On the Controversy of Bloom Filter False Positives - An Information Theoretical Approach to Optimizing Bloom Filter Parameters

***Abstract*—Although Bloom Filters (BF) have been widely used in many networking applications and beyond, the fundamental issue of how to calculate the false positive probability remains elusive. Properly calculating the false positive probability of BF is critical because it is used to calculate the optimal number of hash functions k. Since Bloom gave the false positive formula in 1970, in 2008, Bose *et al.* pointed out that Bloom's formula for calculating the false positive probability is flawed and gave a new false positive formula; and in 2010, Christensen *et al.* further pointed out that Bose's formula is also flawed and gave another new false positive formula. Although Christensen's formula is perfectly accurate, it is time-consuming to calculate the false positive probability, and it is impossible to calculate the derivation of the optimal value of k. While the conventional wisdom is to derive the optimal value of k from a false positive formula, in this paper, we propose the first approach to calculating the optimal k without any false positive formula. Our approach is based on the following observation: for a BF with m bits and n elements, if and only if its entropy is the largest, its false positive probability is the smallest, according to information entropy theory. Furthermore, we propose a new and more accurate upper bound for the false positive probability. When the size of a Bloom filter becomes infinitely large, our upper bound turns equal to the lower bound, which becomes Bloom's formula. This deepens our understanding of Bloom's formula: it is perfectly accurate when m is infinitely large, and it is practically accurate when m is sufficiently large.**

I. Introduction

A. Motivation

A Bloom Filter (BF) is a compact data structure used for quickly checking whether an element belongs to a set or not [1]. Given a set S of n elements, we create a bit array A of length m as follows. First, we initialize each bit of A to 0, then for each element $x \in \mathcal{S}$, we use k hash functions to compute k hash values: $h_1(x), h_2(x), ..., h_k(x)$ where each hash value is in the range $[1, m]$. Second, for each $1 \leqslant i \leqslant k$, we let $A[h_i(x)] = 1$. The resulting bit array A is called the BF for set S. To query whether $y \in S$, we first use the same k hash functions to compute k hash values: $h_1(y), h_2(y), ..., h_k(y)$. Second, we check whether the corresponding k bits in A are all $1s$ (*i.e.*, whether $A[h_1(y)] \wedge A[h_2(y)] \wedge \cdots, A[h_k(y)] = 1$ holds); if yes, then $y \in S$ may probably hold and we can further check whether $y \in S$; if no, then $y \in S$ definitely does not hold. The cases that the BF shows that $y \in S$ may hold but actually $y \notin S$ are called false positives (FP). The FP probability f can be calculated from n, k, and m. Thus, given a set of n elements and the required FP probability f, we can calculate the relationship between k and m. Based on the calculated relationship between k and m, we can properly trade off between space and speed: smaller m means smaller space, and smaller k means the smaller number of hash function calculations. In typical BF applications, as m is determined by the memory budget for the BF, with known values of n and f, we calculate the optimal value for the only unknown parameter k.

As set membership query is a fundamental operation in many networking applications, and BFs have the advantages of small memory consumption, fast query speed, and no false negatives, BFs have been widely used in a wide variety of networking applications, such as web caching [2], [3], IP lookup [4]–[7], packet classification [8], [9], regular expression matching [10], [11], multicast [12], [13], content distribution networks [14], [15], routing [16], [17], P2P networks [18], [19], overlay networks [20], [21], name based networks [22], [23], queue management [24], [25], Internet measurement [26], [27], IP traceback [28], [29], sensor networks [30], [31], data center networks [32]–[34], cloud computing [35], [36], cellular networks [37], [38], and more [39], [40]. Most applications with set membership query can potentially be optimized using BFs.

Although BFs have been widely used in many applications, the fundamental issue of how to calculate FP probability remains elusive. Properly calculating the FP probability of BF is critical because it is used to calculate the optimal value of the important parameter k, the number of hash functions. In [1], Bloom gave a formula for calculating the FP probability with known parameters n, k, and m. Based on Bloom's formula, we can also easily compute the optimal value of the parameter k when m and n are known. This formula has been believed to be correct until 2008 when Prosenjit Bose *et al.* pointed out that Bloom's formula is flawed and gave a new FP formula [41]. Interestingly, two years later, Ken Christensen *et al.* pointed out that Bose's formula is also flawed and gave a new FP formula [42]. So far, it is believed that Christensen's formula is perfectly accurate. However, Both Bose's and Christensen's FP formulas are too complicated to calculate the optimal value of k from given values of n and m.

B. Main Contributions

While the conventional wisdom is to derive the optimal value of BF parameter k from the FP probability, in this paper, we propose the first approach to calculating the optimal k without any FP formula. We first observe that for a BF

with m bits and n elements, if and only if its entropy is the largest, its false positive probability is the smallest, according to information entropy theory. Based on this observation, our approach is to derive a formula for calculating the optimal k by letting the entropy equal to 1.

Furthermore, we propose another method to calculate FP probability. First, we derive the left and right limit expressions of FP probability. Second, we prove that when m goes to infinity, the left and right limits are the same, which is essentially the FP probability. Interestingly, our derived FP formula is the same as Bloom's formula in [1]. This deepens our understanding of Bloom's formula: it is perfectly accurate when m is infinitely large, and it is practically accurate when m is sufficiently large.

In summary, we make three key contributions in this paper. First, we propose an information theoretical approach to calculating the optimal value of BF parameter k without calculating FP probability. Second, we propose a new upper bound which is much more accurate than state-of-the-art. When m is infinitely large, our upper bound becomes the same as the lower bound. This result formally proves that Bloom's formula is practically accurate when m is sufficiently large. Third, we conducted experiments to validate our findings. In particular, we show that the error of Bloom's formula is negligibly small when m is large.

