Lecture 13: How to construct LDPC codes

Course instructor: Alexey Frolov

al.frolov@skoltech.ru

Teaching Assistant: Stanislav Kruglik

stanislav.kruglik@skolkovotech.ru

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Outline

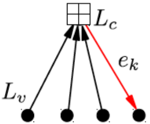
We know how to decode LDPC codes

- Messages are **log-likelihood ratios (LLRs)**:

$$L_{ch} = \log \frac{\mathbb{P}(r|v=0)}{\mathbb{P}(r|v=1)}$$

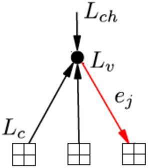
BSC: $r \in \{0,1\}$
AWGNC: $r \in \mathbb{R}$

Check node update:



$$L_c(e_k) = 2\,\mathrm{atanh}\left(\prod_{k' \neq k} \tanh\left(\frac{L_v(e_{k'})}{2}\right)\right)$$

Variable node update:



$$L_v(e_j) = L_{ch} + \sum_{j' \neq j} L_c(e_{j'})$$

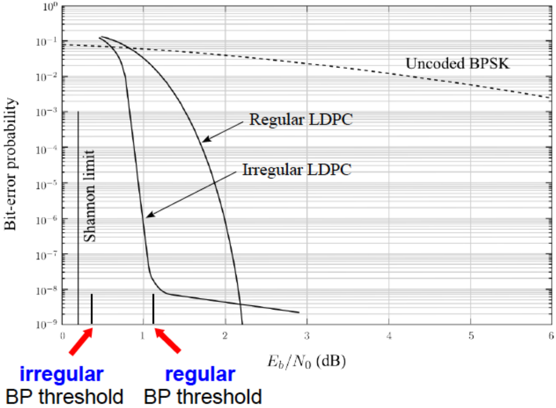
Problems

In what follows the decoding algorithm is fixed.

The decoding algorithm is suboptimal (there are cycles in the Tanner graph).

How to optimize LDPC parity-check matrices for this decoder?

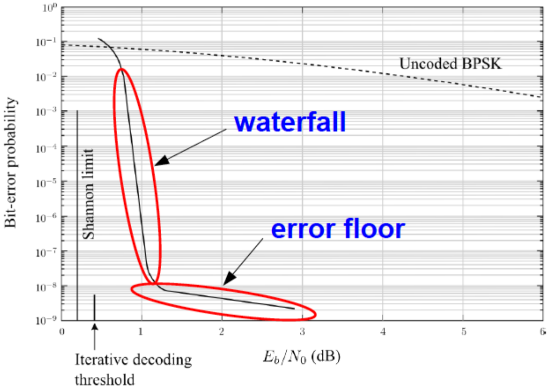
Regular vs Irregular LDPC codes



- **Irregular** LDPC code ensembles can have optimized thresholds **close to capacity**
- **Regular** LDPC code ensembles are asymptotically good and have good graph properties, resulting in a **low error floor**

Waterfall vs error floor

- The **Shannon limit** defines capacity and is a property of the physical channel.



- The **iterative decoding threshold** depends on the code structure and iterative decoding algorithm in use.
- Capacity-approaching LDPC codes typically display a **waterfall** (related to their threshold) and an **error-floor** (related to their graph/distance properties).

How to define the ensemble of irregular LDPC codes?

Degree (weight if we consider PCM) distribution polynomials

$$\Lambda(x) = \sum_{i=1}^{l_{\max}} \Lambda_i x^i \text{ (variable nodes)}$$

and

$$P(x) = \sum_{i=1}^{r_{\max}} P_i x^i \text{ (check nodes)},$$

where Λ_i and P_i are numbers of variable/check nodes of degree i.

Properties:

$$\Lambda(1) = n, P(1) = (1-R)n, R = 1 - \frac{P(1)}{\Lambda(1)}$$

How to define the ensemble of irregular LDPC codes?

Degree (weight if we consider PCM) distribution polynomials

$$L(x) = \frac{\Lambda(x)}{\Lambda(1)}$$

and

$$Q(x) = \frac{P(x)}{P(1)},$$

where L_i and Q_i are *fractions* of variable/check nodes of degree i.

