Tsinghua-Berkeley Shenzhen Institute
Learning From Data
Fall 2020

Writing Assignment 5

Issued: Tuesday 15th December, 2020 **Due:** Wednesday 30th December, 2020

5.1. (2 points) (Bellman's equation) In a dynamic decision problem, given a policy π, the value function satisfies the Bellman equation:

$$V^{\pi}(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(a)}(s')V^{\pi}(s') \tag{1}$$

Now we play a simple game in a 3x3 block square. Our goal is to move the red object from the upper left $(0, 0)$ to the bottom right corner $(2, 2)$ (See Figure 1). The state s

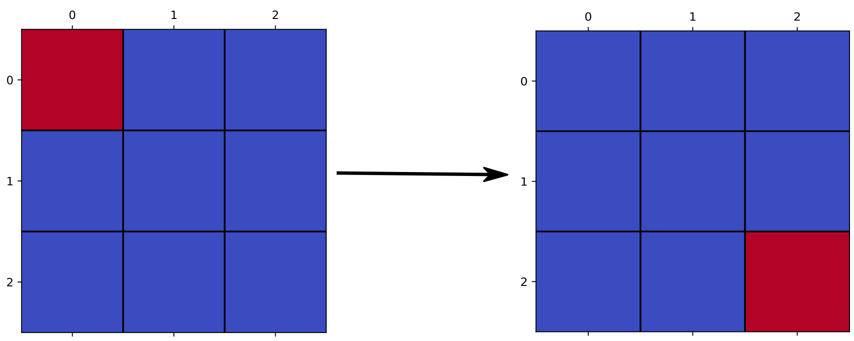

Figure 1: Moving a red object from upper left to bottom right

is represented by a tuple (x, y) where $x, y \in \{0, 1, 2\}$. Choosing $\gamma = 0.8$. The reward matrix satisfies $R((2, 2)) = 1$ and $R(s) = 0$ for other state s. There are four actions possible for each state $\mathcal{A} = \{\mathsf{up}, \mathsf{down}, \mathsf{left}, \mathsf{right}\}$, which deterministically cause the corresponding state transitions, except that actions that would take the agent off the grid in fact leave the state unchanged. For example $P_{s=(1,1),a=\mathsf{right}}(s' = (1, 2)) = 1$ and $P_{s=(1,1),a=\mathsf{right}}(s' = (0, 2)) = 0$. Suppose π is a policy defined by

$$\pi((i, j)) = \begin{cases} (i+1, j) & i < 2 \\ (i, j+1) & i = 2, j < 2 \\ (2, 2) & i = 2 \text{ and } j = 2 \end{cases}$$

Compute numerically the value function $V^{\pi}(s)$ for each s by solving the Bellman's equation.

Solution:

Let $p_{ij} = P(\pi((i, j)) = (i + 1, j))$. Applying the equation (1) we have:

$$V^{\pi}((2, 2)) = 1 + 0.8V^{\pi}((2, 2)) \qquad \cdots\cdots 0.5\,\text{points}$$
$$V^{\pi}((2, 1)) = 0.8V^{\pi}((2, 2)),\ V^{\pi}((2, 0)) = 0.8V^{\pi}((2, 1))$$
$$V^{\pi}((1, 2)) = 0.8V^{\pi}((2, 2)),\ V^{\pi}((0, 2)) = 0.8V^{\pi}((1, 2))$$
$$V^{\pi}((1, 1)) = 0.8V^{\pi}((2, 1)),\ V^{\pi}((0, 1)) = 0.8V^{\pi}((1, 1))$$
$$V^{\pi}((1, 0)) = 0.8V^{\pi}((2, 0)),\ V^{\pi}((0, 0)) = 0.8V^{\pi}((1, 0)) \qquad \cdots\cdots 1\,\text{points}$$

Therefore, we can solve out:

$$V^\pi = \begin{bmatrix} 2.048 & 2.56 & 3.2 \\ 2.56 & 3.2 & 4 \\ 3.2 & 4 & 5 \end{bmatrix} \quad \cdots\cdots 2\,\text{points}$$

5.2. (2 points) (Convergence of Value Iteration) You have learned in class value iteration algorithm updates the value function $V^{t+1}(s) = BV^t(s)$ for every state s, where B is the Bellman backup operator:

$$B\,V(s) = R(s) + \max_{a\in A} \gamma \sum_{s'\in S} P_{sa}(s')V(s') \tag{2}$$

(a) (1 point) Show that Bellman backup operator is a contraction operator. That is, for any value function V_1, V_2,

$$\max_{s\in S} |B\,V_1(s) - B\,V_2(s)| \le \gamma \max_{s\in S} |V_1(s) - V_2(s)| \tag{3}$$

(b) (1 point) Assuming $R_{\max} = \max_{s\in S} R(s)$ and $V^0(s) = 0$ for all $s \in S$, show that

$$\max_{s\in S} |V^t(s) - V^*(s)| \le \frac{\gamma^t R_{\max}}{1-\gamma} \tag{4}$$

From (4), we can see that $V^t(s)$ converges to $V^*(s)$.

