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二维浅水波方程的连续伴随方程以及它的直接离散形式

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Abstract. Insert your abstract here.

1 显式离散

原方程:

$$\text{u-动量:}\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial z}{\partial x}-fv+\gamma u=0 \tag{1}$$

$$\text{v-动量:}\quad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial z}{\partial y}+fu+\gamma v=0 \tag{2}$$

$$\text{连续方程:}\quad \frac{\partial z}{\partial t}+\frac{\partial (Hu)}{\partial x}+\frac{\partial (Hv)}{\partial y}=0 \tag{3}$$

$$\text{其中}\quad H=h+z$$

离散:
u-动量:

$$\begin{aligned} u_{i,j}^{n+1} =& u_{i,j}^n-\Delta t\left(u_{i,j}^n\frac{u_{i+1,j}^n-u_{i-1,j}^n}{2\Delta x}+v_{i+1/2,j}^n\frac{u_{i,j+1}^n-u_{i,j-1}^n}{2\Delta y}\right)\\ &-g\frac{\Delta t}{\Delta x}(z_{i+1,j}^n-z_{i,j}^n)-\Delta t\gamma u_{i,j}^n+\Delta t f v_{i,j}^n \end{aligned}\tag{4}$$

其中:

$$v_{i+1/2,j}^n=\frac{1}{4}(v_{i,j}^n+v_{i,j-1}^n+v_{i+1,j}^n+v_{i+1,j-1}^n)$$

v-动量:

$$\begin{aligned} v_{i,j}^{n+1} =& v_{i,j}^n-\Delta t\left(u_{i,j+1/2}^n\frac{v_{i+1,j}^n-v_{i-1,j}^n}{2\Delta x}+v_{i,j}^n\frac{v_{i,j+1}^n-v_{i,j-1}^n}{2\Delta y}\right)\\ &-g\frac{\Delta t}{\Delta y}(z_{i,j+1}^n-z_{i,j}^n)-\Delta t\gamma v_{i,j}^n-\Delta t f u_{i,j}^n \end{aligned}\tag{5}$$

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其中:

$$u^n_{i,j+1/2} = \frac{1}{4}(u^n_{i-1,j} + u^n_{i,j} + u^n_{i-1,j+1} + u^n_{i,j+1})$$

连续方程:

$$\begin{aligned} z^{n+1}_{i,j} =& z^n_{i,j} - \frac{\Delta t}{\Delta x}\left(H^n_{i+1/2,j}u^n_{i,j} - H^n_{i-1/2,j}u^n_{i-1,j}\right) \\ & - \frac{\Delta t}{\Delta y}\left(H^n_{i,j+1/2}v^n_{i,j} - H^n_{i,j-1/2}v^n_{i,j-1}\right) \end{aligned} \tag{6}$$

其中

$$H^n_{i\pm1/2,j} = \frac{1}{2}(H^n_{i\pm1,j} + H^n_{i,j}) \quad 和 \quad H^n_{i,j\pm1/2} = \frac{1}{2}(H^n_{i,j\pm1} + H^n_{i,j})$$

2 连续伴随方程推导

构造目标函数和拉格朗日函数

假设目标函数 J 是一个时空积分，它度量了模型解与观测值或约束之间的差异。

$$J(u,v,z) = \iiint F(u,v,z,x,y,t)\,dx\,dy\,dt$$

其中 F 是目标函数的密度。

拉格朗日函数 $\mathcal{L}$ 是目标函数 J 加上原始方程 (1)–(3) 与其对应伴随变量 (z^*, u^*, v^*) 的乘积在时空上的积分。

$$\mathcal{L} = J + \iiint [z^* \cdot (3) + u^* \cdot (1) + v^* \cdot (2)]\,dx\,dy\,dt$$

其中，我们将 (1)–(3) 重写为 $\mathrm{RHS} - \mathrm{LHS} = 0$ 的形式。

$$\begin{aligned} \mathcal{L} = J + \iiint \Bigg\{ & z^*\left(\frac{\partial z}{\partial t} + \frac{\partial(Hu)}{\partial x} + \frac{\partial(Hv)}{\partial y}\right) \\ & + u^*\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial z}{\partial x} + \gamma u - fv\right) \\ & + v^*\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial z}{\partial y} + \gamma v + fu\right)\Bigg\}\,dx\,dy\,dt \end{aligned}$$

