

# Solution to factored Eikonal equation in VTI medium

王宇航

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## 1 VTI 程函方程

$$V_{nmo}^2 (1 + 2\eta) (p_x^2 + p_y^2) + V_0^2 p_z^2 (1 - 2\eta V_{nmo}^2 (p_x^2 + p_y^2)) = 1 \quad (1)$$

其中, 射线参数  $p_x = \frac{\partial T}{\partial x}, p_y = \frac{\partial T}{\partial y}, p_z = \frac{\partial T}{\partial z}$ ;  $T(x, y, z)$  为旅行时。其哈密顿量为:

$$H = V_{nmo}^2 (1 + 2\eta) (p_x^2 + p_y^2) + V_0^2 p_z^2 (1 - 2\eta V_{nmo}^2 (p_x^2 + p_y^2))$$

## 2 乘法分解

令

$$T = T_0 \tau \quad (2)$$

其中  $T_0$  为给定的已知函数, 则

$$\nabla T = \nabla T_0 \tau + T_0 \nabla \tau \quad (3)$$

将公式 (1) 中 4 次项移到等式右端得到:

$$V_{nmo}^2 (1 + 2\eta) (p_x^2 + p_y^2) + V_0^2 p_z^2 = 1 + 2\eta V_0^2 V_{nmo}^2 p_z^2 (p_x^2 + p_y^2) \quad (4)$$

将 (3) 代入到公式 (4) 左端得到:

$$\begin{aligned} & V_{nmo}^2 (1 + 2\eta) \left( \left( \frac{\partial \tau}{\partial x} \right)^2 T_0^2 + 2 \frac{\partial \tau}{\partial x} \frac{\partial T_0}{\partial x} T_0 \tau + \left( \frac{\partial T_0}{\partial x} \right)^2 \tau^2 \right) \\ & + V_{nmo}^2 (1 + 2\eta) \left( \left( \frac{\partial \tau}{\partial y} \right)^2 T_0^2 + 2 \frac{\partial \tau}{\partial y} \frac{\partial T_0}{\partial y} T_0 \tau + \left( \frac{\partial T_0}{\partial y} \right)^2 \tau^2 \right) \\ & + V_0^2 \left( \left( \frac{\partial \tau}{\partial z} \right)^2 T_0^2 + 2 \frac{\partial \tau}{\partial z} \frac{\partial T_0}{\partial z} T_0 \tau + \left( \frac{\partial T_0}{\partial z} \right)^2 \tau^2 \right) = 1 + 2\eta V_0^2 V_{nmo}^2 p_z^2 (p_x^2 + p_y^2) \end{aligned} \quad (5)$$

对  $\nabla\tau$  采用一阶差分近似:

$$\frac{\partial\tau}{\partial x} \approx \left( \frac{\tau_{i,j,k} - \tau_x}{\Delta x} \right) s_x, \quad \frac{\partial\tau}{\partial y} \approx \left( \frac{\tau_{i,j,k} - \tau_y}{\Delta y} \right) s_y, \quad \frac{\partial\tau}{\partial z} \approx \left( \frac{\tau_{i,j,k} - \tau_z}{\Delta z} \right) s_z \quad (6)$$

其中

$$\tau_x = \min \{ \tau_{i+1,j,k}, \tau_{i-1,j,k} \}, \quad \tau_y = \min \{ \tau_{i,j+1,k}, \tau_{i,j-1,k} \}, \quad \tau_z = \min \{ \tau_{i,j,k+1}, \tau_{i,j,k-1} \} \quad (7)$$

$s_x, s_y, s_z$  为在取值 1 或-1 的符号变量, 取值取决于差分采用网格点的相对位置。将 (6) 代入公式 (5) 中得到关于  $\tau$  ( $\tau = \tau_{i,j,k}$ ) 的一元二次方程:

$$a\tau^2 + b\tau + c = 1 + 2\eta V_0^2 V_{nmo}^2 p_z^2 (p_x^2 + p_y^2) \quad (8)$$

