Chapter 8

Problems

1. $P\{0 \le X \le 40\} = 1 - P\{|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let Z be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i / n - 75\right| > 5\right\} \approx P\{|Z| > \sqrt{n}\} \le .1 \text{ when } n = 3$$

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b)
$$\begin{aligned} P\left\{\sum_{i=1}^{20} X_i > 15\right\} &= P\left\{\sum_{i=1}^{20} X_i > 15.5\right\} \\ &\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} \\ &= P\{Z > -1.006\} \\ &\approx .8428 \end{aligned}$$

5. Letting X_i denote the i^{th} roundoff error it follows that $E\left[\sum_{i=1}^{50} X_i\right] = 0$,

   $\text{Var}\left(\sum_{i=1}^{50} X_i\right) = 50 \text{ Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform (0, 1) and so $\text{Var}(X) = \text{Var}(.5 + X) = 1/12$. Hence,

$$\begin{aligned} P\left\{\left|\sum X_i\right| > 3\right\} &\approx P\{|N(0, 1)| > 3(12/50)^{1/2}\} \text{ by the central limit theorem} \\ &= 2P\{N(0, 1) > 1.47 = .1416 \end{aligned}$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2 \quad \text{Var}(X_i) = 35/12$ and so

$$\begin{aligned} P\left\{\sum_{i=1}^{79} X_i \le 300\right\} &= P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\} \\ &\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429 \end{aligned}$$

7. $$P\left\{\sum_{i=1}^{100} X_i > 525\right\} \approx P\left\{N(0,1) > \frac{525-500}{\sqrt{(100\times 25)}}\right\} = P\{N(0,1) > .5\} = .3085$$

where the above uses that an exponential with mean 5 has variance 25.

8. If we let X_i denote the life of bulb i and let R_i be the time to replace bulb i then the desired probability is $P\left\{\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \le 550\right\}$. Since $\sum X_i + \sum R_i$ has mean $100 \times 5 + 99 \times .25 =$ 524.75 and variance 2500 + 99/48 = 2502 it follows that the desired probability is approximately equal to $P\{N(0, 1) \le [550 - 524.75]/(2502)^{1/2}\} = P\{N(0, 1) \le .505\} = .693$ It should be noted that the above used that

$$\text{Var}(R_i) = \text{Var}\left(\frac{1}{2}\text{Unif}[0,1]\right) = 1/48$$

9. Use the fact that a gamma $(n, 1)$ random variable is the sum of n independent exponentials with rate 1 and thus has mean and variance equal to n, to obtain:

$$\begin{aligned} P\left\{\left|\frac{X-n}{n}\right| > .01\right\} &= P\left\{|X-n|/\sqrt{n} > .01\sqrt{n}\right\} \\ &\approx P\left\{|N(0,1)| > .01\sqrt{n}\right\} \\ &= 2P\left\{N(0,1) > .01\sqrt{n}\right\} \end{aligned}$$

Now $P\{N(0, 1) > 2.58\} = .005$ and so $n = (258)^2$.

10. If W_n is the total weight of n cars and A is the amount of weight that the bridge can withstand then $W_n - A$ is normal with mean $3n - 400$ and variance $.09n + 1600$. Hence, the probability of structural damage is

$$P\{W_n - A \ge 0\} \approx P\left\{Z \ge (400-3n)/\sqrt{.09n+1600}\right\}$$

Since $P\{Z \ge 1.28\} = .1$ the probability of damage will exceed .1 when n is such that

$$400 - 3n \le 1.28\sqrt{.09n+1600}$$

The above will be satisfied whenever $n \ge 117$.

12. Let L_i denote the life of component i.

$$E\left[\sum_{i=1}^{100} L_i\right] = 1000 + \frac{1}{10}50(101) \ = 1505$$

$$\text{Var}\left(\sum_{i=1}^{100} L_i\right) = \sum_{i=1}^{100}\left(10 + \frac{i}{10}\right)^2 = (100)^2 + (100)(101) + \frac{1}{100}\sum_{i=1}^{100} i^2$$

Now apply the central limit theorem to approximate.

