# Stochastic Modeling and Computations HW#1

Evgeny Marshakov

## Problem 1

i) We have the following distribution

$$p(x) = \begin{cases} Ae^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

To find the constant $A$ we check the normalization

$$1 = \int_{-\infty}^{\infty} p(x)dx = \underbrace{\int_{-\infty}^{0} 0dx}_{=0} + A\int_{0}^{\infty} e^{-\lambda x}dx = -\frac{A}{\lambda}e^{-\lambda x}\Big|_{0}^{\infty} = \frac{A}{\lambda} \Leftrightarrow \quad (1)$$

$$\Leftrightarrow A = \lambda \quad (2)$$

So our distribution is the following

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$$

ii)

$$\mathbb{E}[x] = \lambda\int_{0}^{\infty} xe^{-\lambda x}dx = \underbrace{-xe^{-\lambda x}\Big|_{0}^{\infty}}_{=0} + \int_{0}^{\infty} e^{-\lambda x}dx = -\lambda^{-1}\underbrace{e^{-\lambda x}\Big|_{0}^{\infty}}_{=-1} = \lambda^{-1} \quad (3)$$

$$\mathbb{E}[x^2] = \lambda\int_{0}^{\infty} x^2e^{-\lambda x}dx = \underbrace{-x^2e^{-\lambda x}\Big|_{0}^{\infty}}_{=0} + 2\int_{0}^{\infty} xe^{-\lambda x}dx = 2\lambda^{-2} \quad (4)$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - (\mathbb{E}[x])^2 = \lambda^{-2} \quad (5)$$

iii)

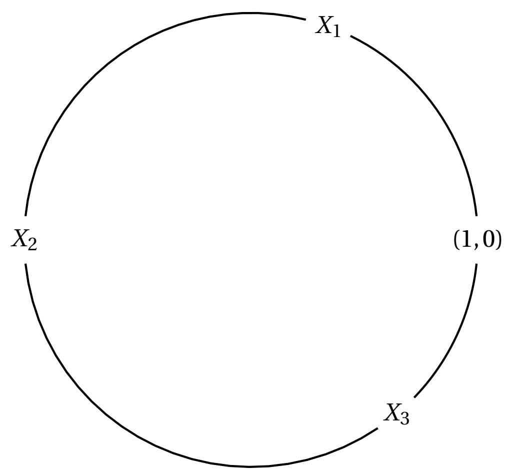
$$G(k) = \lambda \int_0^\infty e^{ikx} e^{-\lambda x} = \frac{\lambda}{ik-\lambda} \underbrace{e^{ikx-\lambda x}\Big|_0^\infty}_{=-1} = \frac{\lambda}{\lambda - ik} \tag{6}$$

iv)

$$\mu_m = \frac{1}{i^m} \frac{\partial^m G(k)}{\partial k^m}\Big|_{k=0} = \frac{1}{i^m} \frac{\lambda i^m m!}{(\lambda - ik)^{m+1}}\Big|_{k=0} = m!\lambda^{-m} \tag{7}$$

## Problem 2

i) We have three random variables $X_i$ on the circle. And we want to find the expected value of the interval that contains the point $(1,0)$.




We can divide the circle at the point $(1,0)$ and consider the interval $[1, 1+2\pi]$ with three random variables on it. In this case the problem of finding the expected value of arc containing the point $(1,0)$ is similar to the problem of finding the expected value of $2\min\{X_1, X_2, X_3\}$ on the interval $[1, 1+2\pi]$, because arcs $[(1,0), X_1]$ and $[X_3, (1,0)]$ has the same expected value. So we reduce our problem to the following

$$2\mathbb{E}[\min\{X_1, X_2, X_3\}], \quad X_i \sim U[1, 1+2\pi] \tag{8}$$

or the same

$$2\mathbb{E}[\min\{X_1, X_2, X_3\}], \quad X_i \sim U[0, 2\pi] \tag{9}$$

The CDF $F_{X_i}(x)$ is the following

$$F_{X_i}(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2\pi}, & x \in [0, 2\pi] \\ 1, & x > 1 \end{cases} \tag{10}$$

