Clifford Algebras and Spin Groups

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Abstract

Clifford algebra $\mathfrak{Cl}(V, Q)$ is a unital associative algebra generated by a vector space V with a quadratic form Q. These algebras are applied to a wide variety of fields, such as geometry and physics. Spin group, which is the double cover of special orthogonal group, can also be constructed from Clifford algebra. In this article, the definition, structures and properties of Clifford algebras as well as spin groups will be discussed.

Contents

1 Clifford Algebras[1]

1.1 Definition and Examples

Definition 1.1. Clifford algebra $\mathfrak{Cl}(V, Q)$ is the freest unital associative algebra over field $\mathbb{F}$, which is generated by a vector space V equipped with a quadratic form $Q : V \to \mathbb{F}$, and

$$v^2 = Q(v)\mathbf{1}, \forall v \in V. \tag{1}$$

To be more specific, $\mathfrak{Cl}(V, Q) = K(V)/L$, where $K(V) = \oplus_{n\geq 0}(\otimes^n V)$ and L is the two sided ideal generated by $v \otimes v - Q(v)\mathbf{1}, \forall v \in V$.

As $(v + w)^2 = v^2 + w^2 + vw + wv$, the fundamental equation (1) can be rewritten as

$$\{v, w\} = (Q(v + w) - Q(v) - Q(w))\mathbf{1}, \tag{2}$$

where $\{v, w\} = vw + wv$ is the anticommutator of v and w.

Below are several examples of Clifford algebra:

Example 1.1. Exterior algebra. If $Q(v) = 0, \forall v \in V$, then $vw = -wv, \forall v, w \in V$. Therefore the Clifford algebra with a zero quadratic form is just the exterior algebra $\Lambda(V)$.

Proposition 1.1. *In fact, if* char $\mathbb{F} \neq 2$*, for nonzero Q there is also a natural isomorphism between* $\mathfrak{Cl}(V, Q)$ *and* $\Lambda(V)$ *as vector spaces.*

Further relation to exterior algebra will be discussed in 1.2.2.

Example 1.2. Dirac algebra. In relativistic quantum mechanics, the idea of Dirac equation is to reduce the Klein-Gordon equation

$$(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\psi = 0 \tag{3}$$

into order-1 form

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \tag{4}$$

where $\gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{\partial t} + \sum_{i=i}^{3} \gamma^i \frac{\partial}{\partial x^i}$. The lowest dimension of appropriate representation of γ^μ is 4. Explicitly, the standard form is

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{5}$$

where σ_i are the Pauli matrices, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We can easily check that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4$, $g^{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$. Therefore the Dirac matrices γ^μ indeed generate a Clifford algebra. Instead of doing direct matrix multiplication, physicist tend to generate new matrices by keeping calculating commutators with the original Dirac matrices. By counting number of γ^μ's used to produce commutator, we see that the dimension of Dirac algebra is $\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 2^4$.

1.2 Properties

1.2.1 Basis and Dimension

Definition 1.2. Inner product on V:

$$\langle v, w \rangle := \frac{1}{2}(Q(v+w) - Q(v) - Q(w)). \tag{6}$$

From the definition v being orthogonal to w indicates $\{v, w\} = 0$. If the dimension of V is n, we are still able to choose an orthogonal basis $\{e_1, \cdots, e_n\}$ of V by Gram–Schmidt orthogonalization.

Afterwards, a basis of $\mathfrak{Cl}(V, Q)$ can be written as

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} | 1 \leq i_1 < \cdots < i_k, 0 \leq k \leq n\}. \tag{7}$$

Notice that the $k = 0$ element is indeed the identity. Therefore we get

$$\dim \mathfrak{Cl}(V, Q) = 2^n. \tag{8}$$

1.2.2 Relation to Exterior Algebra

The canonical isomorphism between Clifford algebra and exterior algebra as vector spaces, which is mentioned in example 1.1, can now be written explicitly with the orthogonal basis above:

$$\begin{aligned} \phi : \mathfrak{Cl}(V, Q) &\to \Lambda(V) \\ e_{i_1} \cdots e_{i_k} &\to e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned} \tag{9}$$

The isomorphism is clearly independent of the choice of basis.