The rest of this paper proceeds as follows. In Section II, we introduce the controversy on FP probability. In Section III, we show the derivation of the optimal number of hash functions using the information entropy theory. In Section IV, we present a new upper bound of the false positive probability of Bloom Filters. In Section V, we conduct experiments to evaluate the error of Bloom Filters. We conclude the paper in Section VI.

II. Prior Art on BF False Positives

In this section, we review prior art on calculating the false positive probability of Bloom filters. Table I summarizes the notations used in this paper.

TABLE I
NOTATIONS AND ABBREVIATION USED IN THIS PAPER

Symbol	Description
$\mathcal{S}$	Set of elements
m	BF size
n	Number of elements in $\mathcal{S}$
k	Number of hash functions
k^*	Optimal number of hash functions
f	False positive probability
f_{bloom}	False positive probability calculated by Bloom
f_{bose}	False positive probability calculated by Bose
f_{christ}	False positive probability calculated by Christensen
f_{true}	True false positive probability of BF
FP	false positive
BF	Bloom filter

A. Bloom's False Positive Formula

In 1970, Bloom calculated the false positive probability of a Bloom filter as follows [1]. Given a set $\mathcal{S}$ of elements, let n be the number of elements in $\mathcal{S}$, k be the number of hash functions, and m be the number of bits in the Bloom filter A constructed from set $\mathcal{S}$. In querying an element x, the false positive happens when the Bloom filter reports that $x \in \mathcal{S}$ (*i.e.*, $A[h_i(x)] = 1$ holds for each $1 \leqslant i \leqslant k$), but actually $x \notin \mathcal{S}$. Consider an arbitrary bit $A[b]$ in A. For any element in $\mathcal{S}$ and any hash function h_i ($1 \leqslant i \leqslant k$), the probability that this element is not hashed to bit $A[b]$ by h_i is $1 - 1/m$. As $\mathcal{S}$ has n elements and each element is hashed k times, the probability of $A[b] = 0$ is p':

$$p' = \left(1 - \frac{1}{m}\right)^{kn} \tag{1}$$

thus, the probability of $A[b] = 1$ is $1 - (1 - 1/m)^{kn}$. For any element $x \notin \mathcal{S}$, the probability that the false positive happens for x, *i.e.*, the probability of $A[h_1(y)] \wedge A[h_2(y)] \wedge \cdots, A[h_k(y)] = 1$, is calculated as follows:

$$f_{bloom} = \left(1 - \left(1 - \frac{1}{m}\right)^{kn}\right)^{k} \tag{2}$$

This formula can be approximated by $(1 - e^{\frac{-nk}{m}})^k$, based on which we can trade off between space indicated by m and time indicated by k.

B. Bose's Derivation

In 2008, Bose *et al.* pointed out that the last step in Bloom's derivation is flawed because for any element $x \notin \mathcal{S}$, which is hashed into k bits $A[h_1(x)], A[h_2(x)], \cdots, A[h_k(x)]$, the k events $A[h_1(x)] = 1, A[h_2(x)] = 1, \cdots, A[h_k(x)] = 1$ are not actually independent [41]. Although for each bit $A[h_i(x)]$ ($1 \leqslant i \leqslant k$), after inserting n elements into array A, the probability of $A[h_i(x)] = 1$ is $1 - (1 - 1/m)^{kn}$, for the probability of $A[h_1(y)] \wedge A[h_2(y)] \wedge \cdots, A[h_k(y)] = 1$ to be $(1 - (1 - 1/m)^{kn})^k$, the k events $A[h_1(x)] = 1, A[h_2(x)] = 1, \cdots, A[h_k(x)] = 1$ need to be independent. We now analyze the reason that the k events $A[h_1(x)] = 1, A[h_2(x)] = 1, \cdots, A[h_k(x)] = 1$ are not independent. Let us first consider the probability of $A[h_1(x)] = 1$, which is $1 - (1 - 1/m)^{kn}$. Second, we consider the probability of $A[h_2(x)] = 1$ based on the condition that $A[h_1(x)] = 1$. There are two possibilities for $h_2(x)$: (1) $h_2(x) \neq h_1(x)$, and (2) $h_2(x) = h_1(x)$. For the first case, the probability of $A[h_2(x)] = 1$ is $1-(1-1/(m-1))^{kn}$. For the second case, the probability of $A[h_2(x)] = 1$ is 1. Similarly, we can analyze the $A[h_3(x)] = 1$ based on the condition that $A[h_2(x)] = 1$ and $A[h_1(x)] = 1$, etc.

Observing the dependency of the k events $A[h_1(x)] = 1, A[h_2(x)] = 1, \cdots, A[h_k(x)] = 1$, Bose *et al.* derive the following false positive formula:

$$f_{bose} = \frac{1}{m^{k(n+1)}} \sum_{i=1}^{m} i^k i! \binom{i}{m} \left(\frac{1}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} j^{kn} \right) \tag{3}$$

这里的p=1-p'

Bose *et al.* derived asymptotically closed forms for the upper and lower bounds of the above formula:

$$f_{bloom} < f_{bose} \leqslant f_{bloom} \times \left(1 + \mathcal{O}\left(\frac{k}{p}\sqrt{\frac{\ln m - k \ln p}{m}}\right)\right) \tag{4}$$

These bounds hold under the condition that

$$\frac{k}{p}\sqrt{\frac{\ln m - k \ln p}{m}} \leqslant c \tag{5}$$

for some constant $c < 1$. Bose *et al.* further showed that for $k \geqslant 2$, f_{bose} is strictly larger than f_{bloom}, and the lower bound converges to f_{bloom} when m becomes infinitely large.

