

## Review

# Complexity testing techniques for time series data: A comprehensive literature review

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## ARTICLE INFO

## Article history:

Received 11 April 2015

Accepted 7 September 2015

Available online 30 September 2015

## Keywords:

Complexity test

Time series data

Literature review

Fractality

Chaos

Entropy

## ABSTRACT

Complexity may be one of the most important measurements for analysing time series data; it covers or is at least closely related to different data characteristics within nonlinear system theory. This paper provides a comprehensive literature review examining the complexity testing techniques for time series data. According to different features, the complexity measurements for time series data can be divided into three primary groups, i.e., fractality (mono- or multi-fractality) for self-similarity (or system memorability or long-term persistence), methods derived from nonlinear dynamics (via attractor invariants or diagram descriptions) for attractor properties in phase-space, and entropy (structural or dynamical entropy) for the disorder state of a nonlinear system. These estimations analyse time series dynamics from different perspectives but are closely related to or even dependent on each other at the same time. In particular, a weaker self-similarity, a more complex structure of attractor, and a higher-level disorder state of a system consistently indicate that the observed time series data are at a higher level of complexity. Accordingly, this paper presents a historical tour of the important measures and works for each group, as well as ground-breaking and recent applications and future research directions.

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## 1. Introduction

To investigate and understand time series data, data characteristic testing has created increasing interest in terms of both the theoretical and application perspectives. Typically, characteristics of time series data include nonstationarity (or stationarity) [1], nonlinearity (or linearity) [2], complexity [3], chaos [4], fractality [5], cyclicity [6], seasonality [7], saltation [8], stochasticity [9] and so on. Amongst them, complexity can be viewed as one of the most important measurements in time series data analysis, as it covers or is closely related to other various data characteristics within nonlinear system theory. On the one hand, complexity can provide another vivid description for diverse nonlinear

characteristics, e.g., chaoticity, fractality, irregularity, long-range memorability and so on [10,11]. In particular, various nonlinear characteristics attempt to describe the intrinsic structure of the target data dynamics, for which complexity can present a generic quantitative estimation. Generally, higher-level complexity indicates a more intricate, disordered, irregular dynamic system. On the other hand, the complexity characteristic also determines the features of various inner factors (e.g., central trend and cyclical, seasonal and mutable patterns) and their interrelationship [12]. Lower-level complexity indicates the larger impact or governing power of regular factors or inner rules (e.g., central trend and cyclical patterns) on the whole data dynamics [11,12].

Therefore, complexity has become an increasingly dominant estimator in analysing time series dynamics, especially for economic data to evaluate market efficiency [13] and for physical and biological data to capture the hidden rules

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[14,15]. Actually, complex systems are special cases of nonlinear systems, and most natural systems are complex systems [10]. In particular, the term ‘complex’ describes an intermediate state between a completely regular system and a completely random system [16]. Generally, a lower-level complexity indicates that the observed system is more likely to follow a deterministic process that can be finely captured and predicted, while a higher-level complexity represents less regular rules controlling the data dynamics, which might otherwise be more unpredictable and more difficult to understand [11].

According to the existing literature, numerous complexity testing techniques have been developed for time series data. Based on different features, they can be generally divided into three categories: fractality, methods derived from nonlinear dynamics and entropy [17]. Each category tends to describe the dynamic system of time series data from a different perspective. Although they have different focusses, these three groups are closely related to or even dependent on each other. In particular, a weaker self-similarity (or system memorability) measured via fractality, a more complex structure of attractor via methods derived from nonlinear dynamics, and a higher-level disorder state of the system via entropy consistently indicate that the observed time series data are at a higher-level of complexity.

Regarding the literature reviews on complexity testing techniques for time series data, there are some important studies. For example, Lopes and Betrouni [18] focussed on the fractality estimation for medical signal data and divided the existing tools into mono- and multi-fractal techniques. Sun et al. [19] provided a survey of the popular methods for estimating fractal dimension and their applications to remote sensing problems. Kantelhardt [20] gave an introduction to mono- and multi-fractality analysis methods for stationary and non-stationary time series data. Zanin et al. [21] analysed the theoretical foundations of PE, as well as recent applications to economic markets and biomedical systems. Alcaraz and Rieta [22] provided a review of sample entropy (SampEn) in the context of non-invasive analysis of atrial fibrillation. Marwan et al. [23] presented a comprehensive review of recurrence-based methods and their applications.

However, the existing literature reviews of complexity testing techniques for time series data were insufficient, and improvements can be made from two main perspectives. First, most literature reviews only concentrate on one group of complexity testing techniques while ignoring some other important types. For example, in Refs. [18–20], only the methods for fractality exploration were considered; Zanin et al. [21] only concentrated on PE, and Alcaraz et al. [22] only considered SampEn; Marwan et al. [23] only focussed on recurrence-based methods. However, according to existing methods, three main groups are included in complexity analysis, i.e., fractality, methods derived from nonlinear dynamics and entropy, which describe data dynamics from different perspectives. That is, some important testing techniques were ignored in the existing literature reviews. Second, some studies presented a survey only within a certain application field. For example, Lopes and Betrouni [18] focussed on medical signal data, Sun et al. [19] focussed on remote sensing problems, and Zanin et al. [21] focussed on economic markets and biomedical systems. However, because most natural

**Table 1**

The abbreviations for different complexity testing techniques for time series data.

Abbreviation	Full meaning
ApEn	Approximate entropy
BMA	Backward moving average
CMA	Centred moving average
DFA	Detrended fluctuation analysis
DMA	Detrended moving average
FuzzyEn	Fuzzy entropy
MF-DFA	Multifractal detrended fluctuation analysis
MF-DMA	Multifractal detrended moving average analysis
PE	Permutation entropy
R/S	Rescaled range
RP	Recurrence plot
RQA	Recurrence qualification analysis
SampEn	Sample entropy
SE	Spectral entropy
SPMF	Standard partition function multi-fractal formalism
WE	Wavelet entropy
WTMM	Wavelet transform modulus maxima

systems belong to complex systems, complexity testing has been widely applied to various research fields for different data. To the best of our knowledge, there are few studies on literature reviews of complexity testing techniques that address not only different types of methods but also various application fields. Given such a background, this paper tries to fill in this gap and provide a comprehensive literature review on complexity testing techniques for time series data in terms of measures and important works in each group of fractality, methods derived from nonlinear dynamics and entropy explorations, as well as respective ground-breaking and recent applications and future research directions.

The main aim of this paper is to provide a comprehensive literature review of the complexity testing techniques for time series data. The remaining part of the paper is organised as follows. The analytical framework for complexity testing techniques is given in Section 2. Sections 3–5 present a historical overview of the measures and important works in fractality, methods derived from nonlinear dynamics and entropy, together with respective ground-breaking and recent applications and future research directions. Section 6 summarises the main ideas of the three groups of complexity testing. Finally, Section 7 presents the paper’s conclusions.

## 2. Analytical framework for literature review

According to the literature, numerous techniques have been developed and used to estimate the complexity of time series data. Based on different features, these methods can be divided into three main categories, i.e., fractality, methods derived from nonlinear dynamics and entropy. Accordingly, the abbreviations for different measures are listed in Table 1, and the analytical framework of the literature review is illustrated in Fig. 1.

Fractality theory mainly focusses on self-similarity (or system memorability or long-term persistence) by analysing the metric scaling behaviour of a data system, which mainly includes the techniques of R/S analysis [2], DFA [24–28], DMA [29], SPMF [30], WTMM [31], MF-DFA [32] and MF-DMA [33].

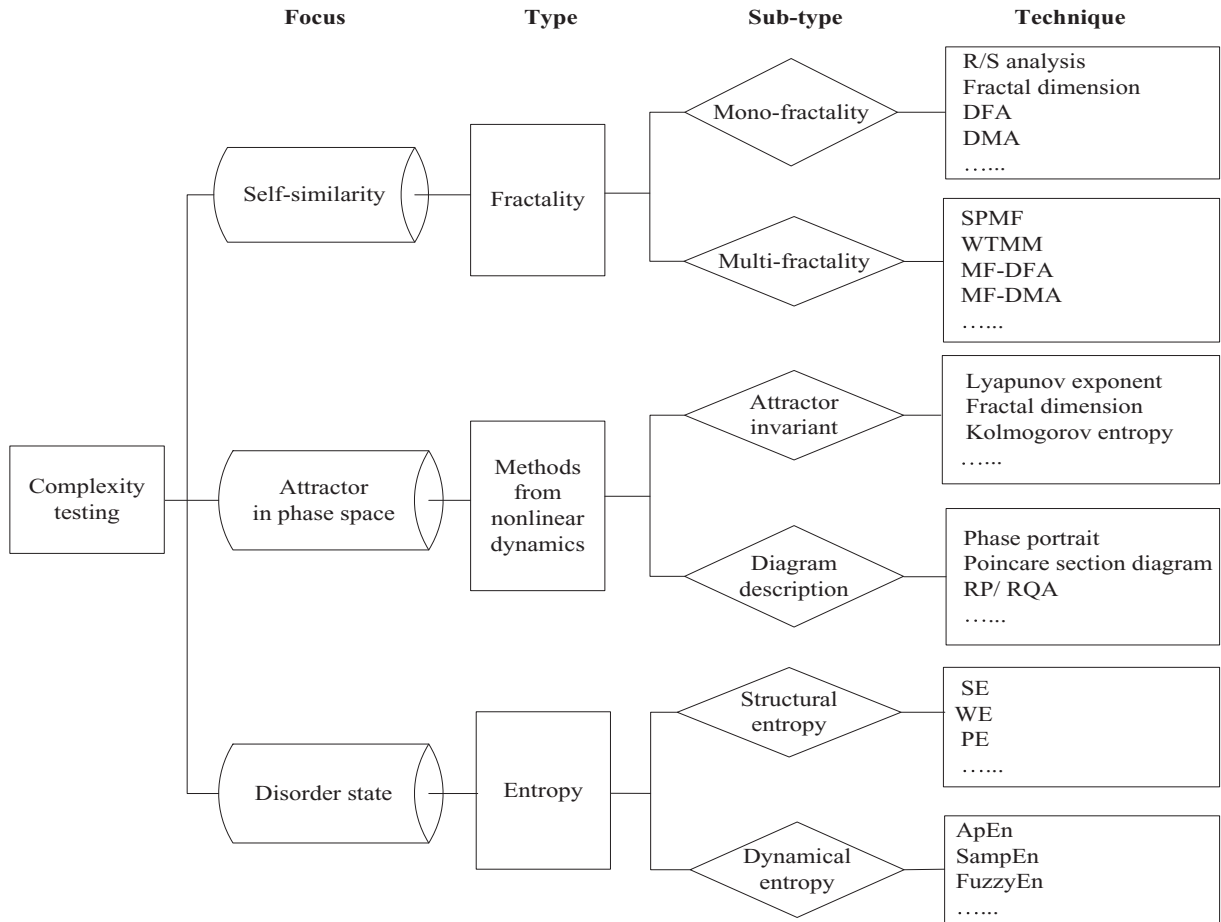


Fig. 1. Analytical framework for complexity testing techniques for time series data.

