

Chapter 5

Problems

1. (a) $c \int_{-1}^1 (1-x^2) dx = 1 \Rightarrow c = 3/4$

(b) $F(x) = \frac{3}{4} \int_{-1}^x (1-x^2) dx = \frac{3}{4} \left(x - \frac{x^3}{3} + \frac{2}{3} \right), -1 < x < 1$

2. $\int x e^{-x/2} dx = -2x e^{-x/2} - 4e^{-x/2}$. Hence,

$$c \int_0^{\infty} x e^{-x/2} dx = 1 \Rightarrow c = 1/4$$

$$\begin{aligned} P\{X > 5\} &= \frac{1}{4} \int_5^{\infty} x e^{-x/2} dx = \frac{1}{4} [10e^{-5/2} + 4e^{-5/2}] \\ &= \frac{14}{4} e^{-5/2} \end{aligned}$$

3. No. $f(5/2) < 0$

4. (a) $\int_{20}^{\infty} \frac{10}{x^2} dx = \frac{-10}{x} \Big|_{20}^{\infty} = 1/2$.

(b) $F(y) = \int_{10}^y \frac{10}{x^2} dx = 1 - \frac{10}{y}, y > 10. F(y) = 0$ for $y < 10$.

(c) $\sum_{i=3}^6 \binom{6}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{6-i}$ since $\bar{F}(15) = \frac{10}{15}$. Assuming independence of the events that the devices exceed 15 hours.

5. Must choose c so that

$$.01 = \int_c^1 5(1-x)^4 dx = (1-c)^5$$

so $c = 1 - (.01)^{1/5}$.

$$6. \quad (a) \quad E[X] = \frac{1}{4} \int_0^{\infty} x^2 e^{-x/2} dx = 2 \int_0^{\infty} y^2 e^{-y} dy = 2\Gamma(3) = 4$$

(b) By symmetry of $f(x)$ about $x=0$, $E[X] = 0$

$$(c) \quad E[X] = \int_5^{\infty} \frac{5}{x} dx = \infty$$

$$7. \quad \int_0^1 (a + bx^2) dx = 1 \text{ or } a + \frac{b}{3} = 1$$

$$\int_0^1 x(a + bx^2) dx = \frac{3}{5} \text{ or } \frac{a}{2} + \frac{b}{4} = 3/5. \text{ Hence,}$$

$$a = \frac{3}{5}, \quad b = \frac{6}{5}$$

$$8. \quad E[X] = \int_0^{\infty} x^2 e^{-x} dx = \Gamma(3) = 2$$

9. If s units are stocked and the demand is X , then the profit, $P(s)$, is given by

$$\begin{aligned} P(s) &= bX - (s - X)P && \text{if } X \leq s \\ &= sb && \text{if } X > s \end{aligned}$$

Hence

$$\begin{aligned} E[P(s)] &= \int_0^s (bx - (s - x)\ell) f(x) dx + \int_s^{\infty} sbf(x) dx \\ &= (b + \ell) \int_0^s xf(x) dx - s\ell \int_0^s f(x) dx + sb \left[1 - \int_0^s f(x) dx \right] \\ &= sb + (b + \ell) \int_0^s (x - s) f(x) dx \end{aligned}$$

Differentiation yields

$$\begin{aligned} \frac{d}{ds} E[P(s)] &= b + (b + \ell) \frac{d}{ds} \left[\int_0^s xf(x) dx - s \int_0^s f(x) dx \right] \\ &= b + (b + \ell) \left[sf(s) - sf(s) - \int_0^s f(s) dx \right] \\ &= b - (b + \ell) \int_0^s f(x) dx \end{aligned}$$

Equating to zero shows that the maximal expected profit is obtained when s is chosen so that

$$F(s) = \frac{b}{b + \ell}$$

where $F(s) = \int_0^s f(x)dx$ is the cumulative distribution of demand.

10. (a) $P\{\text{goes to } A\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60\}$.
 $= 2/3$ since X is uniform $(0, 60)$.

(b) same answer as in (a).

11. X is uniform on $(0, L)$.

$$\begin{aligned} & P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < 1/4\right\} \\ &= 1 - P\left\{\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > 1/4\right\} \\ &= 1 - P\left\{\frac{X}{L-X} > 1/4, \frac{L-X}{X} > 1/4\right\} \\ &= 1 - P\{X > L/5, X < 4L/5\} \\ &= 1 - P\left\{\frac{L}{5} < X < 4L/5\right\} \\ &= 1 - \frac{3}{5} = \frac{2}{5}. \end{aligned}$$

13. $P\{X > 10\} = \frac{2}{3}, P\{X > 25 \mid X > 15\} = \frac{P\{X > 25\}}{P\{X > 15\}} = \frac{5/30}{15/30} = 1/3$

where X is uniform $(0, 30)$.