C. Christensen's Derivation

In 2010, Christensen *et al.* pointed out that Bose's formula has a mistake that the term $(-1)^j$ should be $(-1)^{i-j}$. Christensen *et al.* derived the finally correct false positive formula for Bloom filters as follows:

$$f_{christ} = \frac{m!}{m^{k(n+1)}} \sum_{i=1}^{m} \sum_{j=0}^{i} (-1)^{i-j} \frac{j^{kn} i^k}{(m-i)!j!(i-j)!} \tag{6}$$

Although Christensen's formula is perfectly accurate, it is not much useful in practice. First, given the Bloom filter parameters n, m, and f, it is difficult to calculate the optimal k value as Christensen's formula does not give a closed form expression for calculating the optimal k. Second, given the Bloom filter parameters n, m, and k, the algorithm by Christensen *et al.* takes $\mathcal{O}(knm)$ time to calculate the false positive probability f, which is time-consuming.

III. Computing the Optimal k

Classically, the optimal number of hash functions is derived through finding the extrema of the asymptotic formula of f_{bloom} given in Eq. 2 as :

$$k^*_{bloom} = \frac{m}{n} \ln 2 \tag{7}$$

However, as we said, the underlying formula for FP probability is not fully correct. In this section, we first show the theory of information entropy, then we propose the important theorems which links information entropy and the false positive rate of Bloom filters. Third, we propose a novel method of deriving the optimal value of k, minimizing the FP probability given the value of BF size m and number of inserted elements n.

A. Information Entropy Basis

Information Entropy: In information theory, *information entropy* is used to measure the uncertainty of a random variable. In this paper, it refers to the *Shannon entropy* [43], which measures the value of the information contained in a variable. Entropy is typically measured in bits, nats, or bans [44]. For a variable with s events with the probabilities of p_1, p_2, ..., p_s. The information entropy E is defined as:

$$E = \sum_{i=1}^{s} p_i log_2 \frac{1}{p_i} \tag{8}$$

Property of information entropy: For any variable or message, if its information entropy is not at the maximum, it can be compressed without information loss. Suppose a m-bit string variable is compressed to m'-bit string variable, the entropy becomes E' from E. Then we have $mE = m'E'$.

According to the information entropy formula (Eq. 8), we can obtain the information entropy E of the m-bit variable of a BF:

P'是啥，上一页的概念，有点远
这是怎么推出来的：将"任意一个bit为0"刻画为一个事件，然后才能推出这个公式

$$E = -\left(p' \log_2 p' + (1-p') \log_2 (1-p')\right) \tag{9}$$

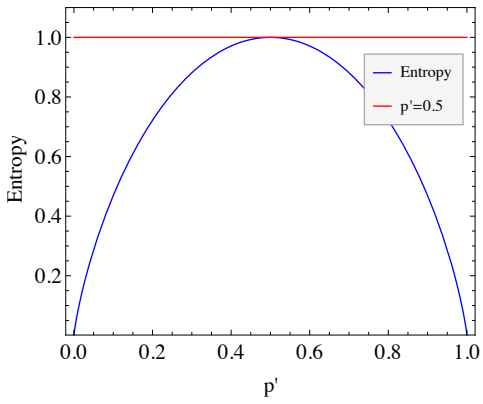


Fig. 1. Information entropy of a BF

To be more vivid, we draw the value of E with the increase of p' in Figure 1. Obviously, the maximum value of entropy of an m-bit sequence is obtained when the probability that each bit in the BF is 1 (or 0) is 0.5.

B. Relation Between False Positive Probability and Information Entropy

In this section, we show the relation between false positive probability and information entropy. To better understand it, we first present a lemma and several definitions.

Lemma I: Given a BF, suppose n and k keep unchanged, when m becomes larger, the FP probability gets smaller.

Proof. This lemma is obviously correct. When n and k are fixed, for larger m, the probability of each bit being 1 in the BF becomes smaller, *i.e.*, p' becomes smaller, thus the FP probability gets smaller. □

Definition I: *Bloom filter variable.* The false positive rate of Bloom filters is determined by m, n, and k. For different n elements, the m-bit string varies. Thus when the values of m, n, k are given, the m-bit string is a *random variable*. When the n elements are given, the m-bit string is a random variable instance. Therefore, when the values of m, n, k are given, we call it a Bloom Filter Variable (BFR). Since it is a random variable, we can compute its information entropy.

Definition II: *Equivalent Bloom filter variables.* Given two Bloom filters variables v_1 and v_2, for the same n elements, there are a pair of BFR instances. Given a set with n elements,

if these pairs of BFR instances always report the same result: true or false for any inputting element, we say v_1 and v_2 are equivalent.

Theorem I: Given a Bloom filter *variable* v_1 with parameters m, n, and k, if its information entropy is not at the maximum, there must exist a smaller equivalent Bloom filter variable v_2 with parameters m', n and k, where $m' < m$.

Proof. For v_1, the parameters are m, n, k. Suppose its k hash functions are $h_1(\cdot), h_2(\cdot), \ldots, h_k(\cdot)$. Since the assumption is that the information entropy of v_1 is not at the maximum, according to the property of information entropy, v_1 can be compressed *without information loss*. After compression, suppose the new random variable has a length of $m'(m' < m)$[1], we name it v_2.

v_1 and v_2 are two variables consisting of bits, and we can regard them as two integer variables. We use $In(v_1)$ and $In(v_2)$ to represent the integer value of v_1 and v_2. Furthermore, we use $|In(v_1)|$ represents the length of v_1, then we have $|In(v_1)| = m$, $|In(v_2)| = m'$. Because we compress v_1 and get v_2, this can be regarded as a function $g(\cdot)$. In other words, $g(In(v_1)) = In(v_2)$. We can also obtain v_1 by equation $In(v_1) = g^{-1}(In(v_2))$.

