Machine Learning HW#5

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Problem 1

- By definition

$$\hat{f}_n(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right) \tag{1}$$

So it can be easily seen that

$$\mathbb{E}\hat{f}_n(x) = \mathbb{E}\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)\right] = \frac{1}{h}\mathbb{E}K\left(\frac{x-x_i}{h}\right) = \frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy \tag{2}$$

and

$$\mathbb{V}\hat{f}_n(x) = \mathbb{V}\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right) = \frac{1}{nh^2}\mathbb{V}\left[K\left(\frac{x-x_i}{h}\right)\right] = \tag{3}$$

$$= \frac{1}{nh^2}\left(\mathbb{E}K^2\left(\frac{x-x_i}{h}\right) - \left[\mathbb{E}K\left(\frac{x-x_i}{h}\right)\right]^2\right) \tag{4}$$

as K is an indicator function we can get rid of the square on the first summand of (4), hence we obtain

$$\frac{1}{nh^2}\left(\mathbb{E}K\left(\frac{x-x_i}{h}\right) - \left[\mathbb{E}K\left(\frac{x-x_i}{h}\right)\right]^2\right) = \frac{1}{nh^2}\left(\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy - \left[\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy\right]^2\right) \tag{5}$$

and the result follows.

- To prove this we need to check a few properties:

1. The first one is

$$\lim_{h\to 0} \mathbb{E}\hat{f}_n(x) = f(x) \tag{6}$$

Indeed,

$$\lim_{h\to 0} \mathbb{E}\hat{f}_n(x) = \lim_{h\to 0} \frac{\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy}{h} = \lim_{h\to 0} \frac{\left(\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy\right)'}{h'} = \tag{7}$$

$$= \frac{1}{2} \lim_{h\to 0} \left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right] = f(x) \tag{8}$$

2. The second is

$$\lim_{n,nh\to\infty,h\to 0} \mathbb{V}\hat{f}_n(x) = 0 \tag{9}$$

Indeed,

$$\lim_{n,nh\to\infty,h\to 0} \mathbb{V}\hat{f}_n(x) = \lim_{n,nh\to\infty,h\to 0} \frac{1}{nh^2} \left(\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy - \left[\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy\right]^2 \right) = \tag{10}$$

$$= \lim_{n,nh\to\infty,h\to 0} \left[\frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy\right] \cdot \left[\frac{1}{nh} - \frac{1}{n}\cdot\frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(y)dy\right] \tag{11}$$

Using the first property we can simplify (11) as follows

$$\lim_{n,nh\to\infty} f(x)\left[\frac{1}{nh} - \frac{1}{n}f(x)\right] = 0 \tag{12}$$

Now we can apply Chebyshev inequality. More precisely, we have the following expression (all conditions of Chebyshev inequality are satisfied)

$$\mathbb{P}\left(|\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)| > \epsilon\right) \le \frac{\mathbb{V}\hat{f}_n(x)}{\epsilon^2} \to 0 \tag{13}$$

And after taking the limit with $n, nh \to \infty, h \to 0$ we obtain

$$\mathbb{P}\left(|\hat{f}_n(x) - f(x)| > \epsilon\right) = \lim_{n,nh\to\infty,h\to 0} \mathbb{P}\left(|\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)| > \epsilon\right) \le \lim_{n,nh\to\infty,h\to 0} \frac{\mathbb{V}\hat{f}_n(x)}{\epsilon^2} = 0 \tag{14}$$

Problem 2