Methods derived from nonlinear dynamics explore data dynamics by investigating the strange attractor in phase-space [34] based on attractor invariants and some useful diagrams. Popular techniques include the Lyapunov exponent [35], fractal dimension [36], Kolmogorov entropy [37], phase portrait [38], Poincare section diagram [39] and RP [40].

Entropy, a thermodynamic quantity, describes the disorder state of the data dynamics [41], which mainly includes SE [42], WE [43], PE [44], ApEn [45], SampEn [46], and FuzzyEn [47].

Although they are from different perspectives, these complexity testing techniques of the three groups are closely related to or even dependent on each other. In particular, a weaker self-similarity (or system memorability or long-term persistence), a more complex structure of strange attractor, and a higher-level disorder state of the system are all related to a higher-level complexity characteristic of the observed time series data. Furthermore, the testing approaches for different types of fractality, methods derived from nonlinear dynamics and entropy are sometimes similar or even mixed with each other. For example, fractal dimension and Kolmogorov entropy can also be introduced into the methods derived from nonlinear dynamics as important attractor invariants to describe attractor properties.

Accordingly, Sections 3–5 present important measures and works in each group, as well as ground-breaking and recent applications and future research directions, respectively.

### 3. Fractality testing

#### 3.1. Measures

The term 'fractal' comes from the Latin adjective 'fractus', which means 'irregular or fragmented' [48]. Mathematically, a fractal series can be defined as a set in which the Hausdorff dimension strictly exceeds the topological dimension [49]. Actually, the fractal phenomenon can be widely found in reality, and popular examples include measuring coastline, island chains, patches of vegetables and coral reefs [50]. Accordingly, the fractality characteristic can be used in complexity testing to discover the metric scaling behaviours of the target dynamics, e.g., the metrics of length and area [19]. Furthermore, fractality can also describe the relationship between the partial components and the whole system, i.e., self-similarity (or system memorability or long-term persistence) [51]. For time series data, fractality measures the system memorability or long-term persistence in terms of scaling behaviours [52]. The existing fractality measurements can be generally divided into mono- and

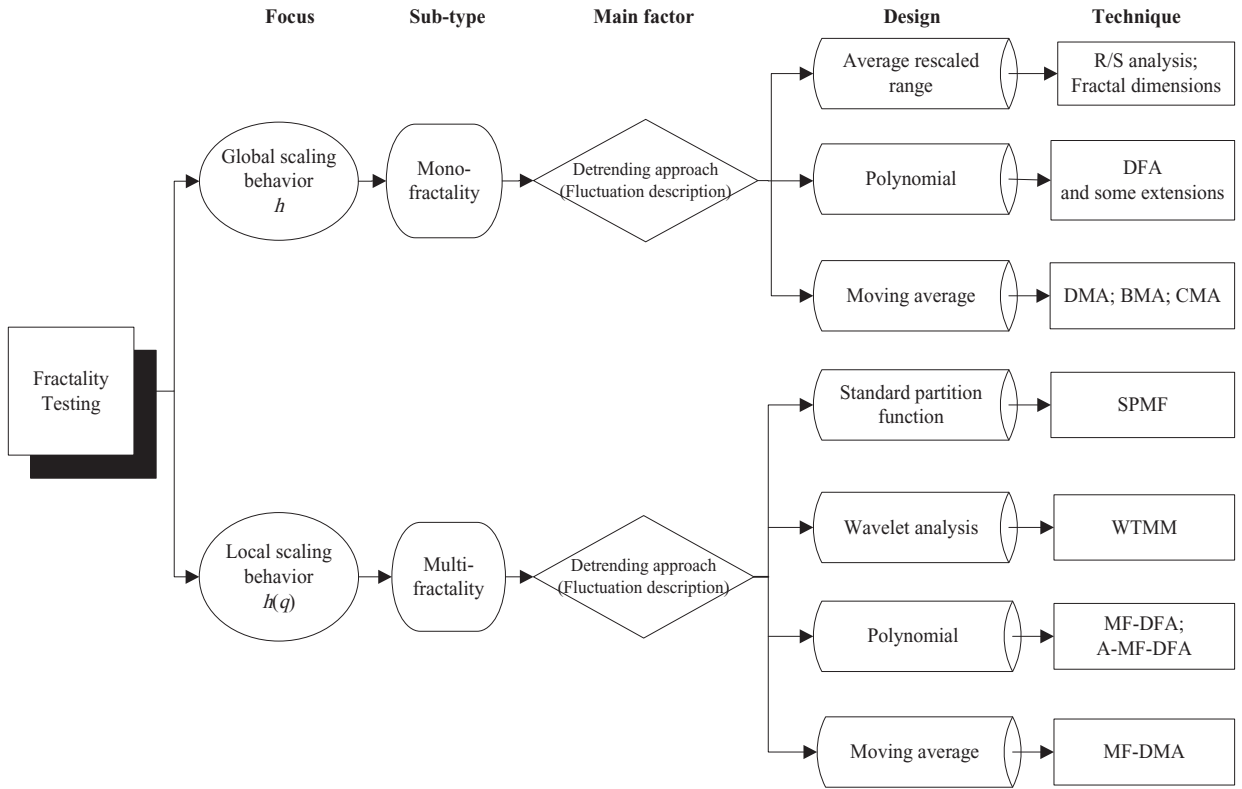


Fig. 2. The measures of fractality testing techniques for time series data.

multi-fractality analyses [18], as the corresponding measures listed in Fig. 2.

### 3.1.1. Mono-fractality analysis

Since the seminal work of the Hurst exponent proposed by Hurst [2], various mono-fractal exponents have been developed and applied to the complexity measurements for time series data. Mono-fractality analysis describes the global structure of time series dynamics in terms of the scaling exponents representing long-term persistence, i.e., self-similarity or system memorability. In particular, a scaling exponent far from the disordered level of 0.5 represents a strong system memorability or self-similarity, i.e., the observed data are at a low level of complexity. In fractality analysis, two main factors are included, i.e., the detrending approach (to describe the fluctuation in time series data) and scaling exponent (to estimate the scaling behaviour by fitting the power-law relationship between the fluctuation function and time scales). Accordingly, by using different detrending approaches (or fluctuation descriptions), various mono-fractality testing methods can be formulated, amongst which R/S analysis, DFA and DMA may be the most conventional and popular methods.

R/S analysis can be observed as the seminal work of mono-fractality analysis. In R/S analysis, the original time series  $x(i)$  ( $i = 1, 2, \dots, N$ ) is first split into  $N_s = [N/s]$  non-overlapping segments with size  $s$ ; then, the profile (integrated data)  $Y_v(j)$  ( $j = 1, 2, \dots, s$ ) in segment  $v$  is obtained by subtracting the local average:

$$\begin{aligned}
 Y_v(j) &= \sum_{i=1}^j (x(vs+i) - \langle x(vs+i) \rangle_s) \\
 &= \sum_{i=1}^j x(vs+i) - \frac{j}{s} \sum_{i=1}^s x(vs+i).
 \end{aligned} \quad (1)$$

The fluctuation in time series data can be calculated via the average rescaled range over all segments:

$$F(s) = \frac{1}{N_s} \sum_{v=0}^{N_s-1} \frac{R_v(s)}{S_v(s)}, \quad (2)$$

where  $R_v(s)$  is the difference between the minimum and maximum of  $Y_v(j)$  in each segment, and  $S_v(s)$  is the standard deviation.

To measure the scaling exponent, repeat the above steps with different sizes of  $s$  to fit the power-law relationship between the fluctuation function and time scale, i.e.,  $F(s) \sim s^H$ . In practice, the Hurst exponent  $H$  can be obtained by fitting the slope of the log-log plot of  $F(s)$  versus  $s$  based on least squares (LS) estimation. While the scaling exponent is less than 0.5, i.e.,  $0 < H < 0.5$ , the time series might hold a long-term anti-persistence behaviour; while  $0.5 < H < 1$ , a long-term persistence behaviour can be investigated; and if  $H = 0.5$ , the series displays a random walk behaviour with no long-term persistence or self-similarity.

For a self-similar process, the fractal dimension is tightly connected to the Hurst exponent, i.e.,  $D = 2 - H$ . Under such a theory, various fractal dimensions have been developed to estimate system complexity in terms of the roughness of

time series [53]. Although specific parts of time series data are differently rough (or smooth), they may be otherwise correlated on a global level, and such correlation might sometimes vanish soon and be somewhat unobservable. Based on phase-space construction, fractal dimensions measure such roughness in terms of the number of active degrees of freedom of the system, including the correlation dimension [37], box-counting dimension [54], information dimension [36], point wise dimension [36], etc.

By utilising other detrending approaches to capture the fluctuation in time series data, various mono-fractality testing methods have been developed. An attractive work is the DFA method [24] using a polynomial function to fit the central trend and extract the fluctuation in time series data. In particular, the polynomial with different orders can effectively eliminate linear, quadratic or higher-order trends of the profile. Due to the obvious drawback of the DFA method, i.e., potential occurrence of abrupt jumps in the profile at the boundaries between segments, other methods have been proposed. For example, the DMA technique was proposed by Alessio et al. [29], which utilises the moving average as the detrending approach rather than the polynomial function. Alvarez-Ramirez et al. [55] improved upon the detrending approach of the BMA method [56] by transforming it into the CMA method for fractality analysis.

Recently, many modified DFA methods have been developed, e.g., the modified DFA (MDFA) [26], a mixture model of original fluctuation analysis and DFA, and the asymmetric DFA (A-DFA) method [25], which considers the asymmetries in the scaling behaviours of time series. Moreover, various effective data analysis techniques have been introduced to preprocess time series to formulate diverse DFA extensions. For example, based on the Fourier transform for eliminating slow oscillatory trends in time series data, the Fourier DFA (FDFA) was proposed by Chianca et al. [28]; using empirical mode decomposition (EMD) for eliminating quasi-periodic or irregular oscillating trends, János and Müller developed the EMD-based DFA [57]; using the singular value decomposition (SVD) for periodic and quasi-periodic trends, Nagarajan and Kavasseri formulated the SVD-based DFA [58]; with a digital high-pass filter, a novel DFA version was presented by Rodriguez et al. [59]; and by introducing unnormalised fluctuation functions for increasing lengths of data, Staudacher et al. built the continuous DFA (CDFA) method [27].

