Wess-Zumino-Witten Model as a Conformal Field Theory

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Abstract

Conformal field theory (CFT) refers to quantum field theory that is invariant under conformal transformations. Two-dimensional CFT is especially important, for there is an infinite-dimensional algebra of local conformal transformations. Wess-Zumino-Witten (WZW) model is just a a two-dimensional CFT with an affine Lie algebra as its symmetry algebra. In this report, we will give a brief introduction to CFT and discuss a little further on WZW models. The main reference book is *Conformal Field Theory* by Philippe Francesco, Pierre Mathieu and David Sénéchal[1].

Contents

1 Backgrounds

1.1 Conformal Symmetry

Definition 1.1. In d-dimensional space time, by $g_{\mu\nu}$ we denote the flat metric tensor of signature (p, q). A *conformal transformation* is an invertible mapping of coordinates $x \to x'$, which leaves the metric tensor invariant up to a scale,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{1}$$

Locally, a conformal transformation is equivalent to a (pseudo) rotation and a dilation which preserve angles, and the set of such transformations apparently form a group. We call it *conformal group*. It indeed has Poincaré group as its subgroup, corresponding to $\Lambda(x) \equiv 1$ situation.

To derive the general form, we introduce an infinitesimal coordinate transformation: $x^\mu \to (x')^\mu = x^\mu + \epsilon^\mu$, and under which

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial (x')^\mu} \frac{\partial x^\beta}{\partial (x')^\nu} g_{\alpha\beta}(x) \\ &= (\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \epsilon^\beta) g_{\alpha\beta}(x) \\ &= g_{\mu\nu}(x) - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)(x). \end{aligned} \tag{2}$$

Therefore it is required that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \gamma(x) g_{\mu\nu}. \tag{3}$$

We can take a trace by multiply $g^{\mu\nu}$ to each side and do contraction, which gives

$$\gamma = \frac{2}{d} \partial \cdot \epsilon \tag{4}$$

By applying an extra derivative ∂_ρ on Eq.(3) and doing permutations, we get

$$\begin{aligned} \partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu &= g_{\mu\nu} \partial_\rho \gamma(x) \\ \partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\rho \epsilon_\mu &= g_{\mu\rho} \partial_\nu \gamma(x) \\ \partial_\rho \partial_\mu \epsilon_\nu + \partial_\mu \partial_\nu \epsilon_\rho &= g_{\rho\nu} \partial_\mu \gamma(x). \end{aligned} \tag{5}$$

Take a linear combination, we arrive at

$$2\partial_\mu \partial_\nu \epsilon_\rho = g_{\mu\rho} \partial_\nu \gamma(x) + g_{\rho\nu} \partial_\mu \gamma(x) - g_{\mu\nu} \partial_\rho \gamma(x). \tag{6}$$

Similarly take a trace, and the equation turns into

$$2\Box\epsilon_\rho = (2-d)\partial_\rho\gamma. \tag{7}$$

Again apply another derivative ∂_σ and take a trace, and the final equations we get are

$$\begin{aligned} (g_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)(\partial\cdot\epsilon) = 0 \\ (d-1)\Box(\partial\cdot\epsilon) = 0 \end{aligned} \tag{8}$$

For $d = 1$ situation, there is no constraint on ϵ, and hence any smooth transformation is conformal in 1-dimensional case.

1.1.1 Conformal Group in Three or Higher Dimensions

When $d \geq 3$, Eq.(8) implies that the third derivatives of ϵ vanish, so ϵ is at most quadratic in coordinates. The general expression should be

$$\epsilon^\mu = a^\mu + b^\mu_\nu x_\nu + c^\mu_{\nu\rho}x^\nu x^\rho, \qquad c^\mu_{\nu\rho} = c^\mu_{\rho\nu}. \tag{9}$$

Substituting the linear term into Eq.(3) yields

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}g_{\mu\nu}b^\rho_\rho, \tag{10}$$

which indicates that $b_{\mu\nu}$ is sum of an antisymmetric part and a metric part:

$$b_{\mu\nu} = \alpha g_{\mu\nu} + m_{\mu\nu}, \qquad m_{\mu\nu} = -m_{\nu\mu}. \tag{11}$$

While substituting the quadratic term into Eq.(6) yields

$$c_{\rho\mu\nu} = g_{\rho\mu}c_\nu + g_{\rho\nu}c_\mu - g_{\mu\nu}c_\rho, \qquad c_\mu := c^\sigma_{\sigma\mu}. \tag{12}$$

