Chapter 6

Problems

2. (a) $p(0, 0) = \dfrac{8 \cdot 7}{13 \cdot 12} = 14/39$,

$p(0, 1) = p(1, 0) = \dfrac{8 \cdot 5}{13 \cdot 12} = 10/39$

$p(1, 1) = \dfrac{5 \cdot 4}{13 \cdot 12} = 5/39$

(b) $p(0, 0, 0) = \dfrac{8 \cdot 7 \cdot 6}{13 \cdot 12 \cdot 11} = 28/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = \dfrac{8 \cdot 7 \cdot 5}{13 \cdot 12 \cdot 11} = 70/429$

$p(0, 1, 1) = p(1, 0, 1) = p(1, 1, 0) = \dfrac{8 \cdot 5 \cdot 4}{13 \cdot 12 \cdot 11} = 40/429$

$p(1, 1, 1) = \dfrac{5 \cdot 4 \cdot 3}{13 \cdot 12 \cdot 11} = 5/143$

3. (a) $p(0, 0) = (10/13)(9/12) = 15/26$

$p(0, 1) = p(1, 0) = (10/13)(3/12) = 5/26$

$p(1, 1) = (3/13)(2/12) = 1/26$

(b) $p(0, 0, 0) = (10/13)(9/12)(8/11) = 60/143$

$p(0, 0, 1) = p(0, 1, 0) = p(1, 0, 0) = (10/13)(9/12)(3/11) = 45/286$

$p(i, j, k) = (3/13)(2/12)(10/11) = 5/143 \qquad \text{if } i + j + k = 2$

$p(1, 1, 1) = (3/13)(2/12)(1/11) = 1/286$

4. (a) $p(0, 0) = (8/13)^2$, $p(0, 1) = p(1, 0) = (5/13)(8/13)$, $p(1, 1) = (5/13)^2$

(b) $p(0, 0, 0) = (8/13)^3$

$p(i, j, k) = (8/13)^2(5/13)$ if $i + j + k = 1$

$p(i, j, k) = (8/13)(5/13)^2$ if $i + j + k = 2$

5. $p(0, 0) = (12/13)^3(11/12)^3$

$p(0, 1) = p(1, 0) = (12/13)^3[1 - (11/12)^3]$

$p(1, 1) = (2/13)[(1/13) + (12.13)(1/13)] + (11/13)(2/13)(1/13)$

8. $f_Y(y) = c\int_{-y}^{y}(y^2 - x^2)e^{-y}dx$

$= \frac{4}{3}cy^3e^{-y}$, $-0 < y < \infty$

$\int_0^{\infty} f_Y(y)dy = 1 \Rightarrow c = 1/8$ and so $f_Y(y) = \frac{y^3e^{-y}}{6}$, $0 < y < \infty$

$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy$

$= \frac{1}{4}e^{-|x|}(1+|x|)$ upon using $-\int y^2e^{-y} = y^2e^{-y} + 2ye^{-y} + 2e^{-y}$

9. (b) $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}(2x^2 + x)$

(c) $P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}dydx\right) = \frac{15}{56}$

(d) $P\{Y > 1/2 \mid X < 1/2\} = P\{Y > 1/2, X < 1/2\}/P\{X < 1/2\}$

$$= \frac{\int_{1/2}^{2}\int_0^{1/2}\left(x^2 + \frac{xy}{2}dxdy\right)}{\int_0^{1/2}(2x^2 + x)dx}$$

10. (a) $f_X(x) = e^{-x}$, $f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

$P\{X < Y\} = 1/2$

(b) $P\{X < a\} = 1 - e^{-a}$

11. $\frac{5!}{2!1!2!}(.45)^2(.15)(.40)^2$

12. $e^{-5} + 5e^{-5} + \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5}$

14. Let X and Y denoted respectively the locations of the ambulance and the accident of the moment the accident occurs.

$$P\{\,|Y-X|<a\} = P\{Y<X<Y+a\} + P\{X<Y<X+a\}$$

$$= \frac{2}{L^2}\int_0^L \int_y^{\min(y+a,L)} dxdy$$

$$= \frac{2}{L^2}\left[\int_0^{L-a}\int_y^{y+a} dxdy + \int_{L-a}^{L}\int_y^{L} dxdy\right]$$

$$= 1 - \frac{L-a}{L} + \frac{a}{L^2}(L-a) = \frac{a}{L}\left(2-\frac{a}{L}\right),\ \ 0<a<L$$

15. (a) $1 = \iint f(x,y)dydx = \iint_{(x,y)\in R} c\, dydx = cA(R)$

where $A(R)$ is the area of the region R.

(b) $f(x, y) = 1/4, -1 \le x, y \le 1$
$= f(x)f(y)$
where $f(v) = 1/2, -1 \le v \le 1$.

(c) $P\{X^2 + Y^2 \le 1\} = \frac{1}{4}\iint_c dydx$ = (area of circle)/4 = $\pi/4$.

16. (a) $A = \cup A_i$,
(b) yes
(c) $P(A) = \sum P(A_i) = n(1/2)^{n-1}$

17. $\frac{1}{3}$ since each of the 3 points is equally likely to be the middle one.

