Please provide a solution for all the problems and send a single pdf file via Canvas by the due date. Points for all the problems are placed before problem description. If you have any questions do not hesitate to contact the course instructor or TA by e-mail or personally. Good luck!

1. (1 point) Let X, Y, Z be continuous random variables, such that $I(X;Z|Y) = H(X|Y) - H(X|Y,Z)$. Proof, that $I(X,Y;Z) = I(Y;Z) + I(X;Z|Y)$.

2. (1 point) Prove, that

$$\sum_{i=1}^{n} H(X_{\bar{i}}) \geq (n-1)H(X^n),$$

where $H(X_{\bar{i}}) = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$.

3. (2 point) Let $\mathcal{X} = \{1, 2, 3, 4\}$. A source produces i.i.d. symbols with $\Pr(X = 1) = p_1$, $\Pr(X = 2) = p_2$, $\Pr(X = 3) = p_3$ and $\Pr(X = 4) = p_4$. You are given a file

$$F = 1133334424222231142341312114 32,$$

generated by this source. Answer the following questions:

   (a) What is your guess for p_1, p_2, p_3, p_4? Why?
   (b) What is the entropy (in bits) of the probability distribution you guessed?
   (c) Construct a binary Huffman code for the probability distribution that you found? What is the average number of bits per symbol?
   (d) Encode the file F with the Huffman code you have designed. What is the length of the encoded binary file? What is the average number of bits that have been used for a symbol in this file?

4. (2 point) Consider a discrete memoryless channel (DMC) with $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and a probability transition matrix

$$P_{Y|X} = \begin{pmatrix} 1 & 0 \\ \delta & 1-\delta \end{pmatrix},$$

where rows and columns correspond to elements of $\mathcal{X}$ and $\mathcal{Y}$ accordingly.

   (a) Find the capacity of the channel.
   (b) Find the limit of capacity and the capacity achieving distributions P_X when $\delta \to 1$.

5. (1 point) Consider a binary symmetric channel (BSC) with transition probability p. The output of this channel is fed to the input of a binary erasure channel (BEC) with erasure probability ε. What is the capacity of the resulting channel?

6. (1 point) Find the capacity of 500 parallel independent discrete time Gaussian channels, such that 499 have a noise variance $N_1 = E[Z^2] = 100$ and one channel has a noise variance $N_2 = E[Z^2] = 25$. The total power constraint is equal to $P = 50$.

7. (1 point) Find the capacity region for noiseless adder modulo two multiple access channel $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$, $Y = X_1 \oplus X_2$.

8. (1 point) Find the capacity region for noiseless adder multiple access channel $\mathcal{X}_1 = \mathcal{X}_2 = \{-1, 1\}$, $Y = X_1 + X_2$.