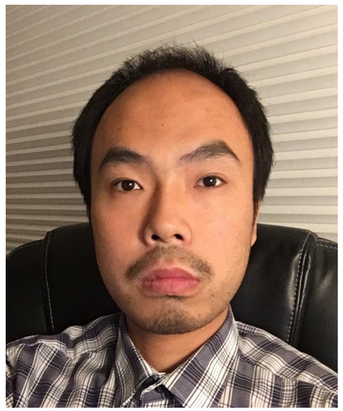
Dynamic Maximum Depth of Geometric Objects

Xiongxin Yang

NYU Shanghai ⟶ HKU

Based on joint work with

Subhash Suri
UCSB



Jie Xue
NYU Shanghai

Jiumu Zhu
NYU Shanghai

The 41st International Symposium on Computational Geometry (SoCG 2025)
Kanazawa, Japan
25 June, 2025

Depth Problem

#objects that contain a point

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MaxDepth

Input: a set of geometric objects $\mathcal{O}$ (in $\mathbb{R}^d$)

Output: a point p^* s.t.

$$\mathrm{dep}(p^*, \mathcal{O}) = \mathrm{dep}(\mathcal{O}) = \max_{p \in \mathbb{R}^d} \mathrm{dep}(p, \mathcal{O})$$

and the value of $\mathrm{dep}(p^*, \mathcal{O})$

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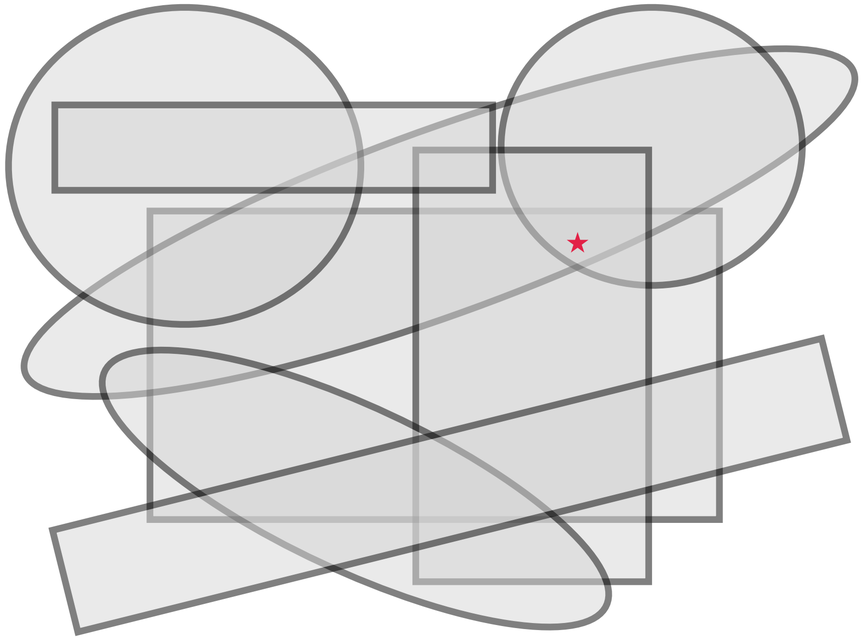

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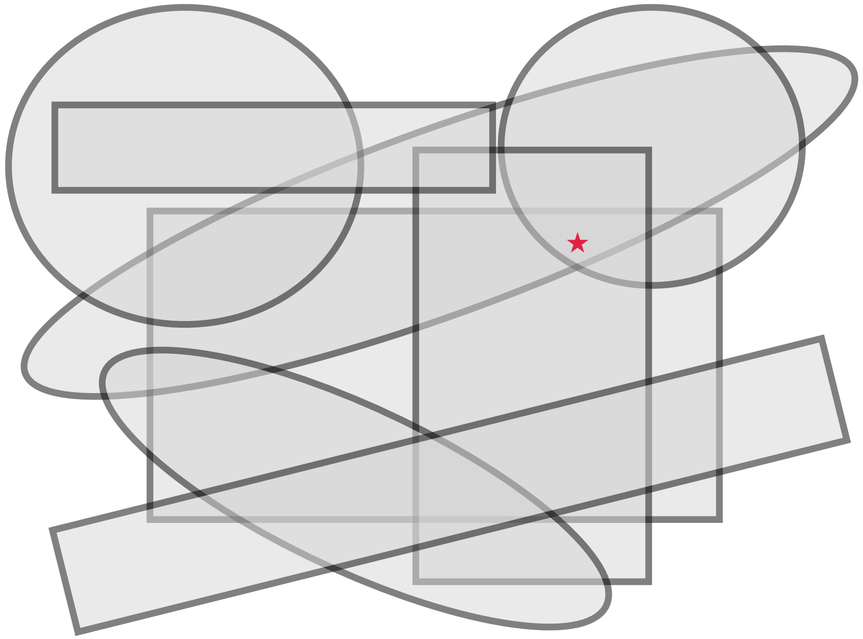

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interval, half-plane, rectangle (box), disk (ball), etc.

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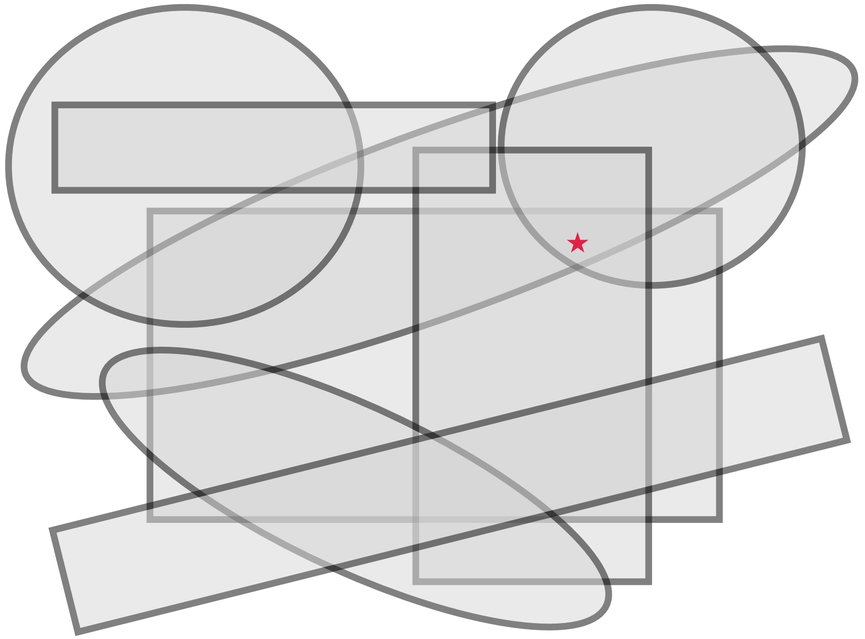

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Dynamic version

Insert or delete an object is allowed
Goal: fast maintain the maximum depth

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Dynamic data structure is an active topic in CG

Dynamic + range search, point location, nearest neighbor, bichomatic closest pair, union of intervals, convex hull, width, geometric connectivity …

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Dynamic data structure for geometric optimization problem

On geometric intersection graph:

- minimum set cover [Agarwal, Chang, Suri, Xiao and Xue 2020; Chan and He 2021; Chan, He, Suri and Xue 2022]
- minimum hitting set [Agarwal, Xie, Yang and Yu 2005; Agarwal, Chang, Suri, Xiao and Xue 2020]
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They are all NP-hard,
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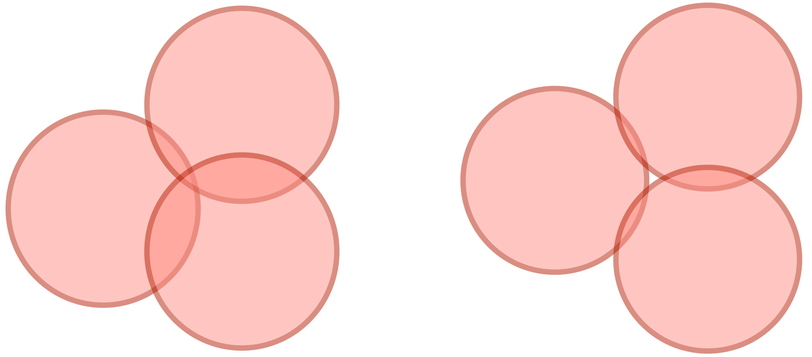
Dynamic Maximum Depth

- Static maximum depth is polynomial time solvable
- Maximum depth is a constant approx. of maximum clique



Known Results for MaxDep

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Static MaxDep Problem

For d-dimensional (axis-parallel) boxes:

- best known running time: $O((n^{d/2}/\log^{d/2} n)(\log\log n)^{O(1)})$ [Chan 2013]
- $\tilde{O}(n^{d/2})$ might be tight [Chan 2010]

For disks: MaxDep is 3SUM-hard [Aronov and Har-Peled 2008]

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Unlikely have sublinear update time for exact depth

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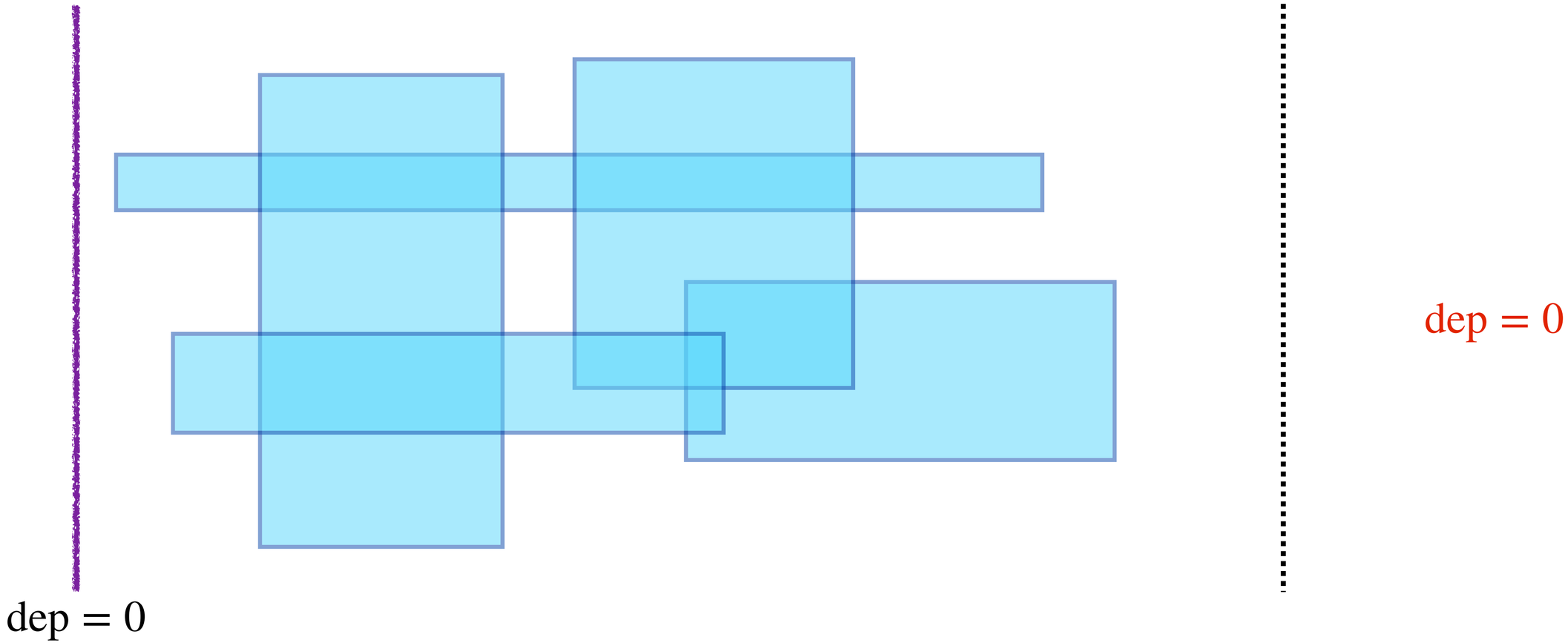

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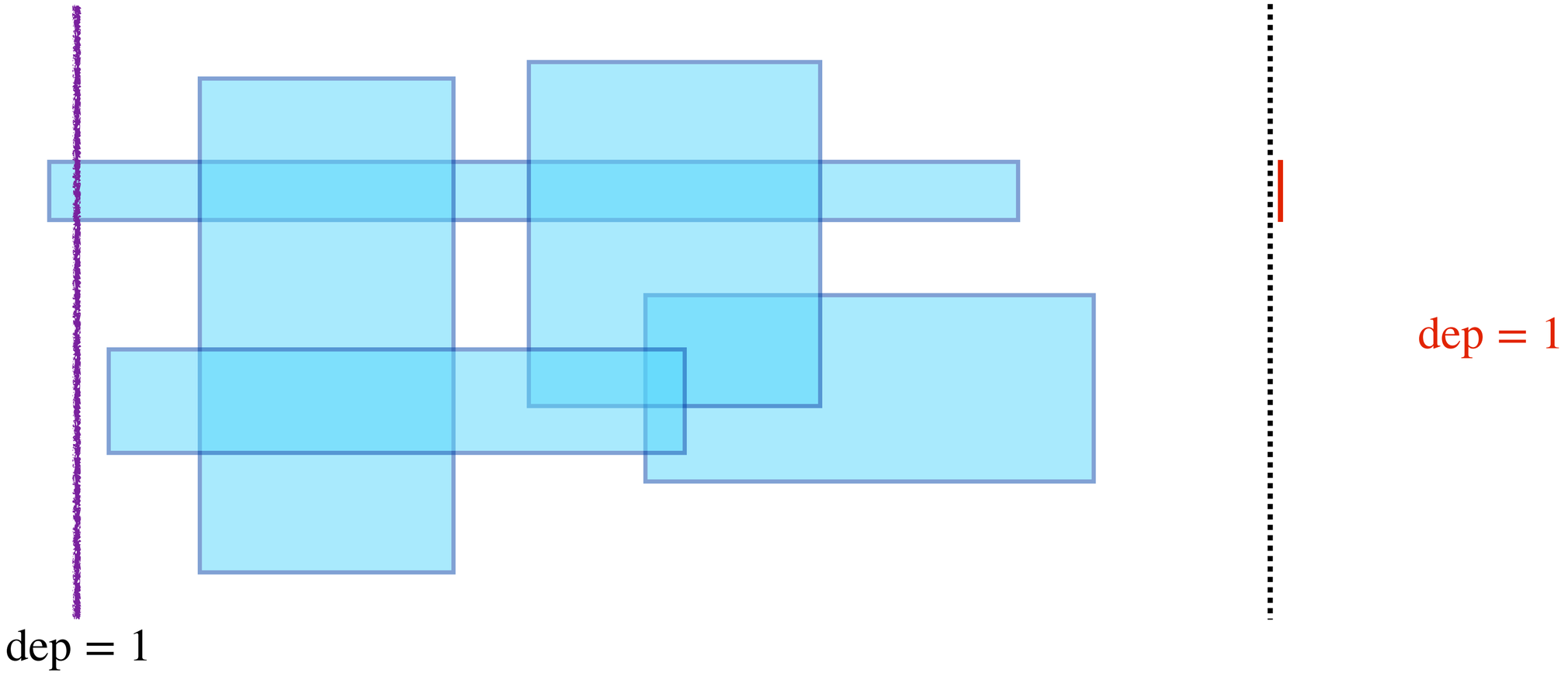

