Beyond Windability: Approximability of the Four-Vertex Model

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Based on joint work with

Tianyu Liu

Stanford ⟶ Meta

Counting Complexity

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Counting problem

U: ground set (with a weight function $w : U \to \mathbb{R}^+(\mathbb{C})$)

Ω: a collection of subsets of U with a certain property

Goal: compute $|\Omega|$ or $Z = \sum_{S \in \Omega} \prod_{i \in S} w(i)$

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This talk!

Holant Problem

[Cai, Lu, and Xia, 2009]

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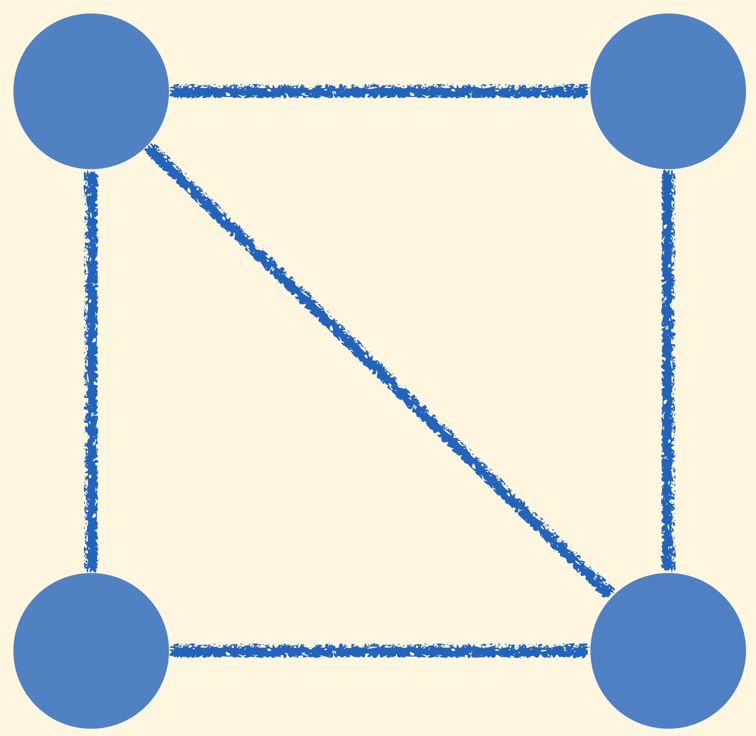

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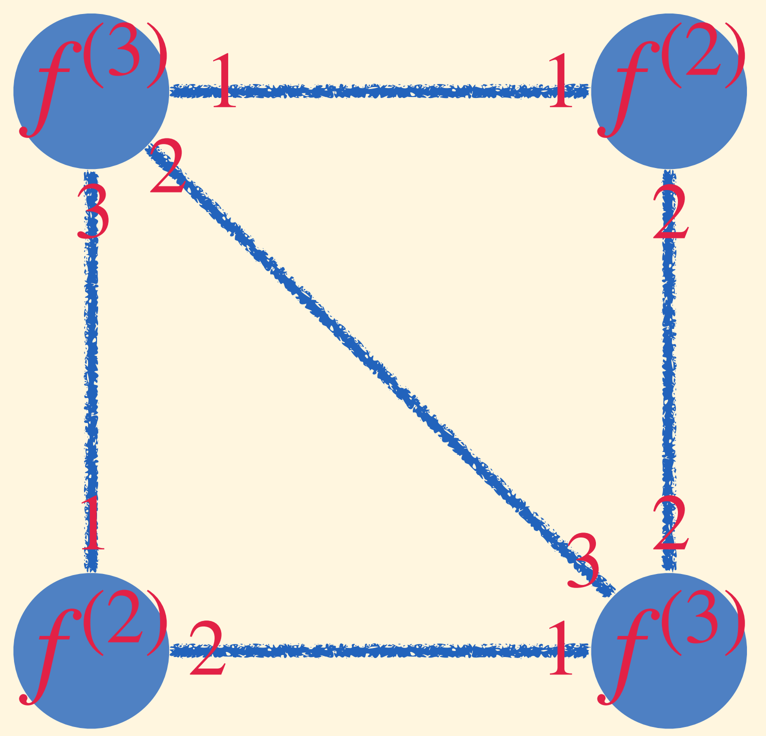

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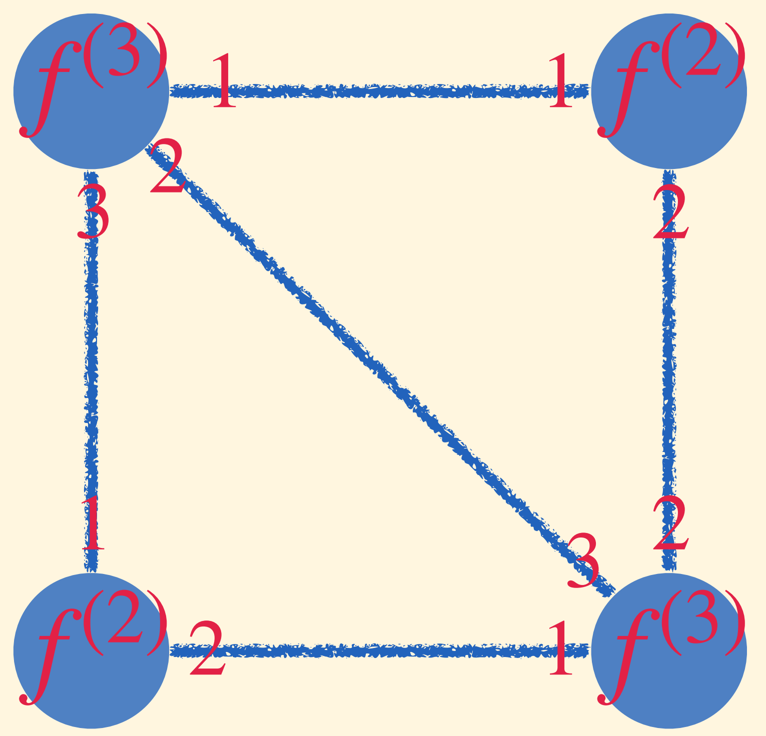

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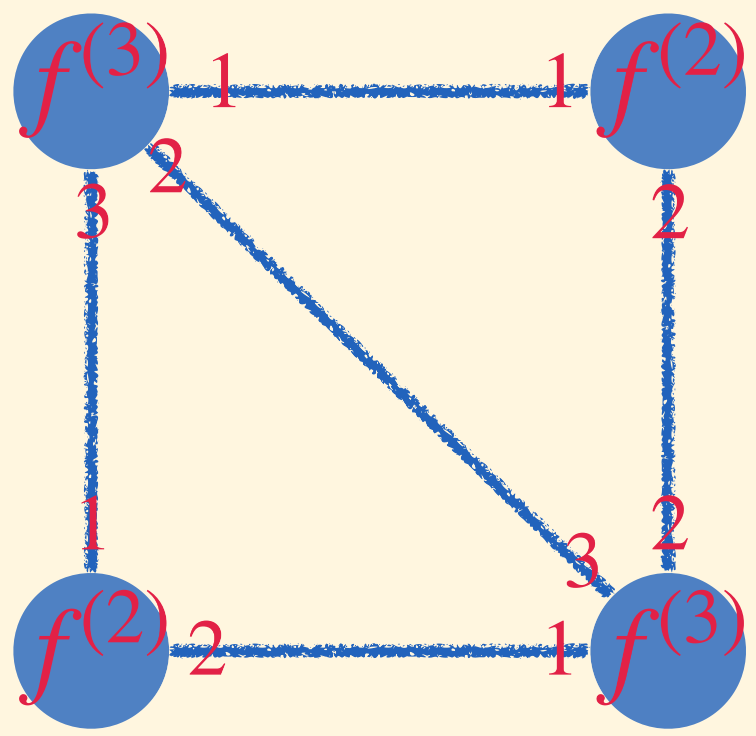

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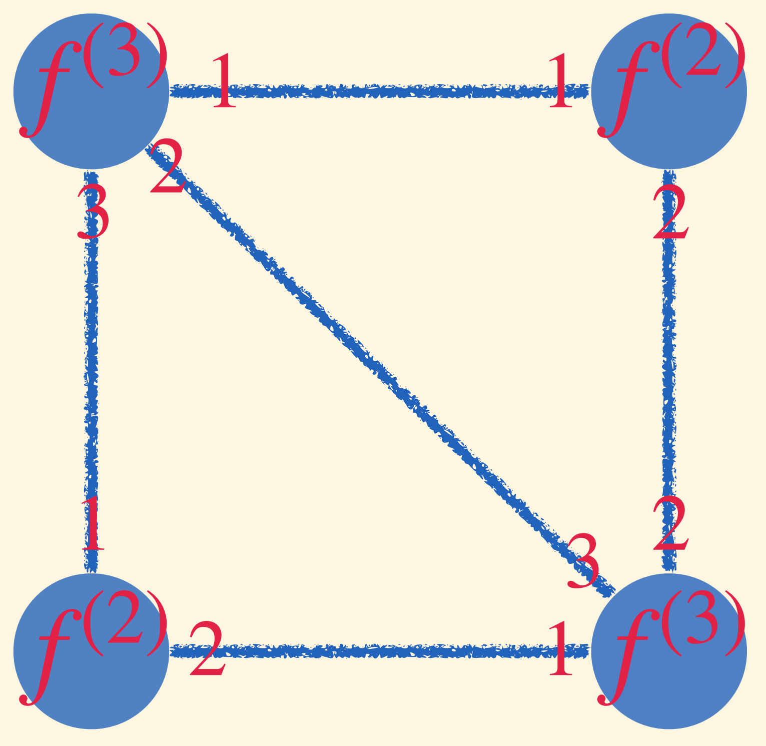

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Other examples:

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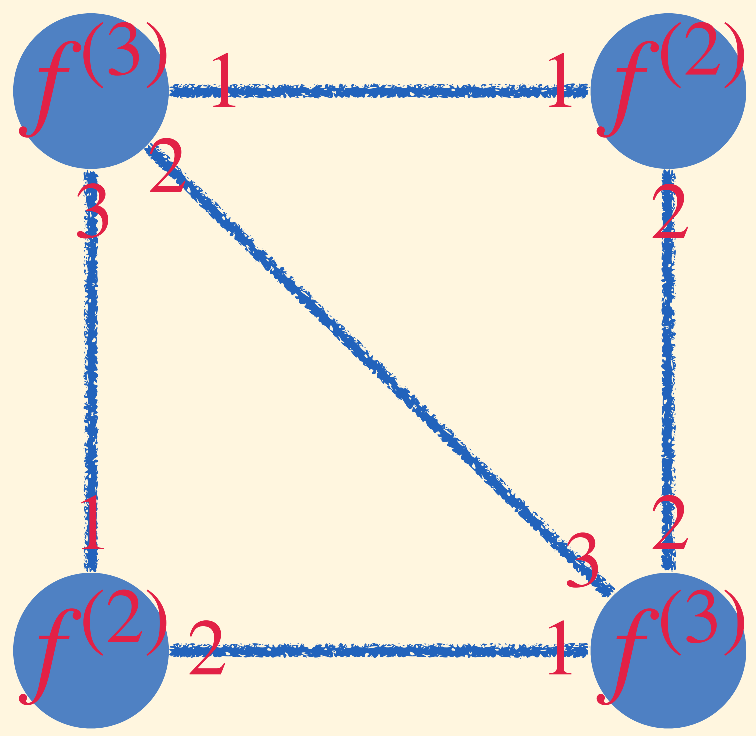

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Dichotomy theorem

find explicit dichotomy criterion for $\mathscr{F}$

computing the Holant value is

- solvable in polynomial time
- #P-hard

Real Boolean case was done!
[Shao and Cai, 2020]

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Deterministic algorithms that solves the problem in time poly$(n, \frac{1}{\varepsilon})$

Output a number $\hat{Z}$ such that

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(**f**ully **poly** … imation scheme)

Rand … solves the … $\frac{1}{\varepsilon})$

can be boosted to $1-\delta$

using $O(\log \delta)$ repeated trials

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Approaches:

- decay of correlation [Weitz, 2006]
- zero-freeness [Barvinok, 2016]
- liner programming [Moitra, 2019]
- cluster expansion [Helmuth, Perkins and Regts, 2019]

