# Two-Period Consumption–Saving Problem with Lognormal Income Risk

## 1 Setup

The representative agent maximizes expected lifetime utility

$$\max_{s \in [0, y_1]} \log(c_1) + \beta\,\mathbb{E}[\log(c_2)], \tag{1}$$

where

$$c_1 = y_1 - s, \qquad c_2 = (1+\lambda)x z_2 + s,$$

and $z_2 \sim \log\mathcal{N}(0, \sigma^2)$, so that $\mathbb{E}[z_2] = e^{\sigma^2/2}$.

The parameter $\lambda$ scales the second-period endowment, $x > 0$ is a baseline level, and $\beta \in (0, 1)$ is the discount factor.

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## 2 Exact First-Order Condition

The first-order condition (Euler equation) for an interior optimum is

$$\frac{1}{y_1 - s} = \beta\,\mathbb{E}\left[\frac{1}{s + (1+\lambda)x z_2}\right]. \tag{2}$$

Because the left-hand side is increasing and the right-hand side is decreasing in $s$, the solution is unique whenever it lies in the interior $(0, y_1)$.

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# 3 Second-Order Approximation

Define

$$\mu \equiv \mathbb{E}[z_2] = e^{\sigma^2/2}, \qquad v \equiv \mathrm{Var}\big((1+\lambda)xz_2\big) = (1+\lambda)^2 x^2 e^{\sigma^2}\big(e^{\sigma^2}-1\big).$$

Using a second-order (delta-method) approximation,

$$\mathbb{E}[\log(s+(1+\lambda)xz_2)] \approx \log\big(s+(1+\lambda)x\mu\big) - \frac{v}{2\big(s+(1+\lambda)x\mu\big)^2}.$$

The approximate objective is then

$$\max_{0\le s\le y_1} \log(y_1-s) + \beta\left[\log\big(s+(1+\lambda)x\mu\big) - \frac{v}{2\big(s+(1+\lambda)x\mu\big)^2}\right]. \tag{3}$$

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## First-Order Condition (Approximate)

The first-order condition becomes

$$\frac{1}{y_1-s} = \beta\left[\frac{1}{m}+\frac{v}{m^3}\right], \qquad m \equiv s+(1+\lambda)x\mu. \tag{4}$$

Let $a=(1+\lambda)x\mu$ and $b=y_1+a$. If we set $v=0$, we obtain the certainty-equivalent (CE) solution:

$$m_0 = \frac{\beta b}{1+\beta}, \qquad s_{\mathrm{CE}} = \frac{\beta y_1 - a}{1+\beta}. \tag{5}$$

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## Precautionary Correction

Linearizing FOC around $m_0$ for small $v$ gives

$$\delta m = \frac{v}{(1+\beta)m_0}, \qquad s^\star \approx s_{\mathrm{CE}} + \frac{v}{(1+\beta)m_0}.$$

Substituting $m_0 = \frac{\beta(y_1+a)}{1+\beta}$ yields

$$\boxed{s^\star(\lambda) \;\approx\; \frac{\beta y_1 - (1+\lambda)xe^{\sigma^2/2}}{1+\beta} \;+\; \frac{(1+\lambda)^2x^2e^{\sigma^2}\big(e^{\sigma^2}-1\big)}{\beta\big[y_1+(1+\lambda)xe^{\sigma^2/2}\big]}.} \tag{6}$$

The first term is the certainty-equivalent saving, and the second term is the *precautionary saving correction*, which is positive and increasing in $\sigma$.

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# 4 Comparative Statics with Respect to $\lambda$

Let

$$A \equiv xe^{\sigma^2/2}, \qquad K \equiv e^{\sigma^2}\big(e^{\sigma^2} - 1\big), \qquad t \equiv 1 + \lambda, \qquad B \equiv \frac{x^2 K}{\beta}.$$

Then equation (6) can be written compactly as

$$s^\star(\lambda) = \frac{\beta y_1 - tA}{1+\beta} + B \cdot \frac{t^2}{y_1 + tA}. \tag{7}$$

The derivative with respect to $\lambda$ is

$$\frac{ds^\star}{d\lambda} = -\frac{A}{1+\beta} + B \cdot \frac{t(2y_1 + tA)}{(y_1 + tA)^2}. \tag{8}$$

- The first term is negative (higher mean future income lowers saving).
- The second term is positive (larger income scale increases risk and hence precautionary saving).

Thus, the total effect is *ambiguous*, depending on the magnitude of risk and the discount factor.

## Convexity in $\lambda$

Differentiating again,

$$\frac{d^2 s^\star}{d\lambda^2} = B \cdot \frac{2y_1^2}{(y_1 + tA)^3} > 0, \tag{9}$$

so $s^\star(\lambda)$ is **convex** in $\lambda$. This means the marginal effect of $\lambda$ on saving becomes less negative (or more positive) as $\lambda$ rises.

