Smoothie

Solutions to Riemannian Geometry, do Carmo

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[1]This is for making an acknowledgement.

Abstract

This is a joint project by the graduate students taking 21-759 Differential Geometry offered in Spring 2016 at Carnegie Mellon University. Participants type up solutions to the exercises at the end of each chapter of the text book Riemannian Geometry by Manfredo Perdigao do Carmo, and supplement or elaborate the materials in the main body, hoping to better understand the subject matter, to create a document for future references, and to learn document collaboration with Git.

Contents

PREFACE

Chapter 0

DIFFERENTIABLE MANIFOLDS

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EXERCISES

1 Product manifold and quotient manifold by group action

Proof.

(a) Clearly, $M \times N$ is Hausdorff and second countable, and $\{U_\alpha \times V_\beta\}$ covers $M \times N$, with $\mathbf{z}_{\alpha,\beta} = (\mathbf{x}_\alpha, \mathbf{y}_\beta)$ being homeomorphic.

Next, we shall show that the atlas is differentiable. Fix $(U_{\alpha_i} \times V_{\beta_i}, \mathbf{z}_{\alpha_i,\beta_i}) =: (W_i, \phi_i)$, where $i = 1, 2$, such that $O := \cap_{i=1}^{2} \phi_i(W_i) \neq \emptyset$. Then the map

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(O) \ni (p, q) \mapsto \left(\mathbf{x}_{\alpha_2}^{-1} \circ \mathbf{x}_{\alpha_1}(p), \mathbf{y}_{\beta_2}^{-1} \circ \mathbf{y}_{\beta_1}(q)\right) \in \phi_2^{-1}(O), p \in \mathbb{R}^{\dim M}, q \in \mathbb{R}^{\dim N}$$

is clearly differentiable, by definition of $\{(U_\alpha, \mathbf{x}_\alpha)\}$ and $\{(V_\beta, \mathbf{y}_\beta)\}$ being differentiable structures on M and N, respectively. Hence by definition, $\{U_\alpha \times V_\beta\}$ is a differentiable atlas on $M \times N$, and thus induces a unique differentiable structure on $M \times N$ (note that $\{(U_\alpha \times V_\beta, \mathbf{z}_{\alpha,\beta})\}$ is not maximal). So $M \times N$ equipped with the given atlas above is a differentiable manifold.

Finally, we shall show that the projection onto M,

$$\pi_1 : M \times N \ni (x, y) \mapsto x \in M, x \in M, y \in N,$$

is differentiable. Fix $(x, y) \in M \times N$, and take, for convenience, $(U_\alpha, \mathbf{x}_\alpha)$ and $(V_\beta, \mathbf{y}_\beta)$ as the charts, with $p := \mathbf{x}_\alpha^{-1}(x) \in U_\alpha$ and $q := \mathbf{y}_\beta^{-1}(y) \in V_\beta$. Then the local expression of π_1 in the chosen charts is

$$\left[x_\alpha^{-1} \circ \pi_1 \circ (x_\alpha, y_\beta)\right](p, q) = x_\alpha^{-1}[\pi_1(x, y)] = x_\alpha^{-1}(x) = p,$$

which is obviously differentiable. Thus $\pi_1 \in C^\infty(M \times N; M)$. By symmetry, the projection onto N, π_2, is also differentiable.

(b) We shall prove a slightly more general result: $\prod_{n=1}^N (M_n/G_n) \overset{\text{diff.}}{\cong} \left(\prod_{n=1}^N M_n\right) / \left(\prod_{n=1}^N G_n\right)$, where each group G_n is a properly discontinuous action on the differentiable manifold M_n. In view of Example 4.8 and Part (a), $\prod_{n=1}^N (M_n/G_n)$ and $M := \prod_{n=1}^N M_n$ are differentiable manifolds with the associated product differentiable structures. We shall show that $G := \prod_{n=1}^N G_n$ acts on M properly discontinuously and that the associated quotient differentiable structure coincides with the product differentiable structure of $\prod_{n=1}^N (M_n/G_n)$.

As a corollary, $(S^1)^n \overset{\text{diff.}}{\cong} T^n$ follows from the fact that $S^1 \overset{\text{diff.}}{\cong} \mathbb{R}/\mathbb{Z} \overset{\text{diff.}}{\cong} T^1$.