At this stage, we consider the new Bloom filter variable v_2, the parameters are m', n, and k. Note that we use k `different` hash functions, and the k hash functions are

$$\begin{aligned} &g^{-1}(In(v_2)) \ll h_1(y) \gg (|g^{-1}(In(v_2))| - h_1(y) - 1),\\ &g^{-1}(In(v_2)) \ll h_2(y) \gg (|g^{-1}(In(v_2))| - h_2(y) - 1),\\ &\ldots\\ &g^{-1}(In(v_2)) \ll h_k(y) \gg (|g^{-1}(In(v_2))| - h_k(y) - 1) \end{aligned} \tag{10}$$

Where '$\ll$' means left shift and '$\gg$' means right shift.

Given an inputting element y, we can compute above k values ONLY using v_2 and $h_i(\cdot)$ without v_1. Then we need to prove that for any incoming element y, v_2 reports the same k-bit value. Let focus on formula 10, we use equation $v_1 = g^{-1}(In(v_2))$ and $m = |g^{-1}(In(v_2))|$, these k hash functions are simplified as

$$\begin{aligned} &v_1 \ll h_1(y) \gg (m - h_1(y) - 1),\\ &v_1 \ll h_2(y) \gg (m - h_2(y) - 1),\\ &\ldots\\ &v_1 \ll h_k(y) \gg (m - h_k(y) - 1) \end{aligned} \tag{11}$$

Here $1 \leqslant i \leqslant k$ and $0 \leqslant h_i(y) \leqslant m - 1$. $v_1 \ll h_i(y) \gg (m - h_i(y) - 1)$ actually means the value of $h_i(y)$-th bit of v_1. This is the same as the k hash functions as v_1. Therefore, v_1 and v_2 are equivalent. □

Theorem II: Given a Bloom filter variable, *if and only if* its information entropy is at the maximum, its FP probability is at the minimum.

[1]Note that during the compression, the information value $m * E$ keeps unchanged, while $E's$ maximum value is 1, thus the length of the compressed message has a minimum value.

Proof. First, we prove "if the FP probability is at the minimum, then its information entropy must be at the maximum".

Given a Bloom filter variable v_1 with parameters m, n, k. Since the assumption is that the information entropy of v_1 is not at the maximum, according to Theorem I, there exists a smaller Bloom filter variable v_2 with parameters m', n, k. Since v_2 and v_1 are equivalent, their FP probabilities are the same, we name it f. At this stage, we enlarge the size of v_2 a little from m' to m'', where $m' < m'' < m$. According to Lemma I, we know the FP probability of v_2 becomes smaller than f. This means for v_1, there exists a BF variable with smaller size, but with a smaller FP probability. Therefore, the FP probability of v_1 is not at the minimum.

This means that if its information entropy is not at the maximum, then the FP probability is definitely not at the minimum. The contrapositive is: if the FP probability is at the minimum, then its information entropy must be at the maximum.

Second, we prove "if its information entropy is at the maximum, the FP probability is at the minimum".

Given a Bloom filter variable v_1 with parameters m, n, k. Since the assumption is that the FP probability of v_1 is not at the maximum, there exists an optimal Bloom filter variable v_0 with parameters m, n, k', where $k' \neq k$. According to ←, the information entropy of v_0 is at the maximum. According to Eq. 9 and Figure 1, the p' of v_0 is 0.5, while the p' of v_1 is not, because they have different value of k. Therefore, the information entropy of BF_1 is not at the maximum.

This means if the FP probability is not at the minimum, its information entropy must be not at the maximum. The contrapositive is if its information entropy is at the maximum, the FP probability is at the minimum. □

C. Computing the Optimal k

According to Theorem II, when the information entropy of the Bloom filter variable is at the maximum, the FP probability is at the minimum. Recalling the definition of p' in Eq. 1, one can use this interpretation to find k^*, *i.e.*, the optimal number of hash functions. From Figure 1, we know that when p' is 0.5, E reaches the maximum value 1. By setting the value of p' to 0.5:

$$p' = \left(1 - \frac{1}{m}\right)^{k^* n} = 0.5 \tag{12}$$

which is derived previously with the first independence assumption that is not disputed, we obtain:

$$k^* = -\frac{\ln 2}{n} / \ln\left(1 - \frac{1}{m}\right) \tag{13}$$

This formula is very close to the formula of k^* obtained by Bloom. As we have, as for small values of x, the approximation $\ln(1 + x) \approx x$, and therefore $-1/\ln(1 - \frac{1}{m}) \approx m$ yielding the same term as in Eq. 7.

Theorem III: Given any BF variable, when m and n are fixed, the FP probability f is a function of k, we represent it $f(k)$. We claim that $f(k)$ is a *convex function*.

Note that in this paper a convex function means that it has only one minimum value.

Proof. Given a Bloom filter variable v_1 with parameters m, n, k_1, its entropy is E_1. Given another Bloom filter variable v_2 with parameters m, n, k_2, its entropy is E_2.

1) For any $k_1 < k_2 \leqslant k^*$, according to Eq. 1, Eq. 9 and Figure 1, we known $p'_1 < p'_2 \leqslant 0.5$. We compress v_1 to v_3 with parameters m_3, n, k_1. In order to make v_3's entropy equal to E_2, m_3 should be mE_1/E_2. In this case, the entropy of v_3 is equal to that of v_2. When the entropy of BF variables is less than 0.5, same entropy leads to same p'. In other words, $p'_3 = p'_2$. Because v_2 has more hash functions ($k_2 > k_1$), thus the FP probability of v_2 is smaller than that of v_3. While v_3 and v_1 have the same FP probability, therefore, the FP probability of v_2 is smaller than that of v_1. In other words, for any $k (k < k^*)$ increasing, the FP probability of BFs decreases.

