Properties of Gaussian Distribution. Law of Large Numbers

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03 April 2016

One-Dimensional Normal Distribution

$$\boxed{p(x) = \mathcal{N}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

$x \in (-\infty, +\infty)$ - random variable

$\mu = \int\limits_{-\infty}^{+\infty} xp(x)dx$ - mean value

$\sigma^2 = \int\limits_{-\infty}^{+\infty} (x-\mu)^2 p(x)dx$ - standard deviation

Characteristic Function

$$\mathcal{G}(k) = \int_{-\infty}^{+\infty} e^{ikx} p(x) dx = \sum_{n=0}^{+\infty} \frac{i^n k^n}{n!} \langle x^n \rangle.$$

Characteristic function of the Gaussian distribution $p(x) = \mathcal{N}(\mu, \sigma^2)$ is

$$\boxed{\mathcal{G}(k) = \exp\left(i\mu k - \frac{\sigma^2 k^2}{2}\right)}$$

For $\mu = 0$, the high-order moments are given by

$$\langle x^{2n} \rangle = \frac{(2n)!}{2^n n!} \sigma^{2n}, \qquad \langle x^{2n+1} \rangle = 0$$

Central Limit Theorem (CLT)

Consider the variable

$$X_n = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where $x_1, x_2, \ldots, x_n$ are independently chosen from the PDF $p(x)$ with well-defined mean μ_x and variance σ_x^2.

CLT states that

$$\boxed{P_n(X_n) \to \mathcal{N}(\mu_x, \frac{\sigma_x^2}{n})}$$

as $n \to \infty$.

Proof of CLT

- Pass to the new variables

$$z_i = \frac{\sqrt{n}(x_i - \mu_x)}{\sigma_x}, \quad Z_n = \frac{1}{n}\sum_{i=1}^{n} z_i.$$

  Check that $\mu_z = 0$ and $\sigma_z^2 = n$.
- Let $g_n(k)$ and $\mathcal{G}(k)$ be the characteristic functions of $P_n(Z_n)$ and $p(z)$, respectively. Prove the relation

$$g_n(k) = \mathcal{G}^n(k/n)$$

- Show that as $n \to \infty$

$$g_n(k) \to \exp\left(-\frac{k^2}{2}\right) \implies P_n(Z_n) \to \mathcal{N}(0,1)$$

Exercise 1

Derive an exact expression for the probability distribution $P_n(X_n)$ of the random variable $X_n = n^{-1}\sum_{i=1}^{n} x_i$, where $x_1, x_2, \dots, x_n$ are independently chosen from the normal distribution $\mathcal{N}(\mu, \sigma^2)$.

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Solution:

$$P_n(X_n) = \mathcal{N}(\mu, \frac{\sigma^2}{n}) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{n(X_n-\mu)^2}{2\sigma^2}\right).$$

Exercise 2

Calculate the probability distribution $P_n(X_n)$ of the random variable $X_n = n^{-1}\sum_{i=1}^{n} x_i$, where $x_1, x_2, \ldots, x_n$ are independently chosen from a Cauchy distribution

$$p(x) = \frac{\gamma}{\pi}\frac{1}{x^2+\gamma^2}.$$

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Solution:

$$P_n(X_n) = p(x) = \frac{\gamma}{\pi}\frac{1}{x^2+\gamma^2}.$$

CLT does not apply to distributions that do not have a finite variance!

Multivariate Normal Distribution

$$P(x_1, \ldots, x_M) = \frac{1}{N} \exp\left(-\frac{x_i A_{ij} x_j}{2}\right),$$

$x_i \in \mathbb{R}$ - random variables
$\hat{A}$ - symmetric positive definite matrix
$N = (2\pi)^{M/2}/\sqrt{\det A}$ - normalization constant

Quadratic moments:

$$\mathbf{E}[x_i x_j] = A^{-1}_{ij}.$$

High order moments (Isserlis' theorem):

$$\mathbf{E}[x_1 x_2 \ldots x_{2n}] = \sum \prod \mathbf{E}[x_i x_j],$$
$$\mathbf{E}[x_1 x_2 \ldots x_{2n+1}] = 0.$$

Exercise 3

The joint probability distribution of two random variables x_1 and x_2 is

$$P(x_1, x_2) = \frac{1}{N} \exp(-x_1^2 - x_1 x_2 - x_2^2).$$

1) Calculate the normalization constant N.
2) Calculate the marginal probability $P(x_1)$.
3) Calculate the statistical moment $\mathbf{E}[x_1^2 x_2^2]$.