We first check that G is a properly discontinuous group action on M. Fix $g_n \in G_n$ for every n, since $M_n \ni p_n \mapsto g_n p_n \in M_n$ is a diffeomorphic automorphism on M_n,

$$M \ni p := (p_1, \ldots, p_N) \mapsto (g_1 p_1, \ldots, g_N p_N) =: gp \in M, p_n \in M_n$$

is also a diffeomorphic automorphism on M. Now fix $p_n \in M_n$, by proper discontinuity of G_n on M_n, we can find an open subset of M_n, $U_n \in p_n$, so as $U_n \cap g_n(U_n) = \emptyset, \forall g_n \in G_n \backslash \{e_n\}$. Note that $V := \prod_{n=1}^N U_n$ is and open neighborhood of $p \in M$, and that

$$V \cap g(V) = \left[\prod_{n=1}^N U_n\right] \cap \left[\prod_{n=1}^N g_n(U_n)\right] = \prod_{n=1}^N \left[U_n \cap g_n(U_n)\right] = \emptyset, \forall g \in G \backslash \{e\}.$$

So G is a properly discontinuous action on M.

Now we shall show that the product differentiable structure is the same as the quotient differentiable structure, which means finding a diffeomorphism $\phi : M/G \to \prod_{n=1}^N (M_n/G_n)$, and this completes the proof. For each n, let $\{(O_n^{\alpha_n}, \mathbf{x}_n^{\alpha_n})\}$ be a differentiable atlas on M_n with $\mathbf{x}_n^{\alpha_n}(O_n^{\alpha_n}) \cap g_n\left(\mathbf{x}_n^{\alpha_n}(O_n^{\alpha_n})\right) = \emptyset, \forall g_n \in G_n \backslash \{e_n\}, \alpha^n$, and $q_n : M_n \ni p_n \mapsto [p_n]_{G_n} \in M_n/G_n$ be the quotient map (projection). Then we know from the construction of quotient manifold by group action, $\{(O_n^{\alpha_n}, \mathbf{y}_n^{\alpha_n} := q_n \circ \mathbf{x}_n^{\alpha_n})\}$ induces the quotient differentiable structure on M_n/G_n. By Part (a), $\{(O^\alpha := \prod_{n=1}^N O_n^{\alpha_n}, \mathbf{y}^\alpha := (\mathbf{y}_1^{\alpha_1}, \ldots, \mathbf{y}_N^{\alpha_N})\}$ gives a differentiable atlas on $\prod_{n=1}^N (M_n/G_n)$. Similarly, $\{(O^\alpha, \mathbf{x}^\alpha := (\mathbf{x}_1^{\alpha_1}, \ldots, \mathbf{x}_N^{\alpha_N})\}$ is a differentiable atlas on M, and from the proof of proper discontinuity, $\mathbf{x}^\alpha(O^\alpha) \cap g\left(\mathbf{x}^\alpha(O^\alpha)\right) = \emptyset, \forall g \in G \backslash \{e\}$.

Hence, $\{O^\alpha, \mathbf{z}^\alpha := q \circ \mathbf{x}^\alpha\}$ determines the quotient differentiable structure on M/G, where $q := (q_1, \dots, q_N)$. Consider $\phi : M/G \ni [p]_G \mapsto ([p_1]_{G_1}, \dots, [p_N]_{G_N}) \in \prod_{n=1}^N M_n/G_n$, which is obviously well-defined and bijective. For $[p]_G \in M/G$, let $(O^\alpha, \mathbf{z}^\alpha)$ be such that $O^\alpha \ni z := (\mathbf{z}^\alpha)^{-1}([p]_G)$, Observe that

$$(\mathbf{y}^\alpha)^{-1} \circ \phi \circ \mathbf{z}^\alpha(z) = (\mathbf{y}^\alpha)^{-1} [\phi([p]_G)] = (\mathbf{y}^\alpha)^{-1} (([p_1]_{G_1}, \dots, [p_N]_{G_N})) = (z_1, \dots, z_N) = z$$

and hence $\phi \in C^\infty(M/G; \prod_{n=1}^N (M_n/G_n))$ as claimed. Thus, $\prod_{n=1}^N (M_n/G_n) \overset{\text{diff.}}{\cong} M/G$.