2) For any $k^* \leqslant k_1 < k_2$, according to Eq. 1, Eq. 9 and Figure 1, we known $p'_1 > p'_2 \geqslant 0.5$. Using the similar derivation, we can derive that the FP probability of v_2 is larger than that of v_1.

According to the above two cases, we know that given any BF variable, when m and n are fixed, the FP probability f is a function of k, we represent it $f(k)$. $f(k)$ is a *convex function*. □

IV. Asymptotic Form of the FP Probability

We will derive a new and very simple approach to computing asymptotic form for the FP probability of BFs. The new derivation is based on *partitioned Bloom Filters* (pBF) that are used frequently to carry out parallel queries. Its principle is simple: the BF is divided into k even partitions, and each hash function only acts on one of the partitions, respectively. The probability that one bit of the BF array remains 0 after inserting n elements in the BF becomes

$$p'_{partition} = \left(1 - \frac{k}{m}\right)^n \tag{14}$$

需不需要说原因啥的，比如hash的random

as now each hash maps into $\frac{m}{k}$ separate bits.

It is intuitive that the FP probability of partitioned BF is a little bigger than that of BF. Unfortunately, there is no strict proof. Here we show one proof method, which is based on the following Lemma.

Lemma II: For $m > 1$, $k > 1$, $n > 1$, $m > k$,

$$\left(1 - \frac{k}{m}\right)^n < \left(1 - \frac{1}{m}\right)^{kn} \tag{15}$$

This formula always holds. Note that when m is large, the left expression approximate to the right one. Below we give the derivation details.

Proof. Let $g(k) = (1-\frac{1}{m})^k - (1-\frac{k}{m})$. For $m > 1$ and $k > 1$, $g(k)$ is a continuous and derivable function, and we can obtain the following inequality in terms of its derivative:

$$g'(k) > \left(1 - \frac{1}{m}\right) \ln\left(1 - \frac{1}{m}\right) + \frac{1}{m} \tag{16}$$

Let $f(m) = \left(1 - \frac{1}{m}\right) \ln\left(1 - \frac{1}{m}\right) + \frac{1}{m}$, we can find $f'(m) = \frac{1}{m^2} \ln\left(1 - \frac{1}{m}\right) < 0$, meaning that the function $f(m)$ is strictly decreasing. When m goes to infinity, we have $\lim_{m\to\infty} f(m) = 0$. Therefore, we know that $f(m) \geqslant 0$, and $g'(k) > f(m) \geqslant 0$, meaning that the function $g(k)$ is strictly increasing. We have therefore

$$\left(1 - \frac{k}{m}\right)^n < \left(1 - \frac{1}{m}\right)^{kn} \tag{17}$$

□

The above lemma shows that $p'_{partition} < p'_{true}$ or equivalently $1 - p'_{partition} > 1 - p'_{true}$. Thus, we know that with the same parameters, the FP probability of the pBF will be larger than that of the standard BF f_{true}, *i.e.*, $f_{partition} > f_{true}$. In addition, Christensens bounds in Eq. 4 state that the precise value of FP probability for a BF f_{true} is larger than f_{bloom}, *i.e.*, $f_{true} > f_{bloom}$. Therefore, we have the following upper and lower bounds:

似乎不对：这个lower bound是怎么出来的?

$$f_{partition} > f_{true} > f_{bloom} \tag{18}$$

For partitioned BF, the probability that one bit of the array is still 0 p' is shown in Eq. 14. Different from standard Bloom Filter, for partitioned Bloom Filter, the event "$E(h_1 = 1), E(h_2 = 1), E(h_3 = 1), ..., E(h_{i-1} = 1)$" is independent with the event "$E(h_{i-1} = 1)$", where $E(h_{i-1} = 1)$ means the event that the position of $h_{i-1}(x)$ is 1, because each hash function is responsible for one partition, and has no effect with each other. Therefore,

$$f_{partition} = (1 - p'_{partition})^k = \left(1 - \left(1 - \frac{k}{m}\right)^n\right)^k \tag{19}$$

Then the formula 18 becomes

$$\left(1 - \left(1 - \frac{k}{m}\right)^n\right)^k > f_{true} > \left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k \tag{20}$$

Then we use the well known limit formula:

$$\lim_{x\to\infty} \left(1 - \frac{1}{x}\right)^{-x} = e \tag{21}$$

Asymptotically when m becomes large, we already know that f_{bloom} converges to the term in Eq. 2. Nevertheless, the upper bound has also an asymptotic behaviour as :

$$\lim_{m\to\infty} \left(1 - \left(1 - \frac{1}{m}\right)^{nk}\right)^k = \left(1 - e^{-nk/m}\right)^k \tag{22}$$

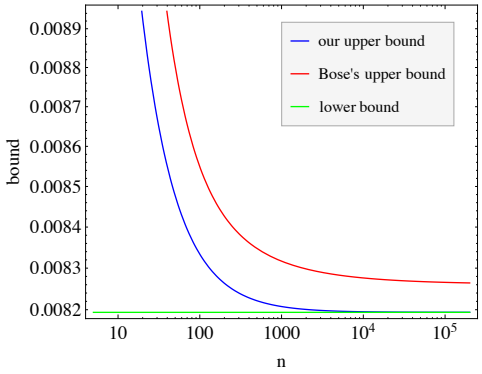


Fig. 2. Upper and lower bound for f_{true} for $k = 7$ and $m = 10n$.

