# Chapter 4

## Problems

1. $P\{X=4\} = \dfrac{\binom{4}{2}}{\binom{14}{2}} = \dfrac{6}{91}$ $\qquad$ $P\{X=0\} = \dfrac{\binom{2}{2}}{\binom{14}{2}} = \dfrac{1}{91}$

$P\{X=2\} = \dfrac{\binom{4}{2}\binom{2}{1}}{\binom{14}{2}} = \dfrac{8}{91}$ $\qquad$ $P\{X=-1\} = \dfrac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \dfrac{16}{91}$

$P\{X=1\} = \dfrac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \dfrac{32}{91}$ $\qquad$ $P\{X=-2\} = \dfrac{\binom{8}{2}}{\binom{14}{2}} = \dfrac{28}{91}$

2.

| | | | | |
|---|---|---|---|---|
| $p(1) = 1/36$ | $p(5) = 2/36$ | $p(9) = 1/36$ | $p(15) = 2/36$ | $p(24) = 2/36$ |
| $p(2) = 2/36$ | $p(6) = 4/36$ | $p(10) = 2/36$ | $p(16) = 1/36$ | $p(25) = 1/36$ |
| $p(3) = 2/36$ | $p(7) = 0$ | $p(11) = 0$ | $p(18) = 2/36$ | $p(30) = 2/36$ |
| $p(4) = 3/36$ | $p(8) = 2/36$ | $p(12) = 4/36$ | $p(20) = 2/36$ | $p(36) = 1/36$ |

4. $P\{X=1\} = 1/2$, $P\{X=2\} = \dfrac{5}{10}\dfrac{5}{9} = \dfrac{5}{18}$, $P\{X=3\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{5}{8} = \dfrac{5}{36}$,

$P\{X=4\} = \dfrac{5}{10}\dfrac{4}{9}\dfrac{3}{8}\dfrac{5}{7} = \dfrac{10}{168}$, $P\{X=5\} = \dfrac{5\cdot4\cdot3\cdot2}{10\cdot9\cdot8\cdot7}\dfrac{5}{6} = \dfrac{5}{252}$,

$P\{X=6\} = \dfrac{5\cdot4\cdot3\cdot2\cdot1}{10\cdot9\cdot8\cdot7\cdot6} = \dfrac{1}{252}$

5. $n - 2i$, $i = 0, 1, \ldots, n$

6. $P(X = 3\} = 1/8$, $P\{X = 1\} = 3/8$, $P\{X = -1\} = 3/8$, $P\{X = -3\} = 1/8$

8. (a) $p(6) = 1 - (5/6)^2 = 11/36$, $p(5) = 2\ 1/6\ \ 4/6 + (1/6)^2 = 9/36$
$p(4) = 2\ 1/6\ \ 3/6 + (1/6)^2 = 7/36$, $p(3) = 2\ 1/6\ \ 2/6 + (1/6)^2 = 5/36$
$p(2) = 2\ 1/6\ \ 1/6 + (1/6)^2 = 3/36$, $p(1) = 1/36$

(d) $p(5) = 1/36$, $p(4) = 2/36$, $p(3) = 3/36$, $p(2) = 4/36$, $p(1) = 5/36$
$p(0) = 6/36$, $p(-j) = p(j)$, $j > 0$



Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

11. (a) $P\{\text{divisible by } 3\} = \dfrac{333}{1000}$ $P\{\text{divisible by } 105\} = \dfrac{9}{1000}$

$P\{\text{divisible by } 7\} = \dfrac{142}{1000}$

$P\{\text{divisible by } 15\} = \dfrac{66}{1000}$

In limiting cases, probabilities converge to 1/3, 1/7, 1/15, 1/10

(b) $P\{\mu(N) \neq 0\} = P\{N \text{ is not divisible by } p_i^2, i \geq 1\}$

$= \prod_i P\{N \text{ is not divisible by } p_i^2\}$

$= \prod_i (1 - 1/p_i^2) = 6/\pi^2$

13. $p(0) = P\{\text{no sale on first and no sale on second}\}$
$= (.7)(.4) = .28$
$p(500) = P\{1 \text{ sale and it is for standard}\}$
$= P\{1 \text{ sale}\}/2$
$=[P\{\text{sale, no sale}\} + P\{\text{no sale, sale}\}]/2$
$= [(.3)(.4) + (.7)(.6)]/2 = .27$

$p(1000) = P\{2 \text{ standard sales}\} + P\{1 \text{ sale for deluxe}\}$
$= (.3)(.6)(1/4) + P\{1 \text{ sale}\}/2$
$= .045 + .27 = .315$

$p(1500) = P\{2 \text{ sales, one deluxe and one standard}\}$
$= (.3)(.6)(1/2) = .09$

$p(2000) = P\{2 \text{ sales, both deluxe}\} = (.3)(.6)(1/4) = .045$

14. $P\{X = 0\} = P\{1 \text{ loses to } 2\} = 1/2$

$P\{X = 1\} = P\{\text{of } 1, 2, 3\text{: 3 has largest, then 1, then 2}\}$
$= (1/3)(1/2) = 1/6$

$P\{X = 2\} = P\{\text{of } 1, 2, 3, 4\text{: 4 has largest and 1 has next largest}\}$
$= (1/4)(1/3) = 1/12$