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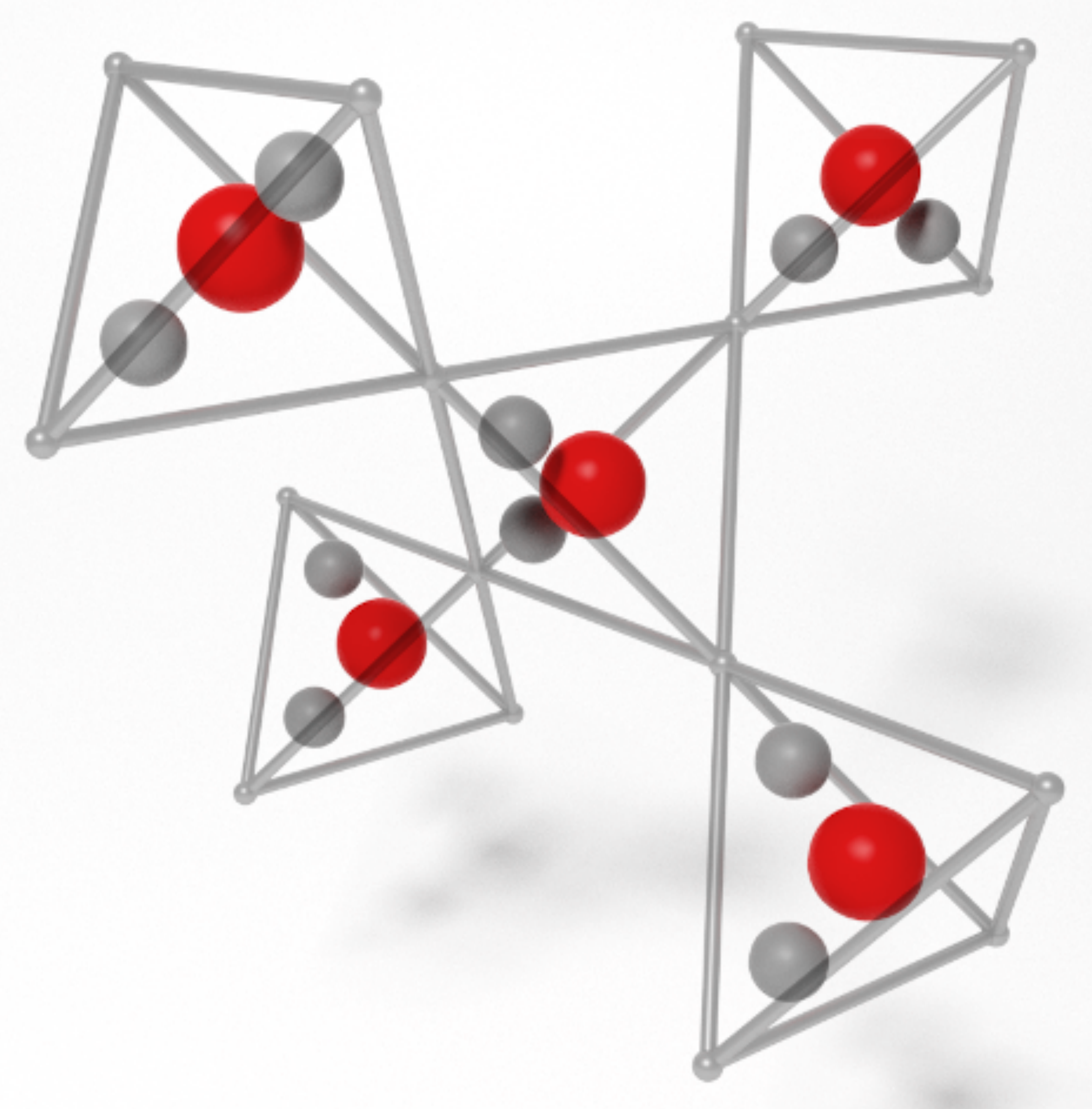
Six-Vertex Model

[Pauling, 1935]



(Figure by Jonas Greitemann)

A model to describe the properties of ice H_2O

ice condition

- Each O has 4 nearest neighbors with O-H-O bonds
- Each H is in 2 possible positions (closer to one O or another)
- Each O must be surrounded by two H's near to it and two on the far side

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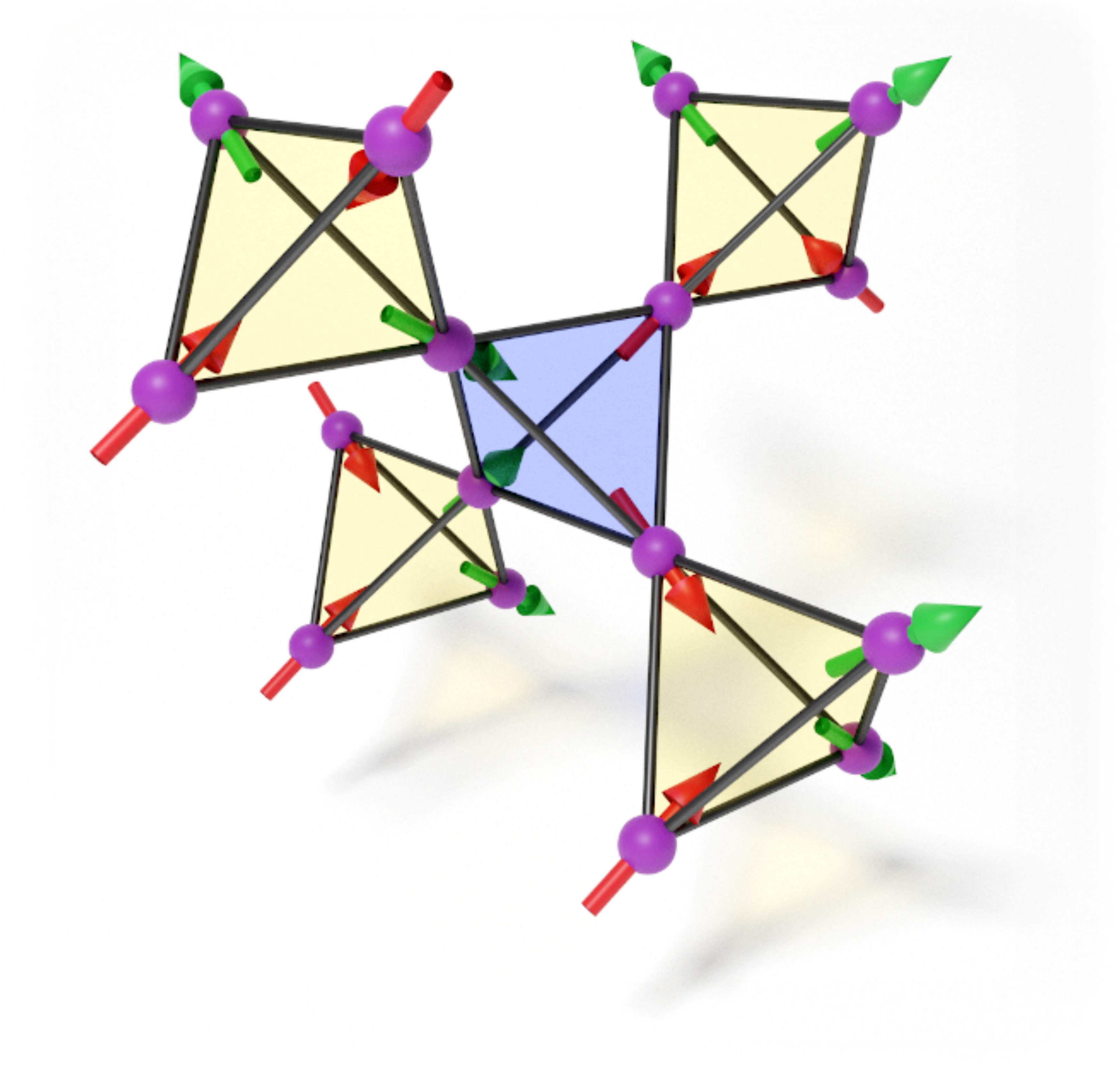

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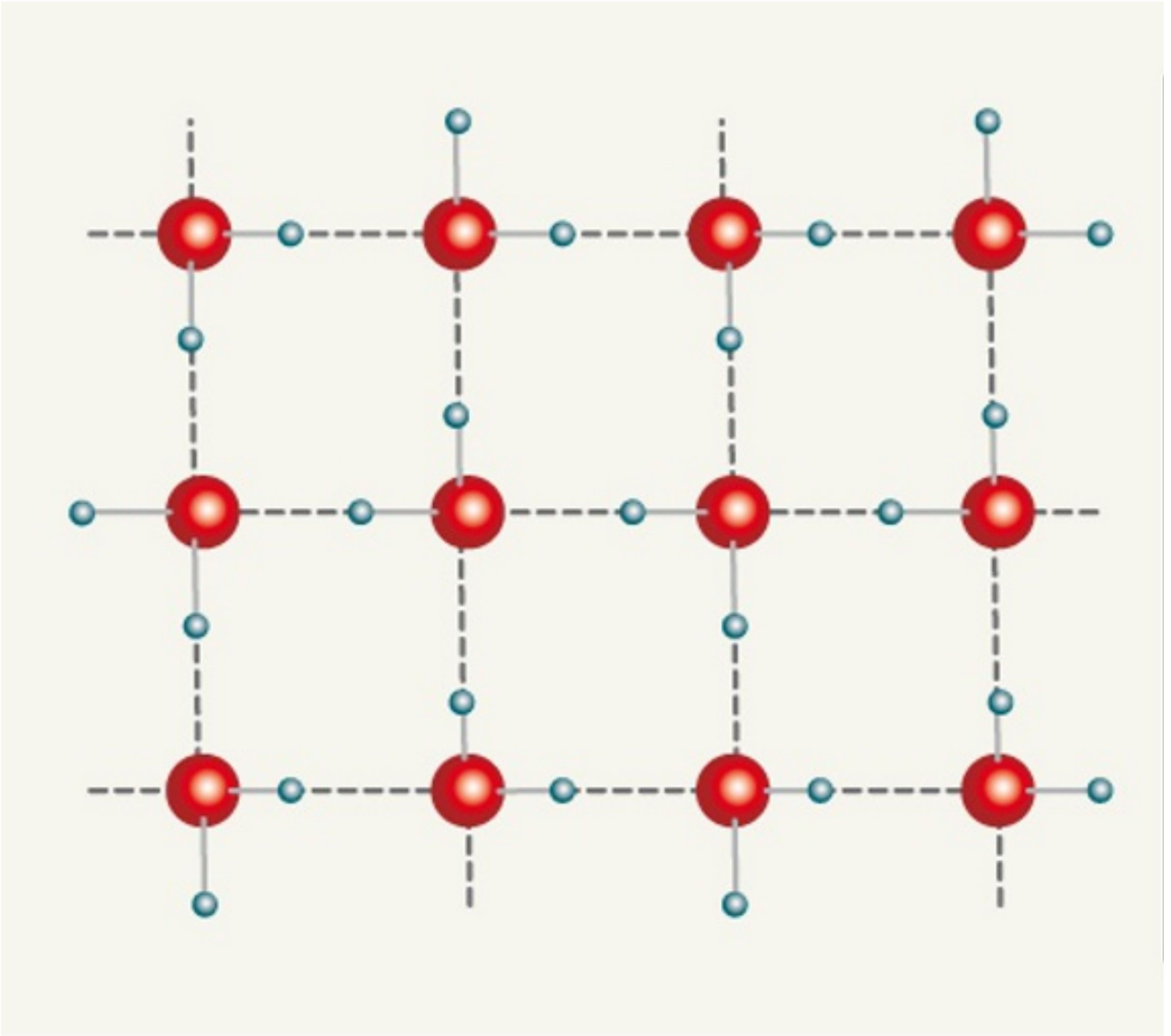
Exact Solvable Model: Square Ice

[Lieb, 1967]



(Figure by Mark Peplow)

two-dimensional version of real ice:
square lattice with ice condition

Lieb's square ice constant

N: the number of O's

Z: the ***partition function***

The *exact solution* (partition function peer vertex)

$$W = \lim_{N\to\infty} Z^{1/N} = \left(\frac{4}{3}\right)^{3/2} \approx 1.5396007\cdots$$

