Finite difference solution to 3D factored VTI Eikonal equation

Yuhang Wang

December 6, 2025

1 Finite Difference Solution for the 3D VTI Eikonal Equation

The eikonal equation for vertical transverse isotropy (VTI) media is given by:

$$V_{\text{nmo}}^2(1+2\eta)(p_x^2+p_y^2)+V_0^2p_z^2(1-2\eta V_{\text{nmo}}^2(p_x^2+p_y^2))=1 \tag{1}$$

The corresponding Hamiltonian function is defined as:

$$H(p_x,p_y,p_z)=V_{\text{nmo}}^2(1+2\eta)(p_x^2+p_y^2)+V_0^2p_z^2(1-2\eta V_{\text{nmo}}^2(p_x^2+p_y^2)) \tag{2}$$

The components of the Hamiltonian gradient are given by:

$$\frac{\partial H}{\partial p_x}=2p_x\left(V_{\text{nmo}}^2(1+2\eta)-2V_0^2V_{\text{nmo}}^2\eta p_z^2\right) \tag{3a}$$

$$\frac{\partial H}{\partial p_y}=2p_y\left(V_{\text{nmo}}^2(1+2\eta)-2V_0^2V_{\text{nmo}}^2\eta p_z^2\right) \tag{3b}$$

$$\frac{\partial H}{\partial p_z}=2V_0^2p_z\left(1-2V_{\text{nmo}}^2\eta(p_x^2+p_y^2)\right) \tag{3c}$$

To implement the finite difference scheme, the computational domain around each grid point is divided into eight octants. Each octant is further subdivided into six tetrahedral elements for numerical discretization. Figure 1 illustrates the tetrahedral decomposition within a single octant, with identical patterns applied to all octants.

Considering Figure 1a as a representative case, we define the spatial increments:

$$dx=x_M-x_A,\quad dy=y_C-y_B,\quad dz=z_B-z_A$$

Finite difference approximations for the directional derivatives along MA, MB, and MC directions yield:

$$T_A-T_M=-dx\cdot p_x,\quad T_B-T_M=-dx\cdot p_x+dy\cdot p_y,\quad T_C-T_M=-dx\cdot p_x+dy\cdot p_y+dz\cdot p_z$$

Solving this system gives the slowness components:

$$p_x=\frac{T_M-T_A}{dx},\quad p_y=\frac{T_C-T_B}{dy},\quad p_z=\frac{T_B-T_A}{dz} \tag{4}$$

Among these, only p_x remains unknown. Substituting (4) into (1) yields a quadratic equation in p_x. Solving this equation and back-substituting the result into (4) provides the traveltime solution.

This solution procedure may produce up to two real roots. Each root must be verified against the causality condition by tracing a ray backward along the characteristic direction from point M and checking for intersection with the interior of triangle ABC. Roots satisfying this condition are deemed valid.

If two valid roots are obtained, the smaller value is selected as the solution for the tetrahedral element. If only one valid root exists, it is adopted directly. In cases where no valid roots are found, solutions are computed separately across the three tetrahedral faces passing through point M, with the minimum value selected as the final solution.