Solution:

(a) Using the definition of Bellman backup operator,

$$\begin{aligned} B\,V_1(s) - B\,V_2(s) &= \gamma(\max_{a\in A} \sum_{s'\in S} P_{sa}(s')V_1(s') - \max_{a\in A} \sum_{s'\in S} P_{sa}(s')V_2(s')) \\ &\le \gamma \max_{a\in A} \sum_{s'\in S} P_{sa}(s')(V_1(s') - V_2(s')) \\ &\le \gamma \max_{s\in S} |V_1(s) - V_2(s)| \end{aligned}$$

Similarly $B\,V_2(s) - B\,V_1(s) \le \gamma \max_{s\in S} |V_1(s) - V_2(s)|$. Therefore $|B\,V_2(s) - B\,V_1(s)| \le \gamma \max_{s\in S} |V_1(s) - V_2(s)|$. Taking the maximum on the left hand side we can show (3). $\cdots\cdots$ 1 points

(b) For V^* we have $B\,V^*(s) = V^*(s)$ using (3) we have

$$\max_{s\in S} |V^{t+1}(s) - V^*(s)| \le \gamma \max_{s\in S} |V^t(s) - V^*(s)| \le \gamma^t \max_{s\in S} V^*(s)$$

$\cdots\cdots$ 0.5 points

From Bellman equation (1), we have $V^*(s) \le R_{\max} + \gamma \max_{s\in S} V^*(s)$. Taking the maximum on the left hand side we have $V^*(s) \le \frac{R_{\max}}{1-\gamma}$. Therefore, (4) holds. $\cdots\cdots$ 1 points

5.3. (3 points) (Mean Square Error) We mentioned Bias-Variance Tradeoff in class. We define the MSE of $\hat{X}$, an estimator of X as $\mathrm{MSE}(\hat{X}) \triangleq \mathbb{E}[(\hat{X}-X)^2]$. The variance of $\hat{X}$ is defined as $\mathrm{Var}(\hat{X}) \triangleq \mathbb{E}[(\hat{X}-\mathbb{E}[\hat{X}])^2]$ and the bias is defined as $\mathrm{Bias}(\hat{X}) \triangleq \mathbb{E}[\hat{X}]-X$.

(a) (1 point) Please prove that

$$\mathrm{MSE}(\hat{X}) = \mathrm{Var}(\hat{X}) + (\mathrm{Bias}(\hat{X}))^2$$

(b) (2 points) Our data are added with an independent Gaussian noise, say, $X+N$, where $\mathbb{E}[N] = 0$ and $\mathbb{E}[N^2] = \sigma^2$ and the estimator is $\hat{X}$. We define the empirical MSE as $\mathbb{E}[(\hat{X} - X - N)^2]$. Please prove that

$$\mathbb{E}[(\hat{X} - X - N)^2] = \mathrm{MSE}(\hat{X}) + \sigma^2$$

The equation tells us that the empirical error is a good estimation of the true error. Thus, we can minimize the empirical error in order to properly minimize the true error.

Solution:

(a)

$$\begin{aligned}
\mathbb{E}[(\hat{X} - X)^2] &= \mathbb{E}[(\hat{X} - \mathbb{E}[\hat{X}] + \mathbb{E}[\hat{X}] - X)^2] \\
&= \mathbb{E}[(\hat{X} - \mathbb{E}[\hat{X}])^2] + \mathbb{E}[(\mathbb{E}[\hat{X}] - X)^2] + 2\,\mathbb{E}[(\hat{X} - \mathbb{E}[\hat{X}])(\mathbb{E}[\hat{X}] - X)] \\
&= \mathrm{Var}(\hat{X}) + (\mathrm{Bias}(\hat{X}))^2
\end{aligned}$$

, since that the expectation is only about $\hat{X}$

$$\mathbb{E}[(\hat{X} - \mathbb{E}[\hat{X}])(\mathbb{E}[\hat{X}] - X)] = (\mathbb{E}[\hat{X}] - \mathbb{E}[\hat{X}])(\mathbb{E}[\hat{X}] - X) = 0$$