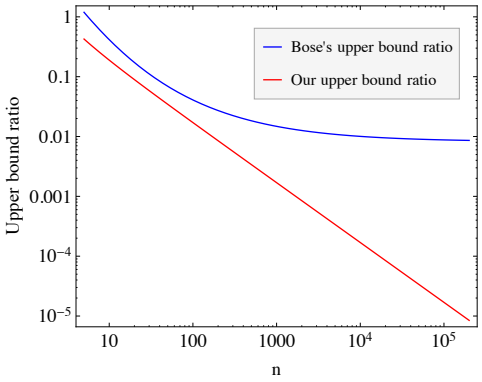


Fig. 3. Bounds error ratio for Bose's bound and the bound derived in this paper for $k = 7$ and $m = 10n$.

that is the same term as the lower bound limit. Through the Sandwich Theorem (also known as squeeze theorem) we get, similarly to Christensen and to Bose, that :

$$\lim_{m\to\infty} f_{true} = \left(1 - e^{-nk/m}\right)^k \tag{23}$$

This means that when m is large, the Bloom's formula can be used with negligible error. However, we still need to evaluate what means m *being large*. We will do this by comparing the two bounds we have in hand: the one from Bose and the one we derived in this paper.

We show in Figure 2 the two upper bounds along with the lower bound obtained for $k = 7$ and $m = 10n$ as a function of n, the number of elements inserted in the BF. As can be seen, the upper bound derived in this paper and the lower bound f_{bloom} converge relatively fast for $n = 9$, while the upper bound derived by Bose has a much slower convergence. We can see this better by looking at the behavior of the bounds error ratio β, defined as $\beta = \frac{upperbound - lowerbound}{lowerbound}$, for the two bounds in Figure 3. As can be seen, the gap between our derived upper bound and f_{bloom} is decreasing polynomially at a constant speed, while Bose's bound has a lower speed of convergence. In order to extend this observation, we show

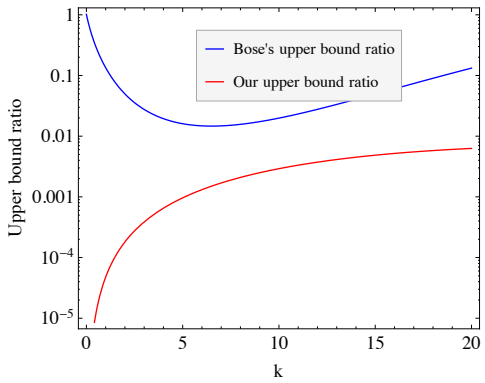


Fig. 4. Bounds error ratio for Bose's bound and the bound derived in this paper for $m = 10000$, $n = 1000$ and varying k.

in Figure 4 the evolution of the bounds error ratio for a BF with $m = 10000$, $n = 1000$ and varying k. As expected, error involved with using f_{bloom} increases with the number of hash functions k increases. However, it can be seen that the convergence behavior of the bounds derived in this paper is much better than the one obtained by Bose.

V. Experimental Results

The goal of this section is to validate the accuracy and correctness of our proposed bound and formula. In this section, we first validate the formula f_{bloom}, and then validate our proposed formula of optimal k – equation 13.

A. Experimental Setup

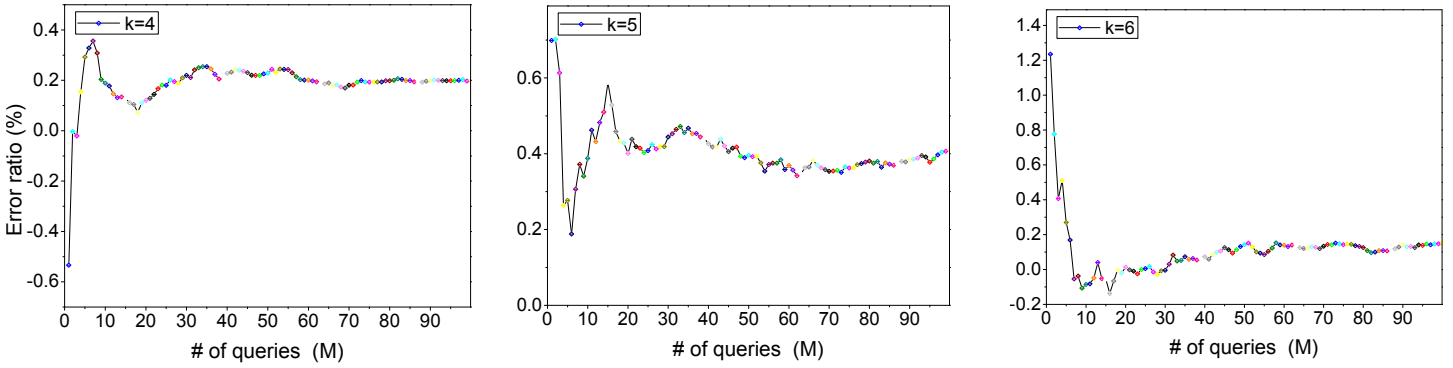
In order to evaluate experimentally the FP probability of a set of entries, we wrote a C++ program to generate a large set of unique entries with 20 characters in each. It takes around 1 hour on an Intel(R) Core(TM) i7-920 with a 4-core computer working at 2.67GHz to generate 100M unique entries and more than 3 hours to get 300M entries which occupy more than 6GB memory. These two sets are used to evaluate FP probability and optimal values of k. We collect 20 hash functions, mainly from [45], and the name of hash functions are shown in Table II. We choose different hash functions for the experiments several times, and the experimental results show little differences and thus we only show the experimental results when using the first k hash functions.

TABLE II
HASH FUNCTIONS USED IN OUR EXPERIMENTS.

