Principle Component Analysis

Rayleigh Quotients

Physical meaning of SVD!

Given a symmertric matrix $A \in S^n$,

$$\boxed{\lambda_{\min}(A) \leq \frac{x^T A x}{x^T x} \leq \lambda_{\max}(A), \forall x \neq 0}$$

$$\lambda_{max}(A) = \max_{x:\|x\|_2=1} x^T A x$$

$$\lambda_{min}(A) = \min_{x:\|x\|_2=1} x^T A x$$

The maximum and minimum are attained for $x = u_1$ and for $x = u_n$, respectively, where u_1 and u_n are the largest and smallest eigenvector of A, respectively.

Rayleigh Quotients – Proof:

- Apply the spectral theorem, U is orthogonal, Λ is diagonal

$$x^T A x = x^T U \Lambda U^T x = \bar{x}^T \Lambda \bar{x} = \sum_{i=1}^{n} \lambda_i \bar{x}_i^2$$

- Obviously,

$$\lambda_{\min} \sum_{i=1}^{n} \bar{x}_i^2 \le \sum_{i=1}^{n} \lambda_i \bar{x}_i^2 \le \lambda_{\max} \sum_{i=1}^{n} \bar{x}_i^2$$

- Also, orthogonal matrix U doesn't change the norm of any vector

$$\sum_{i=1}^{n} x_i^2 = x^T x = x^T U U^T x = (U^T x)^T (U^T x) = \bar{x}^T \bar{x} = \sum_{i=1}^{n} \bar{x}_i^2$$

- Combining the above 3 equations,

$$\lambda_{\min} x^T x \le x^T A x \le \lambda_{\max} x^T x$$