■

2 Orientability of tangent bundle

Proof. To show that TM is orientable, we have to check that the determinant of every transition map in a certain atlas is always positive.

Let $\{(U, \varphi)\}$ be a chart on M, then define $\tilde{\varphi} : U \times \mathbb{R}^n \to TM$ via $\tilde{\varphi}(x, a) := \left(\varphi(x), a^i \frac{\partial}{\partial x_i}\right)$, with $x, a \in \mathbb{R}^n, x \in U$. Clearly $(U \times \mathbb{R}^n, \tilde{\varphi})$ is a chart on TM.

Fix two charts $(U \times \mathbb{R}^n, \tilde{\varphi})$ and $\left(V \times \mathbb{R}^n, \tilde{\psi}\right)$ on TM with $\varphi(U) \cap \psi(V) \neq \emptyset$. Let $x \in U, y \in V, a, b \in \mathbb{R}^n$ be such that $\tilde{\varphi}(x, a) = \left(\varphi(x), a^i \frac{\partial}{\partial x_i}\right) = \left(\psi(y), b^i \frac{\partial}{\partial y_i}\right) = \tilde{\psi}(y, b)$. Thus, $x = \varphi^{-1} \circ \psi(y)$.

Note that $t \mapsto \varphi(\varphi^{-1}(p) + ta) = \varphi(x + ta)$ is a representative of $a^i \frac{\partial}{\partial x_i}$, where $\varphi(x) =: p \in M$. Thus,

$$a = \left.\frac{\mathrm{d}}{\mathrm{d}\, t}\right|_{t=0} \varphi^{-1}(\varphi(x + ta)) = \left.\frac{\mathrm{d}}{\mathrm{d}\, t}\right|_{t=0} \varphi^{-1}(\psi(y + tb)) = D\left(\varphi^{-1} \circ \psi\right)(y) b.$$

Hence, $\tilde{\varphi}^{-1} \circ \tilde{\psi}(y, b) = (x, a) = (\varphi^{-1} \circ \psi(y), \mathrm{D}\,(\varphi^{-1} \circ \psi)(y) b)$. Then,

$$\mathrm{D}\left(\tilde{\varphi}^{-1} \circ \tilde{\psi}\right)(y, b) = \begin{bmatrix} \mathrm{D}\,(\varphi^{-1} \circ \psi)(y) & 0 \\ \cdots & \mathrm{D}\,(\varphi^{-1} \circ \psi)(y) \end{bmatrix},$$

which obviously has positive determinant $\det \mathrm{D}\left(\tilde{\varphi}^{-1} \circ \tilde{\psi}\right)(y, b) = \left[\det \mathrm{D}\,(\varphi^{-1} \circ \psi)(y)\right]^2 > 0$.

So by definition, TM is orientable.

■

3 Orientability of Regular Surface

Proof.

(a)

(b)

■

4 Embeddings of $\mathbb{RP}^2$ in $\mathbb{R}^4$

Proof.

(a) First note that as φ is even, $\tilde{\varphi}$ is well-defined. Consider the following charts:

$$\pi_3 : B^2(0,1) \ni (x,y) \mapsto (x, y, \sqrt{1-x^2-y^2}) \in \mathbb{S}^2 \cap [\mathbb{R} \times \mathbb{R} \times (0,\infty)] \subset \mathbb{R}^3$$
$$\pi_2 : B^2(0,1) \ni (x,z) \mapsto (x, \sqrt{1-x^2-z^2}, z) \in \mathbb{S}^2 \cap [\mathbb{R} \times (0,\infty) \times \mathbb{R}] \subset \mathbb{R}^3$$
$$\pi_1 : B^2(0,1) \ni (y,z) \mapsto (\sqrt{1-y^2-z^2}, y, z) \in \mathbb{S}^2 \cap [(0,\infty) \times \mathbb{R} \times \mathbb{R}] \subset \mathbb{R}^3$$

It is clear that $\cup_{i=1}^3 q \circ \pi_i(B^2(0,1)) = \mathbb{RP}^2$, where $q : \mathbb{S}^2 \to \mathbb{RP}^2$ is the usual projection. So to show that $\tilde{\varphi}$ is an immersion, we only need to verify that $f_i := \phi \circ \pi_i$ is differentiable with df_i being injective for each i.

