THE UNIVERSITY OF TEXAS AT ARLINGTON, TEXAS
DEPARTMENT OF ELECTRICAL ENGINEERING

EE 5321 - 001

OPTIMAL CONTROL

Extended HW 1

by

SOUTRIK MAITI

1001569883

Presented to

Prof. Michael Niestroy

March 9,2018

Pledge of honor:

“On my honor I have neither given nor received aid on Problems 1 and 2 in this assignment”

Signature: ________________Soutrik Maiti_______________________________

Problem 1

Problem 1:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

$$J = \frac{1}{2} W_1 (x_{1f} - 100)^2 + \frac{1}{2} W_2 x_{2f}^2 + \int_0^8 \frac{1}{2} u^2 dt$$

$$t_0 = 0, \; t_f = 8 \quad \Big| \quad x_1(0) = x_2(0) = 0 \qquad \& \quad \phi(x) = \begin{bmatrix} \frac{1}{2} W_1 (x_1 - 100)^2 \\ \frac{1}{2} W_2 x_2^2 \end{bmatrix}$$

The Hamiltonian is as follows:-

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 u$$

Costate eqn matrix:-

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -\partial H/\partial x_1 \\ -\partial H/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix}$$

optimal control eqn:-

$$\frac{\partial H}{\partial u} = u + \lambda_2 = 0$$

$$\Rightarrow \boxed{u = -\lambda_2}$$

Boundary conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \quad \begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix} = \begin{bmatrix} W_1 x_1(0) - 100 \\ W_2 x_2(0) \end{bmatrix}$$

using $(\phi_x + \psi_x^T \nu - \lambda)^T \big|_T dx(T) + (\phi_t + \psi_t^T \nu + H)\big|_T dT = 0$

$\because$ It is a fixed time problem

$\therefore$ $dT \to 0$ & $\psi_t = 0$ ($\because$ No hard constraints)

$\lambda_1(0)$ & $\lambda_2(0)$ is calculated from $(\phi_x + \psi_x^T \nu - \lambda)^T \big|_T dx(T) = 0$

Now,

$$z \triangleq [x_1 \; x_2 \; \lambda_1 \; \lambda_2]^T$$

$$\dot{z} = Az$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$z(t) = e^{At} c$$

At $t = 0$

$e^{At} = I$ & $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (from 1st boundary condition) use boundary condition at $t_f = 8 \rightarrow x_1(8) \rightarrow x_{1f}$ & $x_2(8) \rightarrow x_{2f}$

$$\begin{bmatrix} x_{1f} \\ x_{2f} \\ w_1 x_{1f} - 100 \\ w_2 x_{2f} \end{bmatrix} = e^{8A} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{1f} \\ x_{2f} \\ W_1 x_{1f} - 100 \\ W_2 x_{2f} \end{bmatrix} = \begin{bmatrix} 1 & 8 & 85.33 & -32 \\ 0 & 1 & 32 & -8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ c_3 \\ c_4 \end{bmatrix}$$

Putting $W_1 = W_2 = 1$ & solving for x_{1f}, x_{2f}, c_3 & c_4.

$x_{1f} = 85.33c_3 - 32c_4$ — (i)

$x_{2f} = 32c_3 - 8c_4$ — (ii)

$x_{1f} - c_3 = 100$ — (iii)

$x_{2f} = -8c_3 + c_4$ — (iv)

From (i) & (iii) & (ii) & (iv), we get:-

$100 + c_3 = 85.33c_3 - 32c_4$

$\Rightarrow 84.33c_3 - 32c_4 = 100$ — (v)

$-8c_3 + c_4 = 32c_3 - 8c_4$

$\Rightarrow 40c_3 - 9c_4 = 0$ — (vi)

$c_3 = -1.72$

$c_4 = -7.677$

Putting the value of c_3 & c_4 in (i) & (ii), we get:-

$$\boxed{\begin{aligned} x_{1f} &= 98.8964 \\ x_{2f} &= 6.083 \end{aligned}}$$

Now,

$$Z(t) = e^{At}c$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 0 \\ -1.72 \\ -7.677 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & t & t^3/6 & -t^2/2 \\ 0 & 1 & t^2/2 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1.72 \\ -7.677 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= -0.286t^3 + 3.838t^2 \\ x_2 &= -0.86t^2 + 7.677t \end{aligned}} \rightarrow \text{Final state eqns}$$

$$\lambda_1 = -1.72$$

$$\lambda_2 = 1.72t - 7.677$$

$$\boxed{u = -\lambda_2 = 7.677 - 1.72t} \rightarrow \text{optimal control i/p.}$$

Now,

the cost function $J = \frac{1}{2}W_1(x_1 - 100)^2 + \frac{1}{2}W_2 x_2^2 + \int_0^8 \frac{1}{2}u^2\,dt$

Putting $W_1 = W_2 = 1$

$x_1 = x_{1f}$

$x_2 = x_{2f}$

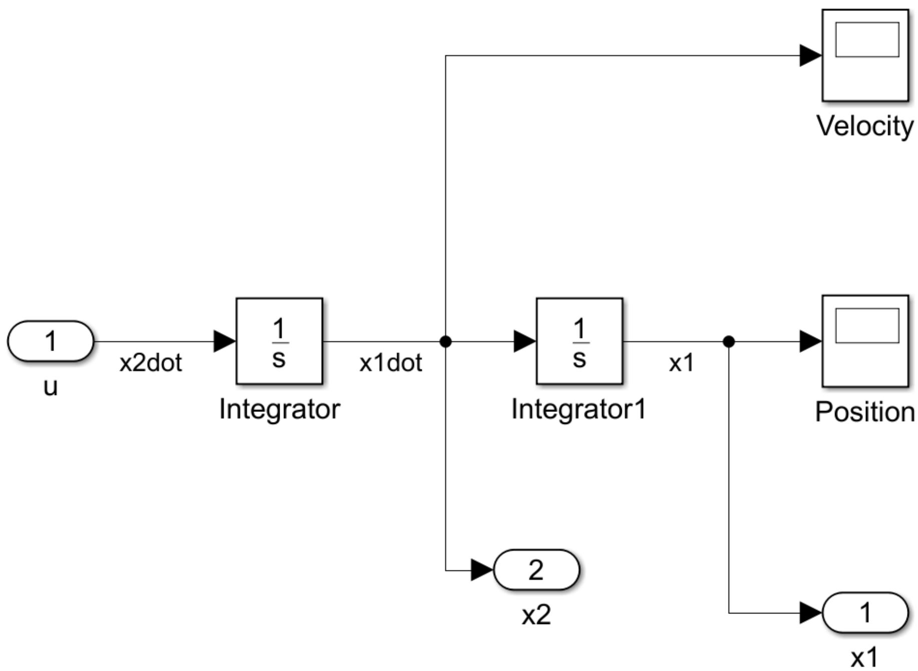
$$\Rightarrow J = 0.5\,(98.8964 - 100)^2 + 0.5 \times (6.083)^2 + \int_0^8 0.5\,(7.677 - 1.72t)^2\,dt$$

$$= 84.7835$$

Problem 2

a)

Simulink diagram:



MATLAB Code:

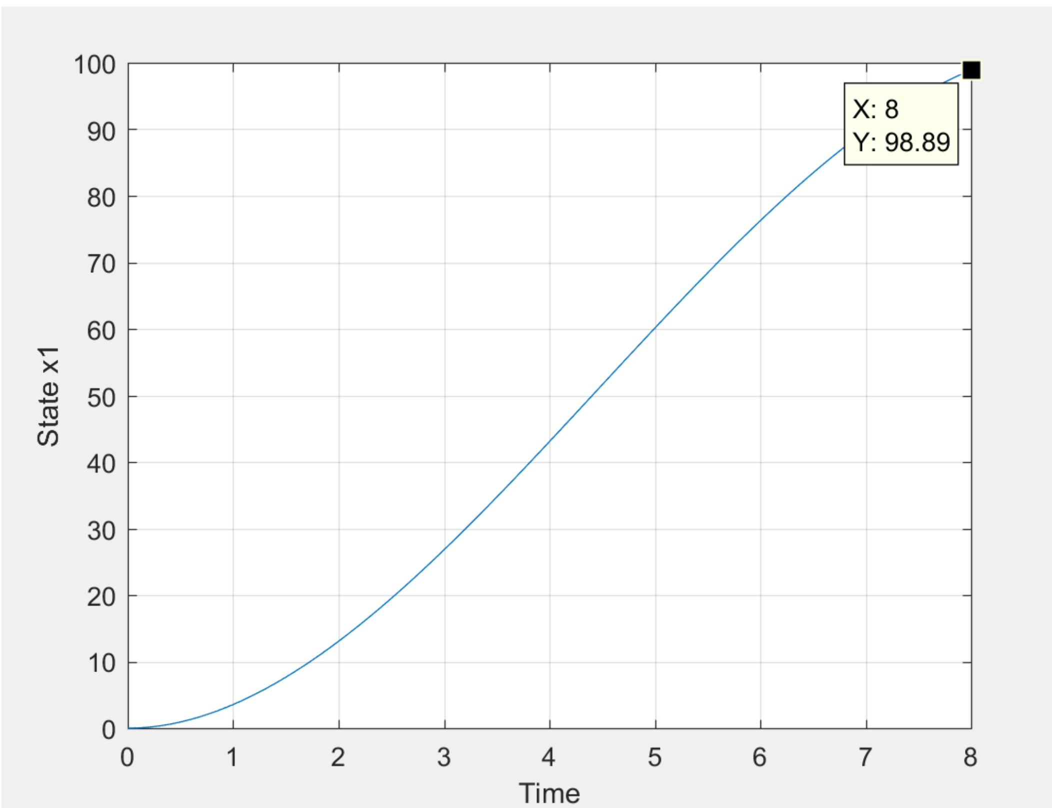
```
%% sim with initial control

u = 0.1*ones(length(t),1);
[t0,y0] = sim('e1',t',[],[t' u]);

%% Unconstrained optimization

options = optimoptions('fminunc','Display','iter','Algorithm','quasi-
newton');
[xopt,optimal_cost] = fminunc('e1cf',u,options);

%% Simulation with optimal control
[t_opt,y_opt] = sim('e1',t',[],[t' xopt]);

%% Plotting State time histories
figure;
plot(t_opt,y_opt(:,1));
grid; xlabel('Time'); ylabel('State x1');
figure;
```

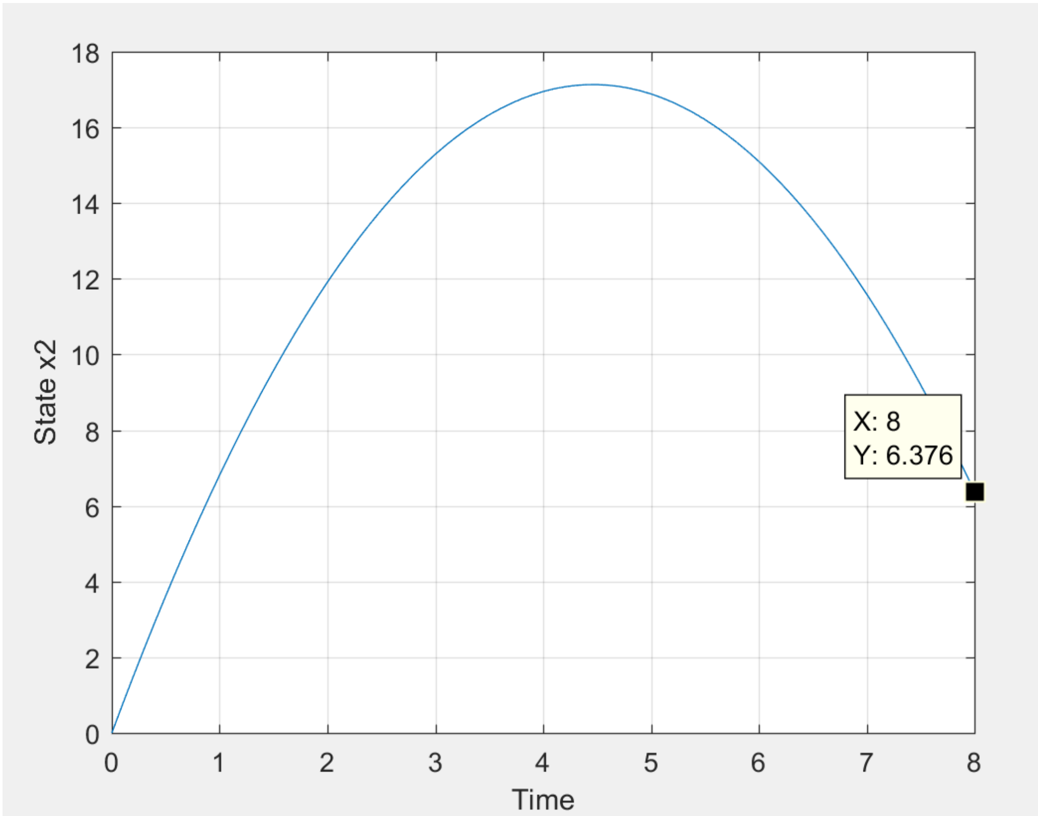
```
plot(t_opt,y_opt(:,2));
grid; xlabel('Time'); ylabel('State x2');
figure;
plot(t_opt,xopt);
grid; xlabel('Time'); ylabel('Control u*');
```

