Part(a)

Runtime of rhelp(int m) is

the base case $T(1) = \Theta(1)$

general expression: $T(m) = \Theta(1) + T(m-1)$

$= \Theta(1) + [\Theta(1) + T(m-2)]$

$= \Theta(1) + \Theta(1) + T(m-2)$

$= 2\times\Theta(1) + [\Theta(1) + T(m-3)]$

$= 3\times\Theta(1) + T(m-3)$

......

$= \Theta(k) + T(m-k)$

when $m-k=1$, $\Rightarrow k = m-1$

thus $T(m) = \Theta(m-1) + \Theta(1) = \Theta(m)$

Runtime of rfunc(int n, int m) is

the base case $T(0,n) = \Theta(1)$

general expression: $T(n,m) = \Theta(1) + \Theta(n) + T(n-m, m)$

$= \Theta(1) + \Theta(n) + [\Theta(1) + \Theta(n-m) + T(n-2\times m, m)]$

$= \Theta(1)\times 2 + \Theta(n) + \Theta(n-m) + [\Theta(1) + \Theta(n-2m) + T(n-3m, m)]$

$= \Theta(k) + \Theta(n) + \cdots \Theta(n-km+m) + T(n-km, m)$

when $n-km=0 \Rightarrow k = \frac{n}{m}$

thus $T(n,m) = \Theta(\frac{n}{m}) + \Theta(n) + \Theta(n-m) + \cdots + \Theta(m) + T(0,m)$

when $m = \sqrt{n}$,

$T(n,\sqrt{n}) = \Theta(\sqrt{n}) + \Theta(n) + \Theta(n-\sqrt{n}) + \cdots + \Theta(\sqrt{n})$

$= \Theta(\sqrt{n}) + \Theta(\frac{1}{2}(n\sqrt{n}+n))$

$= \Theta(n^{\frac{3}{2}})$

$\boxed{= \Theta(n\sqrt{n})}$

Part (b)

when (x % (int) sqrt(n)) == 0, the runtime of f1 is

$$T(n) = 2\times\Theta(1) + \sum_{i=1}^{n}\left(\sum_{j=0}^{i} \cancel{\Theta} O(1)\right)$$

$$= 2\times\Theta(1) + \sum_{i=1}^{n}\Theta(i)$$

$$= 2\times\Theta(1) + \Theta(n^2)$$

$$= \Theta(n^2)$$

In other cases, the runtime of f1 is

$$T = O(1)$$

According to the code, the initial value of x is n, and after each call of function f1, x is deducted by 1.

When $n=4$, the ~~average~~ amortized runtime of f1 is

$$T = \frac{1}{4}\left[2\times\Theta(n^2) + 2\times\Theta(1)\right]$$

when $n=9$, the ~~average~~ amortized runtime of f1 is

$$T = \frac{1}{9}\left[3\times\Theta(n^2) + 3\times\Theta(1)\right]$$

when $n=k$, the ~~average~~ amortized runtime of f1 is

$$T = \frac{1}{k}\left[\sqrt{k}\,\Theta(k^2) + (k-\sqrt{k})\,\Theta(1)\right]$$

Thus when $x=n$, the amortized runtime of f1 is

$$T = \frac{1}{n}\left[\sqrt{n}\,\Theta(n^2) + (n-\sqrt{n})\,\Theta(1)\right]$$

$$= \frac{1}{n}\Theta(n^2\sqrt{n}) + \frac{1}{n}\Theta(n-\sqrt{n})$$

$$= \Theta(n\sqrt{n})$$