Solution to factored VTI Eikonal equation

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1 VTI 程函方程

$$V_{nmo}^2(1+2\eta)\left(p_x^2+p_y^2\right)+V_0^2p_z^2\left(1-2\eta V_{nmo}^2\left(p_x^2+p_y^2\right)\right)=1 \tag{1}$$

其中，射线参数 $p_x=\frac{\partial T}{\partial x},p_y=\frac{\partial T}{\partial y},p_z=\frac{\partial T}{\partial z}$; $T(x,y,z)$ 为旅行时。其哈密顿量为:

$$H=V_{nmo}^2(1+2\eta)\left(p_x^2+p_y^2\right)+V_0^2p_z^2\left(1-2\eta V_{nmo}^2\left(p_x^2+p_y^2\right)\right) \tag{2}$$

2 乘法分解

令

$$T=T_0\tau \tag{3}$$

其中 T_0 为给定的已知函数，则

$$\nabla T=\nabla T_0\tau+T_0\nabla\tau \tag{4}$$

将公式(1) 中 4 次项移到等式右端得到：

$$V_{nmo}^2(1+2\eta)\left(p_x^2+p_y^2\right)+V_0^2p_z^2=1+2\eta V_0^2V_{nmo}^2p_z^2\left(p_x^2+p_y^2\right) \tag{5}$$

将(4)代入到公式(5)左端得到：

$$\begin{aligned}&V_{nmo}^2(1+2\eta)\left(\left(\frac{\partial\tau}{\partial x}\right)^2T_0^2+2\frac{\partial\tau}{\partial x}\frac{\partial T_0}{\partial x}T_0\tau+\left(\frac{\partial T_0}{\partial x}\right)^2\tau^2\right)\\&+V_{nmo}^2(1+2\eta)\left(\left(\frac{\partial\tau}{\partial y}\right)^2T_0^2+2\frac{\partial\tau}{\partial y}\frac{\partial T_0}{\partial y}T_0\tau+\left(\frac{\partial T_0}{\partial y}\right)^2\tau^2\right)\\&+V_0^2\left(\left(\frac{\partial\tau}{\partial z}\right)^2T_0^2+2\frac{\partial\tau}{\partial z}\frac{\partial T_0}{\partial z}T_0\tau+\left(\frac{\partial T_0}{\partial z}\right)^2\tau^2\right)=1+2\eta V_0^2V_{nmo}^2p_z^2\left(p_x^2+p_y^2\right)\end{aligned} \tag{6}$$

对 $\nabla\tau$ 采用一阶差分近似:

$$\frac{\partial\tau}{\partial x} \approx \left(\frac{\tau_{i,j,k}-\tau_x}{\Delta x}\right)s_x, \quad \frac{\partial\tau}{\partial y} \approx \left(\frac{\tau_{i,j,k}-\tau_y}{\Delta y}\right)s_y, \quad \frac{\partial\tau}{\partial z} \approx \left(\frac{\tau_{i,j,k}-\tau_z}{\Delta z}\right)s_z \tag{7}$$

其中

$$\tau_x = \min\left\{\tau_{i+1,j,k}, \tau_{i-1,j,k}\right\}, \quad \tau_y = \min\left\{\tau_{i,j+1,k}, \tau_{i,j-1,k}\right\}, \quad \tau_z = \min\left\{\tau_{i,j,k+1}, \tau_{i,j,k-1}\right\} \tag{8}$$

s_x, s_y, s_z 为在取值 1 或-1 的符号变量，取值取决于差分采用网格点的相对位置。将(7)代入公式(6)中得到关于 $\tau(\tau=\tau_{i,j,k})$ 的一元二次方程:

$$a\tau^2 + b\tau + c = 1 + 2\eta V_0^2 V_{nmo}^2 p_z^2\left(p_x^2+p_y^2\right) \tag{9}$$