In other words, the crowd-in effect (precautionary saving) of $\lambda$ on saving becomes stronger as $\lambda$ rises. As the AI shock reduces $\lambda$ for the mover from low to middle sector, the crowd-in effect is weaker and therefore a reduction of saving for the middle sector. On the other hand, the AI shock enlarges $\lambda$ for the mover from middle to high sector, the crowd-in effect is stronger and therefore an increase of saving for the high sector.

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# 5 Comparative Statics with Respect to $y_1$

From the approximate solution (6), we can derive the effect of first-period income on optimal saving:

$$\frac{\partial s^\star}{\partial y_1} = \frac{\beta}{1+\beta} - B \cdot \frac{t^2}{(y_1 + tA)^2}. \tag{10}$$

This derivative has two components:

- **Direct consumption-smoothing effect** (first term): $\frac{\beta}{1+\beta} > 0$
  - Higher $y_1$ increases saving to smooth consumption across periods
  - This is the standard life-cycle saving motive
- **Precautionary adjustment** (second term): $-B \cdot \frac{t^2}{(y_1+tA)^2} < 0$
  - Higher $y_1$ reduces the relative importance of precautionary saving
  - With more first-period resources, the agent is less concerned about second-period risk

## Net Effect

The net effect $\frac{\partial s^\star}{\partial y_1}$ is **ambiguous** and depends on parameter values and the level of $y_1$:

$$\frac{\partial s^\star}{\partial y_1} > 0 \quad \Leftrightarrow \quad \frac{\beta}{1+\beta} > B \cdot \frac{t^2}{(y_1 + tA)^2} \tag{11}$$

If the inequality for $\frac{\partial s^\star}{\partial y_1} > 0$ holds at $y_1 = 0$, it is a sufficient condition for it to hold for all $y_1 \geq 0$. Plugging $y_1 = 0$ into the condition gives:

$$\frac{\beta}{1+\beta} > B \cdot \frac{1}{A^2} \tag{12}$$

Recall that $A = (1+\lambda)xe^{\sigma^2/2}$ and $B = \frac{(1+\lambda)^2 x^2 e^{\sigma^2}(e^{\sigma^2}-1)}{\beta}$, so

$$B \cdot \frac{1}{A^2} = \frac{(1+\lambda)^2 x^2 e^{\sigma^2}(e^{\sigma^2}-1)}{\beta} \cdot \frac{1}{(1+\lambda)^2 x^2 e^{\sigma^2}} = \frac{e^{\sigma^2}-1}{\beta}$$

Thus, the sufficient condition becomes

$$\frac{\beta}{1+\beta} > \frac{e^{\sigma^2}-1}{\beta} \tag{13}$$

or equivalently,

$$e^{\sigma^2} < \frac{\beta^2}{1+\beta} + 1 \tag{14}$$

Taking logs yields the sufficient upper bound on $\sigma$:

$$\sigma^2 < \log\left(\frac{\beta^2}{1+\beta} + 1\right) \tag{15}$$

Therefore, if $\sigma^2$ satisfies this bound, then $\frac{\partial s^\star}{\partial y_1} > 0$ for all $y_1 \geq 0$.

The second derivative is always positive:

$$\frac{\partial^2 s^\star}{\partial y_1^2} = B \cdot \frac{2t^2}{(y_1 + tA)^3} > 0, \tag{16}$$

so $s^\star(y_1)$ is **convex** in $y_1$.

**Case Analysis:**

- **High $y_1$ or low risk**: $\frac{\partial s^\star}{\partial y_1} > 0$ (consumption smoothing dominates)
- **Low $y_1$ or high risk**: $\frac{\partial s^\star}{\partial y_1}$ could be negative (precautionary adjustment dominates)
- **Asymptotic behavior**: As $y_1 \to \infty$, $\frac{\partial s^\star}{\partial y_1} \to \frac{\beta}{1+\beta} > 0$
- **Convexity**: The marginal effect becomes less negative (or more positive) as $y_1$ increases

## Economic Interpretation

The convexity of $s^\star(y_1)$ captures the interaction between consumption smoothing and precautionary motives:

- At **low wealth levels**, precautionary concerns may dominate, potentially leading to *dissaving* (negative marginal propensity to save) as agents prioritize current consumption when future income is risky
- At **high wealth levels**, standard consumption smoothing dominates, and saving increases with income
- The convex shape reflects **diminishing precautionary motives**: as first-period wealth increases, agents become less sensitive to second-period income risk, consistent with decreasing absolute risk aversion under logarithmic utility