Furthermore, the dynamic fractality changing over time may be another hot issue in the recent study of time series analysis. For example, Tabak and Cajueiro [60] estimated the time-varying degrees of long-range dependence based on R/S analysis by using a subsample rolling window. Similarly, Alvarez-Ramirez et al. [61] proposed the moving window based DFA technique to analyse time-varying autocorrelations in a short receding horizon. Wang and Liu [62] similarly extended the DFA method into a dynamic DFA method with rolling windows.

### 3.1.2. Multi-fractality analysis

Unlike mono-fractality analysis focussing on the global structure, multi-fractality explores the data system from the perspective of local structures. In multi-fractality analysis, an order  $q$  is introduced into the fluctuation function to allow modelling of local scaling behaviour, which can provide a

much more comprehensive and specific analysis. Similarly, the generalised scaling exponent far from the disordered level 0.5 represents a strong system memorability, i.e., the observed data are at a low level of complexity with a high long-term persistence or self-similarity. Similar to mono-fractality analysis, two main factors are included in multi-fractality analysis, i.e., the detrending approach (or fluctuation description) and scaling exponent. Based on different detrending approaches to describe fluctuations, various multi-fractality testing methods can be proposed. Some popular methods are SPMF, WTMM, MF-DFA and MF-DMA.

The most basic model of multi-fractality analysis might be the SPMF. Similar to mono-fractality analysis, the first step in SPMF is to divide the time series into  $N_s = [N/s]$  boxes with an equal size  $s$ ; then, the sum of each segment is calculated by:

$$u(n, s) = \sum_{t=1}^s x[(n-1)s + t], \quad n = 1, 2, \dots, N_s. \quad (3)$$

The fluctuation function of time series data is expressed by the standard partition function  $\chi_q$ :

$$F_q(s) = \chi_q(s) = [\mu(n)]^q, \quad (4)$$

where

$$\mu(n) = \frac{u(n, s)}{\sum_{m=1}^{N_s} u(m, s)}, \quad (5)$$

where  $q$  is the order of fluctuation function set to a series of positive and negative values to capture local scaling behaviours. When considering positive values of  $q$ , the segments with large variances will dominate the whole fluctuation function  $F_q(s)$ , and the scaling exponent  $h(q)$  describes the local scaling behaviours of the segments with large fluctuations. On the contrary, for negative values of  $q$ , the local scaling behaviours of the segments with small fluctuations can be captured.

Unfortunately, the detrending approach of the standard partition function in SPMF might be inefficient for non-stationary time series that are affected by trends or that cannot be normalised. By introducing some other effective detrending approaches, a series of multi-fractality analysis methods have been formulated. For example, Muzy et al. [31] introduced the wavelet transformation as the detrending approach and proposed WTMM, in which the maxima lines in the continuous wavelet transform over all scales are investigated as fluctuations. Based on a generalisation of DFA considering fluctuations with different levels, Kantelhardt et al. [32] advanced the MF-DFA method for non-stationary time series, which does not require the modulus maxima procedure. Furthermore, inspired by A-DFA, the asymmetric MF-DFA (A-MF-DFA) was proposed by Cao et al. [63] to examine the asymmetric multi-fractal scaling behaviours of time series data with uptrends and downtrends. Similarly, the MF-DMA method [33] was extended from DMA, removing the local trends by subtracting the local means.

As for the scaling exponent, the other important factor in multi-fractality analysis, the power-law relationship between the fluctuation function and time scale  $F_q(s) \sim s^{h(q)}$  is first fitted to calculate the local scaling exponent  $h(q)$  for



each value  $q$ . Second, the Renyi scaling exponent  $\tau(q) = q \cdot h(q) - 1$  [33] and singularity spectrum  $f(\alpha)$  transformed from  $\tau(q)$  through a Legendre function [64] can be computed by:

$$\begin{cases} f(\alpha) = q\alpha - \tau(q) \\ \alpha = d\tau(q)/dq \end{cases} \quad (6)$$

where  $\alpha$  is the singularity or Hölder exponent, and  $f(\alpha)$  denotes the dimension of the subseries that is characterised by  $\alpha$ .

In particular, the time series data presents a multi-fractality characteristic when the Renyi scaling exponent  $\tau(q)$  and value  $q$  follow a nonlinear relationship or the generalised Hurst exponent  $h(q)$  is dependent on  $q$  [20]. Furthermore, the multi-fractality characteristic can also be estimated in terms of  $\Delta h(q)$  or  $\Delta\alpha$ :

$$\begin{aligned} \Delta h(q) &= \max[h(q)] - \min[h(q)] \\ \Delta\alpha &= \max(\alpha) - \min(\alpha). \end{aligned} \quad (7)$$

Generally, the larger the difference  $\Delta h(q)$  or  $\Delta\alpha$  that is measured, the higher the degree of multi-fractality the time series data might exhibit.

Recently, multi-scale multi-fractality analysis on different scale ranges has been an interesting direction to improve multi-fractality testing techniques. For example, Wang and Liu [62] analysed the fractal behaviours of large fluctuations and small fluctuations on different scales, i.e., short-, medium-, and long-term multi-fractality.

### 3.2. Applications

According to the existing literature, both mono- and multi-fractality analyses have been widely used for investigating various time series data, and the corresponding applications mainly addressed the research fields of economics [13,65–69], life science [71–73], earth science [2,74–77], engineering [78,79] and so on. In the following paragraphs, we show in detail some exemplary applications of fractality analysis in each field, and some typical works are listed in Table 2.

Fractality testing techniques have been widely used for economic data analysis, especially to evaluate market efficiency. For example, Yuan et al. [65] tested the multi-fractality of the Shanghai stock market in China by the MF-DFA method; it was found that there were two main factors leading to the multi-fractality feature, i.e., fat-tailed probability distributions and non-linear temporal correlations. Wang et al. [66] investigated the multi-fractal behaviour of the US dollar (USD) exchange rate by MF-DMA, and found that the major factors for multi-fractality were the long-range correlations of small and large fluctuations. Alvarez-Ramirez et al. [67] investigated the fractal behaviours of the West Texas Intermediate (WTI), Brent and Dubai crude oil markets by R/S analysis and height-height correlation analysis, and the results demonstrated that these crude oil markets were consistent with the random-walk assumption only for the short-term time-scale (within weeks). Alvarez-Ramirez et al. [13] implemented the lagged DFA to analyse the WTI market, and the results indicated that the crude oil market obviously deviated from market efficiency. Gu et al. [68] discovered the multi-fractality property in the WTI and Brent markets by employing MF-DFA and showed that the two markets became increasingly efficient in the long term. When comparing the MF-DFA and WTMM methods, Oświęcimka et al. [69] preferred the MF-DFA with a much higher accuracy level when having no prior knowledge about the fractal properties of stock market data.

In the research field of life science, one ground-breaking work is Peng et al. [24], in which the DFA method was introduced to analyse deoxyribose nucleic acid (DNA) data. Recently, Galaska et al. [70] compared the MF-DFA with the WTMM for differentiating patients with left ventricular systolic dysfunction based on heart rate data and argued that the former was more sensitive. Humeau et al. [71] studied the peripheral cardiovascular system (CVS) through the SPMF test on laser Doppler flowmetry (LDF) and heart rate variability (HVR) fluctuations, and the results showed that the multi-fractality property in the two fluctuation series appeared quite similar. Dick [72] investigated electroencephalography (EEG) data based on WTMM and demonstrated that the

**Table 2**  
Typical works on the application of fractality testing techniques to time series data.

Field	Authors	Data	Technique
Economics	Yuan et al. [65]	Shanghai stock price index return	MF-DFA
	Wang et al. [66]	US dollar exchange rate	MF-DMA
	Alvarez-Ramirez et al. [67]	WTI, Brent & Dubai spot prices	R/S
	Alvarez-Ramirez et al. [13]	WTI spot closing price	DFA
	Gu et al. [68]	WTI & Brent futures prices	MF-DFA
	Oświęcimka et al. [69]	Stock price	MF-DFA; WTMM
Life science	Peng et al. [24]	DNA data	DFA
	Galaska et al. [70]	Heart rate data	MF-DFA; WTMM
	Humeau et al. [71]	LDF & HVD fluctuations	SPMF; WTMM
	Dick [72]	EEG	WTMM
	Mercy Cleetus and Singh [73]	ECG	MF-DFA
Earth science	Hurst [2]	Nile river water flow	R/S
	Telesca et al. [74]	Geoelectrical data	MF-DFA
	Movahed and Hermanis [75]	River flow fluctuations	MF-DFA; FDFA
	Hu et al. [76]	Sunspot time series	MF-DFA
	Deng et al. [77]	Gold concentration distributions	SPMF
Engineering	Shang et al. [78]	Traffic speed fluctuations	MF-DFA
	Hu et al. [79]	Machine vibration signals	SPMF

changes in multi-fractality can effectively estimate the psycho relaxation efficiency for both healthy and pathological human brains. Mercy Cleetus and Singh [73] implemented MF-DFA to test the fractality feature in an electrocardiogram (ECG) to measure the adaptability of physiological processes and to further identify the pathological conditions.

As for earth science, Hurst [2] first introduced R/S analysis to investigate the fractality property of the Nile river water flow in terms of the relationship between the trend and noise, and Telesca et al. [74] first applied the multi-fractality analysis of MF-DFA to geoelectrical data analysis. Recently, Movahed and Hermanis [75] analysed the multi-fractality of river flow fluctuations by the MF-DFA and FDFA and observed an obvious multi-fractal feature. Hu et al. [76] implemented the MF-DFA method to analyse sunspot time series and found that sunspot activity was driven by multi-fractal scaling laws. Deng et al. [77] analysed gold concentration distributions by SPMF, and the results suggested that gold enrichment processes in both mineralised and barely mineralised areas followed similar processes despite their different mineral intensities.

Regarding engineering and other data, Shang et al. [78] used the MF-DFA to study traffic speed fluctuations and found that long-term correlation was the dominant factor leading to multi-fractal behaviours. Hu et al. [79] proposed an incipient mechanical fault detection method combining the SPMF and Mahalanobis-Taguchi system (MTS), and the experimental results proved that the SPME-based method could improve the accuracy of fault state identification.