应用变分原理

我们从拉格朗日函数 $\mathcal{L}$ 的一阶变分开始。我们要求 $\delta\mathcal{L} = 0$，这是因为在最佳轨迹（满足原始方程）上，模型状态 u, v, z 的微小变化 $\delta u, \delta v, \delta z$ 必须使目标函数 J 停止变化。

拉格朗日函数 $\mathcal{L}$ 如下（其中 Ω 是空间域，$[0,T]$ 是时间域）:

$$\begin{aligned}\mathcal{L}(u,v,z,u^*,v^*,z^*) &= J(u,v,z)\\ &\quad + \int_0^T \int_\Omega [R_z \cdot z^* + R_u \cdot u^* + R_v \cdot v^*]\, dx\, dy\, dt\end{aligned}$$

其中，R 是原始方程的残差（$\mathrm{LHS}-\mathrm{RHS}=0$ 的形式）:

$$\begin{aligned}R_z &= \frac{\partial z}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} \quad (H = h+z)\\ R_u &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial z}{\partial x} + \gamma u - fv\\ R_v &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial z}{\partial y} + \gamma v + fu\end{aligned}$$

(1)：求拉格朗日函数的变分

我们对 $\mathcal{L}$ 中所有状态变量 (u,v,z) 求一阶变分 $\delta\mathcal{L}$:

$$\begin{aligned}\delta\mathcal{L} = \delta J + \int_0^T \int_\Omega \Big[&\delta(R_z)\cdot z^* + \delta(R_u)\cdot u^* + \delta(R_v)\cdot v^*\\ &+ R_z\cdot \delta z^* + R_u \cdot \delta u^* + R_v \cdot \delta v^*\Big] dx\, dy\, dt\end{aligned}$$

根据约束，原始方程 $R_z = R_u = R_v = 0$ 成立。因此，拉格朗日乘子（即伴随变量 u^*, v^*, z^*）的变分项为零:

$$\delta\mathcal{L} = \delta J + \int_0^T \int_\Omega [\delta(R_z)\cdot z^* + \delta(R_u)\cdot u^* + \delta(R_v)\cdot v^*]\, dx\, dy\, dt$$

(2)：对 δR 项进行分部积分 (IBP)

我们对所有包含 $\delta u, \delta v, \delta z$ 导数的项进行分部积分，将导数从状态变量的变分转移到伴随变量 u^*, v^*, z^* 上。

A. 转移时间导数 $\frac{\partial}{\partial t}$

考虑所有时间导数项:

$$\begin{aligned}\int_0^T \int_\Omega &\left[z^*\frac{\partial}{\partial t}(\delta z) + u^*\frac{\partial}{\partial t}(\delta u) + v^*\frac{\partial}{\partial t}(\delta v)\right] dx\, dy\, dt\\ &= \int_\Omega [z^*\delta z + u^*\delta u + v^*\delta v]_0^T\, dx\, dy\\ &\quad - \int_0^T \int_\Omega \left[\frac{\partial z^*}{\partial t}\delta z + \frac{\partial u^*}{\partial t}\delta u + \frac{\partial v^*}{\partial t}\delta v\right] dx\, dy\, dt\end{aligned}$$

- **终端项 (Terminal term)**: 在 $t = T$ 时，我们要求 δJ 对 $u(T), v(T), z(T)$ 的贡献为零，因此我们设置伴随变量在最终时刻 T 的条件：

$$z^*(T) = -\left.\frac{\partial J}{\partial z}\right|_T, \quad u^*(T) = -\left.\frac{\partial J}{\partial u}\right|_T, \quad v^*(T) = -\left.\frac{\partial J}{\partial v}\right|_T$$

  这被称为伴随终端条件 (Adjoint Terminal Condition)。如果 J 不依赖于 T 时刻的状态，则伴随变量在 T 时刻为零。
- **初始项 (Initial term)**: 在 $t = 0$ 时，我们得到 $\int_\Omega [z^*(0)\delta z(0) + u^*(0)\delta u(0) + v^*(0)\delta v(0)]\,dx\,dy$ 这一项至关重要，它将用于计算梯度。

B. 转移空间导数 $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

这一步涉及到质量方程中的通量项，以及动量方程中的压力梯度项和平流项。我们使用通用的 IBP 公式：$\int \phi \frac{\partial \psi}{\partial x} dx = \phi\psi\big|_{\text{边界}} - \int \frac{\partial \phi}{\partial x} \psi dx$。

1. 质量方程的变分 $\delta(R_z) \cdot z^*$ (通量项):

$$\int_\Omega z^* \left[\frac{\partial}{\partial x}(H\delta u + u\delta z) + \frac{\partial}{\partial y}(H\delta v + v\delta z)\right] dx\,dy$$