Edge perspective

For the asymptotic analysis it is more convenient to take on an edge perspective. Define:

$$\lambda(x) = \sum_i \lambda_i x^{i-1}$$

and

$$\rho(x) = \sum_i \rho_i x^{i-1},$$

where λ_i and ρ_i are *fractions* of *edges* that connect to variable(check) nodes of degree i.

Properties:

$$\lambda(x) = \frac{L'(x)}{L'(1)}, \rho(x) = \frac{Q'(x)}{Q'(1)}.$$

Example

Consider $[7,4]$ Hamming code

$$\mathbf{H}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \Lambda(x) &= 3x + 3x^2 + x^3 \\ L(x) &= \frac{3}{7}x + \frac{3}{7}x^2 + \frac{1}{7}x^3 \\ \lambda(x) &= \frac{1}{4} + \frac{1}{2}x + \frac{1}{4}x^2 \end{aligned}$$

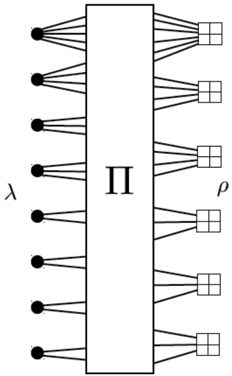
How to define the ensemble of irregular LDPC codes?

- **Node degrees**: random variables [Luby, et al., '97]

$$\lambda(x) = \sum_k \lambda_k x^{k-1}$$ ← variable node distribution

$$\rho(x) = \sum_k \lambda_k x^{k-1}$$ ← check node distribution

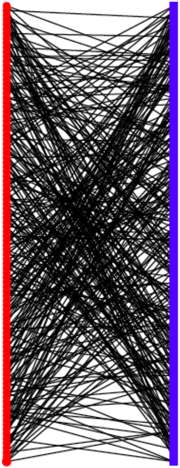
LDPC Ensemble:



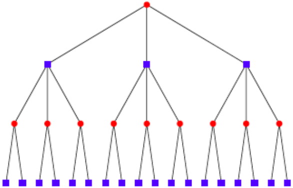
variable nodes **check nodes**

Outline

Computational graph



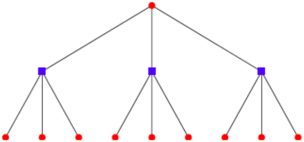
$$\lim_{n\to\infty} \mathbb{E}[P_b(\mathrm{G}, n, \ell)]$$



probability that computation graph of fixed depth becomes tree tends to 1 as n tends to infinity

Density evolution, BEC

Luby, Mitzenmacher, Shokrollahi, Spielman, and Steman ‘97



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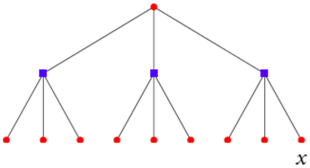

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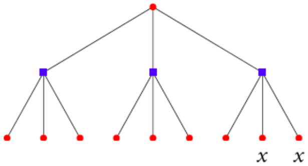

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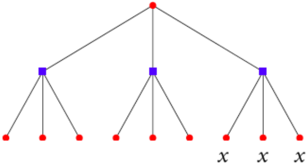

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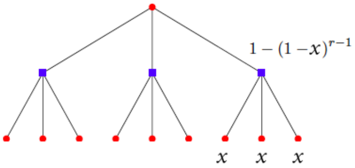

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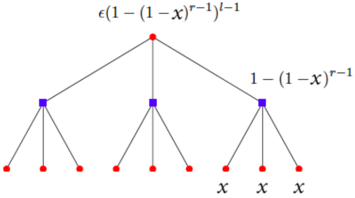

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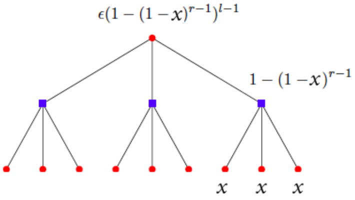
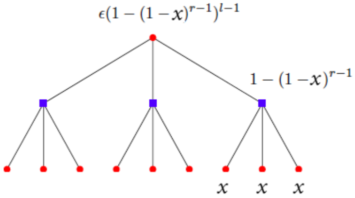