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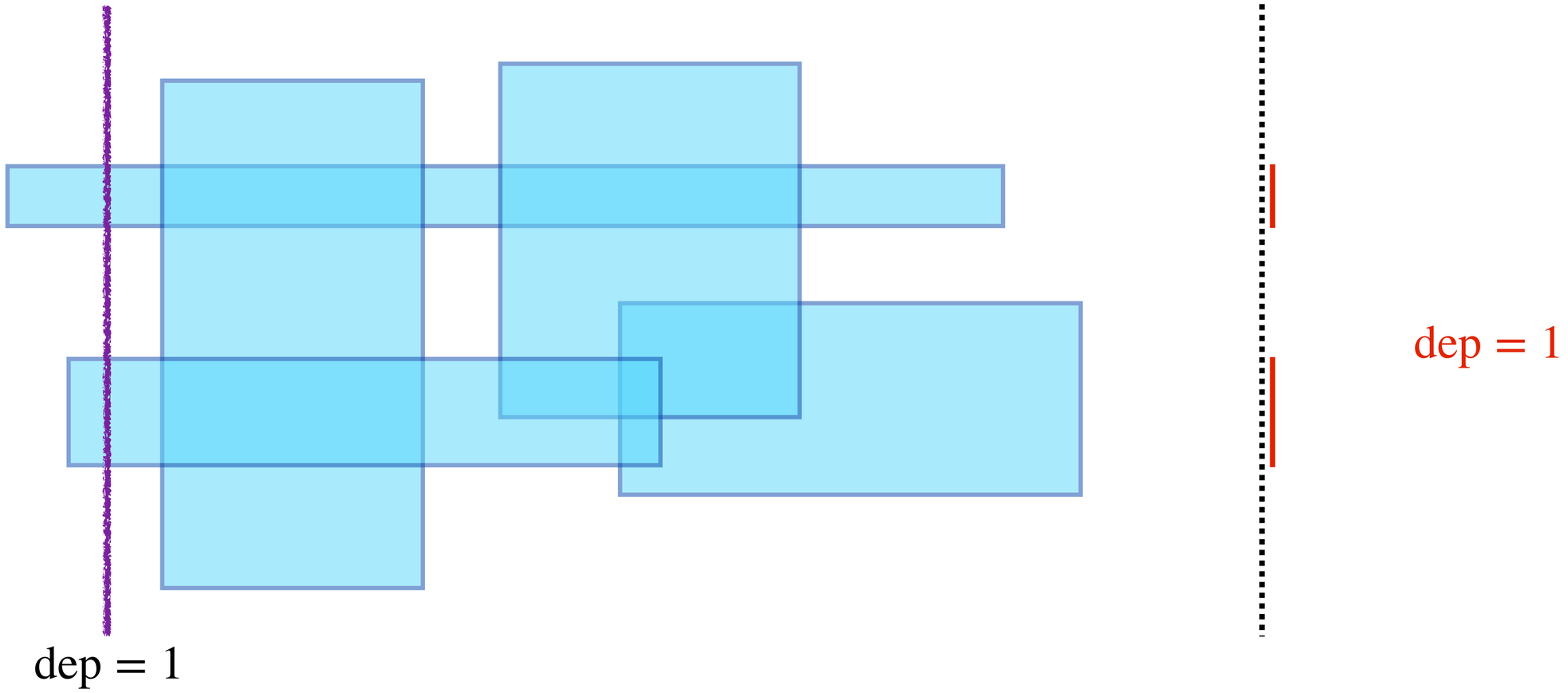

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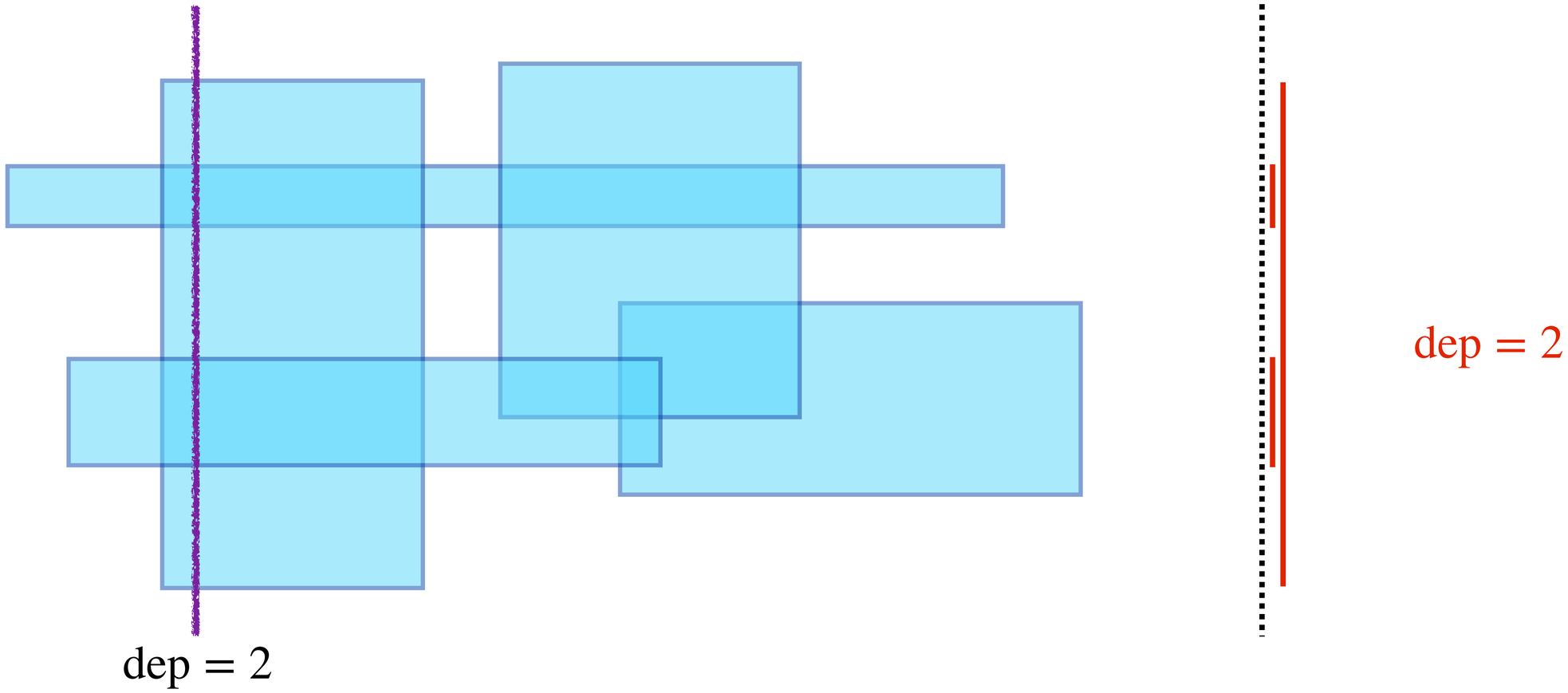

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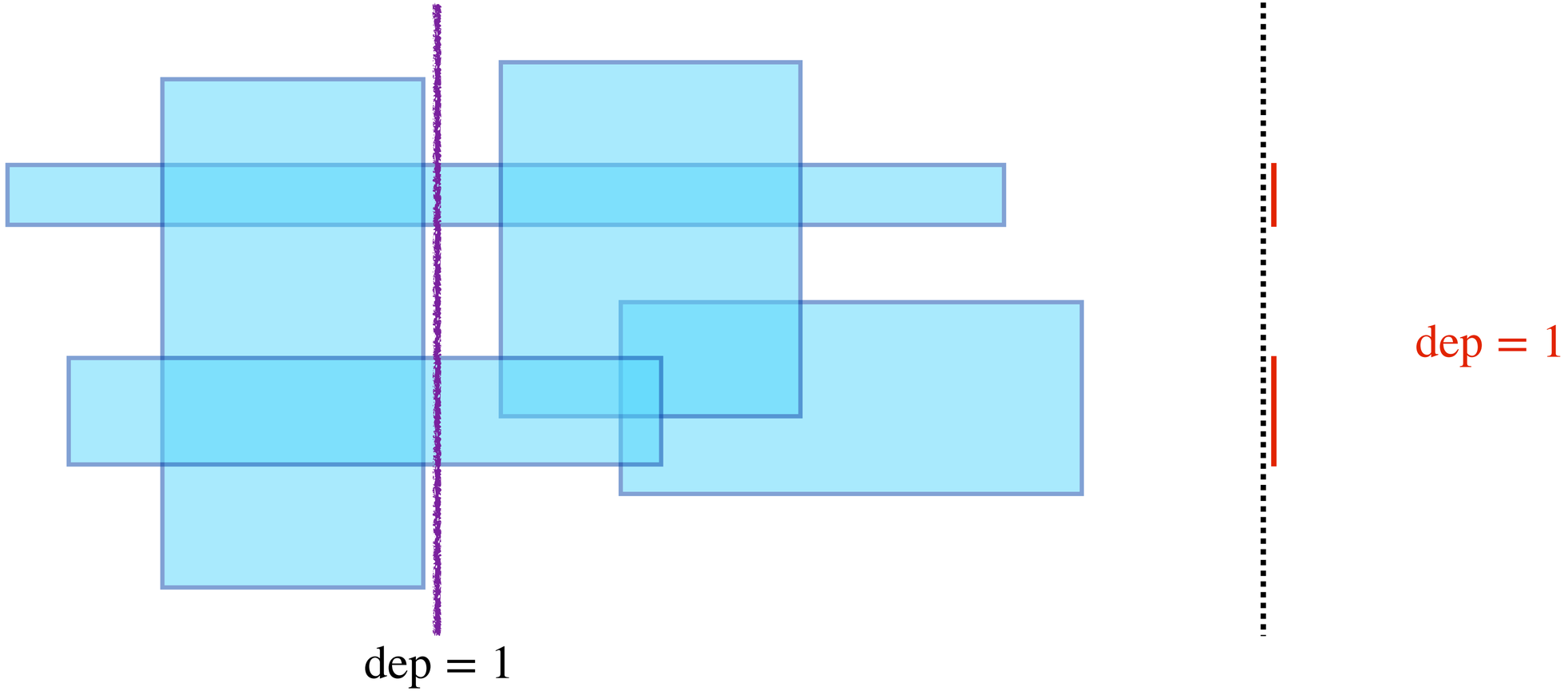

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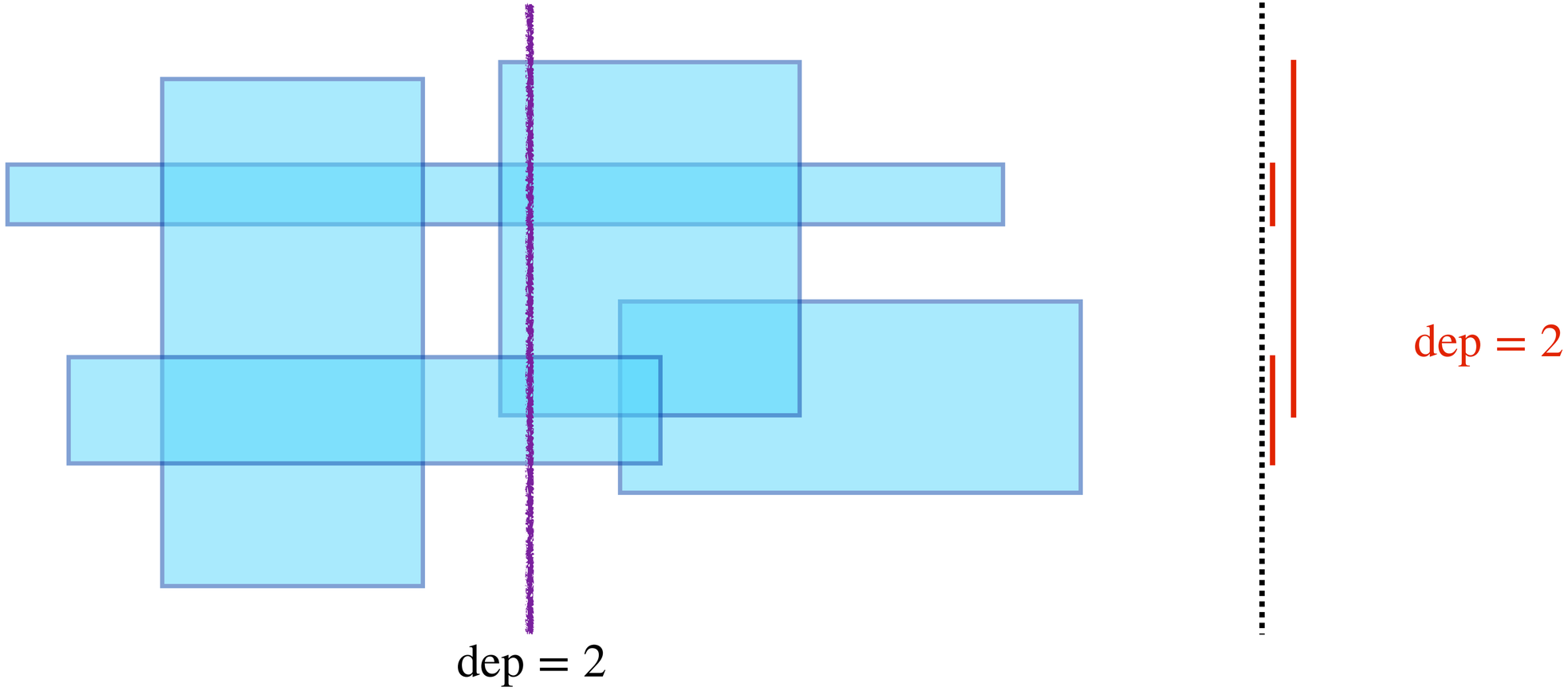

