psi\rangle)$

e.g.1 Combining two qubits

$|+\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $|1\rangle_B$

① $|+\rangle_A \otimes |1\rangle_B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes|1\rangle_B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\cdot|1\rangle_B\\1\cdot|1\rangle_B\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

② $|+\rangle_A \otimes |1\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\otimes|1\rangle_B$

$= \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |1\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |11\rangle_{AB})$

$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

e.g.2 Reconstructing equal superposition

$|+\rangle_A$, $|+\rangle_B$

$|+\rangle_A \otimes |+\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A+|1\rangle_A)\otimes\frac{1}{\sqrt{2}}(|0\rangle_B+|1\rangle_B)$

$= \frac{1}{2}(|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} + |11\rangle_{AB})$

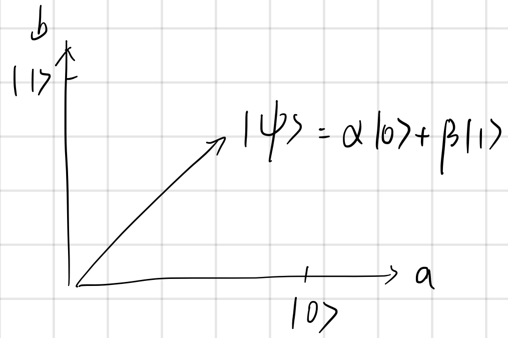
$= \frac{1}{2}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$

$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

0.3 Measuring quantum bits

Lecture 1. Measuring qubits in standard basis

Measured in standard basis ↓



$P_0 = |\langle\psi|0\rangle|^2 = \left|(\alpha^* \ \beta^*)\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \alpha^2 \Rightarrow$ outcome "$|0\rangle$" $\Rightarrow$ collapse to $|0\rangle$

$P_1 = |\langle\psi|1\rangle|^2 = \left|(\alpha^* \ \beta^*)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 = \beta^2 \Rightarrow$ outcome "$|1\rangle$" $\Rightarrow$ collapse to $|1\rangle$

$P_0 + P_1 = 1 = |\alpha|^2 + |\beta|^2$

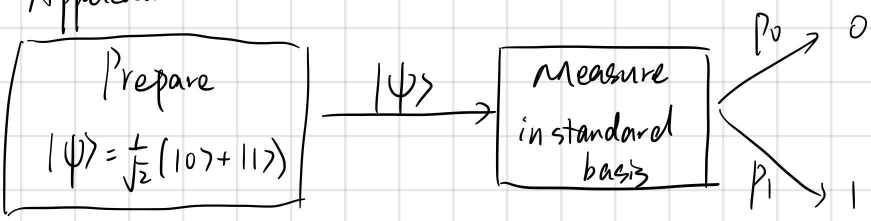
We can generate genuine randomness from a deterministic process.
(classical world 不可能)

⇑

lost the information of α and β

⇑

Application:



$$p_0 = |\langle \psi | 0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)|0\rangle\right|^2 = \frac{1}{2}\left(\underbrace{\langle 0|0\rangle}_{=1} + \underbrace{\langle 1|0\rangle}_{=0}\right)^2 = \frac{1}{2}$$

(because of standard basis, orthogonal.

Similarly, $p_1 = |\langle \psi | 1\rangle|^2 = \cdots = \frac{1}{2}$

Lecture 2. Measuring qubits in another basis