Moreover, if char $\mathbb{F}=0$, another way to establish the isomorphism is:

$$\begin{aligned}\psi_k:\Lambda^k(V)&\to\mathfrak{Cl}(V,Q)\\ v_1\wedge\cdots\wedge v_k&\to\frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)v_{\sigma(1)}\cdots v_{\sigma(k)}.\end{aligned}\tag{10}$$

It can be checked with the orthogonal basis.

1.2.3 Grading

Proposition 1.2. *Clifford algebras are $\mathbb{Z}_2$ graded algebras, or superalgebras.*

Proof. We can introduce a reflection map on V: $v\to -v$, which can be extended to $\mathfrak{Cl}(V,Q)$ naturally:

$$\eta:\mathfrak{Cl}(V,Q)\to\mathfrak{Cl}(V,Q).\tag{11}$$

We can decompose $\mathfrak{Cl}(V,Q)$ with regard to different eigenvalues (± 1) of η, i.e.

$$\mathfrak{Cl}(V,Q)=\mathfrak{Cl}^0(V,Q)\oplus\mathfrak{Cl}^1(V,Q),\tag{12}$$

where $\mathfrak{Cl}^i(V,Q)=\{v\in\mathfrak{Cl}(V,Q)|\eta(v)=(-1)^i v\}$. Using the notations above,

$$\begin{aligned}\mathfrak{Cl}^0(V,Q)&=\oplus_{k\text{ even}}\psi_k(\Lambda^k(V)),\\ \mathfrak{Cl}^1(V,Q)&=\oplus_{k\text{ odd}}\psi_k(\Lambda^k(V)).\end{aligned}\tag{13}$$

Consequently we can achieve the relation

$$\mathfrak{Cl}^i(V,Q)\mathfrak{Cl}^j(V,Q)=\mathfrak{Cl}^{i+j}(V,Q)\ (i,j\in\mathbb{Z}_2).\tag{14}$$

□

1.3 Structures of Clifford Algebras over $\mathbb{R}$ and $\mathbb{C}$

1.3.1 $\mathbb{F}=\mathbb{R}$ Situation[2]

Any nondegenerate quadratic form on a real vector space can be diagonalized as

$$Q(v)=v_1^2+v_2^2+\cdots v_p^2-v_{p+1}^2-\cdots v_{p+q}^2\tag{15}$$

where $p+q=n$ according to Sylvester's Law of Inertia. Such real vector space is denoted as $\mathbb{R}^{p,q}$, and the relating Clifford algebra as $\mathfrak{Cl}_{p,q}(\mathbb{R})$. To figure out their structures, we will start with small p,q and try to find out recurrence relationships.

Lemma 1.3. *Below are some useful tensor product isomorphisms. It is a little bit complex but routine to clarify all of them, hence only the results are stated here.*

- $M_m(\mathbb{R}) \otimes_{\mathbb{R}} M_n(\mathbb{R}) \cong M_{mn}(\mathbb{R})$.
- $M_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{K} \cong M_n(\mathbb{K})$*, for* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*, or* $\mathbb{H}$.
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$.
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$.
- $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$.

Proposition 1.4. *Structures of real Clifford algebras with small* p, q.

$$\begin{aligned} \mathfrak{Cl}_{0,0}(\mathbb{R}) &= \mathbb{R}, \\ \mathfrak{Cl}_{1,0}(\mathbb{R}) &= \mathbb{R} \oplus \mathbb{R}, \\ \mathfrak{Cl}_{0,1}(\mathbb{R}) &\cong \mathbb{C}. \end{aligned} \tag{16}$$

Proof. $\mathfrak{Cl}_{1,0}(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^{1,0} = \mathbb{R} \oplus \mathbb{R}$, while $\mathfrak{Cl}_{0,1}(\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}^{0,1} \cong \mathbb{R} \oplus i\mathbb{R} = \mathbb{C}$. □

Proposition 1.5. $\forall p, q \in \mathbb{N}$*, we have*

$$\begin{aligned} \mathfrak{Cl}_{0,q+2}(\mathbb{R}) &\cong \mathfrak{Cl}_{q,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}, \\ \mathfrak{Cl}_{p+2,0}(\mathbb{R}) &\cong \mathfrak{Cl}_{0,p}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R}), \\ \mathfrak{Cl}_{p+1,q+1}(\mathbb{R}) &\cong \mathfrak{Cl}_{p,q}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R}). \end{aligned} \tag{17}$$

Proof.