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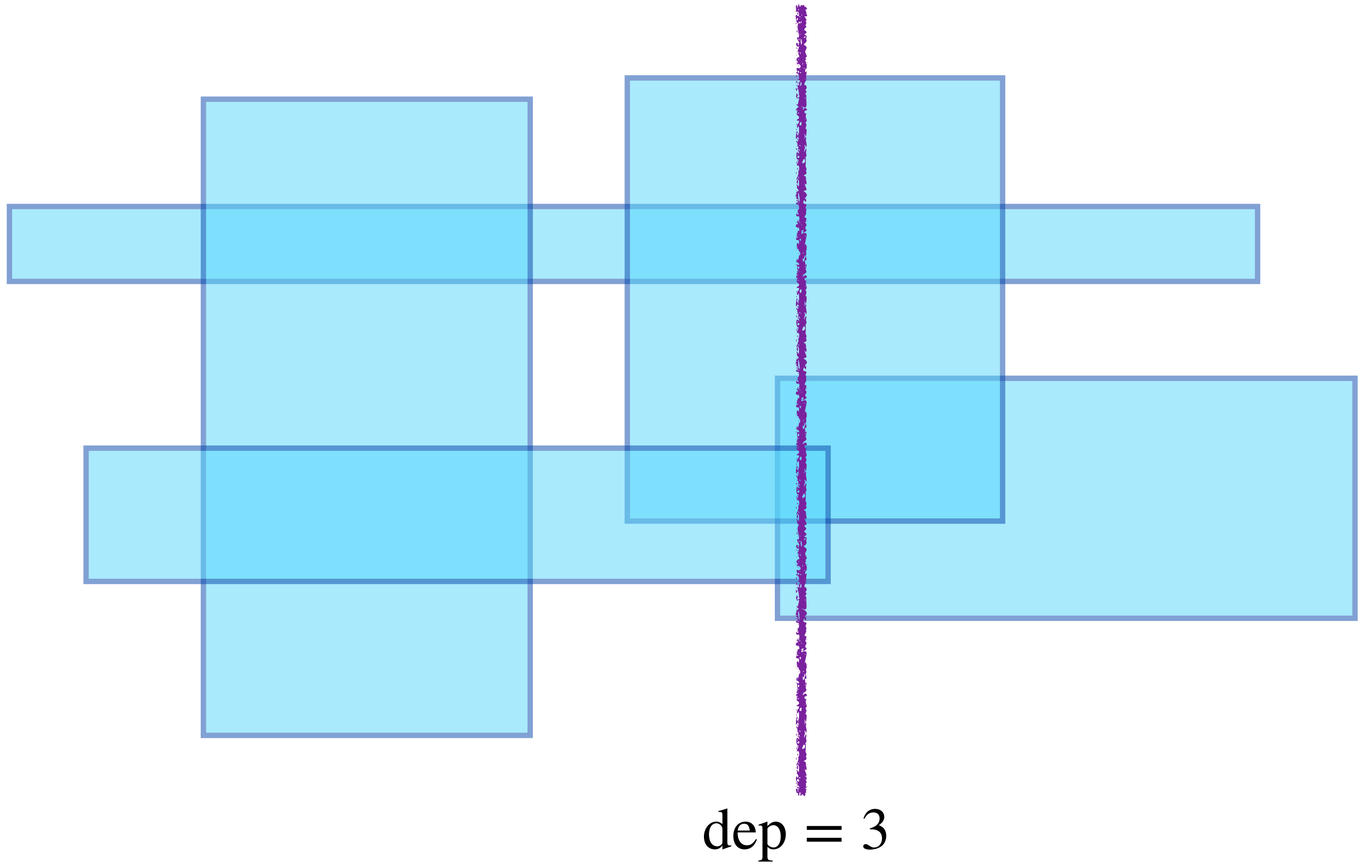

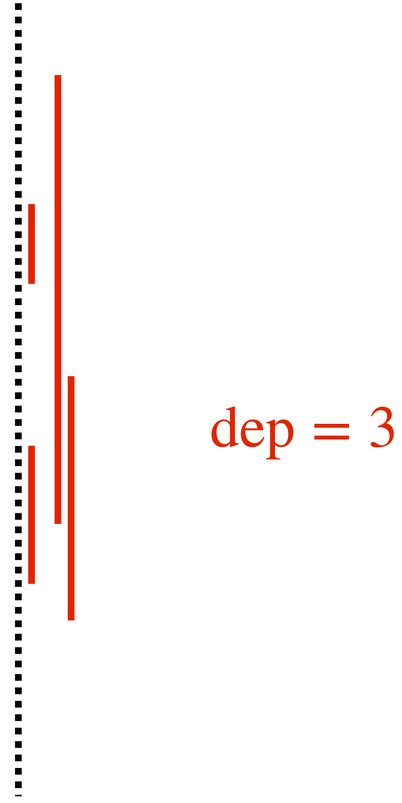

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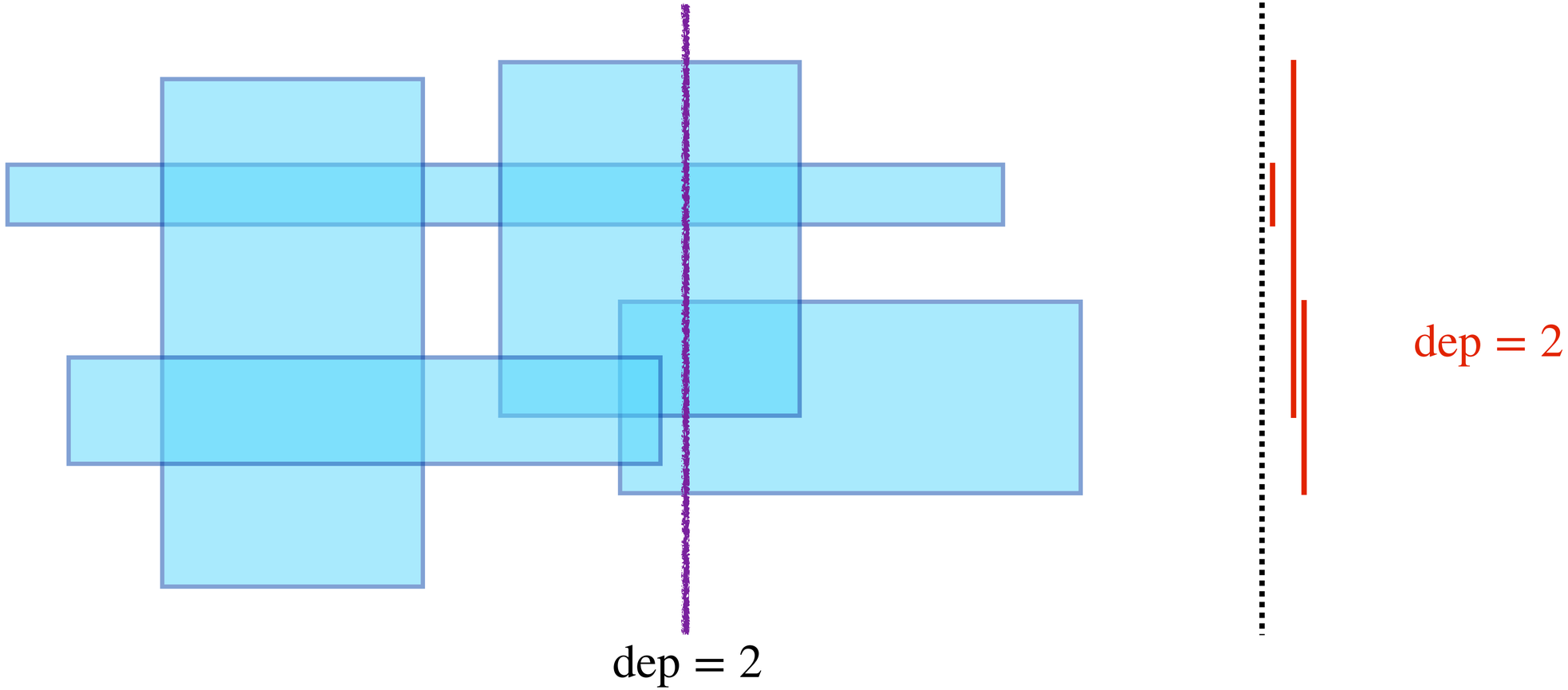

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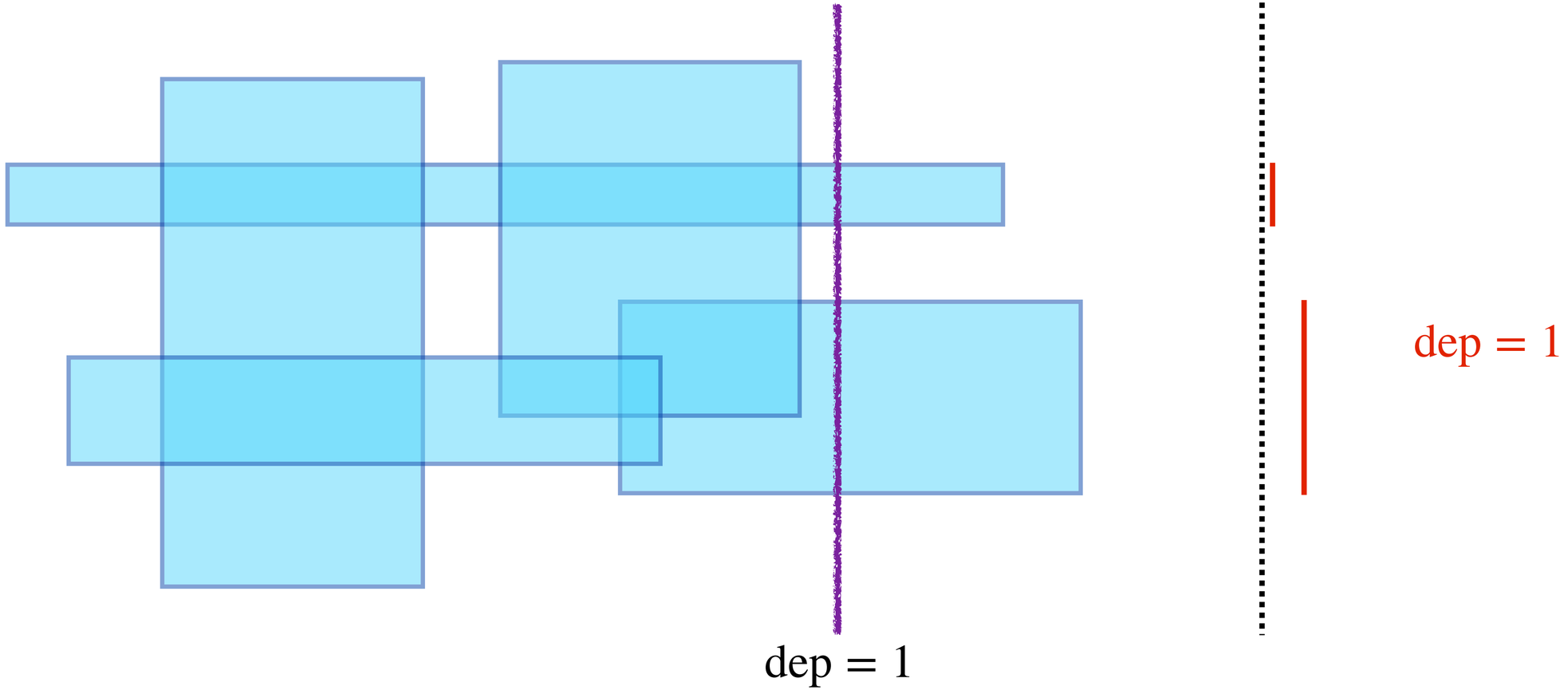

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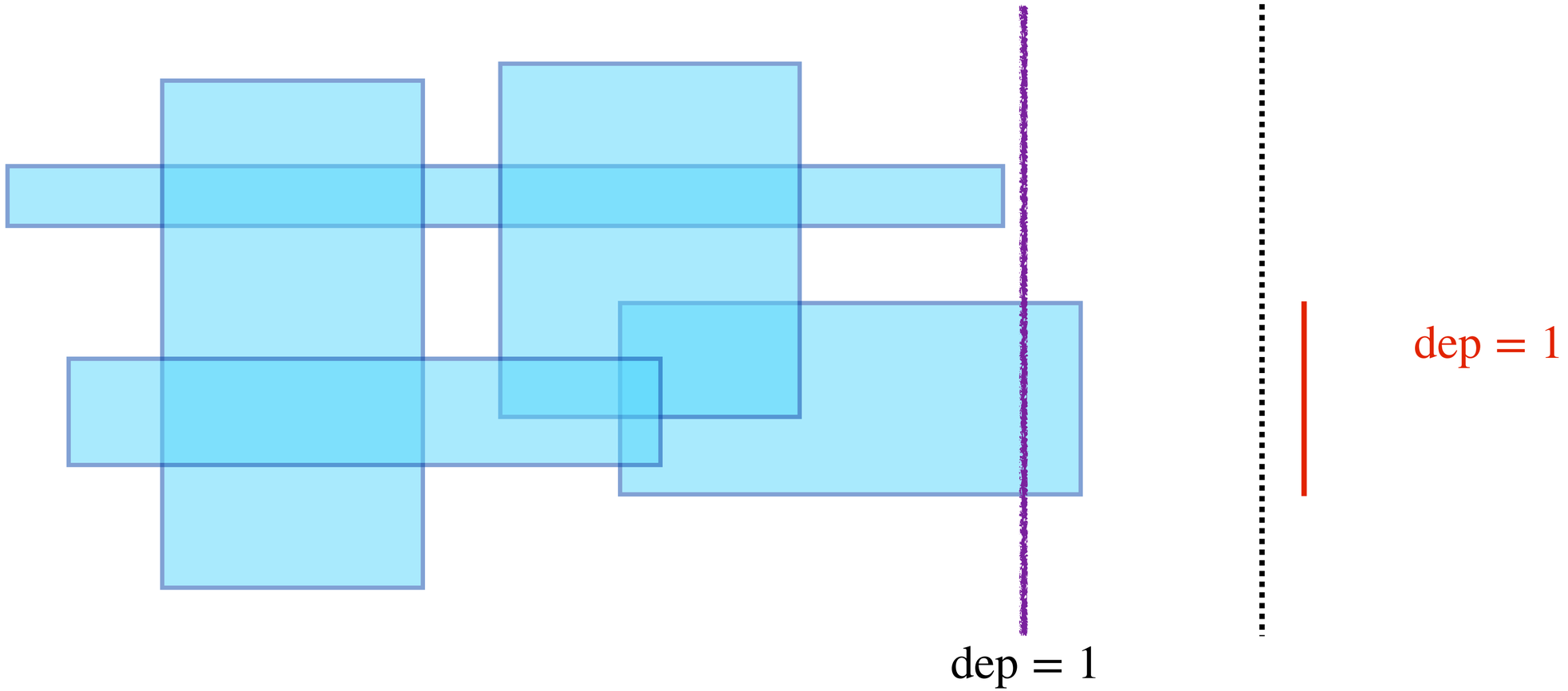

