矩阵

矩阵的加减法

矩阵的乘法

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix}$$

$$\underset{n \times m}{A} \qquad \underset{m \times p}{B} \qquad \underset{n \times p}{C}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

$$\begin{bmatrix} 3 & 5 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 8 \\ 7 & 8 & 9 & 3 \end{bmatrix} * \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 1 & 6 \\ 2 & 4 & 6 & 2 \\ 6 & 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$$\underset{4 \times 4}{A} \qquad \underset{4 \times 4}{B} \qquad \underset{4 \times 4}{C}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

brute force

矩阵分块乘法

$$A=\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\begin{matrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{sr} \end{matrix}}^{\text{子块}A_{11}}} & \multicolumn{3}{c}{\overbrace{\begin{matrix} a_{1(r+1)} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{s(r+1)} & \cdots & a_{sn} \end{matrix}}^{\text{子块}A_{12}}} \\ \hline \multicolumn{3}{c|}{\underbrace{\begin{matrix} a_{(s+1)1} & \cdots & a_{(s+1)r} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mr} \end{matrix}}_{\text{子块}A_{21}}} & \multicolumn{3}{c}{\underbrace{\begin{matrix} a_{(s+1)(r+1)} & \cdots & a_{(s+1)n} \\ \vdots & & \vdots \\ a_{m(r+1)} & \cdots & a_{mn} \end{matrix}}_{\text{子块}A_{22}}} \end{array}\right)$$

因此 A 可以如下改写，其中每个元素都是子块（矩阵）：

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

矩阵分块乘法

$$\begin{bmatrix} \overset{a}{\boxed{\begin{matrix} 2 & 3 & 5 & 4 \\ 4 & 6 & 7 & 8 \\ 5 & 1 & 7 & 7 \\ 8 & 4 & 6 & 2 \end{matrix}}} & \overset{b}{\boxed{\begin{matrix} 7 & 8 & 1 & 3 \\ 1 & 2 & 9 & 5 \\ 5 & 4 & 1 & 3 \\ 1 & 5 & 3 & 7 \end{matrix}}} \\ \underset{c}{\boxed{\begin{matrix} 4 & 2 & 5 & 6 \\ 0 & 4 & 1 & 7 \\ 5 & 0 & 1 & 4 \\ 2 & 1 & 8 & 0 \end{matrix}}} & \underset{d}{\boxed{\begin{matrix} 1 & 7 & 8 & 2 \\ 5 & 4 & 3 & 2 \\ 6 & 1 & 0 & 5 \\ 5 & 4 & 3 & 2 \end{matrix}}} \end{bmatrix} * \begin{bmatrix} \overset{e}{\boxed{\begin{matrix} 5 & 2 & 1 & 7 \\ 1 & 3 & 9 & 8 \\ 9 & 3 & 4 & 5 \\ 9 & 2 & 0 & 1 \end{matrix}}} & \overset{f}{\boxed{\begin{matrix} 6 & 2 & 7 & 5 \\ 1 & 4 & 5 & 3 \\ 2 & 4 & 1 & 3 \\ 4 & 8 & 2 & 6 \end{matrix}}} \\ \underset{g}{\boxed{\begin{matrix} 5 & 1 & 2 & 3 \\ 2 & 0 & 3 & 5 \\ 0 & 9 & 7 & 2 \\ 4 & 3 & 9 & 6 \end{matrix}}} & \underset{h}{\boxed{\begin{matrix} 8 & 6 & 1 & 7 \\ 9 & 6 & 4 & 1 \\ 6 & 1 & 0 & 5 \\ 1 & 2 & 4 & 8 \end{matrix}}} \end{bmatrix} = \begin{bmatrix} \overset{ae + bg}{\boxed{}} & \overset{af + bh}{\boxed{}} \\ \underset{ce + dg}{\boxed{}} & \underset{cf + dh}{\boxed{}} \end{bmatrix}$$

$$
\begin{array}{cccccccc}
\multicolumn{4}{c}{a} & & & & \\
\end{array}
\qquad\qquad
\begin{array}{cccccccc}
\multicolumn{4}{c}{e} & & & & \\
\end{array}
$$