其中

$$a = V_{nmo}^2\left(1+2\eta\right)\left(\frac{T_0^2}{\Delta x^2} + \left(\frac{\partial T_0}{\partial x}\right)^2 + 2s_xT_0\frac{1}{\Delta x}\frac{\partial T_0}{\partial x}\right) \tag{10}$$

$$+ V_{nmo}^2\left(1+2\eta\right)\left(\frac{T_0^2}{\Delta y^2} + \left(\frac{\partial T_0}{\partial y}\right)^2 + 2s_yT_0\frac{1}{\Delta y}\frac{\partial T_0}{\partial y}\right) \tag{11}$$

$$+ V_0^2\left(\frac{T_0^2}{\Delta z^2} + \left(\frac{\partial T_0}{\partial z}\right)^2 + 2s_zT_0\frac{1}{\Delta z}\frac{\partial T_0}{\partial z}\right) \tag{12}$$

$$\begin{aligned} b = &-2V_{nmo}^2\left(1+2\eta\right)\tau_x\left(\frac{T_0^2}{\Delta x^2} + s_xT_0\frac{1}{\Delta x}\frac{\partial T_0}{\partial x}\right) \\ &- 2V_{nmo}^2\left(1+2\eta\right)\tau_y\left(\frac{T_0^2}{\Delta y^2} + s_yT_0\frac{1}{\Delta y}\frac{\partial T_0}{\partial y}\right) \\ &- 2V_0^2\tau_z\left(\frac{T_0^2}{\Delta z^2} + s_zT_0\frac{1}{\Delta z}\frac{\partial T_0}{\partial z}\right) \\ c = &\ V_{nmo}^2\left(1+2\eta\right)\tau_x^2\frac{T_0^2}{\Delta x^2} + V_{nmo}^2\left(1+2\eta\right)\tau_y^2\frac{T_0^2}{\Delta y^2} + V_0^2\tau_z^2\frac{T_0^2}{\Delta z^2} \end{aligned}$$

3 Causality Condition

相速度和群速度的计算 (See Appendix A and B):

$$v_{\text{相}} = \left(\frac{p_x}{p_x^2+p_y^2+p_z^2}, \frac{p_y}{p_x^2+p_y^2+p_z^2}, \frac{p_z}{p_x^2+p_y^2+p_z^2}\right)$$

$$v_{\text{群}} = (v_x, v_y, v_z), \quad \text{其中} v_i = \left(p_x\frac{\partial H}{\partial p_x} + p_y\frac{\partial H}{\partial p_y} + p_z\frac{\partial H}{\partial p_z}\right)^{-1}\frac{\partial H}{\partial p_i}, i = x, y, z$$

计算得

$$
\begin{aligned}
\frac{\partial H}{\partial p_x} &= 2p_x\left(V_{nmo}^2\left(1+2\eta\right)-2V_0^2V_{nmo}^2\eta p_z^2\right)\\
\frac{\partial H}{\partial p_y} &= 2p_y\left(V_{nmo}^2\left(1+2\eta\right)-2V_0^2V_{nmo}^2\eta p_z^2\right)\\
\frac{\partial H}{\partial p_z} &= 2V_0^2p_z\left(1-2V_{nmo}^2\eta\left(p_x^2+p_y^2\right)\right)
\end{aligned}
$$

记

$$
\begin{aligned}
\alpha &= p_x\frac{\partial H}{\partial p_x}+p_y\frac{\partial H}{\partial p_y}+p_z\frac{\partial H}{\partial p_z}\\
&= 2\left(p_x^2+p_y^2\right)\left(V_{nmo}^2\left(1+2\eta\right)-2V_0^2V_{nmo}^2\eta p_z^2\right)\\
&\quad +2V_0^2p_z^2\left(1-2V_{nmo}^2\eta\left(p_x^2+p_y^2\right)\right)
\end{aligned}
$$

现计算 p_x,p_y,p_z 的取值范围，记 $p_r^2=p_x^2+p_y^2$，则程函方程化为

$$
p_z^2=\frac{1-V_{nmo}^2\left(1+2\eta\right)p_r^2}{V_0^2\left(1-2\eta V_{nmo}^2p_r^2\right)}
$$

由上公式知，当 $p_r=0,p_z^2$ 取最大值，此时 $p_z^2=\frac{1}{V_0^2}$，对应群速度方向沿 z 轴情况；当 $p_r^2=\frac{1}{(1+2\eta)V_{nmo}^2}$，$p_z^2$ 取最小值，此时 $p_z^2=0$，对应群速度在 z 轴分速度为 0 情况。由上述取值范围容易得到

$$
\alpha>0
$$

经计算易得

$$
v_i\geq 0\Leftrightarrow\frac{\partial H}{\partial p_i}\geq 0\Leftrightarrow p_i\geq 0,i=x,y,z \tag{13}
$$