18. $$P\{Y - X > L/3\} = \iint_{y-x>L/3} \frac{4}{L^2} dydx$$

$$\frac{L}{2} < y < L$$

$$0 < x < \frac{L}{2}$$

$$= \frac{4}{L^2}\left[\int_0^{L/6}\int_{L/2}^{L} dydx + \int_{L/6}^{L/2}\int_{x+L/3}^{L} dydx\right]$$

$$= \frac{4}{L^2}\left[\frac{L^2}{12} + \frac{5L^2}{24} - \frac{7L^2}{72}\right] = 7/9$$

19. $\int_0^1\int_0^x \frac{1}{x}\,dydx = \int_0^1 dx = 1$

(a) $\int_y^1 \frac{1}{x}dx = -\ln(y)$, $0 < y < 1$

(b) $\int_0^x \frac{1}{x}dy = 1$, $0 < y < 1$

(c) $\frac{1}{2}$

(d) Integrating by parts gives that

$$\int_0^1 y\ \ln(y)dy = -1 - \int_0^1 (y\ln(y) - y)dy$$

yielding the result

$$E[Y] = -\int_0^1 y\ln(y)dy = 1/4$$

20. (a) yes: $f_X(x) = xe^{-x}, f_Y(y) = e^{-y}$, $0 < x < \infty$, $0 < y < \infty$

(b) no: $f_X(x) = \int_x^1 f(x,y)dy = 2(1-x)$, $0 < x < 1$

$$f_Y(y) = \int_0^y f(x,y)dx = 2y,\ 0 < y < 1$$

21. (a) We must show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$. Now,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^1\int_0^{1-y} 24xy\ dxdy \\ &= \int_0^1 12y(1-y)^2 dy \\ &= \int_0^1 12(y - 2y^2 + y^3)dy \\ &= 12(1/2 - 2/3 + 1/4) = 1\end{aligned}$$

(b)
$$\begin{aligned}E[X] &= \int_0^1 xf_X(x)dx \\ &= \int_0^1 x\int_0^{1-x} 24xy\ dydx \\ &= \int_0^1 12x^2(1-x)^2 dx = 2/5\end{aligned}$$

(c) 2/5

22. (a) No, since the joint density does not factor.

(b) $f_X(x) = \int_0^1 (x+y)dy = x + 1/2,\ 0 < x < 1.$

(c) $P\{X + Y < 1\} = \int_0^1 \int_0^{1-x} (x+y)dydx$

$$= \int_0^1 [x(1-x) + (1-x)^2/2]dx = 1/3$$

23. (a) yes

$$f_X(x) = 12x(1-x)\int_0^1 ydy = 6x(1-x),\ 0 < x < 1$$

$$f_Y(y) = 12y\int_0^1 x(1-x)dx = 2y,\ 0 < y < 1$$

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 1/2$

(c) $E[Y] = \int_0^1 2y^2 dy = 2/3$

(d) $\text{Var}(X) = \int_0^1 6x^3(1-x)dx - 1/4 = 1/20$

(e) $\text{Var}(Y) = \int_0^1 2y^3 dy - 4/9 = 1/18$

24. $P\{N = n\} = p_0^{n-1}(1-p_0)$

(b) $P\{X = j\} = p_j/(1-p_0)$

(c) $P\{N = n, X = j\} = p_0^{n-1} p_j$

25. $\dfrac{e^{-1}}{i!}$ by the Poisson approximation to the binomial.

26. (a) $F_{A,B,C}(a, b, c) = abc \quad 0 < a, b, c < 1$

(b) The roots will be real if $B^2 \ge 4AC$. Now

$$P\{AC \le x\} = \iint\limits_{\substack{c \le x/a \\ 0 \le a \le 1 \\ 0 \le c \le 1}} dadc = \int_0^x \int_0^1 dcda + \int_x^1 \int_0^{x/a} dcda$$

$$= x - x\log x.$$

Hence, $F_{AC}(x) = x - x\log x$ and so

$$f_{AC}(x) = -\log x,\ 0 < x < 1$$

$$P\{B^2/4 \geq AC\} = -\int_0^1 \int_0^{b^2/4} \log x dx db$$
$$= \int_0^1 \left[\frac{b^2}{4} - \frac{b^2}{4} \log(b^2/4) \right] db$$
$$= \frac{\log 2}{6} + \frac{5}{36}$$

where the above uses the identity

$$\int x^2 \log x dx = \frac{x^3 \log x}{3} - \frac{x^3}{9}.$$

27. $$P\{X_1/X_2 < a\} = \int_0^\infty \int_0^{ay} \lambda_1 e^{-\lambda} 1^x \lambda_2 e^{-\lambda_2 y} dx dy$$
$$= \int_0^\infty \left(1 - e^{-\lambda_1 ay}\right) \lambda_2 e^{-\lambda_2 y} dy$$
$$= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 a} = \frac{\lambda_1 a}{a\lambda_1 + \lambda_2}$$

$$P\{X_1/X_2 < 1\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

28. (a) $\frac{1}{2}e^{-t}$, since e^{-t} is the probability that AJ is still in service when MJ arrives, and 1/2 is the conditional probability that MJ then finishes first.

(b) Using that the time at which MJ finishes is gamma with parameters 2, 1 yields the result: $1 - 3e^{-2}$

29. (a) If $W = X_1 + X_2$ is the sales over the next two weeks, then W is normal with mean 4,400 and standard deviation $\sqrt{2(230)^2} = 325.27$. Hence, with Z being a standard normal, we have

$$P\{W > 5000\} = P\left\{Z > \frac{5000 - 4400}{325.27}\right\}$$
$$= P\{Z > 1.8446\} = .0326$$

(b) $P\{X > 2000\} = P\{Z > (2000 - 2200)/230\}$
$= P\{Z > -.87\} = P\{Z < .87\} = .8078$

Hence, the probability that weekly sales exceeds 2000 in at least 2 of the next 3 weeks $p^3 + 3p^2(1 - p)$ where $p = .8078$.

We have assumed that the weekly sales are independent.

