Beyond windability: Approximability of the four-vertex model

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Six-vertex Model

The six-vertex model is an abstraction of the crystal lattice with hydrogen bonds [4]. From a graph-theoretic perspective, the six-vertex model is defined on a 4-regular graph G as follows:

- the four incident edges of each vertex are labeled from 1 to 4;
- the state space of the six-vertex model consist of all the Eulerian orientations of G;
- six local configurations 1 to 6 are associated with six weights ω_1 to ω_6.

Figure 1. Valid local configurations of the six-vertex model.

Zero field assumption: $\omega_1 = \omega_2 = a, \omega_3 = \omega_4 = b, \omega_5 = \omega_6 = c$ and $a, b, c \geq 0$.

The partition function of the six-vertex model is defined as

$$Z_{6V}(G; a, b, c) := \sum_{\tau \in \mathcal{EO}(G)} a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6},$$

where $\mathcal{EO}(G)$ is the set of all Eulerian orientations of G and n_i is the number of vertices of type i $(1 \leq i \leq 6)$ in the graph under a particular Eulerian orientation $\tau \in \mathcal{EO}(G)$.

Phase Transition vs. Approximability

View of statistical physics:

The six-vertex model exhibits phase transition phenomena:

- Disordered phase (DO): $a \leq b + c$, $b \leq a + c$ and $c \leq a + b$.
- Ferroelectric phase (FE): $a > b + c$ or $b > a + c$;
- Anti-ferroelectric phase (AFE): $c > a + b$.

(a) Phase diagram of the six-vertex model (b) Approximability of the six-vertex model

Figure 2. The approximability of the six-vertex model is similar to the phase transition phenomenon.

View of computational complexity:

- Computing the partition function of the six-vertex model is a counting problem.
- A complexity dichotomy for exact computation has been established [1].
- For approximability, [2] shows that, there is no FPRAS in the entire order phase (FE and AFE) unless RP = NP, whereas an FPRAS was only given in a subregion of DO with *windable constraint function* via MCMC.

How to overcome the windability to establish the fully approximability?

Main Result: Four-vertex Model

The four-vertex model is a sub-model of the six-vertex model in which two vertex configurations are disallowed. We may assume $b = 0$, i.e., 3 and 4 in Fig. 1 are disallowed. The constraint function of the four-vertex model f^* has the (normalized) form

$$M(f^*) = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}. \quad \text{(WLOG, } \beta > 1.\text{)}$$

Unwindability

The constraint function of the four-vertex model is *unwindable*.

Known Results:

- Exact computation of the partition function is #P-hard [1].
- There is no FPRAS for it *in general* [2].

Our Result

There is an FPRAS for the four-vertex model if a certain system of linear equations over $\mathrm{GF}(2)$ has a solution.

First FPRAS for the six-vertex model with unwindable constraint functions!

Key Reduction: Circuit Decomposition

Basic idea: Reduce the four-vertex model to the Ising model.

Key observation: The four variables of the constraint function can be divided into two pairs, where the variables in the same pair must take opposite values in each valid configuration.

By this observation, for any instance of the four-vertex model, we can decompose the edge-vertex incidence graph of its underlying graph into circuits and construct a graph of circuit.