Cost function:

```
function cost = e1cf(u)
t=0:0.1:8;
w1=0;
w2=1;
[tf,xf]=sim('e1',t',[],[t' u]);
cost = 0.5*w1*(xf(end,1)-100).^2 + 0.5*w2*(xf(end,2)).^2 +
0.5*0.1*trapz(u.*u);
end
```

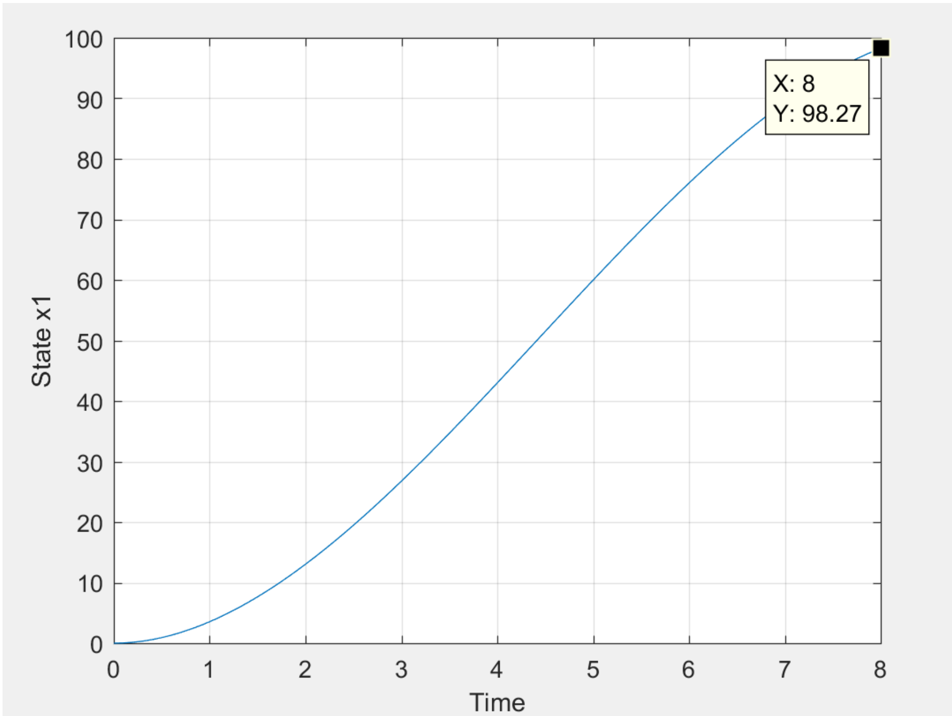
When the state equations from the analytical solution are plotted in MATLAB, we get the following graphs:

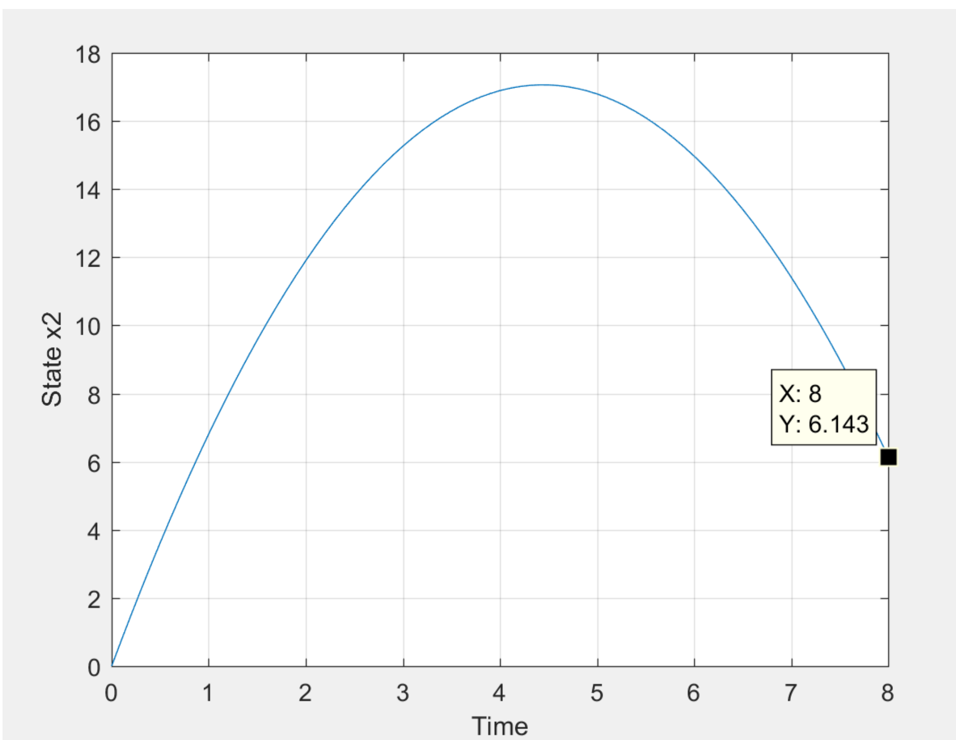




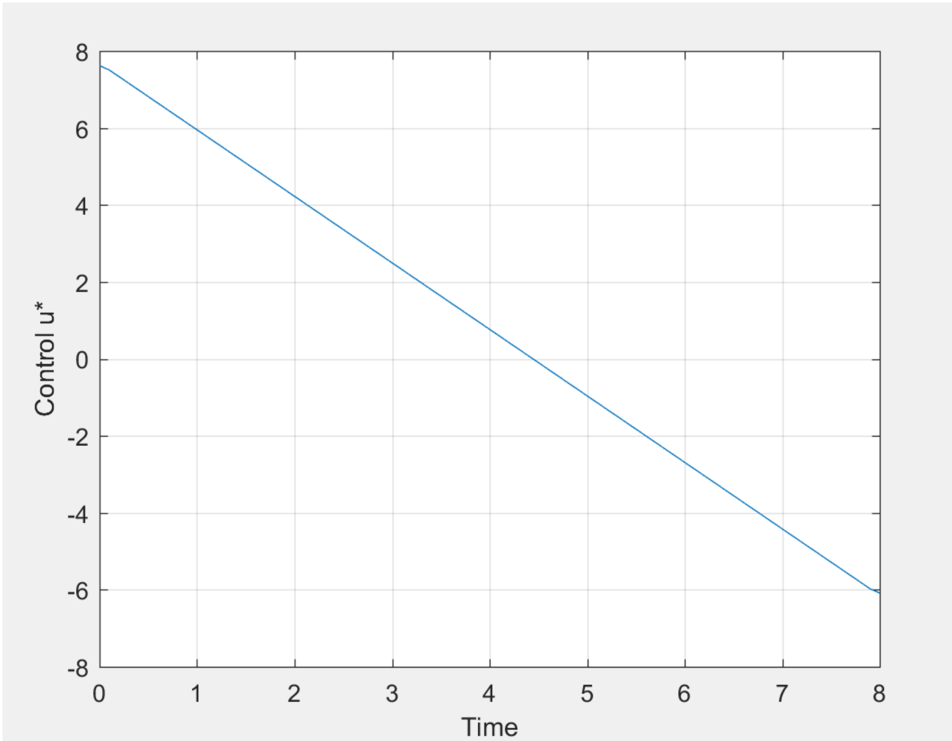
The final value of State 1 is *98.89* and that of State 2 is *6.376*.

When we compute the value of the states numerically(for W1=W2=1) using the MATLAB code as given above, we get the following results as shown below:



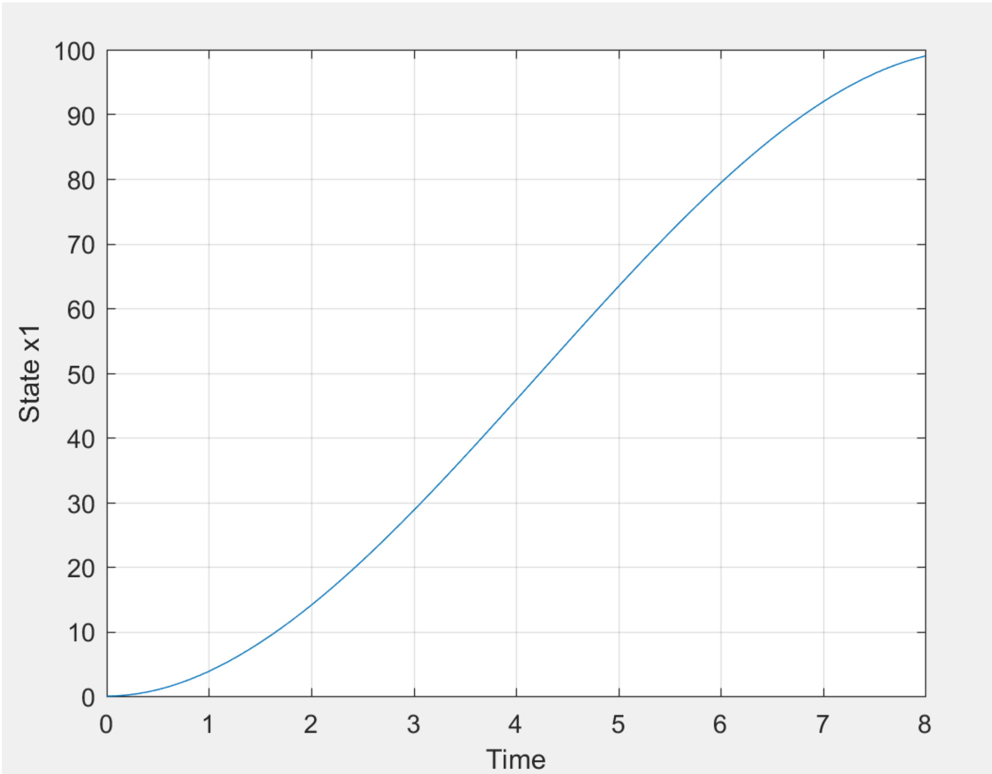


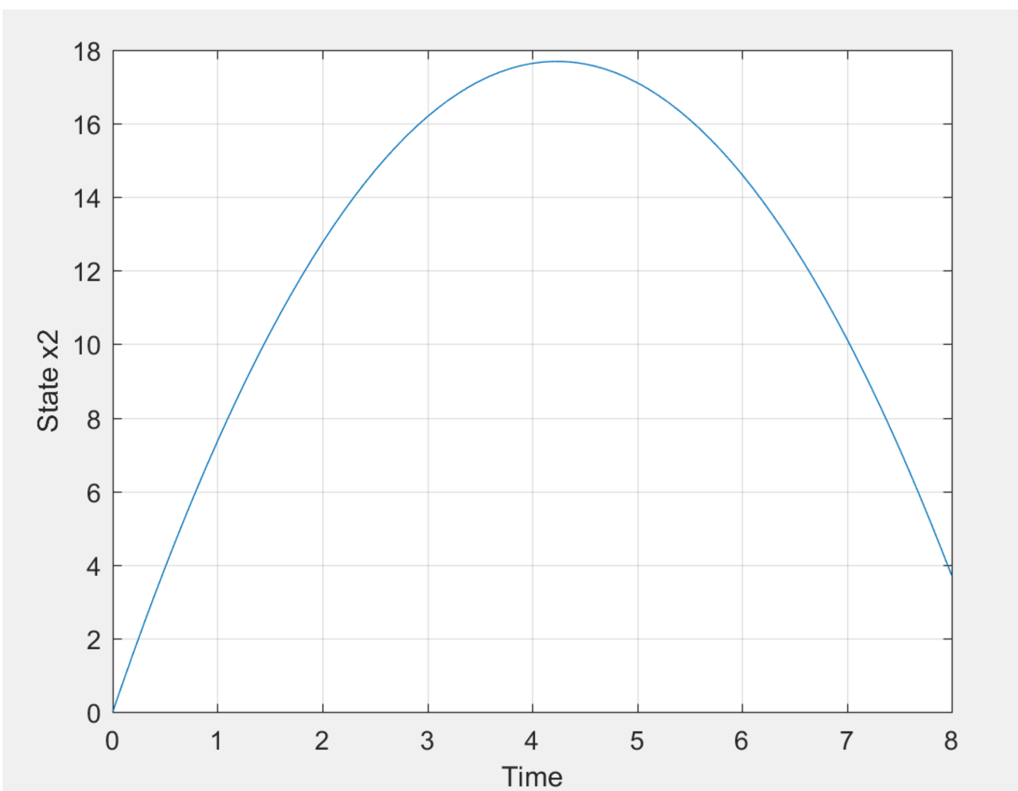
We can see that the final value of state 1 is *98.27* and that of state 2 is *6.143* which is very close to the values obtained analytically.