$$x_t = \epsilon(1-(1-x_{t-1})^{r-1})^{l-1}$$

Density evolution, BEC

Luby, Mitzenmacher, Shokrollahi, Spielman, and Steman '97



$$x_\ell = \epsilon\lambda\left(1-\rho\left(1-x_{\ell-1}\right)\right)$$

EXAMPLE 3.52 (DENSITY EVOLUTION FOR $(\lambda(x) = x^2, \rho(x) = x^5)$). For the degree distribution pair $(\lambda(x) = x^2, \rho(x) = x^5)$ we have $x_0 = \epsilon$ and for $\ell \geq 1$, $x_\ell = \epsilon(1 - (1 - x_{\ell-1})^5)^2$. For example, for $\epsilon = 0.4$ the sequence of values of x_ℓ is 0.4, 0.34, 0.306, 0.2818, 0.2617, 0.2438, and so forth. ◇

Density evolution, BEC

EXAMPLE 3.52 (DENSITY EVOLUTION FOR $(\lambda(x) = x^2, \rho(x) = x^5)$). For the degree distribution pair $(\lambda(x) = x^2, \rho(x) = x^5)$ we have $x_0 = \epsilon$ and for $\ell \geq 1$, $x_\ell = \epsilon(1 - (1 - x_{\ell-1})^5)^2$. For example, for $\epsilon = 0.4$ the sequence of values of x_ℓ is 0.4, 0.34, 0.306, 0.2818, 0.2617, 0.2438, and so forth. $\diamond$

Does this sequence converge to 0 ?

Density evolution, BEC

LEMMA 3.53 (MONOTONICITY OF $f(\cdot,\cdot)$). For a given degree distribution pair (λ, ρ) define $f(\epsilon, x) = \epsilon\lambda(1 - \rho(1 - x))$. Then $f(\epsilon, x)$ is increasing in both its arguments for $x, \epsilon \in [0, 1]$.

LEMMA 3.54 (MONOTONICITY WITH RESPECT TO CHANNEL). Let (λ, ρ) be a degree distribution pair and $\epsilon \in [0, 1]$. If $\mathrm{P}^{\mathrm{BP}}_{\mathcal{T}_\ell}(\epsilon) \xrightarrow{\ell\to\infty} 0$ then $\mathrm{P}^{\mathrm{BP}}_{\mathcal{T}_\ell}(\epsilon') \xrightarrow{\ell\to\infty} 0$ for all $0 \le \epsilon' \le \epsilon$.

phase transition: ϵ^{BP} so that

$$x_t \to 0 \text{ for } \epsilon < \epsilon^{\mathrm{BP}}$$

$$x_t \to x_\infty > 0 \text{ for } \epsilon > \epsilon^{\mathrm{BP}}$$

Threshold

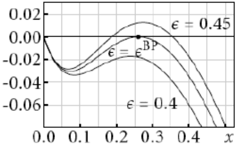
Definition 3.56 (Threshold of Degree Distribution Pair). The *threshold* associated with the degree distribution pair (λ, ρ), call it $\epsilon^{\mathrm{BP}}(\lambda, \rho)$, is defined as

$$\epsilon^{\mathrm{BP}}(\lambda, \rho) = \sup\{\epsilon \in [0,1] : \mathrm{P}^{\mathrm{BP}}_{\mathcal{T}_\ell(\lambda,\rho)}(\epsilon) \xrightarrow{\ell \to \infty} 0\}. \qquad \triangledown$$

Example 3.57 (Threshold of $(\lambda(x) = x^2, \rho = x^5)$). Numerical experiments show that $\epsilon^{\mathrm{BP}}(3,6) \approx 0.42944$. $\diamond$

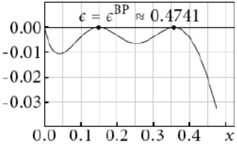
Fixed point characterization, BEC

$f(\epsilon, x) = \epsilon\lambda(1 - \rho(1 - x))$.



