***Stochastic Modeling and Computations***
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# Exam

1. **(15 points)** *Diffusion in a circle*

   Particle moves around a circle, $\varphi \in [0, 2\pi]$, according to the following stochastic equation

   $$\frac{d\varphi}{dt} = \omega + \xi, \tag{1}$$

   where $\omega$ is a constant velocity and $\xi(t)$ is a random correction, modeled as the white noise, i.e. $\langle \xi(t) \rangle = 0$ and $\langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2)$. Assuming that $\varphi(t = 0) = 0$,

   (i) Calculate evolution of the mean value $\langle \varphi(t) \rangle$.

   (ii) Calculate probability to discover particle in state $\varphi$ at the moment of time, $t$.

2. **(25 points)** *Queue with finite buffer*

   Jobs arrive at the single server queuing station according to an exponential (Poisson) distribution with rate $\lambda$. The waiting room of the server has finite capacity, $N$. If the waiting room is full, newly arrived particle is rejected (leaves the system), otherwise the particle is placed in the queue. Server picks up the jobs for processing from the queue one by one, according to the first-come-firsts-served protocol. Assuming that the probability distribution of the service processing time is exponential (Poisson) with rate $\mu$,

   (i) Compute the steady-state probability distribution $p(n)$ of observing $n$ jobs in the queue. What is the condition for a steady-state existence?

   Assuming additionally that the system is in the steady-state,

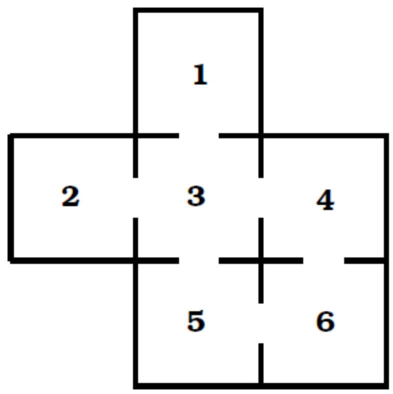
   (ii) Compute the expected number of customers in the queue.

   (iii) Compute probability of (job's) rejection.

   (iv) For $\lambda = 15$ hour$^{-1}$, $\mu = 10$ hour$^{-1}$ and $N = 10$, compute the expected number of customers which are served between 10 am and 11 am.

3. **(25 points)** *Labyrinth with a mousetrap*

A mouse lives in the labyrinth shown in the Figure. At each time step the mouse chooses at random one of the doors and leaves the room trough this door. The process repeats. Formally, mouse dynamics is described fully by a Markov chain of transitions between six states.




(i) Write down the transition matrix $P$ for this Markov chain. Is it irreducible, aperiodic, ergodic?

By definition, the stationary distribution $\pi^*$ is an eigenvector of $P$, which corresponds to the eigenvalue $\lambda = 1$, i.e. it satisfies the equation $P\pi^* = \pi^*$.

(ii) Find the stationary distribution. Does the detailed balance hold?

Now suppose that initially at $t = 0$ the mouse was in the room 1.

(iii) What is the probability to find the mouse in the room 5 in 4 steps? In 5 steps?

(iv) Do the probabilities of finding the mouse in different rooms converge to the stationary distribution $\pi^*$?

Suppose one places a mousetrap in room 5 when the mouse is in room 1.

(v) Find the expected number of steps leading the mouse to the trap, i.e. the expected number of steps till the mouse enters the room 5 for the first time.

*Hint:* One (of many) ways of answering q. (v) is to consider the function $p(i)$ — the expected number of steps leading to the trap given that the mice is in the room $i$, and attempt to relate $p(i)$ with different $i$ to each other. The resulting system of equations will be akin to (analog of) the Bellman equations describing theory behind the dynamic programming.