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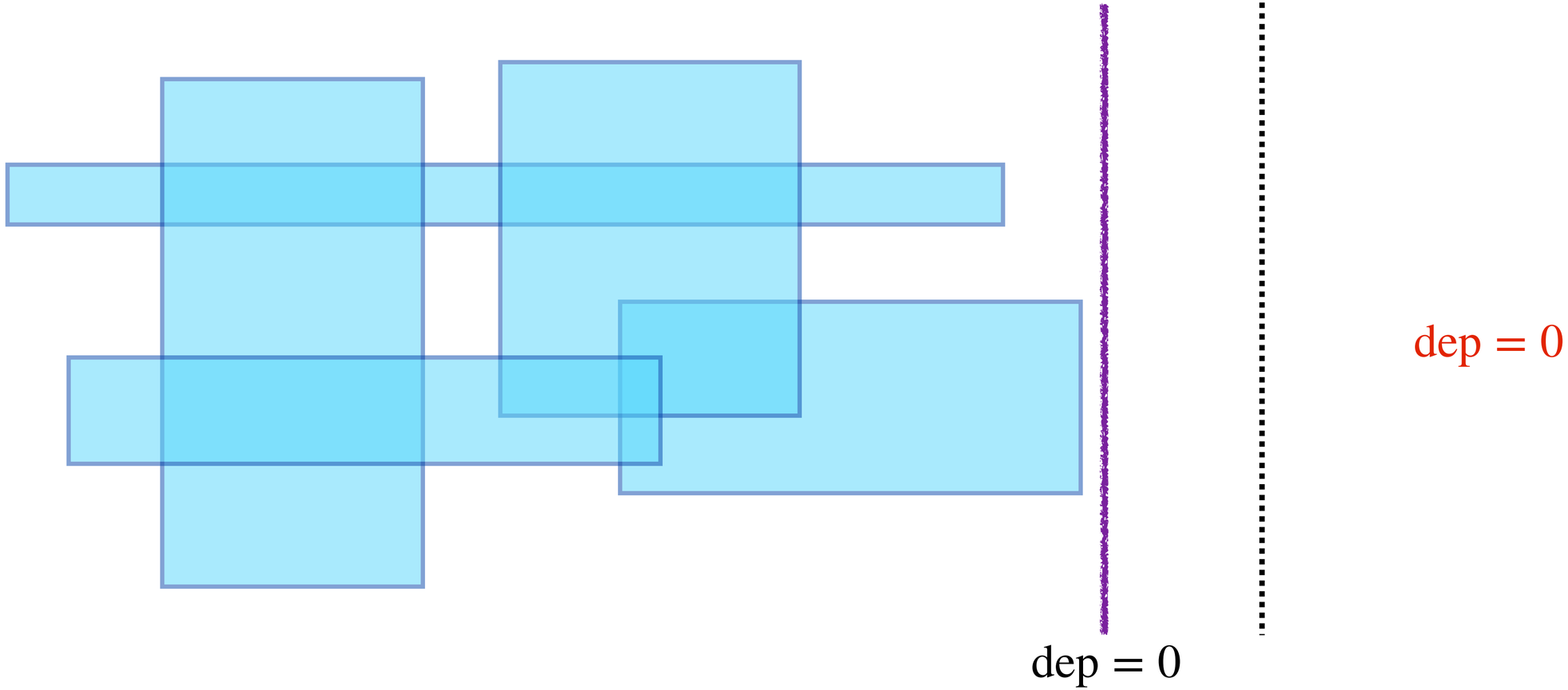

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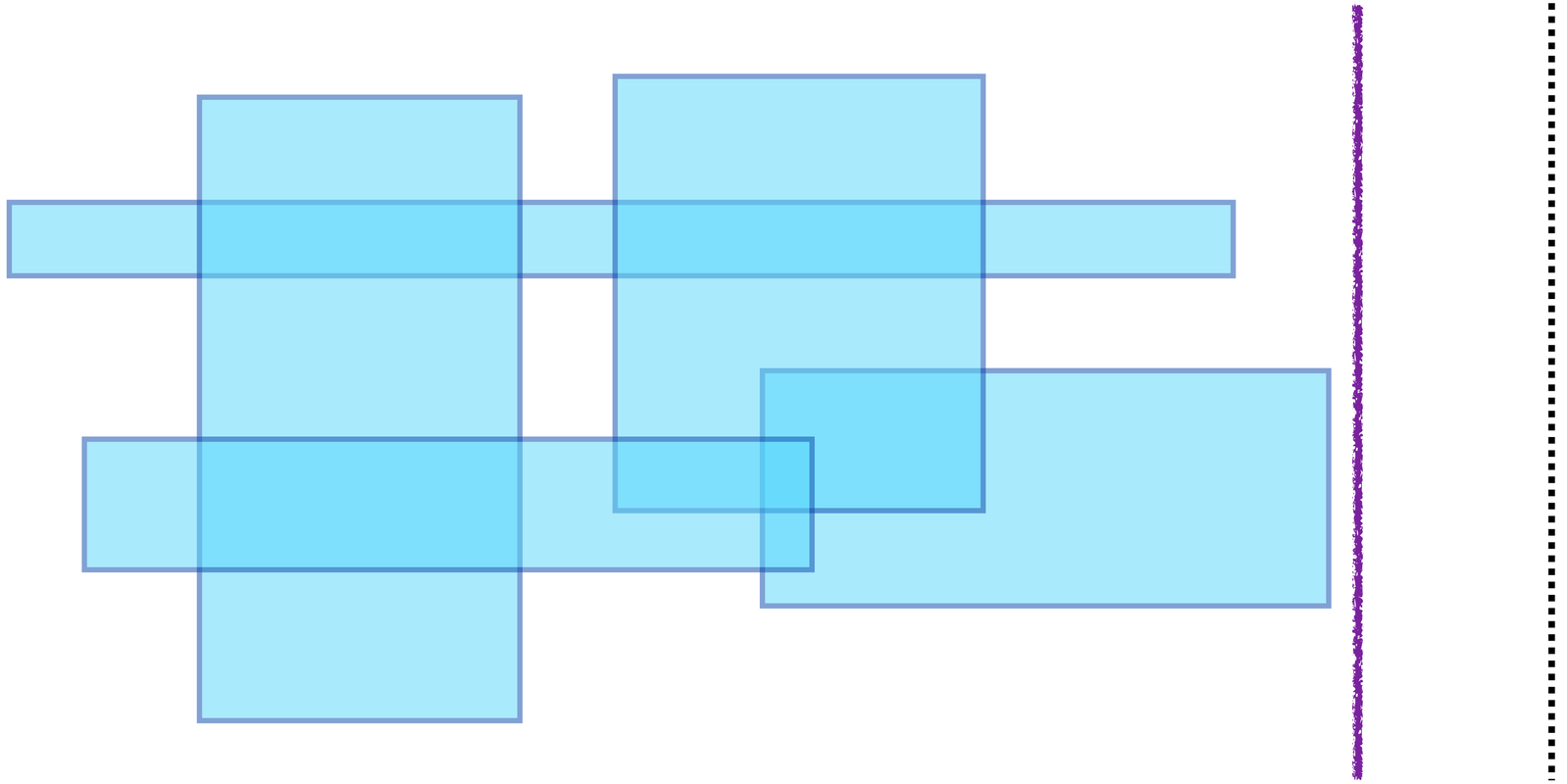

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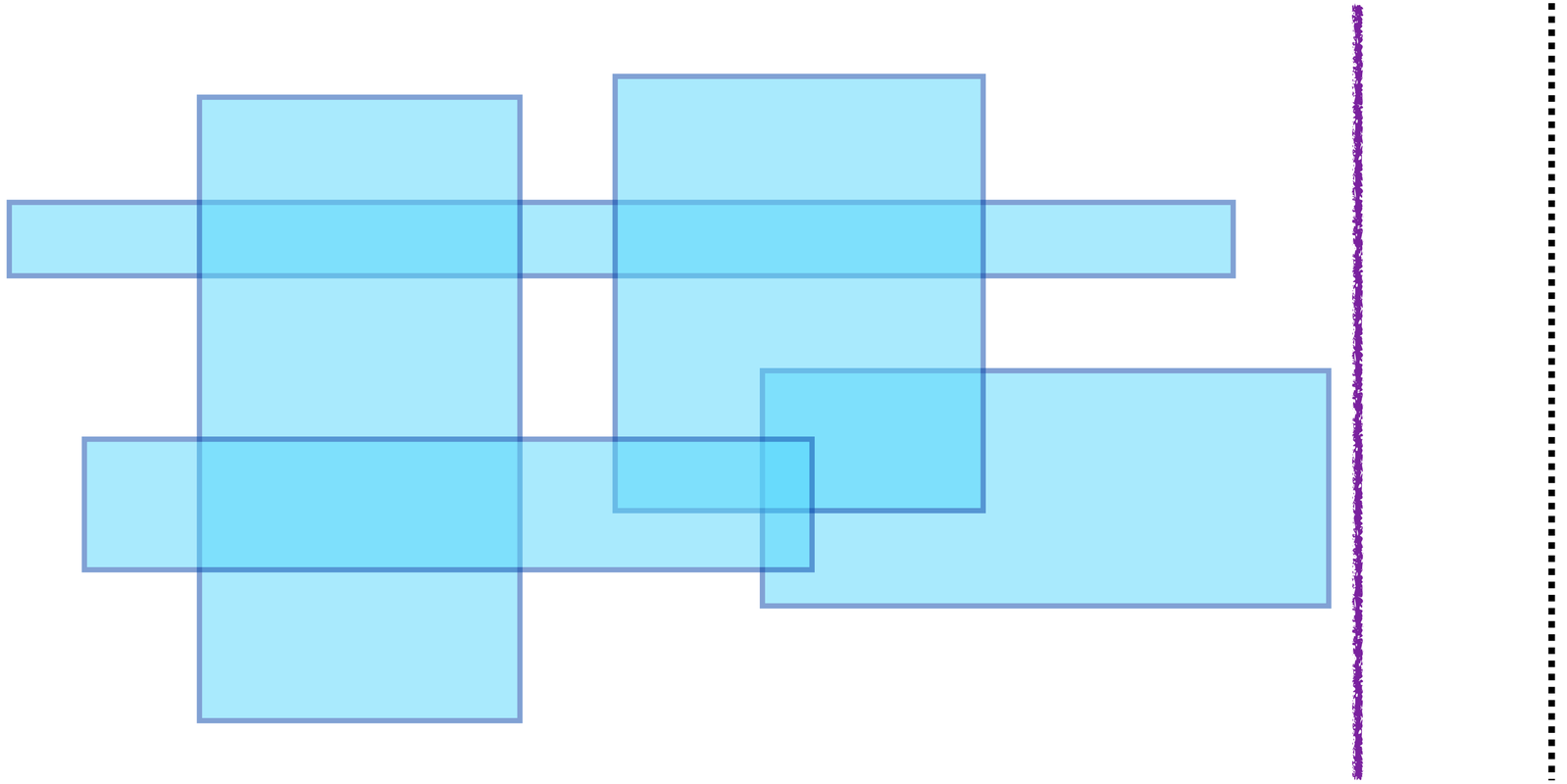

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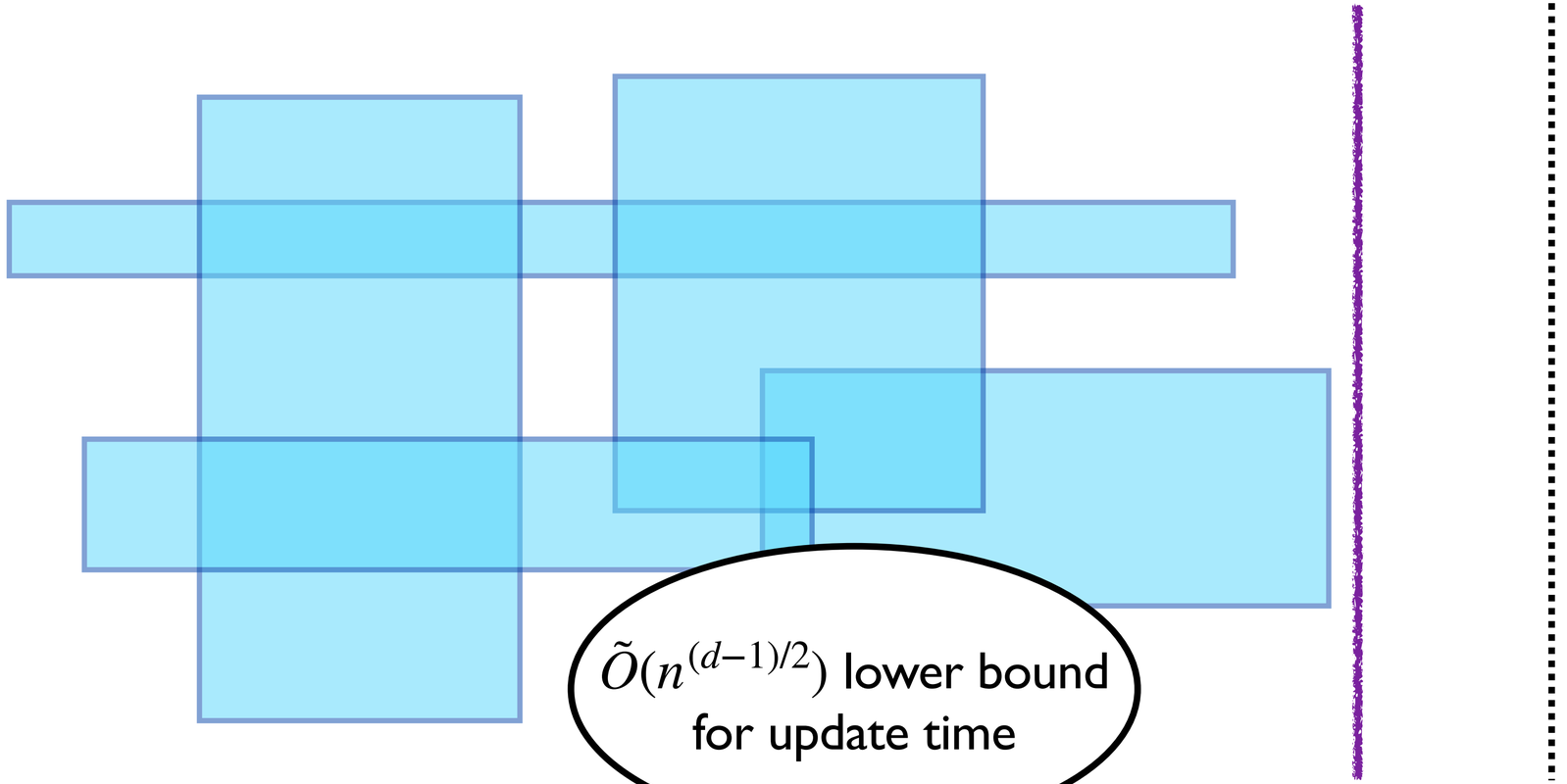