应用 IBP：

$$= \int_{\partial\Omega} z^* \left[(H\delta u + u\delta z)n_x + (H\delta v + v\delta z)n_y\right] ds - \int_\Omega \left[\frac{\partial z^*}{\partial x}(H\delta u + u\delta z) + \frac{\partial z^*}{\partial y}(H\delta v + v\delta z)\right] dx\,dy$$

- **空间边界项 (Boundary term)**: 这一项必须通过设置伴随边界条件 (Adjoint Boundary Conditions) 为零。

2. 动量方程的变分 $\delta(R_u) \cdot u^*$ 和 $\delta(R_v) \cdot v^*$ (压力梯度项):

压力梯度项（重力项）：

$$\int_\Omega \left[u^* g \frac{\partial}{\partial x}(\delta z) + v^* g \frac{\partial}{\partial y}(\delta z)\right] dx\,dy$$

应用 IBP：

$$= \int_{\partial\Omega} g\delta z \left[u^* n_x + v^* n_y\right] ds - \int_\Omega g\delta z \left[\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right] dx\,dy$$

- **空间边界项**: 同样需要通过伴随边界条件消去。

3. 动量方程的变分 (平流项):

u 动量方程中的平流项:

$$\int_\Omega u^* \left[\delta u \frac{\partial u}{\partial x} + u \frac{\partial}{\partial x}(\delta u) + \delta v \frac{\partial u}{\partial y} + v \frac{\partial}{\partial y}(\delta u)\right] dx\, dy$$

应用 IBP:

$$\int_\Omega u^* u \frac{\partial}{\partial x}(\delta u) dx\, dy \xrightarrow{\text{IBP}} -\int_\Omega \frac{\partial}{\partial x}(u^* u) \delta u\, dx\, dy + \text{边界项}$$

$$\int_\Omega u^* v \frac{\partial}{\partial y}(\delta u) dx\, dy \xrightarrow{\text{IBP}} -\int_\Omega \frac{\partial}{\partial y}(u^* v) \delta u\, dx\, dy + \text{边界项}$$

v 动量方程中的交叉平流项:

$$\int_\Omega v^* \left[\delta u \frac{\partial v}{\partial x} + u \frac{\partial}{\partial x}(\delta v) + \delta v \frac{\partial v}{\partial y} + v \frac{\partial}{\partial y}(\delta v)\right] dx\, dy$$

应用 IBP:

$$\int_\Omega v^* u \frac{\partial}{\partial x}(\delta v) dx\, dy \xrightarrow{\text{IBP}} -\int_\Omega \frac{\partial}{\partial x}(v^* u) \delta v\, dx\, dy + \text{边界项}$$

$$\int_\Omega v^* v \frac{\partial}{\partial y}(\delta v) dx\, dy \xrightarrow{\text{IBP}} -\int_\Omega \frac{\partial}{\partial y}(v^* v) \delta v\, dx\, dy + \text{边界项}$$

(3): 导出伴随方程（收集合并项）

我们将所有 $\delta u, \delta v, \delta z$ 的系数（残差）收集起来，并令其为零:

i) δz 的残差为零

将所有 δz 的系数收集起来:

$$\left(-\frac{\partial z^*}{\partial t} - u\frac{\partial z^*}{\partial x} - v\frac{\partial z^*}{\partial y} - g\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) + \frac{\partial J}{\partial z}\right)\delta z = 0$$

伴随 z^* 方程:

$$-\frac{\partial z^*}{\partial t} = u\frac{\partial z^*}{\partial x} + v\frac{\partial z^*}{\partial y} + g\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) + \frac{\partial J}{\partial z}$$

ii) 收集 δu 的系数

- **来自时间项 (IBP 后)**:

$$u^*\frac{\partial}{\partial t}(\delta u) \xrightarrow{\text{IBP}} -\frac{\partial u^*}{\partial t}\delta u$$

系数: $-\dfrac{\partial u^*}{\partial t}$

- **来自质量方程 (R_z) 的质量通量项 (IBP 后)**:

$$z^*\frac{\partial}{\partial x}(H\delta u) \xrightarrow{\text{IBP}} -H\frac{\partial z^*}{\partial x}\delta u$$

系数: $-H\dfrac{\partial z^*}{\partial x}$ (其中 $H = h + z$)