14. $E[X^n] = \int_0^1 x^n dx = \frac{1}{n+1}$
 $P\{X^n \leq x\} = P\{X \leq x^{1/n}\} = x^{1/n}$
 $E[X^n] = \int_0^1 x \frac{1}{n} x^{\left(\frac{1}{n}-1\right)} dx = \frac{1}{n} \int_0^1 x^{1/n} dx = \frac{1}{n+1}$

15. (a) $\Phi(.8333) = .7977$
 (b) $2\Phi(1) - 1 = .6827$
 (c) $1 - \Phi(.3333) = .3695$
 (d) $\Phi(1.6667) = .9522$
 (e) $1 - \Phi(1) = .1587$

$$16. \quad P\{X > 50\} = P\left\{\frac{X - 40}{4} > \frac{10}{4}\right\} = 1 - \Phi(2.5) = 1 - .9938$$

Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$

$$17. \quad E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$$

$$18. \quad .2 = P\left\{\frac{X - 5}{\sigma} > \frac{9 - 5}{\sigma}\right\} = P\{Z > 4/\sigma\} \text{ where } Z \text{ is a standard normal. But from the normal table } P\{Z < .84\} \approx .80 \text{ and so}$$

$$.84 \approx 4/\sigma \text{ or } \sigma \approx 4.76$$

That is, the variance is approximately $(4.76)^2 = 22.66$.

$$19. \quad \text{Letting } Z = (X - 12)/2 \text{ then } Z \text{ is a standard normal. Now, } .10 = P\{Z > (c - 12)/2\}. \text{ But from Table 5.1, } P\{Z < 1.28\} = .90 \text{ and so}$$

$$(c - 12)/2 = 1.28 \text{ or } c = 14.56$$

$$20. \quad \text{Let } X \text{ denote the number in favor. Then } X \text{ is binomial with mean 65 and standard deviation } \sqrt{65(.35)} \approx 4.77. \text{ Also let } Z \text{ be a standard normal random variable.}$$

$$(a) \quad P\{X \geq 50\} = P\{X \geq 49.5\} = P\{X - 65\}/4.77 \geq -15.5/4.77 \\ \approx P\{Z \geq -3.25\} \approx .9994$$

$$(b) \quad P\{59.5 \leq X \leq 70.5\} \approx P\{-5.5/4.77 \leq Z \leq 5.5/4.77\} \\ = 2P\{Z \leq 1.15\} - 1 \approx .75$$

$$(c) \quad P\{X \leq 74.5\} \approx P\{Z \leq 9.5/4.77\} \approx .977$$

$$22. \quad (a) \quad P\{.9000 - .005 < X < .9000 + .005\} \\ = P\left\{-\frac{.005}{.003} < Z < \frac{.005}{.003}\right\} \\ = P\{-1.67 < Z < 1.67\} \\ = 2\Phi(1.67) - 1 = .9050.$$

Hence 9.5 percent will be defective (that is each will be defective with probability $1 - .9050 = .0950$).

$$(b) \quad P\left\{-\frac{.005}{\sigma} < Z < \frac{.005}{\sigma}\right\} = 2\Phi\left(\frac{.005}{\sigma}\right) - 1 = .99 \text{ when}$$

$$\Phi\left(\frac{.005}{\sigma}\right) = .995 \Rightarrow \frac{.005}{\sigma} = 2.575 \Rightarrow \sigma = .0019.$$

$$\begin{aligned}
 23. \quad (a) \quad P\{149.5 < X < 200.5\} &= P\left\{\frac{149.5 - \frac{1000}{6}}{\sqrt{1000 \frac{1}{6} \frac{5}{6}}} < Z < \frac{200.5 - \frac{1000}{6}}{\sqrt{1000 \frac{1}{6} \frac{5}{6}}}\right\} \\
 &= \Phi\left(\frac{200.5 - 166.7}{\sqrt{5000/36}}\right) - \Phi\left(\frac{149.5 - 166.7}{\sqrt{5000/36}}\right) \\
 &\approx \Phi(2.87) + \Phi(1.46) - 1 = .9258.
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad P\{X < 149.5\} &= P\left\{Z < \frac{149.5 - 800(1/5)}{\sqrt{800 \frac{1}{5} \frac{4}{5}}}\right\} \\
 &= P\{Z < -.93\} \\
 &= 1 - \Phi(.93) = .1762.
 \end{aligned}$$