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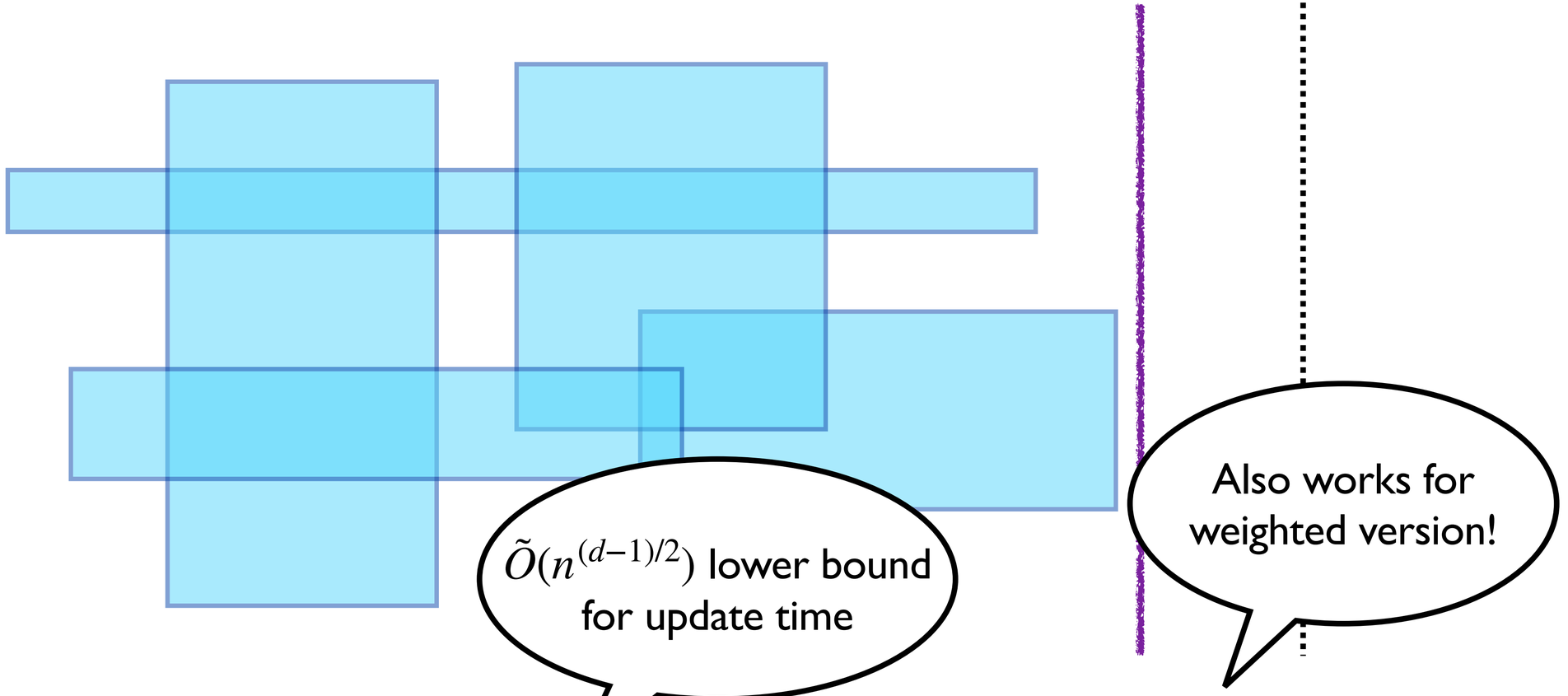

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Partition the plane

- Partition the plane into r slabs
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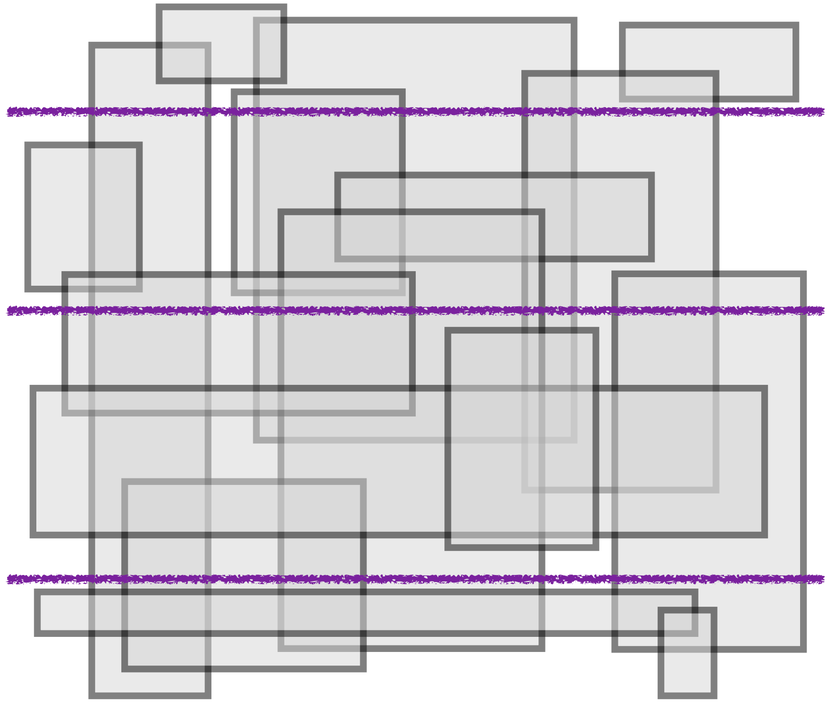

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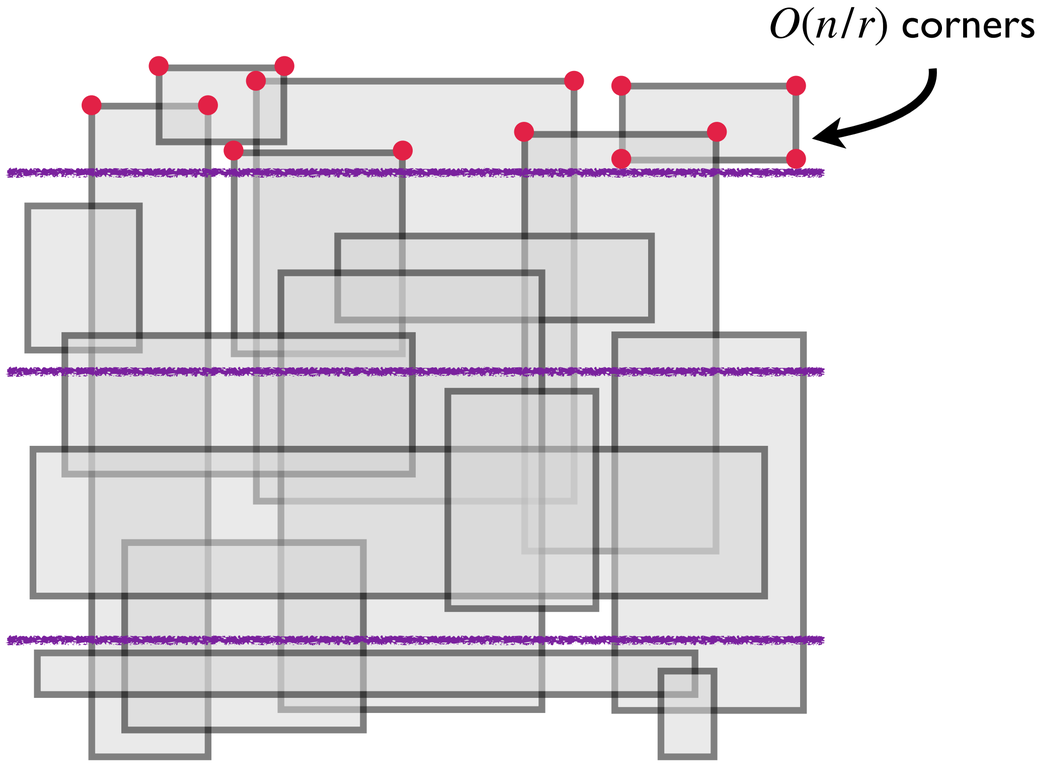

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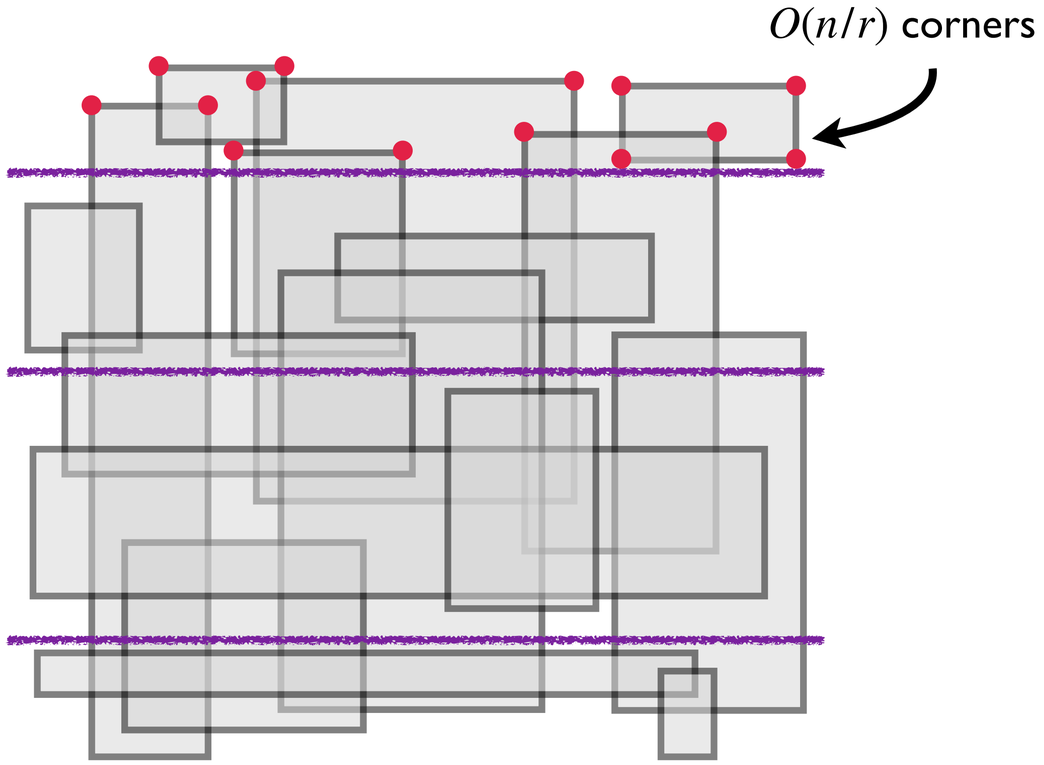
Maintain the max depth in each slab
Pick the largest one



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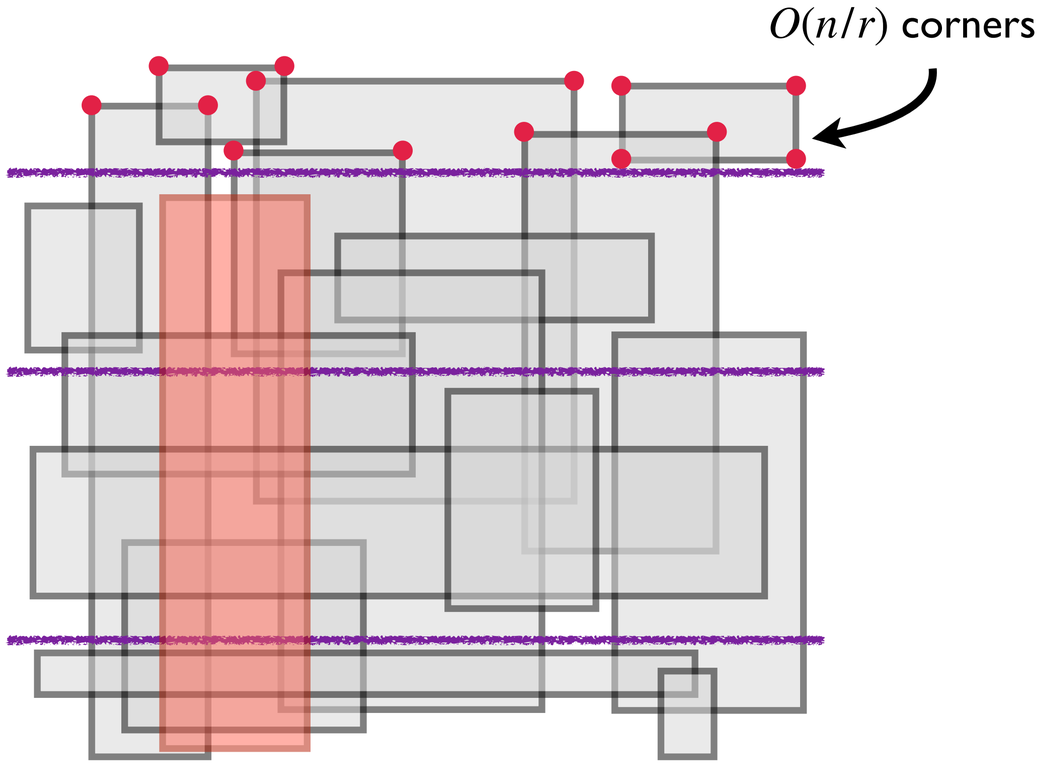

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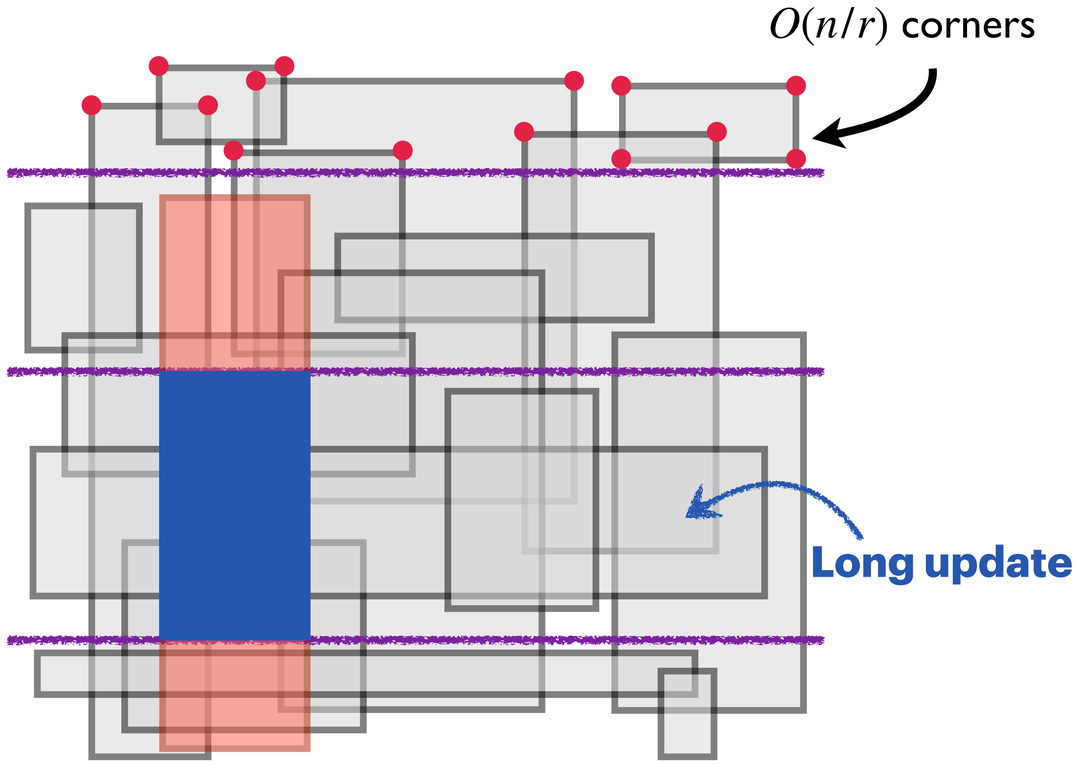