迎风格式要求：

$$
s_xv_x>0,\ s_yv_y>0,\ s_zv_z>0
$$

将(13)代入得

$$
s_xp_x>0,\ s_yp_y>0,\ s_zp_z>0 \tag{14}
$$

将(4)代入(14)得到

$$
\frac{\tau-\tau_x}{h_1}T_0+\frac{\partial T_0}{\partial x}\tau s_x>0,\ \frac{\tau-\tau_y}{h_2}T_0+\frac{\partial T_0}{\partial y}\tau s_y>0,\ \frac{\tau-\tau_z}{h_3}T_0+\frac{\partial T_0}{\partial z}\tau s_z>0 \tag{15}
$$

4 单点更新

以公式(8)中 $\tau_x=\tau_W$, $\tau_y=\tau_S$, $\tau_z=\tau_D$ 为例，求解方程 (8)，有以下 3 种情况：

1. 方程有两个根满足 Causality Condition：取最小的根作为解；

2. 方程有且仅有一个根满足 Causality Condition：取该根作为解；

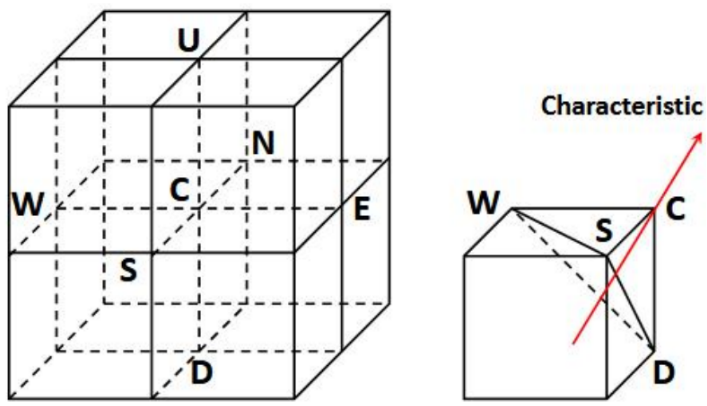

图 1: Example of a finite-difference grid

3. 方程无根满足 Causality Condition：分别在三角形 WSC, DWC, DSC 上求解，并取其最小解。

记所得解为 root，更新 C 点值：$\tau_c^{new} = \min\{root, \tau_c^{old}\}$

5 平面求解

以三角形 DWC 内求解为例，在方程 (4) 中令 $p_y = 0$，解差分方程 (8) 有以下 3 种情况：

1. 方程有两个根满足 Causality Condition：取最小的根作为解；
2. 方程有且仅有一个根满足 Causality Condition：取该根作为解；
3. 方程无根满足 Causality Condition，分别在 WC, DC 上求解，并取最小解。

以 WC 上求解为例，方程 (4) 退化为：

$$V_{nmo}^2(1+2\eta)p_x^2 = f(\tau)$$

so that

$$|p_x| = \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}$$

假设当波场沿 x 轴正方向传播时, 背景波场也沿 x 轴正方向传播, 即 p_x 与 $\frac{\partial T_0}{\partial x}$ 同号). 注意到 $p_x = \tau\frac{\partial T_0}{\partial x} + T_0\frac{\tau-\tau_x}{\Delta x}s_x$. 若 $p_x > 0$, 则 $s_x > 0$, so $s_x = 1$.

$$\tau\frac{\partial T_0}{\partial x} + T_0\frac{\tau-\tau_x}{\Delta x} = \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}} \Leftrightarrow \tau = \frac{T_0\tau_x + \Delta x\sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 + \Delta x\frac{\partial T_0}{\partial x}}$$

若 $p_x < 0$, 则 $s_x < 0$, so $s_x = -1$.

$$\tau\frac{\partial T_0}{\partial x} - T_0\frac{\tau-\tau_x}{\Delta x} = -\sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}} \Leftrightarrow \tau = \frac{T_0\tau_x + \Delta x\sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 - \Delta x\frac{\partial T_0}{\partial x}}$$

In summary, we obtain

$$\tau = \frac{T_0\tau_x + \Delta x\sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 + \Delta x\left|\frac{\partial T_0}{\partial x}\right|}$$

记所得解为 root，更新 C 点值：$\tau_C^{new} = \min\{root, \tau_C^{old}\}$

6 求解算法

记公式 (4) 右端项为

$$f(\tau) = 1 + 2\eta V_0^2 V_{nmo}^2(p_x^2 + p_y^2)$$

Algorithm 1 程函方程求解算法

1: 初始化 $f(\tau) = 1$;

