Chapter 7

Problems

1. Let $X = 1$ if the coin toss lands heads, and let it equal 0 otherwise. Also, let Y denote the value that shows up on the die. Then, with $p(i, j) = P\{X = i, Y = j\}$

$$E[\text{return}] = \sum_{j=1}^{6} 2jp(1, j) + \sum_{j=1}^{6} \frac{j}{2} p(0, j)$$
$$= \frac{1}{12}(42 + 10.5) = 52.5/12$$

2. (a) $6 \cdot 6 \cdot 9 = 324$

(b) $X = (6 - S)(6 - W)(9 - R)$

(c)
$$\begin{aligned} E[X] &= 6(6)(6)P\{S = 0, W = 0, R = 3\} + 6(3)(9)P\{S = 0, W = 3, R = 0\} \\ &+ 3(6)(9)P\{S = 3, W = 0, R = 0\} + 6(5)(7)P\{S = 0, W = 1, R = 2\} \\ &+ 5(6)(7)P\{S = 1, W = 0, R = 2\} + 6(4)(8)P\{S = 0, W = 2, R = 1\} \\ &+ 4(6)(8)P\{S = 2, W = 0, R = 1\} + 5(4)(9)P\{S = 1, W = 2, R = 0\} \\ &+ 4(5)(9)P\{S = 2, W = 1, R = 0\} + 5(5)(8)P\{S = 1, W = 1, R = 1\} \end{aligned}$$

$$= \frac{1}{\binom{21}{3}}\left[216\binom{9}{3} + 324\binom{6}{3} + 420 \cdot 6\binom{9}{2} + 384\binom{6}{2}9 + 360\binom{6}{2}6 + 200(6)(6)(9)\right]$$
$$\approx 198.8$$

3. If the first win is on trial N, then the winnings is $W = 1 - (N - 1) = 2 - N$. Thus,

(a) $P(W > 0) = P(N = 1) = 1/2$
(b) $P(W < 0) = P(N > 2) = 1/4$
(c) $E[W] = 2 - E[N] = 0$

4.
$$E[XY] = \int_0^1 \int_0^y xy \frac{1}{y} dxdy = \int_0^1 y^2 / 2dy = 1/6$$
$$E[X] = \int_0^1 \int_0^y x \frac{1}{y} dxdy = \int_0^1 y / 2dy = 1/4$$
$$E[Y] = \int_0^1 \int_0^y y \frac{1}{y} dxdy = \int_0^1 ydy = 1/2$$

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5. The joint density of the point (X, Y) at which the accident occurs is

$$f(x, y) = \frac{1}{9}, -3/2 < x, y < 3/2$$
$$= f(x) f(y)$$

where

$$f(a) = 1/3, -3/2 < a < 3/2.$$

Hence we may conclude that X and Y are independent and uniformly distributed on $(-3/2, 3/2)$ Therefore,

$$E[\,|X| + |Y|\,] = 2\int_{-3/2}^{3/2} \frac{1}{3} x\, dx = \frac{4}{3}\int_{0}^{3/2} x dx = 3/2\,.$$

6. $E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = 10(7/2) = 35.$

8. $E[\text{number of occupied tables}] = E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$

Now,

$E[X_i] = P\{i^{\text{th}}$ arrival is not friends with any of first $i - 1\}$
$= (1 - p)^{i-1}$

and so

$$E[\text{number of occupied tables}] = \sum_{i=1}^{N} (1-p)^{i-1}$$

7. Let X_i equal 1 if both choose item i and let it be 0 otherwise; let Y_i equal 1 if neither A nor B chooses item i and let it be 0 otherwise. Also, let W_i equal 1 if exactly one of A and B choose item i and let it be 0 otherwise. Let

$$X = \sum_{i=1}^{10} X_i\,, \quad Y = \sum_{i=1}^{10} Y_i\,, \quad W = \sum_{i=1}^{10} W_i$$

(a) $E[X] = \sum_{i=1}^{10} E[X_i] = 10(3/10)^2 = .9$

(b) $E[Y] = \sum_{i=1}^{10} E[Y_i] = 10(7/10)^2 = 4.9$

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(c) Since $X + Y + W = 10$, we obtain from parts (a) and (b) that

$$E[W] = 10 - .9 - 4.9 = 4.2$$

Of course, we could have obtained $E[W]$ from

$$E[W] = \sum_{i=1}^{10} E[W_i] = 10(2)(3/10)(7/10) = 4.2$$

9. Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$E[X_j] = P\{\text{ball } i \text{ is not in urn } j,\ i \geq j\} = \prod_{i=j}^{n}(1-1/i)$

Hence,

(a) $E[\text{number of empty urns}] = \sum_{j=1}^{n}\sum_{i=j}^{n}(1-1/i)$

(b) $P\{\text{none are empty}\} = P\{\text{ball } j \text{ is in urn } j, \text{ for all } j\}$

$$= \prod_{j=1}^{n} 1/j$$

10. Let X_i equal 1 if trial i is a success and 0 otherwise.

(a) .6. This occurs when $P\{X_1 = X_2 = X_3\} = 1$. It is the largest possible since $1.8 = \sum P\{X_i = 1\} = 3P\{X_i = 1\}$. Hence, $P\{X_i = 1\} = .6$ and so

$$P\{X = 3\} = P\{X_1 = X_2 = X_3 = 1\} \leq P\{X_i = 1\} = .6.$$

(b) 0. Letting

$$X_1 = \begin{cases} 1 \text{ if } U \leq .6 \\ 0 \text{ otherwise} \end{cases}, \qquad X_2 = \begin{cases} 1 \text{ if } U \leq .4 \\ 0 \text{ otherwise} \end{cases}, \qquad X_3 = \begin{cases} 1 \text{ if } U \leq .3 \\ 0 \text{ otherwise} \end{cases}$$

Hence, it is not possible for all X_i to equal 1.

11. Let X_i equal 1 if a changeover occurs on the i^{th} flip and 0 otherwise. Then

$$\begin{aligned} E[X_i] &= P\{i-1 \text{ is } H, i \text{ is } T\} + P\{i-1 \text{ is } T, i \text{ is } H\} \\ &= 2(1-p)p, \quad i \geq 2. \end{aligned}$$

$E[\text{number of changeovers}] = E\left[\sum X_i\right] = \sum_{i=1}^{n} E[X_i] = 2(n-1)(1-p)$

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12. (a) Let X_i equal 1 if the person in position i is a man who has a woman next to him, and let it equal 0 otherwise. Then

$$E[X_i] = \begin{cases} \frac{1}{2}\frac{n}{2n-1}, & \text{if } i = 1, 2\text{n} \\ \frac{1}{2}\left[1-\frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right], & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{2n} E[X_i] \\ &= \frac{1}{2}\left(\frac{2n}{2n-1} + (2n-2)\frac{3n}{4n-2}\right) \\ &= \frac{3n^2-n}{4n-2} \end{aligned}$$

(b) In the case of a round table there are no end positions and so the same argument as in part (a) gives the result

$$n\left[1-\frac{(n-1)(n-2)}{(2n-1)(2n-2)}\right] = \frac{3n^2}{4n-2}$$

where the right side equality assumes that $n > 1$.

13. Let X_i be the indicator for the event that person i is given a card whose number matches his age. Because only one of the cards matches the age of the person i

$$E\left[\sum_{i=1}^{1000} X_i\right] = \sum_{i=1}^{1000} E[X_i] = 1$$

14. The number of stages is a negative binomial random variable with parameters m and $1 - p$. Hence, its expected value is $m/(1 - p)$.

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15. Let $X_{i,j}$, $i \neq j$ equal 1 if i and j form a matched pair, and let it be 0 otherwise.