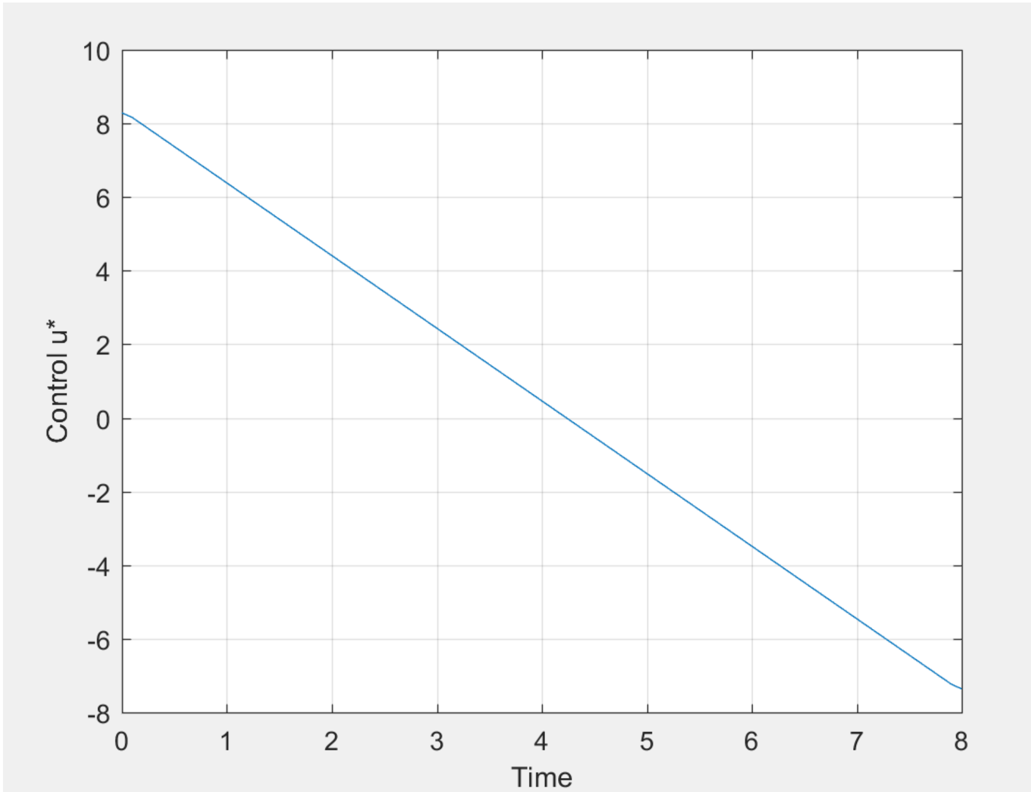



```
    21          1886             86.3921                     1          1.43e-05

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.
```

b) When W1=W2=2, we get the following results:





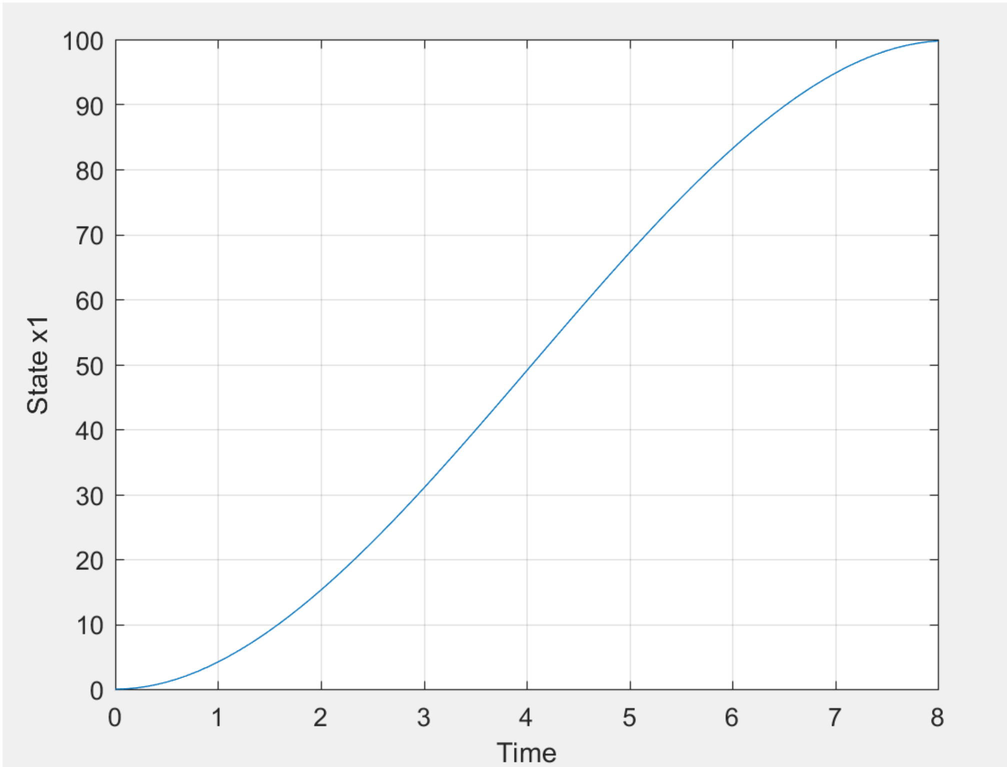


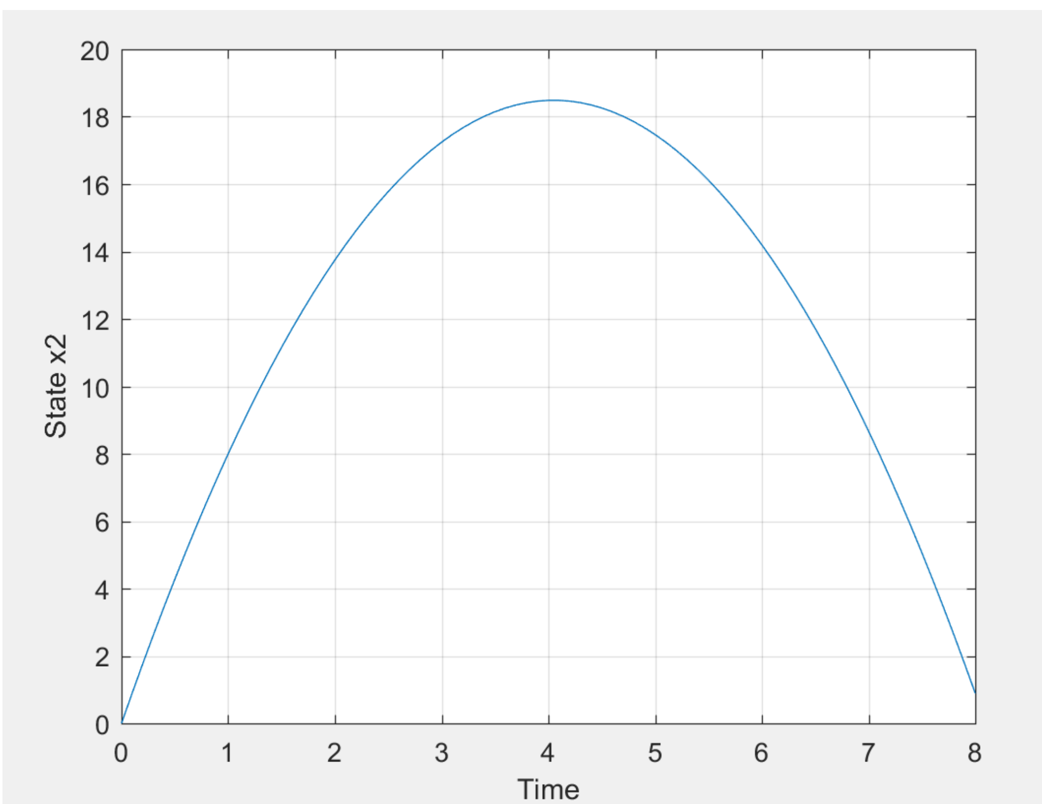
16	1476	98.6527	1	7.3e-05

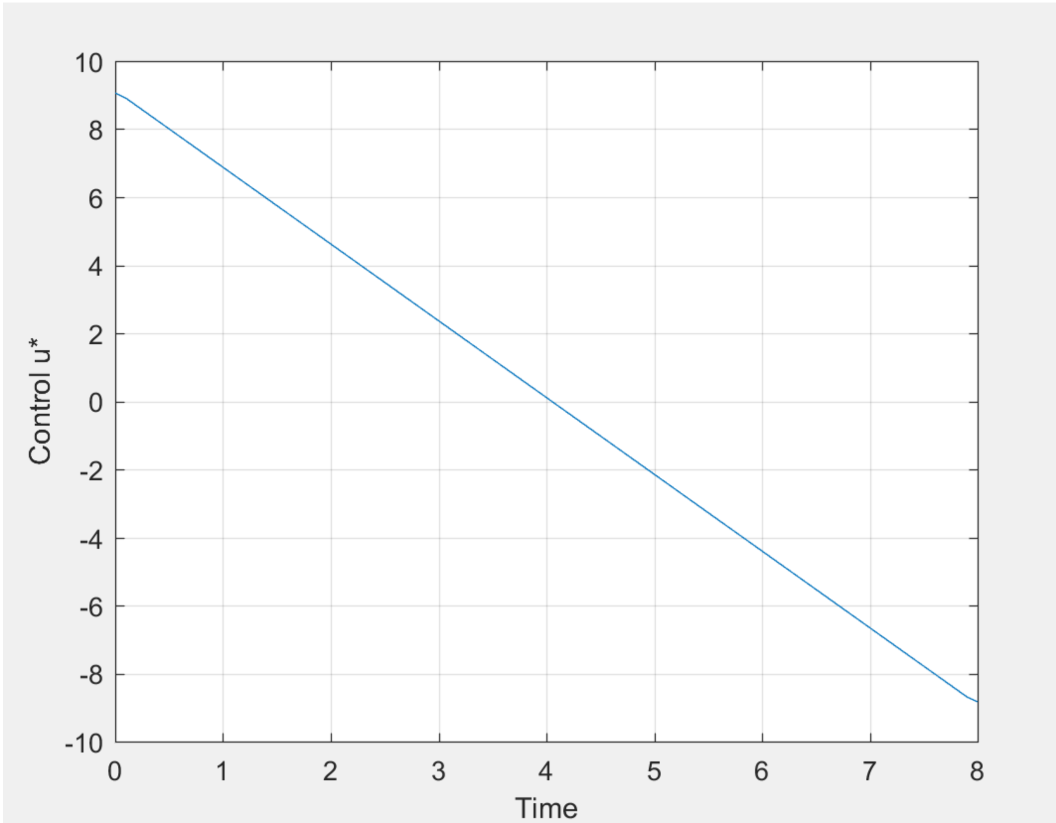
Local minimum found.

Optimization completed because the size of the gradient is less than the default value of the optimality tolerance.

