Stochastic Modeling and Computations
Skolkovo Institute of Science and Technology April 3, 2017
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Problem Set I

This problem set is due by **Sun Apr 23**, 11:59 pm Moscow time. Solutions should be turned in through the course web-site in an electronic format. Full credit will be given only to the correct solution which is described clearly.

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1. **(5 points)** *Exponential Distribution*

   The probability density function of an exponential distribution is

   $$p(x) = \begin{cases} Ae^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1}$$

   where the parameter $\lambda > 0$.

   (i) Calculate the normalization constant A of the distribution.
   (ii) Calculate the *mean value* and the *variance* by direct integration.

   The *characteristic function* of a distribution is

   $$G(k) = \int_{-\infty}^{+\infty} e^{ikx} p(x) dx. \tag{2}$$

   It can be used to calculate high-order moments of the distribution.

   (iii) Calculate the characteristic function $G(k)$ of the exponential distribution.
   (iv) Using the function $G(k)$, calculate the $m-$th moment of the distribution.

2. **(20 points)** *Splitting the circle*

   Randomly choose three points on a circle $x^2 + y^2 = 1$. These points divide the circle into three arcs.

   (i) Calculate analytically the expected length of the arc containing the point $(1, 0)$.
   (ii) Confirm your analytical result by numerical simulations.

3. **(10 points)** *Dice game*

   Assume that you play a dice game 50 times. Awards for the game are as follows

   1, 3 or 5: 0$

   2 or 4: 2$

   6: 26$

   (i) Estimate expected value of winnings

(ii) Estimate standard deviation of winnings

(iii) Estimate probability of winning at least 200$

(iv) Estimate the probability of winning at least 50$ more than your friend who is playing the same dice game.

[hint: use central limit theorem]

4. **(10 points)** *Z channel*

   For Z channel both input and output alphabets are binary. If input is $x = 0$, what comes out is $y = 0$ with unit probability. When the input is $x = 1$, the output is $y = 0$ or $y = 1$ with probabilities f and $1 - f$, respectively. Consider the Z channel with $f = 0.1$ and the following probability distribution of the input symbols: $P(x = 0) = 0.8$, $P(x = 1) = 0.2$.

   (i) Compute the probability distribution of output $P(y)$.

   (ii) Compute the probability of $x = 1$ given $y = 0$.

   (iii) Compute the mutual information $I(X;Y)$.

   (iv) What is the capacity of the channel?

5. **(15 points)** *Hardy-Weinberg Law*

   Consider an experiment with rabbits matting. Let us follow evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid — the order is irrelevant, so gG is the same as Gg) or gg (recessive). In the result of a single mating the offspring inherits a gene from each of its parents with equal probability. Thus, if a dominant parent (GG) mates with a hybrid parent (Gg), the offspring is dominant with probability $1/2$ or hybrid with probability $1/2$. Start with a rabbit of given character (GG, Gg, or gg) and assume that she mates with a hybrid. The offspring produced again mates with a hybrid, and the process is repeated for a number of generations.

   (i) Write down the transition matrix P of the Markov chain thus defined. Is the Markov chain irreducible and aperiodic?

   (ii) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\mu_n(GG)$, $\mu_n(Gg)$, $\mu_n(gg)$ are the probabilities that the n-th generation rabbit is GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Is there a some kind of law/rule emerging?

   (iii) Calculate P^n for general n. How does the moment, μ_n, depend on n?

   (iv) Calculate the stationary distribution of the Markov chain. Is detailed balance hold?

   Note: The first experiment of such kind was conducted in 1858 by Gregor Mendel. He started to breed garden peas in his monastery garden and analysed the offspring of these matings.

6. **(10 points)** *Splitting of Poisson process*

   Customers arrive at a store with the Poisson rate of 20 per hour. Each is either male or female with probability p and $1-p$, respectively.

   (i) Compute probability that at least 50 customers have entered between 9 and 11 am.

   (ii) Compute probability that exactly 20 men entered between 1 pm and 2 pm.

   (iii) Compute the mean inter-arrival time of women.

   (iv) Compute the probability that there will be no male customers between 2 pm and 5 pm.