Dynamic Maximum Depth and Klee's Measure

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joint work with

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Subhash Suri (UCSB)
Xiongxin Yang (UCSB)
Jiumu Zhu (NYU Shanghai)

Background

- **Maximum depth**

  $\mathcal{O}$ = a set of geometric objects in $\mathbb{R}^d$

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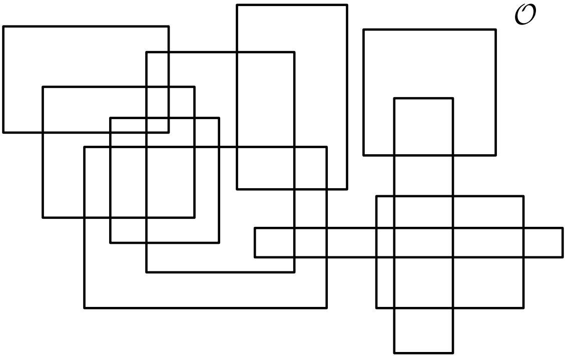

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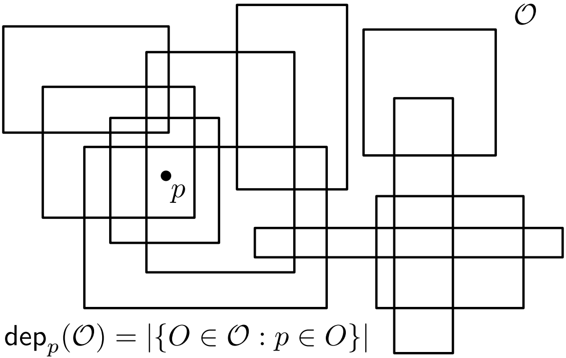

$$\mathrm{dep}_p(\mathcal{O}) = |\{O \in \mathcal{O} : p \in O\}|$$

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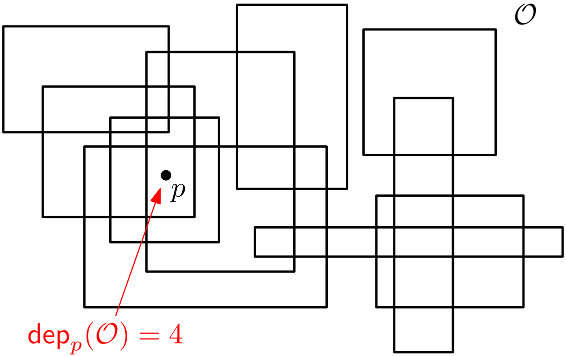

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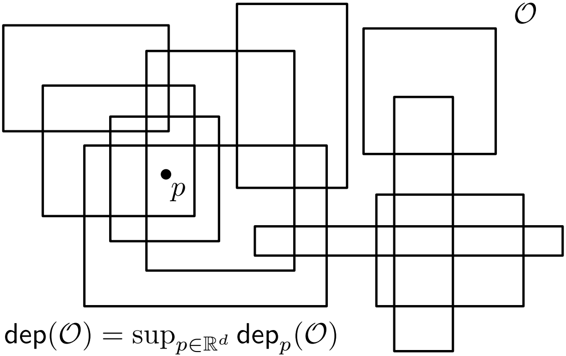

$\text{dep}(\mathcal{O}) = \sup_{p \in \mathbb{R}^d} \text{dep}_p(\mathcal{O})$

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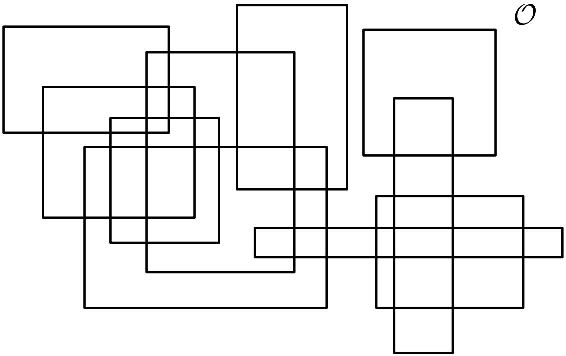