When W1=W2=10, we get the following results:





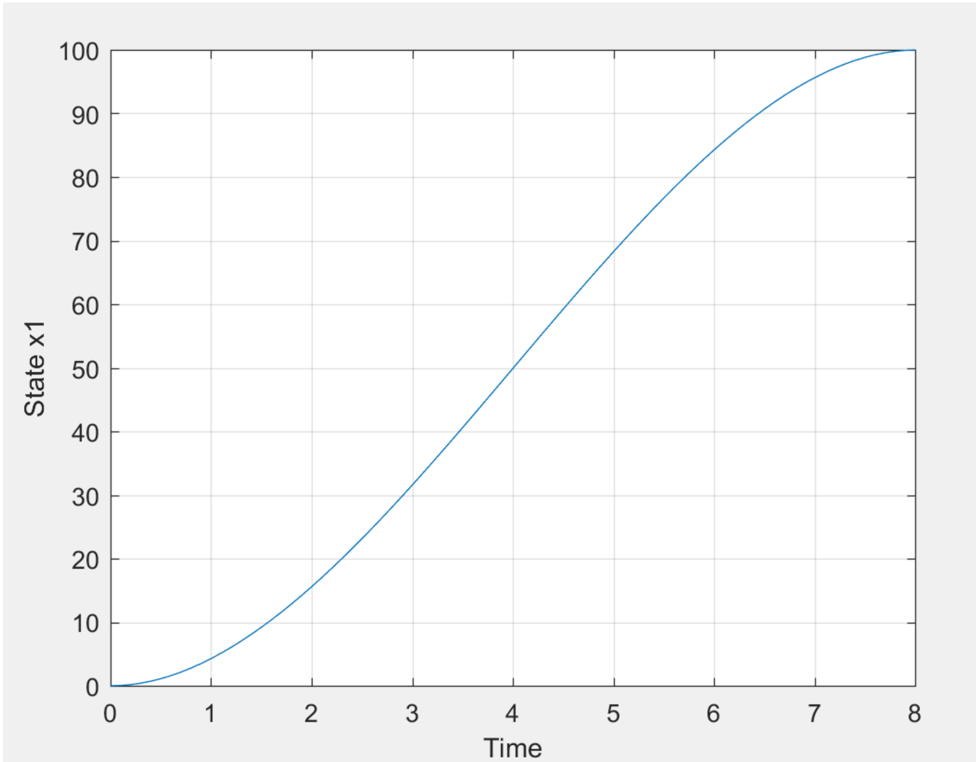


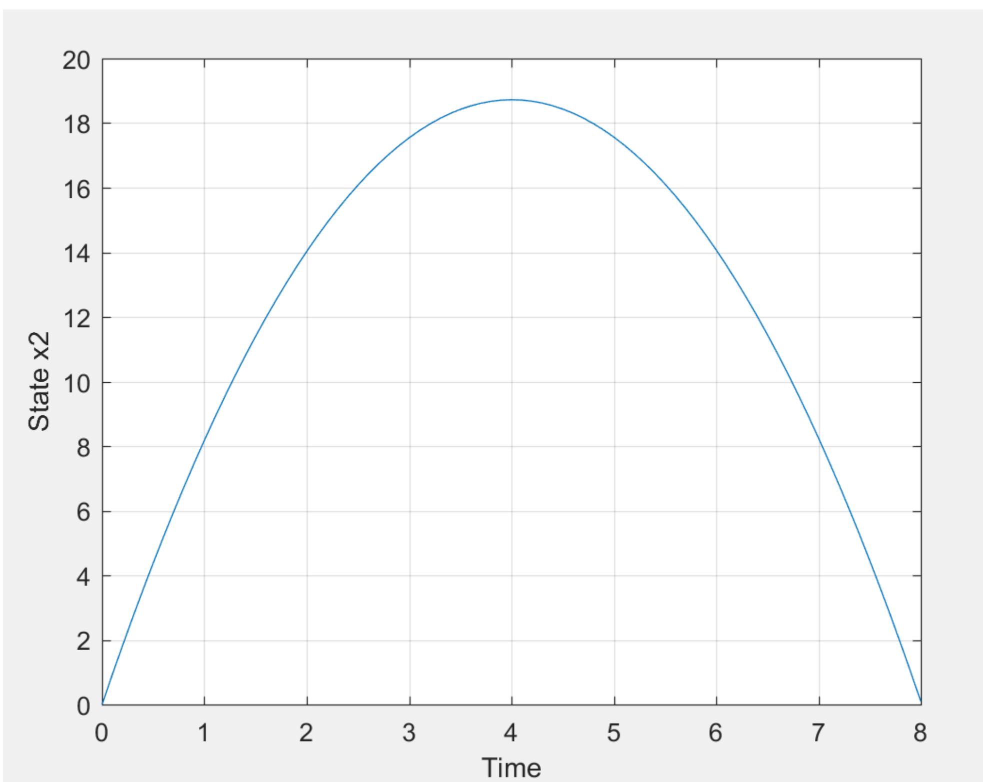

```
    18          1640           112.781                  1         0.000645
```

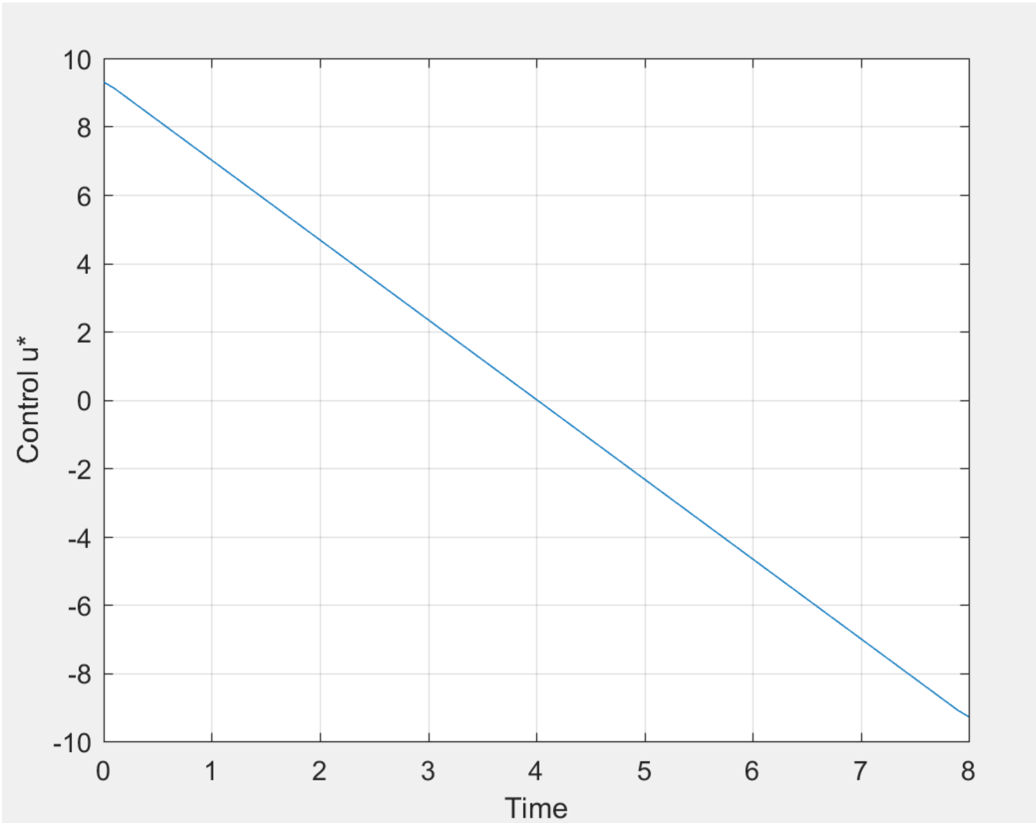
Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.

When W1=W2=100, we get the following results:





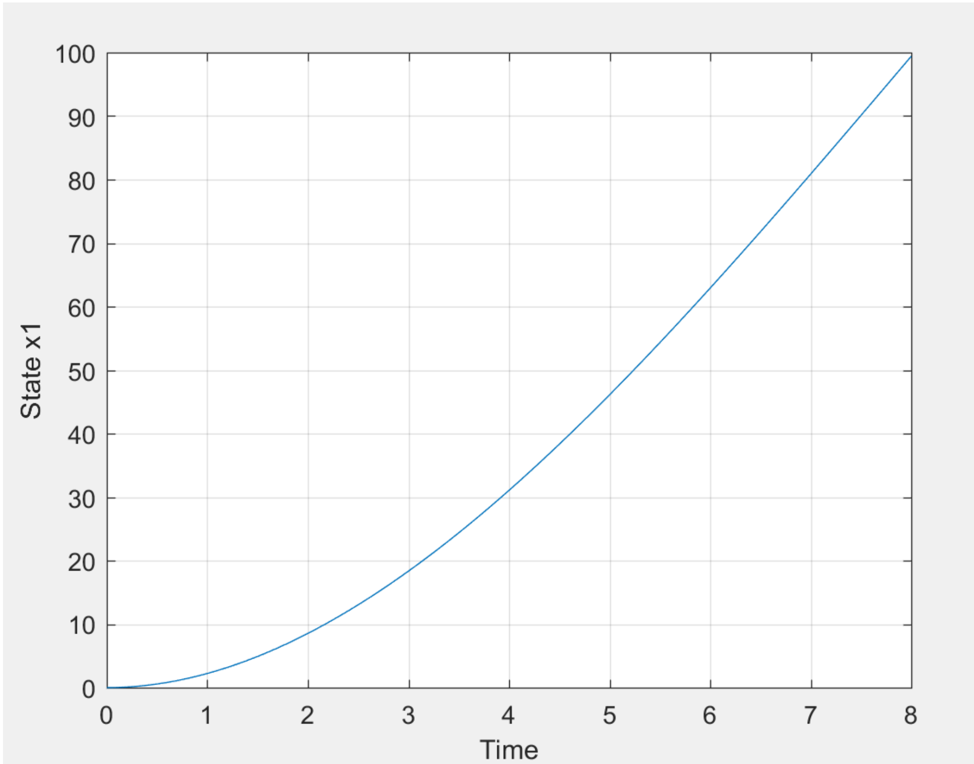


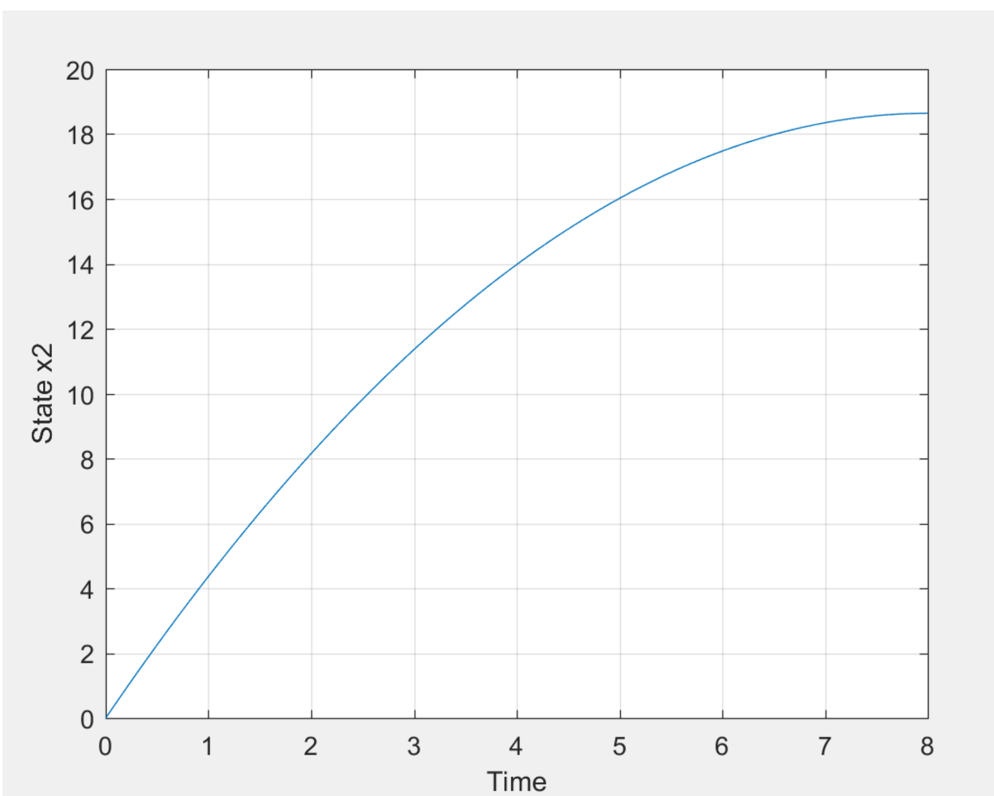

```
    15          1558           116.759              1          0.00528

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.
```