13. (a) $P\{\bar{X} > 80\} = P\left\{\dfrac{\bar{X}-74}{14/5} > 15/7\right\} \approx P\text{P}Z > 2.14\} \approx .0162$

(b) $P\{\bar{Y} > 80\} = P\left\{\dfrac{\bar{Y}-74}{14/8} > 24/7\right\} \approx P\{Z > 3.43\} \approx .0003$

(c) Using that $SD(\bar{Y} - \bar{X}) = \sqrt{196/64 + 196/25} \ \approx 3.30$ we have

$$\begin{aligned} P\{\bar{Y} - \bar{X} > 2.2\} &= P\{\bar{Y} - \bar{X}\}/3.30 > 2.2/3.30\} \\ &\approx P\{Z > .67\} \approx .2514 \end{aligned}$$

(d) same as in (c)

14. Suppose n components are in stock. The probability they will last for at least 2000 hours is

$$p = P\left\{\sum_{i=1}^{n} X_i \ge 2000\right\} \approx P\left\{Z \ge \frac{2000 - 100n}{30\sqrt{n}}\right\}$$

where Z is a standard normal random variable. Since

$.95 = P\{Z \ge -1.64\}$ it follows that $p \ge .95$ if

$$\frac{2000 - 100n}{30\sqrt{n}} \le -1.64$$

or, equivalently,

$$(2000 - 100n)/\sqrt{n} \ \le -49.2$$

and this will be the case if $n \ge 23$.

15. $P\left\{\sum_{i=1}^{10,000} X_i > 2,700,000\right\} \approx P\{Z \ge (2,700,000 - 2,400,000)/(800 \cdot 100)\} = P\{Z \ge 3.75\} \approx 0$

16. (a) Number AJ's jobs, let X_i be the time it takes to do job i, and let $X_A = \sum_{i=1}^{20} X_i$ be the time that it takes AJ to finish all 20 jobs. Because

$$E[X_A] = 20(50) = 1000, \qquad \text{Var}(X_A) = 20(100) = 2000$$

the central limit theorem gives that

$$\begin{aligned} P\{X_A \leq 900\} &= P\left\{\frac{X_A - 1000}{\sqrt{2000}} \leq \frac{900 - 1000}{\sqrt{2000}}\right\} \\ &\approx P\{Z \leq -2.236\} \\ &= 1 - \Phi(2.236) = .013 \end{aligned}$$

(b) Similarly, if we let X_M be the time that it takes MJ to finish all of her 20 jobs, then by the central limit theorem X_M is approximately normal with mean and variance

$$E[X_M] = 20(52) = 1040, \qquad \text{Var}(X_M) = 20(225) = 4500$$

Thus

$$\begin{aligned} P\{X_M \leq 900\} &= P\left\{\frac{X_M - 1040}{\sqrt{4500}} \leq \frac{900 - 1040}{\sqrt{4500}}\right\} \\ &\approx P\{Z \leq -2.087\} \\ &= 1 - \Phi(2.087) = 0.18 \end{aligned}$$

(c) Because the sum of independent normal random variables is also normal, $D \equiv X_M - X_A$ is approximately normal with mean and variance

$$E[D] = 1040 - 1000 = 40, \qquad \text{Var(D)} = 4500 + 2000 = 6500$$

Hence,

$$\begin{aligned} P\{D > 0\} &= P\left\{\frac{D - 40}{\sqrt{6500}} \geq \frac{-40}{\sqrt{6500}}\right\} \\ &\approx P\{Z \geq -.4961\} \\ &= \Phi(.4961) = .691 \end{aligned}$$

Thus even though AJ is more likely than not to finish earlier than MJ, MJ has the better chance to finish within 900 minutes.

19. Let Y_i denote the additional number of fish that need to be caught to obtain a new type when there are at present i distinct types. Then Y_i is geometric with parameter $\frac{4-i}{4}$.

$$E[Y] = E\left[\sum_{i=0}^{3} Y_i\right] = 1 + \frac{4}{3} + \frac{4}{2} + 4 = \frac{25}{3}$$

$$\text{Var}[Y] = \text{Var}\left(\sum_{i=0}^{3} Y_i\right) = \frac{4}{9} + 2 + 12 = \frac{130}{9}$$

Hence,

$$P\left\{\left|Y - \frac{25}{3}\right| > \frac{25}{3}\sqrt{\frac{1300}{9}}\right\} \le \frac{1}{10}$$

and so we can take $a = \dfrac{25 - \sqrt{1300}}{3}$, $b = \dfrac{25 + \sqrt{1300}}{3}$.

Also,

$$P\left\{Y - \frac{25}{3} > a\right\} \le \frac{130}{130 + 9a^2} = \frac{1}{10} \text{ when } a = \frac{\sqrt{1170}}{3}.$$

Hence $P\left\{Y > \dfrac{25 + \sqrt{1170}}{3}\right\} \le .1$.