30. Let X denote Jill's score and let Y be Jack's score. Also, let Z denote a standard normal random variable.

(a) $$\begin{aligned} P\{Y > X\} &= P\{Y - X > 0\} \\ &\approx P\{Y - X > .5\} \\ &= P\left\{\frac{Y - X - (160-170)}{\sqrt{(20)^2 + (15)^2}} > \frac{.5 - (160-170)}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .42\} \approx .3372 \end{aligned}$$

(b) $$\begin{aligned} P\{X + Y > 350\} &= P\{X + Y > 350.5\} \\ &= P\left\{\frac{X + Y - 330}{\sqrt{(20)^2 + (15)^2}} > \frac{20.5}{\sqrt{(20)^2 + (15)^2}}\right\} \\ &\approx P\{Z > .82\} \approx .2061 \end{aligned}$$

31. Let X and Y denote, respectively, the number of males and females in the sample that never eat breakfast. Since

$$E[X] = 50.4,\ \mathrm{Var}(X) = 37.6992,\ \ E[Y] = 47.2,\ \mathrm{Var}(Y) = 36.0608$$

it follows from the normal approximation to the binomial that X is approximately distributed as a normal random variable with mean 50.4 and variance 37.6992, and that Y is approximately distributed as a normal random variable with mean 47.2 and variance 36.0608. Let Z be a standard normal random variable.

(a) $$\begin{aligned} P\{X + Y \geq 110\} &= P\{X + Y \geq 109.5\} \\ &= P\left\{\frac{X + Y - 97.6}{\sqrt{73.76}} \geq \frac{109.5 - 97.6}{\sqrt{73.76}}\right\} \\ &\approx P\{Z > 1.3856\} \approx .0829 \end{aligned}$$

(b) $$\begin{aligned} P\{Y \geq X\} &= P\{Y - X \geq -.5\} \\ &= P\left\{\frac{Y - X - (-3.2)}{\sqrt{73.76}} \geq \frac{-.5 - (-3.2)}{\sqrt{73.76}}\right\} \\ &\approx P\{Z \geq .3144\} \approx .3766 \end{aligned}$$

32. (a) e^{-2}

(b) $1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$

The number of typographical errors on each page should approximately be Poisson distributed and the sum of independent Poisson random variables is also a Poisson random variable.

33. (a) $1 - e^{-2.2} - 2.2e^{-2.2} - e^{-2.2}(2.2)^2/2!$

(b) $1 - \sum_{i=0}^{4} e^{-4.4}(4.4)^i / i!$, (c) $1 - \sum_{i=0}^{5} e^{-6.6}(6.6)^i / i!$

The reasoning is the same as in Problem 26.

34. Use the distribution of the sum of independent geometric random variables to obtain the result: $4(.7)^{12} - 3(.6)^{12}$

35. (a) $P\{X_1 = 1 \mid X_2 = 1\} = 5/13 = 1 - P\{X_1 = 0 \mid X_2 = 1\}$

(b) same as in (a)

36. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = 2/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = 3/12 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$

37. (a) $P\{Y_1 = 1 \mid Y_2 = 1\} = p(1, 1)/[1 - (12/13)^3] = 1 - P\{Y_1 = 0 \mid Y_2 = 1\}$

(b) $P\{Y_1 = 1 \mid Y_2 = 0\} = p(1, 0)/(12/13)^3 = 1 - P\{Y_1 = 0 \mid Y_2 = 0\}$
where $p(1, 1)$ and $p(1, 0)$ are given in the solution to Problem 5.

38. (a) $P\{X = j, Y = i\} = \frac{1}{5}\frac{1}{j}$, $j = 1, \ldots, j$, $i = 1, \ldots, j$

(b) $P\{X = j \mid Y = i\} = \frac{1}{5j} \Big/ \sum_{k=i}^{5} 1/5\,k = \frac{1}{j} \Big/ \sum_{k=i}^{5} 1/k,\ 5 \ge j \ge i.$

(c) No.

39. For $j = i$: $P\{Y = i \mid X = i\} = \dfrac{P\{Y = i, X = i\}}{P\{X = i\}} = \dfrac{1}{36P\{X = i\}}$

For $j < i$: $P\{Y = j \mid X = i\} = \dfrac{2}{36P\{X = i\}}$

Hence

$$1 = \sum_{j=1}^{i} P\{Y = j \mid X = i\} = \frac{2(i-1)}{36P\{X = i\}} + \frac{1}{36P\{X = i\}}$$

and so, $P\{X = i\} = \dfrac{2i-1}{36}$ and

$$P\{Y = j \mid X = i\} = \begin{cases} \dfrac{1}{2i-i} & j = i \\ \dfrac{2}{2i-1} & j < i \end{cases}$$

41. (a) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dx} = (y + 1)^2 xe^{-x(y+1)}$, $0 < x$

(b) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int xe^{-x(y+1)}dy} = xe^{-xy}$, $0 < y$

$$\begin{aligned} P\{XY < a\} &= \int_0^\infty \int_0^{a/x} xe^{-x(y+1)} dy dx \\ &= \int_0^\infty (1 - e^{-a}) e^{-x} dx = 1 - e^{-a} \end{aligned}$$

$f_{XY}(a) = e^{-a}$, $0 < a$

42. $$\begin{aligned} f_{Y|X}(y \mid x) &= \frac{(x^2 - y^2)e^{-x}}{\int_{-x}^{x} (x^2 - y^2)e^{-x} dx} \\ &= \frac{3}{4x^3}(x^2 - y^2), \quad -x < y < x \end{aligned}$$

$$\begin{aligned} F_{Y|X}(y \mid x) &= \frac{3}{4x^3} \int_{-x}^{y} (x^2 - y^2) dy \\ &= \frac{3}{4x^3}(x^2 y - y^3/3 + 2x^3/3), \quad -x < y < x \end{aligned}$$

43. $$f(\lambda \mid n) = \frac{P\{N = n \mid \lambda\}g(\lambda)}{P\{N = n\}}$$
$$= C_1 e^{-\lambda}\lambda^n \alpha e^{-\alpha\lambda}(\alpha\lambda)^{s-1}$$
$$= C_2 e^{-(\alpha+1)\lambda}\lambda^{n+s-1}$$

where C_1 and C_2 do not depend on λ. But from the preceding we can conclude that the conditional density is the gamma density with parameters $\alpha + 1$ and $n + s$. The conditional expected number of accidents that the insured will have next year is just the expectation of this distribution, and is thus equal to $(n + s)/(\alpha + 1)$.