Due to the fact that

$$\mathbb{P}(\min\{X_1, X_2, X_3\} \leq t) = 1 - \mathbb{P}(\min\{X_1, X_2, X_3\} \geq t) = 1 - \mathbb{P}(X_1 \geq t)\mathbb{P}(X_2 \geq t)\mathbb{P}(X_3 \geq t) = \tag{11}$$

$$= 1 - (1 - \mathbb{P}(X_1 \leq t))(1 - \mathbb{P}(X_2 \leq t))(1 - \mathbb{P}(X_3 \leq t)) = 1 - (1 - F_{X_i}(x))^3, \quad x \in [0, 2\pi] \tag{12}$$

More precisely,

$$F_{\min\{X_1, X_2, X_3\}}(x) = \begin{cases} 0, & x < 0 \\ 1 - \left(1 - \frac{x}{2\pi}\right)^3, & x \in [0, 2\pi] \\ 1, & x > 1 \end{cases} \tag{13}$$

So we want to calculate

$$2\mathbb{E}[\min\{X_1, X_2, X_3\}] = 2\int_{\mathbb{R}} x dF_{\min\{X_1, X_2, X_3\}}(x) = 2\int_0^{2\pi} x dF_{\min\{X_1, X_2, X_3\}}(x) = 2\frac{\pi}{2} = \pi \tag{14}$$

ii) In the Figure 1 you can see simulation.

```
In [3]: import numpy as np
        import networkx as nx
        import matplotlib.pyplot as plt
```

```
In [26]: for i in range(20):
             np.random.seed(i)
             max_iter = 100500
             xs = np.zeros(max_iter)
             for iter in range(max_iter):
                 xs[iter] = 2 * np.pi * np.min(np.random.rand(1, 3))
             print(2 * xs.sum()/max_iter)
```

```
3.14000392578
3.13711891213
3.12307646533
3.14410653649
3.1260909926
3.15231786387
3.13542433232
3.14322615435
3.13243715291
3.14711565395
3.14736730935
3.12585256591
3.14384380748
3.14401416373
3.13228913715
3.13677574635
3.14921997046
3.14237725671
3.1424128492
3.13983745014
```

Figure 1: Problem 2 simulation

# Problem 3

i) Let $X_i$ be the $i$-th dice throw. Set

$$X_{50} = \sum_{i=1}^{50} X_i$$

We know that

$$\mathbb{E}[X_i] = \sum_i p_i x_i = \frac{3}{6} \cdot 0 + \frac{2}{6} \cdot 2 + \frac{1}{6} \cdot 26 = 5 \tag{15}$$

So the expected value of $X_{50}$ is the following

$$\mathbb{E}[X_{50}] = 50\mathbb{E}[X_i] = 250 \tag{16}$$

ii) The variance and standard deviation of $X_i$ can be calculated as follows

$$\mathbb{E}[X_i^2] = \sum_i p_i x_i^2 = \frac{3}{6} \cdot 0^2 + \frac{2}{6} \cdot 2^2 + \frac{1}{6} \cdot 26^2 = 114 \Rightarrow \tag{17}$$

$$\mathbb{V}[X_i] = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = 114 - 25 = 89 \Rightarrow \tag{18}$$

$$\sigma = \sqrt{89} \approx 9.43 \tag{19}$$

So the variance and the standard deviation of $X_{50}$ are the following

$$\mathbb{V}[X_{50}] = 50\mathbb{V}[X_i] = 4450 \tag{20}$$

$$\sigma_{50} = \sqrt{\mathbb{V}[X_{50}]} = 5\sqrt{178} \approx 66.7 \tag{21}$$

iii) We want to find probability of the event $\{X_{50} > 200\}$. Set

$$Z_{50} = \frac{X_{50} - 250}{\sigma_{50}}$$

In our estimation we assume that $Z_{50} \sim \mathcal{N}(0,1)$, so

$$\mathbb{P}(X_{50} > 200) \approx \mathbb{P}\left(Z_{50} > \frac{200 - 250}{\sqrt{4450}}\right) \approx 0.77323 \tag{22}$$

iv) Let $Y_{50}$ be the random variable with the same distribution as $X_{50}$ (result of my friend). We know that

$$\mathbb{E}[X_{50} - Y_{50}] = 0 \tag{23}$$

$$\mathbb{V}[X_{50} - Y_{50}] = 2 \cdot \sigma_{50}^2 = 8900 \Rightarrow \tilde{\sigma}_{50} = 10\sqrt{89} \tag{24}$$