1: APHash	2: BKDR	3: BOB	4: CRC32	5: DEKHash
6: DJBHash	7: FNV32	8: Hsieh	9: JSHash	10: OCaml
11: OAAT	12: PJWHash	13: RSHash	14: SBOX	15: SDBM
16: Simple	17: SML	18: STL	19: MD5	20: SHA-1

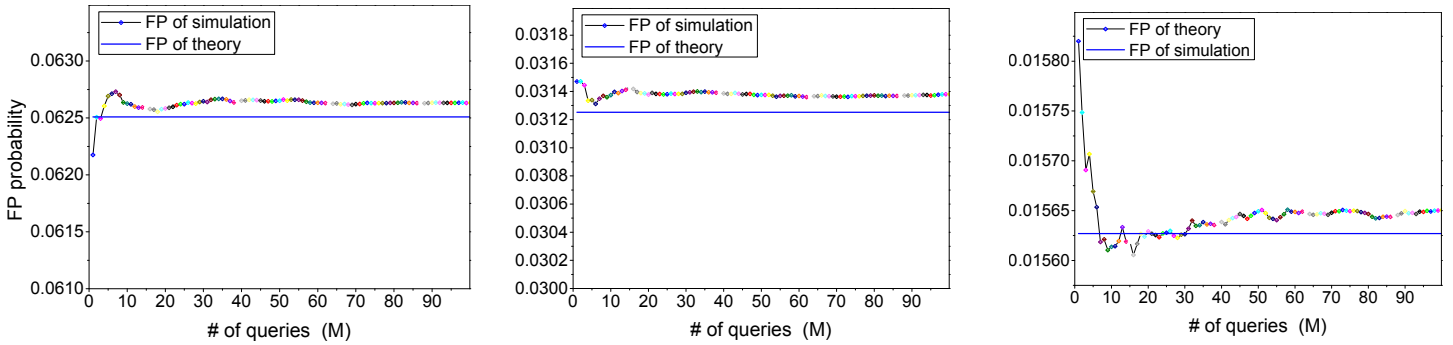
B. False Positive Probability Test

1) Basic Test: Our experimental results show that the error ratio shows that the experimental results of FP probability follow the predictions of f_{bloom} very well. In this experiment,



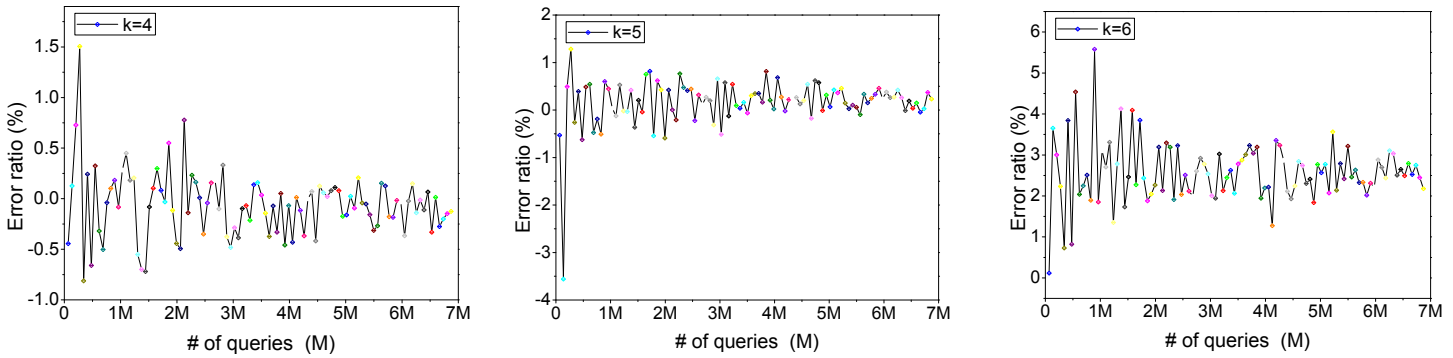
(a) k=4 (b) k=5 (c) k=6

Fig. 5. FP probability error ratio vs. # of queries with different k.



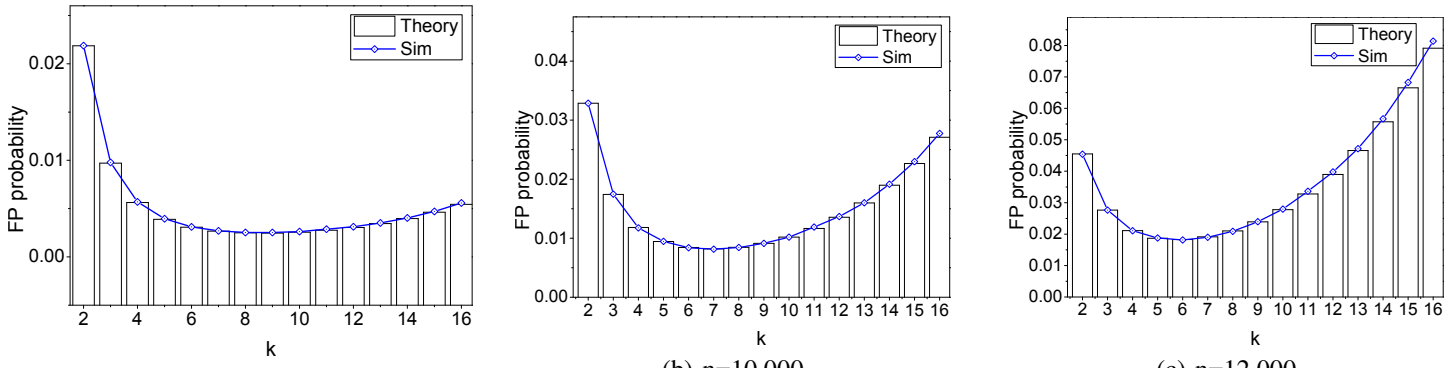
(a) k=4 (b) k=5 (c) k=6

Fig. 6. FP probability error vs. # of queries with different k.



(a) k=4 (b) k=5 (c) k=6

Fig. 7. FP probability error vs. # of queries with independent queries.