44. $P\{X_1 > X_2 + X_3\} + P\{X_2 > X_1 + X_3\} + P\{X_3 > X_1 + X_2\}$

$$= 3P\{X_1 > X_2 + X_3\}$$

$$= 3\iiint\limits_{\substack{x_1 > x_2 > x_3 \\ 0 \le x_i \le 1 \\ i = 1, 2, 3}} dx_1 dx_2 dx_3 \qquad \text{(take } a = 0,\ b = 1\text{)}$$

$$= 3\int_0^1 \int_0^{1-x_3} \int_{x_2+x_3}^1 dx_1 dx_2 dx_3 = 3\int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3) dx_2 dx_3$$
$$= 3\int_0^1 \frac{(1-x_3)^2}{2} dx_3 = 1/2 .$$

45. $$f_{X_{(3)}}(x) = \frac{5!}{2!2!}\left[\int_0^x xe^{-x}dx\right]^2 xe^{-x}\left[\int_x^\infty xe^{-x}dx\right]^2$$
$$= 30(x + 1)^2 e^{-2x} xe^{-x}[1 - e^{-x}(x + 1)]^2$$

46. $$\left(\frac{L-2d}{L}\right)^3$$

47. $$\int_{1/4}^{3/4} f_{X_{(3)}}(x)dx = \frac{5!}{2!2!}\int_{1/4}^{3/4} x^2(1-x)^2 dx$$

48. (a) $P\{\min X_i \le a\} = 1 - P\{\min X_i > a\} = 1 - \prod P\{X_i > a\} = 1 - e^{-5\lambda a}$

(b) $P\{\max X_i \le a\} = \prod P\{X_i \le a\} = (1 - e^{-\lambda a})^5$

49. It is uniform on $(s_{n-1}, 1)$

50. Start with

$$f_{z_1,z_2}(z_1,z_2) = \frac{1}{2\pi}e^{-\left(z_1^2+z_2^2\right)/2}$$

Making the transformation – using that its Jacobian is 1 – yields that

$$f_{X,Y}(x,y) = f_{Z_1Z_2}(x, y-x) = \frac{1}{2\pi}e^{-\left(x^2+(y-x)^2\right)/2}$$

51. $$f_{X_{(1)},X_{(4)}}(x,y) = \frac{4!}{2!}2x\left(\int_X^Y 2z\,dz\right)^2 2y, \ x<y$$
$$= 48xy(y^2-x^2).$$

$$P(X_{(4)}-X_{(1)} \le a\} = \int_0^{1-a}\int_0^{a+x} 48xy(y^2-x^2)dydx + \int_{1-a}^{1}\int_0^{1} 48xy(y^2-x^2)dydx$$

52. $$f_{R_1}(r,\theta) = \frac{r}{\pi} = 2r\frac{1}{2\pi},\ 0 \le r \le 1,\ 0 \le \theta < 2\pi.$$

Hence, R and θ are independent with θ being uniformly distributed on $(0, 2\pi)$ and R having density $f_R(\text{r}) = 2r$, $0 < r < 1$.

53. $f_{R,\theta}(r,\theta) = r,\quad 0 < r\sin\theta < 1,\quad 0 < r\cos\theta < 1,\quad 0 < \theta < \pi/2,\quad 0 < r < \sqrt{2}$

54. $$J = \begin{vmatrix} \frac{1}{2}x^{-1/2}\cos u\sqrt{2} & \frac{1}{2}z^{-1/2}\sin u\sqrt{2} \\ -\sqrt{2z}\sin u & \sqrt{2z}\cos u \end{vmatrix} = \cos^2 u + \sin^2 u = 1$$

$f_{u,z}(u, z)$ - $\frac{1}{2\pi}e^{-z}$. But $x^2 + y^2 = 2z$ so

$f_{X,Y}(x, y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$

55. (a) If $u = xy$, $v = xy$, then $J = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -2\frac{x}{y}$ and

$y = \sqrt{u/v}$, $x = \sqrt{vu}$. Hence,

(b) $f_{u,v}(u, v) = \frac{1}{2v} f_{X,Y}\left(\sqrt{vy}, \sqrt{u/v}\right) = \frac{1}{2vu^2}$, $u \ge 1$, $\frac{1}{u} < v < u$

$$f_u(u) = \int_{1/u}^{u} \frac{1}{2vu^2} dv = \frac{1}{u^2} \log u \text{ , } u \ge 1.$$

For $v > 1$

$$f_V(v) = \int_{v}^{\infty} \frac{1}{2vu^2} du = \frac{1}{2v^2} \text{ , } v > 1$$

For $v < 1$

$$f_V(v) = \int_{1/2}^{\infty} \frac{1}{2vu^2} du = \frac{1}{2}, \ 0 < v < 1.$$

56. (a) $u = x + y$, $v = x/y \Rightarrow y = \frac{u}{v+1}$, $x = \frac{uv}{v+1}$

$$J = \begin{vmatrix} 1 & 1 \\ 1/y & -x/y^2 \end{vmatrix} = -\left(\frac{x}{y^2} + \frac{1}{y}\right) = \frac{-1}{y^2}(x + y) = \frac{-(v+1)^2}{u}$$

$$f_{u,v}(u, v) = \frac{u}{(v+1)^2} \text{ , } 0 < uv < 1 + v, \ 0 < u < 1 + v$$

58. $y_1 = x_1 + x_2$, $y_2 = e^{x_1}$. $J = \begin{vmatrix} 1 & 1 \\ e^{x_1} & 0 \end{vmatrix} = -e^{x_1} = -y_2$