According to the abovementioned studies, fractality has been widely utilised as one helpful measure for system memorability or long-term persistence of various time series data. When comparing mono-fractality with multi-fractality, the mono-fractality measurements are relatively simple, investigating the data system from a whole perspective, while multi-fractality methods can provide much more information by dividing complex systems into sub-systems [18]. Therefore, most existing studies, especially those involving economic data analysis, preferred multi-fractality for a full description of the market volatilities [65, 66]. Furthermore, when comparing the most popular multi-fractality measures of the MF-DFA and the WTMM, the MF-DFA was strongly suggested in the existing studies especially when there is no a prior knowledge about the fractal properties [69].

### 3.3. Directions for future research

Based on the above analysis, there is still ample opportunity to improve the models and applications for fractality testing techniques. As for model improvement, because the detrending approach (or fluctuation description) is a key factor in both mono- and multi-fractality analyses, some other effective regression tools (e.g., various powerful artificial intelligence (AI) algorithms) or signal processing techniques (e.g., ensemble EMD (EEMD) and compressed sensing) can be utilised to capture the central trend in time series data and to effectively extract the fluctuations. Another interesting direction is to extend the existing fractality analysis techniques from univariate time series into a bivariate or multivariate time series framework for investigating the inter-relations across series. For example, based on exist-

ing detrended cross-correlation analysis (DCCA) [80], the multi-fractal DCCA was proposed by Zhou [81] to explore multi-fractal cross-correlation.

Regarding the applications, economic data analysis may be one of the hottest issues in fractality analysis, in which the price (or return) series are the main focus, while trading volumes are neglected. However, the fractality investigation of both price and volume series can provide much more interesting insights into market analysis. Furthermore, extreme events or emergencies might cause structural changes to the multi-fractality property, and accurate estimation for such an impact might also provide a new perspective for both data analysis and emergency management. In addition, it is still possible to extend the applications of fractality analysis to other interesting research fields, e.g., chemistry, computer and IT network, ecology and so on.

## 4. Methods derived from nonlinear dynamics

### 4.1. Measures

Methods derived from nonlinear dynamics investigate time series data mainly based on the chaos theory. The term 'chaos' describes an apparently unpredictable behaviour driven by various interactive inner factors that actually obey certain inner rules but are supersensitive to the initial conditions [17]. A famous case is the 'butterfly effect': A weak wing beat by a butterfly can trigger a storm thousands of miles away [82]. Because of these properties, chaotic behaviour can be predictable in the short term but unpredictable in the long term. Li and Yorke [4] first provided the mathematical definition for chaos.

For time series data, the chaos property is tested based on the phase-space reconstruction by investigating the attractor in phase space [83]. In a chaotic process driven by the inner force, the data system will evolve towards a particular state or behaviour. Based on the phase-space reconstruction, the states of a system can be represented by points in an  $m$ -dimensional space, and the dynamic evolution of the system refers to a series of consecutive states (or vectors). For a sufficient time, the states of the data system will finally converge towards a subspace (or a particular set of points); and such a subspace can be defined as the attractor of a system because it 'attracts' trajectories (i.e., the line connecting the consecutive states in phase space) from all possible initial conditions [84]. In addition, methods derived from nonlinear dynamics estimate the complexity state of data systems, especially through exploring the structure of the attractor. In particular, the attractor may be a fixed point in a deterministic non-chaotic system, a limit cycle in a periodic system, or a limit torus in a quasi-periodic system. In contrast, in a random system, no attractor can be found [85]. The existing methods derived from nonlinear dynamics investigate the attractor properties based on two main helpful tools: attractor invariants and diagram descriptions, as described in Fig. 3.

#### 4.1.1. Attractor invariants

To quantify the complexity of time series data, methods derived from nonlinear dynamics tend to analyse the dynamic system states based on the phase-space

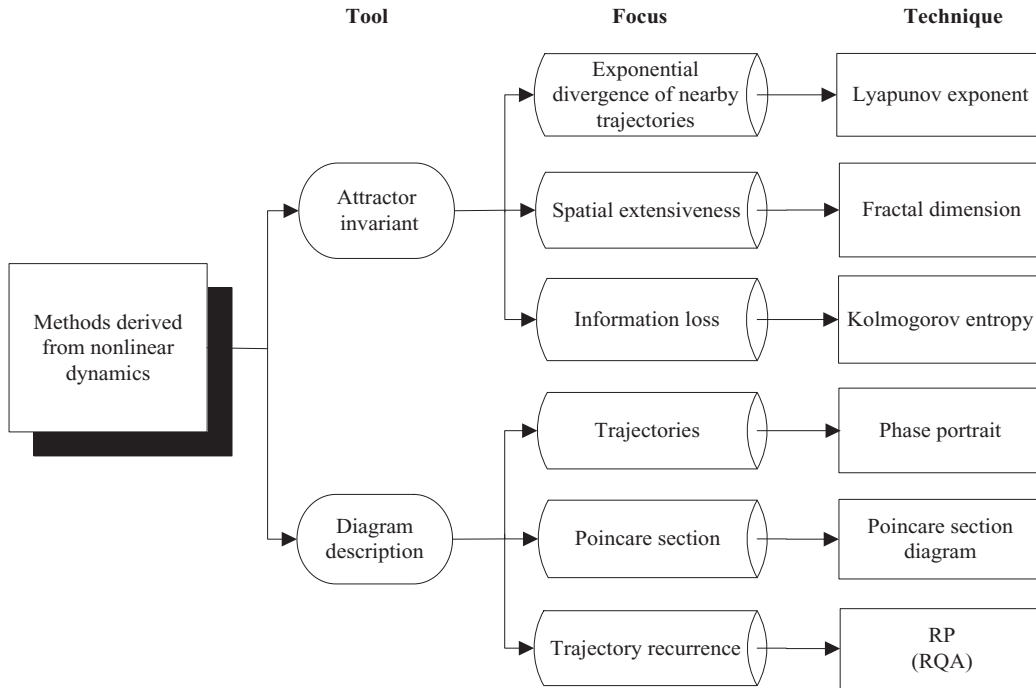


Fig. 3. The measures derived from nonlinear dynamics for time series data.

reconstruction theory proposed by Packard et al. [86] and proved by Takens [87]. Then, three main attractor invariants can be introduced to analyse the attractor properties, i.e., the Lyapunov exponent, fractal dimension, and Kolmogorov entropy. A smaller attractor invariant refers to a simpler structure of the attractor (or final subspace or points set) in phase space, i.e., the data system can be tested at a lower level of complexity. Accordingly, two main steps are involved in the methods derived from nonlinear dynamics, i.e., the phase-space reconstruction and attractor properties exploration.

According to Takens [87], an  $m$  embedding dimension phase space can be constructed from the original time series  $x(i)$  ( $i = 1, 2, \dots, N$ ) with a time lag  $\tau$ :

$$X(j) = \{x(j), x(j + \tau), \dots, x(j + (m - 1)\tau)\}, \quad (8)$$

where  $X(j)$  ( $j = 1, 2, \dots, N - (m - 1)\tau$ ) are the vectors in phase space representing system states. As for attractor properties exploration, three main types of attractor invariants can be introduced to identify the system states from different perspectives. First, the Lyapunov exponent investigates how the system states change over time in terms of the exponential divergence (or convergence) of initially nearby trajectories, and the growing rate of the separation between nearby trajectories reflects the sensitivity of the system to initial conditions [88]. In particular, a positive Lyapunov exponent reveals the exponential divergence of the nearby trajectories due to sensitive dependence on initial conditions; the negative one is associated with exponential convergence of the trajectories, and a zero value indicates a balance between expansive and contractive dynamics. Accordingly, although there exist  $m$  Lyapunov exponents in a given  $m$ -dimensional phase space, we can only focus on the largest one to investigate the variations in the system states.

Regarding the measure development of the Lyapunov exponent, Pesin [89] first provided the theoretic definition of the Lyapunov exponent, and Wolf et al. [35] offered the mathematical form of the Lyapunov exponent for exploring the chaotic property. In Lyapunov exponent estimation, for each trajectory  $X(j)$ , the nearest neighbour  $X(\hat{j})$  (within a given threshold distance  $e$ ) in phase space is first found at initial time  $t = 0$ . The Lyapunov exponent can then be defined by the average growth rate  $\lambda_j$  of the initial distance, i.e.,  $\lambda_j = \lim_{t \rightarrow \infty} \{\log[d_j(t)/d_j(0)]/t\}$ , where  $d_j(t)$  is the distance between trajectories  $X(j)$  and  $X(\hat{j})$  at time  $t$ . Since the seminal work of the Lyapunov exponent proposed by Wolf et al. [35], various modifications have been extended. For example, to reduce computational complexity, a much simpler and faster version was developed by Rosenstein et al. [90] to effectively ensure reliable values even for small-size datasets. McCaffrey et al. [91] introduced non-parametric regression to improve the estimation for the Lyapunov exponent. Kowalik and Elbert [92] proposed a modification in which the largest exponent is computed in a time-dependent way.

Second, various fractal dimensions in fractal theory can also be used to estimate the spatial extensiveness of phase space in terms of dimensional complexity. As for dimension, Hausdorff [93] first provided a rigorous theoretic definition, i.e., the Hausdorff dimension. The dimension of a geometric object is a measure of its spatial extensiveness, and the dimension of the attractor in phase space can reflect the degrees of freedom (or the complexity) of the system. In particular, a higher dimension of attractor represents more spatial extensiveness in a system, i.e., a higher level of complexity of the target time series data. By introducing such a concept of dimensional complexity, various fractal dimensions have been introduced to estimate the system structure in phase



space, e.g., the correlation dimension [37], box-counting dimension [54], information dimension [36], point wise dimension [36] and Rényi dimensions (i.e., generalised dimensions) [94]. For example, the most popular fractal dimension, i.e., the correlation dimension, can be defined based on the correlation integral. Given a threshold distance  $r$ , the correlation integral can be defined as:

$$C(m, r) = \frac{2 \sum_{i < j} H(r - \|X(i) - X(j)\|)}{[N - (m - 1)\tau][N - (m - 1)\tau - 1]}, \quad (9)$$

where  $\|X(i) - X(j)\|$  is the distance between the two vectors, and  $H(z)$  is the Heaviside function:

$$H(z) = \begin{cases} 0, & z \leq 0 \\ 1, & z > 0 \end{cases} \quad (10)$$

Then, the correlation dimension can be calculated as follows:

$$D_m = \lim_{r \rightarrow 0} \left[ \frac{\ln C(m, r)}{\ln r} \right]. \quad (11)$$

Third, Kolmogorov entropy can also be introduced to explore the chaos property in terms of information loss [37]. In phase space, two trajectories might be too close to be identified initially, but they will diverge as time passes; and Kolmogorov entropy can effectively measure the speed at which this takes place. Accordingly, Kolmogorov entropy  $K_2$  can be defined by:

$$K_2 \cong \lim_{m \rightarrow \infty} \lim_{r \rightarrow 0} K_2^m(r), \quad (12)$$

where

$$K_2^m(r) = \frac{1}{\Delta t} \log \frac{C(m, r)}{C(m + 1, r)}, \quad (13)$$

where  $\Delta t$  is the time interval between two successive observations. Specifically, the observed system can be tested to be a regular or ordered system when Kolmogorov entropy  $K_2 = 0$ , a purely random system when  $K_2 \rightarrow \infty$ , or a chaotic system when  $0 < K_2 < \infty$ . A larger  $K_2$  corresponds to a more complex system. When comparing Kolmogorov entropy with other entropies in Group 3, although all entropies can effectively measure the structure complexity of a time series, Kolmogorov entropy, as an important attractor invariant, investigates the attractor property in terms of the rate of information loss (or the degree of predictability of points) along the attractor in phase space [95], while other entropies otherwise focus on the system disorder of times series data in terms of power concentration in frequency domain or similarity changes between embedding vectors [41].

