Quantum Key Distribution (QKD) 量子密钥分发

△ Key Distribution

usually
key: One-time pad technique — Data is encrypted using a truly random key of the same length as the data being encrypted.

plain text —encryption→ cipher text

甲方 —公开信道→ 乙方

信息(明文) —key1→ 密文 —key2→ 明文

if key1 == key2: 对称密码/私密密码 (symmetric cryptography)

else: 非对称密码/公开密码.

△ Current key distribution methods:

usually use public key ciphers { RSA, Diffie-Hellman, ECC

These ciphers are based on upon mathematical calculations that are simple to compute, but require an infeasible amount of processing power to invert.

比如: 求2个质数的乘法很简单,但做大数质因式分解就比较麻烦.

△ { BB84 (Charles H.B, Gilles B., 1984): the first quantum cryptography protocol. 基于海森堡不确定原则
E91 (Artur E., 1991) 基于量子纠缠

1. BB84 protocol: communicating a private key in one-time pad encryption.

⇓

Make the right information publicly know at the right times, and keep the secret information secret.

一次性密码本 OTP { 用以加密的文本,也就是一次性密码本,必须是随机产生的.
它至少需和被加密文件等长
用以加密的文本只能用一次,而且必须对非关系人非常保密.不再使用时应销毁密码本,以防重复使用.

如果密码比加密信息短, We will be forced to use parts of the key more than once.

加密算法举例: 加密 "This is an example".

可以用加密而一次性密码本: MASKL NSFLD FKJPQ

This is an example ⇒ 19 7 8 18 | 8 18 | 0 13 | 4 23 0 12 15 11 4
MASKL NSFLD FKJPQ ⇒ 12 0 18 10 | 11 13 | 18 5 | 11 3 5 10 9 15 16

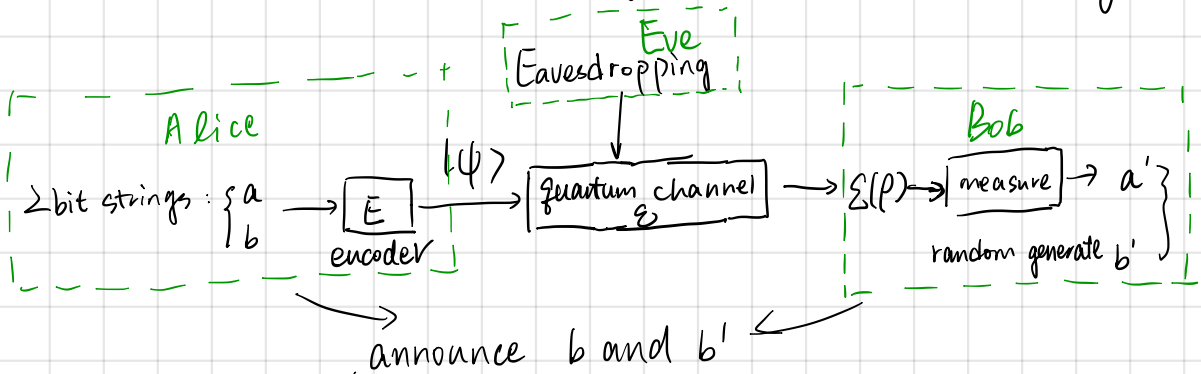
相加后得: 31 7 26 28 | 19 31 | 18 18 | 15 26 5 32 24 26 20

因为一共26个字母⇒模26: 5 7 0 2 | 19 5 | 18 18 | 15 0 5 22 24 0 20

得到b加密后的: F H A C | T F | S S | P A F W Y A U

BB84:

Part 1. 构建一个Alice与Bob之间的shared secret key.



both Alice and Bob discard the qubits in a and a' where b and b' don't match
(用概率来看,平均下来b和b'应有一半是一致的)

以下是对这框图的具体解释:

1.1 Encoder: $|\psi\rangle = \bigotimes_{i=1}^{n} |\psi_{a_i b_i}\rangle \Rightarrow a_i, b_i$: i-th bits of a, b ; and a_i, b_i follows:

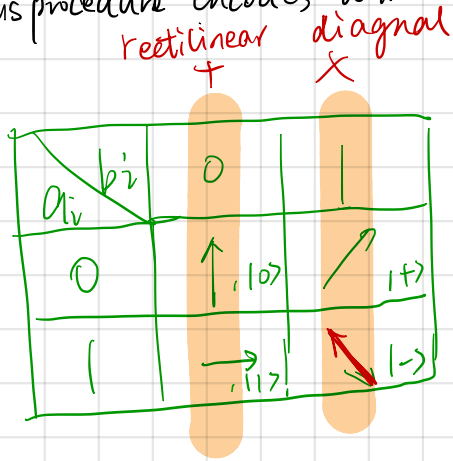
对光源偏振

0	↑	$\|\psi_{00}\rangle = \|0\rangle$
1	→	$\|\psi_{10}\rangle = \|1\rangle$
0	↗	$\|\psi_{01}\rangle = \|+\rangle = \frac{\sqrt{2}\|0\rangle + \sqrt{2}\|1\rangle}{2}$
1	↖	$\|\psi_{11}\rangle = \|-\rangle = \frac{\sqrt{2}\|0\rangle - \sqrt{2}\|1\rangle}{2}$

→ BB84用2 pairs of states. each pair conjugate to the other.
two states within a pair orthogonal with each other.
↓ called "basis"

→ wiki给出的方向定义:应为135°,而非-45°.

This procedure encodes a in the basis "+" or "X", as determined by b.



a_i: random bit created by Alice

b_i: for each a_i, Alice randomly chooses one of the two bases according to b_i and transmit a_i in.
↓
根据左边的表

The qubits are now in states that aren't mutually orthogonal, so it's impossible to distinguish all of them with certainty without knowing b.
→ 2个 vectors orthogonal: 它们相互垂直,点积得0.
↘ a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_n\}$ mutually orthogonal: every pair of vectors is orthogonal

1.2 $\mathcal{E}(\rho)$: quantum channel 后 Bob收到的 state.

$\mathcal{E}(\rho) = \mathcal{E}(|\psi\rangle\langle\psi|)$ // $\mathcal{E}$ represents both the noise and eavesdropping by a 3rd party, Eve //
// $|\psi\rangle$ is what Alice sent over a public and authenticated quantum channel //

注: $|\psi\rangle$ (ket)是 $\langle\psi|$ (bra)的共轭转置

inner product of ket and bra: $\langle\phi|\psi\rangle = (\langle\phi|)(|\psi\rangle)$

outer product: $|\phi\rangle\langle\psi|$, $(|\phi\rangle\langle\psi|)(|\chi\rangle) = \langle\psi|\chi\rangle|\phi\rangle$

For a finite-dimensional vector space, the outer product can be understood as simple matrix multiplication:

$$|\phi\rangle\langle\psi| \doteq \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_N\end{pmatrix}\begin{pmatrix}\psi_1^* & \psi_2^* & \cdots & \psi_N^*\end{pmatrix} = \begin{pmatrix}\phi_1\psi_1^* & \phi_1\psi_2^* & \cdots & \phi_1\psi_N^*\\ \phi_2\psi_1^* & \phi_2\psi_2^* & \cdots & \phi_2\psi_N^*\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N\psi_1^* & \phi_N\psi_2^* & \cdots & \phi_N\psi_N^*\end{pmatrix}$$

The outer product is an $N \times N$ matrix, as expected for a linear operator.

在Bob收到他的states后，Alice、Bob、Eve都有他们的states. 但只有Alice知道 b(Alice decides which basis Alice is encoded in)，所以Bob和Eve不可能distinguish the states of qubits.

而且，根据不可克隆原理(no-cloning theorem)，除非Eve做出measurement，否则她不能拥有一个发到Bob那里的qubits的copy. 做measurement有主可能性得到wrong answer.

no-cloning theorem: it is impossible to create an identical copy of an arbitrary unknow quantum state.

即不可能构造一个能够完全复制任意量子比特，却不对原始量子位产生干扰的系统.

↓

Quantum Information 和 Quantum Cryptography 的基础.

量子原理：量子力学的线性特征

量子信息在信道中传输，不可能被第三方复制而窃取信息，却不对量子信息产生干扰.

1.3 Bob proceeds to generate a string of random bits b' of the same length as b and then measures the string he has received from Alice, a'. At this point, Bob announces publicly that he has received Alice's transmission. Alice then knows she can now safely announce b. Bob communicates over a public channel with Alice to determine which b_i and b'_i are not equal. Both Alice and Bob now discard the qubits in a and a' where b and b' do not match, which is half on average, leaving half the bits as a shared key.