Then

$$E[X_{i,j}] = P\{i, j \text{ is a matched pair}\} = \frac{1}{n(n-1)}$$

Hence, the expected number of matched pairs is

$$E\left[\sum_{i<j} X_{i,j}\right] = \sum_{i<j} E[X_{i,j}] = \binom{n}{2}\frac{1}{n(n-1)} = \frac{1}{2}$$

16. $$E[X] = \int_{y>x} y\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

17. Let I_i equal 1 if guess i is correct and 0 otherwise.

(a) Since any guess will be correct with probability $1/n$ it follows that

$$E[N] = \sum_{i=1}^{n} E[I_i] = n/n = 1$$

(b) The best strategy in this case is to always guess a card which has not yet appeared. For this strategy, the i^{th} guess will be correct with probability $1/(n - i + 1)$ and so

$$E[N] = \sum_{i=1}^{n} 1/(n-i+1)$$

(c) Suppose you will guess in the order 1, 2, …, n. That is, you will continually guess card 1 until it appears, and then card 2 until it appears, and so on. Let J_i denote the indicator variable for the event that you will eventually be correct when guessing card i; and note that this event will occur if among cards 1 thru i, card 1 is first , card 2 is second, …, and card i is the last among these i cards. Since all $i!$ orderings among these cards are equally likely it follows that

$$E[J_i] = 1/i! \text{ and thus } E[N] = E\left[\sum_{i=1}^{n} J_i\right] = \sum_{i=1}^{n} 1/i!$$

18. $$E[\text{number of matches}] = E\left[\sum_{1}^{52} I_i\right], \quad I_i = \begin{cases} 1 & \text{match on card } i \\ 0 & \text{---} \end{cases}$$

$$= 52\frac{1}{13} = 4 \quad \text{since } E[I_i] = 1/13$$

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19. (a) E[time of first type 1 catch] − 1 $= \frac{1}{p_1} - 1$ using the formula for the mean of a geometric random variable.

(b) Let

$$X_j = \begin{cases} 1 & \text{a type } j \text{ is caught before a type 1} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E\left[\sum_{j \neq 1} X_j\right] = \sum_{j \neq 1} E[X_j]$$
$$= \sum_{j \neq 1} P\{\text{type } j \text{ before type } 1\}$$
$$= \sum_{j \neq 1} P_j / (P_j + P_1),$$

where the last equality follows upon conditioning on the first time either a type 1 or type j is caught to give.

$$P\{\text{type } j \text{ before type } 1\} = P\{j \mid j \text{ or } 1\} = \frac{P_j}{P_j + P_1}$$

20. Similar to (b) of 19. Let

$$X_j = \begin{cases} 1 & \text{ball } j \text{ removed before ball 1} \\ 0 & \text{---} \end{cases}$$

$$E\left[\sum_{j \neq 1} X_j\right] = \sum_{j \neq 1} E[X_j] = \sum_{j \neq 1} P\{\text{ball } j \text{ before ball } 1\}$$
$$= \sum_{j \neq 1} P\{j \mid j \text{ or } 1\}$$
$$= \sum_{j \neq 1} W(j)/W(1) + W(j)$$

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21. (a) $365\binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$

(b) Let $X_j = \begin{cases} 1 & \text{if day } j \text{ is someones birthday} \\ 0 & \text{---} \end{cases}$

$$E\left[\sum_1^{365} X_j\right] = \sum_1^{365} E[X_j] = 365\left[1-\left(\frac{364}{365}\right)^{100}\right]$$

22. From Example 3g, 1 + $\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+6$

23. $E\left[\sum_1^5 X_i + \sum_1^8 Y_i\right] = \sum_1^5 E[X_i] + \sum_1^8 E(Y_i)$

$$= 5\frac{2}{11}\frac{3}{20} + 8\frac{3}{120} = \frac{147}{110}$$

24. Number the small pills, and let X_i equal 1 if small pill i is still in the bottle after the last large pill has been chosen and let it be 0 otherwise, $i = 1, \ldots, n$. Also, let Y_i, $i = 1, \ldots, m$ equal 1 if the i^{th} small pill created is still in the bottle after the last large pill has been chosen and its smaller half returned.

Note that $X = \sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i$. Now,

$E[X_i] = P\{$small pill i is chosen after all m large pills$\}$
$= 1/(m + 1)$

$E[Y_i] = P\{i^{\text{th}}$ created small pill is chosen after $m - i$ existing large pills$\}$
$= 1/(m - i + 1)$

Thus,

(a) $E[X] = n/(m + 1) + \sum_{i=1}^{m} 1/(m-i+1)$

(b) $Y = n + 2m - X$ and thus

$E[Y] = n + 2m - E[X]$

25. $P\{N \geq n\}$ $P\{X_1 \geq X_2 \geq \ldots \geq X_n\} = \frac{1}{n!}$

$$E[N] = \sum_{n=1}^{\infty} P\{N \geq n\} = \sum_{n=1}^{\infty} \frac{1}{n!} = e$$

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26. (a) $E[\max] = \int_0^1 P\{\max > t\}dt$

$$= \int_0^1 (1 - P\{\max \le t)\}dt$$

$$= \int_0^1 (1 - t^n / dt = \frac{n}{n+1}$$

(b) $\text{E}[\min] = \int_0^1 p\{\min > t\}4t$

$$= \int_0^1 (1-t)^n dt = \frac{1}{n+1}$$

27. Let X denote the number of items in a randomly chosen box. Then, with X_i equal to 1 if item i is in the randomly chosen box

$$E[X] = E\left[\sum_{i=1}^{101} X_i\right] = \sum_{i=1}^{101} E[X_i] = \frac{101}{10} > 10$$

Hence, X can exceed 10, showing that at least one of the boxes must contain more than 10 items.

28. We must show that for any ordering of the 47 components there is a block of 12 consecutive components that contain at least 3 failures. So consider any ordering, and randomly choose a component in such a manner that each of the 47 components is equally likely to be chosen. Now, consider that component along with the next 11 when moving in a clockwise manner and let X denote the number of failures in that group of 12. To determine $E[X]$, arbitrarily number the 8 failed components and let, for $i = 1, \ldots, 8$,

$$X_i = \begin{cases} 1, & \text{if failed component } i \text{ is among the group of 12 components} \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$X = \sum_{i=1}^{8} X_i$$

and so

$$E[X] = \sum_{i=1}^{8} E[X_i]$$

Because X_i will equal 1 if the randomly selected component is either failed component number i or any of its 11 neighboring components in the counterclockwise direction, it follows that $E[X_i] = 12/47$. Hence,

$$E[X] = 8(12/47) = 96/47$$

Because $E[X] > 2$ it follows that there is at least one possible set of 12 consecutive components that contain at least 3 failures.

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29. Let X_{ii} be the number of coupons one needs to collect to obtain a type i. Then

$$E[X_{i)}] = 8, \quad i = 1,2$$
$$E\{X_i] = 8/3, \quad i = 3,4$$
$$E[\min(X_1, X_2)] = 4$$
$$E[\min(X_i, X_j)] = 2, \quad i = 1,2, \quad j = 3,4$$
$$E[\min(X_3, X_4)] = 4/3$$
$$E[\min(X_1, X_2, X_j)] = 8/5, \quad j = 3,4$$
$$E[\min(X_i, X_{3,} X_4)] = 8/7, \quad i = 1,2$$
$$E[\min(X_1, X_2, X_3, X_4] = 1$$

(a) $E[\max X_i] = 2 \cdot 8 + 2 \cdot 8/3 - (4 + 4 \cdot 2 + 4/3) + (2 \cdot 8/5 + 2 \cdot 8/7) - 1 = \dfrac{437}{35}$

(b) $E[\max(X_1, X_2)] = 8 + 8 - 4 = 12$

(c) $E[\max(X_3, X_4)] = 8/3 + 8/3 - 4/3 = 4$

(d) Let $Y_1 = \max(X_1, X_2)$, $Y_2 = \max(X_3, X_4)$. Then

$$E[\max(Y_1, Y_2)] = E[Y_1] + E[Y_2] - E[\min(Y_1, Y_2)]$$

giving that

$$E[\min(Y_1, Y_2)] = 12 + 4 - \frac{437}{35} = \frac{123}{35}$$

30. $E[(X - Y)]^2 = \mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = 2\sigma^2$

31. $\mathrm{Var}\left(\sum_{i=1}^{10} X_i\right) = 10\ \mathrm{Var}(X_1)$. Now

$$\begin{aligned}\mathrm{Var}(X_1) &= E[X_1^2] - (7/2)^2\\ &= [1 + 4 + 9 + 16 + 25 + 36]/6 - 49/4\\ &= 35/12\end{aligned}$$

and so $\mathrm{Var}\left(\sum_{i=1}^{10} X_i\right) = 350/12$.