24. With C denoting the life of a chip, and ϕ the standard normal distribution function we have

$$\begin{aligned}
 P\{C < 1.8 \times 10^6\} &= \phi\left(\frac{1.8 \times 10^6 - 1.4 \times 10^6}{3 \times 10^5}\right) \\
 &= \phi(1.33) \\
 &= .9082
 \end{aligned}$$

Thus, if N is the number of the chips whose life is less than 1.8×10^6 then N is a binomial random variable with parameters $(100, .9082)$. Hence,

$$P\{N > 19.5\} \approx 1 - \phi\left(\frac{19.5 - 90.82}{90.82(.0918)}\right) = 1 - \phi(-24.7) \approx 1$$

25. Let X denote the number of unacceptable items among the next 150 produced. Since X is a binomial random variable with mean $150(.05) = 7.5$ and variance $150(.05)(.95) = 7.125$, we obtain that, for a standard normal random variable Z ,

$$\begin{aligned}
 P\{X \leq 10\} &= P\{X \leq 10.5\} \\
 &= P\left\{\frac{X - 7.5}{\sqrt{7.125}} \leq \frac{10.5 - 7.5}{\sqrt{7.125}}\right\} \\
 &\approx P\{Z \leq 1.1239\} \\
 &= .8695
 \end{aligned}$$

The exact result can be obtained by using the text diskette, and (to four decimal places) is equal to .8678.

$$\begin{aligned}
 27. \quad P\{X > 5,799.5\} &= P\left\{Z > \frac{799.5}{\sqrt{2,500}}\right\} \\
 &= P\{Z > 15.99\} = \text{negligible.}
 \end{aligned}$$

28. Let X equal the number of left-handers. Assuming that X is approximately distributed as a binomial random variable with parameters $n = 200$, $p = .12$, then, with Z being a standard normal random variable,

$$\begin{aligned} P\{X > 19.5\} &= P\left\{\frac{X - 200(.12)}{\sqrt{200(.12)(.88)}} > \frac{19.5 - 200(.12)}{\sqrt{200(.12)(.88)}}\right\} \\ &\approx P\{Z > -.9792\} \\ &\approx .8363 \end{aligned}$$

29. Let s be the initial price of the stock. Then, if X is the number of the 1000 time periods in which the stock increases, then its price at the end is

$$su^X d^{1000-X} = sd^{1000} \left(\frac{u}{d}\right)^X$$

Hence, in order for the price to be at least $1.3s$, we would need that

$$d^{1000} \left(\frac{u}{d}\right)^X > 1.3$$

or

$$X > \frac{\log(1.3) - 1000\log(d)}{\log(u/d)} = 469.2$$

That is, the stock would have to rise in at least 470 time periods. Because X is binomial with parameters 1000, .52, we have

$$\begin{aligned} P\{X > 469.5\} &= P\left\{\frac{X - 1000(.52)}{\sqrt{1000(.52)(.48)}} > \frac{469.5 - 1000(.52)}{\sqrt{1000(.52)(.48)}}\right\} \\ &\approx P\{Z > -3.196\} \\ &\approx .9993 \end{aligned}$$

$$\begin{aligned} 30. \quad P\{\text{in black}\} &= \frac{P\{5 \mid \text{black}\}\alpha}{P\{5 \mid \text{black}\}\alpha + P\{5 \mid \text{white}\}(1-\alpha)} \\ &= \frac{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8} \alpha}{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8} \alpha + (1-\alpha) \frac{1}{3\sqrt{2\pi}} e^{-(5-6)^2/18}} \\ &= \frac{\frac{\alpha}{2} e^{-1/8}}{\frac{\alpha}{2} e^{-1/8} + \frac{(1-\alpha)}{3} e^{-1/8}} \end{aligned}$$