其中

$$\begin{aligned} a &= V_{nmo}^2 (1 + 2\eta) \left( \frac{T_0^2}{h_1^2} + \left( \frac{\partial T_0}{\partial x} \right)^2 + 2s_x T_0 \frac{1}{h_1} \frac{\partial T_0}{\partial x} \right) \\ &\quad + V_{nmo}^2 (1 + 2\eta) \left( \frac{T_0^2}{h_2^2} + \left( \frac{\partial T_0}{\partial y} \right)^2 + 2s_y T_0 \frac{1}{h_2} \frac{\partial T_0}{\partial y} \right) \\ &\quad + V_0^2 \left( \frac{T_0^2}{h_3^2} + \left( \frac{\partial T_0}{\partial z} \right)^2 + 2s_z T_0 \frac{1}{h_3} \frac{\partial T_0}{\partial z} \right) \\ b &= -2V_{nmo}^2 (1 + 2\eta) \tau_x \left( \frac{T_0^2}{h_1^2} + s_x T_0 \frac{1}{h_1} \frac{\partial T_0}{\partial x} \right) \\ &\quad - 2V_{nmo}^2 (1 + 2\eta) \tau_y \left( \frac{T_0^2}{h_2^2} + s_y T_0 \frac{1}{h_2} \frac{\partial T_0}{\partial y} \right) \\ &\quad - 2V_0^2 \tau_z \left( \frac{T_0^2}{h_3^2} + s_z T_0 \frac{1}{h_3} \frac{\partial T_0}{\partial z} \right) \\ c &= V_{nmo}^2 (1 + 2\eta) \tau_x^2 \frac{T_0^2}{h_1^2} + V_{nmo}^2 (1 + 2\eta) \tau_y^2 \frac{T_0^2}{h_2^2} + V_0^2 \tau_z^2 \frac{T_0^2}{h_3^2} \end{aligned}$$

### 3 Causality Condition

相速度和群速度的计算 (See Appendix A and B):

$$\begin{aligned} v_{\text{相}} &= \left( \frac{p_x}{p_x^2 + p_y^2 + p_z^2}, \frac{p_y}{p_x^2 + p_y^2 + p_z^2}, \frac{p_z}{p_x^2 + p_y^2 + p_z^2} \right) \\ v_{\text{群}} &= (v_x, v_y, v_z), \quad \text{其中 } v_i = \left( p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z} \right)^{-1} \frac{\partial H}{\partial p_i}, i = x, y, z \end{aligned}$$

计算得

$$\begin{aligned}\frac{\partial H}{\partial p_x} &= 2p_x (V_{nmo}^2 (1 + 2\eta) - 2V_0^2 V_{nmo}^2 \eta p_z^2) \\ \frac{\partial H}{\partial p_y} &= 2p_y (V_{nmo}^2 (1 + 2\eta) - 2V_0^2 V_{nmo}^2 \eta p_z^2) \\ \frac{\partial H}{\partial p_z} &= 2V_0^2 p_z (1 - 2V_{nmo}^2 \eta (p_x^2 + p_y^2))\end{aligned}$$

记

$$\begin{aligned}\alpha &= p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z} \\ &= 2(p_x^2 + p_y^2) (V_{nmo}^2 (1 + 2\eta) - 2V_0^2 V_{nmo}^2 \eta p_z^2) \\ &\quad + 2V_0^2 p_z^2 (1 - 2V_{nmo}^2 \eta (p_x^2 + p_y^2))\end{aligned}$$

现计算  $p_x, p_y, p_z$  的取值范围, 记  $p_r^2 = p_x^2 + p_y^2$ , 则程函方程化为

$$p_z^2 = \frac{1 - V_{nmo}^2 (1 + 2\eta) p_r^2}{V_0^2 (1 - 2\eta V_{nmo}^2 p_r^2)}$$

由上公式知, 当  $p_r = 0, p_z^2$  取最大值, 此时  $p_z^2 = \frac{1}{V_0^2}$ , 对应群速度方向沿  $z$  轴情况;  
当  $p_r^2 = \frac{1}{(1+2\eta)V_{nmo}^2}$ ,  $p_z^2$  取最小值, 此时  $p_z^2 = 0$ , 对应群速度在  $z$  轴分速度为 0 情况。  
由上述取值范围容易得到

$$\alpha > 0$$

经计算易得

$$v_i \geq 0 \Leftrightarrow \frac{\partial H}{\partial p_i} \geq 0 \Leftrightarrow p_i \geq 0, i = x, y, z \quad (9)$$

将 (9) 代入得

$$\begin{aligned}s_x v_x &> 0, s_y v_y > 0, s_z v_z > 0 \\ s_x p_x &> 0, s_y p_y > 0, s_z p_z > 0\end{aligned} \quad (10)$$

将 (3) 代入 (10) 得到

$$\frac{\tau - \tau_x}{h_1} T_0 + \frac{\partial T_0}{\partial x} \tau s_x > 0, \frac{\tau - \tau_y}{h_2} T_0 + \frac{\partial T_0}{\partial y} \tau s_y > 0, \frac{\tau - \tau_z}{h_3} T_0 + \frac{\partial T_0}{\partial z} \tau s_z > 0 \quad (11)$$