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32. Use the notation in Problem 9,

$$X = \sum_{j=1}^{n} X_j$$

where X_j is 1 if box j is empty and 0 otherwise. Now, with

$E[X_j] = P\{X_j = 1\} = \prod_{i=j}^{n}(1-1/i)$, we have that

$$\text{Var}(X_j) = E[X_j](1 - E[X_j]).$$

Also, for $j < k$

$$E[X_jX_k] = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i)$$

Hence, for $j < k$,

$$\text{Cov}(X_j, X_k) = \prod_{i=j}^{k-1}(1-1/i)\prod_{i=k}^{n}(1-2/i) - \prod_{i=j}^{n}(1-1/i)\prod_{i=k}^{n}(1-1/i)$$

$$\text{Var}(X) = \sum_{j=1}^{n} E[X_j](1-E[X_j]) + 2\text{Cov}(X_j, X_k)$$

33. (a) $E[X^2 + 4X + 4] = E[X^2] + 4E[X] + 4 = \text{Var}(X) + E^2[X] + 4E[X] + 4 = 14$

(b) $\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 45$

34. Let $X_j = \begin{cases} 1 & \text{if couple } j \text{ are seated next to each other} \\ 0 & \text{otherwise} \end{cases}$

(a) $E\left[\sum_{1}^{10} X_j\right] = 10\frac{2}{19} = \frac{20}{19}$; $P\{X_j = 1\} = \frac{2}{19}$ since there are 2 people seated next to wife j and so the probability that one of them is her husband is $\frac{2}{19}$.

(b) For $i \neq j$, $E[X_iX_j] = P\{X_i = 1, X_j = 1\}$
$= P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\}$
$= \frac{2}{19}\frac{2}{18}$ since given $X_i = 1$ we can regard couple i as a single entity.

$$\text{Var}\left(\sum_{j=1}^{10} X_j\right) = 10\frac{2}{19}\left(1-\frac{2}{19}\right) + 10\cdot 9\left[\frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2\right]$$

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35. (a) Let X_1 denote the number of nonspades preceding the first ace and X_2 the number of nonspades between the first 2 aces. It is easy to see that

$$P\{X_1 = i,\ X_2 = j\} = P\{X_1 = j,\ X_2 = i\}$$

and so X_1 and X_2 have the same distribution. Now $E[X_1] = \frac{48}{5}$ by the results of Example 3j and so $E[2 + X_1 + X_2] = \frac{106}{5}$.

(b) Same method as used in (a) yields the answer $5\left(\frac{39}{14}+1\right)=\frac{265}{14}$.

(c) Starting from the end of the deck the expected position of the first (from the end) heart is, from Example 3j, $\frac{53}{14}$. Hence, to obtain all 13 hearts we would expect to turn over

$52 - \frac{53}{14} + 1 = \frac{13}{14}(53)$.

36. Let $X_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 1 \\ 0 & \text{otherwise} \end{cases}$, $\quad Y_i = \begin{cases} 1 & \text{roll } i \text{ lands on } 2 \\ 0 & \text{otherwise} \end{cases}$

$$\text{Cov}(X_i, Y_j) = E[X_i Y_j] - E[X_i]E[Y_j]$$
$$= \begin{cases} -\frac{1}{36} & i = j \text{ (since } X_iY_j = 0 \text{ when } i = j \\ \frac{1}{36}-\frac{1}{36}=0 & i \neq j \end{cases}$$

$$\text{Cov}\sum_i X_i, \sum_j Y_j = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$
$$= -\frac{n}{36}$$

37. Let W_i, $i = 1, 2$, denote the i^{th} outcome.

$$\begin{aligned}\text{Cov}(X, Y) &= \text{Cov}(W_1 + W_2,\ W_1 - W_2) \\ &= \text{Cov}(W_1, W_1) - \text{Cov}(W_2, W_2) \\ &= \text{Var}(W_1) - \text{Var}(W_2) = 0\end{aligned}$$

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38. $E[XY] = \int_0^\infty \int_0^x y 2e^{-2x} dy dx$

$$= \int_0^\infty x^2 e^{-2x} dx = \frac{1}{8}\int_0^\infty y^2 e^{-y} dy = \frac{\Gamma(3)}{8} = \frac{1}{4}$$

$E[X] = \int_0^\infty x f_x(x) dx,\ f_x(x) = \int_0^x \frac{2e^{-2x}}{x} dy\ = 2e^{-2x}$

$$= \frac{1}{2}$$

$E[Y] = \int_0^\infty y f_Y(y) dy,\ f_Y(y) = \int_0^\infty \frac{2e^{-2x}}{x} dx$

$$= \int_0^\infty \int_y^\infty y \frac{2e^{-2x}}{x} dx dy$$

$$= \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x} dy dx$$

$$= \int_0^\infty x e^{-2x} dx = \frac{1}{4}\int y e^{-2} dy = \frac{\Gamma(2)}{4} = \frac{1}{4}$$

$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

39. $\text{Cov}(Y_n, Y_n) = \text{Var}(Y_n) = 3\sigma^2$
$\text{Cov}(Y_n, Y_{n+1}) = \text{Cov}(X_n + X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2} + X_{n+3})$
$= \text{Cov}(X_{n+1} + X_{n+2},\ X_{n+1} + X_{n+2}) = \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2$
$\text{Cov}(Y_n, Y_{n+2}) = \text{Cov}(X_{n+2}, X_{n+2}) = \sigma^2$
$\text{Cov}(Y_n, Y_{n+j}) = 0$ when $j \geq 3$

40. $f_Y(y) = e^{-y}\int \frac{1}{y} e^{-x/y} dx = e^{-y}$. In addition, the conditional distribution of X given that $Y = y$ is exponential with mean y. Hence,

$E[Y] = 1,\ E[X] = E[E[X \mid Y]] = E[Y] = 1$

Since, $E[XY] = E[E[XY \mid Y]] = E[YE[X \mid Y]] = E[Y^2] = 2$ (since Y is exponential with mean 1, it follows that $E[Y^2] = 2$). Hence, $\text{Cov}(X, Y) = 2 - 1 = 1$.

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41. The number of carp is a hypergeometric random variable.

$$E[X] = \frac{60}{10} = 6$$

$$\text{Var}(X) = \frac{20(80)}{99}\frac{3}{10}\frac{7}{10} = \frac{336}{99} \text{ from Example 5c.}$$

42. (a) Let $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a man and a woman} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = P\{X_i = 1\} = \frac{10}{19}$$

$$E[X_iX_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1\}P\{X_j = 1 \mid X_2 = 1\}$$
$$= \frac{10}{19}\frac{9}{17}, \; i \neq \text{j}$$

$$E\left[\sum_1^{10} X_i\right] = \frac{100}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{10}{19}\left(1-\frac{10}{19}\right) + 10\cdot 9\left[\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right] = \frac{900}{(19)^2}\frac{18}{17}$$

(b) $X_i = \begin{cases} 1 & \text{pair } i \text{ consists of a married couple} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = \frac{1}{19}, \; E[X_iX_j] = P\{X_i = 1\}P\{X_j = 1 \mid X_i = 1\} = \frac{1}{19}\frac{1}{17}, \; i \neq j$$

$$E\left[\sum_1^{10} X_i\right] = \frac{10}{19}$$

$$\text{Var}\left(\sum_1^{10} X_i\right) = 10\frac{1}{19}\frac{15}{19} + 10\cdot 9\left[\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right] = \frac{180}{(19)^2}\frac{18}{17}$$

43. $E[R] = n(n + m + 1)/2$

$$\text{Var}(R) = \frac{nm}{n+m-1}\left[\frac{\sum_{i=1}^{n+m} i^2}{n+m} - \left(\frac{n+m+1}{2}\right)^2\right]$$

The above follows from Example 3d since when $F = G$, all orderings are equally likely and the problem reduces to randomly sampling n of the $n + m$ values 1, 2, …, $n + m$.

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44. From Example 8l $\dfrac{n}{n+m}+\dfrac{nm}{n+m}$. Using the representation of Example 2l the variance can be computed by using

$$E[I_1 I_{l+j}] = \begin{cases} 0 & , \quad j=1 \\ \dfrac{n}{n+m}\dfrac{m}{n+m-1}\dfrac{n-1}{n+m-2} & , \quad n-1 \le j < 1 \end{cases}$$

$$E[I_i I_{i+j}] = \begin{cases} 0 & , \quad j=1 \\ \dfrac{mn(m-1)(n-1)}{(n+m)(n+m-1)(n+m-2)(n+m-3)} & , \quad n-1 \le j < 1 \end{cases}$$

45. (a) $\dfrac{\text{Cov}(X_1+X_2, X_2+X_3)}{\sqrt{\text{Var}(X_1+X_2)}\sqrt{\text{Var}(X_2+X_3)}} = \dfrac{1}{2}$

(b) 0

46.
$$\begin{aligned} E[I_1 I_2] &= \sum_{i=2}^{12} E[I_1 I_2 \mid \text{bank rolls } i] \text{P}\{\text{bank rolls } i\} \\ &= \sum_i (P\{\text{roll is greater than } i\})^2 P\{\text{bank rolls } i\} \\ &= E[I_1^2] \\ &\ge (E[I_1])^2 \\ &= E[I_1]\, E[I_2] \end{aligned}$$

47. (a) It is binomial with parameters $n-1$ and p.