α is the value that makes preceding equal 1/2

$$\begin{aligned}
 31. \quad (a) \quad E[|X - a|] &= \int_a^A (x - a) \frac{dx}{A} + \int_0^a (a - x) \frac{dx}{A} = \frac{A}{2} - \left(a - \frac{a^2}{A}\right) \\
 \frac{d}{da} \left(\quad \right) &= \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2 \\
 (b) \quad E[|X - a|] &= \int_0^a (a - x) \lambda e^{-\lambda x} dx + \int_a^\infty (x - a) \lambda e^{-\lambda x} dx \\
 &= a(1 - e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a}
 \end{aligned}$$

Differentiation yields that the minimum is attained at \bar{a} where

$$e^{-\lambda \bar{a}} = 1/2 \text{ or } \bar{a} = \log 2/\lambda$$

(c) Minimizing $a = \text{median of } F$

$$\begin{aligned}
 32. \quad (a) \quad &e^{-1} \\
 (b) \quad &e^{-1/2}
 \end{aligned}$$

$$33. \quad e^{-1}$$

$$\begin{aligned}
 34. \quad (a) \quad &P\{X > 20\} = e^{-1} \\
 (b) \quad &P\{X > 30 | X > 10\} = \frac{P\{X > 30\}}{P\{X > 10\}} = \frac{1/4}{3/4} = 1/3
 \end{aligned}$$

$$\begin{aligned}
 35. \quad (a) \quad &\exp\left[-\int_{40}^{50} \lambda(t) dt\right] = e^{-.35} \\
 (b) \quad &e^{-1.21}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad (a) \quad &1 - F(2) = \exp\left[-\int_0^2 t^3 dt\right] = e^{-4} \\
 (b) \quad &\exp[-(.4)^4/4] - \exp[-(1.4)^4/4] \\
 (c) \quad &\exp\left[-\int_1^2 t^3 dt\right] = e^{-15/4}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad (a) \quad &P\{|X| > 1/2\} = P\{X > 1/2\} + P\{X < -1/2\} = 1/2 \\
 (b) \quad &P\{|X| \leq a\} = P\{-a \leq X \leq a\} = a, \quad 0 < a < 1. \text{ Therefore,} \\
 &f_{|X|}(a) = 1, \quad 0 < a < 1
 \end{aligned}$$

That is, $|X|$ is uniform on $(0, 1)$.

38. For both roots to be real the discriminant $(4Y)^2 - 44(Y + 2)$ must be ≥ 0 . That is, we need that $Y^2 \geq Y + 2$. Now in the interval $0 < Y < 5$.

$$Y^2 \geq Y + 2 \Leftrightarrow Y \geq 2 \text{ and so}$$

$$P\{Y^2 \geq Y + 2\} = P\{Y \geq 2\} = 3/5.$$

39. $F_Y(y) = P\{\log X \leq y\}$
 $= P\{X \leq e^y\} = F_X(e^y)$

$$f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y}$$

40. $F_Y(y) = P\{e^X \leq y\}$
 $= F_X(\log y)$

$$f_Y(y) = f_X(\log y) \frac{1}{y} = \frac{1}{y}, 1 < y < e$$

Theoretical Exercises

1. The integration by parts formula $\int u dv = uv - \int v du$ with $dv = -2bx e^{-bx^2}$, $u = -x/2b$ yields that

$$\begin{aligned}\int_0^{\infty} x^2 e^{-bx^2} dx &= \frac{-xe^{-bx^2}}{2b} \Big|_0^{\infty} + \frac{1}{2b} \int_0^{\infty} e^{-bx^2} dx \\ &= \frac{1}{(2b)^{3/2}} \int_0^{\infty} e^{-y^2/2} dy \text{ by } y = x\sqrt{2b} \\ &= \frac{\sqrt{2\pi}}{2} \frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}}\end{aligned}$$

where the above uses that $\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2/2} dy = 1/2$. Hence, $a = \frac{4b^{3/2}}{\sqrt{\pi}}$

$$\begin{aligned}2. \quad \int_0^{\infty} P\{Y < -y\} dy &= \int_0^{\infty} \int_{-\infty}^{-y} f_Y(x) dx dy \\ &= \int_{-\infty}^0 \int_0^{-x} f_Y(x) dy dx = - \int_{-\infty}^0 x f_Y(x) dx\end{aligned}$$

Similarly,

$$\int_0^{\infty} P\{Y > y\} dy = \int_0^{\infty} x f_Y(x) dx$$

Subtracting these equalities gives the result.

$$\begin{aligned}4. \quad E[aX + b] &= \int (ax + b) f(x) dx = a \int x f(x) dx + b \int f(x) dx \\ &= aE[X] + b\end{aligned}$$

$$\begin{aligned}5. \quad E[X^n] &= \int_0^{\infty} P\{X^n > t\} dt \\ &= \int_0^{\infty} P\{X^n > x^n\} n x^{n-1} dx \text{ by } t = x^n, dt = n x^{n-1} dx \\ &= \int_0^{\infty} P\{X > x\} n x^{n-1} dx\end{aligned}$$

6. Let X be uniform on $(0, 1)$ and define E_a to be the event that X is unequal to a . Since $\bigcap_a E_a$ is the empty set, it must have probability 0.