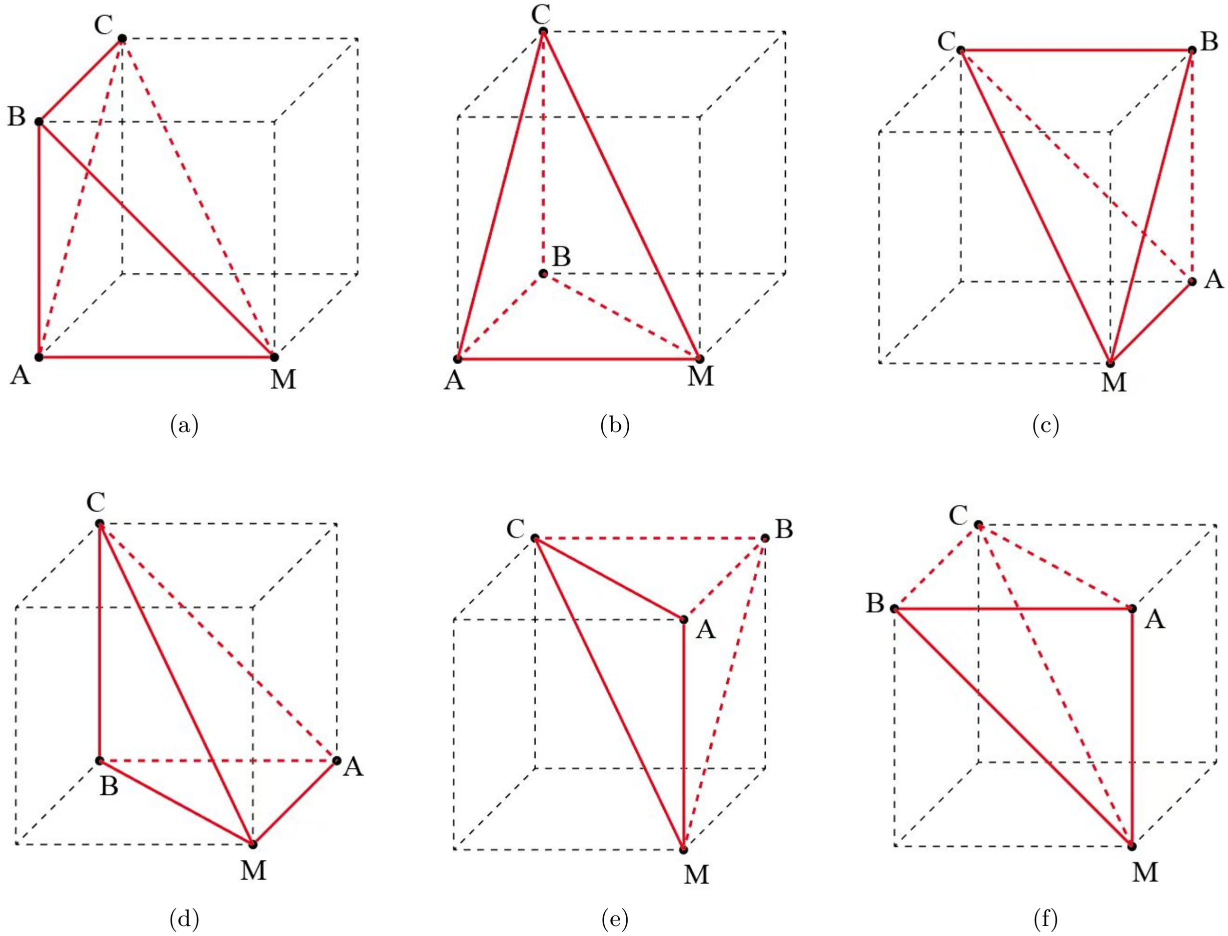


Figure 1: Tetrahedral mesh decomposition within a single octant.

1.1 Solution on Tetrahedral Faces

We now consider cases where characteristic lines lie on tetrahedral faces. While solutions on non-inclined planes are straightforward, inclined planes require special treatment. Figure 2 illustrates three primary scenarios for inclined planes.

1.1.1 Case 1: Horizontal Characteristic Constraint

For the configuration shown in Figure 2a, we define:

$$dx = x_E - x_G, \quad dy = y_E - y_G, \quad dz = z_M - z_G$$

The governing equations are:

$$T_E - T_G = p_x \cdot dx + p_y \cdot dy, \quad T_M - T_G = p_z \cdot dz, \quad -dy\frac{\partial H}{\partial p_x} + dx\frac{\partial H}{\partial p_y} = 0$$

Solving yields:

$$p_x = \frac{(T_E - T_G)dx}{dx^2 + dy^2}, \quad p_y = \frac{(T_E - T_G)dy}{dx^2 + dy^2}$$

Substituting into (1) provides the complete solution, which is then validated against the causality condition using a procedure analogous to the 3D case.