(b) Let $x_{i,j}$ equal 1 if there is an edge between vertices i and j, and let it be 0 otherwise. Then, $D_i = \sum_{k \ne i} X_{i,k}$, and so, for $i \ne j$

$$\begin{aligned} \text{Cov}(D_i, D_j) &= \text{Cov}\left(\sum_{k\ne i} X_{i,k}, \sum_{r\ne j} X_{r,j}\right) \\ &= \sum_{k\ne i}\sum_{r\ne j} Cov(X_{i,k}, X_{r,j}) \\ &= \text{Cov}(X_{i,j}, X_{i,j}) \\ &= \text{Var}(X_{i,j}) \\ &= p(1-p) \end{aligned}$$

where the third equality uses the fact that except when $k=j$ and $r=i$, $X_{i,k}$ and $X_{r,j}$ are independent and thus have covariance equal to 0. Hence, from part (a) and the preceding we obtain that for $i \ne j$,

$$\rho(D_i, D_j) = \frac{p(1-p)}{(n-1)p(1-p)} = \frac{1}{n-1}$$

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48. (a) $E[X] = 6$

(b) $E[X \mid Y = 1] = 1 + 6 = 7$

(c) $1\frac{1}{5} + 2\frac{4}{5}\frac{1}{5} + 3\left(\frac{4}{5}\right)^2\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4(5+6)$

49. Let C_i be the event that coin i is being flipped (where coin 1 is the one having head probability .4), and let T be the event that 2 of the first 3 flips land on heads. Then

$$P(C_1 \mid T) = \frac{P(T \mid C_1)P(C_1)}{P(T \mid C_1)P(C_1) + P(T \mid C_2)P(C_2)}$$
$$= \frac{3(.4)^2(.6)}{3(.4)^2(.6) + 3(.7)^2(.3)} = .395$$

Now, with N_j equal to the number of heads in the final j flips, we have

$$E[N_{10} \mid T] = 2 + E[N_7 \mid T]$$

Conditioning on which coin is being used, gives

$$E[N_7 \mid T] = E[N_7 \mid TC_1]P(C_1T) + E[N_7TC_2]P(C_2 \mid T) = 2.8(.395) + 4.9(.605) = 4.0705$$

Thus, $E[N_{10} \mid T] = 6.0705$.

50. $$f_{X|Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\,dx} = \frac{1}{y}e^{-x/y}, \quad 0 < x < \infty$$

Hence, given $Y = y$, X is exponential with mean y, and so

$$E[X^2 \mid Y = y] = 2y^2$$

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\,dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3\frac{1}{y}dx = y^3/4$$

52. The average weight, call it $E[W]$, of a randomly chosen person is equal to average weight of all the members of the population. Conditioning on the subgroup of that person gives

$$E[W] = \sum_{i=1}^r E\{W \mid \text{member of subgroup } i]\,p_i = \sum_{i=1}^r w_i p_i$$

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53. Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$\begin{aligned} E[X] &= E[X \mid I = 1](.5) + E[X \mid I = 2](.3) + E[X \mid I = 3](.2) \\ &= (2 + E[X])(.5) + (4 + E[X])(.3) + .2 \end{aligned}$$

Therefore,

$$E[X] = 12$$

54. Let R_i denote the return from the policy that stops the first time a value at least as large as i appears. Also, let X be the first sum, and let $p_i = P\{X = i\}$. Conditioning on X yields

$$\begin{aligned} E[R_5] &= \sum_{i=2}^{12} E[R_5 \mid X = i\} p_i \\ &= E[R_5](p_2 + p_3 + p_4) + \sum_{i=5}^{12} ip_i - 7p_7 \\ &= \frac{6}{36} E[R_5] + 5(4/36) + 6(5/36) + 8(5/36) + 9(4/36) + 10(3/36) + 11(2/36) + 12(1/36) \\ &= \frac{6}{36} E[R_5] + 190/36 \end{aligned}$$

Hence, $E[R_5] = 19/3 \approx 6.33$. In the same fashion, we obtain that

$$E[R_6] = \frac{10}{36} E[R_6] + \frac{1}{36} [30 + 40 + 36 + 30 + 22 + 12]$$

implying that

$$E[R_6] = 170/26 \approx 6.54$$

Also,

$$E[R_8] = \frac{15}{36} E[R_8] + \frac{1}{36}(140)$$

or,

$$E[R_8] = 140/21 \approx 6.67$$

In addition,

$$E[R_9] = \frac{20}{26} E[R_9] + \frac{1}{36}(100)$$

or

$$E[R_9] = 100/16 = 6.25$$

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And

$$E[R_{10}] = \frac{24}{36}E[R_{10}] + \frac{1}{36}(64)$$

or

$$E[R_{10}] = 64/12 \approx 5.33$$

The maximum expected return is $E[R_8]$.

55. Let N denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$
$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck i with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right] e^{-6} 6^n / n!$$

56. Let $I_i = \begin{cases} 1 & \text{elevator stops at floor } i \\ 0 & \text{otherwise} \end{cases}$. Let X be the number that enter on the ground floor.

$$E\left[\sum_{i=1}^{N} I_i \mid X = k\right] = \sum_{i=1}^{N} E[I_i \mid X = k] = N\left[1 - \left(\frac{N-1}{N}\right)^k\right]$$
$$E\left[\sum_{i=1}^{N} I_i\right] = N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k e^{-10}\frac{(10)^k}{k!}$$
$$= N - Ne^{-10/N} = N(1 - e^{-10/N})$$

57. $E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 12.5$

58. Let X denote the number of flips required. Condition on the outcome of the first flip to obtain.

$$E[X] = E[X \mid \text{heads}]p + E[x \mid \text{tails}](1 - p)$$
$$= [1 + 1/(1 - p)]p + [1 + 1/p](1 - p)$$
$$= 1 + p/(1 - p) + (1 - p)/p$$

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59. (a) E[total prize shared] = P\{someone wins\} = $1 - (1 - p)^{n+1}$

(b) Let X_i be the prize to player i. By part (a)

$$E\left[\sum_{i=1}^{n+1} X_i\right] = 1 - (1 - p)^{n+1}$$

But, by symmetry all $E[X_i]$ are equal and so

$$E[X] = [1 - (1 - p)^{n+1}]/(n + 1)$$

(c) $E[X] = p\, E[1/(1 + B)]$ where B, which is binomial with parameters n and p, represents the number of other winners.

60. (a) Since the sum of their number of correct predictions is n (one for each coin) it follows that one of them will have more than $n/2$ correct predictions. Now if N is the number of correct predictions of a specified member of the syndicate, then the probability mass function of the number of correct predictions of the member of the syndicate having more than $n/2$ correct predictions is

$$\begin{aligned} P\{i \text{ correct}\} &= P\{N = i\} + P(N = n - i\} \quad i > n/2 \\ &= 2P\{N = i\} \\ &= P\{N = i \mid N > n/2\} \end{aligned}$$

(b) X is binomial with parameters m, 1/2.

(c) Since all of the $X + 1$ players (including one from the syndicate) that have more than $n/2$ correct predictions have the same expected return we see that

$$(X + 1) \cdot \text{Payoff to syndicate} = m + 2$$

implying that

$$E[\text{Payoff to syndicate}] = (m + 2)\, E[(X + 1)^{-1}]$$

(d) This follows from part (b) above and (c) of Problem 56.