7. $SD(aX + b) = \sqrt{\text{Var}(aX + b)} = \sqrt{a^2 \sigma^2} = |a| \sigma$

8. Since $0 \leq X \leq c$, it follows that $X^2 \leq cX$. Hence,

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &\leq E[cX] - (E[X])^2 \\ &= cE[X] - (E[X])^2 \\ &= E[X](c - E[X]) \\ &= c^2[\alpha(1 - \alpha)] \quad \text{where } \alpha = E[X]/c \\ &\leq c^2/4 \end{aligned}$$

where the last inequality first uses the hypothesis that $P\{0 \leq X \leq c\} = 1$ to calculate that $0 \leq \alpha \leq 1$ and then uses calculus to show that $\max_{0 \leq \alpha \leq 1} \alpha(1 - \alpha) = 1/4$.

9. The final step of parts (a) and (b) use that $-Z$ is also a standard normal random variable.

(a) $P\{Z > x\} = P\{-Z < -x\} = P\{Z < -x\}$

(b) $P\{|Z| > x\} = P\{Z > x\} + P\{Z < -x\} = P\{Z > x\} + P\{-Z > x\}$
 $= 2P\{Z > x\}$

(c) $P\{|Z| < x\} = 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\}$ by (b)
 $= 1 - 2(1 - P\{Z < x\})$

10. With $c = 1/(\sqrt{2\pi}\sigma)$ we have

$$f(x) = ce^{-(x-\mu)^2/2\sigma^2}$$

$$f'(x) = -ce^{-(x-\mu)^2/2\sigma^2} (x-\mu)/\sigma^2$$

$$f''(x) = c\sigma^{-4}e^{-(x-\mu)^2/2\sigma^2} (x-\mu)^2 - c\sigma^{-2}e^{-(x-\mu)^2/2\sigma^2}$$

Therefore,

$$f''(\mu + \sigma) = f''(\mu - \sigma) = c\sigma^{-2}e^{-1/2} - c\sigma^{-2}e^{-1/2} = 0$$

11. (a) Integrate

$$E[g'(Z)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(x)e^{-x^2/2} dx$$

by parts $\left(dv = g'(x)dx, u = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right)$ to obtain the result.

(b) Let $g(z) = z^n$ and apply part (a).

(c) Let $n = 3$ in part (b) to obtain $E[Z^4] = 3E[Z^2] = 3$

12. $E[X^2] = \int_0^{\infty} P\{X > x\} 2x^{2-1} dx = 2 \int_0^{\infty} xe^{-\lambda x} dx = \frac{2}{\lambda} E[X] = 2/\lambda^2$

13. (a) $\frac{b+a}{2}$

(b) μ

(c) $1 - e^{-\lambda m} = 1/2$ or $m = \frac{1}{\lambda} \log 2$

14. (a) all values in (a, b)

(b) μ

(c) 0

15. $P\{cX < x\} = P\{X < x/c\} = 1 - e^{-\lambda x/c}$

16. $\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \frac{1/a}{(a-t)/a} = \frac{1}{a-t}, 0 < t < a$

17. If X has distribution function F and density f , then for $a > 0$

$$F_{aX}(t) = P\{aX \leq t\} = F(t/a)$$

and

$$f_{aX} = \frac{1}{a} f(t/a)$$

Thus,

$$\lambda_{aX}(t) = \frac{\frac{1}{a} f(t/a)}{1 - F(t/a)} = \frac{1}{a} \lambda_X(t/a).$$