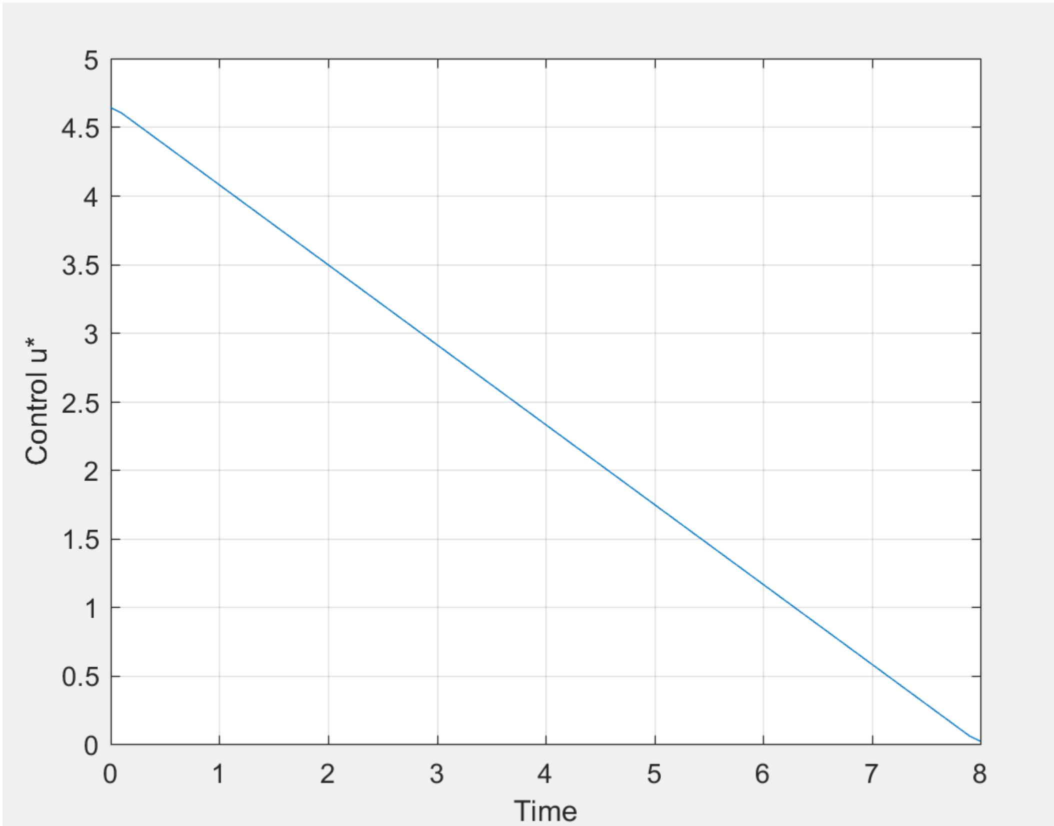
The following table gives us the comparison of optimal cost for the various values of W1 and W2.

Weights	**Optimal Cost**
W1 = W2 = 1	86.3921
W1 = W2 = 2	98.6527
W1 = W2 = 10	112.781
W1 = W2 =100	116.759

c) When W1 = 1 and W2 = 0, we get the following results:



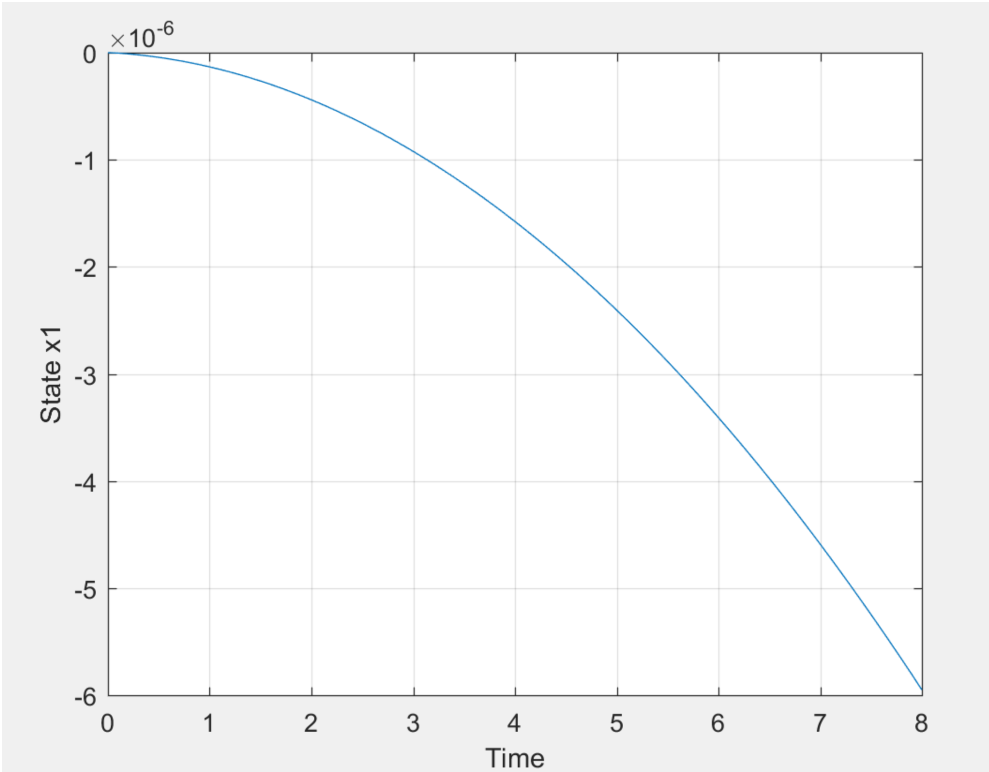


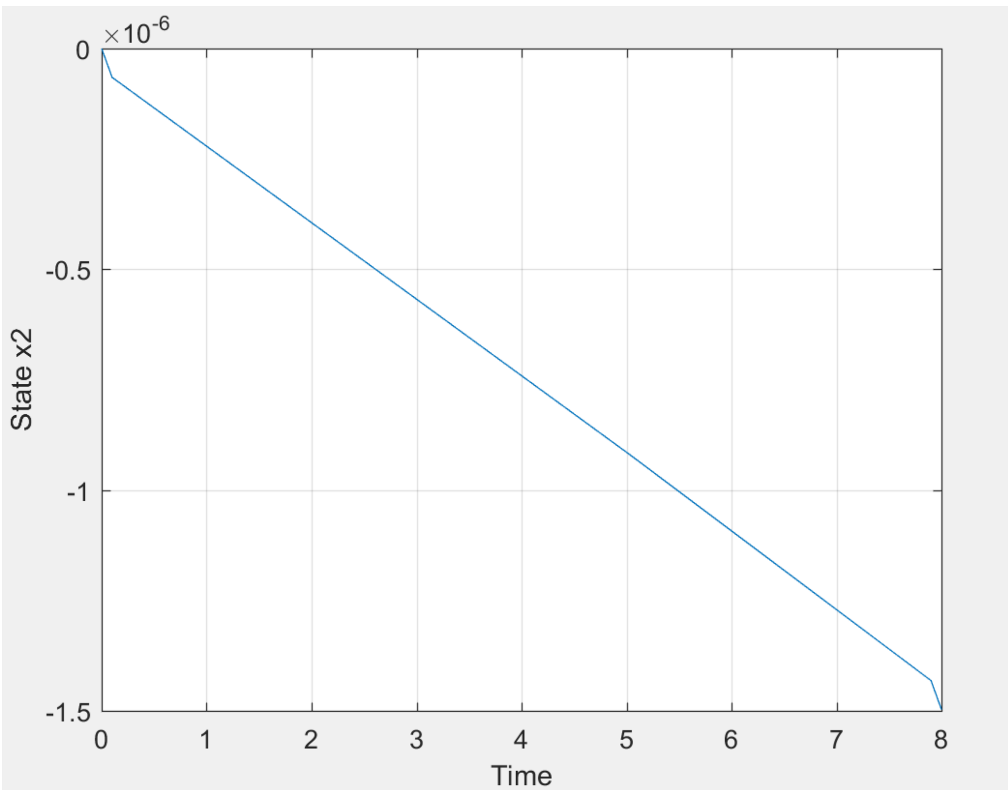


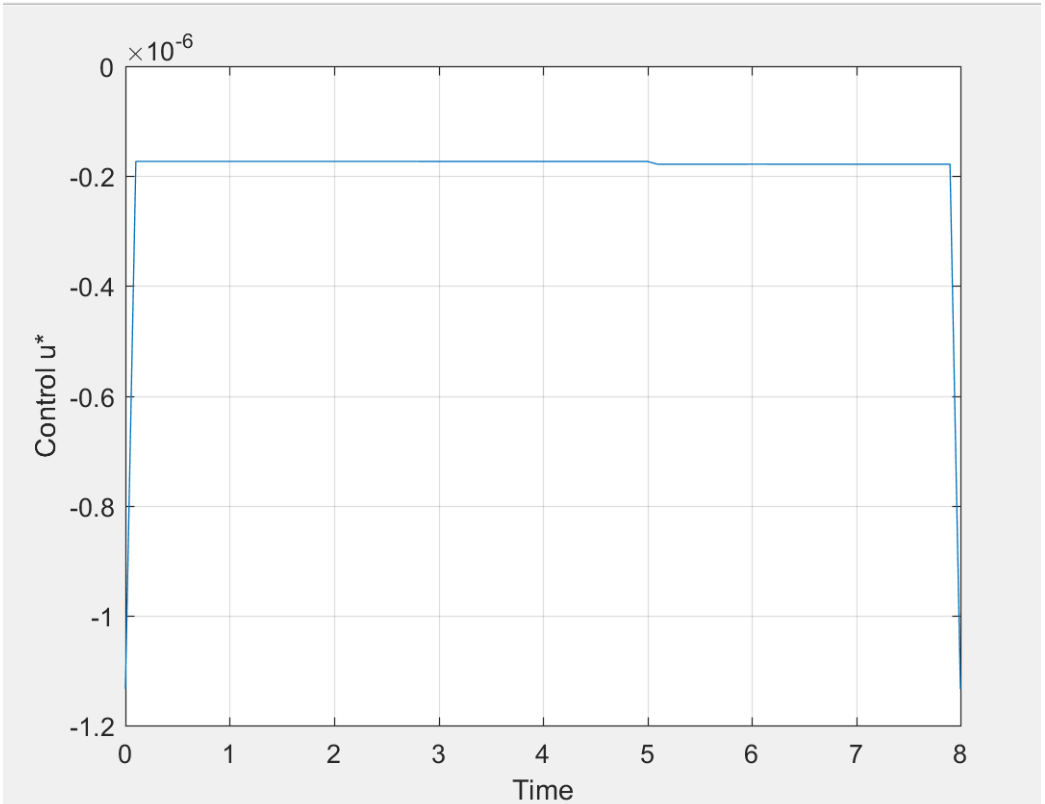

```
    13          1312          29.1285              1        3.41e-05

Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.
```

As we can see from the above plots, the state x2 goes to a arbitrarily large value when there is no constraint on the state 2.

d) When W1 = 0 and W2 = 1, we get the following results:








```
5          574       1.30592e-12                1          1.68e-07
```

```
Local minimum found.

Optimization completed because the size of the gradient is less than
the default value of the optimality tolerance.
```

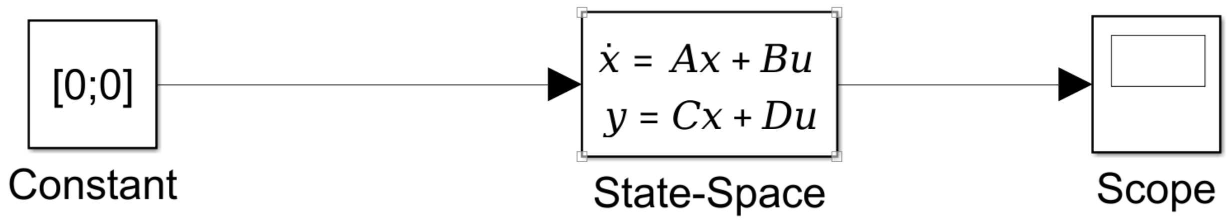
When W1 = 0 and W2 = 1 then the state 1 goes to around -6 and state 2 goes to -1.5 which is not expected.