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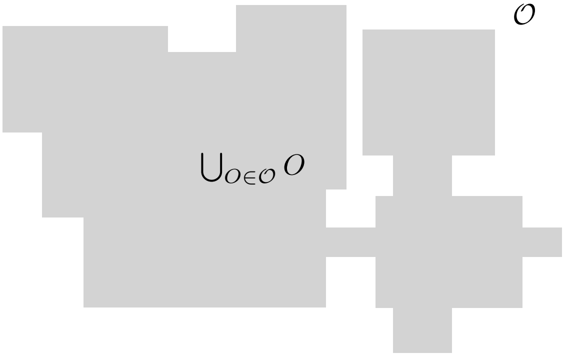

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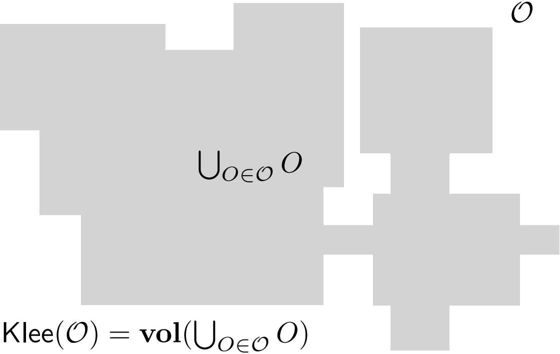

$\mathsf{Klee}(\mathcal{O}) = \mathbf{vol}(\bigcup_{O \in \mathcal{O}} O)$

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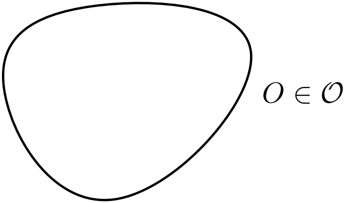

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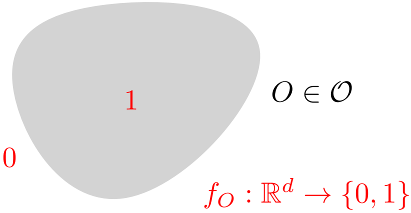

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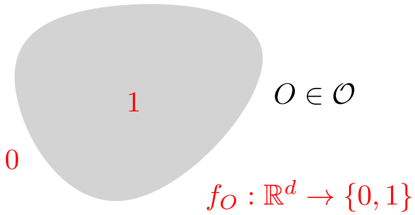

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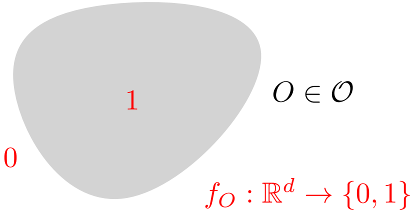

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The maximum depth of n boxes in $\mathbb{R}^d$ can be computed in $\widetilde{O}(n^{d/2})$ time.

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- The bound $\widetilde{O}(n^{d/2})$ is believed to be tight up to logarithmic factors.

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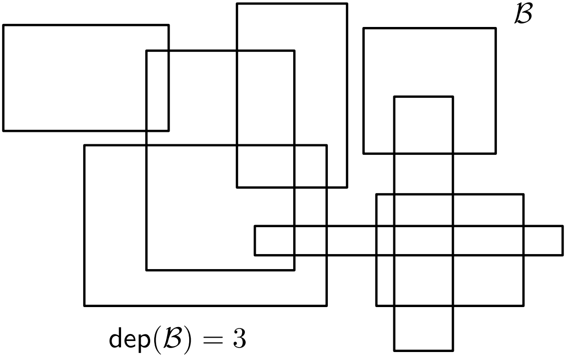

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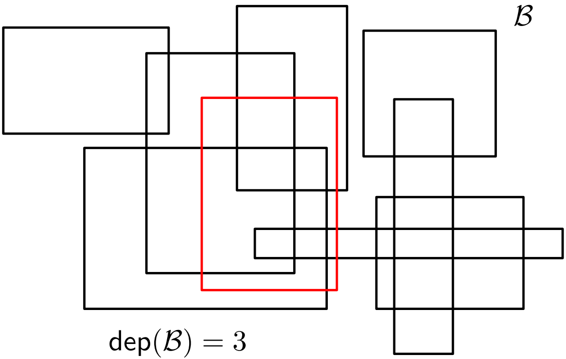