$x_1 = \log y_2$, $x_2 = y_1 - \log y_2$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{y_2} \lambda e^{-\lambda \log y_2} \lambda e^{-\lambda(y_1 - \log y_2)}$$
$$= \frac{1}{y_2} \lambda^2 e^{-\lambda y_1} \text{ , } 1 \le y_2, \ y_1 \ge \log y_2$$

59. $u = x + y,\ v = x + z,\ w = y + z \Rightarrow z = \dfrac{v+w-u}{2}, x = \dfrac{v-w+u}{2}, y = \dfrac{w-v+u}{2}$

$$J = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

$$f(u, v, w) = \frac{1}{2}\exp\left\{-\frac{1}{2}(u+v+w)\right\},\ u + v > w,\ u + w > v,\ v + w + u$$

60. $P(Y_j = i_j, j = 1, \ldots, k + 1\} = P\{Y_j = i_j, j = 1, \ldots, k\}\ P(Y_{k+1} = i_{k+1} \mid Y_j = i_j, j = 1, \ldots, k\}$

$$= \frac{k!(n-k)!}{n!} P\{n+1-\sum_{i=1}^{k} Y_i = i_{k+1} \big| Y_j = i_j, j = 1,\ldots,k\}$$

$$= \begin{cases} k!(n-k)!/n!, & \text{if } \sum_{j=1}^{k+1} i_j = n+1 \\ 0, & \text{otherwise} \end{cases}$$

Thus, the joint mass function is symmetric, which proves the result.

61. The joint mass function is

$$P\{X_i = x_i, i = 1, \ldots, n\} = 1/\binom{n}{k}, x_i \in \{0, 1\}, i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = k$$

As this is symmetric in $x_1, \ldots, x_n$ the result follows.

Theoretical Exercises

1. $$\begin{aligned}P\{X \le a_2,\ Y \le b_2\} = &P\{a_1 < X \le a_2,\ b_1 < Y \le b_2\}\\ &+ P\{X \le a_1,\ b_1 < Y \le b_2\}\\ &+ P\{a_1 < X \le a_2,\ Y \le b_1\}\\ &+ P\{X \le a_1,\ Y \le b_1\}.\end{aligned}$$

The above following as the left hand event is the union of the 4 mutually exclusive right hand events. Also,

$$\begin{aligned}P\{X \le a_1, Y \le b_2\} = &P\{X \le a_1,\ b_1 < Y \le b_2\ \}\\ &+ P\{X \le a_1,\ Y \le b_1\}\end{aligned}$$

and similarly,

$$\begin{aligned}P\{X \le a_2, Y \le b_1\} = &P\{a_1 \le X\ \le a_2, < Y \le b_1\ \}\\ &+ P\{X \le a_1,\ Y \le b_1\}.\end{aligned}$$

Hence, from the above

$$\begin{aligned}F(a_2, b_2) = &P\{a_1 < X \le a_2, b_1 < Y \le b_2\} + F(a_1, b_2) - F(a_1, b_1)\\ &+ F(a_2, b_1) - F(a_1, b_1) + F(a_1, b_1).\end{aligned}$$

2. Let X_i denote the number of type i events, i= 1, …, n.

$$\begin{aligned}P\{X_1 = r_1, \ldots, X_n = r_n\} &= P\left\{X_1 = r_1, \ldots, X_n = r_n \,\middle|\, \sum_{`1}^{n} r_i \text{ events}\right\}\\ &\quad \times e^{-\lambda}\lambda^{\sum_1^n r_i} \Big/ \left(\sum_1^n r_i\right)!\\ &= \frac{\left(\sum_1^n r_i\right)!}{r_1!\ldots r_n!} P_1^{r_1}\ldots p_n^{r_n} \frac{e^{-\lambda}\lambda^{\sum_1^n r_i}}{\left(\sum_1^n r_i\right)!}\\ &= \prod_{i=1}^{n} e^{-\lambda P_i}(\lambda_{p_i})^{r_i} \big/ r_i!\end{aligned}$$

3. Throw a needle on a table, ruled with equidistant parallel lines a distance D apart, a large number of times. Let L, $L < D$, denote the length of the needle. Now estimate π by $\dfrac{2L}{fD}$ where f is the fraction of times the needle intersects one of the lines.

5. (a) For $a > 0$

$$\begin{aligned} F_Z(a) &= P\{X \le aY\} \\ &= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy \\ &= \int_0^{\infty} F_X(ay) f_Y(y)\,dy \end{aligned}$$

$$f_Z(a) = \int_0^{\infty} f_X(ay)\, y f_Y(y)\,dy$$

(b)

$$\begin{aligned} F_Z(a) &= P\{XY < a\} \\ &= \int_0^{\infty}\int_0^{a/y} f_X(x) f_Y(y)\,dx\,dy \\ &= \int_0^{\infty} F_X(a/y) f_Y(y)\,dy \end{aligned}$$

$$f_Z(a) = \int_0^{\infty} f_X(a/y) \frac{1}{y} f_Y(y)\,dy$$

If X is exponential with rate λ and Y is exponential with rate μ then (a) and (b) reduce to

(a) $F_Z(a) = \displaystyle\int_0^{\lambda} \lambda e^{-\lambda a y}\, y \mu e^{-\mu y}\,dy$