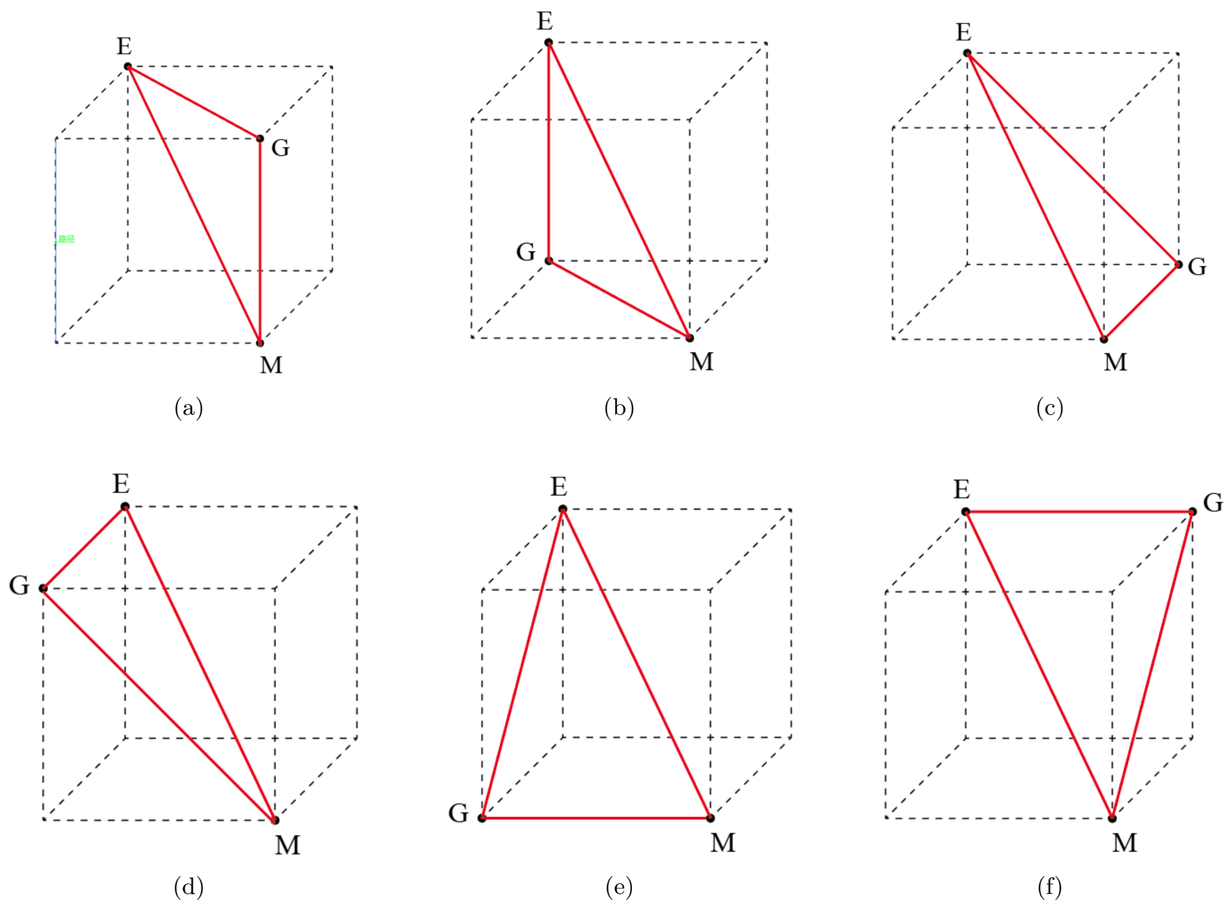


Figure 2: Characteristic configurations on inclined planes.

1.1.2 Case 2: Vertical-Horizontal Mixed Constraint

For Figure 2c, the spatial increments are:

$$dx = x_E - x_G, \quad dy = y_M - y_G, \quad dz = z_E - z_G$$

The system equations are:

$$T_E - T_G = p_x \cdot dx + p_z \cdot dz, \quad T_M - T_G = p_y \cdot dy, \quad -dx\frac{\partial H}{\partial p_z} + dz\frac{\partial H}{\partial p_x} = 0 \tag{5}$$

We define the residual function $f(T_M) = T_E - T_G - p_x \cdot dx - p_z \cdot dz$ and seek roots where $f = 0$. Given the physical constraint $|p_z| \in [0, 1/V_0]$, for each trial value of p_z, we compute $p_x^2 + p_y^2$ from (1). The group velocity components $\frac{\partial H}{\partial p_z}$ and $\frac{\partial H}{\partial p_x}$ are then obtained from (5) and (3), with p_x determined. The bisection method is employed to iteratively converge to the solution satisfying $f = 0$.