## 4 单点更新

以公式 (7) 中  $\tau_x = \tau_W, \tau_y = \tau_S, \tau_z = \tau_D$  为例, 求解方程 (8), 有以下 3 种情况:

1. 方程有两个根满足 Causality Condition: 取最小的根作为解;
2. 方程有且仅有一个根满足 Causality Condition: 取该根作为解;
3. 方程无根满足 Causality Condition: 分别在三角形 WSC, DWC, DSC 上求解, 并取其最小解。

记所得解为  $root$ , 更新 C 点值:  $\tau_c^{new} = \min\{root, \tau_c^{old}\}$

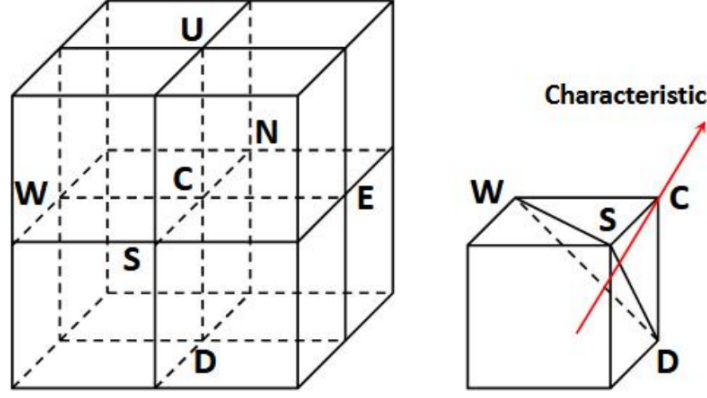


图 1: Example of a finite-difference grid

## 5 平面求解

以三角形 DWC 内求解为例，在方程 (4) 中令  $p_y = 0$ ，解差分方程 (8) 有以下 3 种情况：

1. 方程有两个根满足 Causality Condition：取最小的根作为解；
2. 方程有且仅有一个根满足 Causality Condition：取该根作为解；
3. 方程无根满足 Causality Condition，分别在 WC, DC 上求解，并取最小解。

以 WC 上求解为例，方程 (4) 退化为：

$$V_{nmo}^2(1+2\eta)p_x^2 = f(\tau)$$

so that

$$|p_x| = \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}$$

假设当波场沿  $x$  轴正方向传播时，背景波场也沿  $x$  轴正方向传播，即  $p_x$  与  $\frac{\partial T_0}{\partial x}$  同号). 注意到  $p_x = \tau \frac{\partial T_0}{\partial x} + T_0 \frac{\tau - \tau_x}{\Delta x} s_x$ . 若  $p_x > 0$ ，则  $s_x > 0$ ，so  $s_x = 1$ .

$$\tau \frac{\partial T_0}{\partial x} + T_0 \frac{\tau - \tau_x}{\Delta x} = \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}} \Leftrightarrow \tau = \frac{T_0 \tau_x + \Delta x \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 + \Delta x \frac{\partial T_0}{\partial x}}$$

若  $p_x < 0$ ，则  $s_x < 0$ ，so  $s_x = -1$ .

$$\tau \frac{\partial T_0}{\partial x} - T_0 \frac{\tau - \tau_x}{\Delta x} = -\sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}} \Leftrightarrow \tau = \frac{T_0 \tau_x + \Delta x \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 - \Delta x \frac{\partial T_0}{\partial x}}$$

In summary, we obtain

$$\tau = \frac{T_0 \tau_x + \Delta x \sqrt{\frac{f(\tau)}{V_{nmo}^2(1+2\eta)}}}{T_0 + \Delta x \left| \frac{\partial T_0}{\partial x} \right|}$$

记所得解为 root，更新 C 点值： $\tau_C^{new} = \min\{root, \tau_C^{old}\}$

## 6 求解算法

记公式 (4) 右端项为

$$f(\tau) = 1 + 2\eta V_0^2 V_{nmo}^2 (p_x^2 + p_y^2)$$

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### Algorithm 1 程函方程求解算法

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- 1: 初始化  $f(\tau) = 1$ ;
- 2: 初始化源点网格  $\tau = 1$ , 其余网格  $\tau = \text{HUGE}$ ,

$$T_0(x + x_0, y + y_0, z + z_0) = \sqrt{\frac{b_0 c_0 x^2 + a_0 c_0 y^2 + a_0 b_0 z^2}{a_0 b_0 c_0}}$$

according to [Luo and Qian \(2012, Remark 3.5\)](#), only consider diagonal of  $M(x)$

$$a_0 = b_0 = \frac{V_{nmo}^2 (1 + 2\eta)}{f(\tau)} \quad \text{where } (x, y, z) = (x_0, y_0, z_0)$$

$$c_0 = \frac{V_0^2}{f(\tau)} \quad \text{where } (x, y, z) = (x_0, y_0, z_0)$$

- 3: 依次沿 8 个不同方向扫描更新全部网格点, 并多次重复此过程
  - 4: 用得到的数据计算  $\nabla\tau, \nabla T$  并更新  $f(\tau)$
  - 5: 多次重复步骤 2, 3, 4
  - 6: 由公式 (2) 得到数值解
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The result after solving the Eikonal equation is shown in [Figure 2](#).