To conclude,

Proposition 1.1. *The allowed infinitesimal transformations and corresponding finite conformal transformations are*

Type	*Infinitesimal Transformation* ϵ^μ	*Finite Transformation* x'^μ
Translation	a^μ	$x^\mu + a^\mu$
Dilation	αx^μ	kx^μ
Lorentz Rotation	$m^\mu_\nu x^\nu$	$M^\mu_\nu x^\nu$ ($M^\mu_\nu \in SO(p,q)$)
Special Conformal Transformation (SCT)	$2(c\cdot x)x^\mu - x^2c^\mu$	$\frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}$

We are quite familiar with the first three transformations, whereas the last one is indeed the combination of an inversion, a translation and another inversion:

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - b^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)(\frac{x_\mu}{x^2} - b_\mu)} = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}, \tag{13}$$

and the scale factor is given by

$$\Lambda(x) = (1 - 2b \cdot x + b^2 x^2)^2. \tag{14}$$

The generators of conformal group as a Lie group can be directly achieved from the infinitesimal transformations above:

Type	*Generator*
Translation	$P_\mu = -i\partial_\mu$
Dilation	$D = -ix^\mu \partial_\mu$
Lorentz Rotation	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
Special Conformal Transformation (SCT)	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

If we rewrite the $\frac{1}{2}(d+1)(d+2)$ generators by

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \\ J_{0,\nu} &= \frac{1}{2}(P_\nu + K_\nu) \\ J_{-1,\nu} &= \frac{1}{2}(P_\nu - K_\nu) \\ J_{-1,0} &= D \\ J_{ab} &= -J_{ba} \end{aligned} \tag{15}$$

where $a, b \in \{-1, 0, 1, \cdots, d\}$ and further define $\eta_{ab} = \text{diag}\{-1, 1\} \oplus g_{\mu\nu}$, the commutation relations become

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}), \tag{16}$$

which is identical to $\mathfrak{so}(p+1, q+1)$.

1.1.2 Conformal Group in Two Dimensions

Consider the two dimensional case. By definition, any conformal coordinate transformation $(z^0, z^1) \rightarrow (w^0, w^1)$ should satisfy

$$g^{\mu\nu} \rightarrow (\frac{\partial w^\mu}{\partial z^\alpha})(\frac{\partial w^\nu}{\partial z^\beta})g^{\alpha\beta} \propto g^{\mu\nu}, \tag{17}$$

which indicates

$$(\frac{\partial w^0}{\partial z^0})^2 + (\frac{\partial w^0}{\partial z^1})^2 = (\frac{\partial w^1}{\partial z^0})^2 + (\frac{\partial w^1}{\partial z^1})^2 \tag{18}$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0. \tag{19}$$

These two equations are equivalent to either

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \quad \text{and} \quad \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \tag{20}$$

or

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \quad \text{and} \quad \frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}. \tag{21}$$

Eq.(20) are indeed the Cauchy-Riemann equations. Therefore it is natural to define $z = z^0 + iz^1$ and $w = w^0 + iw^1$. Under this notation, a possible conformal transformation is

$$z \rightarrow w(z) \quad \text{and} \quad \bar{z} \rightarrow \bar{w}(\bar{z}), \tag{22}$$

where w is either holomorphic or anti-holomorphic. Still, we require the transformation to be globally invertible. It is known in complex analysis that the only possible ones are linear fractional transformations

$$w = \frac{az+b}{cz+d}, \qquad ad - bc = 1. \tag{23}$$

If we choose $g^0_{\mu\nu}(z^0, z^1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, with $z^0 = \frac{1}{2}(z+\bar{z})$ and $z^1 = \frac{1}{2i}(z-\bar{z})$, the new metric tensor in terms of z and $\bar{z}$ are

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{24}$$

As any infinitesimal holomorphic transformation can be expressed as

$$z' = z + \epsilon(z), \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}, \tag{25}$$

the generators can be introduced by

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \tag{26}$$

which obey the following commutation relations:

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} \\ \left[\bar{l}_n, \bar{l}_m\right] &= (n-m)\bar{l}_{n+m} \\ \left[l_n, \bar{l}_m\right] &= 0. \end{aligned} \tag{27}$$

It is called Witt algebra.