- $\mathfrak{Cl}_{0,q+2}(\mathbb{R}) \cong \mathfrak{Cl}_{q,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}$:

  Let $V(\mathbb{H})$ be the generation vector space of Clifford algebra $\mathbb{H} \cong \mathfrak{Cl}_{0,2}(\mathbb{R})$. We can choose $V(\mathbb{H}) = \{i, j\}$, where $i, j, k = ij \in \mathbb{H}$. Let $f : \mathbb{R}^{0,q+2} \to \mathbb{R}^{q,0} \otimes_{\mathbb{R}} V(\mathbb{H})$:

$$f(e_n) = \begin{cases} \epsilon_n \otimes ij, & n \leq q \\ 1_{\mathfrak{Cl}_{q,0}(\mathbb{R})} \otimes i, & n = q+1 \\ 1_{\mathfrak{Cl}_{q,0}(\mathbb{R})} \otimes j, & n = q+2 \end{cases} \tag{18}$$

where $\{e_1, \cdots, e_{q+2}\}$ is the natural basis of $\mathbb{R}^{0,q+2}$ and $\{\epsilon_1, \cdots, \epsilon_q\}$ is that of $\mathbb{R}^{q,0}$. It is an isomorphism map, because when $n < m$,

$$\begin{aligned}\{f(e_n), f(e_m)\} &= \begin{cases} \{\epsilon_n, \epsilon_m\} \otimes (ij)^2 = 0_{\mathfrak{Cl}_{q,0}(\mathbb{R})} \otimes 1_{\mathbb{H}}, & n \leq q \\ \epsilon_n \otimes 0_{\mathbb{H}}, & n \leq q \\ 1_{\mathfrak{Cl}_{q,0}(\mathbb{R})} \otimes 0_{\mathbb{H}}, & n \geq q+1 \end{cases} \\ &= 0 \end{aligned} \tag{19}$$

and $\forall v = \sum_{n=1}^{q+2} a_n e_n$,

$$\begin{aligned} f(v)^2 &= \sum_{n=1}^{q+2} \sum_{m=1}^{q+2} a_n a_m f(e_n) f(e_m) \\ &= \sum_{n=1}^{q} a_n^2 \epsilon_n^2 \otimes k^2 + a_{q+1}^2 1 \otimes i^2 + a_{q+2}^2 1 \otimes j^2 + \sum_{n=1, m>n}^{q+1} a_n a_m \{f(e_n), f(e_m)\} \\ &= (-\sum_{n=1}^{q+2} a_n^2) 1_{\mathfrak{Cl}_{q,0}(\mathbb{R})} \otimes 1_{\mathbb{H}} \\ &= Q_{\mathbb{R}^{0,q+2}}(v) 1_{\mathfrak{Cl}_{q,0}(\mathbb{R}) \otimes \mathbb{H}}. \end{aligned} \tag{20}$$

- $\mathfrak{Cl}_{p+2,0}(\mathbb{R}) \cong \mathfrak{Cl}_{0,0}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$:

  The proof is similar to the first equation. We only need to clarify that $V(M_2(\mathbb{R}) = \mathfrak{Cl}_{2,0}(\mathbb{R})) = \{r_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ and $r_3 = r_1 r_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then rewrite the isomorphism map as

$$g(\epsilon_n) = \begin{cases} e_n \otimes r_3, & n \leq q \\ 1_{\mathfrak{Cl}_{0,q}(\mathbb{R})} \otimes r_1, & n = q+1 \\ 1_{\mathfrak{Cl}_{0,q}(\mathbb{R})} \otimes r_2, & n = q+2 \end{cases} \tag{21}$$