For notational ease, let g^σ denote the component functions of $g : U \to \mathbb{R}^n$ in the index vector $\sigma \in [n]^k$. Then,

$$f_3^{(1,2,3)}(x,y) = (x^2 - y^2, xy, xz), z = \sqrt{1-x^2-y^2} > 0$$
$$df_3^{(1,2,3)}(x,y) = \begin{bmatrix} 2x & -2y \\ y & x \\ z - x^2/z & -xy/z \end{bmatrix}$$
$$\det df_3^{(1,2)}(x,y) = 2(x^2+y^2) = 0 \iff x = y = 0, z = 1. \text{ Thus } df_3 \text{ has full rank}$$
$$f_2^{(1,2,3)}(x,z) = (2x^2 + z^2 - 1, xy, xz), y = \sqrt{1-x^2-z^2} > 0$$
$$df_2^{(1,2,3)}(x,z) = \begin{bmatrix} 4x & 2z \\ y - x^2/y & -xz/y \\ z & x \end{bmatrix}$$
$$\text{on } \{z=0\}, \det df_2^{(1,3)}(x,z) = 2(x^2 - z^2) = 0 \iff x = 0, y = 1. \text{ Thus } df_2 \text{ has full rank}$$

The case of f_1 is similar to that of f_2 by symmetry of x and y. By comments above, $\tilde{\varphi}$ is an immersion.

(b) Let $(x,y,z), (a,b,c) \in \mathbb{S}^2$ with $\phi(x,y,z) = \phi(a,b,c)$. We want to show that $(x,y,z) = \pm(a,b,c)$, which is equivalent to φ being injective.

If $x = 0$, then from $xy = ab$, we know that $a = 0$ or $b = 0$. If $b = 0$, then from $x^2 - y^2 = a^2 - b^2$, we have that $a = y = 0$; if $b \neq 0$, then $a = 0$. So either case we always have $a = 0$. By symmetry, we obtain $x = 0 \iff a = 0$. Similarly $y = 0 \iff b = 0$.

If $z = 0$, then from $xz = ac$ and $yz = bc$, we have $c = 0$ or $a = b = 0$. But from above, $a = b = 0 \Rightarrow x = y = 0$, contradicting $(x, y, z) \in \mathbb{S}^2$. So we must have $c = 0$. By symmetry, $z = 0 \iff c = 0$.

Since $(x, y, z) \in \mathbb{S}^2$, $xyz \neq 0$. WLOG, assume that $x \neq 0$ and hence $a \neq 0$. Then from $xy = ab$ and $xz = ac$, we have $y = b(a/x)$ and $z = c(a/x)$. So

$$\begin{aligned} 1 - x^2 = y^2 + z^2 = (b^2 + c^2)(a/x)^2 = (1 - a^2)(a^2/x^2) \Rightarrow x^2(1 - x^2) = a^2(1 - a^2) \\ \Rightarrow x^2 = a^2 \text{ or } x^2 = 1 - a^2 \end{aligned}$$

If $x^2 = a^2$, then it is immediate that $(x, y, z) = \pm(a, b, c)$. If $x^2 = 1 - a^2$, then from $x^2 - y^2 = a^2 - b^2$ we have

$$\begin{aligned} 2a^2 - b^2 = 1 - y^2 = 1 - b^2(a^2/x^2) = 1 - b^2[a^2/(1 - a^2))] \Rightarrow (2a^2 - 1)(1 + b^2/(1 - a^2)) = 0 \\ \Rightarrow 2a^2 - 1 = 0 \Rightarrow x^2 = a^2 \end{aligned}$$

Thus, we always have $(x, y, z) = \pm(a, b, c)$, i.e $\tilde{\varphi}$ is injective.

As $\mathbb{RP}^2$ is compact, and that $\tilde{\varphi}$ is countinous and differentiable, with $d\tilde{\varphi}$ injective, $\tilde{\varphi} : \mathbb{RP}^2 \to \tilde{\varphi}(\mathbb{RP}^2) \subset \mathbb{R}^4$ is then a diffeomorphism, i.e. $\mathbb{RP}^2$ embeds into $\mathbb{R}^4$ via $\tilde{\varphi}$.