- **来自 x-动量方程 (R_u) 的自身平流项**:
  - 第一部分 (无导数): $u^*\left(\delta u\dfrac{\partial u}{\partial x}\right) \to \left(u^*\dfrac{\partial u}{\partial x}\right)\delta u$
  - 第二部分 (有导数 IBP): $u^*\left(u\dfrac{\partial}{\partial x}(\delta u)\right) \xrightarrow{\text{IBP}} -\dfrac{\partial}{\partial x}(uu^*)\delta u = \left(-u\dfrac{\partial u^*}{\partial x} - u^*\dfrac{\partial u}{\partial x}\right)\delta u$
  - 合并: $\left(u^*\dfrac{\partial u}{\partial x}\right) + \left(-u\dfrac{\partial u^*}{\partial x} - u^*\dfrac{\partial u}{\partial x}\right) = -u\dfrac{\partial u^*}{\partial x}$

  系数: $-u\dfrac{\partial u^*}{\partial x}$

- **来自 x-动量方程 (R_u) 的 y-平流交叉项**: 这里原始项是 $v\dfrac{\partial u}{\partial y}$。对 u 变分得到 $v\dfrac{\partial}{\partial y}(\delta u)$。

$$u^*\left(v\frac{\partial}{\partial y}(\delta u)\right) \xrightarrow{\text{IBP}} -\frac{\partial}{\partial y}(vu^*)\delta u = \left(-v\frac{\partial u^*}{\partial y} - u^*\frac{\partial v}{\partial y}\right)\delta u$$

系数: $-v\dfrac{\partial u^*}{\partial y} - u^*\dfrac{\partial v}{\partial y}$

- **来自 y-动量方程 (R_v) 的 x-平流交叉项**: 这里原始项是 $u\dfrac{\partial v}{\partial x}$。对 u 变分得到 $\delta u\dfrac{\partial v}{\partial x}$ (无导数)。

$$v^*\left(\delta u\frac{\partial v}{\partial x}\right) \to \left(v^*\frac{\partial v}{\partial x}\right)\delta u$$

系数: $+v^*\dfrac{\partial v}{\partial x}$

- **来自线性项 (阻尼和科里奥利)**:
  - R_u 中的 γu: $u^*\gamma\delta u \to +\gamma u^*$

– R_v 中的 fu: $v^* f\delta u \to +fv^*$

系数: $\gamma u^* + fv^*$

- **来自目标函数** J: 系数: $+\dfrac{\partial J}{\partial u}$

合并 δu 的系数并令其为 0:

$$\left[-\frac{\partial u^*}{\partial t} - H\frac{\partial z^*}{\partial x} - u\frac{\partial u^*}{\partial x} - v\frac{\partial u^*}{\partial y} - u^*\frac{\partial v}{\partial y} + v^*\frac{\partial v}{\partial x} + \gamma u^* + fv^* + \frac{\partial J}{\partial u}\right]\delta u = 0$$

整理得到 u^* 方程 (移项到 RHS):

$$-\frac{\partial u^*}{\partial t} = u\frac{\partial u^*}{\partial x} + v\frac{\partial u^*}{\partial y} + (h+z)\frac{\partial z^*}{\partial x} - v^*\frac{\partial v}{\partial x} + u^*\frac{\partial v}{\partial y} - \gamma u^* - fv^* - \frac{\partial J}{\partial u}$$

iii) 收集 δv 的系数

找出所有乘以 δv 的项。

- **来自时间项 (IBP 后)**:

$$v^*\frac{\partial}{\partial t}(\delta v) \xrightarrow{\text{IBP}} -\frac{\partial v^*}{\partial t}\delta v$$

系数: $-\dfrac{\partial v^*}{\partial t}$

- **来自质量方程 (R_z) 的质量通量项 (IBP 后)**:

$$z^*\frac{\partial}{\partial y}(H\delta v) \xrightarrow{\text{IBP}} -H\frac{\partial z^*}{\partial y}\delta v$$

系数: $-H\dfrac{\partial z^*}{\partial y}$

- **来自 y-动量方程 (R_v) 的自身平流项**: 类似于 u 的推导，变分后经 IBP 和消去，仅剩平流导数。

$$v^*\left(\delta v\frac{\partial v}{\partial y} + v\frac{\partial}{\partial y}(\delta v)\right) \to -v\frac{\partial v^*}{\partial y}\delta v$$