$P\{X = 3\} = P\{\text{of } 1, 2, 3, 4, 5\text{: 5 has largest then 1}\}$
$= (1/5)(1/4) = 1/20$

$P\{X = 4\} = P\{1 \text{ has largest}\} = 1/5$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

15. $P\{X = 1\} = 11/66$

$$P\{X = 2\} = \sum_{j=2}^{11}\left(\frac{12-j}{66}\right)\left(\frac{11}{54+j}\right)$$

$$P\{X = 3\} = \sum_{\substack{k\neq 1\\ k\neq j}}\sum_{j=2}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+j+k}\right)$$

$$P\{X = 4\} = 1 - \sum_{i=1}^{3} P\{X = 1\}$$

16. $P\{Y_1 = i\} = \dfrac{12-i}{66}$

$$P\{Y_2 = i\} = \sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-i}{54+j}\right)$$

$$P\{Y_3 = i\} = \sum_{\substack{k\neq j\\ k\neq i}}\sum_{j\neq i}\left(\frac{12-j}{66}\right)\left(\frac{12-k}{54+j}\right)\left(\frac{11}{42+k+j}\right)$$

All sums go from 1 to 11, except for prohibited values.

20. (a) $P\{x > 0\} = P\{\text{win first bet}\} + P\{\text{lose, win, win}\}$
$= 18/38 + (20/38)(18/38)^2 \approx .5918$

(b) No, because if the gambler wins then he or she wins \$1.
However, a loss would either be \$1 or \$3.

(c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

21. (a) $E[X]$ since whereas the bus driver selected is equally likely to be from any of the 4 buses, the student selected is more likely to have come from a bus carrying a large number of students.

(b) $P\{X = i\} = i/148$, $i = 40, 33, 25, 50$

$E[X] = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/148 \approx 39.28$
$E[Y] = (40 + 33 + 25 + 50)/4 = 37$

22. Let $N$ denote the number of games played.

(a) $E(N) = 2[p^2 + (1 - p)^2] + 3[2p(1 - p)] = 2 + 2p(1 - p)$

The final equality could also have been obtained by using that $N = 2 + ]$ where $I$ is 0 if two games are played and 1 if three are played. Differentiation yields that

$$\frac{d}{dp}E[N] = 2 - 4p$$

and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$.

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

(b) $E[N] = 3[p^3 + (1-p)^3 + 4[3p^2(1-p)p + 3p(1-p)^2(1-p)]$
$+ 5[6p^2(1-p)^2 = 6p^4 - 12p^3 + 3p^2 + 3p + 3$

Differentiation yields

$$\frac{d}{dp}E[N] = 24p^3 - 36p^2 + 6p + 3$$

Its value at $p = 1/2$ is easily seen to be 0.

23. (a) Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is

$$E[\text{money}] = \frac{1}{2}500 + \frac{1}{2}2000 = 1250$$

(b) Do not immediately buy but use your money to buy after one week. Then

$$E[\text{ounces of commodity}] = \frac{1}{2}1000 + \frac{1}{2}250 = 625$$

24. (a) $p - (1-p)\frac{3}{4} = \frac{7}{4}p - 3/4$, (b) $-\frac{3}{4}p + (1-p)2 = -\frac{11}{4}p + 2$

$\frac{7}{4}p - 3/4 = -\frac{11}{4}p + 2 \Rightarrow p = 11/18$, maximum value = 23.72

(c) $q - \frac{3}{4}(1-q)$ (d) $-\frac{3}{4}q + 2(1-q)$, minimax value = 23/72
attained when $q = 11/18$

25. (a) $P(X = 1) = .6(.3) + .4(.7) = .46$

(b) $E[X] = 1(.46) + 2(.42) = 1.3$

27. $C - Ap = \frac{A}{10} \Rightarrow C = A\left(p + \frac{1}{10}\right)$

28. $3 \cdot \frac{4}{20} = 3/5$

29. If check 1, then (if desired) 2: Expected Cost = $C_1 + (1-p)C_2 + pR_1 + (1-p)R_2$;
if check 2, then 1: Expected Cost = $C_2 + pC_1 + pR_1 + (1-p)R_2$ so 1, 2, best if

$C_1 + (1-p)C_2 \le C_2 + pC_1$, or $C_1 \le \frac{p}{1-p}C_2$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

30. $E[X] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \infty$

(a) probably not

(b) yes, if you could play an arbitrarily large number of games

31. $E[\text{score}] = p^*[1 - (1 - P)^2 + (1 - p^*)(1 - p^2)$

$$\frac{d}{dp} = 2(1 - p)p^* - 2p(1 - p^*)$$
$$= 0 \Rightarrow p = p^*$$

32. If $T$ is the number of tests needed for a group of 10 people, then

$$E[T] = (.9)^{10} + 11[1 - (.9)^{10}] = 11 - 10(.9)^{10}$$

35. If $X$ is the amount that you win, then

$$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$$
$$E[X] = (1.1)4/9 - 5/9 = -.6/9 \approx =-.067$$
$$\text{Var}(X) = (1.1)^2(4/9) + 5/9 - (.6/9)^2 \approx 1.089$$

36. Using the representation

$$N = 2 + I$$

where $I$ is 0 if the first two games are won by the same team and 1 otherwise, we have that

$$\text{Var}(N) = \text{Var}(I) = E[I]^2 - E^2[I]$$

Now, $E[I]^2 = E[I] = P\{I = 1\} = 2p\{1 - p\}$ and so
$\text{Var}(N) = 2p(1 - p)[1 - 2p(1 - p)] = 8p^3 - 4p^4 - 6p^2 + 2p$

Differentiation yields

$$\frac{d}{dp}\text{Var}(N) = 24p^2 - 16p^3 - 12p + 2$$

and it is easy to verify that this is equal to 0 when $p = 1/2$.