2: 初始化源点网格 $\tau = 1$，其余网格 $\tau = \text{HUGE}$，

$$T_0(x+x_0, y+y_0, z+z_0) = \sqrt{\frac{b_0c_0x^2 + a_0c_0y^2 + a_0b_0z^2}{a_0b_0c_0}}$$

according to Luo and Qian (2012, Remark 3.5, only consider diagonal of $M(x)$)

$$a_0 = b_0 = \frac{V_{nmo}^2(1+2\eta)}{f(\tau)} \quad \text{where } (x,y,z) = (x_0, y_0, z_0)$$

$$c_0 = \frac{V_0^2}{f(\tau)} \quad \text{where } (x,y,z) = (x_0, y_0, z_0)$$

3: 依次沿 8 个不同方向扫描更新全部网格点，并多次重复此过程

4: 用得到的数据计算 $\nabla\tau, \nabla T$ 并更新 $f(\tau)$

5: 多次重复步骤 2, 3, 4

6: 由公式 (2) 得到数值解

The result after solving the Eikonal equation is shown in Figure 2.

A 相速度 (Phase velocity)

The slowness vector $\mathbf{p} = (p_x, p_y, p_z) = \nabla T = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z})$

$$|\mathbf{p}|^2 = |\nabla T|^2 = \frac{1}{v^2} \Leftrightarrow p_x^2 + p_y^2 + p_z^2 = \frac{1}{v^2}$$

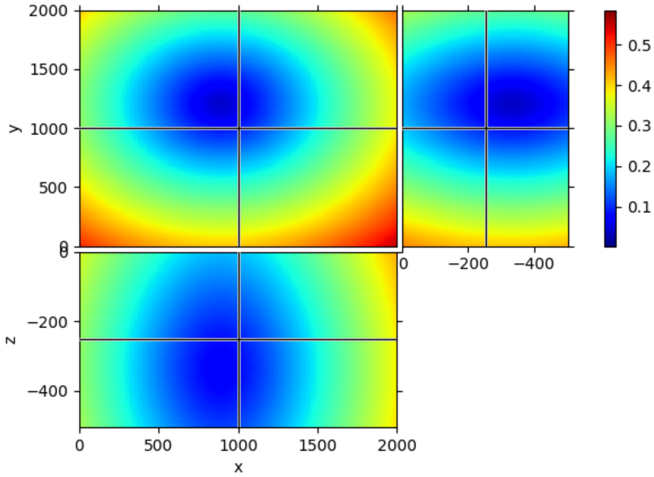

图 2: The contour plot for traveltime computed by soving VTI eikonal equation

相速度矢量 $\mathbf{v}$ 与 $\mathbf{p}$ 同方向，大小为 $v = 1/|\mathbf{p}|$

$$\mathbf{v} = v\frac{\mathbf{p}}{|\mathbf{p}|} = \frac{\mathbf{p}}{|\mathbf{p}|^2} = (\frac{p_x}{p_x^2+p_y^2+p_z^2}, \frac{p_y}{p_x^2+p_y^2+p_z^2}, \frac{p_z}{p_x^2+p_y^2+p_z^2})$$

B 群速度（Group velocity）

在 Hamilton 射线理论中:

$$\frac{d\mathbf{x}}{ds} = \nabla_{\mathbf{p}} H(\mathbf{x}, \mathbf{p}) = (\frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y}, \frac{\partial H}{\partial p_z})$$

这里 s 是弧长。

在高频近似中，波场走时为:

$$t(\mathbf{x}) = \int \mathbf{p} \cdot d\mathbf{x} = \int \mathbf{p} \cdot \frac{d\mathbf{x}}{ds} ds$$

所以:

$$\frac{dt}{ds} = \mathbf{p} \cdot \frac{d\mathbf{x}}{ds} = p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z}.$$

群速度定义为位移关于时间的导数:

$$v_i = \frac{dx_i}{dt} = \frac{dx_i/ds}{dt/ds} = \left(p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z} \right)^{-1} \frac{\partial H}{\partial p_i}, \qquad i = x, y, z.$$

参考文献

Luo, S. and Qian, J. (2012). Fast sweeping method for factored anisotropic eikonal equations: multipicative and additive factors. *Journal of Scientific Computing*, 52:360–382.