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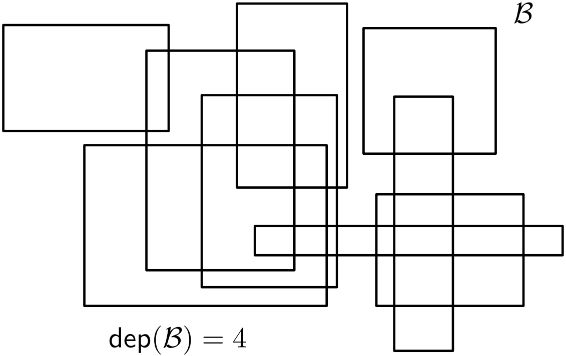

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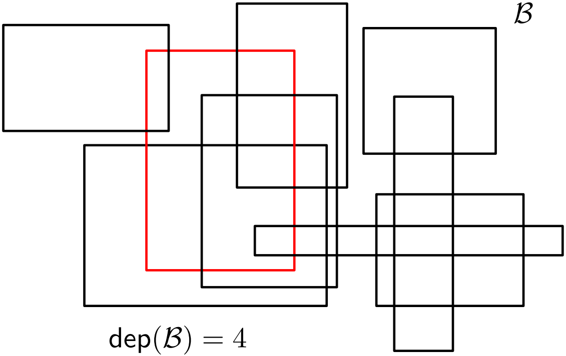

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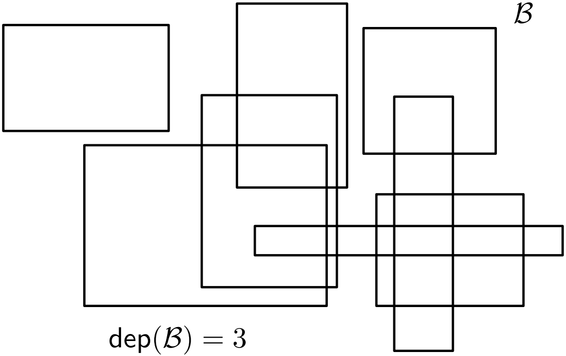

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- For $d = 1$, one can achieve $O(\log n)$ update time using interval trees.

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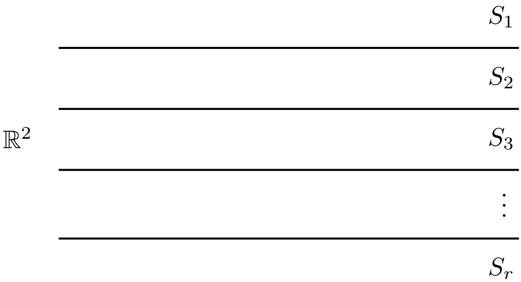

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$O(n/r)$ rectangle corners S_1

$O(n/r)$ rectangle corners S_2

$\mathbb{R}^2$ $O(n/r)$ rectangle corners S_3

$\vdots$

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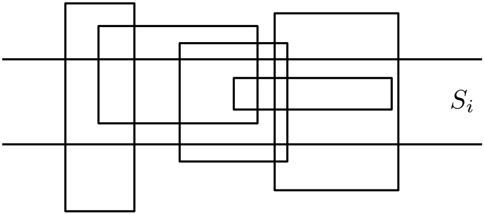

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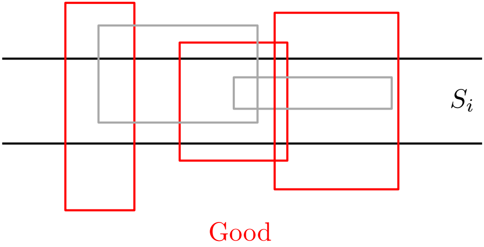

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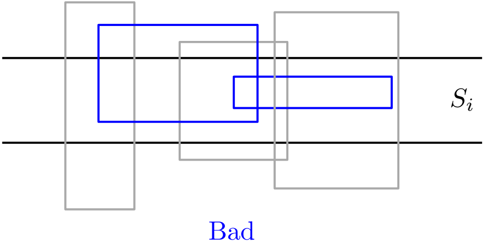