61. (a) $P(M \le x) = \sum_{n=1}^{\infty} P(M \le x \mid N = n) P(N = n) = \sum_{n=1}^{\infty} F^n(x) p(1-p)^{n-1} = \dfrac{pF(x)}{1-(1-p)F(x)}$

(b) $P(M \le x \mid N = 1) = F(x)$

(c) $P(M \le x \mid N > 1) = F(x)P(M \le x)$

(d) $P(M \le x) = P(M \le x \mid N = 1)P(N = 1) + P(M \le x \mid N > 1)P(N > 1)$
$\quad = F(x)p + F(x)P(M \le x)(1 - p)$

again giving the result

$$P(M \le x) = \frac{pF(x)}{1-(1-p)F(x)}$$

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62. The result is true when $n = 0$, so assume that

$$P\{N(x) \geq n\} = x^n/(n-1)!$$

Now,

$$\begin{aligned}
P\{N(x) \geq n+1\} &= \int_0^1 P\{N(x) \geq n+1 \mid U_1 = y\}dy \\
&= \int_0^x P\{N(x-y) \geq n\}dy \\
&= \int_0^x P\{N(u) \geq n\}du \\
&= \int_0^x u^{n-1}/(n-1)!\ du \text{ by the induction hypothesis} \\
&= x^n/n!
\end{aligned}$$

which completes the proof.

(b) $E[N(x)] = \sum_{n=0}^{\infty} P\{N(x) > n = \sum_{n=0}^{\infty} P\{N(x) \geq n+1\} = \sum_{n=0}^{\infty} x^n/n! = e^x$

63. (a) Number the red balls and the blue balls and let X_i equal 1 if the i^{th} red ball is selected and let it by 0 otherwise. Similarly, let Y_j equal 1 if the j^{th} blue ball is selected and let it be 0 otherwise.

$$\text{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \text{Cov}(X_i, Y_j)$$

Now,

$E[X_i] = E[Y_j] = 12/30$

$E[X_iY_j] = P\{\text{red ball } i \text{ and blue ball } j \text{ are selected}\} = \binom{28}{10}\Big/\binom{30}{12}$

Thus,

$$\text{Cov}(X, Y) = 80\left[\binom{28}{10}\Big/\binom{30}{12} - (12/30)^2\right] = -96/145$$

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(b) $E[XY|X] = XE[Y|X] = X(12 - X)8/20$

where the above follows since given X, there are 12-X additional balls to be selected from among 8 blue and 12 non-blue balls. Now, since X is a hypergeometric random variable it follows that

$E[X] = 12(10/30) = 4$ and $E[X^2] = 12(18)(1/3)(2/3)/29 + 4^2 = 512/29$

As $E[Y] = 8(12/30) = 16/5$, we obtain

$$E[XY] = \frac{2}{5}(48 - 512/29) = 352/29,$$

and

$$\text{Cov}(X, Y) = 352/29 - 4(16/5) = -96/145$$

64. (a) $E[X] = E[X|\text{type } 1]p + E[X|\text{type } 2](1 - p) = p\mu_1 + (1 - p)\mu_2$

(b) Let I be the type.

$E[X|I] = \mu_I$, $\text{Var}(X|I) = \sigma_I^2$

$\text{Var}(X) = E[\sigma_I^2] + \text{Var}(\mu_I)$

$= p\sigma_1^2 + (1-p)\sigma_2^2 + p\mu_1^2 + (1-p)\mu_2^2 - [p\mu_1 + (1-p)\mu_2]^2$

65. Let X be the number of storms, and let $G(B)$ be the events that it is a good (bad) year. Then

$$E[X] = E[X|G]P(G) + E[X|B]P(B) = 3(.4) + 5(.6) = 4.2$$

If Y is Poisson with mean λ, then $E[Y^2] = \lambda + \lambda^2$. Therefore,

$$E[X^2] = E[X^2|G]P(G) + E[X^2|B]P(B) = 12(.4) + 30(.6) = 22.8$$

Consequently,

$$\text{Var}(X) = 22.8 - (4.2)^2 = 5.16$$

66. $E[X^2] = \frac{1}{3}\{E[X^2|Y=1] + E[X^2|Y=2] + E[X^2|Y=3]\}$

$= \frac{1}{3}\{9 + E[(5+X)^2] + E[(7+X)^2]\}$

$= \frac{1}{3}\{83 + 24E[X] + 2E[X^2]\}$

$= \frac{1}{3}\{443 + 2E[X^2]\}$ since $E[X] = 15$

Hence,

$$\text{Var}(X) = 443 - (15)^2 = 218.$$

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67. Let F_n denote the fortune after n gambles.

$$\begin{aligned} E[F_n] = E[E[F_n \mid F_{n-1}]] &= E[2(2p-1)F_{n-1}p + F_{n-1} - (2p-1)F_{n-1}] \\ &= (1 + (2p-1)^2)E[F_{n-1}] \\ &= [1 + (2p-1)^2]^2 E[F_{n-2}] \\ &\vdots \\ &= [1 + (2p-1)^2]^n E[F_0] \end{aligned}$$

68. (a) $.6e^{-2} + .4e^{-3}$

(b) $.6e^{-2}\dfrac{2^3}{3!} + .4e^{-3}\dfrac{3^3}{3!}$

(c) $P\{3 \mid 0\} = \dfrac{P\{3,0\}}{P\{0\}} = \dfrac{.6e^{-2}e^{-2}\dfrac{2^3}{3!} + .4e^{-3}e^{-3}\dfrac{3^3}{3!}}{.6e^{-2} + .4e^{-3}}$

69. (a) $\displaystyle\int_0^\infty e^{-x}e^{-x}dx = \frac{1}{2}$

(b) $\displaystyle\int_0^\infty e^{-x}\frac{x^3}{3!}e^{-x}dx = \frac{1}{96}\int_0^\infty e^{-y}y^3dy = \frac{\Gamma(4)}{96} = \frac{1}{16}$

(c) $\dfrac{\displaystyle\int_0^\infty e^{-x}e^{-x}\frac{x^3}{3!}e^{-x}dx}{\displaystyle\int_0^\infty e^{-x}e^{-x}dx} = \dfrac{2}{3^4} = \dfrac{2}{81}$

70. (a) $\displaystyle\int_0^1 pdp = 1/2$

(b) $\displaystyle\int_0^1 p^2dp = 1/3$

71. $$\begin{aligned} P\{X=i\} = \int_0^1 P\{X=i \mid p\}dp &= \int_0^1 \binom{n}{i} p^i(1-p)^{n-i}dp \\ &= \binom{n}{i}\frac{i!(n-i)!}{(n+1)!} = 1/(n+1) \end{aligned}$$

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72. (a) $P\{N \ge i\} = \int_0^1 P\{N \ge i \mid p\}dp = \int_0^1 (1-p)^{i-1}dp = 1/i$

(b) $P\{N = i\} = P\{N \ge i\} - P\{N \ge i+1\} = \dfrac{1}{i(i+1)}$

(c) $E[N] = \sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty} 1/i = \infty$.

73. (a) $E[R] = E[E[R \mid S]] = E[S] = \mu$

(b) $\text{Var}(R \mid S) = 1$, $E[R \mid S] = S$
$\text{Var}(R) = 1 + \text{Var}(S) = 1 + \sigma^2$

(c) $f_R(r) = \int f_S(s)F_{R|S}(r \mid s)ds$

$$= C\int e^{-(s-\mu)^2/2\sigma^2} e^{-(r-s)^2/2} ds$$

$$= K\int \exp\left\{-\left(S - \frac{\mu + r\sigma^2}{1+\sigma^2}\right)\bigg/ 2\left(\frac{\sigma^2}{1+\sigma^2}\right)\right\} ds \exp\{-(ar^2 + br)\}$$

Hence, R is normal.