#### 4.1.2. Diagram descriptions

Based on phase-space reconstruction, the strange attractor can be reconstructed by expanding the 1-dimensional series into higher-dimensional phase space according to Eq. (8), and some useful diagrams can be utilised to provide a vivid visualisation for the system states. From different perspectives, various diagrams can be drawn, e.g., a phase portrait [38], a Poincare section diagram [39] and a RP [40].

The phase portrait, as the simplest and most typical tool, is a 2- or 3-dimensional plot for the system state and attractor in phase space in terms of the trajectories, i.e., the line

connecting consecutive states of a system. From a phase portrait, the structure of the attractor can be directly identified, based on which the complexity state of the system can be approximately investigated. In particular, the attractor may be a fixed point in a deterministic non-chaotic system, a limit cycle in a periodic system, a limit torus in a quasi-periodic system, or an infinite number of lines in a chaotic system. In contrast, in a random system, no attractor can be found [85]. However, the phase portrait, with the simple representation, might only be appropriate for lower-dimensional attractors and inefficient for complex structures.

The Poincare section diagram provides a 2-dimensional section through an  $m$ -dimensional state space to show where the trajectory of the attractor crosses the plane of the section. For example, for a 3-dimensional state space with three variables  $x$ ,  $y$  and  $z$ , a Poincare section can be obtained by plotting the values of  $x$  and  $y$  when  $z$  is set to a given constant. Based on the points in the Poincare section, the chaotic property of a data system can be approximately investigated. In particular, if there is only one fixed point or a few discrete points in the Poincare section, the process can be shown as periodic. In contrast, if there are dense points with a fractal structure, the process might be chaotic, which corresponds to a complex data system.

The RP tends to display the dynamic evolution of data systems in terms of the trajectory recurrence. In particular, the recurrence is defined as the closeness of any two states  $X(i)$  and  $X(j)$  in phase space. The Euclidean norm is frequently adopted as a distance-measure of the closeness between two vectors:

$$D_{i,j} = \|X_i - X_j\|. \quad (14)$$

Given a cut-off distance  $\varepsilon$ , the recurrence matrix can be obtained by:

$$R_{i,j} = H(\varepsilon - D_{i,j}), \quad (15)$$

where  $H(z)$  is the Heaviside function (see Eq. (10)). The RP exactly displays the recurrence matrix  $R_{i,j}$  in which the point  $(i, j)$  is coloured in black when  $R_{i,j} = 1$  or in white when  $R_{i,j} = 0$ . Because  $R_{i,i} = 1$  at point  $(i, i)$ , the RP has a black main diagonal line (i.e., line of identity (LOI)). By investigating the points and lines in the RP, the inner patterns can be identified. In particular, the diagonal lines parallel with the LOI in the RP means that the evolution of states is similar at different times, i.e., the regular or periodic patterns, while the irregular points and lines reflect the complex patterns hidden in the time series. Accordingly, if only diagonal lines exist in the RP, the system might be deterministic; if the diagonal lines are accompanied by some single isolated points, the system might hold a chaotic property; and if only single isolated points can be found, the system may follow a random process.

Furthermore, because the RP is somewhat difficult to interpret, some useful statistics have been accordingly formulated to provide quantitative interpretations of RP, i.e., recurrence quantification analysis (RQA) [96]. The most popular RQA indicators are the recurrence rate (RR) for the density of recurrence points in the RP, determinism (DET) for the ratio of recurrence points forming diagonal structures with all recurrence points, divergence (DIV) for the inverse of the length of the longest diagonal line, and laminarity (LAM) for

**Table 3**

Typical works on the application of methods derived from nonlinear dynamics to time series data.

Field	Authors	Data	Technique
Economics	Brock and Sayers [98]	U.S. macroeconomic data	Tests for deterministic chaos
	Bastos and Caiado [99]	Stock indices in global stock markets	RP; RQA
	Chen [100]	Unemployment rate in Taiwan	RP; RQA
	Niu and Wang [101]	HSI in Hong Kong's stock market	Correlation dimension;
			Lyapunov exponent;
			Kolmogorov entropy
	Adrangi et al. [102]	Nearby futures prices of crude oil, heating oil and unleaded gasoline	Correlation dimension;
			Kolmogorov entropy
	Barkoulas et al. [103]	WTI spot price	Correlation dimension;
Life science	Rapp et al. [104]	Spontaneous neural activity of a monkey	Lyapunov exponent; RP
	Babloyantz [105]	EEG	Information dimension
	Thomasson et al. [106]	EEG	Correlation dimension
	Yılmaz and Güler [107]	Blood flow of ICA	RQA
			Correlation dimension;
			Lyapunov exponent
Earth science	Übeyli [108]	EEG	Lyapunov exponent
	Aboofazeli and Moussavi [109]	Swallowing sound	RP
	Ponyavin [112]	Sunspot numbers, geomagnetic indices and temperature	RP
	Ghorbani et al. [113]	River flow	Correlation dimension;
			Lyapunov exponent
			RP; RQA
Engineering	Nichols et al. [114]	Steel plate response	Correlation dimension
Physics	Yang et al. [115]	Results from transient system simulation platform	Correlation dimension
	Toomey et al. [116]	Laser intensity	Correlation dimension
	Babaei et al. [117]	Pressure fluctuations at superficial gas velocities	RP; RQA

the ratio of the recurrence points forming the vertical structures to all recurrence points. In addition, a time-dependent analysis framework was proposed by Trulla et al. [97] to capture the evolution of these RQA indicators over time.

#### 4.2. Applications

Because the chaotic phenomenon can be widely investigated in reality, the above mentioned effective methods derived from nonlinear dynamics have frequently been applied to various time series data analyses. Similarly, the research fields of economics [98–103], life science [104–109], earth science [112,113], engineering [114,115] and physics [116,117] might be the most predominant application areas of the methods derived from nonlinear dynamics; some notable works are listed in Table 3.

As for economics, the study of the application of methods derived from nonlinear dynamics to economic theory has become a popular issue during the last 20 years; a ground-breaking work is Brock and Sayers [98], in which tests are conducted for the presence of low-dimensional deterministic chaos in the U.S. macroeconomic data, e.g., the unemployment rate, gross national product (GNP) and industrial production. Examples of recent studies are the following. Bastos and Caiado [99] analysed various stock indices in global stock markets using the RP and RQA indicators, and the results suggested that emerging markets were less complex compared with the developed counterparts. Chen [100] applied the RP and RQA to the unemployment rate of Taiwan, and the results showed that these recurrence-based techniques effectively identified different phases in the evolution of the unemployment transition. Niu and Wang [101] studied the complex dynamic behaviours of the Hang Seng index (HSI) in Hong Kong's stock market by the correlation dimension, Lyapunov exponent and Kolmogorov entropy. Adrangi et al. [102] used the correlation dimension and Kolmogorov

entropy to test the presence of low-dimensional chaotic structures in the markets of crude oil, heating oil, and unleaded gasoline. Barkoulas et al. [103] analysed the chaos property of the WTI market by the correlation dimension, Lyapunov exponents and RP, and the empirical evidence suggested that stochastic rather than deterministic rules were present in the crude oil market.

In the research field of life science, some ground-breaking studies are by Rapp et al. [104], which addressed spontaneous neural activity in the motor cortex of a monkey based on 'chaos analysis', and by Babloyantz [105], which addressed human sleep EEG data based on the correlation dimension. Thomasson et al. [106] investigated epileptic EEG activity with the RQA, and the results indicated that both preictal and EEG segments free of seizure activity exhibited significant transients. Yılmaz and Güler [107] evaluated the chaotic behaviour of blood flow obtained from a healthy and stenosed internal carotid artery (ICA) via the correlation dimension and the largest Lyapunov exponent, and the results showed that the two indices were useful for the diagnosis of the ICA stenosis. Übeyli [108] proposed a novel classification method by coupling a probabilistic neural network (PNN) and Lyapunov exponents, and the results demonstrated that the Lyapunov exponents adequately represented the EEG signals and further enhanced the classification accuracies of the PNN. Aboofazeli and Moussavi [109] presented a novel method for swallowing sound detection using a hidden Markov modelling based on the RP, and the experimental results indicated the superiority of the proposed RP-based method. Interestingly, although the chaos methods are derived from nonlinear dynamics, some studies in life science observed the effectiveness of the linear methods. For example, Mormann et al. [110] compared 30 different characterising measures to analyse the EEG data, including both linear approaches (e.g., statistical moments and spectral band power) and nonlinear approaches (e.g., the

correlation dimension and Lyapunov exponent), and observed that the performance of linear measures was similar to or better than nonlinear measures. Similarly, McSharry et al. [111] suggested that some linear methods can detect pre-ictal changes in a manner similar to nonlinear methods, and the combination of linear and non-linear methods would be a good method for reliable seizure anticipation.

Regarding earth science, Ponyavin [112] applied wavelet analysis and RP analysis to geomagnetic and climate records for sunspot numbers, geomagnetic indices and temperature, and the results showed that the solar cycle signal was more pronounced in climatic data during the last 60 years. Ghorbani et al. [113] investigated the daily river flow of the Kizilirmak River by the correlation dimension and Lyapunov exponent, and the results indicated the presence of the chaos property in the examined data.