■

Chapter 1

RIEMANNIAN METRICS

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EXERCISES

Chapter 2

AFFINE CONNECTIONS; RIEMANNIAN CONNECTIONS

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EXERCISES

Chapter 3

GEODESICS; CONVEX NEIGHBORHOODS

NOTES

EXERCISES

Chapter 4

CURVATURE

NOTES

EXERCISES

Chapter 5

JACOBI FIELDS

NOTES

EXERCISES

5 Sectional Curvature and Circumference of Normal Circle

Proof. Parametrize $S_r(p) = \exp_p(S_r(0)) \subseteq M$ by $f(r,s) := \exp_p(rv(s))$, where $|v| \equiv 1$ and $|v'| \equiv 1$. It then follows that $\langle v, v' \rangle \equiv 0$ in T_pM.

Note that

$$L_r = \int_0^{2\pi} \left| \frac{\partial f}{\partial s}(r,s) \right| \mathrm{d}\, s = \int_0^{2\pi} \left| J_{(v,v')(s)}(r) \right| \mathrm{d}\, s,$$

where $J_{(v,v')(s)}$ is the Jacobi field along the curve $r \mapsto f(r,s)$ with initial conditions $J_{(v,v')(s)}(0) = 0$ and $J'_{(v,v')(s)}(0) = v'(s)$.

Thus by a theorem, we know that $\left| J_{(v,v')(s)}(r) \right| = r - \frac{1}{6} K_p\left(v(s), v'(s)\right) r^3 + o(r^3)$. But $\dim M = 2$, so $\forall s \in [0, 2\pi], \operatorname{span}\{v(s), v'(s)\} = \operatorname{span}\{v(0), v'(0)\} = T_pM$, and hence

$$L_r = 2\pi \left(r - \frac{1}{6} K(p) r^3 + o(r^3) \right) \Rightarrow K(p) = \frac{2\pi r - L_r}{\frac{\pi}{3} r^3} + \frac{o(r^3)}{r^3} \Rightarrow K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3}$$

■

Chapter 6

ISOMETRIC IMMERSIONS

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Chapter 7

COMPLETE MANIFOLDS; HOPF-RINOW AND HADAMARD THEOREMS

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Chapter 8

SPACES OF CONSTANT CURVATURE

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THE RAUCH COMPARISON THEOREM

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Chapter 12

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Chapter 14

ILLUSTRATION

14.1 Introduction

The front matter has various entries such as

`\title`, `\author`, `\date`, and `\thanks`

You should replace their arguments with your own.

This text is the body of your article. You may delete everything between the commands

`\begin{document}` ... `\end{document}`

in this file to start with a blank document.

14.2 The Most Important Features

Sectioning commands. The first one is the

`\section{The Most Important Features}`

command. Below you shall find examples for further sectioning commands:

1 Subsection

Subsection text.

Subsubsection

Subsubsection text.

Paragraph Paragraph text.

Subparagraph Subparagraph text.

Select a part of the text then click on the button Emphasize (H!), or Bold (Fs), or Italic (Kt), or Slanted (Kt) to typeset *Emphasize*, **Bold**, *Italics*, *Slanted* texts.

You can also typeset Roman, Sans Serif, Small Caps, and `Typewriter` texts.

You can also apply the special, mathematics only commands $\mathbb{BLACKBOARD}$ $\mathbb{BOLD}$, $\mathcal{CALLIGRAPHIC}$, and $\mathfrak{fraktur}$. Note that blackboard bold and calligraphic are correct only when applied to uppercase letters A through Z.

You can apply the size tags – Format menu, Font size submenu – tiny, scriptsize, footnotesize, small, normalsize, large, Large, LARGE, huge and Huge.

You can use the `\begin{quote} etc. \end{quote}` environment for typesetting short quotations. Select the text then click on Insert, Quotations, Short Quotations:

> The buck stops here. *Harry Truman*
>
> Ask not what your country can do for you; ask what you can do for your country. *John F Kennedy*
>
> I am not a crook. *Richard Nixon*
>
> I did not have sexual relations with that woman, Miss Lewinsky. *Bill Clinton*

The Quotation environment is used for quotations of more than one paragraph. Following is the beginning of *The Jungle Books* by Rudyard Kipling. (You should select the text first then click on Insert, Quotations, Quotation):