$f(\epsilon, x) - x = \epsilon(1 - (1 - x)^5)^2 - x$

$$(\lambda, \rho) = (x^2, x^5)$$



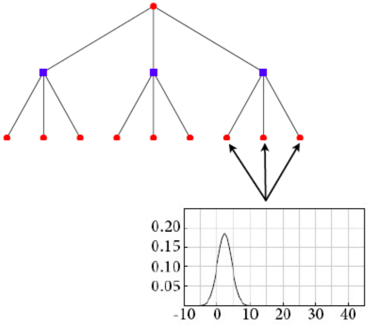
$$\lambda(x) = 0.106257x + 0.486659x^2 + 0.010390x^{10} + 0.396694x^{19}$$

$$\rho(x) = 0.5x^7 + 0.5x^8$$

Density evolution, BEC

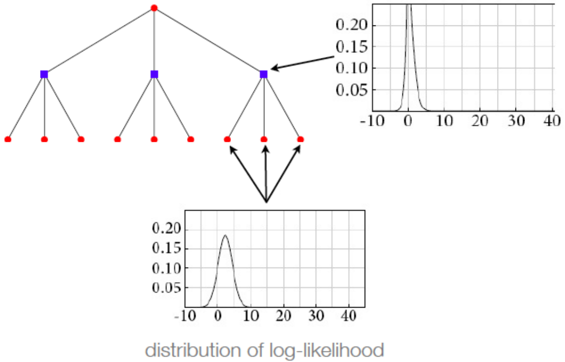
$\mathtt{l}$	$\mathtt{r}$	$r(\mathtt{l},\mathtt{r})$	$\epsilon^{\text{Sha}}(\mathtt{l},\mathtt{r})$	$\epsilon^{\text{BP}}(\mathtt{l},\mathtt{r})$
3	6	$\frac{1}{2}$	$\frac{1}{2} = 0.5$	≈ 0.4294
4	8	$\frac{1}{2}$	$\frac{1}{2} = 0.5$	≈ 0.3834
3	5	$\frac{2}{5}$	$\frac{3}{5} = 0.6$	≈ 0.5176
4	6	$\frac{1}{3}$	$\frac{2}{3} \approx 0.667$	≈ 0.5061
3	4	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	≈ 0.6474

Density evolution, AWGNC

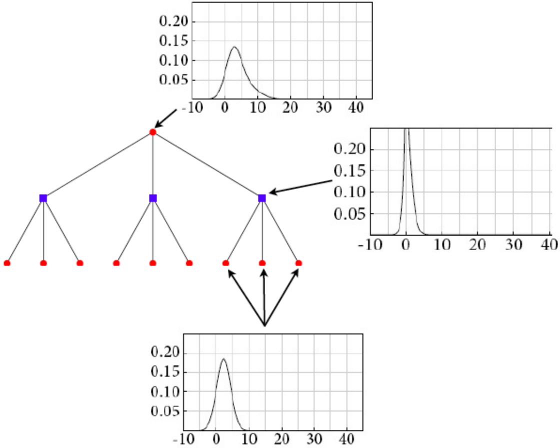


distribution of log-likelihood

Density evolution, AWGNC



Density evolution, AWGNC



distribution of log-likelihood

Outline

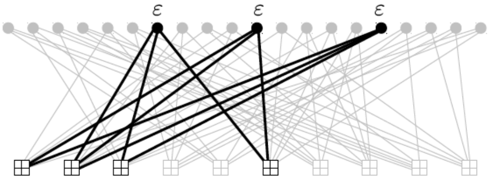
Error floor

- ➡ Error events in the **waterfall** typically result from **large** decoding failures (a large number of symbols decoded incorrectly)
- ➡ Error events in the **error floor** typically result from **small** decoding failures (only a few symbols decoded incorrectly)
- The minimum distance is a **code property**; under ML decoding, a large minimum distance results in a low error floor
- Under sub-optimal **iterative BP decoding**, the error floor is also affected by small failures arising due to **weaknesses in the Tanner graph**
- ➡ These graphical weaknesses have been studied extensively for a variety of channels and are known collectively as **pseudocodewords** [Frey et al '98], **stopping sets** [Di et al '02], **near-codewords** [MacKay & Postol '03], **trapping sets** [Richardson '03], **elementary trapping sets** [Laendner & Milenkovic '05], and **absorbing sets** [Dolecek et al '07].

Error floor, BEC

- On the BEC, the cause of failures is **stopping sets** [Di, et al. '02].

