page 10:

How to prove Proposition 12? or where could I find the proof of this proposition? The following picture is proposition 12

Theorem 11. *The dyadic branching brownian motion with branching rate $\beta > 0$ is Fellerian and its generator is*

$$\mathcal{G}F(t,n,x) := \sum_{i=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_i^2} F(t,n,x) + \sum_{i=1}^{n} \beta[F(t,n+1,\hat{x}_i) - F(t,n,x)]$$

The following is classical:

Proposition 12. *If $F : [0,\infty) \times S \mapsto \mathbb{R}$ is $C^{1,2}$ in t and x respectively and*

$$\left(\mathcal{G} + \frac{\partial}{\partial t}\right) F \equiv 0 \tag{2.7}$$

then $(F(t, \#\mathbb{N}_t, X(t)), t \geq 0)$ is a local martingale.

The next Theorem is often called the McKean representation. It says that solutions of the FKPP equation can be viewed as an expectation with respect to the BBM. It is at heart a Feynman-Kac type of result.

page14(Theorem 16):

Should (3.3) be written as :$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} logt + C + o(1)$? The following picture is Theorem 16

Theorem 16 ([7] Bramson, 1983)**.** *For all initial condition $u(0,x) = g(x)$ increasing sufficiently fast[1] (including the Heaviside initial condition) then*

$$u(t, x + m_t) \to w(x) \quad \textit{uniformly in } x \in \mathbb{R} \textit{ as } t \to \infty \tag{3.3}$$

where $m_t = \sqrt{2}t\frac{3}{2\sqrt{2}} \log t + C + o(1)$

page18-19(Section 3.5):

This section talk about the basis of Bessel-3 process. And there is a measure named Q_x . According to the definition of Q_x, it must relate to both t and x, but it seems that Q_x is independent to t as the latter formula. The following picture is Section 3.5

Let us recall some very basic facts about the Bessel-3 process. If $W_t, t \geq 0$ is a Brownian motion in $\mathbb{R}^3$, then its modulus $|W_t|, t \geq 0$ is called a Bessel-3 process.

Suppose that $B_t, t \geq 0$ is a Brownian motion in $\mathbb{R}$ started from $B_0 = x$ under P_x: then

$$\zeta_t := \frac{B_t}{x} \mathbf{1}_{\{B_s > 0, \forall s \leq t\}}$$

is a non-negative, mean one martingale under P_x, so we can define a new probability measure Q_x by

$$\frac{\mathrm{d}Q_x}{\mathrm{d}P_x}\Big|_{\mathcal{G}_t} := \zeta_t$$

where $\mathcal{G}_t$ is the natural filtration of B_t. Then $B_t; t \geq 0$ is a Bessel process started from x under Q_x. The density of a Bessel process satisfies

$$Q_x(B_t \in \mathrm{d}z) = \frac{z}{x\sqrt{2\pi t}} \left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\right) \mathrm{d}z$$

and it can be checked that

$$Q_x(\cdot) = \lim_{t \to \infty} P_x(\cdot | \tau_0 \geq t) = P_x(\cdot | \tau_0 = \infty)$$

where $\tau_0 = \inf\{t : B_t = 0\}$ (the Bessel process is a Brownian motion conditioned to never hit 0).

page 20-21(section 3.7.1):

Firstly, (3.6) is quite weird since combining the first part of it, $X_u(s) \leq \beta s + 1 \forall s \leq t$ and the second part. $\beta t + y - 1 \leq X_u(t) \leq \beta t + y$, we will get $y \leq 2$. However, in lemma 27, we found $y \in [0, \sqrt{t}]$ and $t \to \infty$. The following picture is the beginning of Section 3.7.1

3.7.1 Bounds on the tail of $M(t)$

Let

$$\beta := \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} \tag{3.5}$$

and define

$$H(y,t) := \#\{u \in \mathcal{N}_t : X_u(s) \leq \beta s + 1 \,\forall s \leq t, \beta t + y - 1 \leq X_u(t) \leq \beta t + y\} \tag{3.6}$$

Secondly, in lemma 27, There is a mistake that the writer chose $\alpha = 1$. I think it should be $\alpha = y$, otherwise, the calculation of $E[H(y,t)]$ will contradict with section 3.6. I doubt that the second line of the calculation is wrong if we set $\alpha = 1$ because the underlined parts in the following picture should be same, Which means $\alpha = y$.

where $\alpha > 0$ s fixed. Observe that the summands are again just path functionals of the X_u. If we wanted a Brownian particle B to follow f we would apply the usual Girsanov martingale transform

$$g_t = e^{\int_0^t f'(s)\,\mathrm{d}B_s - \int_0^t (f'(s))^2\,\mathrm{d}s}$$

i.e. under $\hat{P}_f$ defined by $\mathrm{d}\hat{P}_f/\,\mathrm{d}P|_{\mathcal{G}_t} = g_t$ the process $\hat{B}_t := \underline{\alpha} + f(t) - B_t$ is a Brownian motion started from α. Let us now do a further change of measure and define a new probability Q by

$$\frac{\mathrm{d}Q}{d\hat{P}_f}\Big|_{\mathcal{F}_t} = \frac{\hat{B}_t}{\underline{\alpha}} \mathbf{1}_{\{\hat{B}_s > 0 \forall s \leq t\}}$$

then $\hat{B}$ is a Bessel-3 process under Q started from α.

Combining the many-to-one Lemma with these two changes of measure we get that for a path-functional F

Lemma 27 (First moment for H). *For $t \geq 1$ and $y \in [0, \sqrt{t}]$*

$$\mathbb{E}[H(y,t)] \asymp e^{-\sqrt{2}t}.$$

Proof. We apply our many to one Lemma for counting particles staying under a curve with $f(s) = \beta s$ and $\alpha = 1$ to obtain

$$\begin{aligned}
\mathbb{E}[H(y,t)] &= e^t \mathbb{Q}\Big[\frac{1}{\zeta(t)} \mathbf{1}_{\{\beta t + y - 1 \leq B_t \leq \beta t + y\}}\Big] \\
&= e^t \mathbb{Q}\Big[\frac{y e^{-\beta B_t + \beta^2 t/2}}{\beta t + y - \mathcal{B}_t} \mathbf{1}_{\{\beta t + y - 1 \leq B_t \leq \beta t + y\}}\Big] \\
&\asymp y e^{t - \beta^2 t/2} \mathbb{Q}[\beta t + y - 1 \leq B_t \leq \beta t + y] \\
&\asymp y t^{3/2} e^{-\sqrt{2} y}\ \mathbb{Q}[1 \leq \beta t + y + 1 - B_t \leq 2].
\end{aligned}$$

Since $\beta t + 1 - B_t$ is a Bessel under $\mathbb{Q}$ we have that

$$\mathbb{Q}(1 \leq \beta t + y + 1 - B_t \leq 2) \asymp \int_1^2 \frac{z^2}{t^{3/2}} \mathrm{d}z \asymp t^{-3/2}.$$

□

In conclusion, I think the whole proof in section 3.7.1 is contradictory and I don't know what materials I should refer to, which stops me from going further to read it.