> It was seven o'clock of a very warm evening in the Seeonee Hills when Father Wolf woke up from his day's rest, scratched himself, yawned and spread out his paws one after the other to get rid of sleepy feeling in their tips. Mother Wolf lay with her big gray nose dropped across her four tumbling, squealing cubs, and the moon shone into the mouth of the cave where they all lived. "*Augrh*" said Father Wolf, "it is time to hunt again." And he was going to spring down hill when a little shadow with a bushy tail crossed the threshold and whined: "Good luck go with you, O Chief of the Wolves; and good luck and strong white teeth go with the noble children, that they may never forget the hungry in this world."
>
> It was the jackal—Tabaqui the Dish-licker—and the wolves of India despise Tabaqui because he runs about making mischief, and telling tales, and eating rags and pieces of leather from the village rubbish-heaps. But they are afraid of him too, because Tabaqui, more than any one else in the jungle, is apt to go mad, and then he forgets that he was afraid of anyone, and runs through the forest biting everything in his way.

Use the Verbatim environment if you want LaTeX to preserve spacing, perhaps when including a fragment from a program such as:

```
#include <iostream>          // < > is used for standard libraries.
void main(void)              // ''main'' method always called first.
{
 cout << ''This is a message.'';
                             // Send to output stream.
}
```

(After selecting the text click on Insert, Code Environments, Code.)

2 Mathematics and Text

It holds [1] the following

Theorem 1 *(The Currant minimax principle.) Let T be completely continuous selfadjoint operator in a Hilbert space H. Let n be an arbitrary integer and let $u_1, \ldots, u_{n-1}$ be an arbitrary system of $n-1$ linearly independent elements of H. Denote*

$$\max_{\substack{v \in H, v \neq 0 \\ (v,u_1)=0,\ldots,(v,u_n)=0}} \frac{(Tv,v)}{(v,v)} = m(u_1, \ldots, u_{n-1}) \tag{14.1}$$

Then the n-th eigenvalue of T is equal to the minimum of these maxima, when minimizing over all linearly independent systems $u_1, \ldots u_{n-1}$ in H,

$$\mu_n = \min_{u_1,\ldots,u_{n-1} \in H} m(u_1, \ldots, u_{n-1}) \tag{14.2}$$

The above equations are automatically numbered as equation (14.1) and (14.2).

3 List Environments

You can create numbered, bulleted, and description lists using the tag popup at the bottom left of the screen.

1. List item 1

2. List item 2

    (a) A list item under a list item.

The typeset style for this level is different than the screen style. The screen shows a lower case alphabetic character followed by a period while the typeset style uses a lower case alphabetic character surrounded by parentheses.

(b) Just another list item under a list item.

    i. Third level list item under a list item.

        A. Fourth and final level of list items allowed.

- Bullet item 1
- Bullet item 2
    - Second level bullet item.
        - Third level bullet item.
            - Fourth (and final) level bullet item.

Description List Each description list item has a term followed by the description of that term. Double click the term box to enter the term, or to change it.

Bunyip Mythical beast of Australian Aboriginal legends.

4 Theorem-like Environments

The following theorem-like environments (in alphabetical order) are available in this style.

Acknowledgement 2 *This is an acknowledgement*

Algorithm 3 *This is an algorithm*

Axiom 4 *This is an axiom*

Case 5 *This is a case*

Claim 6 *This is a claim*

Conclusion 7 *This is a conclusion*

Condition 8 *This is a condition*

Conjecture 9 *This is a conjecture*

Corollary 10 *This is a corollary*

Criterion 11 *This is a criterion*

Definition 12 *This is a definition*

Example 13 *This is an example*

Exercise 14 *This is an exercise*

Lemma 15 *This is a lemma*

Proof. This is the proof of the lemma. ■

Notation 16 *This is notation*

Problem 17 *This is a problem*

Proposition 18 *This is a proposition*

Remark 19 *This is a remark*

Solution 20 *This is a solution*

Summary 21 *This is a summary*

Theorem 22 *This is a theorem*

Proof of the Main Theorem. This is the proof. ■

This text is a sample for a short bibliography. You can cite a book by making use of the command `\cite{KarelRektorys}`: [1]. Papers can be cited similarly: [2]. If you want multiple citations to appear in a single set of square brackets you must type all of the citation keys inside a single citation, separating each with a comma. Here is an example: [2, 3, 4].

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.1 The First Appendix

The appendix fragment is used only once. Subsequent appendices can be created using the Section Section/Body Tag.