Engineering studies using the methods derived from nonlinear dynamics are somewhat rare compared with other application fields. Nichols et al. [114] employed the RP and RQA as effective diagnosis tools to detect damage-induced changes in materials based on steel plate response data. Yang et al. [115] proposed a method of fault detection based on the correlation dimension, and the results indicated this was a valid method for detecting fixed and drifting bias faults. As for applications to physics, the correlation dimension has been used to investigate the dynamic complexity of solid state lasers [116]. Babaei et al. [117] investigated the hydrodynamics of a gas–solid fluidised bed using the RP and RQA, and the results showed that the fluidised bed was more complex than the Lorenz system.

According to the above studies, the chaos property can be observed as a useful measure in complexity analysis for time series data. In particular, the information obtained from the methods derived from nonlinear dynamics can be used to distinguish different cases, e.g., epileptic seizures prediction [106,107] and fault detection in mechanical engineering [114,115]. However, although the chaos methods are derived from nonlinear dynamics, the performance of linear measures has often shown similar to or better than nonlinear measures, especially for the EEG data [110,111], and the combination of linear and nonlinear methods would be a good method.

#### 4.3. Directions for future research

Based on the above analyses, improvements in the models and the applications of methods derived from nonlinear dynamics should be explored. As for model improvement, because each attractor invariant explores the chaos property from a distinct perspective, a combination exponent can effectively provide a more comprehensive description for the global system structure of time series data. Furthermore, some powerful forecasting techniques have recently been formulated that consider the chaos features of the sample data; this may be another important way to improve the techniques for both data analysis and prediction.

As for applications, economic data analysis might be the most popular topic in complexity analysis that uses methods derived from nonlinear dynamics, although it lacks analysis of volume series data, e.g., production, consumption and

sales data. However, the volume series should be considered as well as the price series, which can provide much more information for market analysis. In addition, the applications of methods derived from nonlinear dynamics can be further extended to other research fields, e.g., food science, geology, computer networks and so on.

## 5. Entropy measurements

### 5.1. Measures

To evaluate the disorder state of data dynamics, diverse entropies have been introduced into the complexity testing research. Actually, entropy is a thermodynamic statistic for describing system disorder [41]. Generally, a large value of entropy refers to a complex system in a high-level disorder state. Existing entropies can be further divided into two categories: structural entropies and dynamical entropies [43]. While structural entropies measure structure complexity in terms of power concentration, i.e., how concentrated (or widespread) the distribution of a power spectrum of a time series is, by transforming the observed data from the time domain into the frequency or time-frequency domain for further analysis, the dynamical entropies investigate the potential changes in similarity between inner patterns in terms of the conditional probability that two sequences (or patterns) remain to be similar to each other as previous when the embedding dimension of the phase space increases [46]. The entropy measurements, together with the relationships, are described in Fig. 4.

#### 5.1.1. Structural entropy

Structural entropies measure the structural complexity of time series data dynamics in terms of power concentration, a measure of how concentrated (or widespread) the distribution of a power spectrum of a time series is, by transforming the data from the time domain into the frequency or time-frequency domain for further analysis. In particular, the concentration of the frequency spectrum in terms of a single narrow peak corresponds to a low entropy value, indicating low-level complexity of the data dynamics. In contrast, for a highly complex system, a wide band response can be found in the frequency domain [43].

Accordingly, two main steps are included in structural entropy exploration, i.e., domain transformation (to transform the observed data from the time domain to the frequency or time-frequency domain to obtain the power spectrum) and entropy estimation (to investigate the power concentration). The former might be a key factor in structural entropy formulation, and by using different domain transformation methods, e.g., Fourier transformation (FT), wavelet transformation and series permutation, various structural entropies can be formulated, amongst which SE [42], WE [43] and PE [44] may be the most conventional and popular methods.

The SE [42] can be observed as the basic form of structural entropy, from which other novel structural entropies were extended. In the first step of domain transformation, the given time series  $x(i)$  ( $i = 1, 2, \dots, N$ ) is transformed from the time domain to the frequency domain  $\lambda(k)$  ( $k = 1, 2, \dots, N$ ) by the FT:

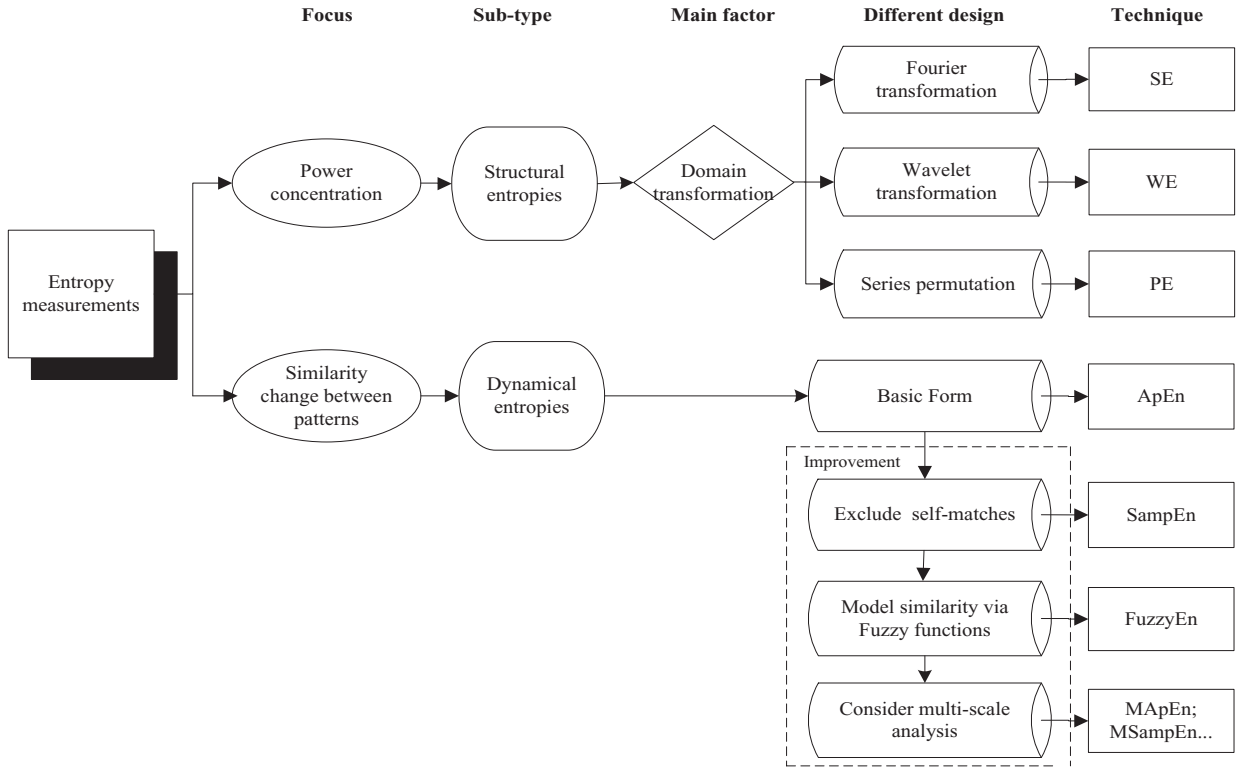


Fig. 4. The measures of entropy measurement techniques for time series data.

$$\lambda(k) = \sum_{n=1}^N x(n) e^{-i \frac{2\pi}{N} nk}, \quad k = 1, 2, \dots, N, \quad (16)$$

where  $\lambda(k)$  is the discrete Fourier transformation of the original series data. The term  $|\lambda(k)|^2$  is the power spectrum.

In SE estimation, the power density is first estimated via the relative contribution of each term to the overall power:

$$p(k) = \frac{|\lambda(k)|^2}{\sum_{k=1}^{N/2} |\lambda(k)|^2}. \quad (17)$$

Then, a certain effective entropy can be introduced to quantitatively estimate the power concentration for analysing the structural complexity. Shannon information entropy might be the most popular statistic to quantify the disorder state [118]:

$$SE = - \sum_{k=1}^{N/2} p(k) \ln(p(k)). \quad (18)$$

For clear representation, the entropy can be normalised on the range of [0,1] by dividing  $\ln(N/2)$ , where  $N$  is the total number of frequencies, which equals the size of the original time series data. Accordingly, a low SE (near 0) corresponds to power concentration in terms of one single narrow peak, indicating an ordered data process, while a large value (near 1) reflects a wide band response in the frequency domain, indicating a complex system.

To overcome the limitations of the FT, i.e., ignoring the time evolution of frequency patterns and holding the data

assumption of stationarity [43], various other structural entropies have been extended from SE by introducing other effective domain transformation methods. For example, Powell and Percival [42] employed the short-time Fourier transformation (STFT) with a Hanning window to effectively capture the dynamic frequencies and to partially relax the stationarity assumption into a quasi-stationarity assumption. Rosso et al. [43] utilised wavelet transformation and proposed the WE, in which the data can be transformed to a time-frequency domain. Based on embedding vectors, the PE was accordingly formulated with its unique merits of time-saving, robustness and stability with respect to nonlinear monotonous transformations [44]. In particular, the PE transforms the time series data by generating the embedding vectors, sorting the values of each vector in an ascending order, and creating permutation patterns with the offset of the permuted values; and the PE is estimated based on the probability distribution of all permutation patterns. Recently, Bian et al. [119] proposed a modified PE, where the equal values in an embedding vector are labelled with the same order in the corresponding permutation pattern, unlike the original PE, which maps them onto different orders according to the time of their appearance.

### 5.1.2. Dynamical entropy

Dynamical entropies explore system complexity from the perspective of similarity changes between inner patterns in data dynamics in terms of the conditional probability that two sequences in phase space remain similar to each other as previous when the embedding dimension increases [46].

Mathematically, dynamical entropy can be commonly defined as a negative natural logarithm of such conditional probability, and a larger value indicates a higher level of complexity of time series data dynamics with a lower probability that the similarity remains when the embedding dimension increases (or a higher probability that the similarity changes). The most predominant algorithms are the ApEn, SampEn, FuzzyEn and so on.