There is another way to define a complex coordinate, which may seem more natural from Minkowski space. Consider a compact one-dimensional space with coordinate x from 0 to L, the space-time is cylinder like. With the map:

$$z = e^{\frac{2\pi(t+ix)}{L}}, \tag{28}$$

The remote past is situated at the origin while remote future is at infinity. With a same t, the new points will share a same radius, and therefore it is also called *radial quantization*.

1.2 Affine Lie Algebras

Definition 1.2. If $\mathfrak{g}$ is a finite dimensional simple Lie algebra, the corresponding *affine Lie algebra* $\hat{\mathfrak{g}}$ is constructed as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}k, \tag{29}$$

where $\mathbb{C}\left[t, t^{-1}\right]$ is the complex vector space of Laurent polynomials in the indeterminate t and $\mathbb{C}k$ is a one-dimensional centre. The Lie bracket is

$$[x \otimes t^n + \alpha k, y \otimes t^m + \beta k] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}\mathcal{K}(x, y)k \tag{30}$$

where $\mathcal{K}(a, b)$ is the Killing form.

Below we use the notation

$$J_n^a = J^a \otimes t^n, \tag{31}$$

where J^a's are the generators of $\mathfrak{g}$. The Killing form occurs from the consideration that $\left[J_0^a, J_n^b\right] = \left[J^a, J^b\right] \otimes t^n$, which means the generators transform into the adjoint representation of $\mathfrak{g}$. Hence the central extensions must be invariant tensors of the adjoint representation. While up to normalization, there is only one choice, the Killing form. We may assume J^a's are orthonormal with respect to the Killing form to rewrite the commutation relation as

$$\left[J_n^a, J_n^b\right] = \sum_c i f_c^{ab} J_{m+n}^c + n\delta_{n+m,0}\delta_{ab} k. \tag{32}$$

2 WZW Models

2.1 Non-linear Sigma Models

WZW model is a two dimensional conformal field theory whose conserved currents generating an affine Lie algebra. In search of theories with such properties, the non-linear sigma model is first considered. Its action is

$$S_0 = \frac{1}{4a^2} \int \mathrm{d}^2 x \mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g), \tag{33}$$

where a^2 is a positive constant and $g(x)$ is a G-value matrix boson field, where group manifold G is associated with the Lie algebra $\mathfrak{g}$. Here a unitary representation is taken so that S_0 is real and positive determined. Besides, Tr' stands for a representation independent normalization, namely $\mathrm{Tr}' = \frac{1}{\chi_{\mathrm{repr}}}\mathrm{Tr}$,where χ_{repr} is the Dynkin index of a representation.

With Euler-Lagrange equation, one can easily figure out the equation of motion to this model:

$$\partial^\mu (g^{-1} \partial_\mu g) = 0, \tag{34}$$

which implies the conservation of currents

$$J_\mu = g^{-1} \partial_\mu g. \tag{35}$$

Thus if we write $\tilde{J}_z = g^{-1}\partial_z g$ and $\tilde{J}_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$, then

$$\partial^z J_z + \partial^{\bar{z}} J_{\bar{z}} = 0, \tag{36}$$

or

$$\partial_{\bar{z}} J_z + \partial_z J_{\bar{z}} = 0. \tag{37}$$

Witten assumed the holomorphic factorization property of a conformal field theory [2], which forces a separate conservation of holomorphic and anti-holomorphic part of the current. However, this does not always hold for non-linear sigma model, because $\partial_z J_{\bar{z}} = 0$ and $\partial_{\bar{z}} J_z$ implies $\partial_z \partial_{\bar{z}} g = \partial_z g g^{-1} \partial_{\bar{z}} g$ and $\partial_{\bar{z}} \partial_z g = \partial_{\bar{z}} g g^{-1} \partial_z g$. Nevertheless, for non-Abelian group G, the r.h.s. of the two equations may not equal to each other.

2.2 Wess-Zumino-Witten Action

A more complicated action must be considered in order to enhance the symmetry and recover the conserved currents. The result turns to be adding a Wess-Zumino term to the action,

$$\Gamma = \frac{-i}{24\pi}\int_B \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\mathrm{Tr}'(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}). \tag{38}$$

It is defined on a three-dimensional manifold B, whose boundary is the compactification of the original two-dimensional space, and $\tilde{g}$ refers to the extension of g to B. However, a compact two-dimensional space delimits two distinct three-manifolds. Therefore there are two choices of B with opposite orientation, and their difference will be integrated over the whole compact three-dimensional space. By path integral,

$$\Delta\Gamma = \frac{-i}{24\pi}\int \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\mathrm{Tr}'(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g}) = n2\pi i, \tag{39}$$

where n is an integer. It shows that any coupling constant multiplying this term must be quantized.