21. $g(x) = x^{n(n-1)}$ is convex. Hence, by Jensen's Inequality

$E[Y^{n/(n-1)}] \ge E[Y])^{n/(n-1)}$ Now set $Y = X^{n-1}$ and so
$E[X^n] \ge (E[X^{n-1}])^{n/(n-1)}$ or $(E[X^n])^{1/n} \ge (E[X^{n-1}])^{1/(n-1)}$

22. No

23. (a) $20/26 \approx .769$

(b) $20/(20 + 36) = 5/14 \approx .357$

(d) $p \approx P\{Z \ge (25.5 - 20)/\sqrt{20}\} \approx P\{Z \ge 1.23\} \approx .1093$

(e) $p = .112184$

Theoretical Exercises

1. This follows immediately from Chebyshev's inequality.

2. $P\{D > \alpha\} = P\{\,|X - \mu| > \alpha\mu\} \le \dfrac{\varsigma^2}{\alpha^2\mu^2} = \dfrac{1}{\alpha^2 r^2}$

3. (a) $\dfrac{\lambda}{\sqrt{\lambda}} = \sqrt{\lambda}$

   (b) $\dfrac{np}{\sqrt{np(1-p)}} = \sqrt{np/(1-p)}$

   (c) answer = 1

   (d) $\dfrac{1/2}{\sqrt{1/12}} = \sqrt{3}$

   (e) answer = 1

   (d) answer = $|\mu|/\sigma$

4. For $\varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(c)| < \varepsilon$ whenever $|x - c| \le \delta$. Also, let B be such that $|g(x)| < B$. Then,

$$E[g(Z_n)] = \int_{|x-c|\le\delta} g(x)dF_n(x) + \int_{|x-c|>\delta} g(x)dF_n(x)$$
$$\le (\varepsilon + g(c))P\{\,|Z_n - c| \le \delta\} + BP\{\,|Z_n - c| > \delta\}$$

In addition, the same equality yields that

$$E[g(Z_n)] \ge (g(c) - \varepsilon)P\{\,|Z_n - c| \le \delta\} - BP\{\,|Z_n - c| > \delta\}$$

Upon letting $n \to \infty$, we obtain that

$$\limsup E[g(Z_n)] \le g(c) + \varepsilon$$
$$\liminf E[g(Z_n)] \ge g(c) - \varepsilon$$

The result now follows since ε is arbitrary.

5. Use the notation of the hint. The weak law of large numbers yields that

$$\lim_{n\to\infty} P\{|(X_1 + \ldots + X_n)/n - c| > \varepsilon\} = 0$$

Since $X_1 + \ldots + X_n$ is binomial with parameters n, x, we have

$$E\left[f\left(\frac{X_1 + \ldots + X_n}{n}\right)\right] = \sum_{k=1}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

The result now follows from Exercise 4.

6. $$E[X] = \sum_{i=1}^{k} i\,P\{X = i\} + \sum_{i=k+1}^{\infty} i\,P\{X = i\}$$
$$\geq \sum_{i=1}^{k} i\,P\{X = k\}$$
$$= P\{X = k\}^{k(k+1)/2}$$
$$\geq \frac{k^2}{2} P\{X = k\}$$

7. Take logs and apply the central limit theorem

8. It is the distribution of the sum of t independent exponentials each having rate λ.

9. 1/2

10. Use the Chernoff bound: $e^{-ti}M(t) = e^{\lambda(e^t - 1) - ti}$ will obtain its minimal value when t is chosen to satisfy

$\lambda e^t = i$, and this value of t is negative provided $i < \lambda$.

Hence, the Chernoff bound gives

$$P\{X \leq i\} \leq e^{i-\lambda}(\lambda/i)^i$$

11. $e^{-ti}M(t) = (pe^t + q)^n e^{-ti}$ and differentiation shows that the value of t that minimizes it is such that

$$npe^t = i(pe^t + q) \text{ or } e^t = \frac{iq}{(n-i)p}$$

Using this value of t, the Chernoff bound gives that

$$P\{X \geq i\} \leq \left(\frac{iq}{n-i} + q\right)^n (n-i)^i\, p^i\, /(iq)^i$$
$$= \frac{(nq)^n (n-i)^i\, p^i}{i^i q^i (n-i)^n}$$

12. $1 = E[e^{\theta X}] \geq e^{\theta E[X]}$ by Jensen's inequality.

Hence, $\theta E[X] \leq 0$ and thus $\theta > 0$.