The ApEn can be observed as the basic dynamical entropy for measuring similarity changes in terms of information generation, and the first step is to create embedding vectors of the given time series  $x(i)$  ( $i = 1, 2, \dots, N$ ) with embedding dimension  $m$ :

$$X(j) = [x(j), x(j+1), \dots, x(j+m-1)], \\ j = 1, \dots, N-m+1, \quad (19)$$

The similarity between any two vectors  $X(i)$  and  $X(j)$  is estimated by a simple form, i.e., the Heaviside function with a predefined threshold  $r$ :

$$D_{i,j}(r) = \begin{cases} 1, & d_{i,j} < r \\ 0, & d_{i,j} \geq r \end{cases}, \quad (20)$$

where  $d_{i,j} = \max_{k=0,1,\dots,m-1} |x(i+k) - x(j+k)|$ . The probability of template matching for all vectors is calculated by:

$$\phi^m(r) = \frac{\sum_{i=1}^{N-m+1} \log B_i^m(r)}{N-m+1}, \quad (21)$$

where  $B_i^m(r)$  is the probability of template matching for  $m$  points of the  $i$ th vector:

$$B_i^m(r) = \frac{\sum_{j=1}^{N-m+1} D_{i,j}(r)}{N-m+1}. \quad (22)$$

Finally, let the embedding dimension increase to  $m = m+1$ , and repeat the above steps to generate probability  $\phi^{m+1}(r)$  on the incremental comparisons. The ApEn( $m, r$ ) can be defined as the negative average natural logarithm of the conditional possibility that the two sequences remain similar to each other when the dimension increases from  $m$  to  $m+1$ :

$$\text{ApEn}(m, r) = \lim_{N \rightarrow \infty} [\phi^m(r) - \phi^{m+1}(r)]. \quad (23)$$

In practice, for a given finite  $N$ -points dataset, the ApEn can be approximate estimated by:

$$\text{ApEn}(m, r, N) = \phi^m(r) - \phi^{m+1}(r). \quad (24)$$

However, the ApEn always overestimates the probability, which covers self-matches. To address such a drawback, Richman and Moorman [46] developed the SampEn. In the SampEn, the correlation sum computation in ApEn (i.e., Eq. (21)) is modified by introducing an additional constraint  $i \neq j$  to exclude self-matches:

$$B_i^m(r) = \frac{\sum_{j=1, j \neq i}^{N-m} D_{i,j}(r)}{N-m-1}. \quad (25)$$

Moreover, the probability of template matching for all vectors in ApEn (i.e., Eq. (21)) is changed to:

$$B^m(r) = \frac{\sum_{i=1}^{N-m} B_i^m(r)}{N-m}. \quad (26)$$

Finally, the SampEn can be redefined as:

$$\text{SampEn}(m, r) = \lim_{N \rightarrow \infty} \left[ -\ln \frac{B^{m+1}(r)}{B^m(r)} \right]. \quad (27)$$

Recently, by introducing various fuzzy membership functions to measure the similarity rather than the simple Heaviside function in ApEn and SampEn, diverse fuzzy entropies were proposed [47]. It is assumed that in reality, the boundaries between various patterns in data dynamics may be ambiguous, and it is difficult to determine their relationship via simple measurement methods, e.g., the Heaviside function. Therefore, various fuzzy membership functions, e.g., the Gaussian function, sigmoid function and bell-shaped function, can be utilised to describe the similarity between two vectors when the following two properties are met: (1) continuity, for ensuring the similarity measurement stable without abrupt change and (2) convexity, for guaranteeing the self-similarity can be maximised.

Furthermore, it is apparent that the above entropies only consider a single scale, but the disorder state might vary across different time scales. Accordingly, various multi-scale entropies have been proposed recently to capture complexity at multiple temporal scales using a coarse-graining procedure [120]. By introducing a scale factor  $\tau$ , the consecutive coarse-grained series can be constructed for further analysis according to the following:

$$x^\tau(j) = \frac{1}{\tau} \sum_{i=(j-1)\tau+1}^{j\tau} x(i), \quad j = 1, \dots, \left\lceil \frac{N}{\tau} \right\rceil. \quad (28)$$

When  $\tau = 1$ , the consecutive coarse-grained series are actually the original time series data. Based on the scale factor  $\tau$ , various multi-scale entropies can be accordingly proposed, i.e., multi-scale ApEn (MApEn) [120], multi-scale SmpEn (MSampEn) [120], and multi-scale PE (MPE) [121].

## 5.2. Applications

There are numerous studies on time series data analysis that use entropy-based methods; many of them are in the research fields of economics [122–125], life science [22,45,46,119,126,127], earth science [128–130], engineering [131–134] and so on. Generally, the most popular entropies might be the PE and various dynamical entropies. Some typical cases are listed in Table 4.

Entropy has been widely used in economic data analyses to quantify the market efficiency in various markets. For example, Pincus and Kalman [122] demonstrated that the ApEn can be used as a helpful tool to analyse the market efficiency of stock markets as a marker of market stability. Zunino et al. [123] implemented the PE to investigate the market efficiency of stock markets, and the results suggested that it was useful to identify different stages of market development and could be easily implemented. Wang et al. [124] measured the market efficiency in foreign exchange (FX) markets using the MApEn, and the results indicated that the developed FX market was more efficient than emerging FX markets. Martina et al. [125] used MApEn to monitor the evolution of crude oil price movements, and the highest market efficiency was found for small time-scales up to one or two weeks. The above studies observed that entropy-based methods could



**Table 4**

Typical works on the application of entropy calculation techniques to time series data.

Field	Authors	Data	Technique
Economics	Pincus and Kalman [122]	Stock indices	ApEn
	Zunino et al. [123]	World stock indices	PE
	Wang et al. [124]	Foreign exchange rates	MApEn
	Martina et al. [125]	WTI closing spot price	MApEn
Life science	Pincus [45]	Heart beat rate	ApEn
	Richman and Moorman [46]	Heart beat rate	SampEn
	Lake et al. [126]	Heart beat rate	SampEn
	Alcaraz and Rieta [22]	EEG	SampEn
	Bian et al. [119]	Heart beat rate	Modified PE
	Deffeyes et al. [127]	Position data measured by force plate	ApEn
Earth science	Fan et al. [128]	ECG	ApEn
	Chou [129]	Rainfall	MSampEn
	Mihailović et al. [130]	River flow	Kolmogorov entropy; SampEn; PE
Engineering	Pérez-Canales et al. [131]	Spindle speed	ApEn
	Wang et al. [132]	Traffic flow	MSampEn; MPE
	Sun et al. [133]	Battery current and voltage	SampEn
	Ei Safty and EL-Zonkoly [134]	Current signals	WE

effectively capture the market structure, even under socio-political extreme events.

As for life science, some ground-breaking works are by Pincus [45], which applied the ApEn to clinical cardiovascular analysis, and Richman and Moorman [46], which compared the ApEn and SampEn and recommended the latter because of the higher accuracy for clinical cardiology. Recently, the ApEn and SampEn methods were still the most popular techniques in the field of life science data analysis. For example, Lake et al. [126] employed the SampEn to investigate the abnormal heart rate of reduced variability and transient deceleration in the course of neonatal sepsis. Alcaraz and Rieta [22] implemented the SampEn to assess the organisation degree of atrial activity obtained from surface EEGs, and the results showed that patients with heart rates above 130 bpm must be handled with care. Bian et al. [119] applied a modified PE to analyse heart rate variability, and the results showed that the modified PE could greatly improve the performance of distinguishing heart rate variability signals under different conditions. Deffeyes et al. [127] applied the ApEn to the postural sway to identify developmental delay, and the postural sway in an early sitting was tested to appear lower-level complexity for infants with developmental delay in the anterior–posterior axis. In particular, Fan et al. [128] compared the ApEn with two traditional linear methods (i.e., 95% spectral edge frequency (SEF95) and bispectral index (BIS)) for measuring the depth of anaesthesia in the pre-anaesthesia, maintenance and recovery stages, and the results indicated that the ApEn was more sensitive than the BIS in detecting the recovery of consciousness from the non-respond to respond stage, which demonstrated the superiority of the nonlinear method.

The applications of entropy in earth science were somewhat insufficient compared with those in other research areas. Some important works are as follows. Chou [129] investigated the complexity of rainfall time series at four stations using the MSampEn; the results provide helpful insights into water resource planning. Mihailović et al. [130] implemented Kolmogorov entropy, SampEn and PE to analyse the river flow time series of two mountain rivers in Bosnia and Herzegovina, and the results showed the

complexity levels of the two flow time series were similar. In addition, the results also indicated that the ApEn with a biased statistic is effective for analysing the complexity of noisy, medium-sized time series data; the SampEn is unbiased and less dependent on data size; and the PE is robust even for noisy, nonlinear data [130].

As for engineering, Pérez-Canales et al. [131] identified chatter instabilities in the milling process by using the ApEn under a time-frequency monitoring method, and the results showed that unstable chattering was associated with entropy increments for a frequency range. In Wang et al. [132], the MSampEn and MPE methods were employed to investigate traffic series, and it was argued that the complexity of weekend traffic time series differed from that of the work-day time series, which could help enhance prediction performance. Sun et al. [133] used the SampEn as an indicator of the state-of-health (SOH) of the lead–acid battery in a health auxiliary diagnosis method. Ei Safty and Ei-Zonkoly [134] implemented wavelet analysis and the WE method to analyse current signals and found the method can effectively identify the type of fault in the power system.

Generally speaking, because entropies can directly reflect the complexity state even with a simple calculation process, this group has been somewhat more widely applied than fractality and methods derived from nonlinear dynamics. However, the entropy measurement might neglect the nonlinear dynamics of data systems [41].

### 5.3. Directions for future research

According to the previous analyses, certain model improvements and application extensions are worth considering. As for model improvement, because some parameters in various entropy measurements should be pre-set, which directly determines the complexity testing results, e.g., the embedding dimension  $m$  and tolerance parameter  $r$  in dynamical entropies, these parameters should be carefully designed. In particular, powerful intelligent optimisation algorithms or AI tools can be introduced as promising tools to search appropriate parameter values. Furthermore, because structural entropy and dynamical entropy perform

analyses from different perspectives, an integrated index coupling both of them might provide a more comprehensive description for the complexity state of time series data.

As for the applications, the applications of entropy in the field of life science were somewhat abundant compared with those that used the other two types of fractality and methods derived from nonlinear dynamics. However, the applications in some research fields were still insufficient, e.g., chemistry, ecology, physics and so on.

## 6. Discussions

This paper provides a comprehensive literature review on the complexity testing techniques for time series data. Compared with existing literature reviews on complexity testing techniques for time series data, the primary contributions of this paper are from the following two perspectives. First, most literature reviews only concentrate on one group of complexity testing techniques while ignoring some other important types; this paper addresses this gap in the literature by providing a comprehensive literature review that includes various complexity testing techniques. Second, unlike some survey studies that only address a certain application field, this literature review addresses complexity testing techniques in different application areas.