19.
$$\begin{aligned} E[X^k] &= \int_0^{\infty} x^k \lambda e^{-\lambda x} dx = \lambda^{-k} \int_0^{\infty} \lambda e^{-\lambda x} (\lambda x)^k dx \\ &= \lambda^{-k} \Gamma(k+1) = k!/\lambda^k \end{aligned}$$

$$\begin{aligned}
20. \quad E[X^k] &= \frac{1}{\Gamma(t)} \int_0^{\infty} x^k \lambda e^{-\lambda x} (\lambda x)^{t-1} dx \\
&= \frac{\lambda^{-k}}{\Gamma(t)} \int_0^{\infty} \lambda e^{-\lambda x} (\lambda x)^{t+k-1} dx \\
&= \frac{\lambda^{-k}}{\Gamma(t)} \Gamma(t+k)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[X] &= t/\lambda, \\
E[X^2] &= \lambda^{-2} \Gamma(t+2)/\Gamma(t) = (t+1)t/\lambda^2
\end{aligned}$$

and thus

$$\text{Var}(X) = (t+1)t/\lambda^2 - t^2/\lambda^2 = t/\lambda^2$$

$$\begin{aligned}
21. \quad \Gamma(1/2) &= \int_0^{\infty} e^{-x} x^{-1/2} dx \\
&= \sqrt{2} \int_0^{\infty} e^{-y^2/2} dy \quad \text{by } x = y^2/2, dx = y dy = \sqrt{2x} dy \\
&= 2\sqrt{\pi} \int_0^{\infty} (2\pi)^{-1/2} e^{-y^2/2} dy \\
&= 2\sqrt{\pi} P\{Z > 0\} \quad \text{where } Z \text{ is a standard normal} \\
&= \sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}
22. \quad 1/\lambda(s) &= \int_{x \geq s} \lambda e^{-\lambda x} (\lambda x)^{t-1} dx / \lambda e^{-\lambda s} (\lambda s)^{t-1} \\
&= \int_{x \geq s} e^{-\lambda(x-s)} (x/s)^{t-1} dx \\
&= \int_{y \geq 0} e^{-\lambda y} (1+y/s)^{t-1} dy \quad \text{by letting } y = x - s
\end{aligned}$$

As the above, equal to the inverse of the hazard rate function, is clearly decreasing in s when $t \geq 1$ and increasing when $t \leq 1$ the result follows.

$$23. \quad \lambda(s) = c(s-v)^{\beta-1}, s > v \text{ which is clearly increasing when } \beta \geq 1 \text{ and decreasing otherwise.}$$

$$24. \quad F(\alpha) = 1 - e^{-1}$$

25. Suppose X is Weibull with parameters v, α, β . Then

$$\begin{aligned} P\left\{\left(\frac{X-v}{\alpha}\right)^\beta \leq x\right\} &= P\left\{\frac{X-v}{\alpha} \leq x^{1/\beta}\right\} \\ &= P\{X \leq v + \alpha x^{1/\beta}\} \\ &= 1 - \exp\{-x\}. \end{aligned}$$

26. We use Equation (6.3).

$$\begin{aligned} E[X] &= B(a+1, b)/B(a, b) = \frac{\Gamma(a+1)}{\Gamma(a+b+1)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{a}{a+b} \\ E[X^2] &= B(a+2, b)/B(a, b) = \frac{\Gamma(a+2)}{\Gamma(a+b+2)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)} \end{aligned}$$

Thus,

$$\text{Var}(X) = \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$

27. $(X-a)/(b-a)$

29.
$$\begin{aligned} P\{F(X) \leq x\} &= P\{X \leq F^{-1}(x)\} \\ &= F(F^{-1}(x)) \\ &= x \end{aligned}$$

30.
$$\begin{aligned} F_Y(x) &= P\{aX + b \leq x\} \\ &= P\left\{X \leq \frac{x-b}{a}\right\} \quad \text{when } a > 0 \\ &= F_X((x-b)/a) \quad \text{when } a > 0. \end{aligned}$$

$$f_Y(x) = \frac{1}{a} f_X((x-b)/a) \quad \text{if } a > 0.$$

When $a < 0$, $F_Y(x) = P\left\{X \geq \frac{x-b}{a}\right\} = 1 - F_X\left(\frac{x-b}{a}\right)$ and so

$$f_Y(x) = -\frac{1}{a} f_X\left(\frac{x-b}{a}\right).$$

31.
$$\begin{aligned} F_Y(x) &= P\{e^X \leq x\} \\ &= P\{X \leq \log x\} \\ &= F_X(\log x) \end{aligned}$$

$$f_Y(x) = f_X(\log x)/x$$

$$= \frac{1}{x\sqrt{2\pi\sigma}} e^{-(\log x - \mu)^2 / 2\sigma^2}$$