$$
\begin{bmatrix}
2 & 3 & 5 & 4 & 7 & 8 & 1 & 3 \\
4 & 6 & 7 & 8 & 1 & 2 & 9 & 5 \\
5 & 1 & 7 & 7 & 5 & 4 & 1 & 3 \\
8 & 4 & 6 & 2 & 1 & 5 & 3 & 7 \\
4 & 2 & 5 & 6 & 1 & 7 & 8 & 2 \\
0 & 4 & 1 & 7 & 5 & 4 & 3 & 2 \\
5 & 0 & 1 & 4 & 6 & 1 & 0 & 5 \\
2 & 1 & 8 & 0 & 5 & 4 & 3 & 2
\end{bmatrix}
*
\begin{bmatrix}
5 & 2 & 1 & 7 & 6 & 2 & 7 & 5 \\
1 & 3 & 9 & 8 & 1 & 4 & 5 & 3 \\
9 & 3 & 4 & 5 & 2 & 4 & 1 & 3 \\
9 & 2 & 0 & 1 & 4 & 8 & 2 & 6 \\
5 & 1 & 2 & 3 & 8 & 6 & 1 & 7 \\
2 & 0 & 3 & 5 & 9 & 6 & 4 & 1 \\
0 & 9 & 7 & 2 & 6 & 1 & 0 & 5 \\
4 & 3 & 9 & 6 & 1 & 2 & 4 & 8
\end{bmatrix}
=
\begin{bmatrix}
\quad & \quad \\
\quad & \quad
\end{bmatrix}
$$

$$
\begin{array}{cc}
a & b \\
\multicolumn{2}{c}{\begin{bmatrix}
\begin{matrix} 2 & 3 \\ 4 & 6 \end{matrix} & \begin{matrix} 5 & 4 \\ 7 & 8 \end{matrix} \\
\begin{matrix} 5 & 1 \\ 8 & 4 \end{matrix} & \begin{matrix} 7 & 7 \\ 6 & 2 \end{matrix}
\end{bmatrix}} \\
c & d
\end{array}
*
\begin{array}{cc}
e & f \\
\multicolumn{2}{c}{\begin{bmatrix}
\begin{matrix} 5 & 2 \\ 1 & 3 \end{matrix} & \begin{matrix} 1 & 7 \\ 9 & 8 \end{matrix} \\
\begin{matrix} 9 & 3 \\ 9 & 2 \end{matrix} & \begin{matrix} 4 & 5 \\ 0 & 1 \end{matrix}
\end{bmatrix}} \\
g & h
\end{array}
=
\begin{array}{cc}
ae + bg & af + bh \\
\multicolumn{2}{c}{\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}} \\
ce + dg & cf + dh
\end{array}
$$

Master method

$$\begin{array}{c} \begin{array}{cc} a \qquad\qquad & \qquad\qquad b \end{array} \\ \left[\begin{array}{cccc|cccc} 2 & 3 & 5 & 4 & 7 & 8 & 1 & 3 \\ 4 & 6 & 7 & 8 & 1 & 2 & 9 & 5 \\ 5 & 1 & 7 & 7 & 5 & 4 & 1 & 3 \\ 8 & 4 & 6 & 2 & 1 & 5 & 3 & 7 \\ \hline 4 & 2 & 5 & 6 & 1 & 7 & 8 & 2 \\ 0 & 4 & 1 & 7 & 5 & 4 & 3 & 2 \\ 5 & 0 & 1 & 4 & 6 & 1 & 0 & 5 \\ 2 & 1 & 8 & 0 & 5 & 4 & 3 & 2 \end{array}\right] \\ \begin{array}{cc} c \qquad\qquad & \qquad\qquad d \end{array} \end{array} * \begin{array}{c} \begin{array}{cc} e \qquad\qquad & \qquad\qquad f \end{array} \\ \left[\begin{array}{cccc|cccc} 5 & 2 & 1 & 7 & 6 & 2 & 7 & 5 \\ 1 & 3 & 9 & 8 & 1 & 4 & 5 & 3 \\ 9 & 3 & 4 & 5 & 2 & 4 & 1 & 3 \\ 9 & 2 & 0 & 1 & 4 & 8 & 2 & 6 \\ \hline 5 & 1 & 2 & 3 & 8 & 6 & 1 & 7 \\ 2 & 0 & 3 & 5 & 9 & 6 & 4 & 1 \\ 0 & 9 & 7 & 2 & 6 & 1 & 0 & 5 \\ 4 & 3 & 9 & 6 & 1 & 2 & 4 & 8 \end{array}\right] \\ \begin{array}{cc} g \qquad\qquad & \qquad\qquad h \end{array} \end{array} = \begin{array}{c} \begin{array}{cc} ae + bg & af + bh \end{array} \\ \left[\begin{array}{c|c} \qquad\qquad & \qquad\qquad \\ \hline \qquad\qquad & \qquad\qquad \end{array}\right] \\ \begin{array}{cc} ce + dg & cf + dh \end{array} \end{array}$$