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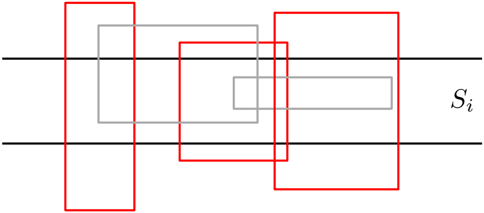

Good rectangles can be viewed as intervals

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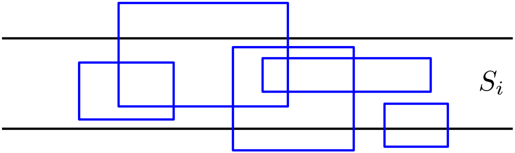

$f : \mathbb{R} \to \mathbb{N}_0 \qquad f(x) = \mathsf{dep}_{\{x\}\times\mathbb{R}}(\mathcal{B}_i^{\mathrm{bad}})$

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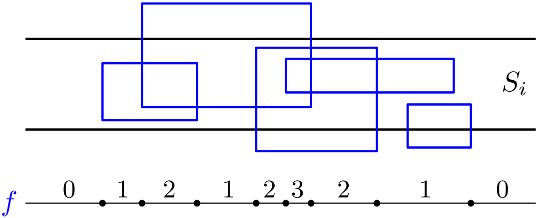

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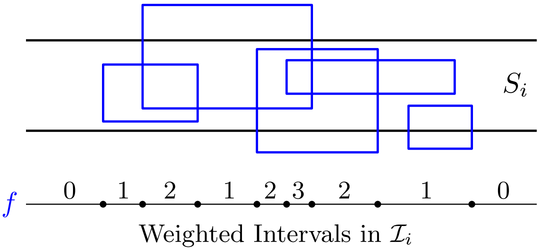

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No →

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Yes → Insert/delete $\mathbf{proj}_x(B)$ to $\mathcal{I}_i$

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No → Recompute the dep function
Delete intervals of the old func.
Insert intervals of the new func.

$O((n/r)\log n)$

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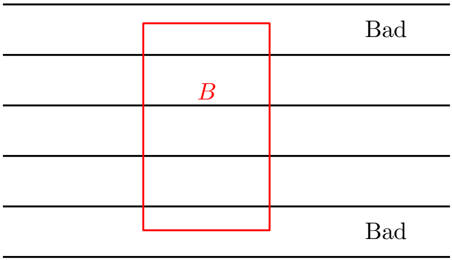

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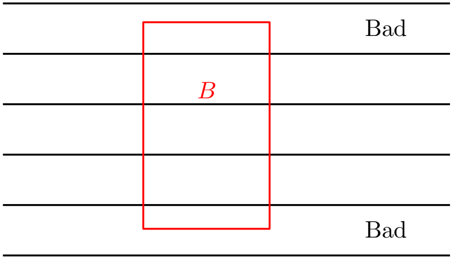

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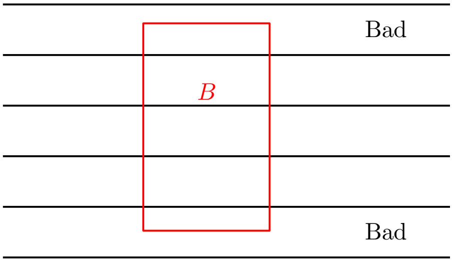

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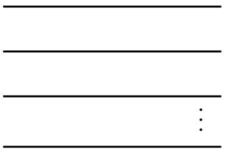

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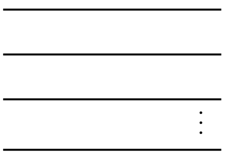

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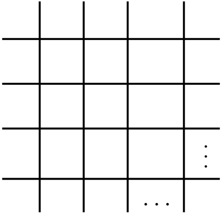

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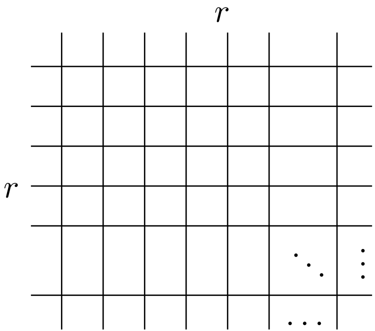