Six-Vertex Model on Graphs

The six-vertex model is defined on 4-*regular* graphs

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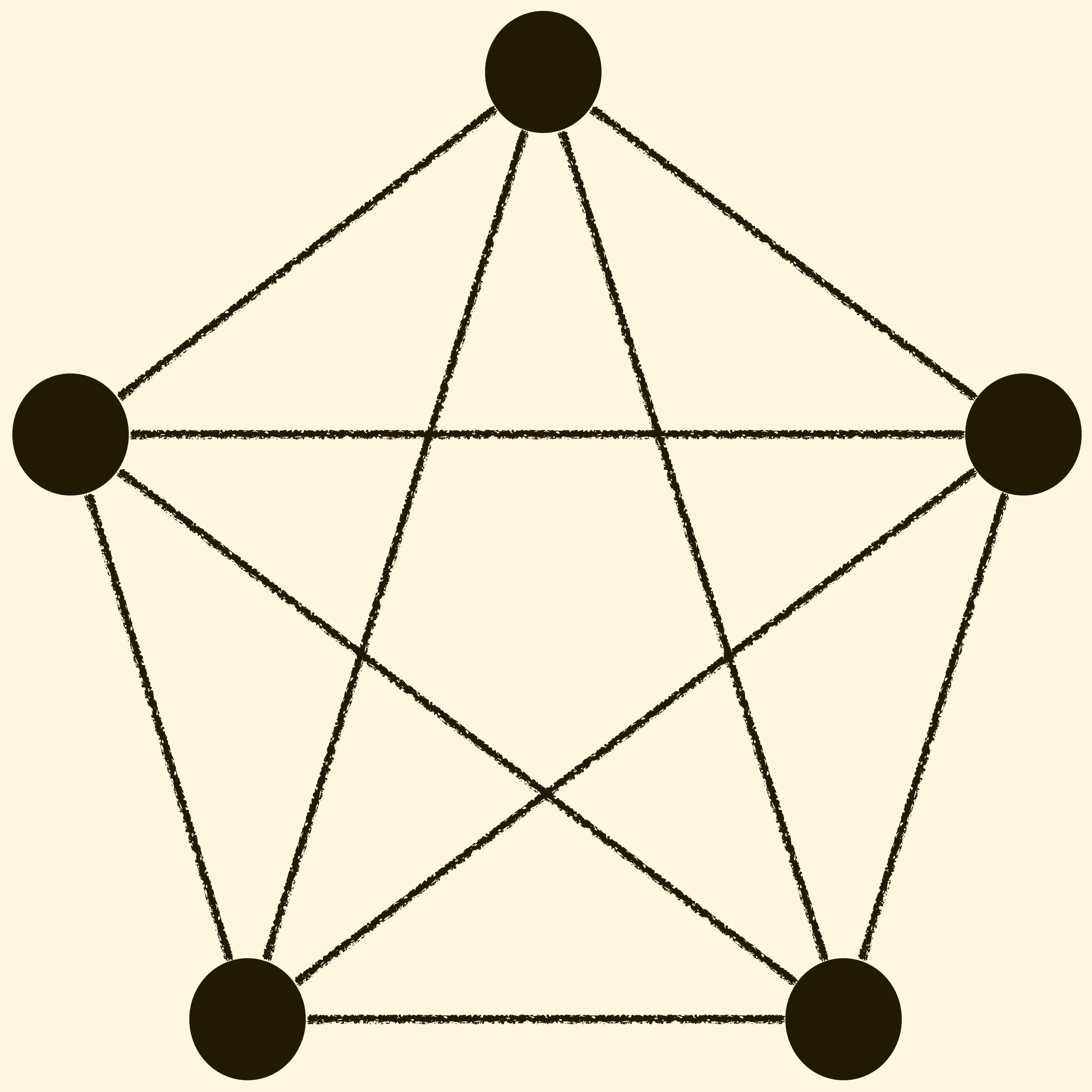

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Valid global configurations

Eulerian orientations of the underlying graph:
each vertex has equal in-degree and out-degree

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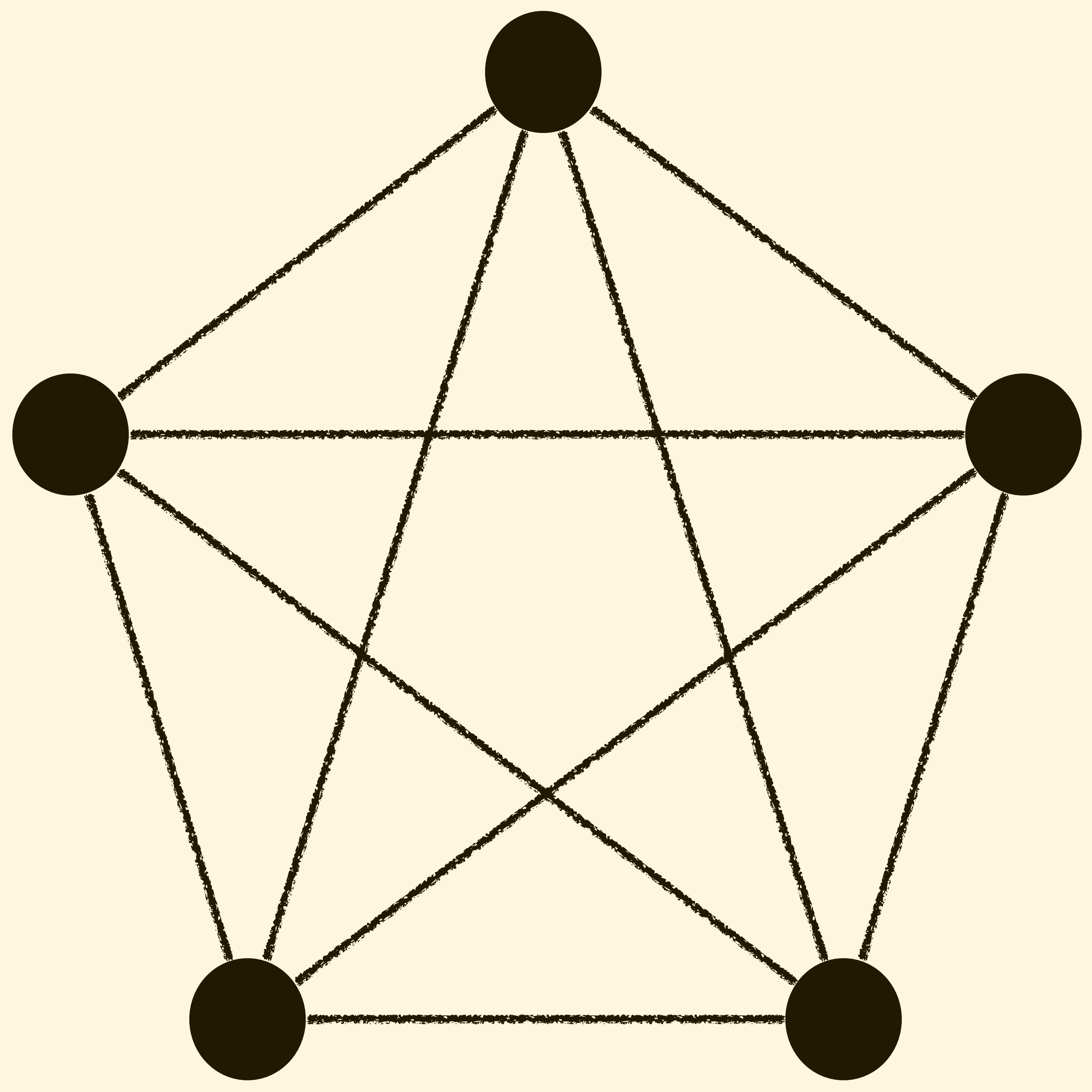

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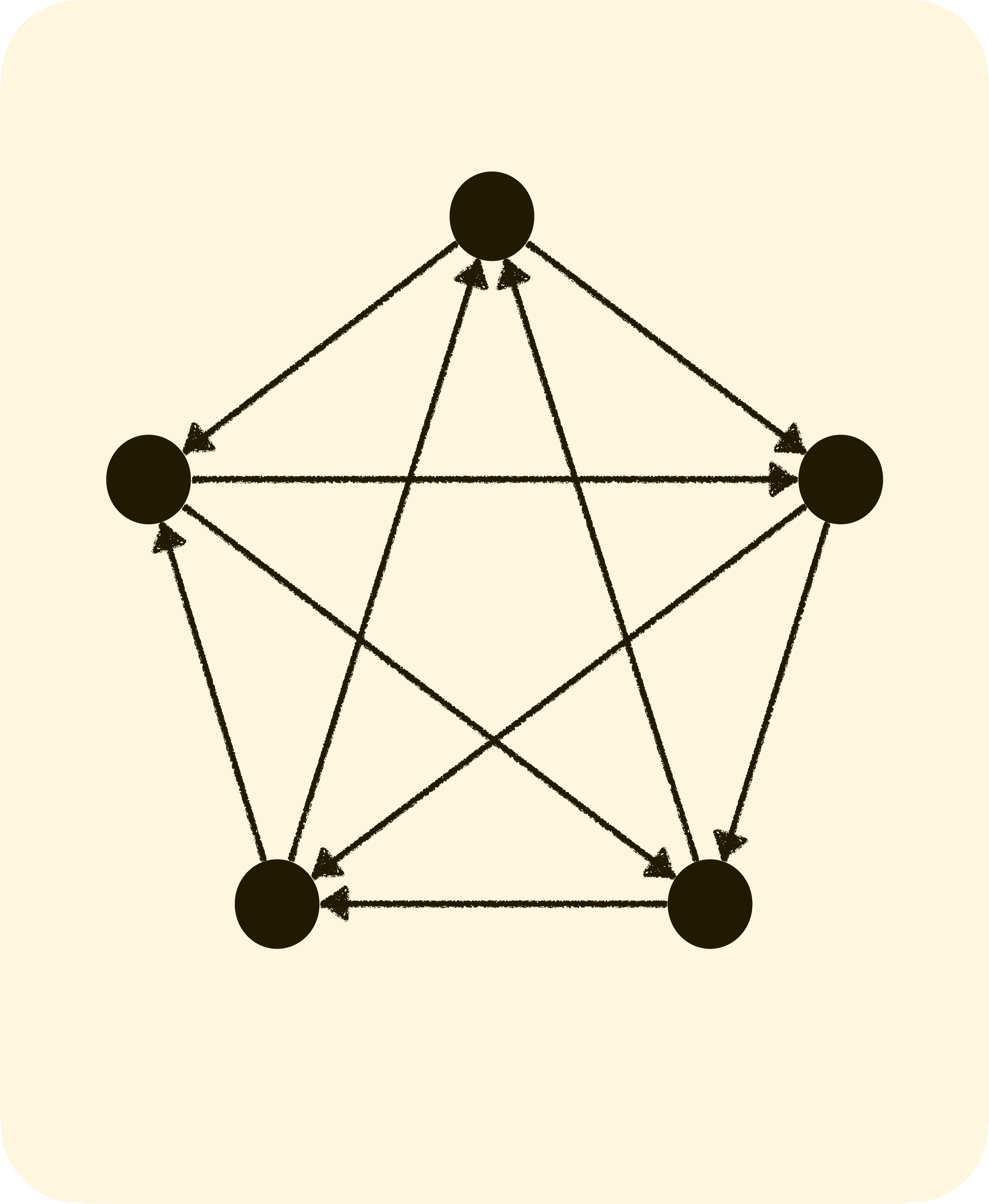

