Chapter 9

Problems and Theoretical Exercises

1. (a) $P(2 \text{ arrivals in } (0, s) \mid 2 \text{ arrivals in } (0, 1)\}$

$$= P\{2 \text{ in } (0, s), 0 \text{ in } (s, 1)\}/e^{-\lambda}\lambda^2/2)$$
$$= [e^{-\lambda s}(\lambda s)^2/2][e^{-(1-s)\lambda}]/(e^{-\lambda}\lambda^2/2) = s^2 = 1/9 \text{ when } s = 1/3$$

(b) $1 - P\{\text{both in last 40 minutes}) = 1 - (2/3)^2 = 5/9$

2. $e^{-3s/60}$

3. $e^{-3s/60} + (s/20)e^{-3s/60}$

8. The equations for the limiting probabilities are:

$$\prod_c = .7\prod_c + .4\prod_s + .2\prod_g$$
$$\prod_s = .2\prod_x + .3\prod_s + .4\prod_g$$
$$\prod_g = .1\prod_c + .3\prod_s + .4\prod_g$$
$$\prod_c + \prod_s + \prod_g = 1$$

and the solution is: $\prod_c = 30/59$, $\prod_s = 16/59$, $\prod_g = 13/59$. Hence, Buffy is cheerful 3000/59 percent of the time.

9. The Markov chain requires 4 states:

0 = RR = Rain today and rain yesterday

1 = RD = Dry today, rain yesterday

2 = DR = Rain today, dry yesterday

3 = DD = Dry today and dry yesterday

with transition probability matrix

$$\underline{P} = \begin{vmatrix} .8 & .2 & 0 & 0 \\ 0 & 0 & .3 & .7 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \end{vmatrix}$$

The equations for the limiting probabilities are:

$\Pi_0 = .8\Pi_0 + .4\Pi_2$
$\Pi_1 = .2\Pi_0 + .6\Pi_2$
$\Pi_2 = .3\Pi_1 + .2\Pi_3$
$\Pi_3 = .7\Pi_1 + .8\Pi_3$
$\Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 = 1$

which gives

$\Pi_0 = 4/15,\ \Pi_1 = \Pi_2 = 2/15,\ \Pi_3 = 7/15.$

Since it rains today when the state is either 0 or 2 the probability is 2/5.

10. Let the state be the number of pairs of shoes at the door he leaves from in the morning. Suppose the present state is i, where $i > 0$. Now after his return it is equally likely that one door will have i and the other $5 - i$ pairs as it is that one will have $i - 1$ ant the other $6 - i$. Hence, since he is equally likely to choose either door when he leaves tomorrow it follows that

$$P_{i,i} = P_{i,5-i} = P_{i,i-1} = P_{i,6-i} = 1/4$$

provided all the states i, $5 - i$, $i - 1$, $6 - i$ are distinct. If they are not then the probabilities are added. From this it is easy to see that the transition matrix P_{ij}, $i, j = 0, 1, \ldots, 5$ is as follows:

$$\underline{P} = \begin{matrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \end{matrix}$$

Since this chain is doubly stochastic (the column sums as well as the row sums all equal to one) it follows that $\Pi_i = 1/6$, $i = 0, \ldots, 5$, and thus he runs barefooted one-sixth of the time.

11. (b) 1/2

(c) Intuitively, they should be independent.

(d) From (b) and (c) the (limiting) number of molecules in urn 1 should have a binomial distribution with parameters (M, 1/2).