(b)

$$\begin{aligned}
\mathbb{E}[(\hat{X} - X - N)^2] &= \mathbb{E}[(\hat{X} - X)^2] + \mathbb{E}[N^2] + 2\,\mathbb{E}[(\hat{X} - X)N] \\
&= \mathrm{MSE}(\hat{X}) + \sigma^2 + 2\,\mathbb{E}[(\hat{X} - X)]\,\mathbb{E}[N] \\
&= \mathrm{MSE}(\hat{X}) + \sigma^2
\end{aligned}$$

5.4. (3 points) Important inequalities in Learning Theory.

(a) (1.5 points) (Markov's Inequality) Let X be a non-negative random variable, then for every positive constant a, please show that

$$P(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

(b) (1.5 points) (Chebyshev's inequality)For random variable X, if its expected value $\mathbb{E}(X)$ and variance $Var(X)$ are both finite, for every positive constant a, please show that

$$P(|X - \mathbb{E}(X)| \geq a) \leq \frac{Var(X)}{a^2}$$

5.5. (3 points) (Bonus question: VC Dimension) Given some finite domain set, $\mathcal{X}$, and a number $k \leq |\mathcal{X}|$,

please figure out the VC-dimension of each of the following classes:

(a) (1.5 points) $\mathcal{H}_k^{\mathcal{X}} = \{h \in \{0,1\}^{\mathcal{X}} : |\{x : h(x) = 1\}| = k\}$. That is, the set of all functions that assign the value 1 to exactly k elements of $\mathcal{X}$.

(b) (1.5 points) $\mathcal{H}_{\leq k}^{\mathcal{X}} = \{h \in \{0,1\}^{\mathcal{X}} : |\{x : h(x) = 1\}| \leq k \text{ or } |\{x : h(x) = 0\}| \leq k\}$

Solution:

(a) $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) = \min\{k, |\mathcal{X}| - k\}$
First, we prove $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) \leq k$. Let $C \subseteq \mathcal{X}$ be a set of size $k+1$. Then, there doesn't exist $h \in \mathcal{H}_k^{\mathcal{X}}$ which satisfies $h(x) = 1$ for all $x \in C$. Analogously, it is easy to verify that $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) \leq |\mathcal{X}| - k$. Hence, $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) \leq \min\{k, |\mathcal{X}| - k\}$
Then, we prove $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) \geq \min\{k, |\mathcal{X}| - k\}$. Let $C = \{x_1, \ldots, x_m\} \subseteq \mathcal{X}$ be a set with of size $m \leq \min\{k, |\mathcal{X}| - k\}$. Let $(y_1, \ldots, y_m) \in \{0,1\}^m$ be a vector of labels. Denote $\sum_{i=1}^{m} y_i$ by s. Pick an arbitrary subset $E \subseteq \mathcal{X} \backslash C$ of $k - s$ elements, and let $h \in \mathcal{H}_k^{\mathcal{X}}$ be the hypothesis which satisfies $h(x_i) = y_i$ for every $x_i \in C$, and $h(x) = \mathbb{1}\{E\}$ for every $x \in \mathcal{X} \backslash C$. We conclude that C is shattered by $\mathcal{H}_k^{\mathcal{X}}$. It follows that $\text{VCdim}(\mathcal{H}_k^{\mathcal{X}}) \geq \min\{k, |\mathcal{X}| - k\}$.

(b) $\text{VCdim}(\mathcal{H}_{\leq k}^{\mathcal{X}}) = k$
First, we prove $\mathcal{H}_{\leq k}^{\mathcal{X}} \leq k$. It is the same as part (a).
Then, we prove $\mathcal{H}_{\leq k}^{\mathcal{X}} \geq k$. Let $C = \{x_1, \ldots, x_m\} \subseteq \mathcal{X}$ be a set with of size $m \leq k$. Let $(y_1, \ldots, y_m) \in \{0,1\}^m$ be a vector of labels. This labeling is obtained by some hypothesis $h \in \mathcal{H}_{\leq k}^{\mathcal{X}}$ which satisfies $h(x_i) = y_i$ for every $x_i \in C$, and $h(x) = 0$ for every $x \in \mathcal{X} \backslash C$. We conclude that C is shattered by $\mathcal{H}_{\leq k}^{\mathcal{X}}$. It follows that $\text{VCdim}(\mathcal{H}_{\leq k}^{\mathcal{X}}) \geq k$.