Problem 3:

a)

Open Loop system:

Simulink diagram



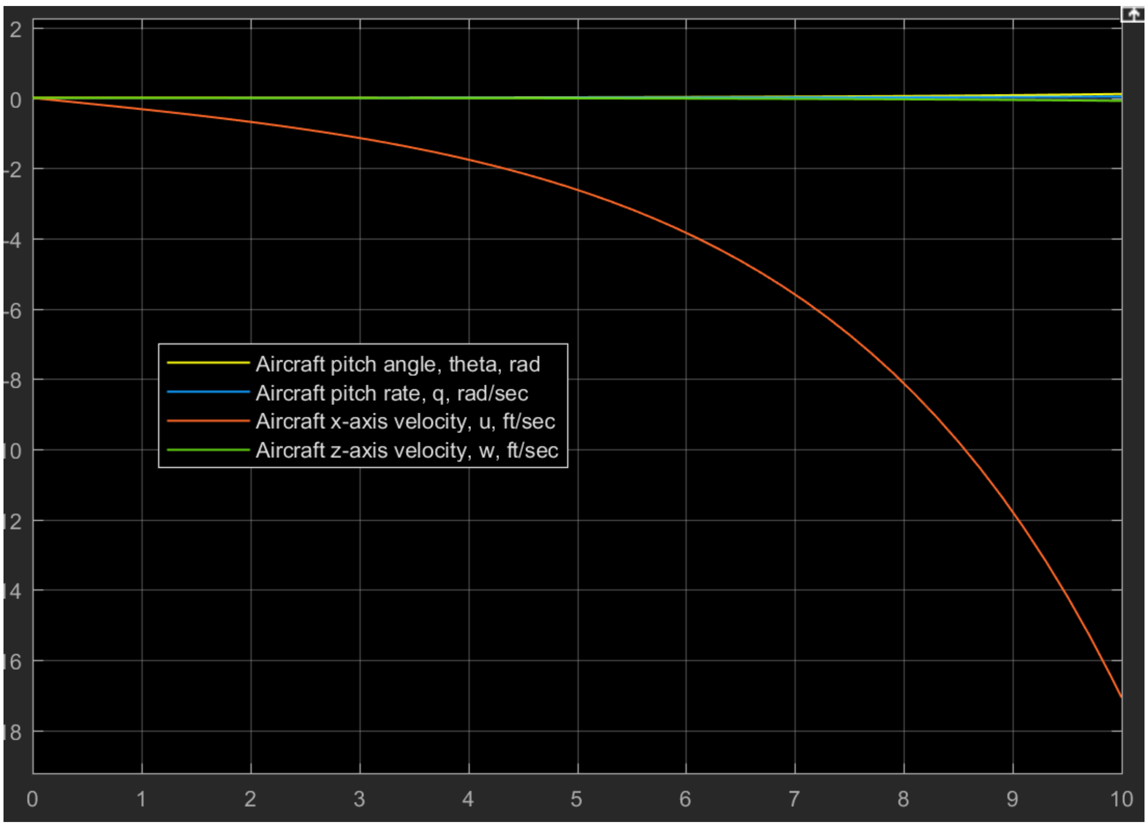
MATLAB Code:

```
%% System matrices

A = [-0.0000 1.0000 -0.0000 0.0000;
-0.0000 -0.5000 -0.0022 0.0015;
-30.8405 -60.240 0.0089 0.0011;
-0.0190 0.0092 0.0017 0.0120];

B = [-0.0000 -0.0000;
-0.0226 -0.0212;
0.0010 -0.0035;
0.0068 -0.0072];

C = eye(4);

D = zeros(4,2);

%% State space representation

system = ss(A,B,C,D);

%% Eigen values of the system

lamda = eig(system)
```

RESULT:

```
lamda =

    0.3704
   -0.4855
   -0.3771
    0.0132
```

The state time histories are as follows:

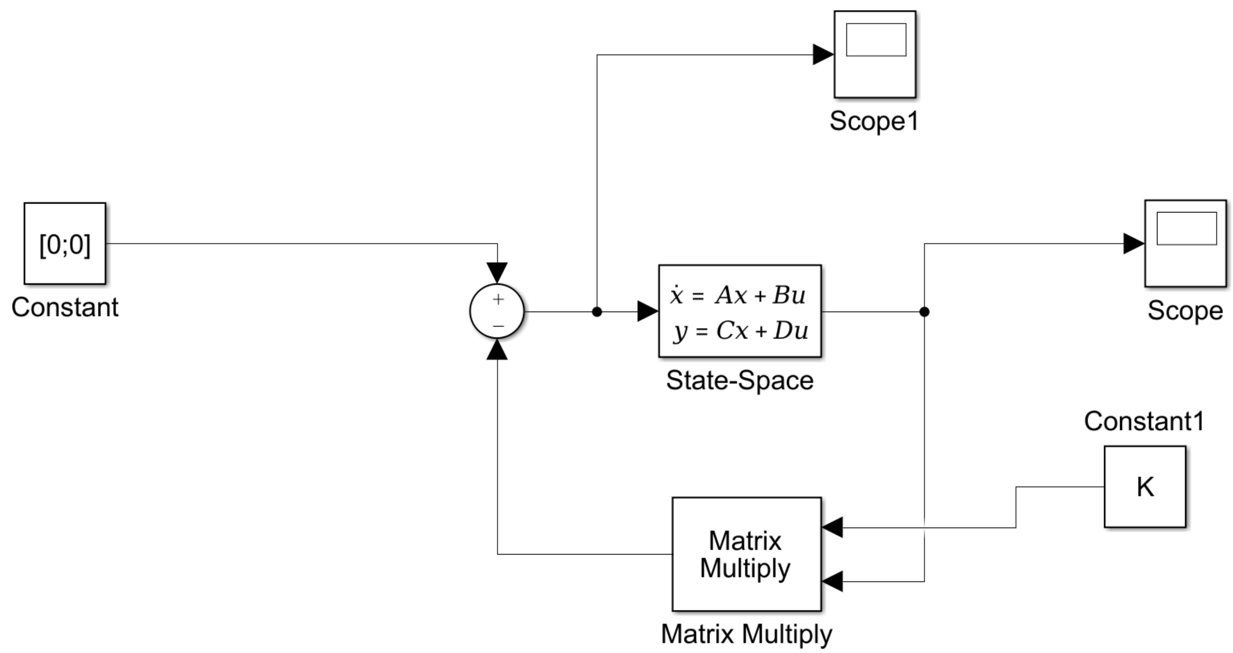


Since two of the eigen values lies in the open right half plane, the open loop system is not stable.

b)

LQR Feedback Controller:

Simulink diagram



MATLAB code:

```
%% LQR Controller feedback design
R = eye(2);
Q = eye(4);

[K,S,e] = lqr(system,Q,R,0)

%% System Matrices (with LQR Feedback)
Alqr = A-B*K;
Blqr = B;
Clqr = C;
Dlqr = D;

system_lqr = ss(Alqr,Blqr,Clqr,Dlqr)

lamda_lqr = eig(system_lqr)
```

RESULT:

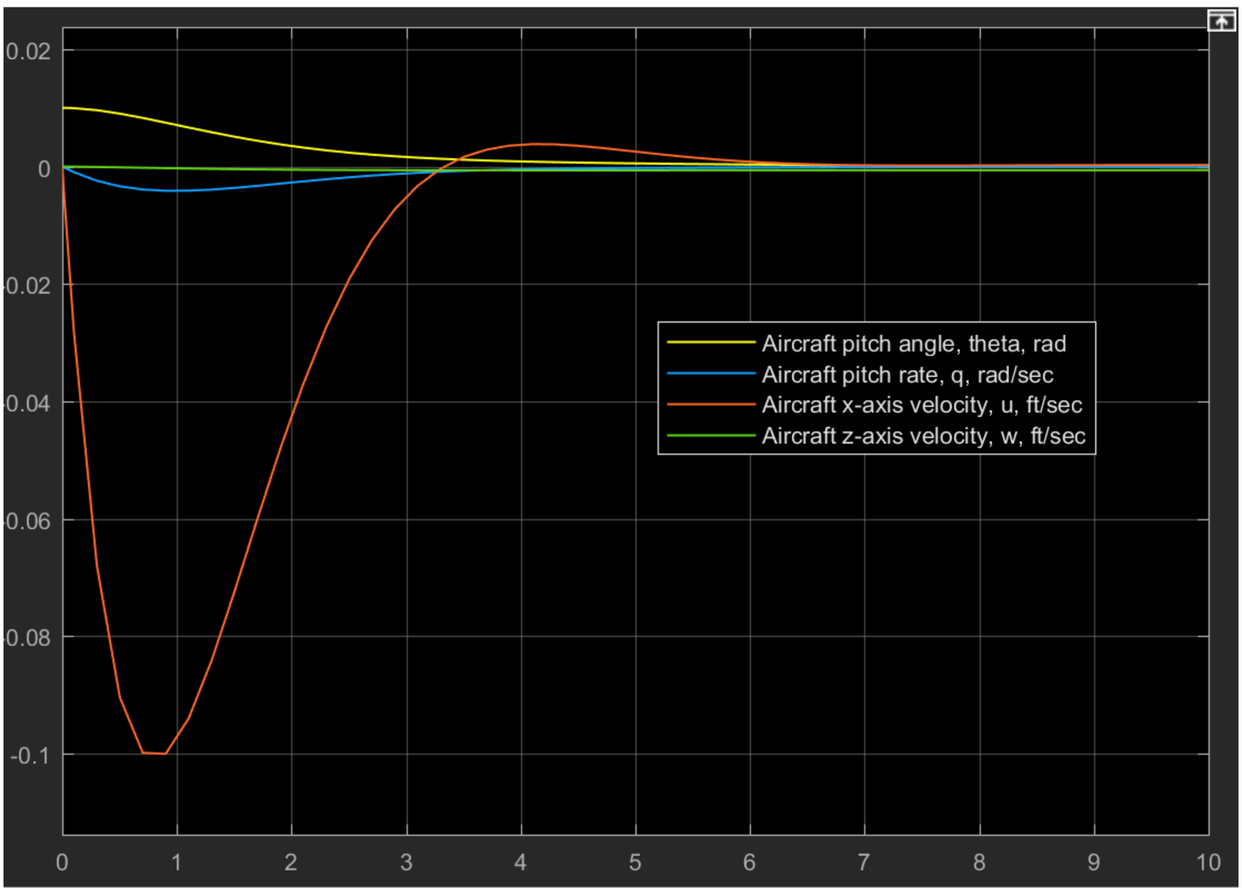
```
K =

  -24.4191  -47.7714    0.7950    2.1501
  -22.5720  -44.4861    0.7365   -1.7104


S =

   1.0e+03 *

    0.5489    1.0740   -0.0179   -0.0188
    1.0740    2.1085   -0.0350   -0.0126
   -0.0179   -0.0350    0.0011    0.0003
   -0.0188   -0.0126    0.0003    0.2744


e =

  -0.9998 + 0.9340i
  -0.9998 - 0.9340i
  -0.5118 + 0.0000i
  -0.0157 + 0.0000i


system_lqr =

  A =
              x1          x2          x3          x4
   x1           0           1           0           0
   x2       -1.03      -2.523     0.03138     0.01383
   x3       -30.9      -60.35     0.01068   -0.007037
   x4    -0.01547     0.01375    0.001597    -0.01494

  B =
            u1        u2
   x1        0         0
   x2  -0.0226   -0.0212
   x3    0.001   -0.0035
   x4   0.0068   -0.0072

  C =
       x1  x2  x3  x4
   y1   1   0   0   0
   y2   0   1   0   0
   y3   0   0   1   0
   y4   0   0   0   1

  D =
       u1  u2
   y1   0   0
   y2   0   0
```

y3   0   0
y4   0   0

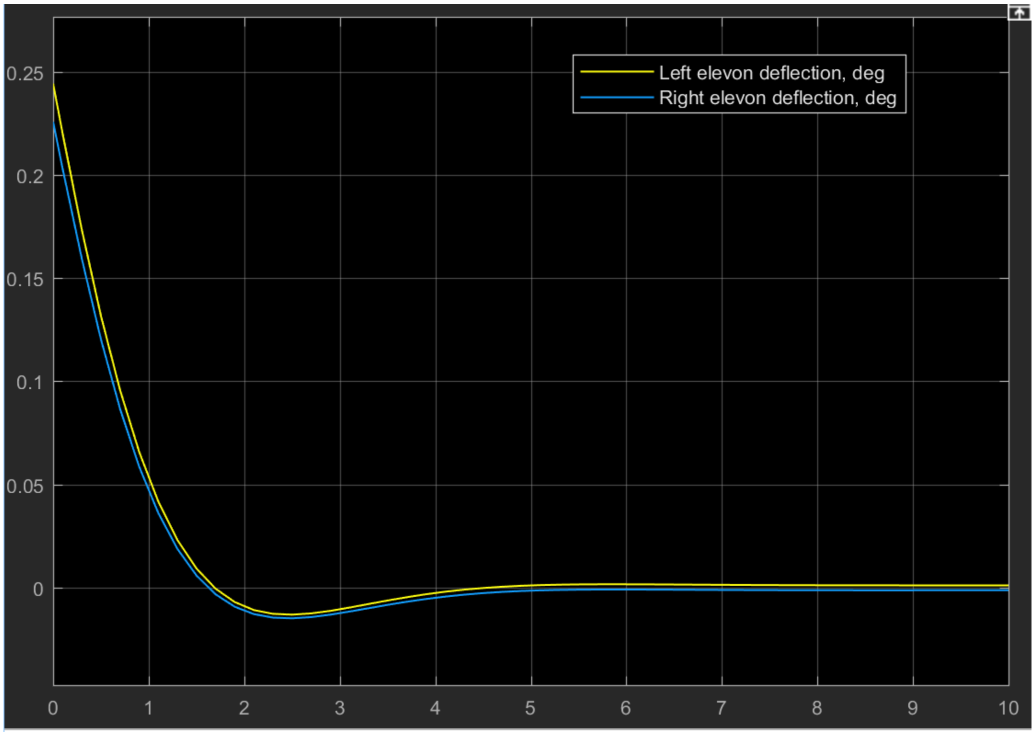
Continuous-time state-space model.