Sections 2–5 accordingly present the development of the measures and applications as well as directions for future research for each type of complexity testing technique. From the detailed discussions above, the following conclusions can be provided concerning the nature and basic ideas of complexity testing techniques, in addition to classification, relationships amongst sub-types, and directions for future research.

### 6.1. Complexity and complexity testing techniques

Complex systems are special cases of nonlinear systems, and the term ‘complex’ describes an intermediate state between a completely regular system and a completely random system. Generally, a lower level of complexity indicates that the observed system is more likely to follow a deterministic process, which can be finely captured and predicted, while a higher level of complexity indicates less regular rules controlling the data dynamics, which might be more unpredictable and more difficult to understand. According to the existing literature, numerous complexity testing techniques have been developed for time series data. According to different features, they can be divided into three general categories: fractality, methods derived from nonlinear dynamics and entropy. Each category tends to describe the dynamic system of time series data from different perspectives.

### 6.2. Fractality testing techniques

Fractality theory mainly focusses on the self-similarity (or system memorability or long-term persistence) by analysing the metric scaling behaviour of a data system, which is grouped into mono- and multi-fractality analyses. The mono-fractality analysis investigates the global structure of time series data in terms of global scaling behaviours via a single statistic, while multi-fractality analysis uses multiple

exponents to discover local scaling behaviours by introducing an order  $q$  into the fluctuation function.

In fractality analysis, two important factors are included, i.e., the detrending approach (to describe the fluctuation of time series data) and scaling exponent (to estimate the scaling behaviour). By using different detrending approaches (or fluctuation descriptions), various fractality testing methods can be formulated. In particular, the mono-fractality analysis techniques include R/S analysis with the detrending approach of the average rescaled range, DFA with a polynomial function, and DMA with a moving average. As for multi-fractality analysis, the SPMF, WTMM, MF-DFA and MF-DMA have been proposed based on the detrending approaches of standard partition function, wavelet transformation, polynomial function and moving average, respectively.

The scaling exponent is measured by fitting the power-law relationship between the fluctuation function and time scale. Generally, a scaling exponent far from the disordered level of 0.5 represents a strong self-similarity (or system memorability or long-term persistence), i.e., the observed data are at a low level of complexity.

Recently, various dynamic and multi-scale fractality testing models have been proposed, which might be interesting ways to improve fractality testing techniques.

### 6.3. Methods derived from nonlinear dynamics

The methods derived from nonlinear dynamics investigate time series data mainly based on the chaos theory. The term ‘chaos’ describes an apparently unpredictable behaviour driven by various interactive inner factors that actually obey certain inner rules but are supersensitive to the initial conditions. For a sufficient time, the states of the data system will finally converge towards a subspace (or a particular set of points); and such a subspace can be defined as the attractor of a system because it ‘attracts’ trajectories from all possible initial conditions. Accordingly, the methods derived from nonlinear dynamics explore the time series data based on the phase-space reconstruction theory and measures complexity by investigating the property of the attractor in phase space. In particular, the attractor may be a fixed point in a deterministic non-chaotic system, a limit cycle in a periodic system, or a limit torus in a quasi-periodic system. In contrast, in a random system, no attractor can be found.

To measure the attractor properties, there are two helpful tools, i.e., attractor invariants and diagram descriptions. Three attractor invariants can be introduced, i.e., the Lyapunov exponent, fractal dimension, and Kolmogorov entropy. In particular, the Lyapunov exponent estimates the average exponential divergence (or convergence) rates of the separation between initially nearby trajectories; fractal dimension for spatial extensiveness in terms of dimensional complexity; and Kolmogorov entropy for information loss. Consistently, a smaller attractor invariant refers to a simpler structure of the attractor (or final subspace or points set) in phase space, i.e., the data system can be tested at a lower level of complexity.

Because 1-dimensional time series data can be expanded into a higher-dimensional phase space based on the phase-space reconstruction, some useful diagrams can be utilised to provide visualisation of the system states. For example, the phase portrait, as the simplest and the most typical tool,

**Table 5**

Comparison and correlation analysis amongst fractality, methods derived from nonlinear dynamics and entropy.

	Fractality	Methods derived from nonlinear dynamics	Entropy
Focus	Self-similarity	Attractor property	Disorder state
Assumption about complexity	Weak self-similarity	Complex structure of Attractor	High-level disorder
Assumption about regularity	Strong self-similarity	A fixed point	Low-level disorder
Main advantages	Describe global and local structure of time series data	Visualise data dynamics in phase space	Provide a direct measure for system disorder; the calculation process is relatively simple
Main disadvantage	The measurement processes are relatively complicated; methods are dependent on the length of the sample data.	The attractor property is somewhat difficult to be quantitatively interpreted and captured; methods are dependent on the length of the sample data.	Neglect nonlinear dynamics of the system; methods are dependent on the length of the sample data.

depicts the trajectories in phase space, i.e., the line connecting consecutive states of a system. The Poincare section diagram provides a 2-dimensional section through an  $m$ -dimensional state space to show where the trajectories of the attractor cross the plane of the section. The RP tends to display the dynamic evolution of data systems in terms of the trajectory recurrence.

#### 6.4. Entropy measurement techniques

Entropy, a thermodynamic quantity, describes the disorder state of the data dynamics. Generally, a large value of entropy refers to a complex system at a high level of disorder state. Existing entropies can be further divided into two categories: structural entropies and dynamical entropies.

Structural entropies measure the structural complexity of time series data in terms of power concentration, a measure of how concentrated (or widespread) the distribution of the power spectrum of a time series is. Two main steps are accordingly included, i.e., domain transformation (to transform the observed data from the time domain to the frequency or time-frequency domain to obtain the power spectrum) and entropy estimation (to investigate the power concentration). By using different domain transformation methods, various structural entropies can be formulated, e.g., the SE, WE and PE, which are based on the domain transformation methods of the Fourier transformation, wavelet transformation and phase reconstruction, respectively. The concentration of the frequency spectrum in terms of one single narrow peak corresponds to a low entropy value, indicating a low-level complexity of the data dynamics. In contrast, for a highly complex system, a wide band response can be found in the frequency domain.

Dynamical entropies explore system complexity from the perspective of similarity changes between inner patterns in data dynamics in terms of the conditional probability that two sequences in phase space remain similar to each other as previous when the embedding dimension increases. The ApEn can be observed as the basic dynamical entropy, from which other various dynamical entropies were extended. For example, because the ApEn always overestimates the probability, which covers self-matches, the SampEn was proposed by introducing an additional constraint to exclude self-matches. By introducing various fuzzy membership functions to measure the similarity rather than the simple Heaviside function in ApEn and SampEn, diverse fuzzy

entropies were proposed. Recently, various multi-scale entropies have been proposed to capture complexity on multiple temporal scales.

#### 6.5. Relationship amongst different types

A comparison and correlation analysis amongst the three groups is presented in Table 5. The three types measure the complexity characteristic of time series data from different perspectives, i.e., self-similarity, attractor property and disorder state, and therefore possess their respective advantages and disadvantages. In particular, the fractality measurement can effectively describe the global and local structure of time series data, but the measurement processes are relatively complicated [18]. The methods derived from nonlinear dynamics can visualise data dynamics in phase space, but the attractor property is somewhat difficult to be quantitatively interpreted and captured. The entropies can provide a direct measurement for system disorder even with a relatively simple calculation process but neglect nonlinear dynamics of data systems [41]. Notably, the issue of finite-size data has been rarely addressed in the existing literature, and most measures are dependent on the length of the sample data.

However, these complexity testing techniques of the three groups are actually closely related to or even dependent on each other. In particular, a weaker self-similarity (or system memorability or long-term persistence), a more complex structure of strange attractor, and a higher-level disorder state of the system are all related to a higher-level complexity characteristic of the observed time series data. Furthermore, the testing techniques for different types of fractality, methods derived from nonlinear dynamics and entropy are sometimes similar or even mixed with each other. For example, the fractal dimension and Kolmogorov entropy can also be introduced into the methods derived from nonlinear dynamics as important attractor invariants to describe attractor properties.

#### 6.6. Future research directions

There is still ample opportunity to improve the models and applications for complexity testing techniques of time series data.

As for model improvement, the following seven directions can be considered. (1) Because each technique explores the complexity state of time series data dynamics from a

distinct perspective, an integrated complexity estimator covering various aspects can be proposed to offer a more comprehensive description for the observed time series data. (2) According to the above discussions about measures, the key factors in each technique can be identified, e.g., the detrending approach in fractality testing techniques and domain transformation in structural entropies; and developing more powerful tools, especially based on AI techniques, might be an important way to improve existing complexity testing techniques. (3) There are certain parameters in each technique that directly determine the complexity testing results, and these parameters should be carefully pre-set based on some powerful optimisation approaches. (4) Recently, various multi-scale complexity testing techniques have been proposed to capture complexity on multiple temporal scales. (5) Complexity analysis techniques for time series can also be extended from univariate time series to a bivariate or multivariate time series framework to investigate the interrelations across series. (6) The complexity concept can also be introduced into the time series forecasting research to formulate some novel powerful data-characteristics-based forecasting techniques, which might be another important way to improve the techniques for both data analysis and prediction. (7) The issue of finite-size data has been rarely addressed in the literature. For instance, the Hurst and scaling exponents are dependent on the length of the data. In such a case, confidence bands against, e.g., randomness, should be considered in the analysis of real time series.

As for applications, economic data analysis may be one of the most popular topics in complexity analysis for evaluating market efficiency, in which the price (or return) series are the main focus, and trading volumes are neglected. However, complexity investigation of both price and volume series can provide more interesting insights into market analysis. Furthermore, extreme events or emergencies might create structural changes in the complexity property, and accurate estimation for such an impact might also provide a new perspective for both data analysis and emergency management. In addition, the application of complexity testing techniques in certain research fields might be insufficient, e.g., chemistry, computers and IT networks, and ecology.

## Acknowledgments

This work is supported by grants from the National Science Fund for Distinguished Young Scholars (NSFC no. 71025005), the [National Natural Science Foundation of China](#) (NSFC no. [91224001](#), NSFC no. [71433001](#) and NSFC no. [71301006](#)), National Program for Support of Top-Notch Young Professionals and the Fundamental Research Funds for the Central Universities in BUCT. Authors would like to express their sincere appreciation to the editor and the two independent reviewers in making valuable comments and suggestions to the paper. Their comments have improved the quality of the paper immensely.

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