(b) $F_Z(a) = \displaystyle\int_0^{\infty} \lambda e^{-\lambda a / y} \frac{1}{y} \mu e^{-\mu y}\,dy$

6.

$$\begin{aligned} F_{X+Y}(t) &= \int_{x+y\le t} f_{X,Y}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{t-x} f_{X,Y}(x,y)\,dy\,dx \end{aligned}$$

Differentiation yields that

$$\begin{aligned} f_{X+Y}(t) &= \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{t-x} f_{X,Y}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty}\frac{d}{dt}\int_{-\infty}^{t-x} f_{X,Y}(x,y)\,dy\,dx \\ &= \int_{-\infty}^{\infty} f_{X,Y}(x, t-x)\,dx \end{aligned}$$

7. (a) $P\{cX \le a\} = P\{X \le a/c\}$ and differentiation yields

$$f_{cX}(a) = \frac{1}{c} f_X(a/c) = \frac{\lambda}{c} e^{-\lambda a/c} (\lambda a/c)^{t-1} \Gamma(t) .$$

Hence, cX is gamma with parameters $(t, \lambda/c)$.

(b) A chi-squared random variable with $2n$ degrees of freedom can be regarded as being the sum of n independent chi-square random variables each with 2 degrees of freedom (which by Example is equivalent to an exponential random variable with parameter λ). Hence by Proposition X_{2n}^2 is a gamma random variable with parameters $(n, 1/2)$ and the result now follows from part (a).

8. (a) $P\{W \le t\} = 1 - P\{W > t\} = 1 - P\{X > t, Y > t\} = 1 - [1 - F_X(t)]\,[1 - F_Y(t)]$

(b) $f_W(t) = f_X(t)[1 - F_Y(t)] + f_Y(t)\,[1 - F_X(t)]$

Dividing by $[1 - F_X(t)][1 - F_Y(t)]$ now yields

$$\lambda_W(t) = f_X(t)/[1 - F_X(t)] + f_Y(t)/[1 - F_Y(t)] = \lambda_X(t) + \lambda_Y(t)$$

9. $$P\{\min(X_1, \ldots, X_n) > t\} = P\{X_1 > t, \ldots, X_n > t\} = e^{-\lambda t} \ldots e^{-\lambda t} = e^{-n\lambda t}$$

thus showing that the minimum is exponential with rate $n\lambda$.

10. If we let X_i denote the time between the i^{th} and $(i + 1)^{\text{st}}$ failure, $i = 0, \ldots, n - 2$, then it follows from Exercise 9 that the X_i are independent exponentials with rate 2λ. Hence, $\sum_{i=0}^{n-2} X_i$ the amount of time the light can operate is gamma distributed with parameters $(n - 1, 2\lambda)$.

11. $$I = \underset{x_1 < x_2 > x_3 < x_4 > x_5}{\int\int\int\int\int} f(x_1) \ldots f(x_5)dx_1 \ldots dx_5$$

$$= \underset{\substack{u_1 < u_2 > u_3 < u_4 > u_5 \\ 0 < u_i < 1}}{\int\int\int\int\int} du_1 \ldots du_5 \qquad \text{by } u_i = F(x_i),\ i = 1, \ldots, 5$$

$$= \int\int\int\int u_2 du_2 ... du_5$$

$$= \int\int\int (1 - u_3^2)/2 \quad du_3 \ldots$$

$$= \int\int [u_4 - u_4^3/3]/2 du_4 du_5$$

$$= \int_0^1 [u^2 - u^4/3]/2 du = 2/15$$

12. Assume that the joint density factors as shown, and let

$$C_i = \int_{-\infty}^{\infty} g_i(x)dx, \; i = 1, \ldots, n$$

Since the n-fold integral of the joint density function is equal to 1, we obtain that

$$1 = \prod_{i=1}^{n} C_i$$

Integrating the joint density over all x_i except x_j gives that

$$f_{X_j}(x_j) = g_j(x_j)\prod_{i \neq j} C_i = g_j(x_j)/C_j$$

If follows from the preceding that

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} f_{X_j}(x_j)$$

which shows that the random variables are independent.

13. No. Let $X_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & -- \end{cases}$. Then

$$f_{X|X_1}, \ldots, X_{n+m}(x \,|\, x_1, \ldots, x_{n+m}) = \frac{P\{x_1, \ldots, x_{n+m} \,|\, X = x\}}{P\{x_1, \ldots, x_{n+m}\}} f_X(x)$$
$$= cx^{\sum x_i}(1-x)^{n+m-\sum x_i}$$

and so given $\sum_{1}^{n+m} X_i = n$ the conditional density is still beta with parameters $n + 1$, $m + 1$.

14. $P\{X = i \,|\, X + Y = n\} = P\{X = i, Y = n - i\}/P\{X + Y = n\}$

$$= \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{\binom{n-1}{1} p^2(1-p)^{n-2}} = \frac{1}{n-1}$$

15. Let X denote the trial number of the kth success, and let $s, s, f, f, s, \ldots, f$ be an outcome of the first $n - 1$ trials that contains a total of $k - 1$ successes and $n - k$ failures. Using that X is a negative binomial random, we have

$$\begin{aligned} P\left(s,s,f,f,s,\ldots,f \middle| X=n\right) &= \frac{P(s,s,f,f,s,\ldots,f,X=n)}{P\{X=n\}} \\ &= \frac{P(s,s,f,f,s,\ldots,f,s)}{\binom{n-1}{k-1}p^k(1-p)^{n-k}} \\ &= \frac{p^k(1-p)^{n-k}}{\binom{n-1}{k-1}p^k(1-p)^{n-k}} \\ &= \frac{1}{\binom{n-1}{k-1}} \end{aligned}$$

and the result if proven.