We will assume that the random variable

$$\tilde{Z}_{50} = \frac{X_{50} - Y_{50} - \mathbb{E}[X_{50} - Y_{50}]}{\tilde{\sigma}_{50}} = \frac{X_{50} - Y_{50}}{10\sqrt{89}} \tag{25}$$

is distributed by $\mathcal{N}(0,1)$. So

$$\mathbb{P}(X_{50} - Y_{50} \geq 50) = \mathbb{P}\left(\tilde{Z}_{50} \geq \frac{50}{10\sqrt{89}}\right) \approx 0.298056 \tag{26}$$

## Problem 4

We want to find the capacity of the $Z$ channel. Let $p_0$ be the probability $\mathbb{P}(X=0)$. Set

$$q_0 = \mathbb{P}(Y=0) = p_0 + (1-p_0)f \tag{27}$$

$$q_1 = \mathbb{P}(Y=1) = (1-p_0)(1-f) \tag{28}$$

Then we have

$$H(Y) = -q_0\log_2(q_0) - q_1\log_2(q_1) = -[p_0+(1-p_0)f]\log_2[p_0+(1-p_0)f]- \tag{29}$$

$$-[(1-p_0)(1-\delta)]\log_2[(1-p_0)(1-f)] \tag{30}$$

$$H(Y|X) = p_0 H(Y|X=0) + (1-p_0)H(Y|X=1) = (1-p_0)(-f\log_2 f - (1-f)\log_2(1-f)) = (1-p_0)\mathbf{H}(f) \tag{31}$$

Since $H(Y|X=0) = -1\log_2 1 = 0$. Here $\mathbf{H}(f) = H(Ber(f)) = -f\log_2 f - (1-f)\log_2(1-f)$

So the the mutual information is the following

$$I(X;Y)(p_0) = H(Y) - H(Y|X) = -[p_0+(1-p_0)f]\log_2[p_0+(1-p_0)f] - [(1-p_0)(1-f)]\log_2[(1-p_0)(1-f)]- \tag{32}$$

$$-(1-p_0)\mathbf{H}(f) = \mathbf{H}\big((1-p_0)(1-f)\big) - (1-p_0)\mathbf{H}(f) \tag{33}$$

After calculating the derivative and putting it to zero we obtain

$$\frac{dI(X;Y)}{dp_0} = -(1-f)\log_2\left[\frac{p_0+(1-p_0)f}{(1-p_0)(1-f)}\right] + \mathbf{H}(f) = 0 \Leftrightarrow \tag{34}$$

$$\Leftrightarrow p_0 = 1 - \frac{1}{(1-f)(1+2^{\mathbf{H}(f)/(1-f)})} \tag{35}$$

Substituting the result into the capacity formula we obtain

$$C(f) = \mathbf{H}\left(\frac{1}{1+2^{\mathbf{H}(f)/(1-f)}}\right) - \frac{1}{(1-f)(1+2^{\mathbf{H}(f)/(1-f)})}\mathbf{H}(f) = \log_2\left[1+2^{-\frac{\mathbf{H}(f)}{1-f}}\right] \tag{36}$$

Now we can calculating everything we want

i) $\mathbb{P}(Y=0) = 0.8 + 0.2\cdot 0.1 = 0.82$

$\mathbb{P}(Y=1) = 1 - 0.82 = 0.18$

ii) $\mathbb{P}(X=1|Y=0) = \frac{\mathbb{P}(Y=0|X=1)\cdot\mathbb{P}(X=1)}{\mathbb{P}(Y=0)} = \frac{0.1\cdot 0.2}{0.82} = 0.02439024$

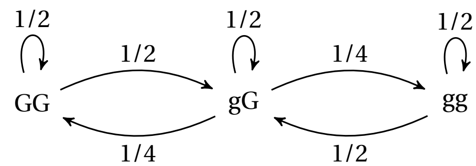
iii) Here we use logarithm w.r.t 2.