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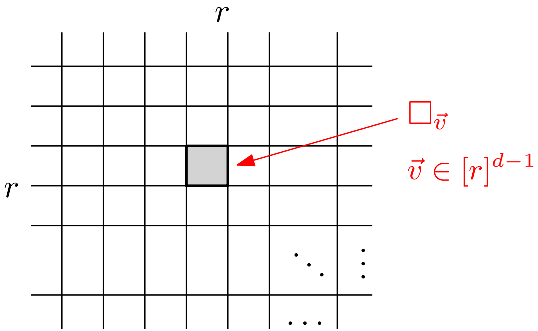

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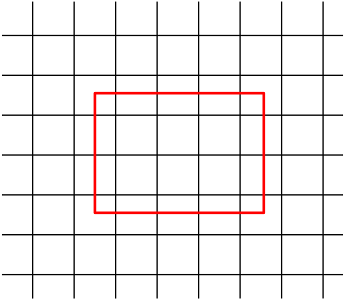

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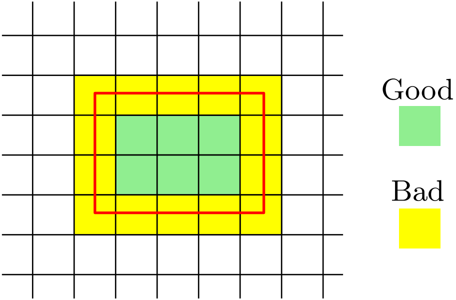

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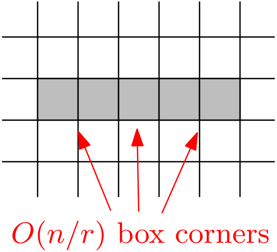

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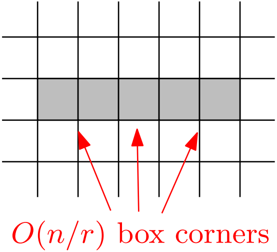

Most bad boxes are only bounded in one dimension

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- We design an offline algorithm whose total update time depends on not only the number of updates but also the structure of the boxes.

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$$\phi_\sigma(B,i) = \begin{cases} 1 & B \text{ is nontrivial in dim } i \text{ and} \\ & B \text{ is nontrivial in dim } j \text{ for some } j <_\sigma i \\ 0 & \text{Otherwise} \end{cases}$$

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- This partitions $\mathbb{R}^{d-1}$ into $O(\Phi_\sigma(\mathcal{B}^*))$ regions. Inside each region, all boxes in $\mathcal{B}^*$ are trivial in all but one dimensions.

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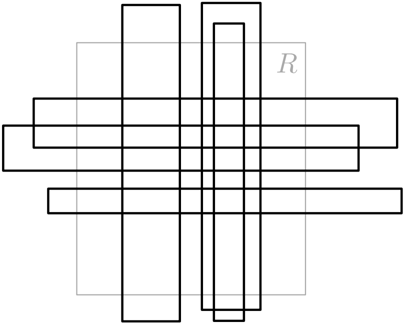

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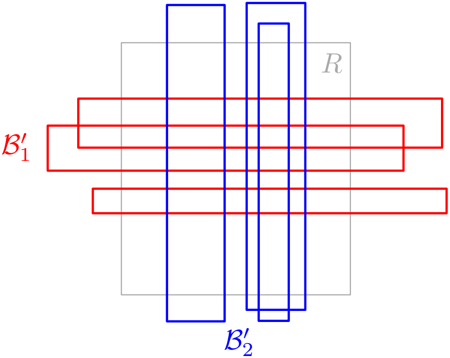

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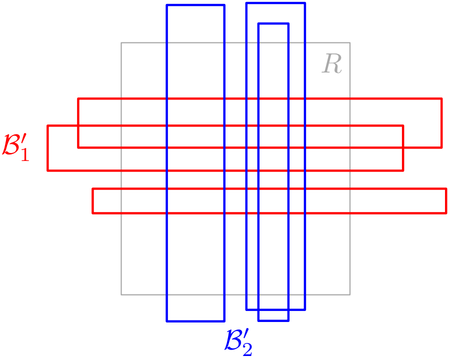

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Q & A