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# 6 Small-Risk Approximation

For small $\sigma^2$, use

$$e^{\sigma^2/2} \approx 1 + \frac{\sigma^2}{2}, \qquad e^{\sigma^2}(e^{\sigma^2} - 1) \approx \sigma^2.$$

This simplifies the solution to

$$s^\star(\lambda) \approx \frac{\beta y_1 - (1+\lambda)x(1+\frac{\sigma^2}{2})}{1+\beta} + \frac{(1+\lambda)^2 x^2 \sigma^2}{\beta\,[y_1 + (1+\lambda)x(1+\frac{\sigma^2}{2})]}. \tag{17}$$

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# 7 Illustration

We consider the second-order approximate solution

$$s^\star(\lambda) = \frac{\beta y_1 - (1+\lambda)x e^{\sigma^2/2}}{1+\beta} + \frac{(1+\lambda)^2 x^2 e^{\sigma^2}\big(e^{\sigma^2} - 1\big)}{\beta\,[\,y_1 + (1+\lambda)x e^{\sigma^2/2}\,]}.$$

Parameters are chosen to produce a symmetric convex pattern

$$s^\star(-0.2) > s^\star(0) < s^\star(0.2),$$

with both gaps of similar magnitude.

## Parameter Values

| Parameter | Symbol | Value |
|---|---|---|
| First-period endowment | $y_1$ | 1.0 |
| Discount factor | $\beta$ | 0.9535 |
| Scale of risky endowment | $x$ | 1.13 |
| Lognormal standard deviation | $\sigma$ | 0.681 |

Table 1: Parameterization for balanced convexity in $s^\star(\lambda)$.

## Results

The pattern confirms that $s^\star(\lambda)$ is convex: both $s^\star(-0.2)$ and $s^\star(0.2)$ exceed $s^\star(0)$, illustrating symmetric precautionary adjustments to small deviations in future endowment scaling $\lambda$.

| | $\lambda = -0.2$ | $\lambda = 0$ | $\lambda = 0.2$ |
|---|---|---|---|
| $s^\star(\lambda)$ | 0.28011 | 0.27650 | 0.28004 |
| Gap from $\lambda = 0$ | +0.00361 | – | +0.00354 |

Table 2: Values of $s^\star(\lambda)$ showing convex pattern.

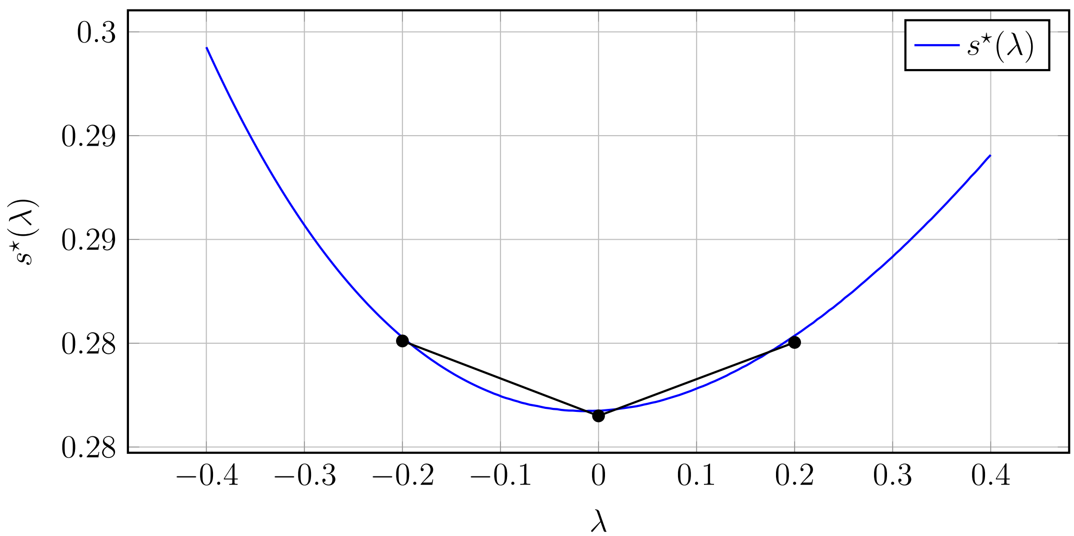


Figure 1: Balanced convexity of $s^\star(\lambda)$ under parameter set in Table 1.

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# 8 Economic Interpretation

- For small $\lambda$, a larger future endowment reduces saving (the income effect dominates).
- For large $\lambda$, risk amplification dominates, raising precautionary saving.
- The convexity of $s^\star(\lambda)$ captures this trade-off: increasing $\lambda$ both raises the mean and magnifies the uncertainty of future consumption.