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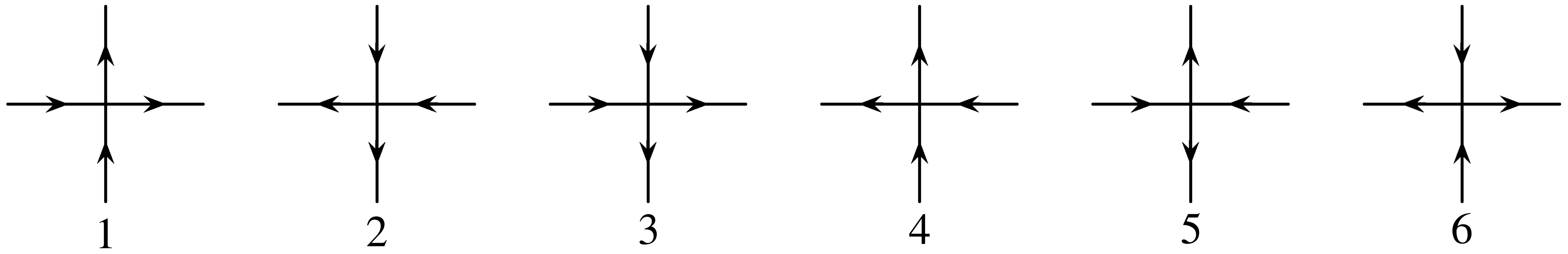
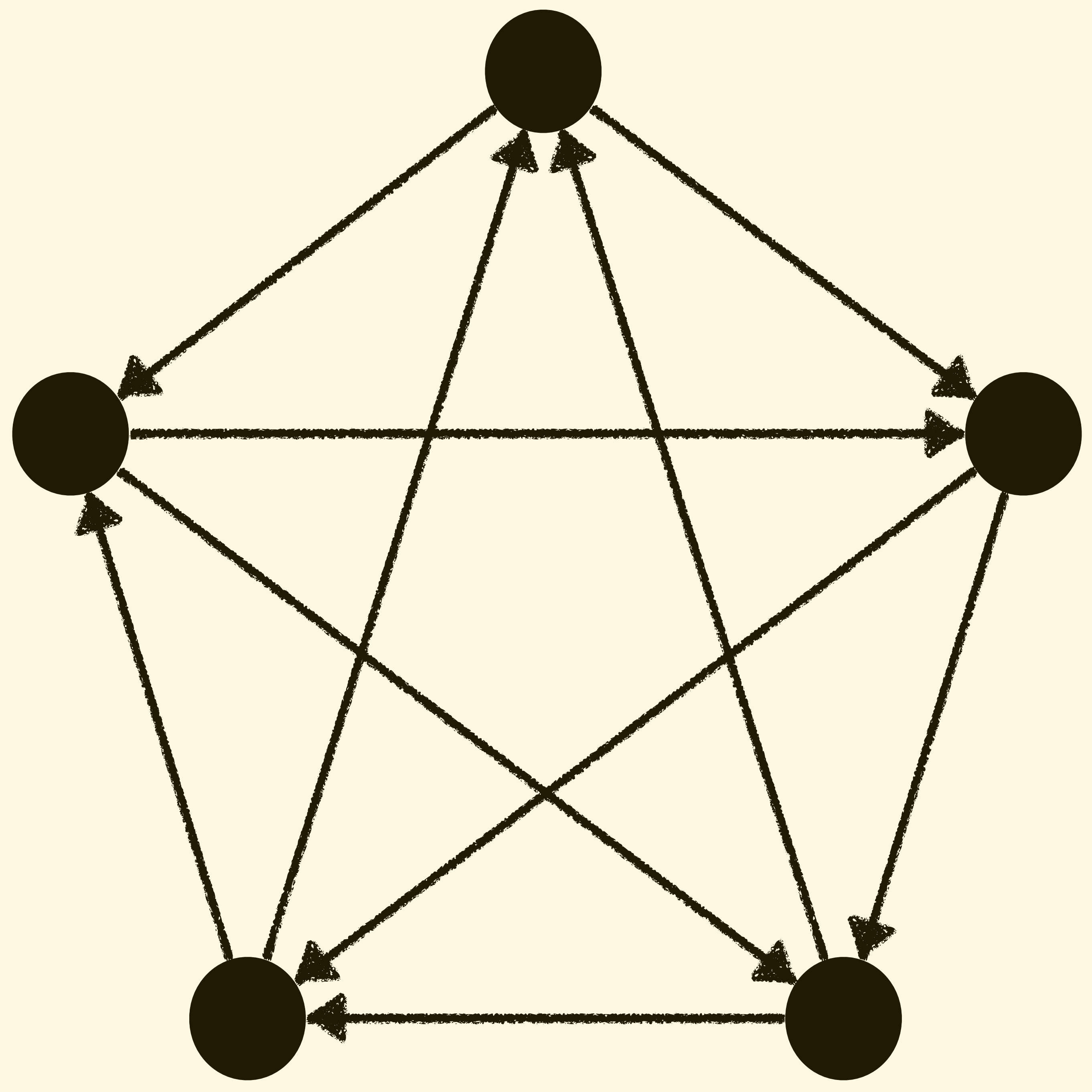
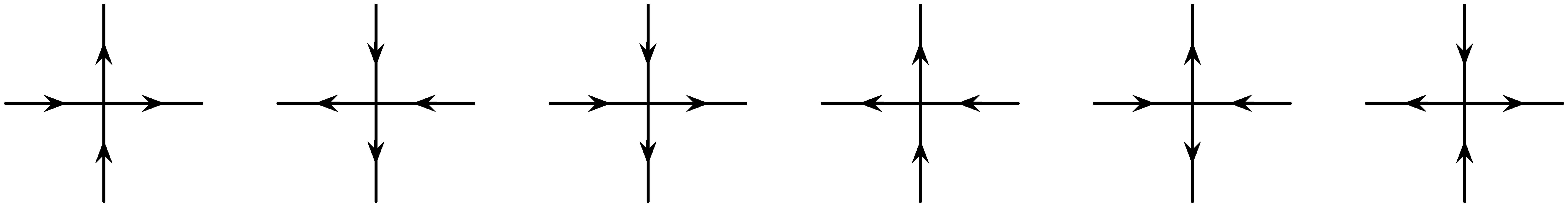
$$\binom{4}{2} = 6 \text{ types}$$





Partition Function



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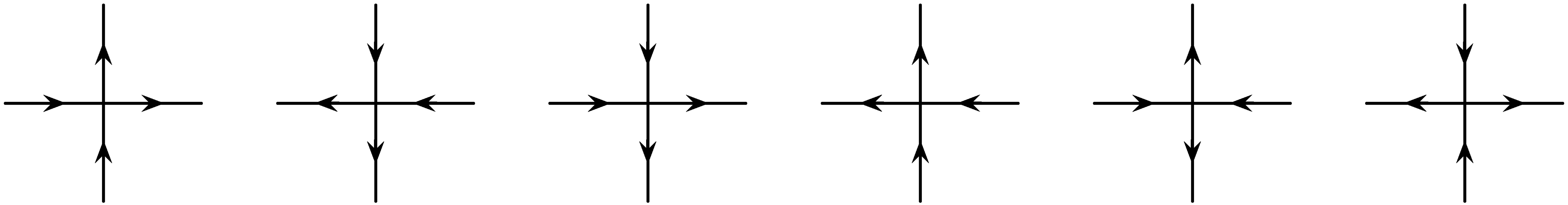

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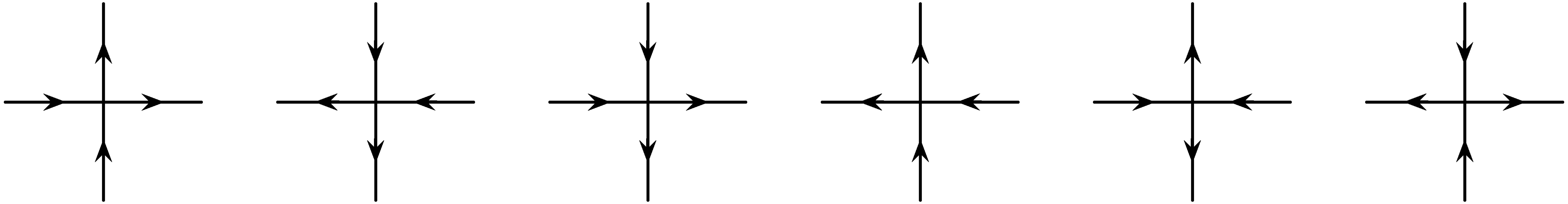

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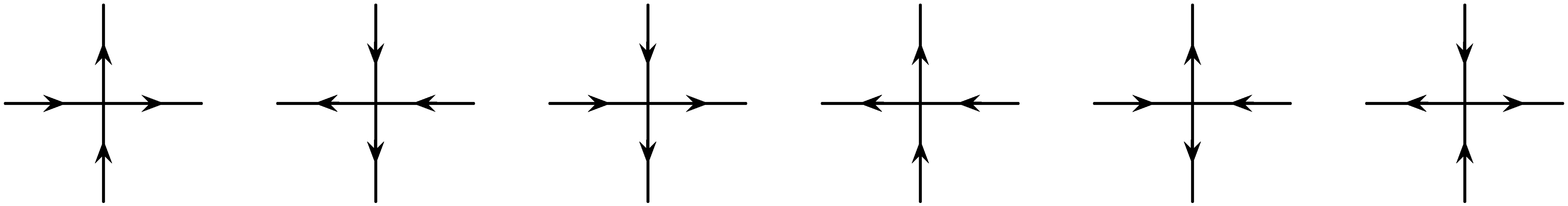

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arrow reversal symmetry
(zero field assumption)

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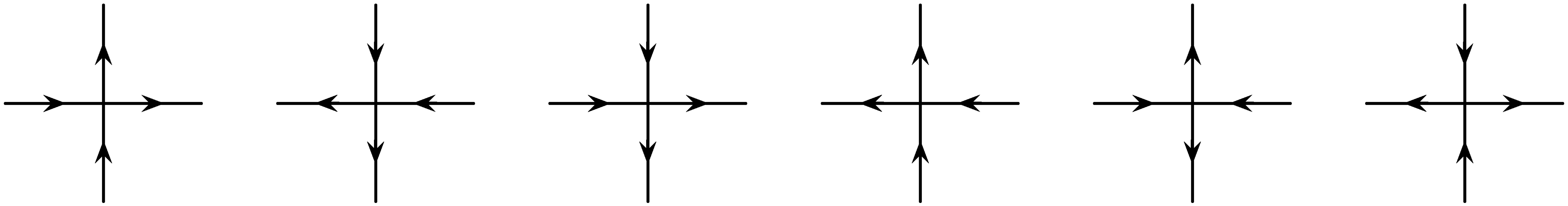

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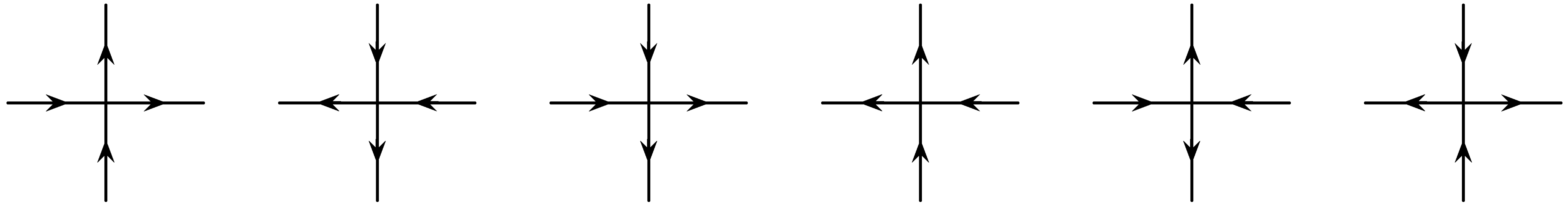

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- Water ice: $a = b = c = 1$
- KDP model: $a = b > 1, c = 1$
- Rys F model: $a = b = 1, c > 1$

Partition Function

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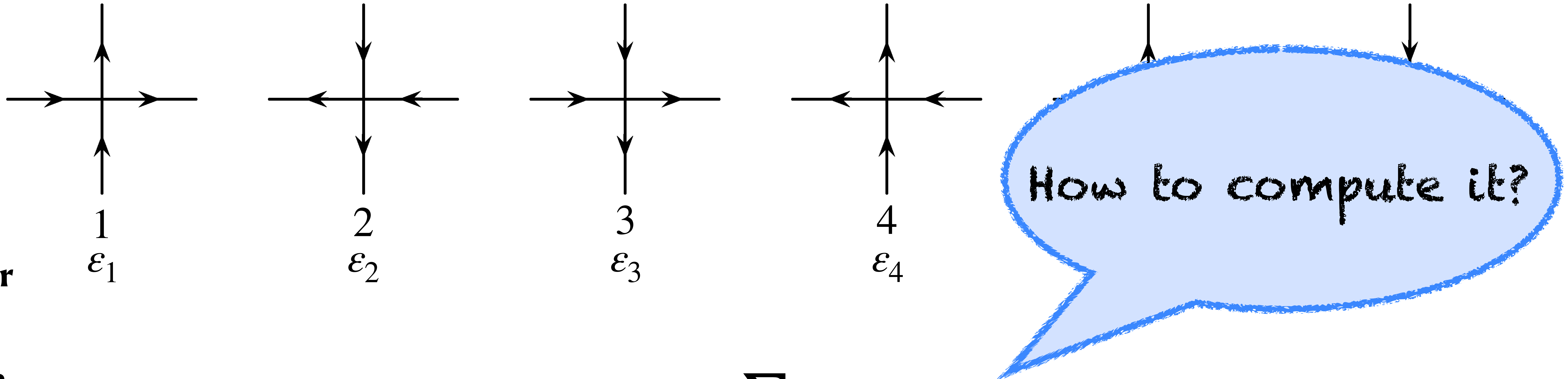

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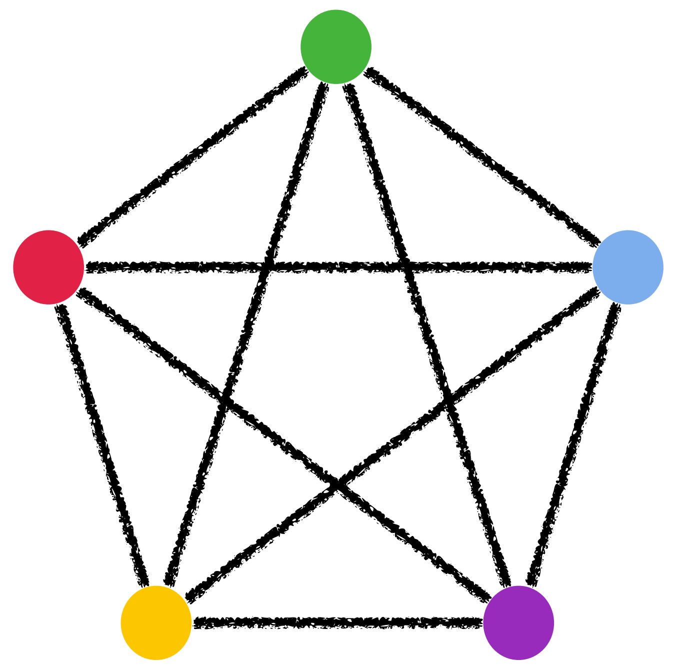

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bipartite graph $G' = (U_E, U_V, E')$