1.4

From the remaining k bits where both Alice and Bob measured in the same basis, Alice randomly chooses $k/2$ bits and discloses her choices over the public channel. Both Alice and Bob announce these bits publicly and run a check to see whether more than a certain number of them agree. If this check passes, Alice and Bob proceed to use information reconciliation and privacy amplification techniques to create some number of shared secret keys. Otherwise, they cancel and start over.

disclose: 透露.

信息协调和隐私放大

整个过程举个例子：

a ⇒	Alice's random bit	0	1	1	0	1	0	0	1
b ⇒	Alice's random sending basis	+	+	×	+	×	×	×	+
a'/a根据b编码后⇒	Photon polarization Alice sends (发送给Bob的光子)	↑	→	↖	↑	↖	↗	↗	→
b' ⇒	Bob's random measuring basis	+	×	×	×	+	×	+	+
根据b'观测a'后. ⇒	Photon polarization Bob measures	↑	↗	↖	↗	→	↗	→	→
	PUBLIC DISCUSSION OF BASIS								
	Shared secret key	0		1			0		1

相当于：

有"×"的地方正做一次Hadamard变换，"+"时不变

有"×"的地方反做一次Hadamard变换，有"+"时不变

↓

下一页会对这个过程详细说明.

但这一行会有出入

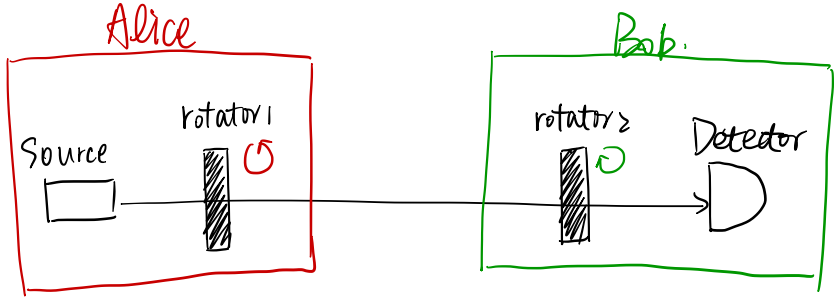
BB84 算法过程详解（依据的是Qiskit的教程，可能与上次最后来自wiki的步骤不一致）

Create shared secret key 的过程

initial key a	0	1	1	0	1	0	0	1
⇓转换成qubit								
initial key a (qubit)	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
randomly Hadamard 变换	Hadmard transform on a if basis is "x". do nothing for "+"							
random rotation string b	0	0	1	0	1	1	1	0
⇓ 0 stands for "+", 1 stands for "x"								
对应的 rotation basis	+	+	×	+	×	×	×	+
传输前的 key (transmit photons) 数学表达	$\|0\rangle$	$\|1\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
方向表达	↑(90°)	→(0°)	↖(135°)	↑(90°)	↖(135°)	↗(45°)	↗(45°)	→(0°)
Quantum Channel								
传输后的 key	$\|0\rangle$	$\|1\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
randomly inverse Hadamard 变换	Inverse Hadmard transform on a if basis is "x". do nothing for "+"							
random rotation string b'	0	1	1	1	0	1	0	0
⇓ 0 stands for "+", 1 stands for "x"								
对应的 rotation basis	+	×	×	×	+	×	+	+
最终的 qubit key	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|1\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
⇓measure a'	0	0或1	1	0或1	0或1	0	0或1	1

b和b'相同位上的a和a'也一定相同，它们组成新的key

Shared secret key 〖以相同字符其长度平均值为 a/a'/b/b' 长度的一半〗



注：

1. 方框图表示：
   - 一：Alice随机产生的 bit string
   - 一：Bob随机产生的 bit string
2. randomly选择 "+" 或 "-" basis 的过程就相当于 randomly 施以 Hadamard 变换

$|0\rangle \xrightarrow{H} \frac{\sqrt{2}|0\rangle+\sqrt{2}|1\rangle}{2}$

$|1\rangle \xrightarrow{H} \frac{\sqrt{2}|0\rangle-\sqrt{2}|1\rangle}{2}$

Hadamard gate 的矩阵表示：$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$

3. Inverse transform of Hadamard gate:

$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}$,

$H^{-1}=\frac{1}{-\frac{1}{2}-\frac{1}{2}}\begin{bmatrix}-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