lamda_lqr =
  -0.9998 + 0.9340i
  -0.9998 - 0.9340i
  -0.5118 + 0.0000i
  -0.0157 + 0.0000i

The state time histories are as follows:



The control input are as follows:

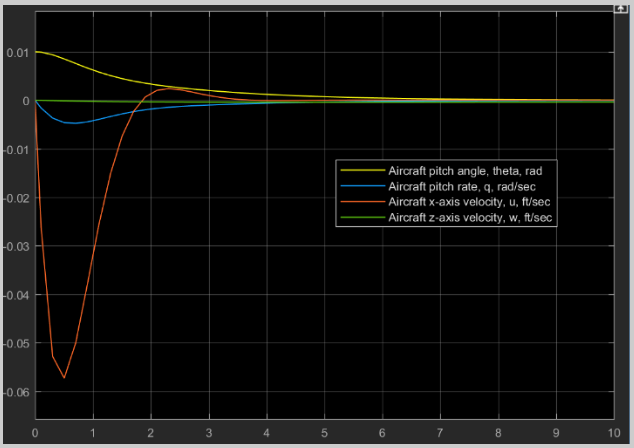


We can see that all the eigen values lie in the open left half plane and the state time history plots tell us the system reaches stability.
Hence the system is stable.

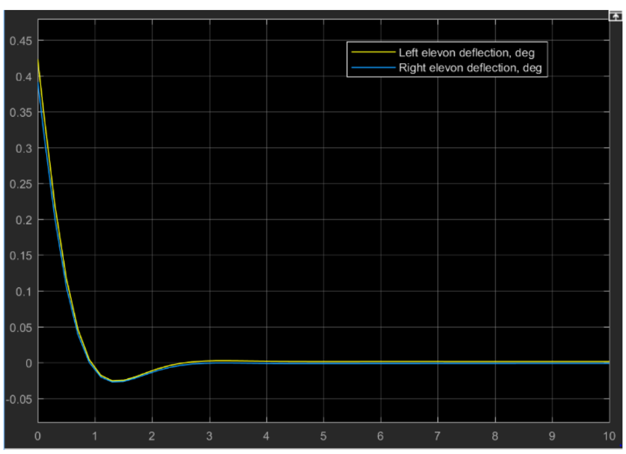
c)

Repeating part b with Q = 10*eye(4) and Q = 100*eye(4)

Q = 10 * eye(4)



*State time history (Q = 10*eye(4))*



*Control time history (Q = 10*eye(4))*

lambda_lqr =

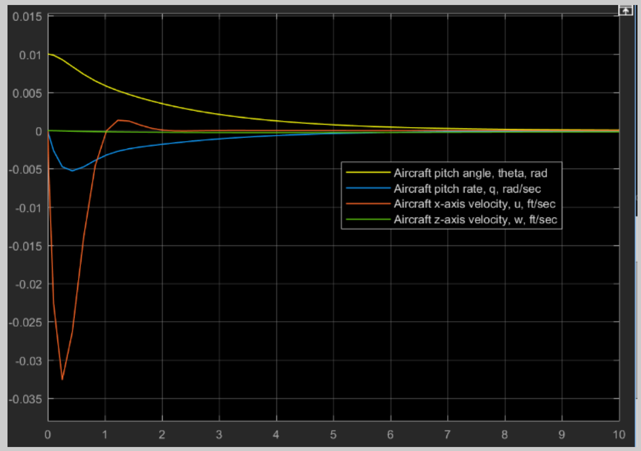
-1.7373 + 1.6991i

-1.7373 - 1.6991i

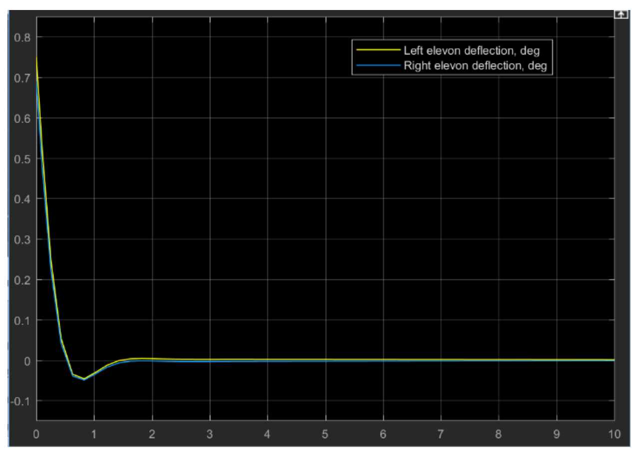
-0.5119 + 0.0000i

-0.0336 + 0.0000i

Q = 100 * eye(4)



*State time history (Q = 100*eye(4))*



*Control time history (Q = 100*eye(4))*

lambda_lqr =

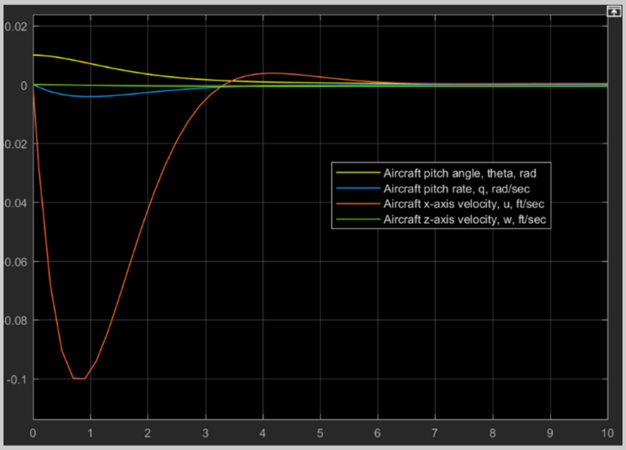
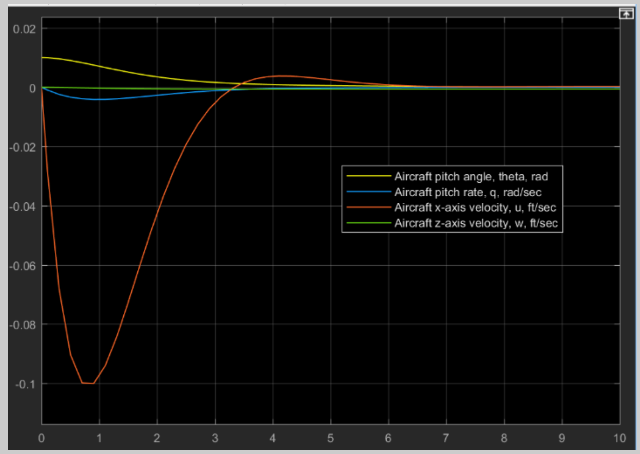
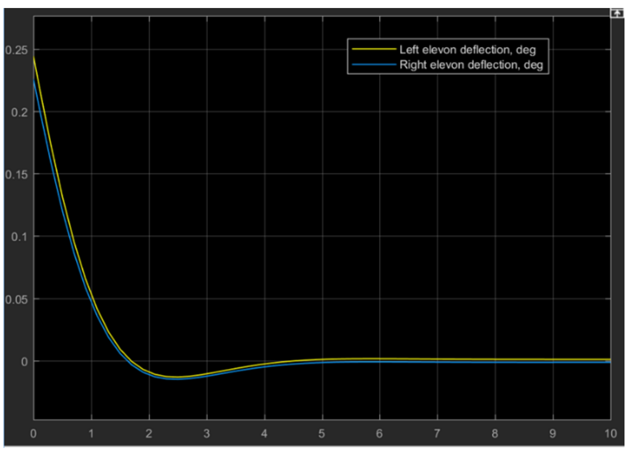
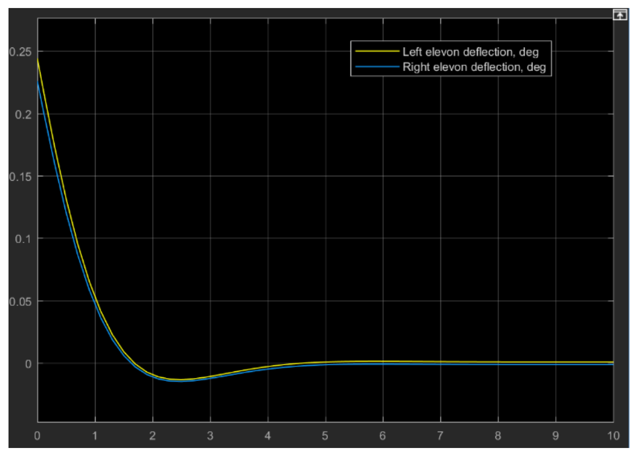
-3.0696 + 3.0409i

-3.0696 - 3.0409i

-0.5119 + 0.0000i

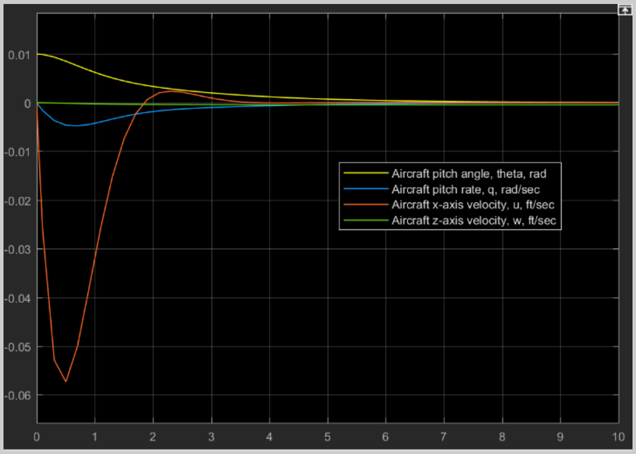
-0.0998 + 0.0000i

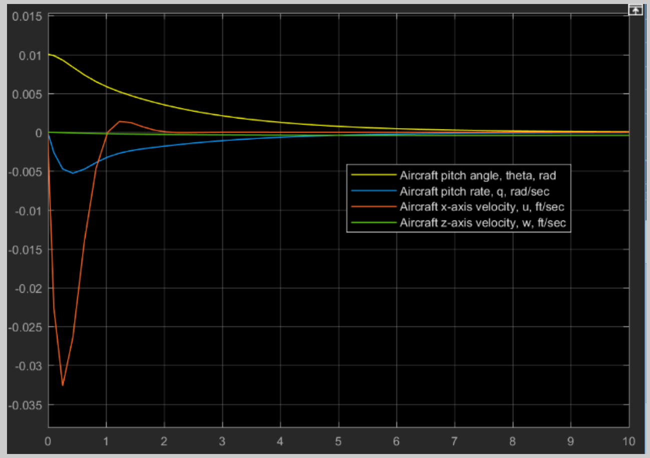
d) With Q for 1st and 3rd state and comparing it with part b

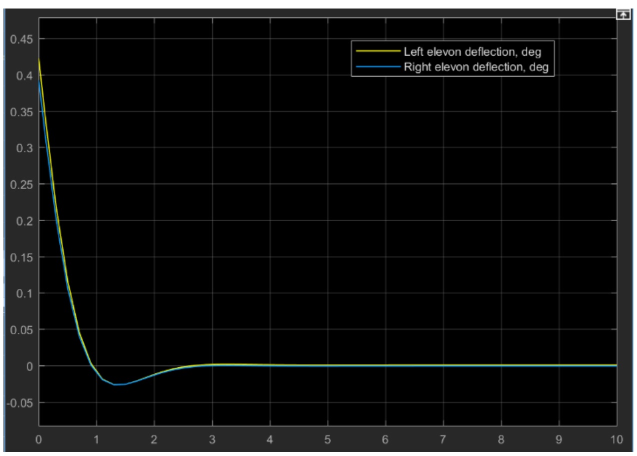
(b) Q = eye(4)	(d) Q = [1 0 0 0;0 0 0 0;0 0 1 0;0 0 0 0]
 *State time history (b)*	 *State time history (d)*
 *Control time history (b)*	 *Control time history (d)*
lambda_lqr = -0.9998 + 0.9340i -0.9998 - 0.9340i -0.5118 + 0.0000i -0.0157 + 0.0000i	lambda_lqr = -0.9997 + 0.9342i -0.9997 - 0.9342i -0.5118 + 0.0000i -0.0120 + 0.0000i