1.1.3 Case 3: Alternative Mixed Constraint

For Figure 2d, the spatial parameters are:

$$dx = x_M - x_G, \quad dy = y_E - y_G, \quad dz = z_M - z_G$$

The governing equations become:

$$T_E - T_G = p_y \cdot dy, \quad T_M - T_G = p_x \cdot dx + p_z \cdot dz, \quad -dx\frac{\partial H}{\partial p_z} + dz\frac{\partial H}{\partial p_x} = 0 \tag{6}$$

Since p_y is known, the range for p_x^2 is constrained by $p_x^2 \in \left(0, \frac{1}{V_{\text{nmo}}^2(1+2\eta)} - p_y^2\right)$. Defining the residual function $f(T_M) = -\frac{\partial H}{\partial p_x} \cdot dz + \frac{\partial H}{\partial p_z} \cdot dx$, we seek roots where $f = 0$. For each trial value of p_x, we compute p_z from (1), then determine $\frac{\partial H}{\partial p_x}$ and $\frac{\partial H}{\partial p_z}$ using (3). The bisection method is again employed to find the solution satisfying the constraint.

2 Multiplicative Factorization

2.1 Solution for the 3D VTI Eikonal Equation

We introduce a multiplicative factorization approach for solving the 3D eikonal equation within a tetrahedral element. The traveltime field T is decomposed into two multiplicative components:

$$T = T_0 \tau \tag{7}$$

where T_0 represents a known reference field and τ denotes the unknown perturbation factor. The gradient of the traveltime field is consequently expressed as:

$$\nabla T = \tau \nabla T_0 + T_0 \nabla \tau \tag{8}$$

Considering Figure 1a as a representative case, we define the spatial increments:

$$dx = x_M - x_A, \quad dy = y_C - y_B, \quad dz = z_B - z_A$$

The components of the gradient of τ at point M, computed using finite difference approximations, are given by:

$$\frac{\partial \tau}{\partial x} = \frac{\tau_M - \tau_A}{dx}, \quad \frac{\partial \tau}{\partial y} = \frac{\tau_C - \tau_B}{dy}, \quad \frac{\partial \tau}{\partial z} = \frac{\tau_B - \tau_A}{dz} \tag{9}$$

Here, τ_A, τ_B, and τ_C are known values at neighboring grid points, while τ_M is the unknown value at point M. Consequently, $\frac{\partial \tau}{\partial y}$ and $\frac{\partial \tau}{\partial z}$ can be directly computed, whereas $\frac{\partial \tau}{\partial x}$ remains unknown.

Substituting (9) into (8) yields the components of the gradient of T at point M:

$$\frac{\partial T}{\partial x} = \tau\left(\frac{\partial T_0}{\partial x} + \frac{T_0}{dx}\right) - \frac{T_0\tau_A}{dx} \tag{10a}$$

$$\frac{\partial T}{\partial y} = \tau\frac{\partial T_0}{\partial y} + T_0\frac{\partial \tau}{\partial y} \tag{10b}$$

$$\frac{\partial T}{\partial z} = \tau\frac{\partial T_0}{\partial z} + T_0\frac{\partial \tau}{\partial z} \tag{10c}$$

From equations (10a)-(10c), it is evident that the partial derivatives of T with respect to x, y, and z are all linear functions of τ. Substituting these expressions into the eikonal equation (1) yields a quartic equation in τ.

For notational convenience, we express the gradient components as:

$$\nabla T = (a_1\tau + b_1,\ a_2\tau + b_2,\ a_3\tau + b_3) \tag{11}$$

The eikonal equation then transforms into:

$$a\tau^4 + b\tau^3 + c\tau^2 + d\tau + e = 0 \tag{12}$$

where the coefficients are given by:

$$a = 2\eta V_0^2 V_{\text{nmo}}^2 a_3^2(a_1^2 + a_2^2) \tag{13a}$$

$$b = 4\eta V_0^2 V_{\text{nmo}}^2 a_3 \left((a_1^2 + a_2^2)b_3 + (a_1b_1 + a_2b_2)a_3\right) \tag{13b}$$

$$\begin{aligned} c = {} & 2\eta V_0^2 V_{\text{nmo}}^2 \left((a_1^2 + a_2^2)b_3^2 + (b_1^2 + b_2^2)a_3^2 + 4(a_1b_1 + a_2b_2)a_3b_3\right) \\ & - V_{\text{nmo}}^2(1 + 2\eta)(a_1^2 + a_2^2) - V_0^2 a_3^2 \end{aligned} \tag{13c}$$

$$\begin{aligned} d = {} & 4\eta V_0^2 V_{\text{nmo}}^2 b_3 \left((a_1b_1 + a_2b_2)b_3 + (b_1^2 + b_2^2)a_3\right) \\ & - 2V_{\text{nmo}}^2(1 + 2\eta)(a_1b_1 + a_2b_2) - 2V_0^2 a_3b_3 \end{aligned} \tag{13d}$$

$$e = 1 + 2\eta V_0^2 V_{\text{nmo}}^2 \left((b_1^2 + b_2^2)b_3^2\right) - V_{\text{nmo}}^2(1 + 2\eta)(b_1^2 + b_2^2) - V_0^2 b_3^2 \tag{13e}$$

The quartic equation (12) is solved using Ferrari's method, which may yield up to four real roots. Among these, the smallest valid root is selected as the solution. The causality condition requires that a ray traced backward from point M along the characteristic direction corresponding to the solution intersects the interior of triangle ABC.