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

37. $E[X^2] = [(40)^3 + (33)^3 + (25)^3 + (50)^3]/148 \approx 1625.4$

$$\text{Var}(X = E[X^2] - (E[X])^2 \approx 82.2$$

$E[Y^2] = = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/4 = 1453.5,$ Varr$(Y) = 84.5$

38. (a) $E[(2 + X)^2] = \text{Var}(2 + X) + (E[2 + X])^2 = \text{Var}(X) + 9 = 14$

(b) $\text{Var}(4 + 3X) = 9\ \text{Var}(X) = 45$

39. $\binom{4}{2}(1/2)^4 = 3/8$

40. $\binom{5}{4}(1/3)^4(2/3)^1 + (1/3)^5 = 11/243$

41. $\sum_{i=7}^{10}\binom{10}{i}(1/2)^{10}$

42. $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 \geq \binom{3}{2}p^2(1-p) + p^3$

$\Leftrightarrow 6p^3 - 15p^2 + 12p - 3 \geq 0$

$\Leftrightarrow 6(p - 1/2)(p - 1)^2 \geq 0$

$\Leftrightarrow p \geq 1/2$

43. $\binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8) + (.2)^5$

44. $\alpha\sum_{i=k}^{n}\binom{n}{i}p_1^i(1-p_1)^{n-i} + (1-\alpha)\sum_{i=k}^{n}\binom{n}{i}p_2^i(1-p_2)^{n-i}$

45. with 3: $P\{\text{pass}\} = \frac{1}{3}\left[\binom{3}{2}(.8)^2(.2) + (.8)^3\right] + \frac{2}{3}\left[\binom{3}{2}(.4)^2(.6) + (.4)^3\right]$

$= .533$

with 5: $P\{\text{pass}\} = \frac{1}{3}\sum_{i=3}^{5}\binom{5}{i}(.8)^i(.2)^{5-i} + \frac{2}{3}\sum_{i=3}^{5}\binom{5}{i}(.4)^i(.6)^{5-i}$

$= .3038$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

46. Let $C$ be the event that the jury is correct, and let $G$ be the event that the defendant is guilty. Then

$$\begin{aligned} P(C) &= P(C|G)P(G) + P(C|G^c)P(G^c) \\ &= \sum_{i=0}^{3} (.2)^i (.8)^{12-i} (.65) + \sum_{i=0}^{8} (.1)^i (.9)^{12-i} (.35) \end{aligned}$$

Let $CV$ be the event that the defendant is convicted. Then

$$\begin{aligned} P(CV) &= P(CV|G)P(G) + P(CV|G^c)P(G^c) \\ &= \sum_{i=0}^{3} (.2)^i (.8)^{12-i} (.65) + \left(1 - \sum_{i=0}^{8} (.1)^i (.9)^{12-i} (.35)\right) \\ &= \sum_{i=0}^{3} (.2)^i (.8)^{12-i} (.65) + \sum_{i=9}^{12} (.1)^i (.9)^{12-i} (.35) \end{aligned}$$

47. (a) and (b): (i) $\sum_{i=5}^{9} \binom{9}{i} p^i (1-p)^{9-i}$, (ii) $\sum_{i=5}^{8} \binom{8}{i} p^i (1-p)^{8-i}$,

(iii) $\sum_{i=4}^{7} \binom{7}{i} p^i (1-p)^{7-i}$ where $p = .7$ in (a) and $p = .3$ in (b).

48. The probability that a package will be returned is $p = 1 - (.99)^{10} - 10(.99)^9(.01)$. Hence, if someone buys 3 packages then the probability they will return exactly 1 is $3p(1-p)^2$.

49. (a) $\frac{1}{2}\binom{10}{7}.4^7.6^3 + \frac{1}{2}\binom{10}{7}.7^7.3^3$

(b) $\dfrac{\frac{1}{2}\binom{9}{6}.4^7.6^3 + \frac{1}{2}.7^7.3^3}{.55}$

50. (a) $P\{H, T, T | 6 \text{ heads}\}$

$$\begin{aligned} &= P(H, T, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\} \\ &= P\{H, T, T\}P\{6 \text{ heads} | H, T, T\}/P\{6 \text{ heads}\} \\ &= pq^2\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4 \\ &= 1/10 \end{aligned}$$

(b) $P\{T, H, T | 6 \text{ heads}\}$

$$\begin{aligned} &= P(T, H, T \text{ and } 6 \text{ heads}\}/P\{6 \text{ heads}\} \\ &= P\{T, H, T\}P\{6 \text{ heads} | T, H, T\}/P\{6 \text{ heads}\} \\ &= q^2p\binom{7}{5}p^5q^2 \Big/ \binom{10}{6}p^6q^4 \\ &= 1/10 \end{aligned}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

51. (a) $e^{-.2}$ (b) $1 - e^{-.2} - .2e^{-.2} = 1 - 1.2e^{-.2}$

Since each letter has a small probability of being a typo, the number of errors should approximately have a Poisson distribution.