- $\mathfrak{Cl}_{p+1,q+1}(\mathbb{R}) \cong \mathfrak{Cl}_{p,q}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$:

  Use the natural basis of $\mathbb{R}^{p+1,q+1}$: $\{\epsilon_1, \cdots, \epsilon_{p+1}, e_1, \cdots, e_{q+1}\}$, and $V(M_2(\mathbb{R}) = \mathfrak{Cl}_{1,1}(\mathbb{R}))' = \{s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$ and $s_3 = s_1 s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the isomorphism map this time is

$$
\begin{aligned}
h(\epsilon_n) &= \begin{cases} \epsilon_n \otimes s_3, & n \leq p \\ 1_{\mathfrak{Cl}_{p,q}(\mathbb{R})} \otimes s_1, & n = p+1 \end{cases} \\
h(e_m) &= \begin{cases} e_m \otimes s_3, & m \leq q \\ 1_{\mathfrak{Cl}_{p,q}(\mathbb{R})} \otimes s_2, & m = q+1 \end{cases}
\end{aligned} \tag{22}
$$

□

In fact, with 1.3 as well as proposition 1.4 and 1.5, we have indeed achieved the structures of all real Clifford algebras. Still we can see more beautiful periodicity from the theorem below:

Theorem 1.6. *(Cartan-Bott Periodicity of 8)*

$$
\begin{aligned}
\mathfrak{Cl}_{0,m+8}(\mathbb{R}) &\cong \mathfrak{Cl}_{0,m}(\mathbb{R}) \otimes M_{16}(\mathbb{R}), \\
\mathfrak{Cl}_{n+8,0}(\mathbb{R}) &\cong \mathfrak{Cl}_{n,0}(\mathbb{R}) \otimes M_{16}(\mathbb{R}).
\end{aligned} \tag{23}
$$

Proof. We only write down the first one.

$$
\begin{aligned}
\mathfrak{Cl}_{0,m+8}(\mathbb{R}) &\cong \mathfrak{Cl}_{m+6,0}(\mathbb{R}) \otimes \mathbb{H} \\
&\cong \mathfrak{Cl}_{0,m+4}(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \\
&\cong \mathfrak{Cl}_{m+2,0}(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \\
&\cong \mathfrak{Cl}_{0,m}(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \\
&\cong \mathfrak{Cl}_{0,m}(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes M_4(\mathbb{R}) \\
&\cong \mathfrak{Cl}_{0,m}(\mathbb{R}) \otimes M_{16}(\mathbb{R}).
\end{aligned} \tag{24}
$$

□

All $p, q \leq 8$ real Clifford algebras have been listed in table 1.3.1.

1.3.2 $\mathbb{F} = \mathbb{C}$ Situation

Any nondegenerate quadratic form on a complex vector space can be diagonalized as

$$
Q(z) = z_1^2 + \cdots z_n^2 \tag{25}
$$

where $n = \dim V$. Hence we can denote the Clifford algebra generated by $\mathbb{C}^n$ as $\mathfrak{Cl}_n(\mathbb{C})$. The general complex cases are much easier than real ones, because

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)^2$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)^2$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)^2$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)^2$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)^2$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)^2$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2)^2$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)^2$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)^2$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)^2$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)^2$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)^2$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)^2$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)^2$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)^2$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R}(1)^2$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)^2$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)^2$
0	$\mathbb{R}(1)$	$\mathbb{C}(1)$	$\mathbb{H}(1)$	$\mathbb{H}(1)^2$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)^2$	$\mathbb{R}(16)$
p \ q	0	1	2	3	4	5	6	7	8

Table 1: Clifford algebras $\mathfrak{Cl}_{p,q}$ when $p, q \leq 8$ ($\mathbb{R}(16) = M_{16}(\mathbb{R})$, $\mathbb{H}(16)^2 = \mathbb{H}(16) \oplus \mathbb{H}(16)$ and so forth.)