Then the whole action can be written as

$$S = S_0 + n\Gamma \tag{40}$$

if we choose a normalized Γ by integrating $\Delta\Gamma$ on the three-sphere S^3. Take a variation of g to the latter term:

$$\delta\Gamma = \frac{i}{8\pi}\int_B \mathrm{d}^3 y \epsilon_{\mu\nu\gamma}\partial^\gamma(\mathrm{Tr}'(g^{-1}\delta g\partial^\mu(g^{-1}\partial^\nu g))) = \frac{i}{8\pi}\int \mathrm{d}^2 x \epsilon_{\mu\nu}\mathrm{Tr}'(g^{-1}\delta g\partial^\mu(g^{-1}\partial^\nu g)), \tag{41}$$

and the whole equation of motion turns into

$$\partial^\mu(g^{-1}\partial_\mu g) + \frac{a^2 in}{4\pi}\epsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) = 0. \tag{42}$$

In terms of the complex variables $z, \bar{z}$, we have $\epsilon_{z\bar{z}} = i/2$ and $\partial^z = 2\partial_{\bar{z}}$. The equation becomes

$$(1 + \frac{a^2 n}{4\pi})\partial_z(g^{-1}\partial_{\bar{z}} g) + (1 - \frac{a^2 n}{4\pi})\partial_{\bar{z}}(g^{-1}\partial_z g) = 0. \tag{43}$$

It is what we desire, because when $a^2 = \pm\frac{4\pi}{n}$, the holomorphic currents (when $n > 0$, or respectively the anti-holomorphic ones when $n < 0$) are separately conserved. Thus the final action is

$$S^{WZW} = \frac{n}{16\pi}\int \mathrm{d}^2 x \mathrm{Tr}'(\partial^\mu g^{-1}\partial_\mu g) + n\Gamma. \tag{44}$$

2.3 Affine Lie Algebra Structure

The conserving equation of current: $\partial_z(g^{-1}\partial_{\bar{z}}g) = 0$ implies that the action should be invariant under the local $G(z) \times G(\bar{z})$ transformation of arbitrary G-valued matrices $\Omega(z)$ and $\bar{\Omega}(\bar{z})$:

$$g(z,\bar{z}) \to \Omega(z)g(z,\bar{z})\bar{\Omega}^{-1}(\bar{z}). \tag{45}$$

In fact, under infinitesimal transformations

$$\Omega(z) = 1 + \omega(z) \qquad \bar{\Omega}(\bar{z}) = 1 + \bar{\omega}(\bar{z}), \tag{46}$$

variation of g turns to be

$$g \to g + \delta_\omega g + \delta_{\bar{\omega}} g = g + \omega g - g\bar{\omega}. \tag{47}$$

Therefore the variation of action is

$$\begin{aligned}\delta S &= \frac{n}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}'(g^{-1}\delta g \left[\partial_z(g^{-1}\partial_{\bar{z}}g)\right]) \\ &= \frac{n}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}'(g^{-1}(\omega g - g\bar{\omega}) \left[\partial_z(g^{-1}\partial_{\bar{z}}g)\right]) \\ &= \frac{n}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}'(\omega\partial_{\bar{z}}(\partial_z g g^{-1}) - \bar{\omega}\partial_z(g^{-1}\partial_{\bar{z}}g)) \\ &= 0.\end{aligned} \tag{48}$$

The last equal sign holds from simple integration by parts.