(d) $E[RS] = E[E[RS \mid S]] = E[SE[R \mid S]] = E[S^2] = \mu^2 + \sigma^2$

$\text{Cov}(R, S) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

75. X is Poisson with mean $\lambda = 2$ and Y is Binomial with parameters 10, 3/4. Hence

(a) $P\{X + Y = 2\} = P\{X = 0)P\{Y = 2\} + P\{X = 1\}P\{Y = 1\} + P\{X = 2\}P\{Y = 0\}$

$$= e^{-2}\binom{10}{2}(3/4)^2(1/4)^8 + 2e^{-2}\binom{10}{1}(3/4)(1/4)^9 + 2e^{-2}(1/4)^{10}$$

(b) $P\{XY = 0\} = P\{X = 0\} + P\{Y = 0\} - P\{X = Y = 0\}$
$= e^{-2} + (1/4)^{10} - e^{-2}(1/4)^{10}$

(c) $E[XY] = E[X]E[Y] = 2 \cdot 10 \cdot \dfrac{3}{4} = 15$

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77. The joint moment generating function, $E[e^{tX+sY}]$ can be obtained either by using

$$E[e^{tX+sY}] = \int\int e^{tX+sY} f(x, y) dy\, dx$$

or by noting that Y is exponential with rate 1 and, given Y, X is normal with mean Y and variance 1. Hence, using this we obtain

$$E[e^{tX+sY} \mid Y] = e^{sY} E[E^{tX} \mid Y] = e^{sY} e^{Yt + t^2/2}$$

and so

$$E[e^{tX+sY}] = e^{t^2/2} E[e^{(s+t)Y}]$$
$$= e^{t^2/2}(1-s-t)^{-1},\ s + t < 1$$

Setting first s and then t equal to 0 gives

$$E[e^{tX}] = e^{t^2/2}(1-t)^{-1},\ t < 1$$
$$E[e^{sY}] = (1-s)^{-1},\ s < 1$$

78. Conditioning on the amount of the initial check gives

$$E[\text{Return}] = E[\text{Return} \mid A]/2 + E[\text{Return} \mid \text{B}]/2$$
$$= \{AF(A) + B[1 - F(A)]\}/2 + \{BF(B) + A[1 - F(B)]\}/2$$
$$= \{A + B + [B - A][F(B) - F(A)]\}/2$$
$$> (A + B)/2$$

where the inequality follows since $[B - A]$ and $[F(B) - F(A)$ both have the same sign.

(b) If $x < A$ then the strategy will accept the first value seen: if $x > B$ then it will reject the first one seen; and if x lies between A and B then it will always yield return B. Hence,

$$E[\text{Return of } x\text{-strategy}] = \begin{cases} B & \text{if } A < x < B \\ (A+B)/2 & \text{otherwise} \end{cases}$$

(c) This follows from (b) since there is a positive probability that X will lie between A and B.

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79. Let X_i denote sales in week i. Then

$$\begin{aligned} E[X_1 + X_2] &= 80 \\ \mathrm{Var}(X_1 + X_2) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2) \\ &= 72 + 2[.6(6)(6)] = 93.6 \end{aligned}$$

(a) With Z being a standard normal

$$\begin{aligned} P(X_1 + X_2 > 90) &= P\left(Z > \frac{90-80}{\sqrt{93.6}}\right) \\ &= P(Z > 1.034) \approx .150 \end{aligned}$$

(b) Because the mean of the normal $X_1 + X_2$ is less than 90 the probability that it exceeds 90 is increased as the variance of $X_1 + X_2$ increases. Thus, this probability is smaller when the correlation is .2.

(c) In this case,

$$\begin{aligned} P(X_1 + X_2 > 90) &= P\left(Z > \frac{90-80}{\sqrt{72+2[.2(6)(6)]}}\right) \\ &= P(Z > 1.076) \approx .141 \end{aligned}$$

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Theoretical Exercises

1. Let $\mu = E[X]$. Then for any a

$$\begin{aligned}E[(X-a)^2 &= E[(X-\mu+\mu-a)^2] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2E[(x-\mu)(\mu-a)] \\ &= E[(X-\mu)^2] + (\mu-a)^2 + 2(\mu-a)E[(X-\mu)] \\ &= E[(X-\mu)^2 + (\mu-a)^2\end{aligned}$$

2. $$\begin{aligned}E[\,|X-a|\,] &= \int_{x<a}(a-x)f(x)dx + \int_{x>a}(x-a)f(x)dx \\ &= aF(a) - \int_{x<a} xf(x)dx + \int_{x>a} xf(x)dx - a[1-F(a)]\end{aligned}$$

Differentiating the above yields
derivative = $2af(a) + 2F(a) - af(a) - af(a) - 1$
Setting equal to 0 yields that $2F(a) = 1$ which establishes the result.

3. $$\begin{aligned}E[g(X,Y)] &= \int_0^\infty P\{g(X,Y) > a\}da \\ &= \int_0^\infty \int_{\substack{x,y:\\ g(x,y)>a}} \int f(x,y)dydxda = \iint \int_0^{g(x,y)} daf(x,y)dydx \\ &= \iint g(x,y)dydx\end{aligned}$$

4. $$\begin{aligned}g(X) &= g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\frac{(X-\mu)^2}{2} + \ldots \\ &\approx g(\mu) + g'(\mu)(X-\mu) + g''(\mu)\frac{(X-\mu)^2}{2}\end{aligned}$$

Now take expectations of both sides.

5. If we let X_k equal 1 if A_k occurs and 0 otherwise then

$$X = \sum_{k=1}^n X_k$$

Hence,

$$E[X] = \sum_{k=1}^n E[X_k] = \sum_{k=1}^n P(A_k)$$

But

$$E[X] = \sum_{k=1}^n P\{X \ge k\} = \sum_{k=1}^n P(C_k)\,.$$

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6. $X = \int_0^\infty X(t)dt$ and taking expectations gives

$$E[X] = \int_0^\infty E[X(t)]\,dt = \int_0^\infty P\{X > t\}dt$$

7. (a) Use Exercise 6 to obtain that

$$E[X] = \int_0^\infty P\{X > t\}dt \geq \int_0^\infty P\{Y > t\}dt = E[Y]$$

(b) It is easy to verify that

$$X^+ \geq_{\text{st}} Y^+ \text{ and } Y^- \geq_{\text{st}} X^-$$

Now use part (a).

8. Suppose $X \geq_{\text{st}} Y$ and f is increasing. Then

$$\begin{aligned} P\{f(X) > a\} &= P\{X > f^{-1}(a)\} \\ &\geq P\{Y > f^{-1}(a)\} \quad \text{since } x \geq_{st} Y \\ &= P\{f(Y) > a\} \end{aligned}$$

Therefore, $f(X) \geq_{st} f(Y)$ and so, from Exercise 7,
$E[f(X)] \geq E[f(Y)]$.

On the other hand, if $E[f(X)] \geq E[f(Y)]$ for all increasing functions f, then by letting f be the increasing function

$$f(x) = \begin{matrix} 1 & \text{if } x > t \\ 0 & \text{otherwise} \end{matrix}$$

then

$$P\{X > t\} = E[f(X)] \geq E[f(Y)] = P\{Y > t\}$$

and so X $>_{\text{st}}$ Y.

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9. Let

$$I_j = \begin{cases} 1 & \text{if a run of size } k \text{ begins at the } j^{\text{th}} \text{ flip} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\text{Number of runs of size } k = \sum_{j=1}^{n-k+1} I_j$$

$$E[\text{Number of runs of size } k = E\left[\sum_{j=1}^{n-k+1} I_j\right]$$

$$= P(I_1 = 1) + \sum_{j=2}^{n-k} P(I_j = 1) + P(I_{n-k+1} = 1)$$

$$= p^k(1-p) + (n-k-1)p^k(1-p)^2 + p^k(1-p)$$

10. $$1 = E\left[\sum_1^n X_i \Big/ \sum_1^n X_i\right] = \sum_1^n E\left[X_i \Big/ \sum_1^n X_i\right] = nE\left[X_1 \Big/ \sum_1^n X_i\right]$$

Hence,

$$E\left[\sum_1^k X_i \Big/ \sum_1^n X_i\right] = k/n$$

11. Let

$$I_j = \begin{cases} 1 & \text{outcome } j \text{ never occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then $X = \sum_1^r I_j$ and $E[X] = \int_{j=1}^{r} (1-p_j)^n$

12. For X having the Cantor distribution, $E[X] = 1/2$, $\text{Var}(X) = 1/8$

13. Let

$$I_j = \begin{cases} 1 & \text{record at } j \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n P\{X_j \text{ is largest of } X_1, \ldots, X_j\} = \sum_1^n 1/j$$

$$\text{Var}\left(\sum_1^n I_j\right) = \sum_1^n \text{Var}(I_j) = \sum_1^n \frac{1}{j}\left(1 - \frac{1}{j}\right)$$

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15. $\mu = \sum_{i=1}^{n} p_i$ by letting Number $= \sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & i \text{ is success} \\ 0 & \text{---} \end{cases}$

Var(Number) $= \sum_{i=1}^{n} p_i(1-p_i)$

maximization of variance occur when $p_i \equiv \mu/n$

minimization of variance when $p_i = 1$, $i = 1, \ldots, [\mu]$, $p_{[\mu]+1} = \mu - [\mu]$

To prove the maximization result, suppose that 2 of the p_i are unequal—say $p_i \neq p_j$. Consider a new p-vector with all other p_k, $k \neq i, j$, as before and with $\overline{p}_i = \overline{p}_j = \dfrac{p_i + p_j}{2}$. Then in the variance formula, we must show

$$2\left(\frac{p_i + p_j}{2}\right)\left(1 - \frac{p_i + p_j}{2}\right) \geq p_i(1 - p_i) + p_j(1 - p_j)$$

or equivalently,

$$p_i^2 + p_j^2 - 2p_i p_j = (p_i - p_j)^2 \geq 0.$$

The maximization is similar.