52. (a) $1 - e^{-3.5} - 3.5e^{-3.5} = 1 - 4.5e^{-3.5}$

(b) $4.5e^{-3.5}$

Since each flight has a small probability of crashing it seems reasonable to suppose that the number of crashes is approximately Poisson distributed.

53. (a) The probability that an arbitrary couple were both born on April 30 is, assuming independence and an equal chance of having being born on any given date, $(1/365)^2$. Hence, the number of such couples is approximately Poisson with mean $80{,}000/(365)^2 \approx .6$. Therefore, the probability that at least one pair were both born on this date is approximately $1 - e^{-.6}$.

(b) The probability that an arbitrary couple were born on the same day of the year is 1/365. Hence, the number of such couples is approximately Poisson with mean $80{,}000/365 \approx 219.18$. Hence, the probability of at least one such pair is $1 - e^{-219.18} \approx 1$.

54. (a) $e^{-2.2}$ (b) $1 - e^{-2.2} - 2.2e^{-2.2} = 1 - 3.2e^{-2.2}$

55. $\frac{1}{2}e^{-3} + \frac{1}{2}e^{-4.2}$

56. The number of people in a random collection of size $n$ that have the same birthday as yourself is approximately Poisson distributed with mean $n/365$. Hence, the probability that at least one person has the same birthday as you is approximately $1 - e^{-n/365}$. Now, $e^{-x} = 1/2$ when $x = \log(2)$. Thus, $1 - e^{-n/365} \geq 1/2$ when $n/365 \geq \log(2)$. That is, there must be at least 365 log(2) people.

57. (a) $1 - e^{-3} - 3e^{-3} - e^{-3}\frac{3^2}{2} = 1 - \frac{17}{2}e^{-3}$

(b) $P\{X \geq 3 \mid X \geq 1\} = \dfrac{P\{X \geq 3\}}{P\{X \geq 1\}} = \dfrac{1 - \frac{17}{2}e^{-3}}{1 - e^{-3}}$

59. (a) $1 - e^{-1/2}$

(b) $\frac{1}{2}e^{-1/2}$

(c) $1 - e^{-1/2} - \frac{1}{2}e^{-1/2} = 1 - \frac{3}{2}e^{-1/2}$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

60. $$P\{\text{beneficial} \mid 2\} = \frac{P\{2 \mid \text{beneficial}\}3/4}{P\{2 \mid \text{beneficial}\}3/4 + P\{2 \mid \text{not beneficial}\}1/4}$$

$$= \frac{e^{-3}\frac{3^2}{2}\frac{3}{4}}{e^{-3}\frac{3^2}{2}\frac{3}{4} + e^{-5}\frac{5^2}{2}\frac{1}{4}}$$

61. $1 - e^{-1.4} - 1.4e^{-1.4}$

62. For $i < j$, say that trial pair $(i, j)$ is a success if the same outcome occurs on trials $i$ and $j$. Then $(i, j)$ is a success with probability $\sum_{k=1}^{n} p_k^2$ . By the Poisson paradigm the number of trial pairs that result in successes will approximately have a Poisson distribution with mean

$$\sum_{i<j}\sum_{k=1}^{n} p_k^2 = n(n-1)\sum_{k=1}^{n} p_k^2 / 2$$

and so the probability that none of the trial pairs result in a success is approximately $\exp\left(-n(n-1)\sum_{k=1}^{n} p_k^2 / 2\right)$.

63. (a) $e^{-2.5}$

(b) $1 - e^{-2.5} - 2.5e^{-2.5} - \frac{(2.5)^2}{2}e^{-2.5} - \frac{(2.5)^3}{3!}e^{-2.5}$

64. (a) $1 - \sum_{i=0}^{7} e^{-4}4^i / i! \equiv p$

(b) $1 - (1 - p)^{12} - 12p(1 - p)^{11}$

(c) $(1 - p)^{i-1}p$

65. (a) $1 - e^{-1/2}$

(b) $P\{X \ge 2 \mid X \ge 1\} = \dfrac{1 - e^{-1/2} - \frac{1}{2}e^{-1/2}}{1 - e^{-1/2}}$

(c) $1 - e^{-1/2}$

(d) $1 - \exp\{-500 - i)/1000\}$

66. Assume $n > 1$.

(a) $\dfrac{2}{2n-1}$

(b) $\dfrac{2}{2n-2}$

(c) $\exp\{-2n/(2n - 1)\} \approx e^{-1}$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

67. Assume $n > 1$.

(a) $\dfrac{2}{n}$

(b) Conditioning on whether the man of couple $j$ sits next to the woman of couple $i$ gives the result: $\dfrac{1}{n-1}\dfrac{1}{n-1}+\dfrac{n-2}{n-1}\dfrac{2}{n-1}=\dfrac{2n-3}{(n-1)^2}$