In cases where no valid roots are found using the tetrahedral element approach, solutions are computed separately across the three tetrahedral faces passing through point M (namely triangles ABM, ACM, and BCM), with the minimum valid value selected as the final solution.

2.2 Solution on Tetrahedral Faces

When characteristic lines lie on non-inclined planes, the solution procedure is similar to that described in Section 2.1. Here we focus on the solution procedure for non-inclined planes.

2.2.1 Case 1: Horizontal Characteristic Constraint

For the configuration shown in Figure 2a, we define:

$$dx = x_E - x_G, \quad dy = y_E - y_G, \quad dz = z_M - z_G$$

Finite difference equations:

$$\tau_E - \tau_G = dx\frac{\partial \tau}{\partial x} + dy\frac{\partial \tau}{\partial y}, \quad \tau_M - \tau_G = dz\frac{\partial \tau}{\partial z}$$

Substituting into (8) yields:

$$\left(dx\frac{\partial T_0}{\partial x} + dy\frac{\partial T_0}{\partial y}\right)\tau + T_0(\tau_E - \tau_G) = dxp_x + dyp_y \tag{14}$$

$$dz\frac{\partial T_0}{\partial z}\tau + T_0(\tau - \tau_G) = dzp_z \tag{15}$$

Characteristic constraint equation:

$$-dy\frac{\partial H}{\partial p_x} + dx\frac{\partial H}{\partial p_y} = 0 \tag{16}$$

Solving within triangle EGM is equivalent to solving the system of equations consisting of (1), (14), (15), and (16) for the unknowns τ, p_x, p_y, p_z. It is evident that equations (14), (15), and (16) are linear equations. Solving these three equations allows us to express p_x, p_y, p_z as first-degree polynomials expressions in term of τ. Substituting these into equation (1) yields a quartic equation, and solving this equation provides the solutions in this case.

2.2.2 Case 2: Vertical-Horizontal Mixed Constraint

For Figure 2c, the spatial increments are:

$$dx = x_E - x_G, \quad dy = y_M - y_G, \quad dz = z_E - z_G$$

Finite difference equations:

$$\tau_E - \tau_G = dx\frac{\partial \tau}{\partial x} + dz\frac{\partial \tau}{\partial z}, \quad \tau_M - \tau_G = dy\frac{\partial \tau}{\partial y}$$

Substituting into (8) yields:

$$\left(dx\frac{\partial T_0}{\partial x} + dz\frac{\partial T_0}{\partial z}\right)\tau + T_0(\tau_E - \tau_G) = dxp_x + dzp_z \tag{17}$$

$$dy\frac{\partial T_0}{\partial y}\tau + T_0(\tau - \tau_G) = dyp_y \tag{18}$$

Characteristic constraint equation:

$$-dx\frac{\partial H}{\partial p_z} + dz\frac{\partial H}{\partial p_x} = 0 \tag{19}$$

The complete system consists of equations (17), (18), (1), and (19), which is solved using the bisection method. For a given value of p_z, $p_x^2 + p_y^2$ is computed from (1), then $\frac{\partial H}{\partial p_z}$ is determined, and p_x is calculated from (19). These values are substituted into (17) to compute the corresponding τ value. Then p_y is solved from (18), and p_z is iteratively adjusted until (1) is satisfied. The resulting τ value is validated against the causality condition; if it fails, it is discarded. Solutions are also computed separately for characteristic directions along EM and GM, with the minimum valid value selected.