Proposition 1.7. *(Bott Periodicity of 2)*

$$\mathfrak{Cl}_{n+2}(\mathbb{C}) \cong \mathfrak{Cl}_n(\mathbb{C}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) \tag{26}$$

Proof. Totally similar to the proof of real cases, while here $V(M_2(\mathbb{C}) = \mathfrak{Cl}_2(\mathbb{C})) = \{s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$ and $s_3 = s_1 s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the isomorphism map is

$$g'(e_n) = \begin{cases} e_n \otimes_{\mathbb{R}} s_3, & n \leq q \\ 1_{\mathfrak{Cl}_q(\mathbb{C})} \otimes_{\mathbb{R}} s_1, & n = q+1 \\ 1_{\mathfrak{Cl}_q(\mathbb{C})} \otimes_{\mathbb{R}} i s_2. & n = q+2 \end{cases} \tag{27}$$

s_1, is_2 and s_3 are exactly Pauli matrices, and $s_1^2 = (is_2)^2 = s_3^2 = \mathbf{1}$. □

Furthermore, clearly $\mathfrak{Cl}_0(\mathbb{C}) = \mathbb{C}$ and $\mathfrak{Cl}_1(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, so

Theorem 1.8. *(Structures of Clifford algebras over $\mathbb{C}$):*

$$\mathfrak{Cl}_n(\mathbb{C}) \cong \begin{cases} M_{2^m}(\mathbb{C}), & n = 2m \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}), & n = 2m+1 \end{cases} \tag{28}$$

Proof. For $n = 2m$, use induction. $m = 0$ situation has been proved. Assume that the isomorphism holds for $m \leq k$, then for $m = k + 1$,

$$\begin{aligned} \mathfrak{Cl}_{2(k+1)}(\mathbb{C}) &= \mathfrak{Cl}_{2k}(\mathbb{C}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) = M_{2^k}(\mathbb{C}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) \\ &= M_{2^k}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = M_{2^{k+1}}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ &= M_{2^{k+1}}(\mathbb{C}). \end{aligned} \tag{29}$$

$n = 2m + 1$ situation as similar. ☐

2 Spin Groups[3]

2.1 Construction

Definition 2.1. Conjugation on $\mathfrak{Cl}(V,Q)$. Again take an orthogonal basis of V: $\{e_1, \cdots, e_n\}$, then the Conjugation on $\mathfrak{Cl}(V,Q)$ is defined by

$$(e_{n_1}e_{n_2}\cdots e_{n_k})^* := (-1)^k(e_{n_k}e_{n_{k-1}}\cdots e_{n_1}). \tag{30}$$

Definition 2.2. Pin groups and spin groups.

$$\begin{aligned}\text{Pin}(V,Q) &:= \{g \in \mathfrak{Cl}(V,Q) | gg^* = \mathbf{1}\};\\ \text{Spin}(V,Q) &:= \text{Pin}(V,Q)\bigcap \mathfrak{Cl}^0(V,Q).\end{aligned} \tag{31}$$

This gives the general construction of a spin group from Clifford algebra.

2.2 Double Covering[4]

By the discussion above, there are fruitful structures of Clifford algebras over $\mathbb{R}$, and here we focus on $(V,Q) = \mathbb{R}^{0,n}$ situations:

$$\text{Spin}(n) := \text{Spin}(\mathbb{R}^{0,n}) \tag{32}$$

In fact, pin groups Pin(n) (respectively, spin groups Spin(n)) are double covers of orthogonal groups $O(n) = O_n(\mathbb{R})$ (respectively, special orthogonal groups $SO(n) = SO_n(\mathbb{R})$), which can be seen from below:

Lemma 2.1. *There exists a covering map $\mathscr{A}$ of* Pin(n) *onto* $O(n)$ *with kernel* $\{\pm 1\}$*, where for any* $g \in$ Pin(n)*,* $x \in \mathbb{R}^n$*,* $(\mathscr{A}g)x = \eta(g)xg^*$*, and hence there is an exact sequence:*

$$\mathbf{1} \to \mathbb{Z}_2 \to \text{Pin}(n) \xrightarrow{\mathscr{A}} O(n) \to \mathbf{1}. \tag{33}$$

Proof. The proof will be done in three steps.