(c) $e^{-2}$

68. $\exp(-10e^{-5}\}$

69. With $P_j$ equal to the probability that 4 consecutive heads occur within $j$ flips of a fair coin, $P_1 = P_2 = P + 3 = 0$, and

$P_4 = 1/16$
$P_5 = (1/2)P_4 + 1/16 = 3/32$
$P_6 = (1/2)P_5 + (1/4)P_4 + 1/16 = 1/8$
$P_7 = (1/2)P_6 + (1/4)P_5 + (1/8)P_4 + 1/16 = 5/32$
$P_8 = (1/2)P_7 + (1/4)P_6 + (1/8)P_5 + (1/16)P_4 + 1/16 = 6/32$
$P_9 = (1/2)P_8 + (1/4)P_7 + (1/8)P_6 + (1/16)P_5 + 1/16 = 111/512$
$P_{10} = (1/2)P_9 + (1/4)P_8 + (1/8)P_7 + (1/16)P_6 + 1/16 = 251/1024 = .2451$

The Poisson approximation gives

$$P_{10} \approx 1 - \exp\{-6/32 - 1/16\} = 1 - e^{-.25} = .2212$$

70. $e^{-\lambda t} + (1 - e^{-\lambda t})p$

71. (a) $\left(\dfrac{26}{38}\right)^5$

(b) $\left(\dfrac{26}{38}\right)^3 \dfrac{12}{38}$

72. $P\{\text{wins in } i \text{ games}\} = \dbinom{i-1}{3}(.6)^4(.4)^{i-4}$

73. Let $N$ be the number of games played. Then

$P\{N = 4\} = 2(1/2)^4 = 1/8,$ $\qquad P\{N = 5\} = 2\dbinom{4}{1}(1/2)(1/2)^4 = 1/4$

$P\{N = 6\} = 2\dbinom{5}{2}(1/2)^2(1/2)^4 = 5/16,$ $\qquad P\{N = 7\} = 5/16$

$E[N] = 4/8 + 5/4 + 30/16 + 35.16 = 93/16 = 5.8125$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

74. (a) $\left(\frac{2}{3}\right)^5$

(b) $\binom{8}{5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^3+\binom{8}{6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^2+\binom{8}{7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)^8$

(c) $\binom{5}{4}\left(\frac{2}{3}\right)^5\frac{1}{3}$

(d) $\binom{6}{4}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^2$

76. $\binom{N_1+N_2-k}{N_1}(1/2)^{N_1+N_2-k}(1/2)+\binom{N_1+N_2-k}{N_2}(1/2)^{N_1+N_2-k}(1/2)$

77. $2\binom{2N-k}{N}(1/2)^{2N-k}$

$2\binom{2N-k-1}{N-1}(1/2)^{2N-k-1}(1/2)$

79. (a) $P\{X=0\}=\dfrac{\binom{94}{10}}{\binom{100}{10}}$

(b) $P\{X>2\}=1-\dfrac{\binom{94}{10}+\binom{94}{9}\binom{6}{1}+\binom{94}{8}\binom{6}{2}}{\binom{100}{10}}$

80. $P\{\text{rejected} \mid 1 \text{ defective}\} = 3/10$

$P\{\text{rejected} \mid 4 \text{ defective}\} = 1-\binom{6}{3}\Big/\binom{10}{3} = 5/6$

$P\{4 \text{ defective} \mid \text{rejected}\} = \dfrac{\frac{5}{6}\frac{3}{10}}{\frac{5}{6}\frac{3}{10}+\frac{3}{10}\frac{7}{10}} = 75/138$

81. $P\{\text{rejected}\} = 1-(.9)^4$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

83. Let $X_i$ be the number of accidents that occur on highway $i$. Then

$$E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 1.5$$

84. Let $X_i$ equal 1 if box $i$ does not have any balls, and let it equal 0 otherwise. Then

$$E\left[\sum_{i=1}^{5} X_i\right] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} P(X_i = 1) = \sum_{i=1}^{5} (1 - p_i)^{10}$$

Let $Y_i$ equal 1 if box $i$ has exactly one ball, and let it equal 0 otherwise. Then

$$E\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} E[Y_i] = \sum_{i=1}^{5} P(Y_i = 1) = \sum_{i=1}^{5} 10 p_i (1 - p_i)^9$$

where we used that the number of balls that go into box $i$ is binomial with parameters 10 and $p_i$.

85. Let $X_i$ equal 1 if there is at least one type $i$ coupon in the set of $n$ coupons. Then

$$E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E[X_i] = \sum_{i=1}^{k} P(X_i = 1) = \sum_{i=1}^{k} \left(1 - (1 - p_i)^n\right) = k - \sum_{i=1}^{k} (1 - p_i)^n$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

## Theoretical Exercises

1. Let $E_i$ = {no type $i$ in first $n$ selections}

$$P\{T > n\} = P\left(\bigcup_{i=1}^{N} E_i\right)$$
$$= \sum_i (1 - P_i)^n - \sum_{I<J}\sum (1 - P_i - P_j)^n + \sum\sum_{i<j<k}\sum (1 - p_{_J} - p_k)^n$$
$$\ldots + (-1)^N \sum_i P_i^n$$

$$P\{T = n\} = P\{\mathrm{T} > n - 1\} - P\{T > n\}$$

2. Not true. Suppose $P\{X = b\} = \varepsilon > 0$ and $b_n = b + 1/n$. Then $\lim\limits_{b_n \to b} P(X < b_n\} = P\{X \le b\} \ne$ $P\{X < b\}$.