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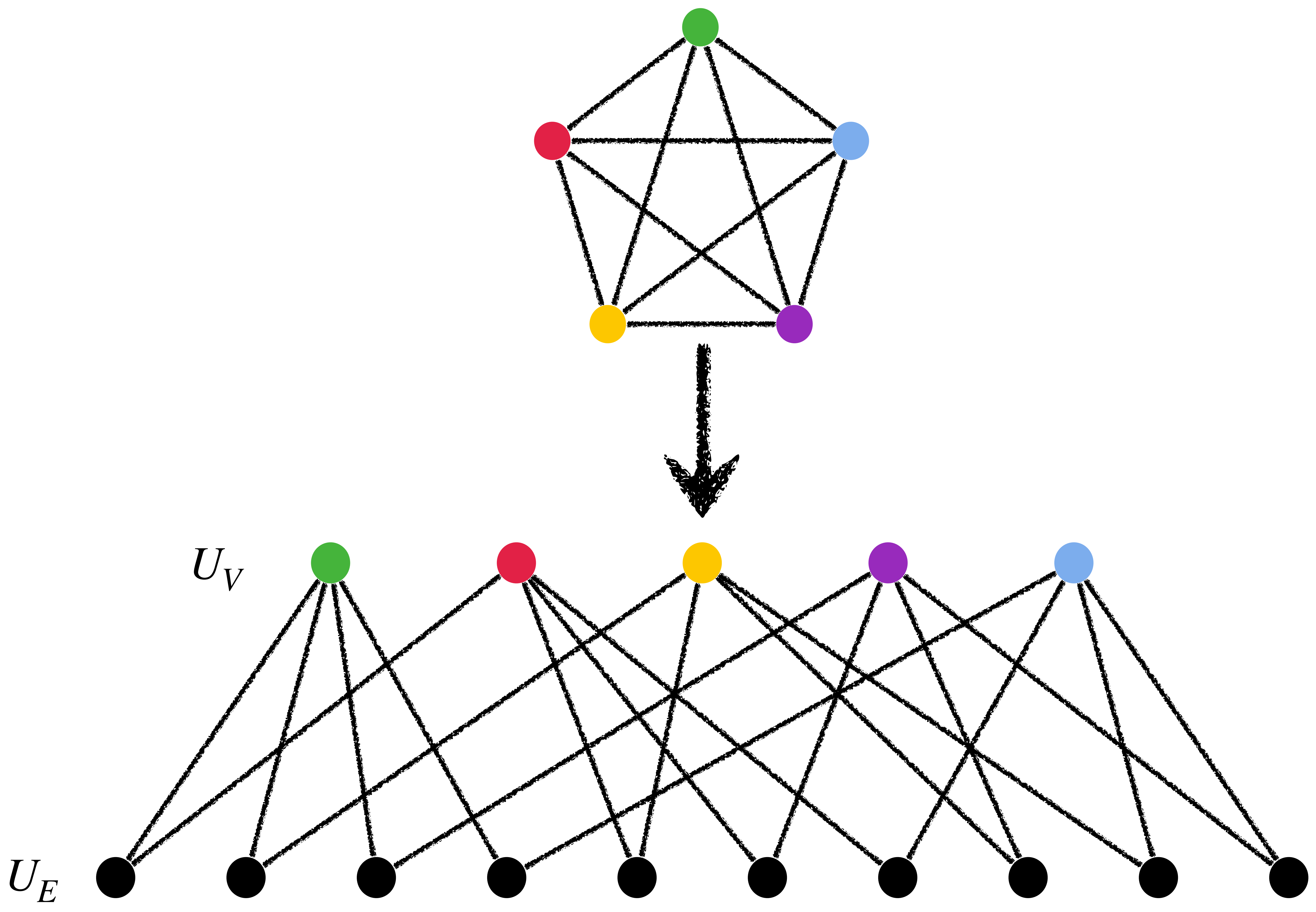

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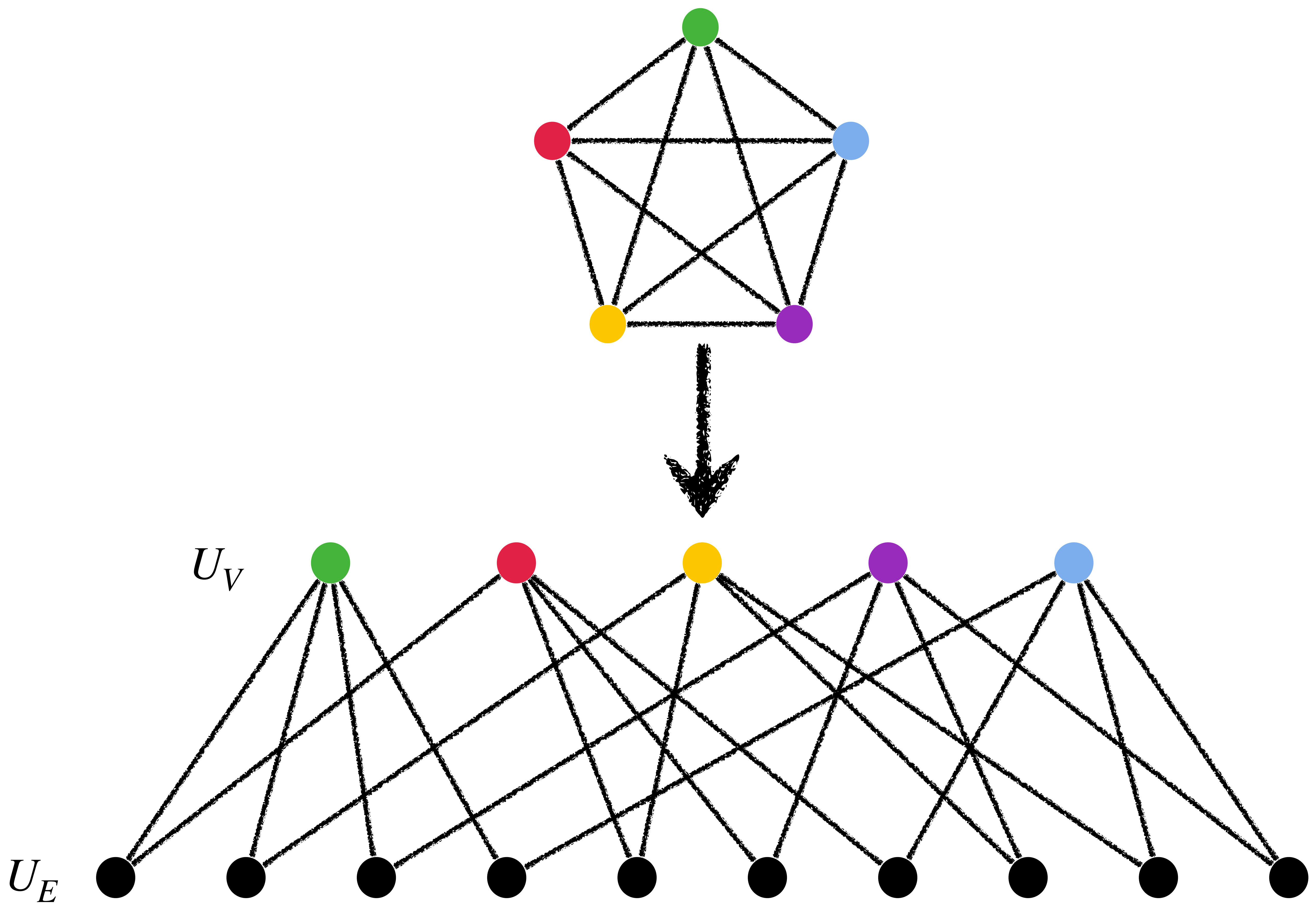

Bipartite Holant problem

when $G = (U, V, E)$ is a bipartite graph

mapping π

- assigns to each vertex $u \in U$ an $f_u \in \mathcal{F}$
- assigns to each vertex $v \in V$ an $f_v \in \mathcal{G}$
- a linear order of the incident edges at v and u.

the problem is written as $\text{Holant}_\Omega(\mathcal{F} \mid \mathcal{G})$

Six-Vertex Model as Holant Problem

Edge-vertex incidence graph

bipartite graph $G' = (U_E, U_V, E')$

edge $(u_e, u_v) \in E'$ iff $e \in E(v)$

Bipartite Holant problem

Assign $f_{6\mathrm{V}}$ to each $u_v \in U_V$, where

$$f_{6\mathrm{V}}\left(\begin{bmatrix} 0000 & 0010 & 0001 & 0011 \\ 0100 & 0110 & 0101 & 0111 \\ 1000 & 1010 & 1001 & 1011 \\ 1100 & 1110 & 1101 & 1111 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$$

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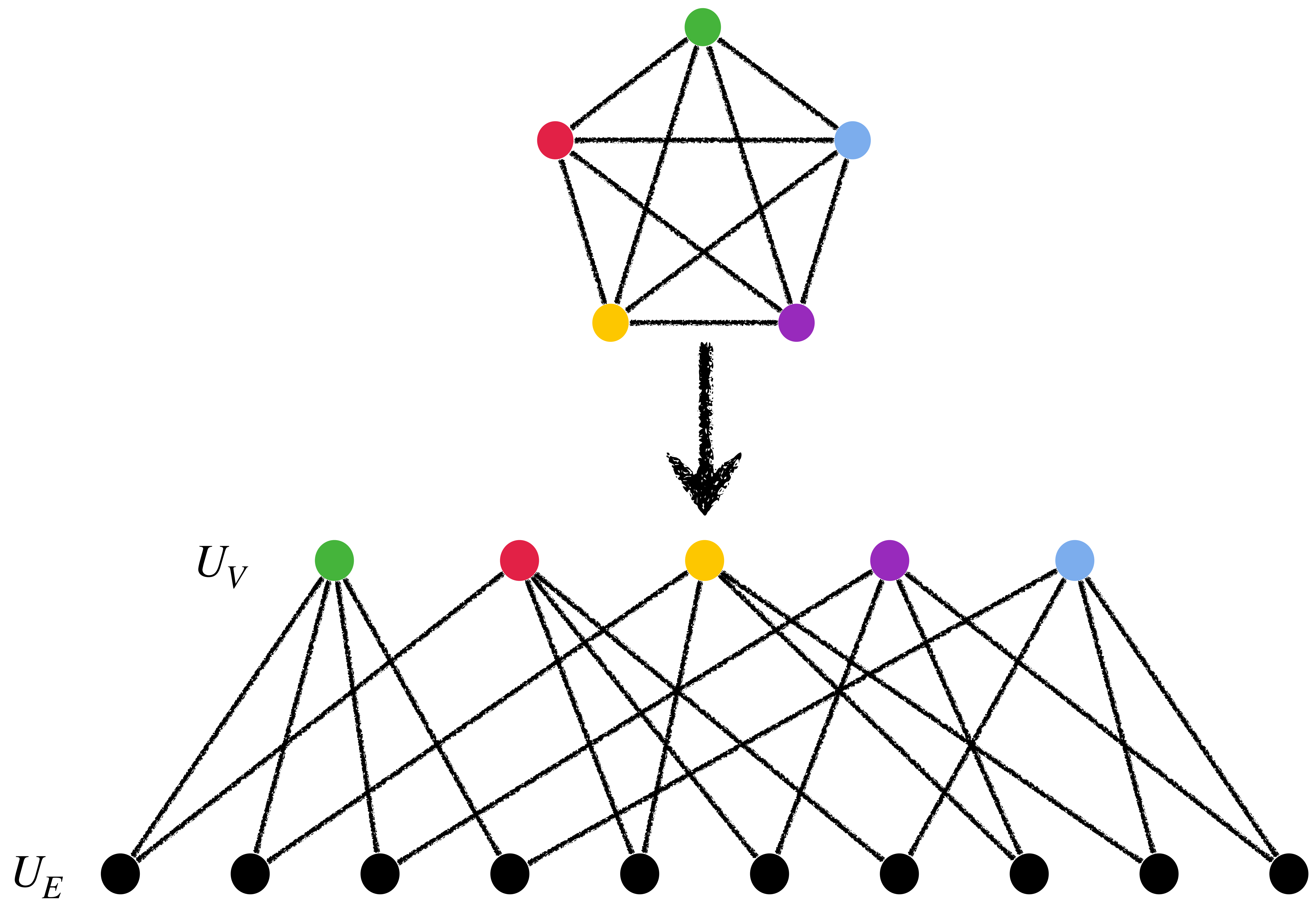