16. $$\begin{aligned} P\{X=k \mid X+Y=m\} &= \frac{P\{X=k, X+Y=m\}}{P\{X+Y=m\}} \\ &= \frac{P\{X=k, Y=m-k\}}{P\{X+Y=m\}} \\ &= \frac{\binom{n}{k}p^k(1-p)^{n-k}\binom{n}{m-k}p^{m-k}(1-p)^{n-m+k}}{\binom{2n}{m}p^m(1-p)^{2n-m}} \\ &= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} \end{aligned}$$

17. $$\begin{aligned} P(X=n, Y=m) &= \sum_i P(X=n, Y=m \mid X_2=i)P(X_2=i) \\ &= e^{-(\lambda_1+\lambda_2+\lambda_3)}\sum_{i=0}^{\min(n,m)} \frac{\lambda_1^{n-i}}{(n-1)!}\frac{\lambda_3^{m-i}}{(m-i)!}\frac{\lambda_2^i}{i!} \end{aligned}$$

18. Starting with

$$p(i|j) = \frac{P(X = i, Y = j)}{P(Y = j)}$$

$$q(j|i) = \frac{P(X = i, Y = j)}{P(X = i)}$$

we see that

$$\frac{p(i|j)}{q(j|i)} = \frac{P(X = i)}{P(Y = j)}$$

which gives that

$$\sum_i \frac{p(i|j)}{q(j|i)} = \frac{1}{P(Y = j)}$$

and the result follows.

19. (a) $P\{X_1 > X_2 \mid X_1 > X_3\} = \dfrac{P\{X_1 = \max(X_1, X_2, X_3)\}}{P\{X_1 > X_3\}} = \dfrac{1/3}{1/2} = 2/3$

(b) $P\{X_1 > X_2 \mid X_1 < X_3\} = \dfrac{P\{X_3 > X_1 > X_2\}}{P\{X_1 < X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(c) $P\{X_1 > X_2 \mid X_2 > X_3\} = \dfrac{P\{X_1 > X_2 > X_3\}}{P\{X_2 > X_3\}} = \dfrac{1/3!}{1/2} = 1/3$

(d) $P\{X_1 > X_2 \mid X_2 < X_3\} = \dfrac{P\{X_2 = \min(X_1, X_2, X_3)\}}{P\{X_2 < X_3\}} = \dfrac{1/3}{1/2} = 2/3$

20.
$$\begin{aligned} P\{U > s \mid U > a\} &= P\{U > s\}/P\{U > a\} \\ &= \frac{1-s}{1-a},\ a < s < 1 \end{aligned}$$

$$\begin{aligned} P\{U < s \mid U < a\} &= P\{U < s\}/P\{U < a\} \\ &= s/a,\ 0 < s < a \end{aligned}$$

Hence, $U \mid U > a$ is uniform on (a, 1), whereas $U \mid U < a$ is uniform over (0, a).

21.
$$\begin{aligned} f_{W|N}(w \mid n) &= \frac{P\{N = n \mid W = w\} f_W(w)}{P\{N = n\}} \\ &= Ce^{-w}\frac{w^n}{n!}\beta e^{-\beta w}(\beta w)^{t-1} \\ &= C_1 e^{-(\beta+1)w} w^{n+t-1} \end{aligned}$$

where C and C_1 do not depend on w. Hence, given $N = n$, W is gamma with parameters ($n + t$, $\beta + 1$).

22. $$f_{W|X_i,i=1,\ldots,n}(w|x_1,\ldots,x_n) = \frac{f(x_1,\ldots,x_n|w)f_w(w)}{f(x_1,\ldots,x_n)}$$
$$= C\prod_{i=1}^{n} we^{-wx_i}e^{-\beta w}(\beta w)^{t-1}$$
$$= Ke^{-w\left(\beta+\sum_1^n x_i\right)}w^{n+t-1}$$

23. Let X_{ij} denote the element in row i, column j.

$P\{X_{ij}$ is s saddle point$\}$

$$= P\left\{\min_{k=1,\ldots,m} X_{ik} > \max_{k\neq i} X_{kj}, X_{ij} = \min_k X_{ik}\right\}$$

$$= P\left\{\min_k X_{ik} > \max_{k\neq i} X_{kj}\right\}P\left\{X_{ij} = \min_k X_{ik}\right\}$$

where the last equality follows as the events that every element in the i^{th} row is greater than all elements in the j^{th} column excluding X_{ij} is clearly independent of the event that X_{ij} is the smallest element in row i. Now each size ordering of the $n + m - 1$ elements under consideration is equally likely and so the probability that the m smallest are the ones in row i is $1\Big/\binom{n+m-1}{m}$. Hence

$$P\{X_{ij} \text{ is a saddlepoint}\} = \frac{1}{\binom{n+m-1}{m}}\frac{1}{m} = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

and so

$$P\{\text{there is a saddlepoint}\} = P\left(\bigcup_{i,j}\{X_{ij} \text{ is a saddlepoint}\}\right)$$
$$= \sum_{i,j} P\{X_{ij} \text{ is a saddlepoint}\}$$
$$= \frac{m!n!}{(n+m-1)!}$$

24. For $0 < x < 1$

$$P([X] = n, X - [X] < x) = P(n < X < n + x) = e^{-n\lambda} - e^{-(n+x)\lambda} = e^{-n\lambda}(1 - e^{-x\lambda})$$

Because the joint distribution factors, they are independent. $[X] + 1$ has a geometric distribution with parameter $p = 1 - e^{-\lambda}$ and $x - [X]$ is distributed as an exponential with rate λ conditioned to be less than 1.

