Chapter 10

1. (a) After stage k the algorithm has generated a random permutation of 1, 2, …, k. It then puts element $k + 1$ in position $k + 1$; randomly chooses one of the positions 1, …, $k + 1$ and interchanges the element in that position with element $k + 1$.

   (b) The first equality in the hint follows since the permutation given will be the permutation after insertion of element k if the previous permutation is $i_1, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}$ and the random choice of one of the k positions of this permutation results in the choice of position j.

2. Integrating the density function yields that that distribution function is

$$F(x) = \begin{matrix} e^{2x}/2 & , \quad x > 0 \\ 1 - e^{-2s}/2, & x > 0 \end{matrix}$$

which yields that the inverse function is given by

$$F^{-1}(u) = \begin{matrix} \log(2u)/2 & \text{if } u < 12 \\ -\log(2[1-u])/2 & \text{if } u > 1/2 \end{matrix}$$

Hence, we can simulate X from F by simulating a random number U and setting $X = F^{-1}(U)$.

3. The distribution function is given by

$$F(x) = \begin{matrix} x^2/4 - x + 1, & 2 \le x \le 3, \\ x - x^2/12 - 2, & 3 \le x \le 6 \end{matrix}$$

Hence, for $u \le 1/4$, $F^{-1}(u)$ is the solution of

$$x^2/4 - x + 1 = u$$

that falls in the region $2 \le x \le 3$. Similarly, for $u \ge 1/4$, $F^{-1}(u)$ is the solution of

$$x - x^2/12 - 2 = u$$

that falls in the region $3 \le x \le 6$. We can now generate X from F by generating a random number U and setting $X = F^{-1}(U)$.

4. Generate a random number U and then set $X = F^{-1}(U)$. If $U \le 1/2$ then $X = 6U - 3$, whereas if $U \ge 1/2$ then X is obtained by solving the quadratic $1/2 + X^2/32 = U$ in the region $0 \le X \le 4$.

5. The inverse equation $F^{-1}(U) = X$ is equivalent to

or

$$1 - e^{-\alpha X^{\beta}} = U$$
$$X = \{-\log(1 - U)/\alpha\}^{1/\beta}$$

Since $1 - U$ has the same distribution as U we can generate from F by generating a random number U and setting $X = \{-\log(U)/\alpha\}^{1/\beta}$.

6. If $\lambda(t) = ct^n$ then the distribution function is given by

$$1 - F(t) = \exp\{-kt^{n+1}\},\ t \geq 0 \quad \text{where } k = c/(n + 1)$$

Hence, using the inverse transform method we can generate a random number U and then set X such that

$$\exp\{-kX^{n+1}\} = 1 - U$$

or

$$X = \{-\log(1 - U)/k\}^{1/(n+1)}$$

Again U can be used for $1 - U$.

7. (a) The inverse transform method shows that $U^{1/n}$ works.

(b) $$\begin{aligned} P\{\text{Max}U_i \leq v\} &= P\{U_1 \leq x, \ldots, U_n \leq x\} \\ &= \prod P\{U_i \leq x\} \quad \text{by independence} \\ &= x^n \end{aligned}$$

(c) Simulate n random numbers and use the maximum value obtained.

8. (a) If X_i has distribution F_i , $i = 1, \ldots, n$, then, assuming independence, F is the distribution of $\text{Max}X_i$. Hence, we can simulate from F by simulating X_i, $i = 1, \ldots, n$ and setting $X = \text{Max}X_i$.

(b) Use the method of (a) replacing Max by Min throughout.

9. (a) Simulate X_i from F_i, $i = 1, 2$. Now generate a random number U and set X equal to X_1 if $U < p$ and equal to X_2 if $U > p$.

(b) Note that

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-3x}, \quad x > 0, \; F_2(x) = x, \quad 0 < x < 1$$

Hence, using (a) let U_1, U_2, U_3 be random numbers and set

$$X = \begin{matrix} -\log(U_1)/3 & \text{if } U_3 < 1/3 \\ U_2 & \text{if } U_3 > 1/3 \end{matrix}$$

where the above uses that $-\log(U_1)/3$ is exponential with rate 3.

10. With $g(x) = \lambda e^{-\lambda x}$

$$\frac{f(x)}{g(x)} = \frac{2e^{-x^2/2}}{\lambda(2\pi)^{1/2}e^{-\lambda x}} = \frac{2}{\lambda(2\pi)^{1/2}}\exp\{-[(x-\lambda)^2 - \lambda^2]/2\}$$
$$\frac{2e^{\lambda^2/2}}{\lambda(2\pi)^{1/2}}\exp\{-(x-\lambda)^2/2\}$$

Hence, $c = 2e^{\lambda^2/2}/[\lambda(2\pi)^{1/2}]$ and simple calculus shows that this is minimized when $\lambda = 1$.

11. Calculus yields that the maximum value of $f(x)/g(x) = 60x^3(1-x)^2$ is attained when $x = 3/5$ and is thus equal to 1296/625. Hence, generate random numbers U_1 and U_2 and set $X = U_1$ if $U_2 \le 3125U_1^3(1-U_1)^2/108$. If not, repeat.

12. Generate random numbers $U_1, \ldots, U_n$, and approximate the integral by $[k(U_1) + \ldots + k(U_n)]/n$. This works by the law of large numbers since $E[k(U)] = \int_0^1 k(x)dx$.

16. $E[g(X)/f(X)] = \int [g(x)/f(x)]f(x)dx = \int g(x)dx$