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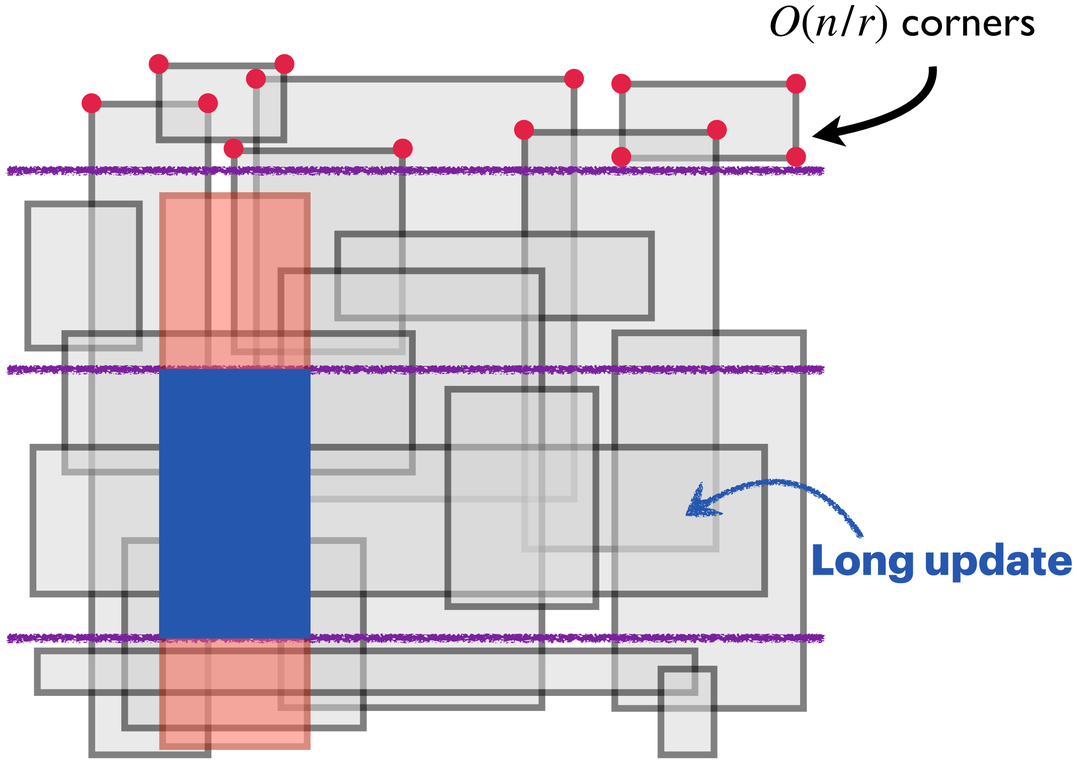

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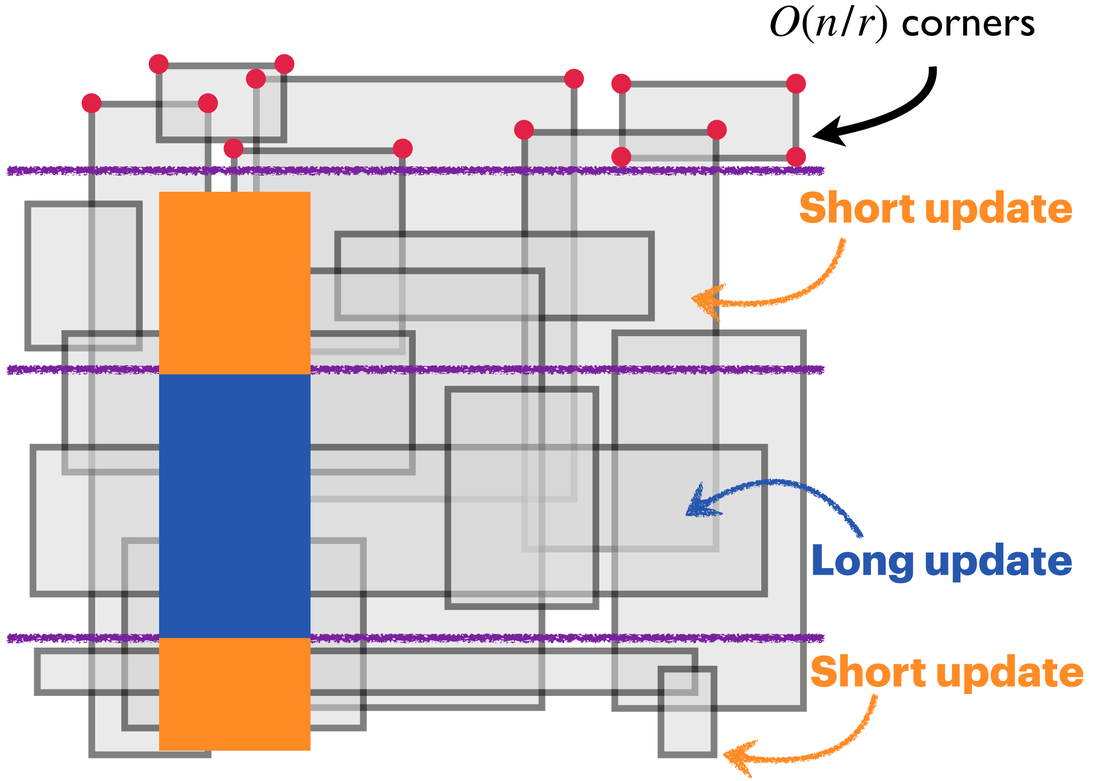
"Short" update



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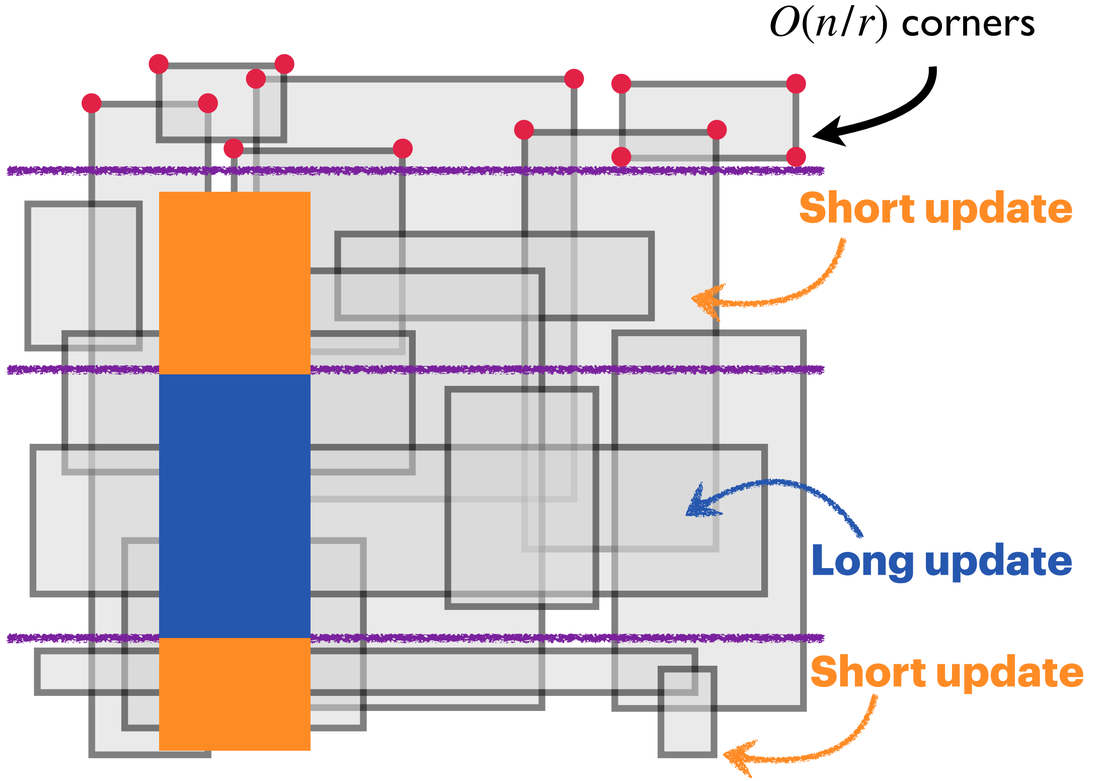
- 😇 easy: $O(\log n)$
- 😈 many: $O(r)$

"Short" update
- 😈 hard: $O((n/r)\log n)$
- 😇 rare: ≤ 2



Rectangle: Optimal Update via Partition

Partition the plane

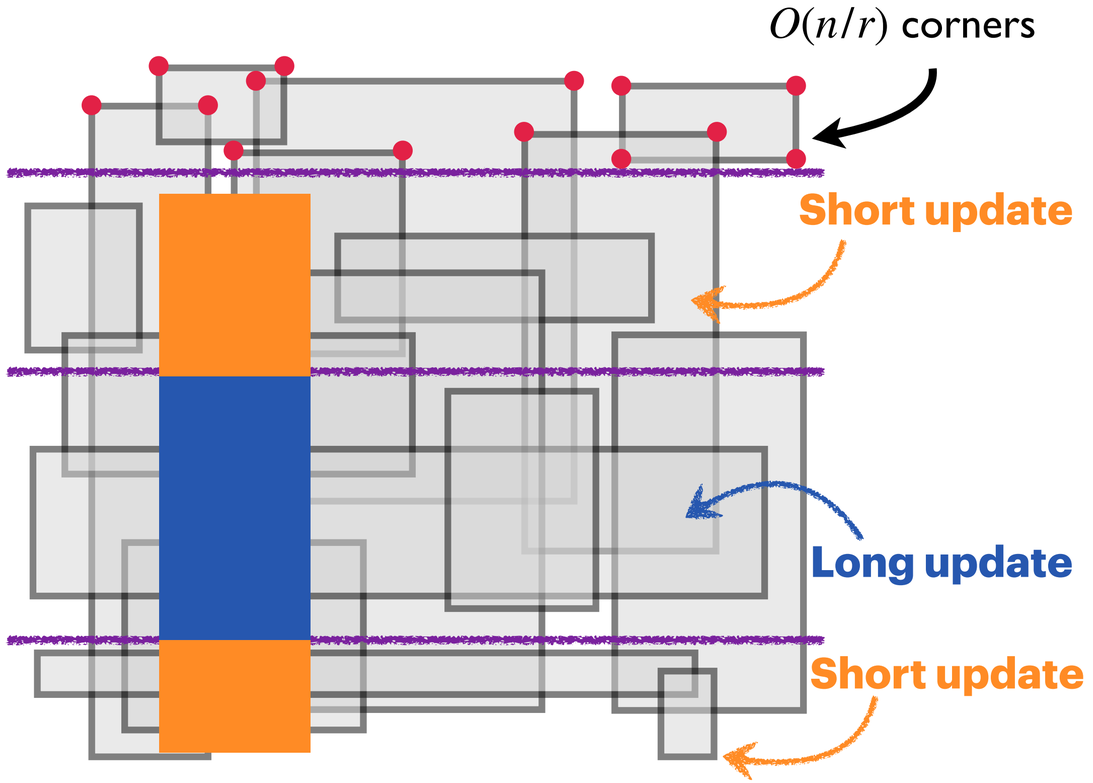
- Partition the plane into r slabs
- Each slab have $O(n/r)$ corners of rectangles

Update in each slab

Maintain the max depth in each slab
Pick the largest one

"Long" update
- 😇 easy: $O(\log n)$
- 😈 many: $O(r)$

"Short" update
- 😈 hard: $O((n/r)\log n)$
- 😇 rare: ≤ 2

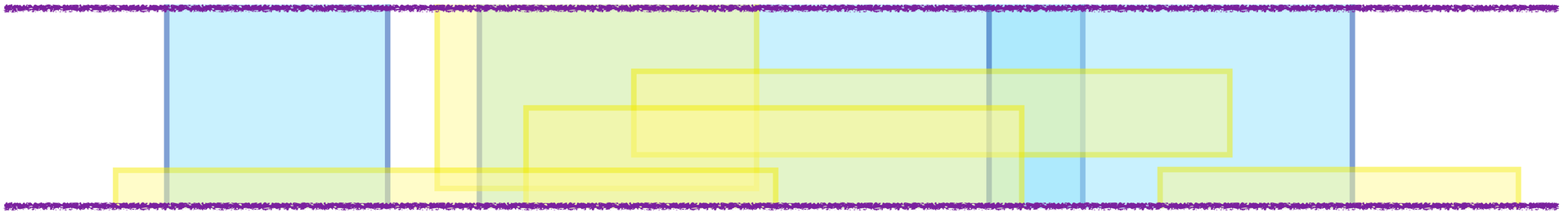


Total update time

$$O\left(2 \cdot \frac{n}{r} \log n + r \cdot \log n\right) = \tilde{O}\left(\sqrt{n}\right)$$

when $r = \sqrt{n}$

Details of Slab Data Structure



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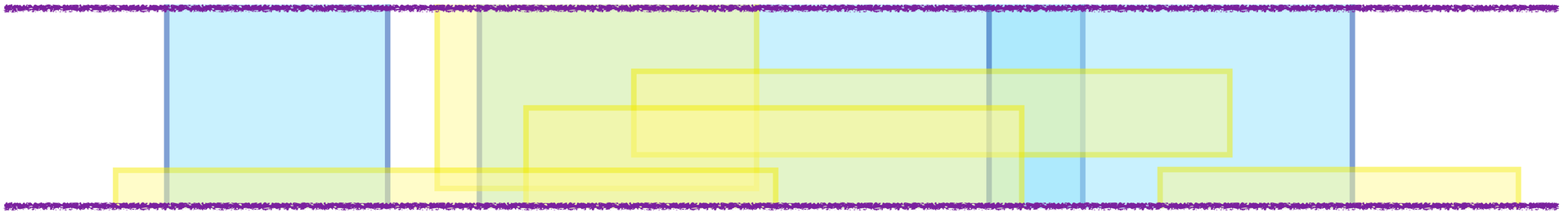

