1 Linearization of the FOC around the certainty-equivalent point

We start from the approximate first-order condition (FOC) derived from the second-order expansion of expected log utility:

$$\frac{1}{y_1 - s} = \beta\left[\frac{1}{m} + \frac{v}{m^3}\right], \qquad m \equiv s + (1+\lambda)x\mu, \tag{1}$$

where $\mu = e^{\sigma^2/2}$ and $v = (1+\lambda)^2 x^2 K$ with $K = e^{\sigma^2}(e^{\sigma^2} - 1)$. The variable m is the mean of second-period consumption, and v scales with its variance.

1. Certainty-equivalent benchmark

When there is no risk ($v = 0$), the FOC becomes

$$\frac{1}{y_1 - s_{\text{CE}}} = \frac{\beta}{s_{\text{CE}} + (1+\lambda)x\mu}.$$

Solving for s_{CE} gives:

$$\begin{aligned} s_{\text{CE}} + (1+\lambda)x\mu &= \beta(y_1 - s_{\text{CE}}), \\ (1+\beta)s_{\text{CE}} &= \beta y_1 - (1+\lambda)x\mu, \\ \boxed{s_{\text{CE}} = \frac{\beta y_1 - (1+\lambda)x\mu}{1+\beta}.} \end{aligned}$$

Let m_0 denote the certainty-equivalent mean of second-period consumption:

$$m_0 \equiv s_{\text{CE}} + (1+\lambda)x\mu.$$

2. Define small deviations

Let the true optimum under risk be $s^\star$ with corresponding mean

$$m = s^\star + (1+\lambda)x\mu, \qquad \delta m \equiv m - m_0.$$

From this definition,

$$s^\star = s_{\text{CE}} + \delta m.$$

We assume both δm and v are small and linearize equation (1) around $(s_{\text{CE}}, m_0, v = 0)$.

3. Linearize the FOC

Equation (1) can be written as

$$\underbrace{\frac{1}{y_1 - s^\star}}_{\text{LHS}} = \underbrace{\beta\left[\frac{1}{m} + \frac{v}{m^3}\right]}_{\text{RHS}}.$$

Left-hand side (LHS). For small deviations $s^\star = s_{\mathrm{CE}} + \delta m$, expand:

$$\begin{aligned}\frac{1}{y_1 - s^\star} &\approx \frac{1}{y_1 - s_{\mathrm{CE}}} + \frac{d}{ds}\left(\frac{1}{y_1 - s}\right)_{s=s_{\mathrm{CE}}} (s^\star - s_{\mathrm{CE}}) \\ &= \frac{1}{y_1 - s_{\mathrm{CE}}} + \frac{1}{(y_1 - s_{\mathrm{CE}})^2}\,\delta m.\end{aligned}$$

Using the certainty-equivalent FOC $\frac{1}{y_1 - s_{\mathrm{CE}}} = \frac{\beta}{m_0}$, we have:

$$\mathrm{LHS} \approx \frac{\beta}{m_0} + \frac{\beta^2}{m_0^2}\,\delta m.$$

Right-hand side (RHS). Substitute $m = m_0 + \delta m$ and expand to first order in δm and v:

$$\begin{aligned}\frac{1}{m} &= \frac{1}{m_0} - \frac{\delta m}{m_0^2}, \\ \frac{1}{m^3} &= \frac{1}{m_0^3} - \frac{3\delta m}{m_0^4}.\end{aligned}$$

Ignoring higher-order terms ($v\delta m$), the RHS becomes:

$$\mathrm{RHS} \approx \beta\left[\frac{1}{m_0} - \frac{\delta m}{m_0^2} + \frac{v}{m_0^3}\right] = \frac{\beta}{m_0} - \frac{\beta}{m_0^2}\,\delta m + \frac{\beta v}{m_0^3}.$$

4. Equate and solve for δm

Equating LHS and RHS and canceling β/m_0 gives:

$$\frac{\beta^2}{m_0^2}\,\delta m = -\frac{\beta}{m_0^2}\,\delta m + \frac{\beta v}{m_0^3}.$$

Combine like terms:

$$\delta m\left(\frac{\beta^2 + \beta}{m_0^2}\right) = \frac{\beta v}{m_0^3}.$$

Hence,

$$\boxed{\delta m = \frac{v}{(1+\beta)m_0}.}$$

5. Approximate optimal saving

Using $s^\star = s_{\mathrm{CE}} + \delta m$, we obtain:

$$\boxed{s^\star \approx s_{\mathrm{CE}} + \frac{v}{(1+\beta)m_0}.}$$

6. Interpretation

- The first term s_{CE} is the optimal saving under certainty.
- The additive correction $\frac{v}{(1+\beta)m_0}$ is positive for $v > 0$, reflecting the precautionary saving motive.
- The correction is smaller when β or m_0 is large: patient agents or richer agents exhibit a smaller marginal precautionary adjustment.

2 Apply a second-order Taylor expansion

We approximate $\mathbb{E}[\log C]$ using a second-order Taylor expansion of the function $g(c) = \log c$ around the mean of the random variable C.

Let

$$C = s + (1+\lambda)xz_2, \qquad m \equiv \mathbb{E}[C] = s + (1+\lambda)x\mu, \qquad v \equiv \mathrm{Var}(C).$$

Then the second-order Taylor expansion of $g(C)$ around m is

$$g(C) \approx g(m) + g'(m)\,(C-m) + \frac{1}{2}g''(m)\,(C-m)^2.$$

Taking expectations and using $\mathbb{E}[C-m] = 0$ yields

$$\mathbb{E}[g(C)] \approx g(m) + \frac{1}{2}g''(m)\,\mathbb{E}\left[(C-m)^2\right].$$

Since $\mathbb{E}[(C-m)^2] = \mathrm{Var}(C) = v$, we have

$$\mathbb{E}[\log C] \approx g(m) + \frac{1}{2}g''(m)\,v.$$

For $g(c) = \log c$,

$$g'(c) = \frac{1}{c}, \qquad g''(c) = -\frac{1}{c^2}.$$

Substituting $g(m)$ and $g''(m)$ gives

$$\mathbb{E}[\log C] \approx \log m - \frac{v}{2m^2}.$$

Finally, recalling that $m = s + (1+\lambda)x\mu$, the delta-method approximation is

$$\boxed{\mathbb{E}[\log(s + (1+\lambda)xz_2)] \approx \log\big(s + (1+\lambda)x\mu\big) - \frac{v}{2\big(s + (1+\lambda)x\mu\big)^2}.}$$