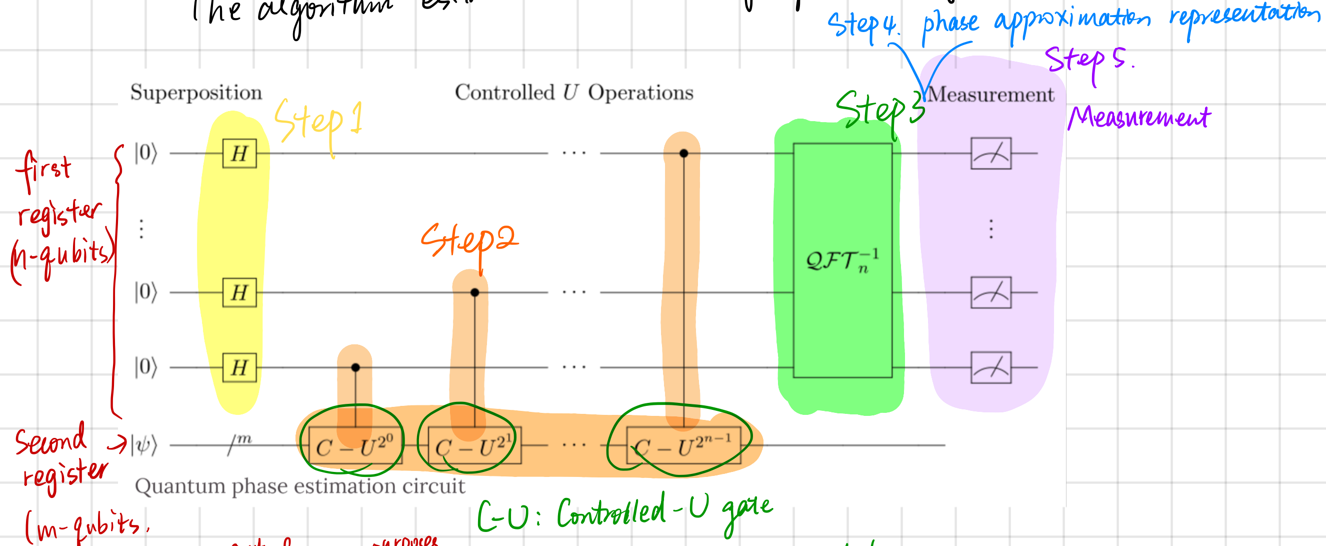
Quantum Phase Estimation Algorithm

- estimate the phase (eigenvalue) of an eigenvector.
- Given unitary operator U and quantum state $|\psi\rangle$,

$$\Rightarrow U\underbrace{|\psi\rangle}_{\text{eigen vector}} = \underbrace{e^{2\pi i\theta}}_{\text{eigen value}}|\psi\rangle$$

The algorithm estimates θ with high probability within error ε, using $O(1/\varepsilon)$ C-U operations.



C-U: Controlled-U gate

the 1st bit serves as a control:

$|00\rangle \xrightarrow{C-U} |00\rangle$

$|01\rangle \xrightarrow{C-U} |1\rangle \otimes U|0\rangle = |1\rangle \otimes (u_{00}|0\rangle + u_{10}|1\rangle)$

$|10\rangle \xrightarrow{C-U} |1\rangle \otimes U|1\rangle = |1\rangle \otimes (u_{01}|0\rangle + u_{11}|1\rangle)$

The matrix of C-U is: $C(U) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{00} & u_{01} \\ 0 & 0 & u_{10} & u_{11} \end{bmatrix}$

- The explanation of whole process:

Step 1 Hadamard gate

$$\underbrace{|0\rangle^{\otimes n}}_{\text{1st register}} \underbrace{|\psi\rangle}_{\text{2nd register}}$$

$$\downarrow H^{\otimes n}$$

$$\frac{1}{2^{\frac{n}{2}}}(|0\rangle + |1\rangle)^{\otimes n}$$

Step 2. Apply C-U.