16. Suppose that each element is, independently, equally likely to be colored red or blue. If we let X_i equal 1 if all the elements of A_i are similarly colored, and let it be 0 otherwise, then $\sum_{i=1}^{r} X_i$ is the number of subsets whose elements all have the same color. Because

$$E\left[\sum_{i=1}^{r} X_i\right] = \sum_{i=1}^{r} E\left[X_i\right] = \sum_{i=1}^{r} 2(1/2)^{|A_i|}$$

it follows that for at least one coloring the number of monocolored subsets is less than or equal to $\sum_{i=1}^{r} (1/2)^{|A_i|-1}$

17. $\text{Var}\left(\lambda X_1 + (1-\lambda)X_2\right) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2$

$$\frac{d}{d\lambda}(\quad) = 2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2 = 0 \Rightarrow \lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

As Var($\lambda X_1 + (1-\lambda)X_2$) $= E\left[(\lambda X_1 + (1-\lambda)X_2 - \mu)^2\right]$ we want this value to be small.

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18. (a. Binomial with parameters m and $P_i + P_j$.

(b) Using (a) we have that $\text{Var}(N_i + N_j) = m(P_i + P_j)(1 - P_i - P_j)$ and thus

$$m(P_i + P_j)(1 - P_i - P_j) = mP_i(1 - P_i) + mP_j(1 - P_j) + 2\,\text{Cov}(N_i, N_j)$$

Simplifying the above shows that

$$\text{Cov}(N_i, N_j) = -mP_iP_j.$$

19.
$$\begin{aligned}\text{Cov}(X + Y, X - Y) &= \text{Cov}(X, X) + \text{Cov}(X, -Y) + \text{Cov}(Y, X) + \text{Cov}(Y, -Y)\\ &= \text{Var}(X) - \text{Cov}(X, Y) + \text{Cov}(Y, X) - \text{Var}(Y)\\ &= \text{Var}(X) - \text{Var}(Y) = 0.\end{aligned}$$

20. (a) $\text{Cov}(X, Y|Z)$

$$\begin{aligned}&= E[XY - E[X|Z]Y - XE[Y|Z] + E[X|Z]E[Y|Z]\ [Z]\\ &= E[XY|Z] - E[X|Z]\ E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]E[Y|Z]\\ &= E[XY|Z] - E[X|Z]E[Y|Z]\end{aligned}$$

where the next to last equality uses the fact that given Z, $E[X|Z]$ and $E[Y|Z]$ can be treated as constants.

(b) From (a)

$$E[\text{Cov}(X, Y|Z)] = E[XY] - E[E[X|Z]E[Y|Z]]$$

On the other hand,

$$\text{Cov}(E[X|Z], E[Y|Z] = E[E[X|Z]E[Y|Z]] - E[X]E[Y]$$

and so

$$\begin{aligned}E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z]) &= E[XY] - E[X]E[Y]\\ &= \text{Cov}(X, Y)\end{aligned}$$

(c) Noting that $\text{Cov}(X, X|Z) = \text{Var}(X|Z)$ we obtain upon setting $Y = Z$ that

$$\text{Var}(X) = E[\text{Var}(X|Z)] + \text{Var}(E[X|Z])$$

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21. (a) Using the fact that f integrates to 1 we see that

$c(n, i) \equiv \int_0^1 x^{i-1}(1-x)^{n-i}\,dx = (i-1)!(n-i)!/n!$. From this we see that

$E[X_{(i)}] = c(n+1, i+1)/c(n, i) = i/(n+1)$

$E[X_{(i)}^2] = c(n+2, i+2)/c(n, i) = \dfrac{i(i+1)}{(n+2)(n+1)}$

and thus

$$\text{Var}(X_{(i)}) = \frac{i(n+1-i)}{(n+1)^2(n+2)}$$

(b) The maximum of $i(n+1-i)$ is obtained when $i = (n+1)/2$ and the minimum when i is either 1 or n.

22. $\text{Cov}(X, Y) = b\,\text{Var}(X)$, $\text{Var}(Y) = b^2\,\text{Var}(X)$

$$\rho(X,Y) = \frac{b\,\text{Var}(X)}{\sqrt{b^2}\,\text{Var}(X)} = \frac{b}{|b|}$$

26. Follows since, given X, $g(X)$ is a constant and so

$E[g(X)Y \mid X] = g(X)E[Y \mid X]$

27. $E[XY] = E[E[XY \mid X]]$
$= E[XE[Y \mid X]]$

Hence, if $E[Y \mid X] = E[Y]$, then $E[XY] = E[X]E[Y]$. The example in Section 3 of random variables uncorrelated but not independent provides a counterexample to the converse.

28. The result follows from the identity

$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]]$ which is obtained by noting that, given X, X may be treated as a constant.

29. $x = E[X_1 + \ldots + X_n \mid X_1 + \ldots + X_n = x] = E\left[X_1 \middle| \sum X_i = x\right] + \ldots + E\left[X_n \middle| \sum X_i = x\right]$
$= nE\left[X_1 \middle| \sum X_i = x\right]$

Hence, $E[X_1 \mid X_1 + \ldots + X_n = x] = x/n$

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30. $E[N_iN_j \mid N_i] = N_iE[N_j \mid N_i] = N_i(n - N_i)\dfrac{p_j}{1-p_i}$ since each of the $n - N_i$ trials no resulting in outcome i will independently result in j with probability $p_j/(1 - p_i)$. Hence,

$$E[N_iN_j] = \frac{p_j}{1-p_i}\left(nE[N_i] - E\left[N_i^2\right]\right) = \frac{p_j}{1-p_i}\left[n^2p_i - n^2p_i^2 - np_i(1-p_i)\right]$$
$$= n(n-1)p_ip_j$$

and

$$\text{Cov}(N_i, N_j) = n(n-1)p_ip_j - n^2p_ip_j = -np_ip_j$$

31. By induction: true when $t = 0$, so assume for $t - 1$. Let $N(t)$ denote the number after stage t.

$$E[N(t) \mid N(t-1)] = N(t-1) - E[\text{number selected}]$$
$$= N(t-1) - N(t-1)\frac{r}{b+w+r}$$
$$E[N(t) \mid N(t-1)] = N(t-1)\frac{b+w}{b+w+r}$$
$$E[N(t)] = \left(\frac{b+w}{b+w+r}\right)^t w$$

32.
$$E[XI_A] = E[XI_A \mid A]P(A) + E[XI_A \mid A^c]P(A^c)$$
$$= E[X \mid A]P(A)$$

34. (a) $E[T_r \mid T_{r-1}] = T_{r-1} + 1 + (1 - p)E[T_r]$

(b) Taking expectations of both sides of (a) gives

$$E[T_r] = E[T_{r-1}] + 1 + (1 - p)E[T_r]$$

or

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$

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(c) Using the result of part (b) gives

$$E[T_r] = \frac{1}{p} + \frac{1}{p}E[T_{r-1}]$$
$$= \frac{1}{p} + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{p}E[T_{r-2}]\right)$$
$$= 1/p + (1/p)^2 + (1/p)^2 E[T_{r-2}]$$
$$= 1/p + (1/p)^2 + (1/p)^3 + (1/p)^3 E[T_{r-3}]$$
$$= \sum_{i=1}^{r}(1/p)^i + (1/p)^r E[T_0]$$
$$= \sum_{i=1}^{r}(1/p)^i \quad \text{since } E[T_0] = 0.$$

35. $$P(Y > X) = \sum_j P(Y > X \mid X = j)p_j$$
$$= \sum_j P(Y > j \mid X = j)p_j$$
$$= \sum_j P(Y > j)p_j$$
$$= \sum_j (1-p)^j p_j$$

36. Condition on the first ball selected to obtain

$$M_{a,b} = \frac{a}{a+b}M_{a-1,b} + \frac{b}{a+b}M_{a,b-1},\ a, b > 0$$

$$M_{a,0} = a, \qquad M_{0,b} = b, \qquad M_{a,b} = M_{b,a}$$

$$M_{2,1} = \frac{4}{3}, \qquad M_{3,1} = \frac{7}{4}, \qquad M_{3,2} = 3/2$$

37. Let X_n denote the number of white balls after the n^{th} drawing

$$E[X_{n+1} \mid X_n] = X_n\frac{X_n}{a+b} + (X_n+1)\left(1-\frac{X_n}{a+b}\right) = \left(1-\frac{1}{a+b}\right)X_n + 1$$

Taking expectations now yields (a).

