Two-Period Consumption–Saving Problem with Lognormal Income Risk

1 Setup

The representative agent maximizes expected lifetime utility

$$\max_{s\in[0,y_1]} \log(c_1) + \beta\,\mathbb{E}[\log(c_2)], \tag{1}$$

where

$$c_1 = y_1 - s, \qquad c_2 = (1+\lambda)xz_2 + s,$$

and $z_2 \sim \log\mathcal{N}(0,\sigma^2)$, so that $\mathbb{E}[z_2] = e^{\sigma^2/2}$.

The parameter λ scales the second-period endowment, $x > 0$ is a baseline level, and $\beta \in (0,1)$ is the discount factor.

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2 Exact First-Order Condition

The first-order condition (Euler equation) for an interior optimum is

$$\frac{1}{y_1 - s} = \beta\,\mathbb{E}\left[\frac{1}{s + (1+\lambda)xz_2}\right]. \tag{2}$$

Because the left-hand side is increasing and the right-hand side is decreasing in s, the solution is unique whenever it lies in the interior $(0, y_1)$.

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3 Second-Order Approximation

Define

$$\mu \equiv \mathbb{E}[z_2] = e^{\sigma^2/2}, \qquad v \equiv \mathrm{Var}\big((1+\lambda)xz_2\big) = (1+\lambda)^2x^2e^{\sigma^2}\big(e^{\sigma^2}-1\big).$$

Using a second-order (delta-method) approximation,

$$\mathbb{E}[\log(s+(1+\lambda)xz_2)] \approx \log\big(s+(1+\lambda)x\mu\big) - \frac{v}{2\big(s+(1+\lambda)x\mu\big)^2}.$$

The approximate objective is then

$$\max_{0\le s\le y_1} \ \log(y_1-s) + \beta\left[\log\big(s+(1+\lambda)x\mu\big) - \frac{v}{2\big(s+(1+\lambda)x\mu\big)^2}\right]. \tag{3}$$

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First-Order Condition (Approximate)

The first-order condition becomes

$$\frac{1}{y_1-s} = \beta\left[\frac{1}{m}+\frac{v}{m^3}\right], \qquad m \equiv s+(1+\lambda)x\mu. \tag{4}$$

Let $a = (1+\lambda)x\mu$ and $b = y_1 + a$. If we set $v = 0$, we obtain the certainty-equivalent (CE) solution:

$$m_0 = \frac{\beta b}{1+\beta}, \qquad s_{\mathrm{CE}} = \frac{\beta y_1 - a}{1+\beta}. \tag{5}$$

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Precautionary Correction

Linearizing around m_0 for small v gives

$$\delta m = \frac{v}{(1+\beta)m_0}, \qquad s^\star \approx s_{\mathrm{CE}} + \frac{v}{(1+\beta)m_0}.$$

Substituting $m_0 = \frac{\beta(y_1+a)}{1+\beta}$ yields

$$\boxed{s^\star(\lambda) \ \approx \ \frac{\beta y_1 - (1+\lambda)xe^{\sigma^2/2}}{1+\beta} \ + \ \frac{(1+\lambda)^2x^2e^{\sigma^2}\big(e^{\sigma^2}-1\big)}{\beta\big[y_1+(1+\lambda)xe^{\sigma^2/2}\big]}.} \tag{6}$$

The first term is the certainty-equivalent saving, and the second term is the *precautionary saving correction*, which is positive and increasing in σ.

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4 Comparative Statics with Respect to λ

Let

$$A \equiv xe^{\sigma^2/2}, \qquad K \equiv e^{\sigma^2}\big(e^{\sigma^2}-1\big), \qquad t \equiv 1+\lambda, \qquad B \equiv \frac{x^2K}{\beta}.$$

Then equation (6) can be written compactly as

$$s^\star(\lambda) = \frac{\beta y_1 - tA}{1+\beta} + B\cdot\frac{t^2}{y_1+tA}. \tag{7}$$

The derivative with respect to λ is

$$\frac{ds^\star}{d\lambda} = -\frac{A}{1+\beta} + B\cdot\frac{t(2y_1+tA)}{(y_1+tA)^2}. \tag{8}$$

- The first term is negative (higher mean future income lowers saving).
- The second term is positive (larger income scale increases risk and hence precautionary saving).

Thus, the total effect is *ambiguous*, depending on the magnitude of risk and the discount factor.