$$\begin{aligned} & ae + bg \\ &= (a+d)(e+h) + d(g-e) - (a+b)h + (b-d)(g+h) \\ &= ae + ah + de + dh + dg - de - ah - bh - bg + bh - dg - dh \\ &= ae + bg \end{aligned}$$

$$
\begin{array}{cc} a & b \end{array}
\qquad
\begin{array}{cc} e & f \end{array}
\qquad
\begin{array}{cc} ae + bg & af + bh \end{array}
$$

$$
\left[\begin{array}{cccc|cccc}
2 & 3 & 5 & 4 & 7 & 8 & 1 & 3 \\
4 & 6 & 7 & 8 & 1 & 2 & 9 & 5 \\
5 & 1 & 7 & 7 & 5 & 4 & 1 & 3 \\
8 & 4 & 6 & 2 & 1 & 5 & 3 & 7 \\
\hline
4 & 2 & 5 & 6 & 1 & 7 & 8 & 2 \\
0 & 4 & 1 & 7 & 5 & 4 & 3 & 2 \\
5 & 0 & 1 & 4 & 6 & 1 & 0 & 5 \\
2 & 1 & 8 & 0 & 5 & 4 & 3 & 2
\end{array}\right]
*
\left[\begin{array}{cccc|cccc}
5 & 2 & 1 & 7 & 6 & 2 & 7 & 5 \\
1 & 3 & 9 & 8 & 1 & 4 & 5 & 3 \\
9 & 3 & 4 & 5 & 2 & 4 & 1 & 3 \\
9 & 2 & 0 & 1 & 4 & 8 & 2 & 6 \\
\hline
5 & 1 & 2 & 3 & 8 & 6 & 1 & 7 \\
2 & 0 & 3 & 5 & 9 & 6 & 4 & 1 \\
0 & 9 & 7 & 2 & 6 & 1 & 0 & 5 \\
4 & 3 & 9 & 6 & 1 & 2 & 4 & 8
\end{array}\right]
=
\left[\begin{array}{c|c}
 & \\
\hline
 &
\end{array}\right]
$$

$$
\begin{array}{cc} c & d \end{array}
\qquad
\begin{array}{cc} g & h \end{array}
\qquad
\begin{array}{cc} ce + dg & cf + dh \end{array}
$$

af + bh

= a(f-h) + (a+b)h

= af - ah + ah + bh

= af + bh

$$
\begin{array}{c}
\begin{array}{cc} \qquad a \qquad & \qquad b \qquad \end{array} \\
\left[\begin{array}{cccc|cccc}
2 & 3 & 5 & 4 & 7 & 8 & 1 & 3 \\
4 & 6 & 7 & 8 & 1 & 2 & 9 & 5 \\
5 & 1 & 7 & 7 & 5 & 4 & 1 & 3 \\
8 & 4 & 6 & 2 & 1 & 5 & 3 & 7 \\
\hline
4 & 2 & 5 & 6 & 1 & 7 & 8 & 2 \\
0 & 4 & 1 & 7 & 5 & 4 & 3 & 2 \\
5 & 0 & 1 & 4 & 6 & 1 & 0 & 5 \\
2 & 1 & 8 & 0 & 5 & 4 & 3 & 2
\end{array}\right] \\
\begin{array}{cc} \qquad c \qquad & \qquad d \qquad \end{array}
\end{array}
*
\begin{array}{c}
\begin{array}{cc} \qquad e \qquad & \qquad f \qquad \end{array} \\
\left[\begin{array}{cccc|cccc}
5 & 2 & 1 & 7 & 6 & 2 & 7 & 5 \\
1 & 3 & 9 & 8 & 1 & 4 & 5 & 3 \\
9 & 3 & 4 & 5 & 2 & 4 & 1 & 3 \\
9 & 2 & 0 & 1 & 4 & 8 & 2 & 6 \\
\hline
5 & 1 & 2 & 3 & 8 & 6 & 1 & 7 \\
2 & 0 & 3 & 5 & 9 & 6 & 4 & 1 \\
0 & 9 & 7 & 2 & 6 & 1 & 0 & 5 \\
4 & 3 & 9 & 6 & 1 & 2 & 4 & 8
\end{array}\right] \\
\begin{array}{cc} \qquad g \qquad & \qquad h \qquad \end{array}
\end{array}
=
\begin{array}{c}
\begin{array}{cc} ae + bg & af + bh \end{array} \\
\left[\begin{array}{c|c}
\qquad\qquad & \qquad\qquad \\
\hline
\qquad\qquad & \qquad\qquad
\end{array}\right] \\
\begin{array}{cc} ce + dg & cf + dh \end{array}
\end{array}
$$

ce + dg

= (c+d)e + d(g-e)