To prove (b), use (a) and the boundary condition $M_0 = a$

(c) $P\{(n + 1)\text{st is white}\} = E[P\{(n + 1)\text{st is white} \mid X_n\}]$

$$= E\left[\frac{X_n}{a+b}\right] = \frac{M_n}{a+b}$$

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40. Let I equal 1 if the first trial is a success and 0 if it is a failure. Now, if $I = 1$, then $X = 1$; because the variance of a constant is 0, this gives

$$\text{Var}\left(X\,|I=1\right)=0$$

On the other hand, if $I = 0$, then the conditional distribution of X given that $I = 0$ is the same as the unconditional distribution of 1 (the first trial) plus a geometric with parameter p (the number of additional trials needed for a success).

Therefore,

$$\text{Var}\left(X\,|I=0\right)=\text{Var}(1+X)=\text{Var}(X)$$

Consequently,

$$\begin{aligned}E\left[\text{Var}\left(X\,|I\right)\right]&=\text{Var}\left(X\,|I=1\right)P(I=1)+\text{Var}\left(X\,|I=0\right)P(I=0)\\&=(1-p)\text{Var}(X)\end{aligned}$$

By the same reasoning used to compute the conditional variances, we have

$$E\left[X\,|I=1\right]=1,\qquad E\left[X\,|I=0\right]=1+E[X]=1+\frac{1}{p}$$

which can be written as

$$E\left[X\,|I\right]=1+\frac{1}{p}(1-I)$$

yielding that

$$\text{Var}\left(E\left[X\,|I\right]\right)=\frac{1}{p^2}\text{Var}(I)=\frac{1}{p^2}\,p(1-p)=\frac{1-p}{p}$$

The conditional variance formula now gives

$$\begin{aligned}Var(X)&=E\left[\text{Var}\left(X\,|I\right)\right]+Var\left(E\left[X\,|I\right]\right)\\&=(1-p)\text{Var}(X)+\frac{1-p}{p}\end{aligned}$$

or

$$\text{Var}(X)=\frac{1-p}{p^2}$$

41. (a) No

(b) Yes, since $f_Y(x\,|\,I = 1) = f_X(x) = f_X(-x) = f_Y(x\,|\,I = 0)$

(c) $f_Y(x) = \dfrac{1}{2}f_X(x)+\dfrac{1}{2}f_X(-x)=f_X(x)$

(d) $E[XY] = E[E[XY\,|\,X]] = E[XE[Y\,|\,X]] = 0$

(e) No, since X and Y are not jointly normal.

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42. If $E[Y|X]$ is linear in X, then it is the best linear predictor of Y with respect to X.

43. Must show that $E[Y^2] = E[XY]$. Now

$$\begin{aligned} E[XY] &= E[XE[X|Z]] \\ &= E[E[XE[X|Z] \ |Z]] \\ &= E[E^2[X|Z]] = E[Y^2] \end{aligned}$$

44. Write $X_n = \sum_{i=1}^{X_{n-1}} Z_i$ where Z_i is the number of offspring of the ith individual of the $(n-1)$st generation. Hence,

$$E[X_n] = E[E[X_n|X_{n-1}]] = E[\mu X_{n-1}] = \mu E[X_{n-1}]$$

so,

$$E[X_n] = \mu E[X_{n-1}] = \mu^2 E[X_{n-2}] \ldots = \mu^n E[X_0] = \mu^n$$

(c) Use the above representation to obtain

$$E[X_n|X_{n-1}] = \mu X_{n-1},\ \text{Var}(X_n|X_{n-1}) = \sigma^2 X_{n-1}$$

Hence, using the conditional Variance Formula,

$$\text{Var}(X_n) = \mu^2\ \text{Var}(X_{n-1}) + \sigma^2\mu^{n-1}$$

(d) $\pi = P\{\text{dies out}\}$

$$= \sum_j P\{\text{dies out}|X_i = j\}p_j$$

$= \sum_j \pi^j p_j$, since each of the j members of the first generation can be thought of as starting their own (independent) branching process.

46. It is easy to see that the n^{th} derivative of $\sum_{j=0}^{\infty}(t^2/2)^j/j!$ will, when evaluated at $t = 0$, equal 0 whenever n is odd (because all of its terms will be constants multiplied by some power of t). When $n = 2j$ the n^{th} derivative will equal $\frac{d^n}{dt^n}\{t^n\}/(j!2^j)$ plus constants multiplied by powers of t. When evaluated at 0, this gives that

$$E[Z^{2j}] - (2j)!/(j!2^j)$$

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47. Write $X = \sigma Z + \mu$ where Z is a standard normal random variable. Then, using the binomial theorem,

$$E[X^n] = \sum_{i=0}^{n} \binom{n}{i} \sigma^i E[Z^i] \mu^{n-i}$$

Now make use of theoretical exercise 46.

48. $\phi_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = e^{tb}E[e^{taX}] = e^{tb}\phi_X(ta)$

49. Let $Y = \log(X)$. Since Y is normal with mean μ and variance σ^2 it follows that its moment generating function is

$$M(t) = E[e^{tY}] = e^{\mu t + \sigma^2 t^2/2}$$

Hence, since $X = e^Y$, we have that

$$E[X] = M(1) = e^{\mu + \sigma^2/2}$$

and

$$E[X^2] = M(2) = e^{2\mu + 2\sigma^2}$$

Therefore,

$$\text{Var}(X) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

50. $\psi(t) = \log \phi(t)$

$\psi'(t) = \phi'(t)/\phi(t)$

$$\psi''(t) = \frac{\phi(t)\phi''(t) - (\phi'(t))^2}{\phi^2(t)}$$

$$\psi''(t)\Big|_{t=0} = E[X^2] - (E[X])^2 = \text{Var}(X).$$

51. Gamma (n, λ)

52. Let $\phi(s, t) = E[e^{sX+tY}]$

$$\frac{\partial^2}{\partial s \partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[XYe^{sX+tY}]\Big|_{\substack{s=0\\t=0}} = E[XY]$$

$$\frac{\partial}{\partial s}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[X], \quad \frac{\partial}{\partial t}\phi(s,t)\Big|_{\substack{s=0\\t=0}} = E[Y]$$

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53. Follows from the formula for the joint moment generating function.

54. By symmetry, $E[Z^3] = E[Z] = 0$ and so $\text{Cov}(Z,Z^3) = 0$.

55. (a) This follows because the conditional distribution of $Y + Z$ given that $Y = y$ is normal with mean y and variance 1, which is the same as the conditional distribution of X given that $Y = y$.

(b) Because $Y + Z$ and Y are both linear combinations of the independent normal random variables Y and Z, it follows that $Y + Z$, Y has a bivariate normal distribution.

(c)
$$\mu_x = E[X] = E[Y + Z] = \mu$$
$$\sigma_x^2 = \text{Var}(X) = \text{Var}(Y + Z) = \text{Var}(Y) + \text{Var}(Z) = \sigma^2 + 1$$
$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(Y+Z,Y)}{\sigma\sqrt{\sigma^2+1}} = \frac{\sigma}{\sqrt{\sigma^2+1}}$$

(d) and (e) The conditional distribution of Y given $X = x$ is normal with mean

$$E[Y \mid X = x] = \mu + \rho\frac{\sigma}{\sigma_x}(x - \mu_x) = \mu + \frac{\sigma^2}{1+\sigma^2}(x - \mu)$$

and variance

$$\text{Var}(Y \mid X = x) = \sigma^2\left(1 - \frac{\sigma^2}{\sigma^2+1}\right) = \frac{\sigma^2}{\sigma^2+1}$$

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