3. When $\alpha > 0$

$$P\{\alpha X + \beta \le x\} = P\left\{x \le \frac{x-\beta}{\alpha}\right\} = F\left(\frac{x-\beta}{\alpha}\right)$$

When $\alpha < 0$

$$P\{\alpha X + \beta \le x\} = P\left\{X \ge \frac{x-\beta}{\alpha}\right\} = 1 - \lim_{h \to 0^+} F\left(\frac{x-\beta}{\alpha} - 1\right).$$

4. $$\sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} P\{N = K\}$$
$$= \sum_{k=1}^{\infty} kP\{N = k\} = E[N].$$

5. $$\sum_{i=0}^{\infty} i\, P\{N > i\} = \sum_{i=0}^{\infty} i \sum_{k=i+1}^{\infty} P\{N = k\}$$
$$= \sum_{k=1}^{\infty} P\{N = k\} \sum_{i=0}^{k-1} i$$
$$= \sum_{k=1}^{\infty} P\{N = k\}(k-1)k/2$$
$$= \left(\sum_{k=1}^{\infty} k^2 P\{N = k\} - \sum_{k=1}^{\infty} kP\{N = k\}\right)\bigg/ 2$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

6. $E[c^X] = cp + c^{-1}(1 - p)$

Hence, $1 = E[c^X]$ if
$cp + c^{-1}(1 - p) = 1$
or, equivalently
$pc^2 - c + 1 - p = 0$
or
$(pc - 1 + p)(c - 1) = 0$

Thus, $c = (1 - p)/p$.

7. $E[Y] = E[X/\sigma - \mu/\sigma] = \dfrac{1}{\sigma}E[X] - \mu/\sigma = \mu/\sigma - \mu/\sigma = 0$
$\text{Var}(Y) = (1/\sigma)^2\ \text{Var}(X) = \sigma^2/\sigma^2 = 1.$

8. Let $I$ equal 1 if $X = a$ and let it equal 0 if $X = b$, then

$$X = a + (b - a)I$$

yielding the result

$$\text{Var}(X) = (b - a)^2\text{Var}(I) = (b - a)^2 p(1 - p)$$

9.
$$1 = \sum_{i=0}^{n} P(X = i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$$

If $x$ and $y$ are positive numbers, then letting $p = \dfrac{x}{x + y}$, gives

$$1 = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{x}{x + y}\right)^n \left(\frac{y}{x + y}\right)^{n-i}$$

or, upon multiplying both sides by $(x + y)^n$, that

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n=i}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

10. $$E[1/(X+1)] = \sum_{i=0}^{n} \frac{1}{i+1} \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i}$$
$$= \sum_{i=0}^{n} \frac{n!}{(n-i)!(i+1)!} p^i (1-p)^{n-i}$$
$$= \frac{1}{(n+1)p} \sum_{i=0}^{n} \binom{n+1}{i+1} p^{i+1} (1-p)^{n-i}$$
$$= \frac{1}{(n+1)p} \sum_{j=1}^{n+1} \binom{n+1}{j} p^{j} (1-p)^{n+1-j}$$
$$= \frac{1}{(n+1)p} \left[1 - \binom{n+1}{0} p^0 (1-p)^{n+1-0}\right]$$
$$= \frac{1}{(n+1)p} [1-(1-p)^{n+1}]$$

11. For any given arrangement of $k$ successes and $n - k$ failures:

$$P\{\text{arrangement} \mid \text{total of } k \text{ successes}\}$$
$$= \frac{P\{\text{arrangement}\}}{P\{k \text{ successes}\}} = \frac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$$

12. Condition on the number of functioning components and then use the results of Example 4c of Chapter 1:

$$\text{Prob} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \left[\binom{i+1}{n-i} \Big/ \binom{n}{i}\right]$$

where $\binom{i+1}{n-i} = 0$ if $n - i > i + 1$. We are using the results of Exercise 11.

13. Easiest to first take log and then determine the $p$ that maximizes $\log P\{X = k\}$.

$$\log P\{X = k\} = \log \binom{n}{k} + k \log p + (n - k) \log (1 - p)$$

$$\frac{\partial}{\partial p} \log P\{x = k\} = \frac{k}{p} - \frac{n-k}{1-p}$$
$$= 0 \Rightarrow p = k/n \text{ maximizes}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

14. (a) $1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \frac{\alpha p}{1-p}$

(b) Condition on the number of children: For $k > 0$

$$\begin{aligned} P\{k \text{ boys}\} &= \sum_{n=1}^{\infty} P\{k \mid n \text{ children}\}\alpha p^n \\ &= \sum_{n=k}^{\infty} \binom{n}{k} (1/2)^n \alpha p^n \end{aligned}$$

$$P\{0 \text{ boys}\} = 1 - \frac{\alpha p}{1-p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n$$