$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$

$\downarrow$ C-U $\quad U^{2^j}|\psi\rangle = U^{2^j-1}\cdot U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = e^{2\pi i 2^j\theta}|\psi\rangle$

$$\underbrace{\frac{1}{2^{\frac{n}{2}}}\underbrace{(|0\rangle + e^{2\pi i 2^{n-1}\theta}|1\rangle)}_{\text{1 st qubit}} \otimes\cdots\otimes \underbrace{(|0\rangle + e^{2\pi i 2^1\theta}|1\rangle)}_{(n-1)^{th}\text{ qubit}} \otimes \underbrace{(|0\rangle + e^{2\pi i 2^0\theta}|1\rangle)}_{n^{th}\text{ qubit}}}_{\text{1st register}}$$

$\Rightarrow$ the implementation method of C-U gate is introduced in Page 3.

$$= \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle$$

$\downarrow$ inverse quantum Fourier transform

Step 3. inverse QFT

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$$\frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle$$

Step 4. iQFT

inverse QFT

$$\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k} e^{\frac{-2\pi i k x}{2^n}}|x\rangle = \underbrace{\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}|x\rangle}_{\text{1st register}}$$

Thus the 2 registers are now:

$$\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}|x\rangle\otimes|\psi\rangle$$

Step 4. phase approximation

Approximate θ by rounding $2^n\theta$ to the nearest integer:

$$\underset{\substack{\downarrow\\ \text{nearest integer}}}{2^n\theta = a} + 2^n\delta,\quad 0\le|2^n\delta|\le\frac{1}{2}$$

$$\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-a)}\cdot e^{2\pi i\delta k}|x\rangle\otimes|\psi\rangle$$

measurement: probability that $|\psi\rangle$ collapse to $|a\rangle$ after measurement $=|\langle a|\psi\rangle|^2$

Step 5. measurement

$$\Pr(a)=\left|\left\langle a\,\middle|\,\underbrace{\frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{\frac{-2\pi i k}{2^n}(x-a)}e^{2\pi i\delta k}\,\middle|\,x}_{\text{State of the first register}}\right\rangle\right|^2=\frac{1}{2^{2n}}\left|\sum_{k=0}^{2^n-1}e^{2\pi i\delta k}\right|^2=\begin{cases}1 & \delta=0\\ \frac{1}{2^{2n}}\left|\frac{1-e^{2\pi i 2^n\delta}}{1-e^{2\pi i\delta}}\right|^2 & \delta\neq 0\end{cases}$$

geometric progression

$$|2^n\theta\rangle\otimes|\psi\rangle$$

for $\delta\neq 0$, since $0\le|2^n\delta|\le\frac{1}{2}\Leftrightarrow|\delta|\le\frac{1}{2^{n+1}}$,

If U is an exact binary fraction, we can measure its phase with probability 1. o.w., we can measure it with probability close to 1.

$$\Pr(a)=\frac{1}{2^{2n}}\left|\frac{1-e^{2\pi i 2^n\delta}}{1-e^{2\pi i\delta}}\right|^2 \quad \text{for } \delta\neq 0$$

$$=\frac{1}{2^{2n}}\left|\frac{\sin(\pi\cdot 2^n\delta)}{\sin(\pi\delta)}\right|^2 \Leftarrow |1-e^{2ix}|^2=4|\sin(x)|^2$$

$$\ge\frac{1}{2^{2n}}\left|\frac{\sin(\pi\cdot 2^n\delta)}{\pi\delta}\right|^2 \Leftarrow$$