(a) n=8,000 (b) n=10,000 (c) n=12,000

Fig. 8. Variation of FP probability as a function of the number of hash functions k for $m = 100,000$ and different number of inserted entries n

TABLE III
FALSE POSITIVE PROBABILITY IN THEORY AND SIMULATION.

k	# of FP	FP from f_{bloom}	FP of simulation	Error ratio (%)
2	24814826	0.25003480	0.24814830	-0.7545290
4	6263312	0.06250841	0.06263312	0.1995030
6	1565056	0.01562703	0.01565056	0.1505790
8	392198	0.00390674	0.00392198	0.3901310
10	95314	0.00097668	0.00095314	-2.4102060
12	24946	0.00024417	0.00024946	2.1670100
14	6166	0.00006104	0.00006166	1.0125530
16	1528	0.00001526	0.00001528	0.1283920

we use the 100M-entries file described above. Each round of experiments is done with a given value of k chosen as an even number from 2 to 16. We insert the first 5000 (n = 5000) elements into the Standard Bloom Filter (SBF), and then calculate the optimal m according to the formula 13, and then get:

$$m = \frac{1}{1 - 2^{-\frac{1}{kn}}} \tag{24}$$

In order to evaluate the FP probability, we query 100M elements from the BF we built, and the query results are shown in Table III. We show experimental results of FP probability of BF as well as the FP probability using the formula of f_{bloom}. It can be seen that as k increases from 2 to 16, the number of false positives decreases from 24814826 to 1528 and the error ratio is 2.41% at most.

2) Error ratio vs. Querying number: We build the same SBF as described previously and use the same 100M-traffic file. We run experiments when k is 4, 5, and 6, respectively. n is still 5000, and m is calculated to be the optimal number according to formula 24. We query 1M up to 100M elements and calculate the error ratio when every next 1M elements are queried.

The results are shown in Figure 5. It shows that at the very beginning, the error ratio is much bigger and also with drastic fluctuation. The more elements we query, the stabilizer the error ratio will be. When the number of false positives in a test is small, the test results can hardly be reliable. In our test, the smallest number of false positives is 15,820, which is the first spot in Figure 5c, when k is 6 and query number is 1M. All in all, the error ratio is actually so small that it can be ignored. They are ranging from -0.533% to +0.356% when $k = 4$, ranging from +0.188% to +0.699% when $k = 5$, and ranging from -0.138% to +1.235% when $k = 6$.

To give a more intuitive result, we plot Figure 6. The blue line is the theoretical FP probability, while the other curve is the evaluated FP probability (defined as FP number/total query number). That error ratio stabilizes with the increase of the querying number still holds.

To give a whole picture, we carry out another experiment. In this experiment, we query the first n elements and sum up the FP number. And then, we pick up the latter n elements and also get the FP number. These two groups of elements are totally different without duplicate elements across the groups. Thus, we query up to $100 * n$ elements and get 100 spots. The 100M-element traffic file is not large enough for this experiment, so we use the 300M-element traffic file which contains more than 300M elements we described in the experimental setup section.

As shown in Figure 7, the smallest number of false positives is 1,075, which is still in the first spot in Figure 7c, when k is 6 and this time query number is 68,712. Although it is smaller than the previous accumulated experimental results, it is still bigger than 1,000, so the result is reasonable. By the way, the 100th query number is $100 * 68712 = 6,871,200$ and the number of FP is 109,713, which is large enough.

It can be seen from Figure 7, the error ratios fluctuate drastically at the beginning and quickly converge to a small range. And also, the error ratios are very small ranging from -0.812% to +1.500% when $k = 4$, ranging from -3.559% to +1.284% when $k = 5$, and ranging from +0.115% to +5.581% when $k = 6$.

C. Optimal k Formula Validation

Our experimental results show that our proposed formula of optimal k is accurate. In this experiment, we validate that the formula Eq. 13 derived for k^*, gives a value minimizing the FP probability. For this purpose, we create large Bloom filters with $m = 100,000$ and with different k varying from 2 to 16, and then insert $n = 8000$, 10000, and 12000 entries into it, respectively. We use the same 100M-entries to evaluate the FP probability. We show the results in Figure 8. We also show the value of FP probability derive using the formula f_{bloom}.

It can be seen in the figure 8 for the given value of m, n, and k, the experimental results and the theoretical values of FP probability match well. Moreover, when using formula in Eq. 13, on the third scenario that is $n = 12,000$, one can derive an optimal value of $k^* = 6$. As shown in Figure 8, it can be seen that this is indeed the value minimizing the FP probability to be 1.816%. Similar observations are made for $n = 8000$ and $10,000$.

It can be seen from Figure 8 that the optimal ks are totally identical between the theoretical and practical results. For example, when $m = 100,000$, $n = 12,000$ and the best k should be 5.78 obtained from our formula 13. Since k is an integer, we set $k = 6$. In this experiment, when k is 6, the FP probability is the lowest, which is 1.816% in the 15 spots ranging from 2 to 16. The same logical also applied to other cases when n is 8,000 and 10,000.

VI. CONCLUSION

In this paper, we discuss the controversy of the formula of Bloom Filter. Three formulas of false positive probability are presented: Bloom's formula, Bose's formula, and Christensen's formula. There is an error in the deduction process of Bloom's formula, and a minor error in Bose's formula. Christensen's formula is correct, but the false positive must be caculated using an iterative table-based algorithm with a time complexity of $O(knm)$. What is worse, it cannot deduce the optimal value of k.

To compute the optimal value of k, we use information and entropy theory to deduce the exact formula of k. To compute the false positive probability, 1) For small m, Christensen's formula can be used; 2) For large m, we propose a new upper bound which is much more accurate than state-of-the-art. Fortunately, when m is infinitely large, our upper bound becomes the same as the lower bound, which is Bloom's formula.

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