- $\mathscr{A}$ maps Pin(n) into $O(n)$:

  Let $g \in$ Pin(n) and $x \in \mathbb{R}^{0,n}$. Use the standard basis of Clifford algebra. As $gg^* = \mathbf{1}$, we can always use anticommutating relations to rewrite gxg^* into the form of a vector in $\mathbb{R}^{0,n}$. Moreover,

$$\begin{aligned}|(\mathscr{A}g)x|^2 &= \sum_{i=1}^{n}(\eta(g)x_ie_ig^*)^2 = -\sum_{i=1}^{n}(\eta(g)x_ie_ig^*)(\eta(g)x_ie_ig^*)^* \\ &= -\sum_{i=1}^{n}\eta(g)x_ie_ig^*gx_ie_i^*\eta(g)^* = -\sum_{i=1}^{n}x_i^2\eta(gg^*) \\ &= -\sum_{i=1}^{n}x_i^2 = |x|^2.\end{aligned} \tag{34}$$

$\therefore \mathscr{A}g \in O(n)$.

- $\mathscr{A}$ maps Pin(n) onto $O(n)$:

  Each element in the orthogonal group is a product of reflections, and hence it suffices to show that each reflection lies in $\mathscr{A}(\text{Pin}(n))$. Assume there is a reflection across a plane perpendicular to $x \in S^{n-1}(|x|^2 = xx^* = 1)$, denote it as r_x. Then $r_xx = -x$ and $r_xy = y$, $\forall y \perp x$. Moreover, $\eta(x)xx^* = -xxx^* = -x$, and $\eta(x)yx^* = -xyx^* = yxx^* = y$. Therefore $\mathscr{A}x = r_x$.

- ker$\mathscr{A} = \{\pm 1\}$:

  Clearly $\pm 1 \in$ ker$\mathscr{A}$, and any other element of $\mathbb{R}$ is not in ker$\mathscr{A}$. Therefore it suffices to show that ker$\mathscr{A} \subset \mathbb{R}$. Suppose that $g \in$ Pin$(\mathbb{R})$ and $\mathscr{A}g = \mathbf{1}$. Under the standard basis, we can write that $g = e_1a + b$, where a and b are linear combinations of elements in basis not containing e_1. Take $x = e_1$, then

$$\begin{aligned}(\mathscr{A}g)e_1 =& \eta(e_1a+b)e_1(e_1a+b)^* = e_1 \\ &\Rightarrow \eta(e_1a+b)e_1 = e_1(e_1a+b) \\ &\Rightarrow -e_1\eta(a)e_1 + \eta(b)e_1 = -a + e_1b \\ &\Rightarrow a + e_1b = -a + e_1b \\ &\Rightarrow a = 0\end{aligned} \tag{35}$$

  Use induction, and it can be similarly showed that g does not contain any $e_k, 1 \leq k \leq n$, and hence $g \in \mathbb{R}$.

□

Lemma 2.2. $\mathrm{Pin}(n)$ *and* $\mathrm{Spin}(n)$ *are compact Lie groups with*

$$\begin{aligned} \mathrm{Pin}(n) &= \{x_1 \cdots x_k | x_i \in S^{n-1}, 1 \le i \le k \le 2n\} \\ \mathrm{Spin}(n) &= \{x_1 \cdots x_{2k} | x_i \in S^{n-1}, 1 \le i \le 2k, 2 \le 2k \le 2n\} \\ &= \mathscr{A}^{-1}(SO(n)) \end{aligned} \tag{36}$$

Proof. Each element of $O(n)$ is a product of at most $2n$ reflections, so with lemma 2.1, the first equation has been achieved. Furthermore, as $\mathrm{Spin}(n) = \mathrm{Pin}(n) \bigcap \mathfrak{Cl}^0(n)$, $\mathrm{Spin}(n) = \{\eta(g) = g | g \in \mathrm{Pin}(n)\}$, which indicates the first half of the second equation. Finally, $\det \mathscr{A}(x_1 \cdots x_k) = (-1)^k$, and therefore $\mathrm{Spin}(n) = \mathscr{A}^{-1}(SO(n))$. □

Theorem 2.3. *(a) When* $n \geq 2$*,* $\mathrm{Pin}(n)$ *has two connected components with* $\mathrm{Spin}(n) = \mathrm{Pin}(n)^0$.