Definition: A stopping set is a subset S of the variable nodes such that all neighboring check nodes are connected to S at least twice



Example stopping set in a $(3,6)$-regular Tanner graph

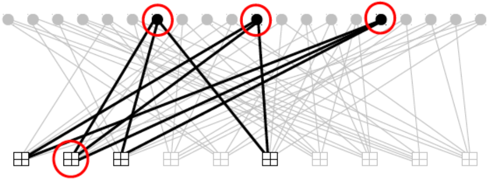
- If the highlighted nodes are all erasures then the BP decoder will fail to correct them

➡ Message-passing decoding is suboptimal! The MAP decoder fails if and only if the set of erasures contains the set of all non-zero positions in the codeword.

Error floor, AWGNC

- On the AWGNC, failures are attributed to **trapping sets** [Richardson '03].

Definition: An (a,b) general trapping set $\tau_{a,b}$ of a bipartite graph is a set of a variable nodes which induce a subgraph with exactly b odd-degree check nodes.



A $(3,1)$ trapping set in a $(3,6)$-regular Tanner graph

- Low connectivity outside the set causes the iterative decoder to become trapped and fail to correct the symbols in the set
- Certain types of trapping sets with small a and b, such as **elementary trapping sets** and **absorbing sets**, are known to be particularly harmful

Progressive-edge grows algorithm

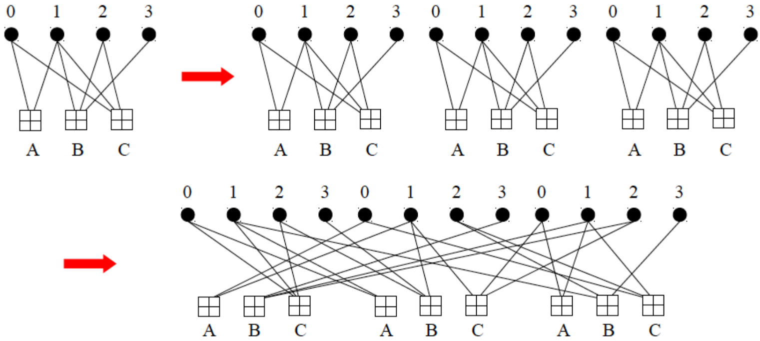
Input: $L(x)$
Output: PCM with "big" girth

Main idea: greedy algorithm, add edges to the graph in a sequential manner. Each time choose the connection, that maximizes the girth.

Outline

Protograph-based LDPC codes

- Codes can be constructed from a **protograph** using a **copy-and-permute** operation



[Tho05] J. Thorpe, “Low-Density Parity-Check (LDPC) codes constructed from protographs”, *Jet Propulsion Laboratory INP Progress Report*, Vol. 42-154 Aug. 2003.

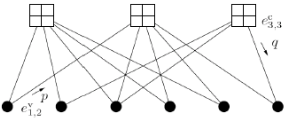
Protograph-based LDPC codes

- **Compact representation** of a permutation matrix based ensemble by a base matrix:

$$\mathbf{H} = \begin{bmatrix} \mathbf{\Pi}_{1,1} & \mathbf{\Pi}_{1,2} & \mathbf{\Pi}_{1,3} & \mathbf{\Pi}_{1,4} & \mathbf{\Pi}_{1,5} & \mathbf{0} \\ \mathbf{\Pi}_{2,1} & \mathbf{0} & \mathbf{\Pi}_{2,3} & \mathbf{\Pi}_{2,4} & \mathbf{\Pi}_{2,5} & \mathbf{\Pi}_{2,6} \\ \mathbf{0} & \mathbf{\Pi}_{3,2} & \mathbf{\Pi}_{3,3} & \mathbf{0} & \mathbf{0} & \mathbf{\Pi}_{3,6} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

base matrix



protograph

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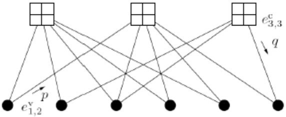
base matrix



protograph

[Tho05] J. Thorpe, "Low-Density Parity-Check (LDPC) codes constructed from protographs", *Jet Propulsion Laboratory INP Progress Report*, Vol. 42-154 Aug. 2003.

Quasi-cyclic LDPC codes

Replace permutation matrices with circulant matrices (usually of weight 1).

Why this code is quasi-cyclic?

Thank you for your attention!