$I(X;Y) = \mathbf{H}(0.2 \cdot 0.9) - 0.2 \cdot \mathbf{H}(0.1) \approx 0.68 - 0.09 = 0.59$ bits.

iv) $C(0.1) \approx 0.76$

## Problem 5

We have the following Markov chain




which is obviously irreducible and aperiodic. The transition matrix of this Markov chain is the following

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{37}$$

Let us prove by induction the following formula for $P^n$

$$P^n = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{n+1}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{n+1}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{n+1}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{n+1}} \end{bmatrix} \tag{38}$$

Indeed, in the case of $n = 1$ we obtain (37). The step is also obvious

$$P^{n+1} = P^n \cdot P = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{n+1}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{n+1}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{n+1}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{n+1}} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{1}{2^{n+2}} & \frac{1}{2} & \frac{1}{4} - \frac{1}{2^{n+2}} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} - \frac{1}{2^{n+2}} & \frac{1}{2} & \frac{1}{4} + \frac{1}{2^{n+2}} \end{bmatrix} \tag{39}$$

We want to find $\mu_1, \mu_2, \mu_3$. From the formula (38) we see that $\mu_i$ does not depend on $n$ and has the following values

$$\mu_i(gg) = \mu_i(GG) = \frac{1}{4}, \;\; i = 1,2,3$$
$$\mu_i(gG) = \frac{1}{2}, \;\; i = 1,2,3$$

It can be easily seen that

$$\pi^* = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \tag{40}$$

is the stationary distribution, because

$$\pi^* P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \pi^* \tag{41}$$

It can be easily seen that detailed balance holds

$$\begin{aligned} \pi_1^* P_{13} &= \frac{1}{4} \cdot 0 = P_{31} \pi_3^* \\ \pi_1^* P_{12} &= \frac{1}{4} \cdot \frac{1}{2} = P_{21} \pi_2^* \\ \pi_2^* P_{23} &= \frac{1}{2} \cdot \frac{1}{4} = P_{32} \pi_3^* \end{aligned}$$

# Problem 6

i) Let $N(t)$ be a Poisson process with the rate $\lambda = 20$

$$\mathbb{P}(N(11-9) \geq 50) = \mathbb{P}(N(2) \geq 50) = \sum_{k=50}^{\infty} \frac{(20 \cdot 2)^k}{k!} e^{-20 \cdot 2} = e^{-40} \left( e^{40} - \sum_{k=0}^{49} \frac{(20 \cdot 2)^k}{k!} \right) \approx 0.0703351 \tag{42}$$

ii) Let $N_1(t)$ and $N_2(t)$ be Poisson processes for male and female arrivals, resp. The rate of $N_1(t)$ is $\lambda p$, the rate of $N_2(t) - \lambda(1-p)$. So

$$\mathbb{P}(N_1(1) = 20) = \frac{(20p)^{20}}{20!} e^{-20p} \tag{43}$$

iii) From recitations we know that for Poisson process with the rate $\lambda$ the pdf of inter-arrival time is equal to

$$P(T) = \begin{cases} \lambda e^{-\lambda T}, & T \geq 0 \\ 0, otherwise \end{cases} \tag{44}$$

It can be easily seen that the mean if this random variable is

$$\mathbb{E}T = \int_0^\infty \lambda T e^{-\lambda T} dT = \frac{1}{\lambda} \tag{45}$$

In the female case we have the Poisson rate $20(1-p)$, so the mean inter-arrival time is $\frac{1}{20(1-p)}$.

iv)

$$\mathbb{P}(N_2(3) = 0) = \frac{(60p)^0}{0!} e^{-60p} = e^{-60p}$$