17. (a) If $X$ is binomial $(n, p)$ then, from exercise 15,

$$\begin{aligned} P\{X \text{ is even}\} &= [1 + (1 - 2p)^n]/2 \\ &= [1 + (1 - 2\lambda/n)^n]/2 \quad \text{when } \lambda = np \\ &\to (1 + e^{-2\lambda})/2 \quad \text{as } n \text{ approaches infinity} \end{aligned}$$

(b) $P\{X \text{ is even}\} = e^{-\lambda} \sum_n \lambda^{2n}/(2n)! = e^{-\lambda}(e^{\lambda} + e^{-\lambda})/2$

18. $\log P\{X = k\} = -\lambda + k \log \lambda - \log (k!)$

$$\begin{aligned} \frac{\partial}{\partial \lambda} \log P\{X = k\} &= -1 + \frac{k}{\lambda} \\ &= 0 \Rightarrow \lambda = k \end{aligned}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

19. $$E[X^n] = \sum_{i=0}^{\infty} i^n e^{-\lambda}\lambda^i / i!$$
$$= \sum_{i=1}^{\infty} i^n e^{-\lambda}\lambda^i / i!$$
$$= \sum_{i=1}^{\infty} i^{n-1} e^{-\lambda}\lambda^i / (i-1)!$$
$$= \sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j+1} / j!$$
$$= \lambda\sum_{j=0}^{\infty} (j+1)^{n-1} e^{-\lambda}\lambda^{j} / j!$$
$$= \lambda E[(X+1)^{n-1}]$$

Hence $[X^3] = \lambda E(X+1)^2]$
$$= \lambda\sum_{i=0}^{\infty} (i+1)^2 e^{-\lambda}\lambda^i / i!$$
$$= \lambda\left[\sum_{i=0}^{\infty} i^2 e^{-\lambda}\lambda^i / i! + 2\sum_{i=0}^{\infty} i e^{-\lambda}\lambda^i / i! + \sum_{i=0}^{\infty} e^{-\lambda}\lambda^i / i!\right]$$
$$= \lambda[E[X^2] + 2E[X] + 1)$$
$$= \lambda(\mathrm{Var}(X) = E^2[X] + 2E[X] + 1)$$
$$= \lambda(\lambda + \lambda^2 + 2\lambda + 1) = \lambda(\lambda^2 + 3\lambda + 1)$$

20. Let $S$ denote the number of heads that occur when all $n$ coins are tossed, and note that $S$ has a distribution that is approximately that of a Poisson random variable with mean $\lambda$. Then, because $X$ is distributed as the conditional distribution of $S$ given that $S > 0$,

$$P\{X = 1\} = P\{S = 1 \mid S > 0\} = \frac{P\{S = 1\}}{P\{S > 0\}} \approx \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}$$

21. (i) 1/365
(ii) 1/365
(iii) 1 The events, though independent in pairs, are not independent.

22. (i) Say that trial $i$ is a success if the $i^{\text{th}}$ pair selected have the same number. When $n$ is large trials 1, …, $k$ are roughly independent.
(ii) Since, $P\{\text{trial } i \text{ is a success}\} = 1/(2n - 1)$ it follows that, when $n$ is large, $M_k$ is approximately Poisson distributed with mean $k/(2n - 1)$. Hence,

$$P\{M_k = 0\} \approx \exp[-k/(2n - 1)]$$

(iii) and (iv) $P\{T > \alpha n\} = P\{M_{\alpha n} = 0\} \approx \exp[-\alpha n/(2n - 1)] \to e^{-\alpha/2}$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

23. (a) $P(E_i) = 1 - \sum_{j=0}^{2} \binom{365}{j} (1/365)^j (364/365)^{365-j}$

(b) $\exp(-365P(E_1)\}$

24. (a) There will be a string of $k$ consecutive heads within the first $n$ trials either if there is one within the first $n - 1$ trials, or if the first such string occurs at trial $n$; the latter case is equivalent to the conditions of 2.

(b) Because cases 1 and 2 are mutually exclusive

$$P_n = P_{n-1} + (1 - P_{n-k-1})(1 - P)p^k$$

25. 
$$\begin{aligned} P(m \text{ counted}) &= \sum_n P(m \mid n \text{ events}) e^{-\lambda}\lambda^n / n! \\ &= \sum_{n=m}^{\infty} \binom{n}{m} p^m (1-p)^{n-m} e^{-\lambda}\lambda^n / n! \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \sum_{n=m}^{\infty} \frac{[\lambda(1-p)]^{n-m}}{(n-m)!} e^{-\lambda(1-p)} \\ &= e^{-\lambda p} \frac{(\lambda p)^m}{m!} \end{aligned}$$

Intuitively, the Poisson λ random variable arises as the approximate number of successes in $n$ (large) independent trials each having a small success probability $\alpha$ (and $\lambda$ $n\alpha$). Now if each successful trial is counted with probability $p$, than the number counted is Binomial with parameters $n$ (large) and $\alpha p$ (small) which is approximately Poisson with parameter $\alpha pn = \lambda p$.

27. 
$$\begin{aligned} P\{X = n + k \mid X > n\} &= \frac{P\{X = n + k\}}{P\{X > n\}} \\ &= \frac{p(1-p)^{n+k-1}}{(1-p)^n} \\ &= p(1 - p)^{k-1} \end{aligned}$$

If the first $n$ trials are fall failures, then it is as if we are beginning anew at that time.