25. Let $Y = \max(X_1, \ldots, X_n)$, $Z = \min(X_1, \ldots, X_n)$

$$P\{Y \le x\} = P\{X_i \le x, i = 1, \ldots, n\} = \prod_1^n P\{X_i \le x\} = F^n(x)$$

$$P\{Z > x\} = P\{X_i > x, i = 1, \ldots, n\} = \prod_1^n P\{X_i > x\} = [1 - F(x)]^n.$$

26. (a) Let $d = D/L$. Then the desired probability is

$$n! \int_0^{1-(n-1)d} \int_{x_1+d}^{1-(n-2)d} \cdots \int_{x_{n-3}+d}^{1-2d} \int_{x_{n-2}+d}^{1-d} \int_{x_{n-1}+d}^{1} dx_n dx_{n-1} \ldots dx_2 dx_1$$
$$= [1 - (n-1)\mathrm{d}]^n.$$

(b) 0

27. $$F_{x_{(j)}}(x) = \sum_{i=j}^n \binom{n}{i} F^i(x)[1 - F(x)]^{n-i}$$

$$f_{X_{(j)}}(x) = \sum_{i=j}^n \binom{n}{i} iF^{i-1}(x) f(x)[1 - F(x)]^{n-i}$$
$$- \sum_{i=j}^n \binom{n}{i} F^i(x)(n-i)[1 - F(x)]^{n-i-1} f(x)$$

$$= \sum_{i=j}^n \frac{n!}{(n-i)!(i-1)!} F^{i-1}(x) f(x)[1 - F(x)]^{n-i}$$
$$- \sum_{k=j+1}^n \frac{n!}{(n-k)!(k-1)!} F^{k-1}(x) f(x)[1 - F(x)]^{n-k} \text{ by } k = i + 1$$
$$= \frac{n!}{(n-j)!(j-1)!} F^{j-1}(x) f(x)[1 - F(x)]^{n-j}$$

28. $$f_{X(n+1)}(x) = \frac{(2n+1)!}{n!n!} x^n (1-x)^n$$

29. In order for $X_{(i)} = x_i$, $X_{(j)} = x_j$, $i < j$, we must have

(i) $i - 1$ of the X's less than x_i

(ii) 1 of the X's equal to x_i

(iii) $j - i - 1$ of the X's between x_i and x_j

(iv) 1 of the X's equal to x_j

(v) $n - j$ of the X's greater than x_j

Hence,

$$f_{X_{(i)},X_{(j)}}(x_i, x_j)$$
$$= \frac{n!}{(i-1)!1!(j-i-1)!1!(n-j)!} F^{i-1}(x_i) f(x_i)[F(x_j) - F(x_i)]^{j-i-1} f(x_j) \times [1 - F(x_j)]^{n-j}$$

31. Let $X_1, \ldots, X_n$ be n independent uniform random variables over $(0, a)$. We will show by induction on n that

$$P\{X_{(k)} - X_{(k-1)} > t\} = \begin{cases} \left(\dfrac{a-t}{a}\right)^n & \text{if } t < a \\ 0 & \text{if } t > a \end{cases}$$

It is immediate when $n = 1$ so assume for $n - 1$. In the n case, consider

$$P\{X_{(k)} - X_{(k-1)} > t \mid X_{(n)} = s\}.$$

Now given $X_{(n)} = s$, $X_{(1)}, \ldots, X_{(n-1)}$ are distributed as the order statistics of a set of $n - 1$ uniform $(0, s)$ random variables. Hence, by the induction hypothesis

$$P\{X_{(k)} - X_{(k-1)} > t \mid X_{(n)} = s\} = \begin{cases} \left(\dfrac{s-t}{s}\right)^{n-1} & \text{if } t < s \\ 0 & \text{if } t > s \end{cases}$$

and thus, for $t < a$,

$$P\{X_{(k)} - X_{(k-1)} > t = \int_t^a \left(\frac{s-t}{s}\right)^{n-1} \frac{ns^{n-1}}{a^n} ds = \left(\frac{a-t}{a}\right)^n$$

which completes the induction. (The above used that $f_{X_{(n)}}(s) = n\left(\dfrac{s}{a}\right)^{n-1} \dfrac{1}{a} = \dfrac{ns^{n-1}}{a^n}$).

32. (a) $P\{X > X_{(n)}\} = P\{X$ is largest of $n + 1\} = 1/(n + 1)$

(b) $P\{X > X_{(1)}\} = P\{X$ is not smallest of $n + 1\} = 1 - 1/(n + 1) = n/(n + 1)$

(c) This is the probability that X is either the $(i + 1)^{\text{st}}$ or $(i + 2)^{\text{nd}}$ or … j^{th} smallest of the $n + 1$ random variables, which is clearly equal to $(j - 1)/(n + 1)$.

35. The Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & 1/y \\ 0 & -x/y^2 \end{vmatrix} = -x/y^2$$

Hence, $|J|^{-1} = y^2/|x|$. Therefore, as the solution of the equations $u = x$, $v = x/y$ is $x = u$, $y = u/v$, we see that

$$f_{u,v}(u, v) = \frac{|u|}{v^2} f_{X,Y}(u, u/v) = \frac{|u|}{v^2}\frac{1}{2\pi}e^{-(u^2+u^2/v^2)/2}$$

Hence,

$$\begin{aligned} f_{V(u)} &= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2(1+1/v^2)/2}du \\ &= \frac{1}{2\pi v^2}\int_{-\infty}^{\infty}|u|e^{-u^2/2\sigma^2}du \text{ , where } \sigma^2 = v^2/(1+v^2) \\ &= \frac{1}{\pi v^2}\int_0^{\infty}ue^{-u^2/2\sigma^2}du \\ &= \frac{1}{\pi v^2}\sigma^2\int_0^{\infty}e^{-y}dy \\ &= \frac{1}{\pi(1+v^2)} \end{aligned}$$