## A 相速度 (Phase velocity)

The slowness vector  $\mathbf{p} = (p_x, p_y, p_z) = \nabla T = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z})$

$$|\mathbf{p}|^2 = |\nabla T|^2 = \frac{1}{v^2} \Leftrightarrow p_x^2 + p_y^2 + p_z^2 = \frac{1}{v^2}$$

相速度矢量  $\mathbf{v}$  与  $\mathbf{p}$  同方向, 大小为  $v = 1/|\mathbf{p}|$

$$\mathbf{v} = v \frac{\mathbf{p}}{|\mathbf{p}|} = \frac{\mathbf{p}}{|\mathbf{p}|^2} = \left( \frac{p_x}{p_x^2 + p_y^2 + p_z^2}, \frac{p_y}{p_x^2 + p_y^2 + p_z^2}, \frac{p_z}{p_x^2 + p_y^2 + p_z^2} \right)$$

## B 群速度 (Group velocity)

在 Hamilton 射线理论中:

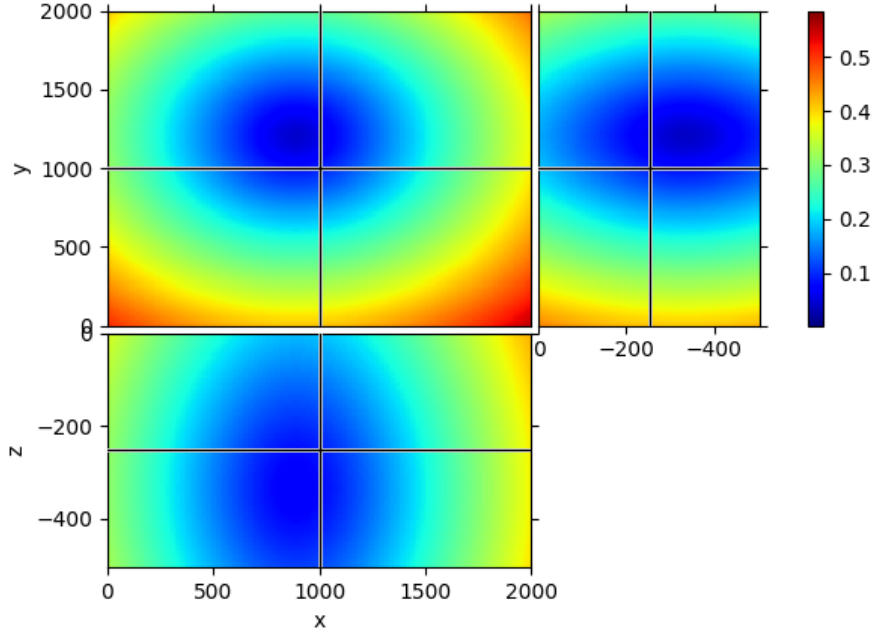


图 2: The contour plot for traveltime computed by solving VTI eikonal equation

$$\frac{d\mathbf{x}}{ds} = \nabla_{\mathbf{p}} H(\mathbf{x}, \mathbf{p}) = \left( \frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y}, \frac{\partial H}{\partial p_z} \right)$$

这里  $s$  是弧长。

在高频近似中，波场走时为：

$$t(\mathbf{x}) = \int \mathbf{p} \cdot d\mathbf{x} = \int \mathbf{p} \cdot \frac{d\mathbf{x}}{ds} ds$$

所以：

$$\frac{dt}{ds} = \mathbf{p} \cdot \frac{d\mathbf{x}}{ds} = p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z}.$$

群速度定义为位移关于时间的导数：

$$v_i = \frac{dx_i}{dt} = \frac{dx_i/ds}{dt/ds} = \left( p_x \frac{\partial H}{\partial p_x} + p_y \frac{\partial H}{\partial p_y} + p_z \frac{\partial H}{\partial p_z} \right)^{-1} \frac{\partial H}{\partial p_i}, \quad i = x, y, z.$$

## 参考文献

Luo, S. and Qian, J. (2012). Fast sweeping method for factored anisotropic eikonal equations: multiplicative and additive factors. *Journal of Scientific Computing*, 52:360–382.