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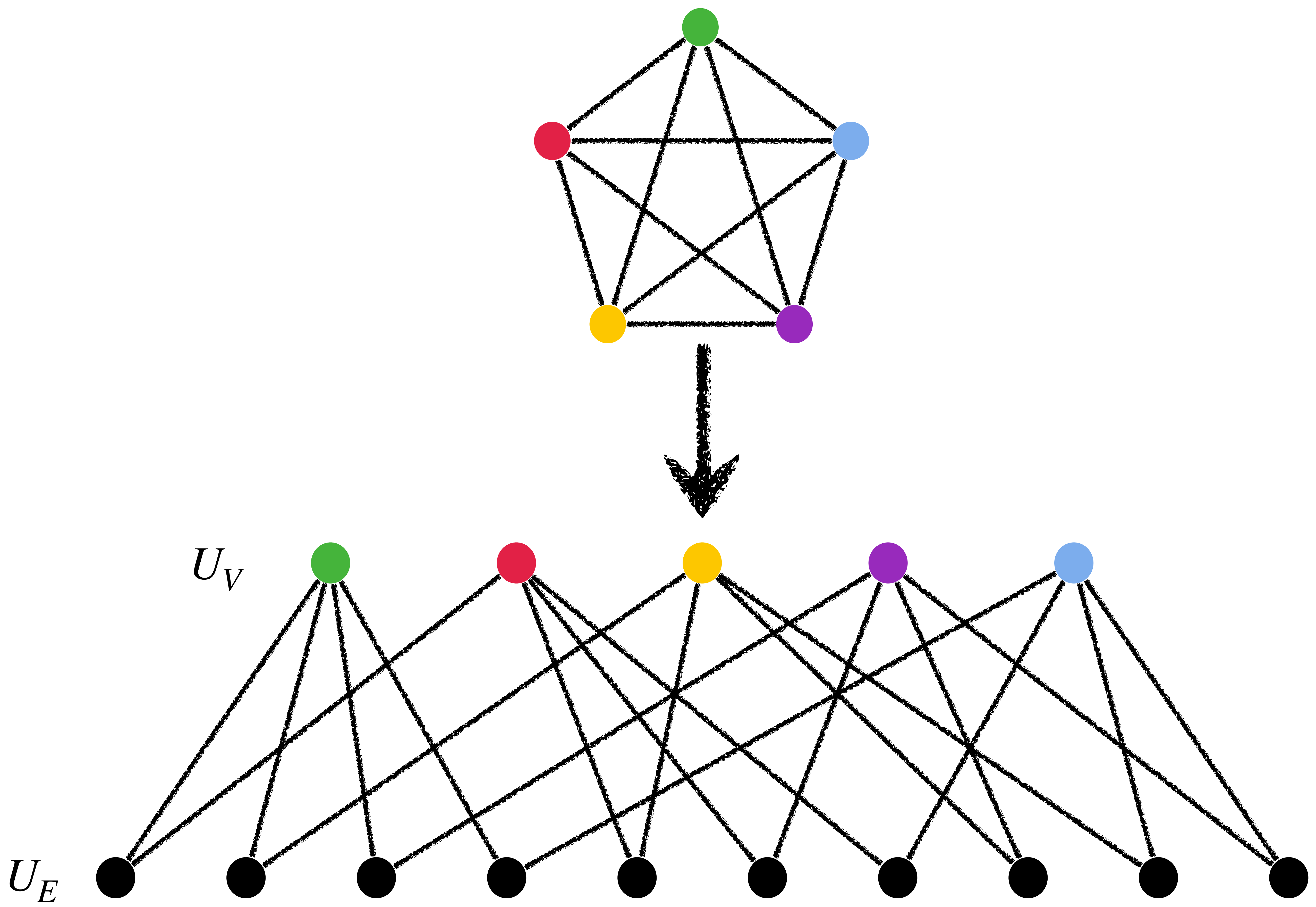

$$Z_{6V}(G; a, b, c) = \text{Hoalnt}_{\Omega}(\neq_2 | f_{6V})$$

Exact Computational Complexity

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For #EulerianOrientation (unweighted case):

It is #P-complete on

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- tractable
- traceable on planar graphs, but #P-hard on general graphs
- #P-hard even on planar graphs

Approximate Counting and Sampling

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There is ***an FPRAS*** for $Z_{6V}(G; a, b, c)$ if $a^2 \leq b^2 + c^2, b^2 \leq a^2 + c^2$, and $c^2 \leq a^2 + b^2$.

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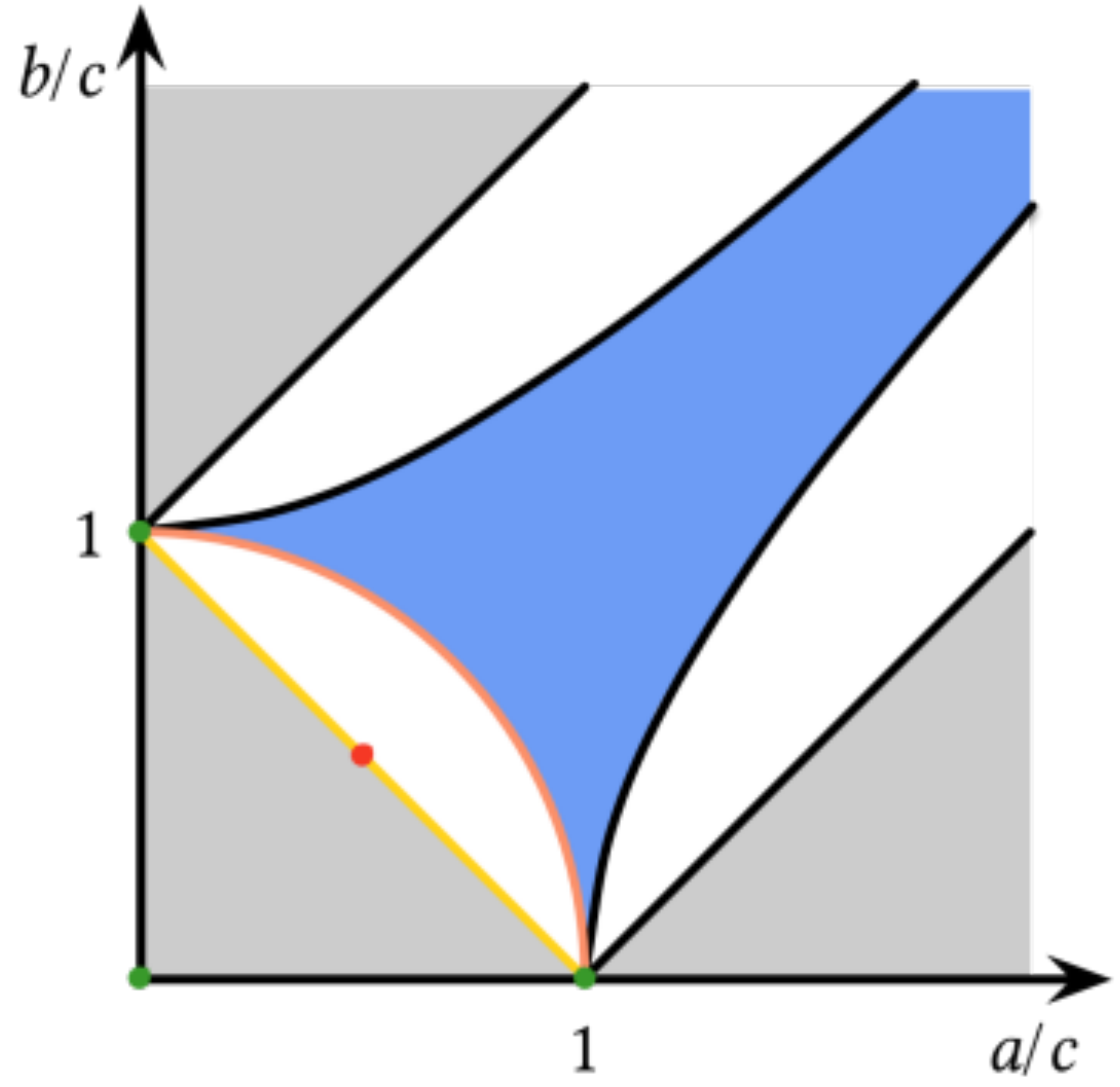

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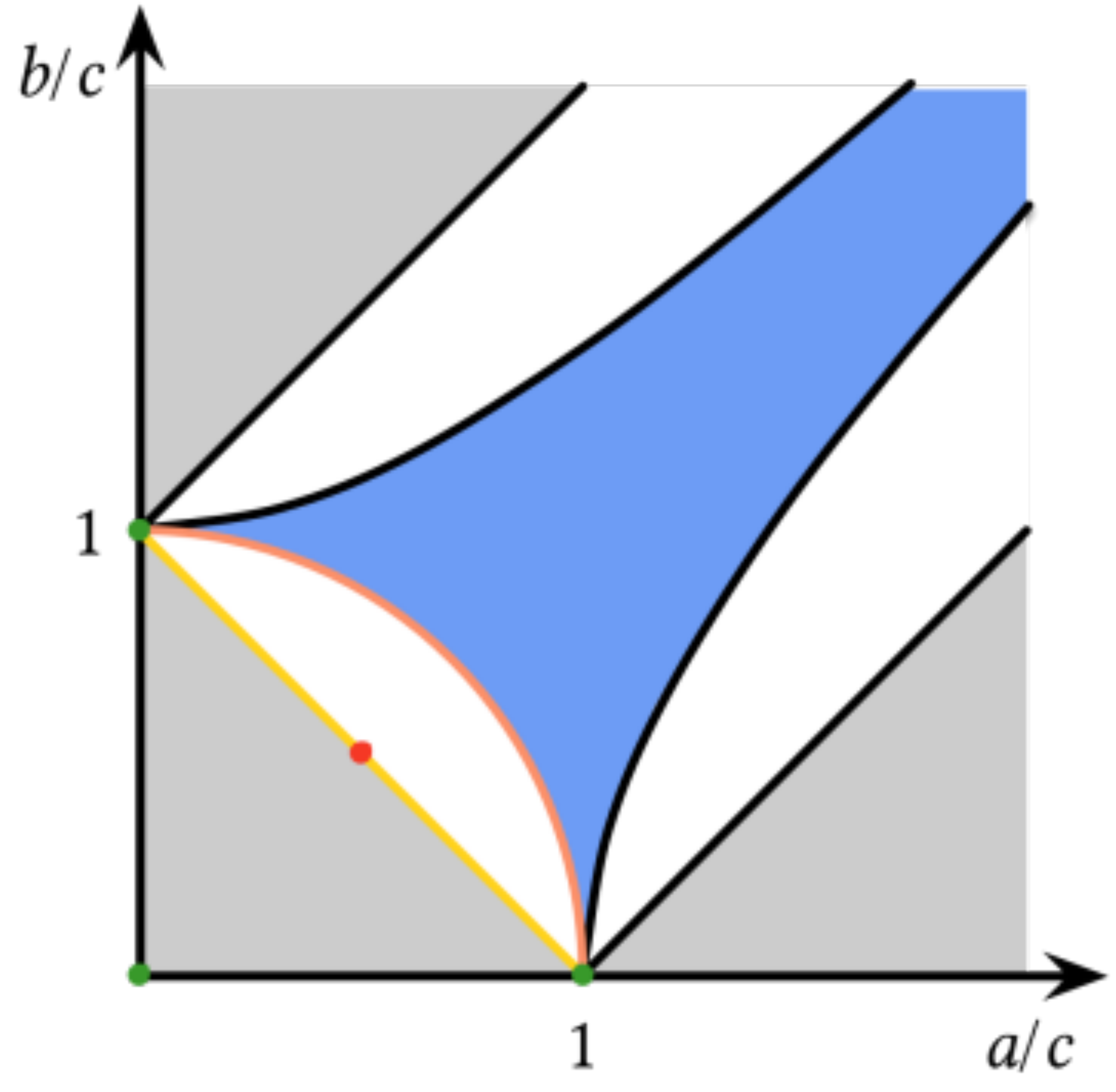

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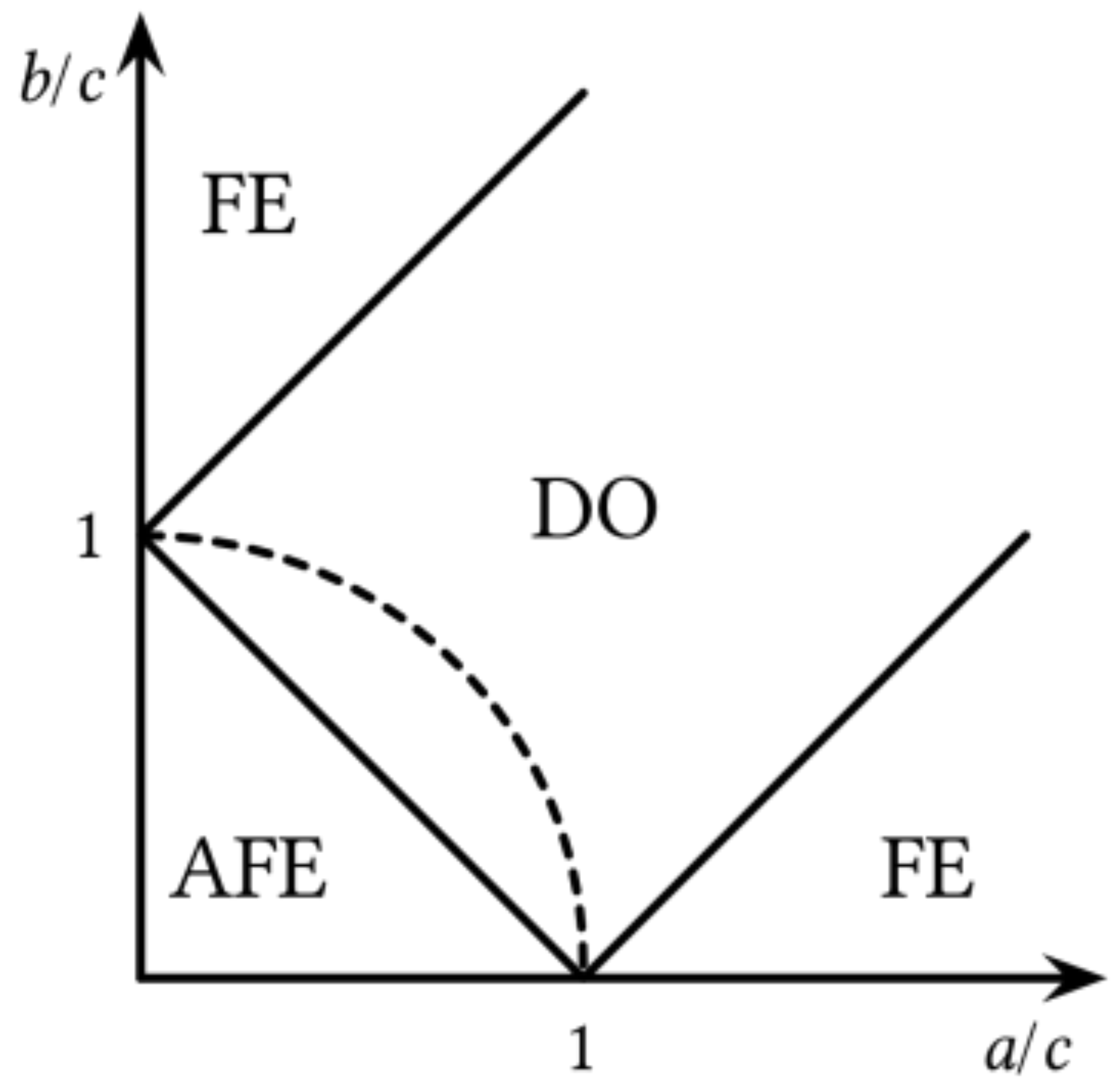