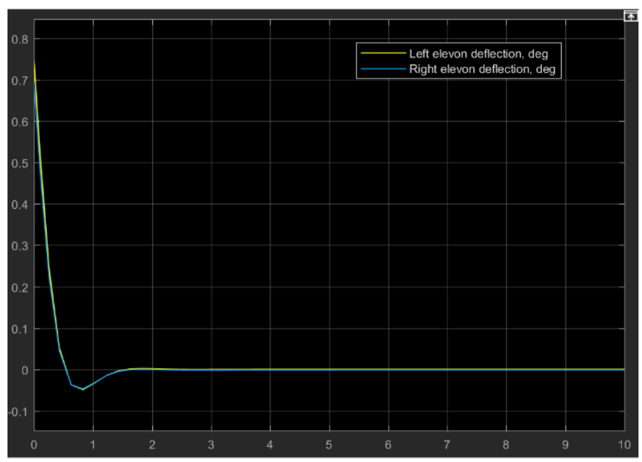
The state and control time histories for both the cases (b) & (d) are almost similar except the eigen values which exhibit very slight difference.

e) Part c with Q = [1 0 0 0;0 0 0 0;0 0 1 0;0 0 0 0]

<table>
<tr><th>Q = 10 *[1 0 0 0;0 0 0 0;0 0 1 0;0 0 0 0]</th><th>Q = 100 *[1 0 0 0;0 0 0 0;0 0 1 0;0 0 0 0]</th></tr>
<tr><td>


State time history
</td><td>


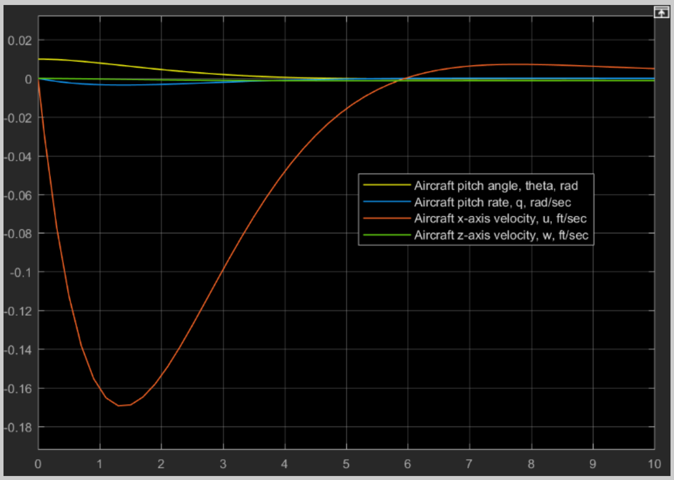
State time history
</td></tr>
<tr><td>


Control time history
</td><td>


Control time history
</td></tr>
<tr><td>
lambda_lqr =
-1.7367 + 1.6998i
-1.7367 - 1.6998i
-0.5119 + 0.0000i
-0.0120 + 0.0000i
</td><td>
lambda_lqr =
-3.0657 + 3.0449i
-3.0657 - 3.0449i
-0.5119 + 0.0000i
-0.0120 + 0.0000i
</td></tr>
</table>

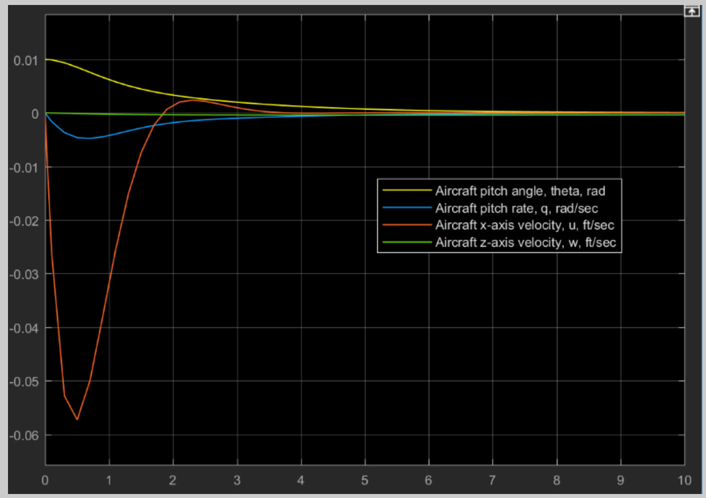
f) With R = 10*eye(2) and R = 0.1*eye(2)

<table>
<tr><th>Q = eye(4); R = 10*eye(2)</th><th>Q = eye(4); R = 0.1*eye(2)</th></tr>
<tr><td>



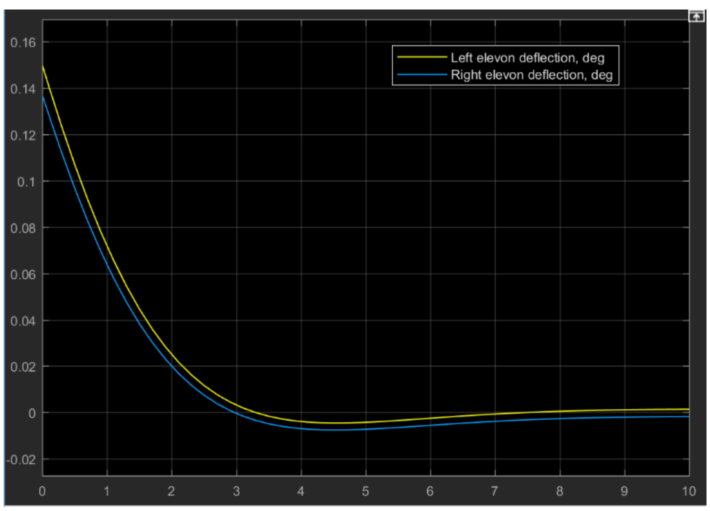
State time history

</td><td>



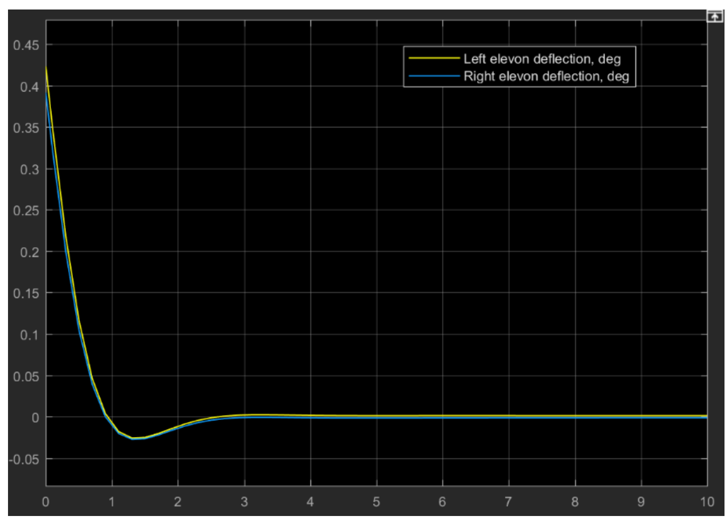
State time history

</td></tr>
<tr><td>



Control time history

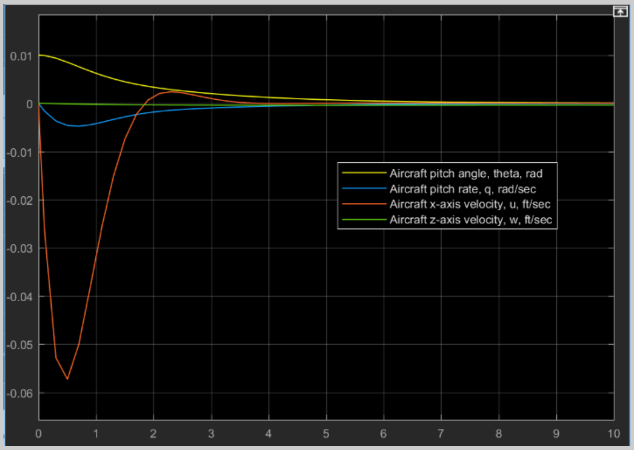
</td><td>

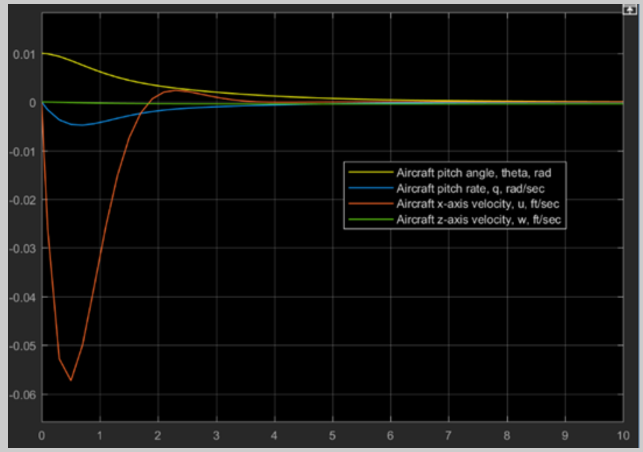


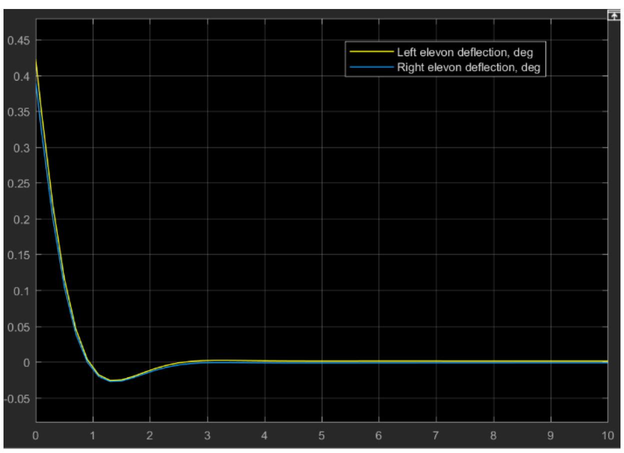
Control time history

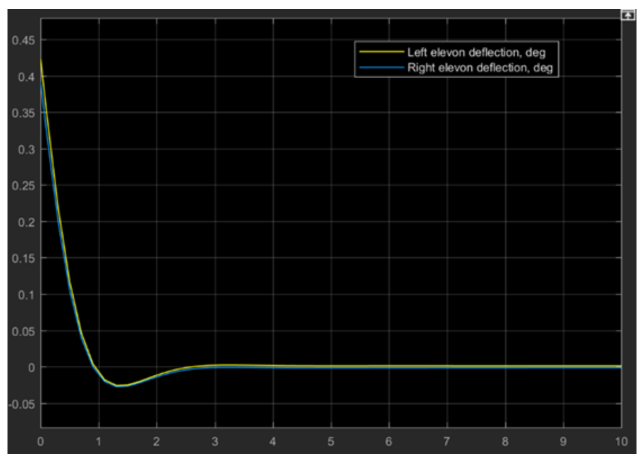
</td></tr>
</table>

Comparing state & control time histories for R=eye(2) & Q = 10*eye(4) and R=0.1*eye(2) & Q = eye(4).

<table>
<tr><th>Q = 10 * eye(4) & R = eye(2)</th><th>Q = eye(4) & R = 0.1*eye(2)</th></tr>
<tr><td>

State time history (Q = 10*eye(4) & R=eye(2))</td><td>

State time history (Q = eye(4) & R = 0.1*eye(2))</td></tr>
<tr><td>

Control time history (Q = 10*eye(4) & R = eye(2))</td><td>

Control time history (Q = eye(4) & R =0.1*eye(2))</td></tr>
<tr><td>lambda_lqr =
-1.7373 + 1.6991i
-1.7373 - 1.6991i
-0.5119 + 0.0000i
-0.0336 + 0.0000i</td><td>lambda_lqr =
-1.7373 + 1.6991i
-1.7373 - 1.6991i
-0.5119 + 0.0000i
-0.0336 + 0.0000i</td></tr>
</table>

Upon comparison, we can see that the state and control time histories are the same as well as the eigen values of the system are also same, hence these systems exhibit same behavior.