= ce + de + dg - de

= ce + dg

$$
\begin{array}{cc}
a & b \\
\left[\begin{array}{cccc|cccc}
2 & 3 & 5 & 4 & 7 & 8 & 1 & 3 \\
4 & 6 & 7 & 8 & 1 & 2 & 9 & 5 \\
5 & 1 & 7 & 7 & 5 & 4 & 1 & 3 \\
8 & 4 & 6 & 2 & 1 & 5 & 3 & 7 \\
\hline
4 & 2 & 5 & 6 & 1 & 7 & 8 & 2 \\
0 & 4 & 1 & 7 & 5 & 4 & 3 & 2 \\
5 & 0 & 1 & 4 & 6 & 1 & 0 & 5 \\
2 & 1 & 8 & 0 & 5 & 4 & 3 & 2
\end{array}\right] & \\
c & d
\end{array}
*
\begin{array}{cc}
e & f \\
\left[\begin{array}{cccc|cccc}
5 & 2 & 1 & 7 & 6 & 2 & 7 & 5 \\
1 & 3 & 9 & 8 & 1 & 4 & 5 & 3 \\
9 & 3 & 4 & 5 & 2 & 4 & 1 & 3 \\
9 & 2 & 0 & 1 & 4 & 8 & 2 & 6 \\
\hline
5 & 1 & 2 & 3 & 8 & 6 & 1 & 7 \\
2 & 0 & 3 & 5 & 9 & 6 & 4 & 1 \\
0 & 9 & 7 & 2 & 6 & 1 & 0 & 5 \\
4 & 3 & 9 & 6 & 1 & 2 & 4 & 8
\end{array}\right] & \\
g & h
\end{array}
=
\begin{bmatrix}
ae + bg & af + bh \\
ce + dg & cf + dh
\end{bmatrix}
$$

cf + dh

= a(f-h) + (a+d)(e+h) - (c+d)e -(a-c)(e+f)

= af - ah + ae + ah + de + dh - ce - de - ae - af + ce + cf

= cf + dh

$$ae + bg = (a+d)(e+h) + d(g-e) - (a+b)h + (b-d)(g+h)$$

$$af + bh = a(f-h) + (a+b)h$$

$$ce + dg = (c+d)e + d(g-e)$$

$$cf + dh = a(f-h) + (a+d)(e+h) - (c+d)e - (a-c)(e+f)$$

$$ae + bg = (a+d)(e+h) + d(g-e) - (a+b)h + (b-d)(g+h)$$

$$af + bh = a(f-h) + (a+b)h$$

$$ce + dg = (c+d)e + d(g-e)$$

$$cf + dh = a(f-h) + (a+d)(e+h) - (c+d)e - (a-c)(e+f)$$

$p1=(a+d)(e+h)$	$p5=a(f-h)$
$p2=d(g-e)$	$p6=(c+d)e$
$p3=(a+b)h$	$p7=(a-c)(e+f)$
$p4=(b-d)(g+h)$	

$$ae + bg = (a+d)(e+h) + d(g-e) - (a+b)h + (b-d)(g+h)$$

$$af + bh = a(f-h) + (a+b)h$$

$$ce + dg = (c+d)e + d(g-e)$$

$$cf + dh = a(f-h) + (a+d)(e+h) - (c+d)e - (a-c)(e+f)$$

$p1=(a+d)(e+h)$	$p5=a(f-h)$	$c11 = ae+bg = p1+p2-p3+p4$
$p2=d(g-e)$	$p6=(c+d)e$	$c12 = af+bh = p5+p3$
$p3=(a+b)h$	$p7=(a-c)(e+f)$	$c21 = ce+dg = p6+p2$
$p4=(b-d)(g+h)$		$c22 = cf+dh = p5+p1-p6-p7$

Master method