系数: $-v\dfrac{\partial v^*}{\partial y}$

- **来自 y-动量方程 (R_v) 的 x-平流交叉项**: 原始项 $u\dfrac{\partial v}{\partial x}$。对 v 变分得到 $u\dfrac{\partial}{\partial x}(\delta v)$。

$$v^*\left(u\frac{\partial}{\partial x}(\delta v)\right) \xrightarrow{\text{IBP}} -\frac{\partial}{\partial x}(uv^*)\delta v = \left(-u\frac{\partial v^*}{\partial x} - v^*\frac{\partial u}{\partial x}\right)\delta v$$

系数: $-u\dfrac{\partial v^*}{\partial x} - v^*\dfrac{\partial u}{\partial x}$

- **来自 x-动量方程 (R_u) 的 y-平流交叉项**: 原始项 $v\dfrac{\partial u}{\partial y}$。对 v 变分得到 $\delta v\dfrac{\partial u}{\partial y}$ (无导数)。

$$u^*\left(\delta v\frac{\partial u}{\partial y}\right) \to \left(u^*\frac{\partial u}{\partial y}\right)\delta v$$

系数: $+u^*\dfrac{\partial u}{\partial y}$

- **来自线性项**:
  - R_u 中的 $-fv$: $u^*(-f)\delta v \to -fu^*$
  - R_v 中的 γv: $v^*\gamma\delta v \to +\gamma v^*$

  系数: $-fu^* + \gamma v^*$

- **来自目标函数** J: 系数: $+\dfrac{\partial J}{\partial v}$

合并 δv 的系数并令其为 0:

$$\left[-\frac{\partial v^*}{\partial t} - H\frac{\partial z^*}{\partial y} - v\frac{\partial v^*}{\partial y} - u\frac{\partial v^*}{\partial x} - v^*\frac{\partial u}{\partial x} + u^*\frac{\partial u}{\partial y} - fu^* + \gamma v^* + \frac{\partial J}{\partial v}\right]\delta v = 0$$

整理得到 v^* 方程 (移项到 RHS):

$$-\frac{\partial v^*}{\partial t} = u\frac{\partial v^*}{\partial x} + v\frac{\partial v^*}{\partial y} + (h+z)\frac{\partial z^*}{\partial y} - u^*\frac{\partial u}{\partial y} + v^*\frac{\partial u}{\partial x} - \gamma v^* + fu^* - \frac{\partial J}{\partial v}$$

如何引导出初始时刻梯度

在完成了 IBP 并在内部区域设置残差为零 (即得到了伴随方程) 并在 $t = T$ 和空间边界设置条件后，$\delta\mathcal{L}$ 唯一剩下的非零项就是初始时刻的项:

$$\delta\mathcal{L} = -\int_\Omega [z^*(0)\delta z(0) + u^*(0)\delta u(0) + v^*(0)\delta v(0)]\,dx\,dy + \int_0^T\int_\Omega\left[\frac{\partial J}{\partial u}\delta u + \frac{\partial J}{\partial v}\delta v + \frac{\partial J}{\partial z}\delta z\right]dx\,dy$$

δJ 的定义是 J 对其输入参数的变分。在变分同化中，我们通常将初始状态 $\mathbf{x}_0 = (u(0), v(0), z(0))$ 视为唯一的优化参数。

因此，δJ 可以写成:

$$\delta J = \int_\Omega\left[\frac{\partial J}{\partial u(0)}\delta u(0) + \frac{\partial J}{\partial v(0)}\delta v(0) + \frac{\partial J}{\partial z(0)}\delta z(0)\right]dx\,dy$$

根据拉格朗日乘子原理，在最佳解处，$\delta\mathcal{L} = 0$ 成立。

$$\begin{aligned}0 = \delta\mathcal{L} = \int_\Omega\Bigg[&\left(-z^*(0) + \frac{\partial J}{\partial z(0)}\right)\delta z(0)\\ &+\left(-u^*(0) + \frac{\partial J}{\partial u(0)}\right)\delta u(0)\\ &+\left(-v^*(0) + \frac{\partial J}{\partial v(0)}\right)\delta v(0)\Bigg]dx\,dy\end{aligned}$$

由于 $\delta u(0), \delta v(0), \delta z(0)$ 是任意的（它们是我们要优化的参数的变分），为了使上式成立，它们的系数必须为零。

所以，我们得到了目标函数 J 对初始条件的梯度：

$$\begin{cases} \dfrac{\partial J}{\partial z(0)} = z^*(0) \\ \dfrac{\partial J}{\partial u(0)} = u^*(0) \\ \dfrac{\partial J}{\partial v(0)} = v^*(0) \end{cases}$$