28. The events $\{X > n\}$ and $\{Y < r\}$ are both equivalent to the event that there are fewer than $r$ successes in the first $n$ trials; hence, they are the same event.

29. 
$$\begin{aligned} \frac{P\{X = k+1\}}{P\{X = k\}} &= \frac{\binom{Np}{k+1}\binom{N-np}{n-k-1}}{\binom{Np}{k}\binom{N-Np}{n-k}} \\ &= \frac{(Np-k)(n-k)}{(k+1)(N-Np-n+k+1)} \end{aligned}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

30. $P\{Y=j\} = \binom{j-1}{n-1}\Big/\binom{N}{n},\ n \le j \le N$

$$E[Y] = \sum_{j=n}^{N} \binom{j-1}{n-1}\Big/\binom{N}{n}$$

$$= \frac{n}{\binom{N}{n}} \sum_{j=n}^{N} \binom{j}{n}$$

$$= \frac{n}{\binom{N}{n}} \sum_{i=n+1}^{N+1} \binom{i-1}{n+1-1}$$

$$= \frac{n}{\binom{N}{n}} \binom{N+1}{n+1}$$

$$= \frac{n(N+1)}{n+1}$$

31. Let $Y$ denote the largest of the remaining $m$ chips. By exercise 28

$$P\{Y=j\} = \binom{j-1}{m-1}\Big/\binom{m+n}{m},\ m \le j \le n+m$$

Now, $X = n + m - Y$ and so

$$P\{X=i\} = P\{Y = m+n-i\} = \binom{m+n-i-1}{m-1}\Big/\binom{m+n}{m},\ i \le n$$

32. $P\{X=k\} = \frac{k-1}{n} \prod_{i=0}^{k-2} \frac{n-i}{n},\ k > 1$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

34. $$E[X] = \sum_{k=0}^{n} \frac{k\binom{n}{k}}{2^n - 1} = \frac{n2^{n-1}}{2^n - 1}$$

$$E[X^2] = \sum_{k=0}^{n} \frac{k^2\binom{n}{k}}{2^n - 1} = \frac{2^{n-2}n(n+1)}{2^n - 1}$$

$$\text{Var}(X) = E[X^2] - \{E[X])^2 = \frac{n2^{2n-2} - n(n+1)2^{n-2}}{(2^n - 1)^2}$$

$$\sim \frac{n2^{2n-2}}{2^{2n}} = \frac{n}{4}$$

$$E[Y] = \frac{n+1}{2},\ E[Y^2] = \sum_{i=1}^{n} i^2 / n \sim \int_1^{n+1} \frac{x^2\,dx}{n} \sim \frac{n^2}{3}$$

$$\text{Var}(Y) \sim \frac{n^2}{3} - \left(\frac{n+1}{2}\right)^2 \sim \frac{n^2}{12}$$

35. (a) $P\{X > i\} = \frac{1}{2}\frac{2}{3}\ldots\frac{i}{i+1} = \frac{1}{i+1}$

(b) $P(X < \infty\} = \lim_{i\to\infty} P\{X \le i\}$

$= \lim_{i}(1 - 1/(i+1)) = 1$

(c) $E[X] = \sum_i iP\{X = i\}$

$= \sum_i i(P\{X > i-1\} - P\{X > i\}$

$= \sum_i i\left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \sum_i \frac{1}{i+1}$

$= \infty$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.

36. (a) This follows because $\{X + Y = z_k\} = \cup_{(i,j)\in A_k} \{X = x_i, Y = y_j\}$, and the events $\{X = x_i, Y = y_j\}$ , $(i, j) \in A_k$, are mutually exclusive.

(b)

$$\begin{aligned}
E[X+Y] &= \sum_k z_k P\{X+Y=z_k\} \\
&= \sum_k z_k \sum_{(i,j)\in A_k} P\{X=x_i, Y=y_j\} \\
&= \sum_k \sum_{(i,j)\in A_k} z_k P\{X=x_i, Y=y_j\} \\
&= \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\}
\end{aligned}$$

(c) This follows from (b) because every pair $i, j$ is in exactly one of the sets $A_k$.

(d) This follows because

$$\{X = x_i\} = \cup_j \{X = x_i, Y = y_j\}$$

(e) From the preceding

$$\begin{aligned}
E[X+Y] &= \sum_k \sum_{(i,j)\in A_k} (x_i+y_j) P\{X=x_i, Y=y_j\} \\
&= \sum_{i,j} (x_i+y_j) P\{X=x_i, Y=y_j\} \\
&= \sum_{i,j} x_i P\{X=x_i, Y=y_j\} + \sum_{i,j} y_j P\{X=x_i, Y=y_j\} \\
&= \sum_i x_i \sum_j P\{X=x_i, y=y_j\} + \sum_j y_j \sum_i P\{X=x_i, Y=y_j\} \\
&= \sum_i x_i P\{X=x_i\} + \sum_j y_j P\{Y=y_j\} \\
&= E[X] + E[Y]
\end{aligned}$$

Copyright © 2010 Pearson Education, Inc. Publishing as Prentice Hall.