Reduce to intervals

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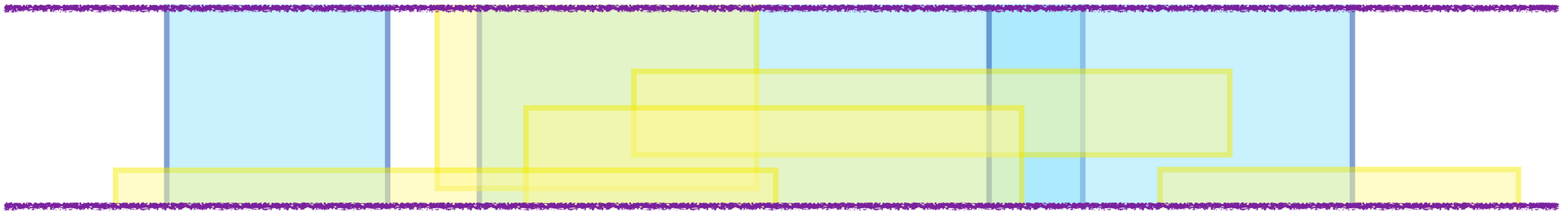
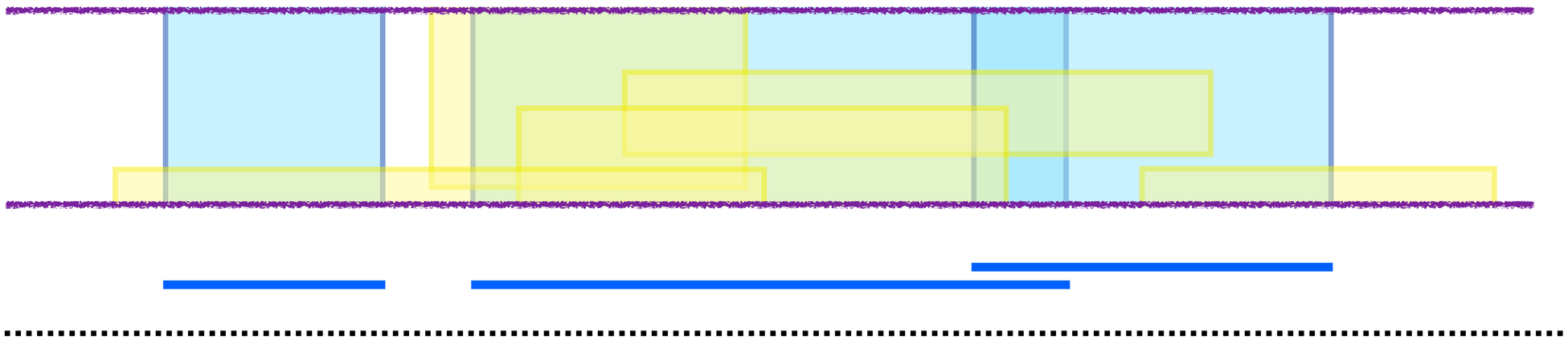


Reduce to intervals

$O(n)$ long rectangles $\xrightarrow[O(n) \text{ time}]{\text{directly}}$ $O(n)$ unweight intervals

Details of Slab Data Structure



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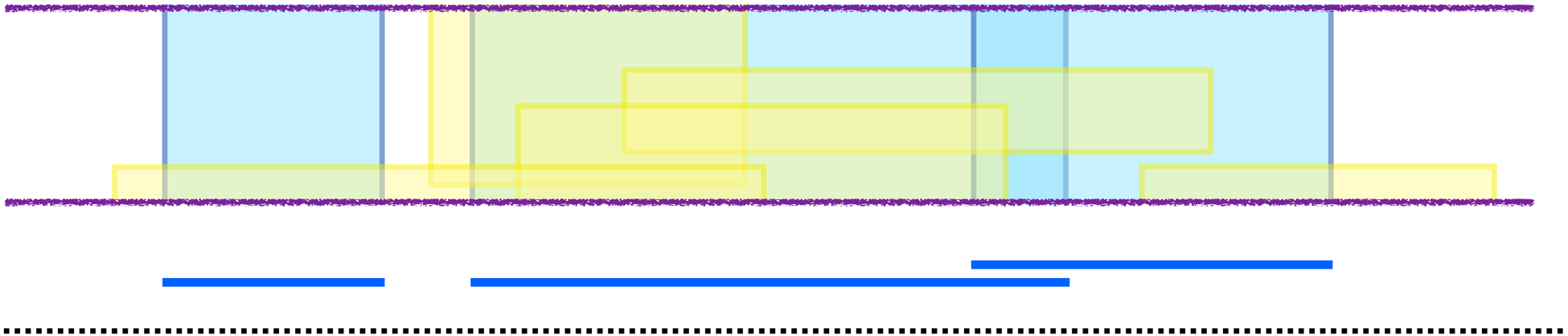

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$O\left(\frac{n}{r}\right)$ short rectangles $\xrightarrow[O\left(\frac{n}{r}\log\frac{n}{r}\right) \text{ time}]{\text{sweep line}}$ $O(n/r)$ weighted intervals

interval: two adjacent boundaries
weight: max dep w.r.t. short rectangles

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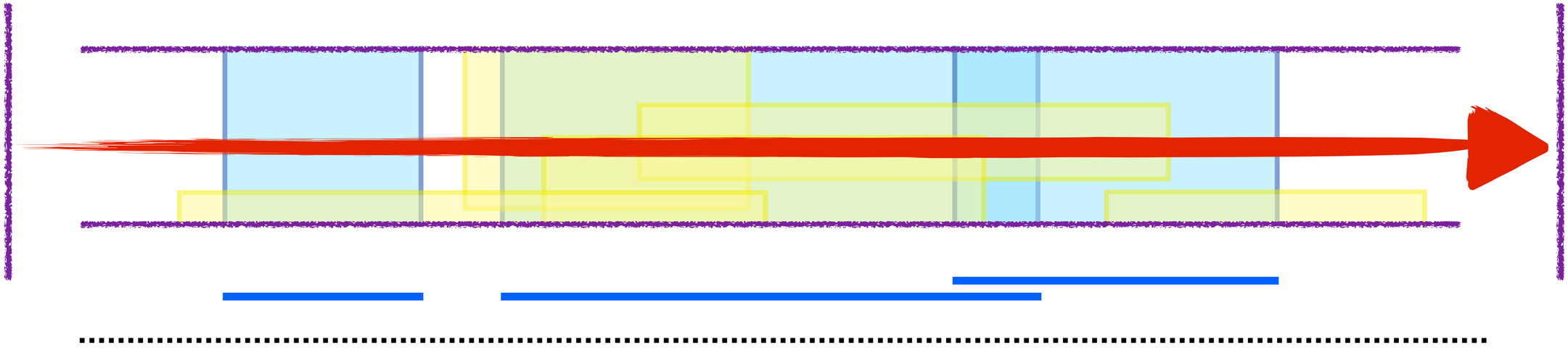

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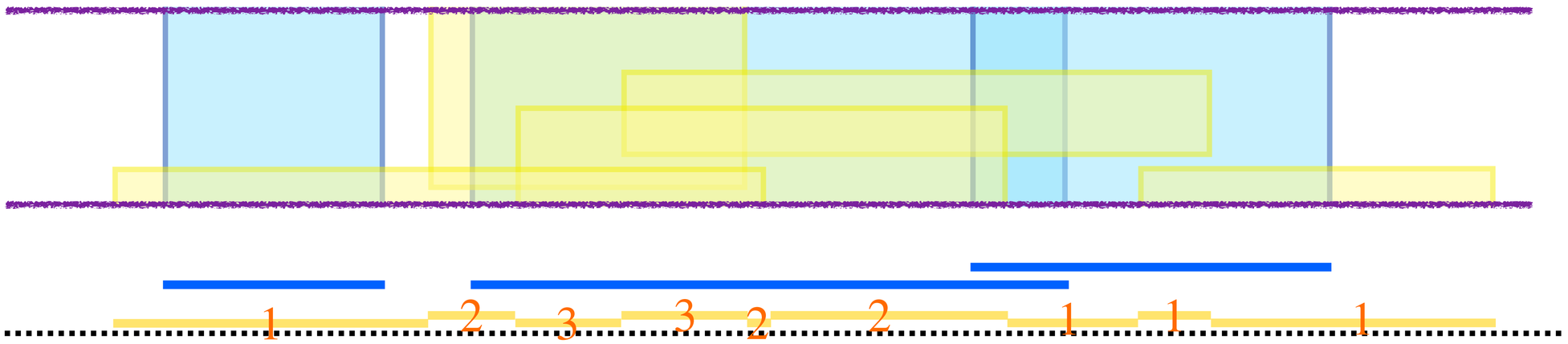

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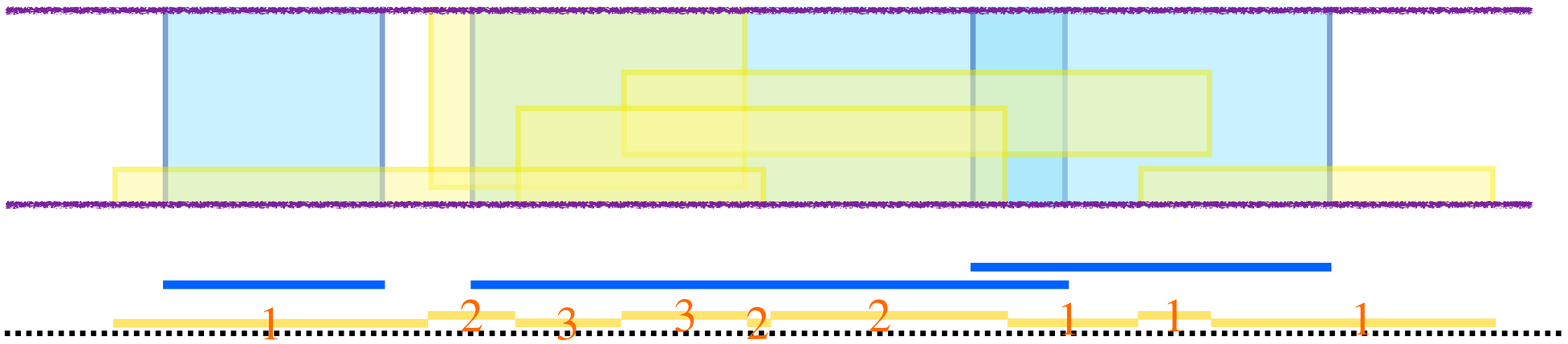

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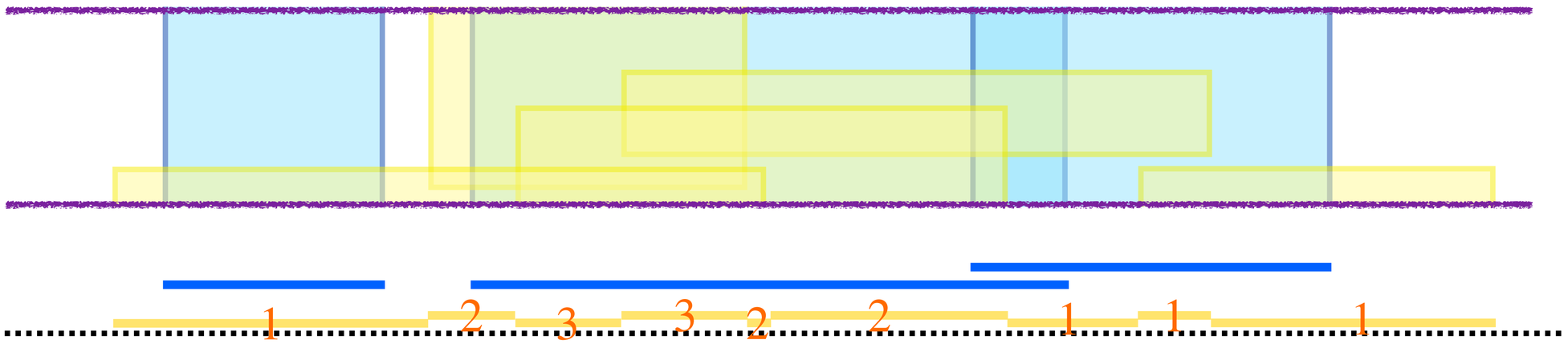

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Update

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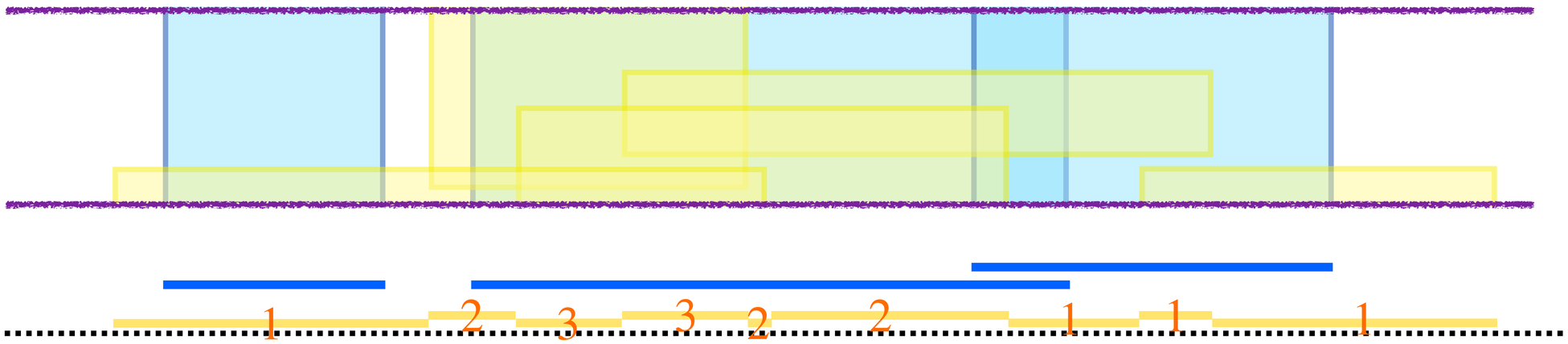

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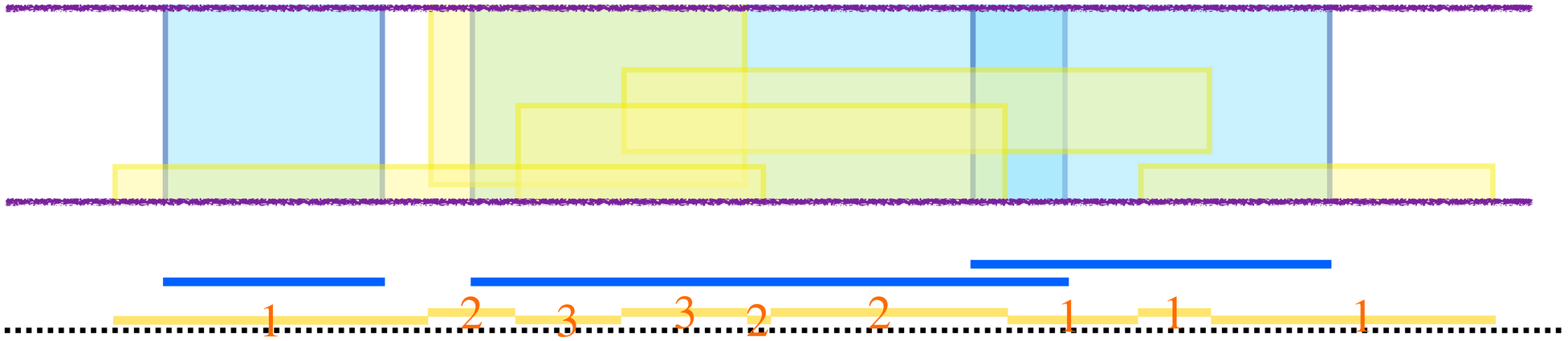

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weight: max dep w.r.t. short rectangles

Total preprocessing time: $O(n \log n)$

Update

Long update: $O(\log n)$

Short update:

- Delete all weighted intervals
- Recompute weighted intervals
- Insert all weighted intervals

$O((n/r) \cdot \log n)$

Approx. Ratio vs Update Time

🤔 Exact data structure: balancing long and short rectangles

Approx. Ratio vs Update Time

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💡 Speed-up for approx.: maintain long and short rectangles separately

Approx. Ratio vs Update Time

$\mathbf{D}_1$: exact data structure
$\mathbf{I}$: interval data structure
$\mathscr{L}$: set of long rect.
$\mathscr{S}$: set of short rect.