结论：初始时刻的梯度（即目标函数 J 对初始状态的敏感度）恰好等于在 $t = 0$ 时，通过逆时间积分伴随方程组得到的伴随变量的值。

3：连续伴随方程的直接离散形式

方程 (1): u-动量伴随方程 (u^*)

连续方程：

$$-\frac{\partial u^*}{\partial t} = u\frac{\partial u^*}{\partial x} + v\frac{\partial u^*}{\partial y} - v^*\frac{\partial v}{\partial x} + u^*\frac{\partial v}{\partial y} + (h+z)\frac{\partial z^*}{\partial x} - \gamma u^* - fv^* - \frac{\partial J}{\partial u}$$

离散更新公式 ($u^*_{i,j}$ 位于 $i + 1/2, j$)：

$$\begin{aligned} u^{*n}_{i,j} = u^{*n+1}_{i,j} + \Delta t\, [\, & \frac{u^n_{i,j}}{2\Delta x}\left(u^{*n+1}_{i+1,j} - u^{*n+1}_{i-1,j}\right) + \frac{v^n_{i,j}}{2\Delta y}\left(u^{*n+1}_{i,j+1} - u^{*n+1}_{i,j-1}\right) \\ & - \frac{v^{*n+1}_{i,j}}{2\Delta x}\left(v^n_{i+1,j} - v^n_{i-1,j}\right) \\ & + \frac{(h_{i,j} + z^n_{i,j})}{2\Delta x}\left(z^{*n+1}_{i+1,j} - z^{*n+1}_{i-1,j}\right) \\ & - \gamma u^{*n+1}_{i,j} - fv^{*n+1}_{i,j} \end{aligned} \tag{7}$$

注：此处省略了 $u^*\frac{\partial v}{\partial y}$ 的离散

方程 (2): v-动量伴随方程 (v^*)

连续方程：

$$-\frac{\partial v^*}{\partial t} = u\frac{\partial v^*}{\partial x} + v\frac{\partial v^*}{\partial y} - u^*\frac{\partial u}{\partial y} + v^*\frac{\partial u}{\partial x} + (h+z)\frac{\partial z^*}{\partial y} - \gamma v^* + fu^* - \frac{\partial J}{\partial v}$$

离散更新公式 ($v^*_{i,j}$ 位于 $i, j+1/2$)：

$$\begin{aligned} v^{*n}_{i,j} = v^{*n+1}_{i,j} + \Delta t \, [\, & \frac{u^n_{i,j}}{2\Delta x}\left(v^{*n+1}_{i+1,j} - v^{*n+1}_{i-1,j}\right) + \frac{v^n_{i,j}}{2\Delta y}\left(v^{*n+1}_{i,j+1} - v^{*n+1}_{i,j-1}\right) \\ & - \frac{u^{*n+1}_{i,j}}{2\Delta y}\left(u^n_{i,j+1} - u^n_{i,j-1}\right) \\ & + \frac{(h_{i,j} + z^n_{i,j})}{2\Delta y}\left(z^{*n+1}_{i,j+1} - z^{*n+1}_{i,j-1}\right) \\ & - \gamma v^{*n+1}_{i,j} + f u^{*n+1}_{i,j} \end{aligned} \tag{8}$$

注：此处省略了 $v^*\frac{\partial u}{\partial x}$ 的离散

方程 (3): 连续性伴随方程 (z^*)

连续方程：

$$-\frac{\partial z^*}{\partial t} = u\frac{\partial z^*}{\partial x} + v\frac{\partial z^*}{\partial y} + g\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) - \frac{\partial J}{\partial z}$$

离散更新公式 ($z^*_{i,j}$ 位于 i, j)：

$$\begin{aligned} z^{*(n)}_{i,j} = z^{*(n+1)}_{i,j} + \Delta t \, [\, & \frac{u^{\text{center}}_{i,j}}{2\Delta x}\left(z^*_{i+1,j} - z^*_{i-1,j}\right) + \frac{v^{\text{center}}_{i,j}}{2\Delta y}\left(z^*_{i,j+1} - z^*_{i,j-1}\right) \\ & + g\left(\frac{\partial x}{\partial u^*}\frac{\Delta x}{u^*_{i,j} - u^*_{i-1,j}} + \frac{\partial y}{\partial v^*}\frac{\Delta y}{v^*_{i,j} - v^*_{i,j-1}}\right) \\ & - \Delta t\left(\frac{\partial J}{\partial z}\right)_{i,j} \end{aligned} \tag{9}$$