2.2.3 Case 3: Alternative Mixed Constraint

For Figure 2d, the spatial parameters are:

$$dx = x_M - x_G, \quad dy = y_E - y_G, \quad dz = z_M - z_G$$

Finite difference equations:

$$\tau_E - \tau_G = dy\frac{\partial \tau}{\partial y}, \quad \tau_M - \tau_G = dx\frac{\partial \tau}{\partial x} + dz\frac{\partial \tau}{\partial z}$$

Substituting into (8) yields:

$$dy\frac{\partial T_0}{\partial y}\tau + T_0(\tau_E - \tau_G) = dyp_y \tag{20}$$

$$\left(dx\frac{\partial T_0}{\partial x} + dz\frac{\partial T_0}{\partial z}\right)\tau + T_0(\tau - \tau_G) = dxp_x + dzp_z \tag{21}$$

The characteristic constraint equation is the same as in Section 2.3:

The complete system consists of equations (20), (21), (1), and (19), which is solved using the bisection method. For a given value of p_z, $p_x^2 + p_y^2$ is computed from (1), then $\frac{\partial H}{\partial p_z}$ is determined, and p_x is calculated from (19). These values are substituted into (21) to compute the corresponding τ value. Then p_y is solved from (20), and p_z is iteratively adjusted until (1) is satisfied. The resulting τ value is validated against the causality condition; if it fails, it is discarded. Solutions are also computed separately for characteristic directions along EM and GM, with the minimum valid value selected.

2.2.4 Existence and Uniqueness of Bisection Solution

Taking Case 2 as an example, when $|p_z|$ increases (the sign of p_z is determined by the causality constraint), the computed value of $p_x^2 + p_y^2$ from (1) decreases, which leads to an increase in $|\frac{\partial H}{\partial p_z}|$ value from (3c). Consequently, the calculated $|p_x|$ value from (19) increases, and the corresponding τ value computed from (17) also increases. Then, the solved $|p_y|$ value from (18) increases, meaning that $|p_x|$, $|p_y|$, and $|p_z|$ all increase simultaneously, causing the left-hand side of (1) to increase while the right-hand side remains constant. Therefore, the existence of a solution can be determined by checking whether the function values computed at both ends of the p_z range have opposite signs. Moreover, when existence is satisfied, uniqueness is also guaranteed.

2.3 Algorithm Steps

The numerical algorithm for solving the VTI eikonal equation using multiplicative factorization proceeds as follows:

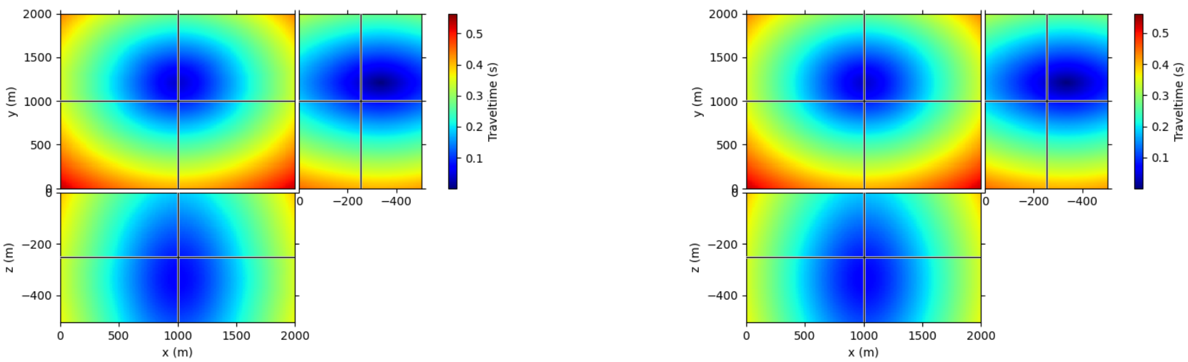
1. **Initialization**: Initialize the reference field T_0 and the perturbation factor τ throughout the computational domain.

2. **Alternating Sweeps**: Perform alternating sweeps along eight different directions across all grid points. For each grid point, employ an upwind finite difference scheme considering computations within a single octant only.

3. **Iteration**: Repeat step 2 until convergence is achieved. The traveltime field T is then computed from the obtained τ and the initialized T_0 using Equation (7).

3 CONCLUSIONS

Numerical experiments are conducted in a homogeneous medium model, and the results are compared with reference solutions obtained by the shooting method. Figure Figure 3a shows the cross-sectional profile obtained using the direct finite difference method, with an absolute error of 6.286e-3. Figure Figure 3b presents the cross-sectional profile obtained using the multiplicative factorization method, with an absolute error of 2.104e-3.



(a)

(b)

Figure 3: Traveltime solutions: (a) Direct FD (6.286e-3) and (b) Multiplicative Factorization (2.104e-3).