🤔 Exact data structure: balancing long and short rectangles

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Construction

$\mathbf{D}_2$: Build $\mathbf{D}_1$ on $\mathscr{S}$, $\mathbf{I}$ on $\mathscr{L}$

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$\mathbf{D}_3$:	Build $\mathbf{D}_2$ on $\mathscr{S}$, $\mathbf{I}$ on $\mathscr{L}$		

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Approx. Ratio vs Update Time

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• • •		• • •	• • •
$\mathbf{D}_k$:	Build $\mathbf{D}_{k-1}$ on $\mathscr{S}$, $\mathbf{I}$ on $\mathscr{L}$	$\max\left\{\frac{1}{k-1}\mathrm{dep}(\mathscr{S}), \mathrm{dep}(\mathscr{L})\right\} \geq \frac{1}{k}\mathrm{dep}(\mathscr{O})$	$\tilde{O}\left(2 \cdot (n/r)^{\frac{1}{k}} + r\right) = \tilde{O}\left(n^{\frac{1}{k+1}}\right)$ $r = n^{1/(k+1)}$

Approx. Ratio vs Update Time

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Summary

$\forall k \in \mathbb{Z}^+, \exists \frac{1}{k}$-approx. data structure for dynamic MaxDep with $\tilde{O}\left(n^{\frac{1}{k+1}}\right)$ update time

Approximate via Discretization

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DisMaxDepth

Input: a set of geometric objects $\mathcal{O}$

a set of points $\mathcal{P}$

Output: a point $p^* \in \mathcal{P}$ s.t.

$$\mathrm{dep}(p^*, \mathcal{O}) = \mathrm{dep}(\mathcal{P}, \mathcal{O}) = \max_{p \in \mathcal{P}} \mathrm{dep}(p, \mathcal{O})$$

and the value of $\mathrm{dep}(p^*, \mathcal{O})$

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Companion points

Insert/delete some "well-chosen" points $\Gamma(O)$
while insert or delete geometric object O

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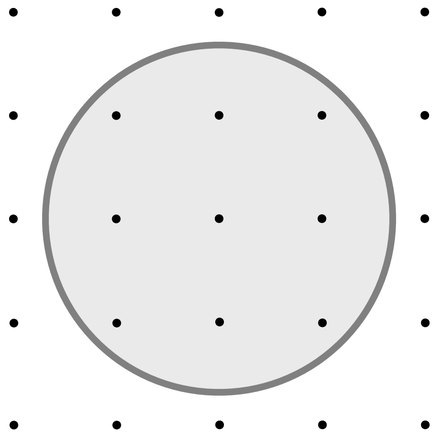

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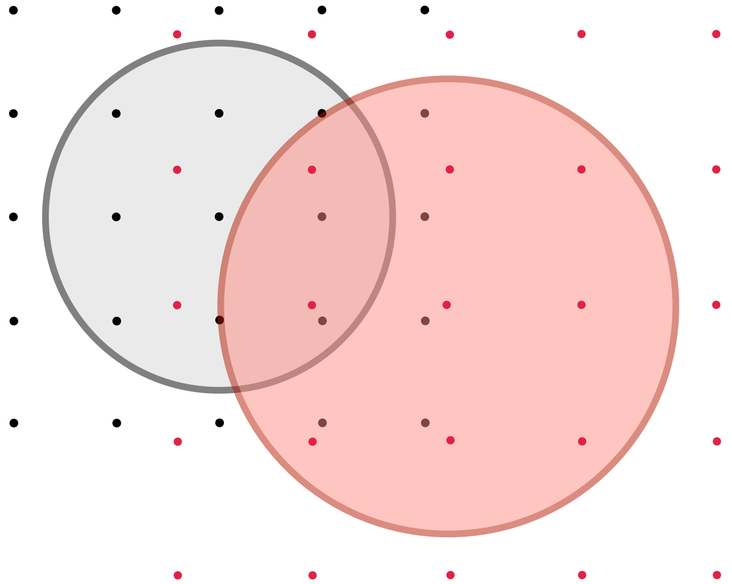

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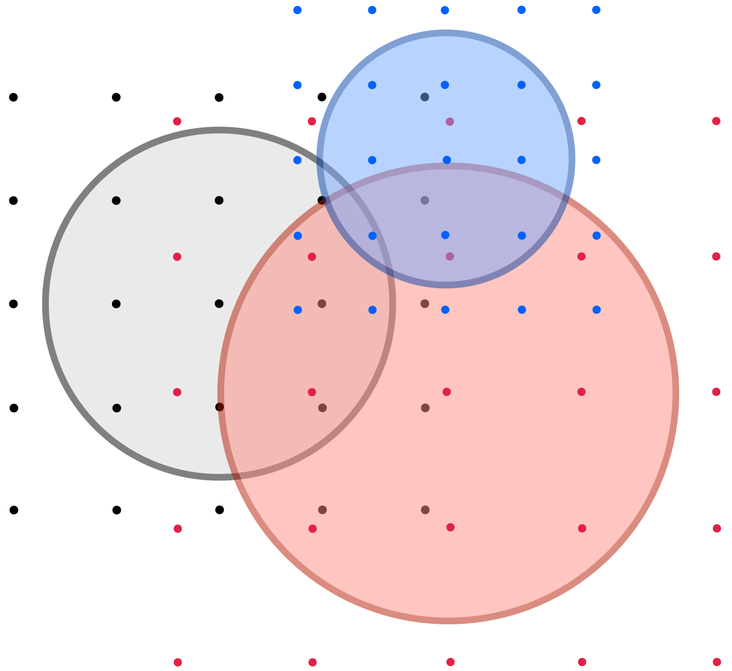

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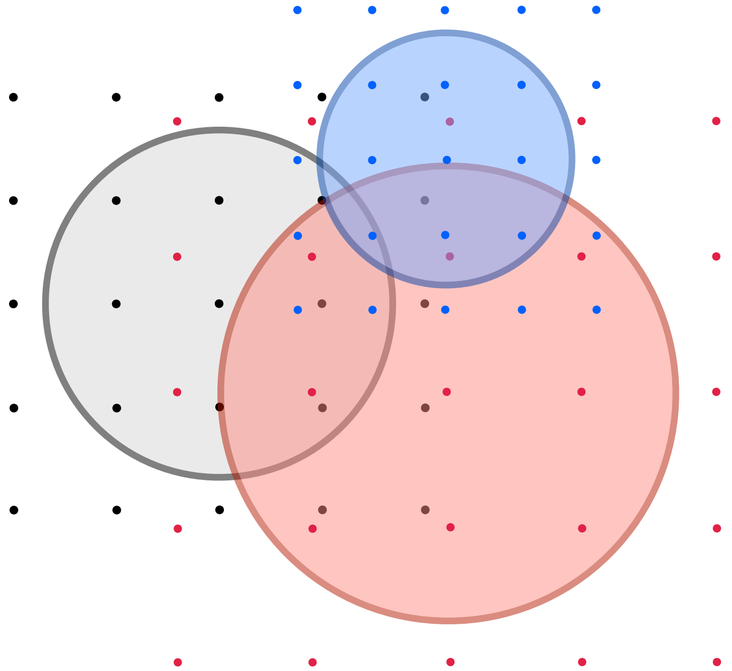

Companion points

Insert/delete some "well-chosen" points $\Gamma(O)$
while insert or delete geometric object O

"Well-chosen"

Hopefully, $\mathrm{dep}(\Gamma, \mathcal{O}) \geq \frac{1}{k} \cdot \mathrm{dep}(\mathcal{O})$

Approximation for Disks and Fat Objects

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Approx. for disks

$$\forall \varepsilon > 0, \exists\, \Gamma, \text{s.t. } \mathrm{dep}(\Gamma, \mathcal{O}) \geq \left(\frac{1}{2} - \varepsilon\right) \mathrm{dep}(\mathcal{O})$$

Approximation for Disks and Fat Objects

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Source of approx. ratio

Given a disk D and a set S of points in D,

$\exists$ circular sector X with angle $(1 - 2\varepsilon)\pi$,

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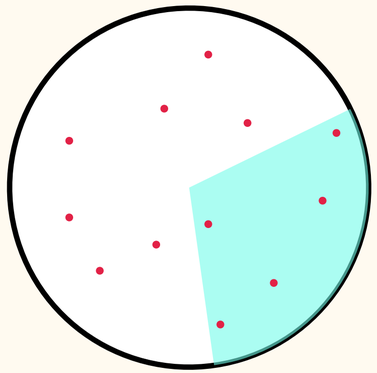

Approximation for Disks and Fat Objects

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$$\forall \varepsilon > 0, \exists\, \Gamma, \text{s.t. } \mathrm{dep}(\Gamma, \mathcal{O}) \geq \left(\frac{1}{2} - \varepsilon\right) \mathrm{dep}(\mathcal{O})$$

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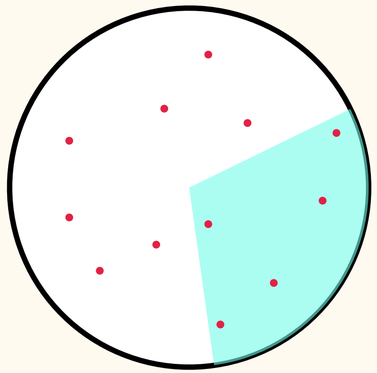

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$\tilde{O}(n^{2/3})$ update time

via dynamic partition tree [Matoušek 1992]

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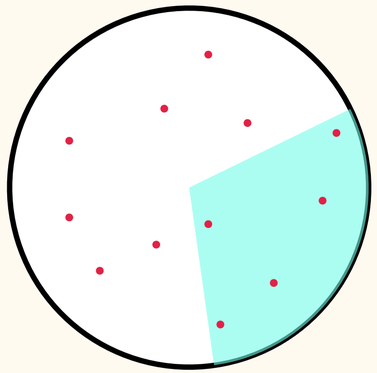

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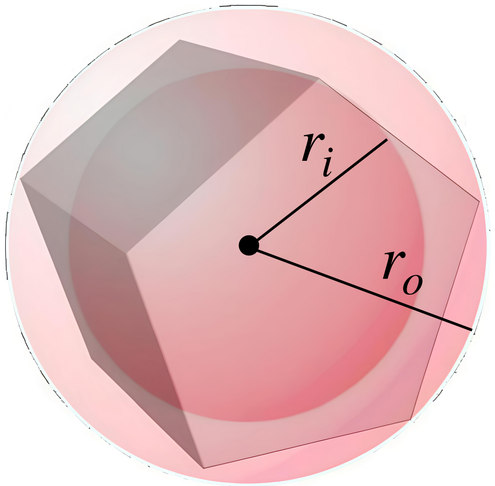

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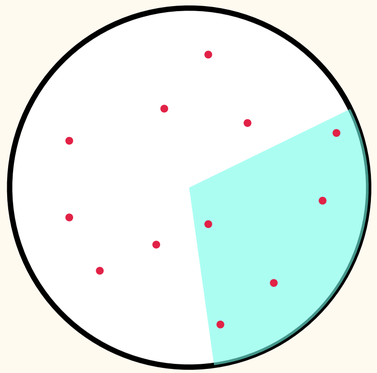

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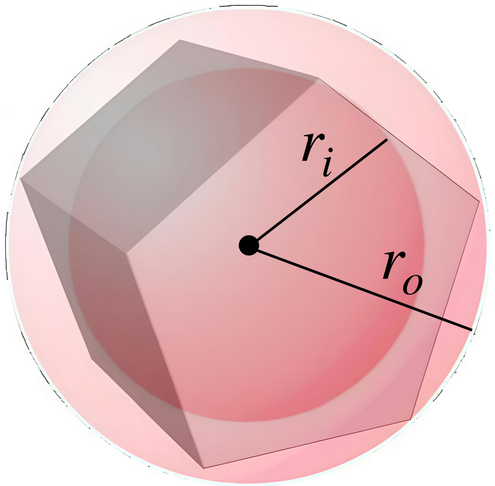

Strategy for fat objects (in $\mathbb{R}^d$)

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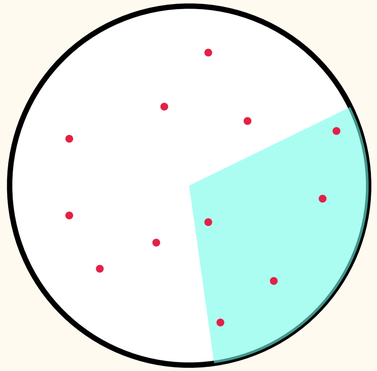

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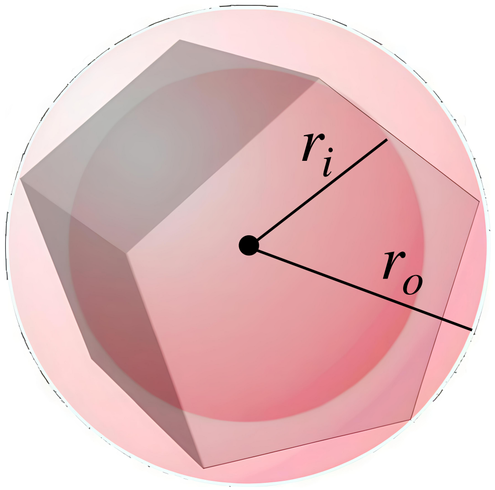

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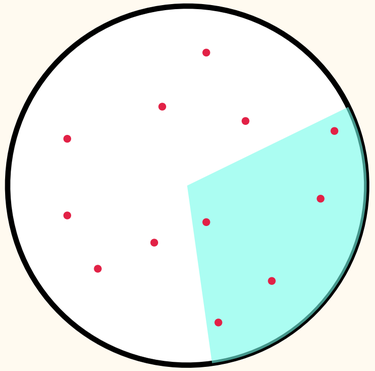

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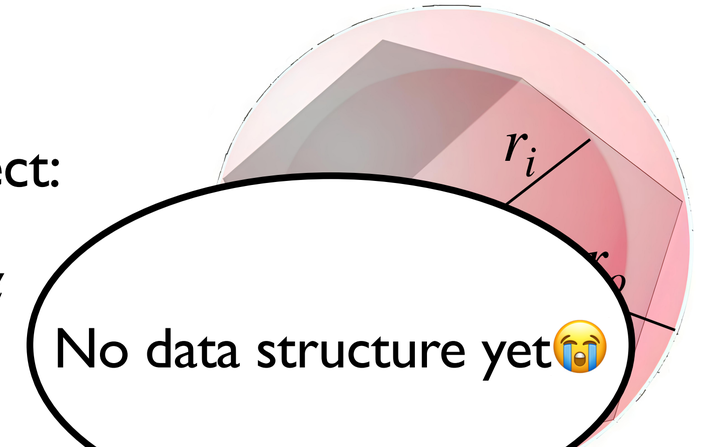

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Thank you!

Any Questions?

Open Problems

- Data structure for d-dim. boxes with $\tilde{O}(n^{(d-1)/2})$ update time
- Improve update time for special objects, e.g., squares
- Data structure for fat objects in discrete setting
- Data structure for other objects, e.g., halfspaces, quadrants