Rescale the conserved currents as

$$\begin{aligned} J(z) &\equiv -nJ_z(z) = -n\partial_z g g^{-1} \\ \bar{J}(\bar{z}) &\equiv nJ_{\bar{z}}(\bar{z}) = ng^{-1}\partial_{\bar{z}}g. \end{aligned} \tag{49}$$

Under this convention,

$$\delta S = -\frac{1}{2\pi}\int \mathrm{d}^2x(\partial_{\bar{z}}(\mathrm{Tr}'\left[\omega(z)J(z)\right]) + \partial_z(\mathrm{Tr}'\left[\bar{\omega}(\bar{z})\bar{J}(\bar{z})\right])). \tag{50}$$

While d^2x can be replaced with $-\frac{i}{2}\mathrm{d}z\mathrm{d}\bar{z}$, so

$$\delta S = \frac{i}{4\pi}\oint \mathrm{d}z \mathrm{Tr}'\left[\omega(z)J(z)\right] - \frac{i}{4\pi}\oint \mathrm{d}\bar{z}\mathrm{Tr}'\left[\bar{\omega}(\bar{z})\bar{J}(\bar{z})\right]. \tag{51}$$

Moreover, the currents in fact lie in the Lie algebra of G, and so do ω. Hence we can write

$$J = \sum_a J^a t^a \qquad \omega = \sum_a \omega^a t^a, \tag{52}$$

t^a are generators of G. With the normalization for Tr$'$, this yields

$$\delta S = \frac{i}{2\pi}\oint \mathrm{d}z \sum_a \omega^a J^a - \frac{i}{2\pi}\oint \mathrm{d}\bar{z} \sum_a \bar{\omega}^a \bar{J}^a. \tag{53}$$

Finally, consider

$$\delta\langle X\rangle = \int \langle \delta s X\rangle, \tag{54}$$

where X stands for a number of identical fields with different coordinates and we take an average by path integral. It is called correlation function. δs represents for the density of δS. The *Ward identity* is achieved:

$$\delta\langle X\rangle = \frac{i}{2\pi}\oint \mathrm{d}z \sum_a \omega^a \langle J^a X\rangle - \frac{i}{2\pi}\oint \mathrm{d}\bar{z} \sum_a \bar{\omega}^a \langle \bar{J}^a X\rangle. \tag{55}$$

One the other hand, the variation of conserved current J is

$$\begin{aligned}
\delta_\omega J &= -n\left[\partial_z(\omega g)g^{-1} - \partial_z g g^{-1}(\omega g)g^{-1}\right] \\
&= -n\partial_z\omega - \omega n\partial_z g g^{-1} + n\partial_z g g^{-1}\omega \\
&= [\omega, J] - n\partial_z\omega,
\end{aligned} \tag{56}$$

which can be rewritten as

$$\delta_\omega J^a = \sum_{b,c} i f_{abc}\omega^b J^c - n\partial_z\omega^a. \tag{57}$$

Substitute it back to Eq.(55), we have

$$\begin{aligned}
\delta_\omega\langle J^a(w)\rangle &= -\frac{1}{2\pi i}\oint \mathrm{d}z \sum_b \omega^b(z)\langle J^b(z)J^a(w)\rangle \\
&= \sum_{b,c} i f_{abc}\omega^b(w)\langle J^c(w)\rangle - n\partial_z\omega^a(w).
\end{aligned} \tag{58}$$

We may assume J^c is non-singular at $w = z$, then this equation will lead to the *operator product expansion (OPE)*:

$$J^a(z)J^b(w) \sim \frac{n\delta_{ab}}{(z-w)^2} + \sum_c i f_{abc}\frac{J^c(w)}{z-w}. \tag{59}$$

Further introducing the modes J^a_k from Laurent expansion

$$J^a_k(z) = \sum_{k\in\mathbb{Z}} z^{-k-1}J^a_k, \tag{60}$$

the commutation relations will be

$$
\begin{aligned}
\left[J_k^a, J_l^b\right] &= \frac{1}{(2\pi i)^2}\oint_0 \mathrm{d}w w^{l+1}\oint_w \mathrm{d}z z^{k+1}\left(\frac{n\delta_{ab}}{(z-w)^2}+\sum_c i f_{abc}\frac{J^c(w)}{z-w}\right) \\
&= \sum_c i f_{abc} J^c_{k+l} + nk\delta_{ab}\delta_{k+l,0}.
\end{aligned}
\tag{61}
$$

We call it a *current algebra*, which proves to be an affine Lie algebra with n as its centre. For $\bar{J}^a$, we have another copy of affine Lie algebra. As $\delta_{\bar{\omega}} J = 0$, the two algebras are independent.

References

[1] Pierre Mathieu Philippe Francesco and David Sénéchal. Conformal field theory. Springer-Verlag New York, 1997.

[2] Edward Witten. On holomorphic factorization of wzw and coset models. Communications in Mathematical Physics, 144(1):189–212, 1992.