∴ Hadamard 逆变换 = Hadamard 变换

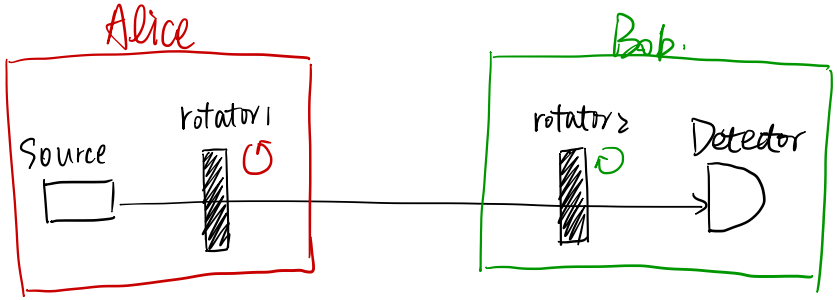
BB84 Protocol (English version)

The process of creating shared secret key

initial key a	0	1	1	0	1	0	0	1
⇓change into qubit								
initial key a (qubit)	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
randomly Hadamard transform	Hadmard transform on a if basis is "x". do nothing for "+"							
random rotatation string b	0	0	1	0	1	1	1	0
⇓ 0 stands for "+", 1 stands for "x"								
corresponding rotation basis	+	+	×	+	×	×	×	+
the qubit key to be transmitted (transmit photons) — dirac notation	$\|0\rangle$	$\|1\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
the qubit key to be transmitted (transmit photons) — direction	↑(90°)	→(0°)	↖(135°)	↑(90°)	↖(135°)	↗(45°)	↗(45°)	→(0°)
Quantum Channel								
the key after transmission	$\|0\rangle$	$\|1\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
randomly inverse Hadamard transform	Inverse Hadmard transform on a if basis is "x". do nothing for "+"							
random rotation string b'	0	1	1	1	0	1	0	0
⇓ 0 stands for "+", 1 stands for "x"								
corresponding rotation basis	+	×	×	×	+	×	+	+
final qubit key	$\|0\rangle$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|1\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\|0\rangle$	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\|1\rangle$
⇓measure a'	0	0 or 1	1	0 or 1	0 or 1	0	0 or 1	1

For all i when a[i] = a'[i]

Shared secret key [[Probabilistically, it's half of the length of a/a'/b/b']]



Attention:

1. For the left table:
   - —: The bit string randomly generated by alice
   - —: The bit string randomly generated by Bob
2. Randomly choosing "+" or "-" as basis, is equivalent to randomly doing Hadamard transform

$|0\rangle \to [H] \to \frac{\sqrt{2}|0\rangle+\sqrt{2}|1\rangle}{2}$

$|1\rangle \to [H] \to \frac{\sqrt{2}|0\rangle-\sqrt{2}|1\rangle}{2}$

Hadamard gate in matrix: $H=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}-\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$

3. Inverse transform of Hadamard gate:

$H=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}$,

$H^{-1}=\frac{1}{-\frac{1}{2}-\frac{1}{2}}\begin{bmatrix}-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

∴ Inverse Hadamard transform = Hadamard transform

Part 2. 窃听者 Eve

Eve 在拦截与 Alice 发送到 Bob 的信息后，必须再发送给 Bob. 否则 Bob 没收到 message 就会有所察觉.

Let's explain further why Eve can be detected. If Eve intercepts a qubit from Alice, she will not know if Alice rotated its state or not. Eve can only measure a 0 or 1. And she can't measure the qubit and then send the same qubit on, because her measurement will destroy the quantum state. Consequently, Eve doesn't know when or when not to rotate to recreate Alice's original qubit. She may as well send on qubits that have not been rotated, hoping to get the rotation right 50% of the time. After she sends these qubits to Bob, Alice and Bob can compare select parts of their keys to see if they have discrepancies in places they should not.

1. Alice 发送给 Bob 她的 qubits，但被 Eve 拦截并测量.
2. 为避免怀疑，Eve 根据她测量得到的 bits 来 prepare qubits，然后发送这些 qubits 给 Bob.
3. Bob 和 Alice 仍按 Part 1 中介绍的那样 create keys.
4. Bob and Alice randomly select the same part of their keys to share publicly
   - If keys don't match ⇒ 他们会发现 Eve 的窃听
   - If keys match ⇒ 他们 confident that Eve 没有窃听
5. They throw away the part of keys they made publicly (in step 4) and 用剩下的那部分 key 来加密和解密消息