$\sin(x)\le x$

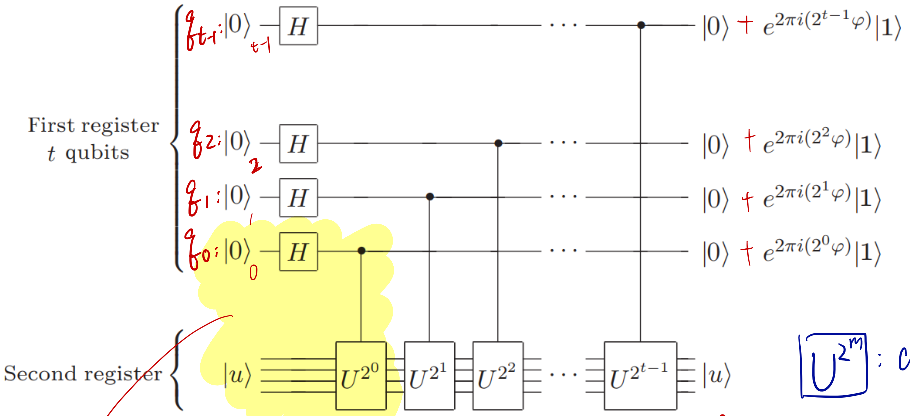
$$\ge\frac{1}{2^{2n}}\left|\frac{2\cdot 2^n\delta}{\pi\delta}\right|^2 \Leftarrow |2\cdot 2^n\delta|\le|\sin(\pi 2^n\delta)| \text{ for } |\delta|\le\frac{1}{2^{n+1}}$$

$$=\frac{4}{\pi^2}$$

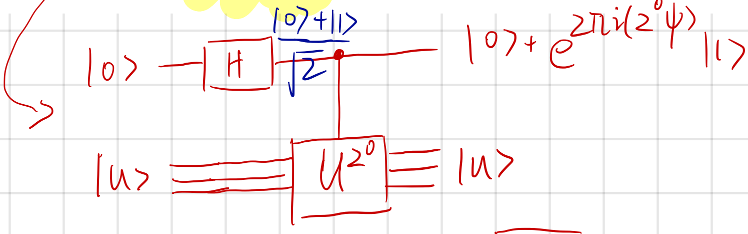
$\therefore$ The probability for getting correct result if $\delta\neq 0$ is:

$\Pr(a)\ge\frac{4}{\pi^2}\approx 0.405$. $\Rightarrow$ Can be increased to $1-\varepsilon$ when increasing the amount of qubits by $O(\log(\frac{1}{\varepsilon}))$

How to understand the Controlled-U gate?

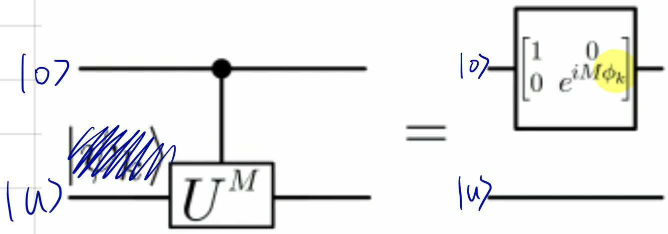


$\boxed{U^{2^m}}$: controlled-U gates

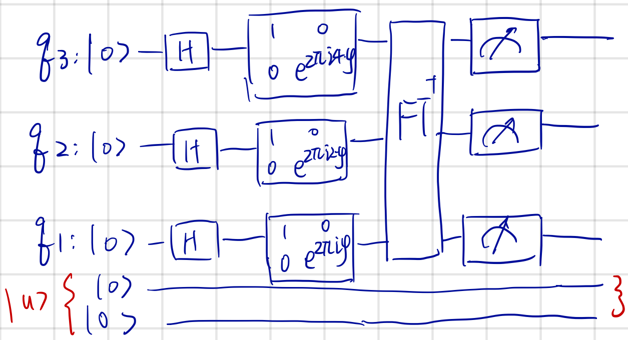


$\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ is the control qubit of gate $\boxed{U^{2^0}}$, and $|u\rangle$ is the target qubits, However, $|u\rangle$ doesn't make any change after the gate, instead, $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ becomes $\frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i(2^0\varphi)}|1\rangle\right]$ after the Controlled-U gate.

Thus the controlled-U gate can be replaced by a controlled phase gate:



Because the above two circuits are the same, when implementing the controlled phase gate, an example circuit is (1st register has 3 qubits, 2nd register has 2 qubits):



} These 2 qubits don't make any change.

[measurement symbol]: measurement

During my implementation, I use $\varphi = \frac{1}{8}$ as an example.

Then after the measurement, I get 001, then divided by 2^t ($t=3$), I get 0.001, which is $\frac{1}{8}$.

q_3 q_2 q_1

same with the φ in the Controlled phase gates.