*Stochastic Modeling and Computations*
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# Problem Set II

This problem set is due by **Sun May 21**, 11:59 pm Moscow time. Solutions should be turned in through the course web-site in an electronic format. Full credit will be given only to the correct solution which is described clearly.

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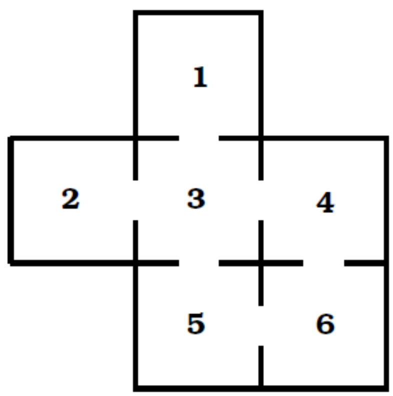
1. **(25 points)** *Labyrinth with a mousetrap*

   A mouse lives in the labyrinth shown below. At each time step the mouse chooses at random one of the doors and leaves the room trough this door. The process repeats. Formally, mouse dynamics is described fully by a Markov chain of transitions between six states.




   (i) Write down the transition matrix $P$ for this Markov chain. Is it irreducible, aperiodic, ergodic?

   By definition, the stationary distribution $\pi^*$ is an eigenvector of $P$, which corresponds to the eigenvalue $\lambda = 1$, i.e. it satisfies the equation $P\pi^* = \pi^*$.

   (ii) Find the stationary distribution. Does the detailed balance hold?

   Now suppose that initially at $t = 0$ the mouse was in the room 1.

   (iii) What is the probability to find the mouse in the room 5 in 4 steps? In 5 steps?

   (iv) Do the probabilities of finding the mouse in different rooms converge to the stationary distribution $\pi^*$?

   Suppose one places a mousetrap in room 5 when the mouse is in room 1.

   (v) Find the expected number of steps leading the mouse to the trap, i.e. the expected number of steps till the mouse enters the room 5 for the first time.

*Hint:* One (of many) ways of answering (v) is to consider the function $p(i)$ — the expected number of steps leading to the trap given that the mice is in the room $i$, and attempt to relate $p(i)$ with different $i$ to each other. The resulting system of equations will be akin to the Bellman equations describing theory behind the dynamic programming.

2. **(25 points)** *Elementary Diffusion.*

   Consider a particle jumping over nodes of the one-dimensional chain, where the states are labeled $n = 0, \pm 1, \pm 2, \ldots$. Left and right jumps are performed with the rates $\mu$ and $\lambda$ respectively. Assume that at the moment of time $t = 0$ the particle was located at the node $n = 0$.

   (i) Using any programming language perform and illustrate a sample of a particle path/trajectory.

   (ii) Find $P(n,t)$ numerically, where $P(n,t)$ is the probability to observe a particle at the position $n$ at the moment of time $t$. In order to simulate the particle motion split the time axis into discrete intervals and for any time step implement decision (on where to move next) according to the rates.

   (iii) Solve (ii) analytically by solving the master equation, which is stated in continuous time as follows (this is a discrete space analog of the Fokker-Planck equation),

   $$\partial_t P(n,t) = -(\lambda + \mu)P(n,t) + \mu P(n+1,t) + \lambda P(n-1,t). \tag{1}$$

   (iv) Replace a discrete variable $n$ in this equation by a continuous variable $x$. Under what assumptions can you do it? Solve the resulting (continuous time, continuous space) equation analytically and compare the result with the simulations performed in (ii). (*Hint:* if $\lambda = \mu$ then the right-hand side is just $\lambda \partial_x^2 P(x,t)$.)

   (v) For the original case of discrete space and setting $\lambda = 0$, solve the problem exactly. Compare the solution with (proper version of) the simulations performed in (ii).

3. **(25 points)** *Mortal Brownian Particle.*

   Unstable Brownian particle moves within the interval $0 < x < L$ between two absorbing walls starting from the initial position $x_0$. The Poisson rate of particle decay is $\alpha$ and the diffusion coefficient is $D$.

   (i) Calculate the survival probability $p(t)$ of the particle analytically.

   (ii) Find the expected lifetime of the particle analytically and by direct numerical simulations for $L = 1$, $x_0 = 0.2$, $D = 1$ and $\alpha = 1$.

   (iii) Assume that $x_0 = L/5$. What is the probability that the particle will be absorbed before it decays? Answer this question analytically or numerically.

   (*Hint:* The probability distribution of an unstable Brownian particle is described by the equation, $\partial_t n = D\partial_x^2 n - \alpha n$.)

4. **(15 points)** *Queue with finite buffer*

   Jobs arrive at the single server queuing station according to an exponential (Poisson) distribution with rate $\lambda$. The waiting room of the server has finite capacity, $N$. If the waiting room is full, newly arrived particle is rejected (leaves the system), otherwise the particle is placed in the queue. Server picks up the jobs for processing from the queue one by one, according to the first-come-firsts-served protocol. Assuming that the probability distribution of the service processing time is exponential (Poisson) with rate $\mu$,

   (i) Compute the steady-state probability distribution $p(n)$ of observing $n$ jobs in the queue. What is the condition for a steady-state existence?

   (ii) Compute the expected number of customers in the steady-state queue.