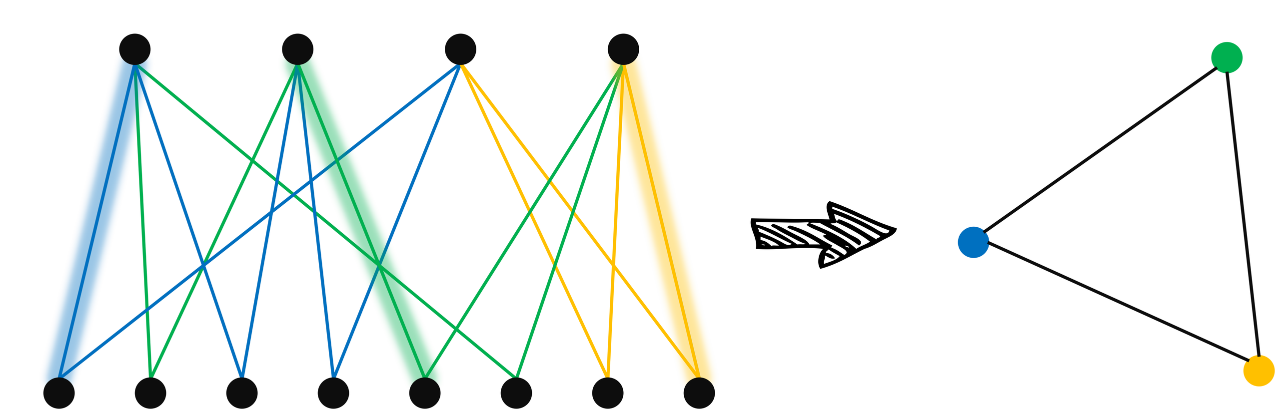

Figure 3. An example of circuit decomposition, where the highlighted edges denote the initial edges.

Arbitrarily choose an edge as the initial edge of each circuit to represent the value of this circuit.

The constraint function of the graph of circuit (at edge (u, v)) f'_{uv} has the form

$$M(f^*) = \begin{bmatrix} \beta^{A_{uv}} & \beta^{D_{uv}} \\ \beta^{D_{uv}} & \beta^{A_{uv}} \end{bmatrix} = \beta^{D_{uv}} \begin{bmatrix} \beta^{A_{uv}-D_{uv}} & 1 \\ 1 & \beta^{A_{uv}-D_{uv}} \end{bmatrix},$$

where A_{uv} and D_{uv} are the number of "agree" and "disagree" vertices between circuit u and v. Thus, f'_{uv} is an Ising-type constraint function.

Adjust the Initial Edges

However, by [3], different situations arise according to the value of A_{uv} and D_{uv}:

- If $A_{uv} > D_{uv}$ for any (u, v), the constraint function is ferro-Ising-type, which can be approximated efficiently.
- If $A_{uv} < D_{uv}$ for some (u, v), the constraint function is antiferro-Ising-type, which is NP-hard to approximate.

Observation: If we change a certain circuit's initial edge, the agree (disagree) vertices between it and its neighbor will become disagree (agree) vertices.

Choose initial edges "*wisely*": adjust initial edges of some circuits to obtain a ferro-Ising-type constraint function.

Let X_u indicate whether circuit u changes its initial edge, this can be formulated as a system of linear equations over $\mathrm{GF}(2)$:

$$\forall (u, v), \quad X_u \oplus X_v = \mathbf{I}(A(u, v) < D(u, v)).$$

Once this system has a solution, we can obtain an FPRAS for the four-vertex model immediately.

Application: Planar Four-vertex Model

When it comes to planar graphs, the dual graph of any 4-regular plane graph G is bipartite. Thus, the faces of G have a 2-coloring.

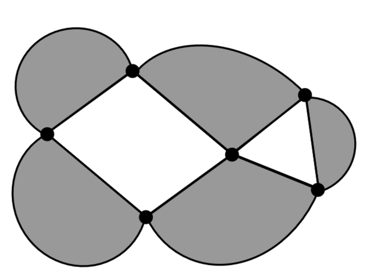

Figure 4. An example of face 2-coloring where the outer face is white.

Based on this fact, we design a "canonical labeling" for the six-vertex model on a plane graph:

- The edges which bound each black face are one-in-one-out, and the directions of the arrow-flows along two black faces are different. These vertices (1 and 2 in Fig. 5) have weight a.
- The edges which bound each black face are two-in or two-out. These vertices (3 and 4 in Fig. 5) have weight b.
- The edges which bound each black face are one-in-one-out, and the directions of the arrow-flows along two black faces are the same. These vertices (5 and 6 in Fig. 5) have weight c.

Figure 5. Valid local configurations of the planar six-vertex model under canonical labeling.

This labeling already encode the staggered ice-type model, a primitive case in statistical physics.

Under this labeling, the system of linear equations we proposed is *always* solvable.

This method also works for *torus grid graphs*.

References

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