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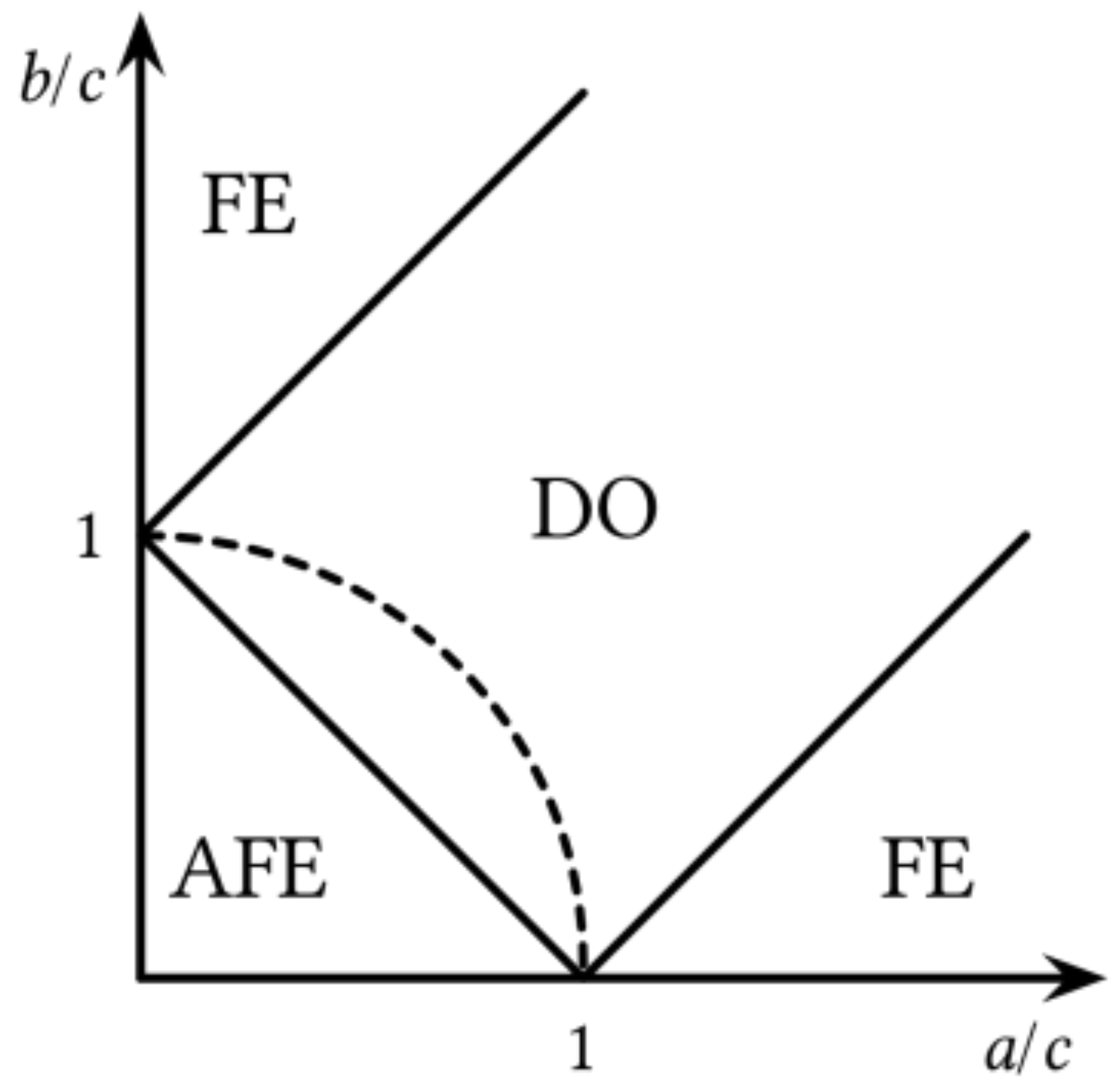

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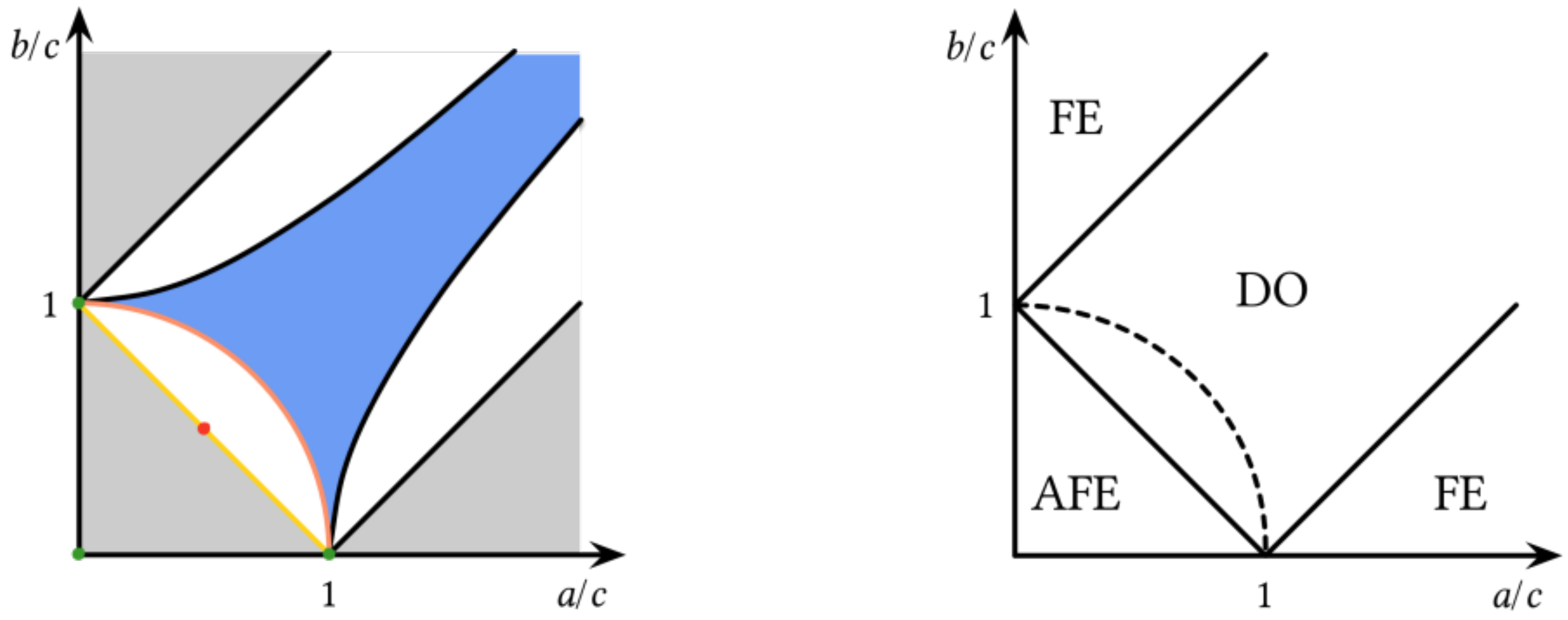
Approximability v.s. Phase Transition



Approximability of the six-vertex model (roughly) conforms the phase transition phenomenon in physics!

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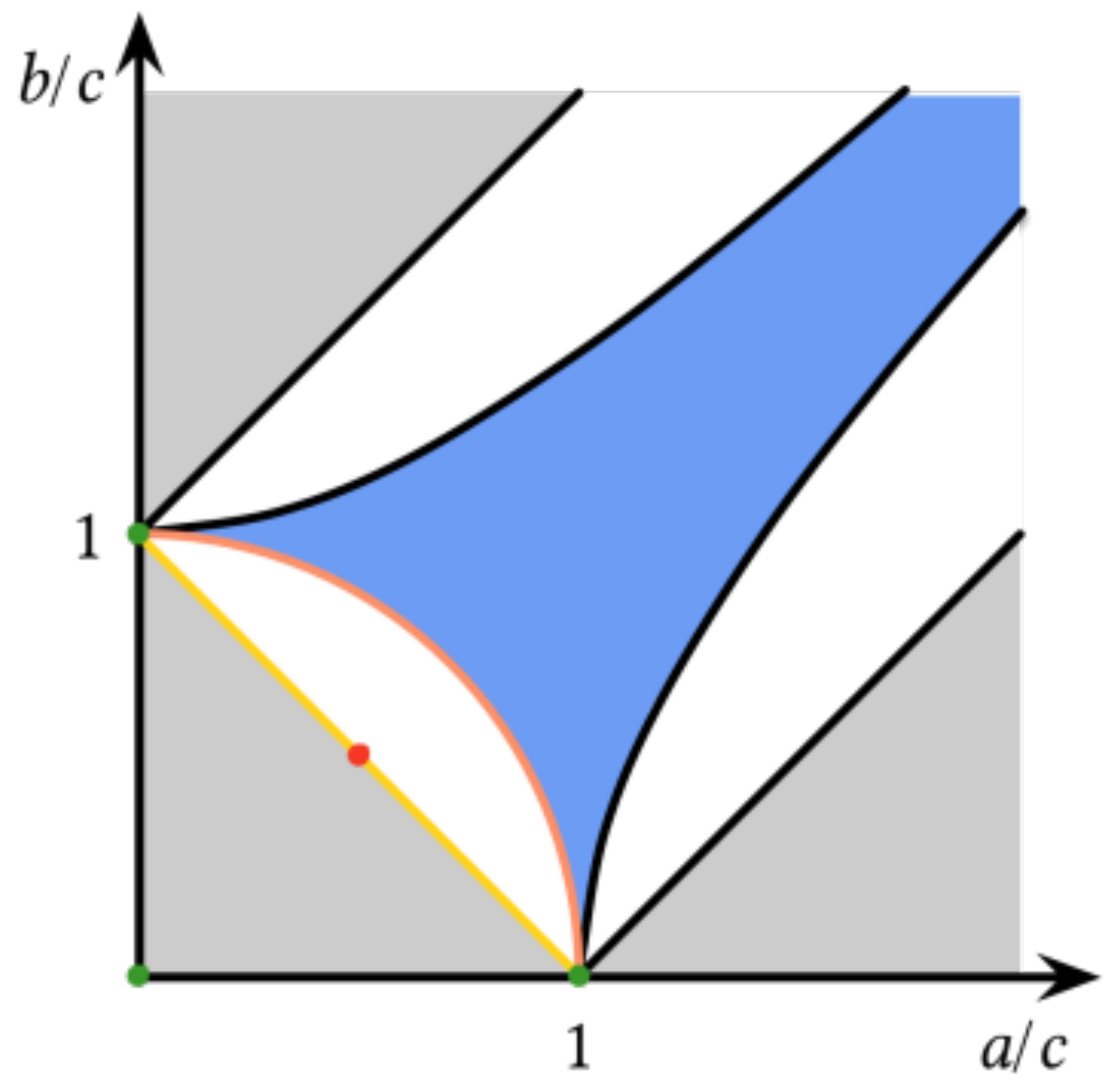