Convexity in λ

Differentiating again,

$$\frac{d^2s^\star}{d\lambda^2} = B\cdot\frac{2y_1^2}{(y_1+tA)^3} > 0, \tag{9}$$

so $s^\star(\lambda)$ is **convex** in λ. This means the marginal effect of λ on saving becomes less negative (or more positive) as λ rises.

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5 Small-Risk Approximation

For small σ^2, use

$$e^{\sigma^2/2} \approx 1+\frac{\sigma^2}{2}, \qquad e^{\sigma^2}(e^{\sigma^2}-1) \approx \sigma^2.$$

This simplifies the solution to

$$s^\star(\lambda) \approx \frac{\beta y_1 - (1+\lambda)x(1+\frac{\sigma^2}{2})}{1+\beta} + \frac{(1+\lambda)^2 x^2 \sigma^2}{\beta\,[y_1 + (1+\lambda)x(1+\frac{\sigma^2}{2})]}. \tag{10}$$

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6 Illustration

We consider the second-order approximate solution

$$s^\star(\lambda) = \frac{\beta y_1 - (1+\lambda)x e^{\sigma^2/2}}{1+\beta} + \frac{(1+\lambda)^2 x^2 e^{\sigma^2}\big(e^{\sigma^2} - 1\big)}{\beta\,\big[\,y_1 + (1+\lambda)x e^{\sigma^2/2}\,\big]}.$$

Parameters are chosen to produce a symmetric convex pattern

$$s^\star(-0.2) > s^\star(0) < s^\star(0.2),$$

with both gaps of similar magnitude.

Parameter Values

Parameter	Symbol	Value
First-period endowment	y_1	1.0
Discount factor	β	0.9535
Scale of risky endowment	x	1.13
Lognormal standard deviation	σ	0.681

Table 1: Parameterization for balanced convexity in $s^\star(\lambda)$.

Results

	$\lambda = -0.2$	$\lambda = 0$	$\lambda = 0.2$
$s^\star(\lambda)$	0.28011	0.27650	0.28004
Gap from $\lambda = 0$	+0.00361	–	+0.00354

Table 2: Values of $s^\star(\lambda)$ showing convex pattern.

The pattern confirms that $s^\star(\lambda)$ is convex: both $s^\star(-0.2)$ and $s^\star(0.2)$ exceed $s^\star(0)$, illustrating symmetric precautionary adjustments to small deviations in future endowment scaling λ.

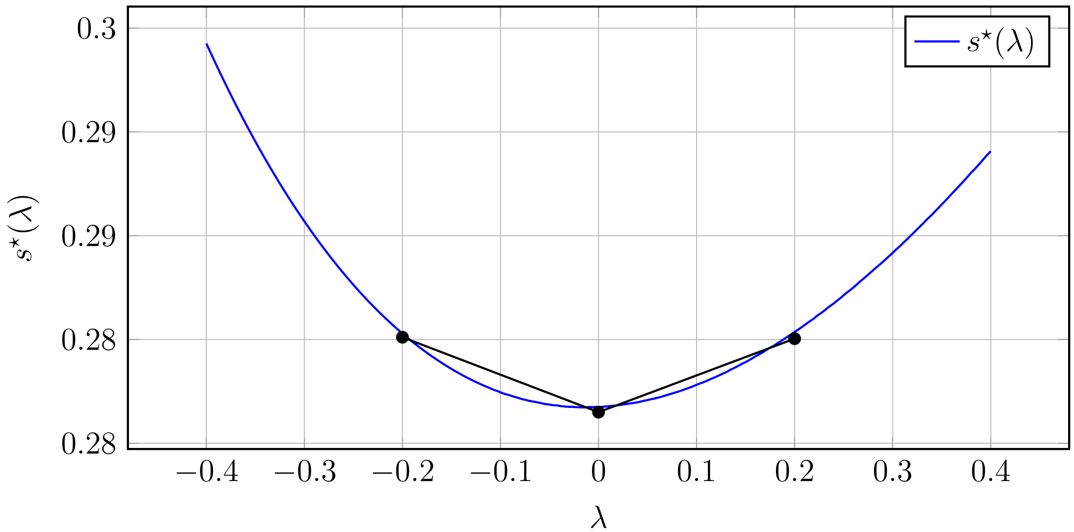

Figure 1: Balanced convexity of $s^\star(\lambda)$ under parameter set in Table 1.

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7 Economic Interpretation

- For small λ, a larger future endowment reduces saving (the income effect dominates).
- For large λ, risk amplification dominates, raising precautionary saving.
- The convexity of $s^\star(\lambda)$ captures this trade-off: increasing λ both raises the mean and magnifies the uncertainty of future consumption.