(b) $\mathrm{Spin}(n)$ *is the (simply when* $n \geq 3$*) connected (when* $n \geq 2$*) two-fold cover of* $SO(n)$*. The homomorphism is given by* $\mathscr{A}$ *with the exact sequence:*

$$\mathbf{1} \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\mathscr{A}} SO(n) \to \mathbf{1}. \tag{37}$$

Proof. (b) When $n \geq 2$, take a path: $t \to \gamma(t) = e_1(-e_1 \cos t + e_2 \sin t) = \cos t + e_1 e_2 \sin t \subset \mathrm{Spin}(n)$. While $\gamma(0) = 1$, $\gamma(\pi) = -1$, so $\{\pm 1\}$ are connected by γ. Also $\mathrm{Spin}(n)$ is the double cover of connect Lie group $SO(n)$, so $\mathrm{Spin}(n)$ is connected. For $n \geq 3$, the standard covering map of $SO(n)$ gives $\pi_1(SO(n)) = \mathbb{Z}_2$. According to the uniqueness of simply connected covering, $\mathrm{Spin}(n)$ is simply connected.

(a) Since $O(n)$ is not connected, $\mathrm{Spin}(n) = \mathscr{A}^{-1}(O(n))$ is also not connected. We know $\mathrm{Spin}(n)$ and $x_0 \mathrm{Spin}(n)(x_0 \in S^{n-1})$ are both connected components of $\mathrm{Pin}(n)$ which cover the whole pin group, so the proof has been done.

□

2.3 Spinors

Definition 2.3. Spinors are vectors that form a representation space of a spin group.

In physics, tensors are used to describe bosons while spinors are used for fermions. An important difference between spinors and tensors is that, when the space is rotated through a complete circle (2π), a spinor will transform to its negative.

Example 2.1. The Dirac algebra is a Clifford algebra. Hence in Dirac equation (Eq.(4)), ψ is in the representation space of then relevant spin group, and it is a spinor. Dirac equation is therefore used to describe fermions.

Example 2.2. In three dimensional case, there is an accidental isomorphism: Spin(3) $\cong SU(2)$. We can parametrize $SO(3)$ with Euler angles α, β, γ:

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

and there will be a natural homomorphism map from $SU(2)$ to $SO(3)$ if we use the same parameters:

$$g(\alpha,\beta,\gamma) = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{pmatrix}. \tag{39}$$

It is the double covering map, for in $SO(3)$, $\alpha \in [0, 2\pi], \beta \in [0, \pi], \gamma \in [0, 2\pi]$, while in $SU(2)$, $\alpha \in [0, 2\pi], \beta \in [0, \pi], \gamma \in [0, 4\pi]$. If we take a rotation: $\gamma \to \gamma + 2\pi$, for a vector v in $\mathbb{R}^3$, $R(0, 0, 2\pi)v = v$. However, for a spinor s, $g(0, 0, 2\pi)s = -s$.

2.4 Indefinite Signature

Indeed, for $(V, Q) = \mathbb{R}^{p,q}$ situation, the spin group is still the double cover of $SO_0(p, q)$, which is connected component of $SO(p, q)$ with identity. Physicists have concerns on especially Spin$(1, q)$, because it is under the Lorentzian signature, which describes space-time. In particular,

$$\text{Spin}(1,3) \cong SL_2(\mathbb{C}), \tag{40}$$

while weights of $sl_2(\mathbb{C})$ are just possible observation results of angular momentum or spins.

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