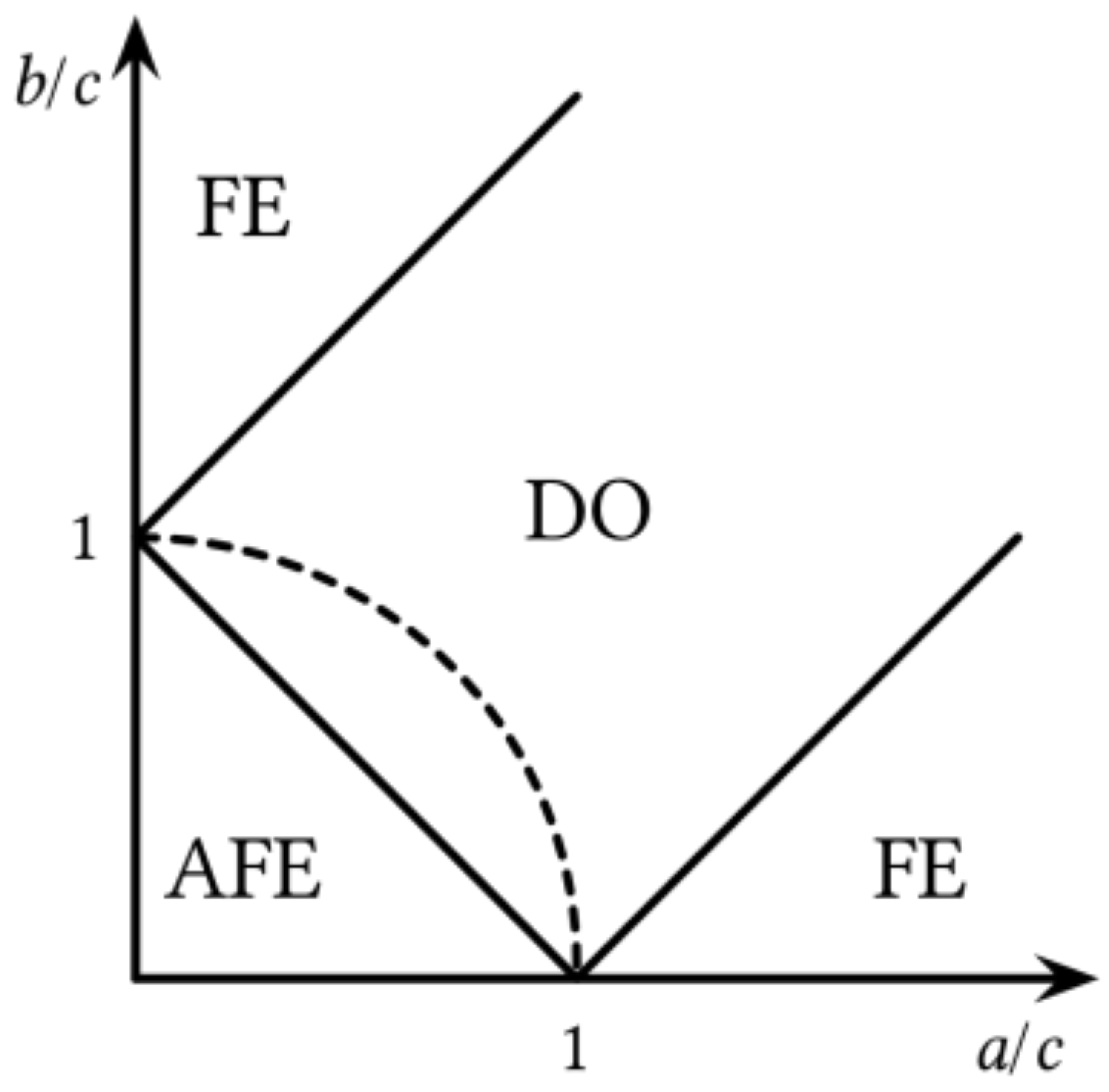

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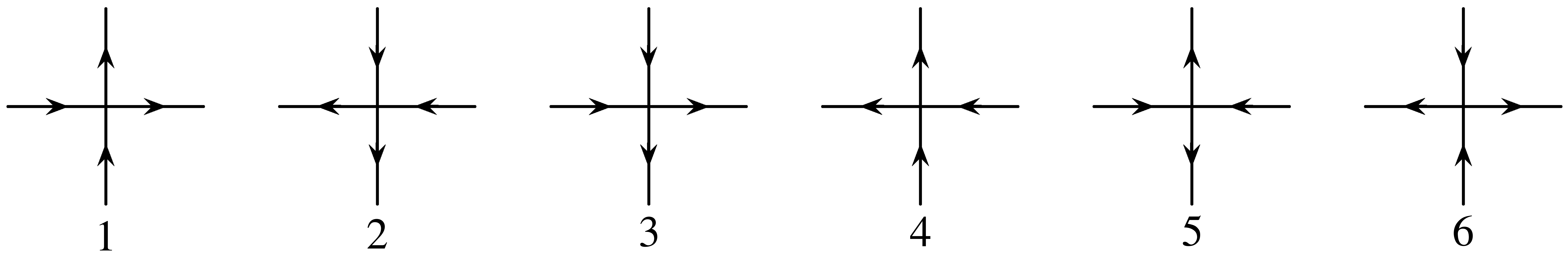
How to bridge the gap?

Attempt: Four-Vertex Model



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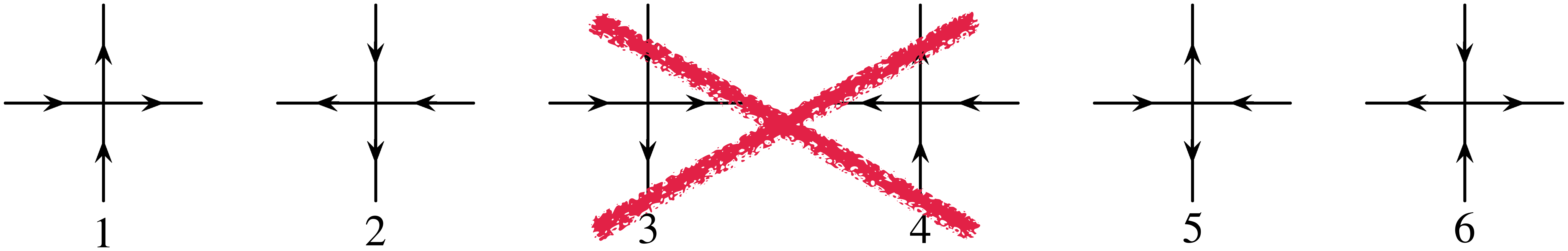

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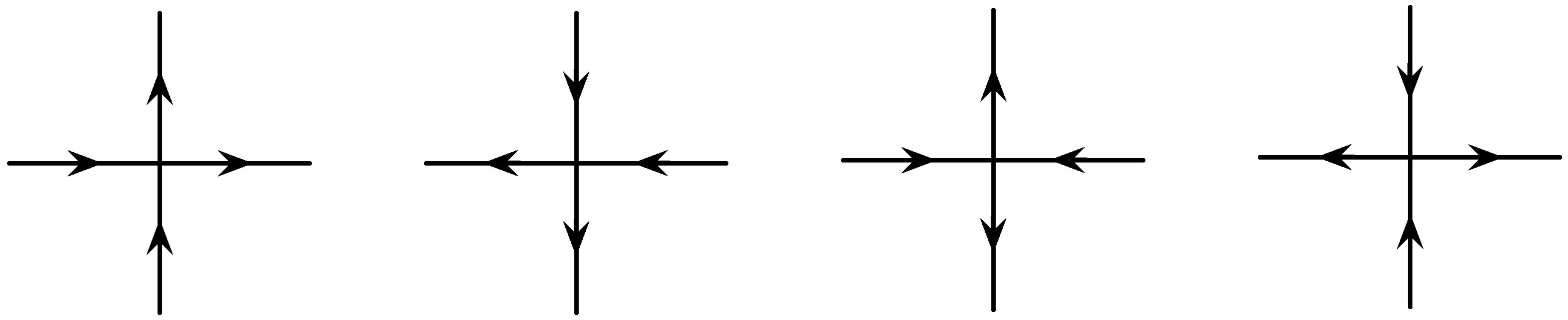

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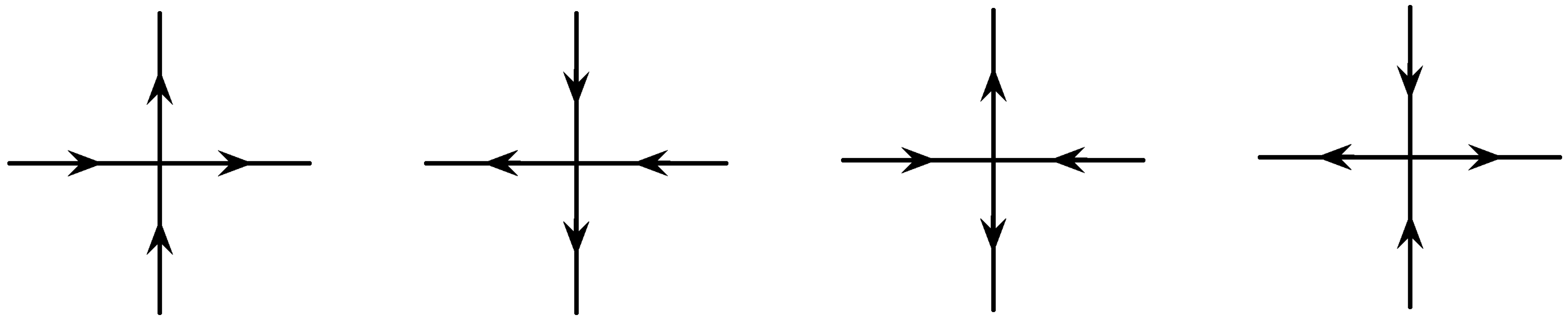

normalized weight	β	β	1	1

(WLOG, $\beta > 1$)

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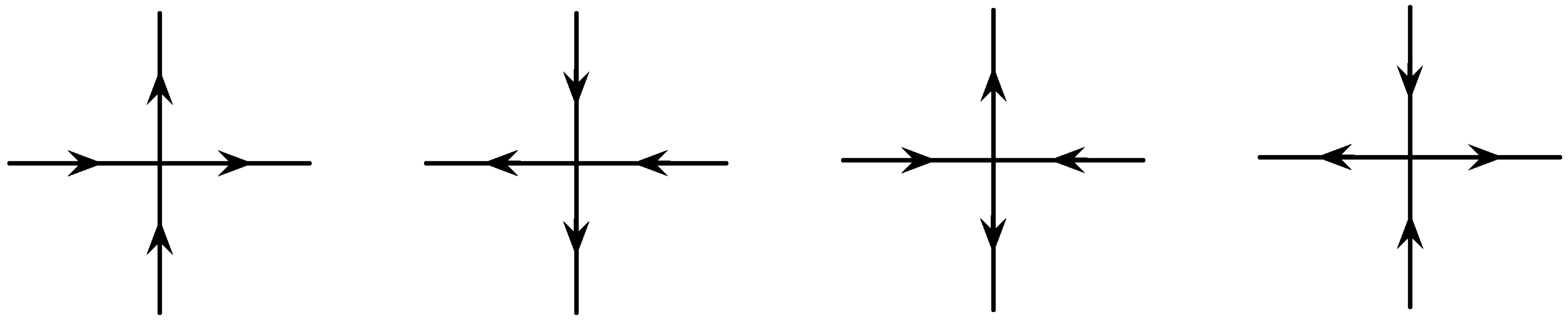

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Goal: compute

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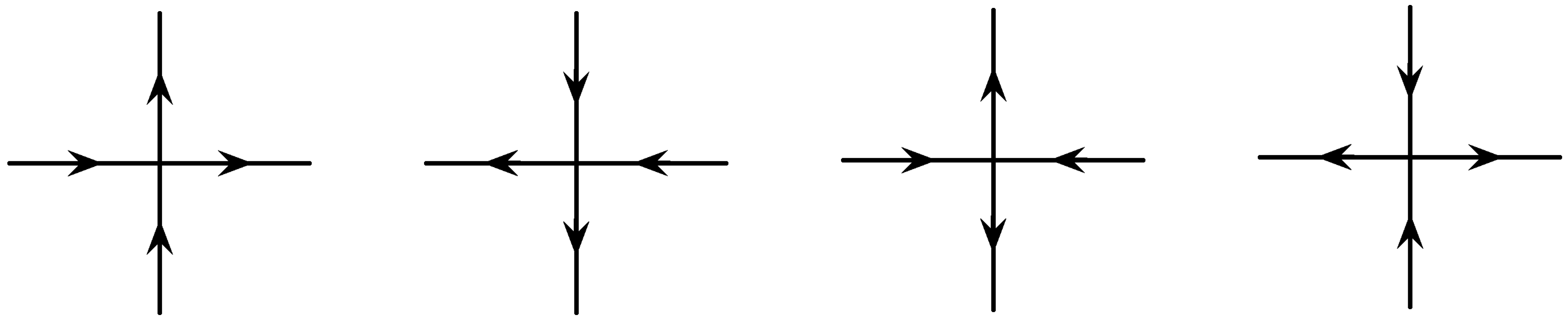

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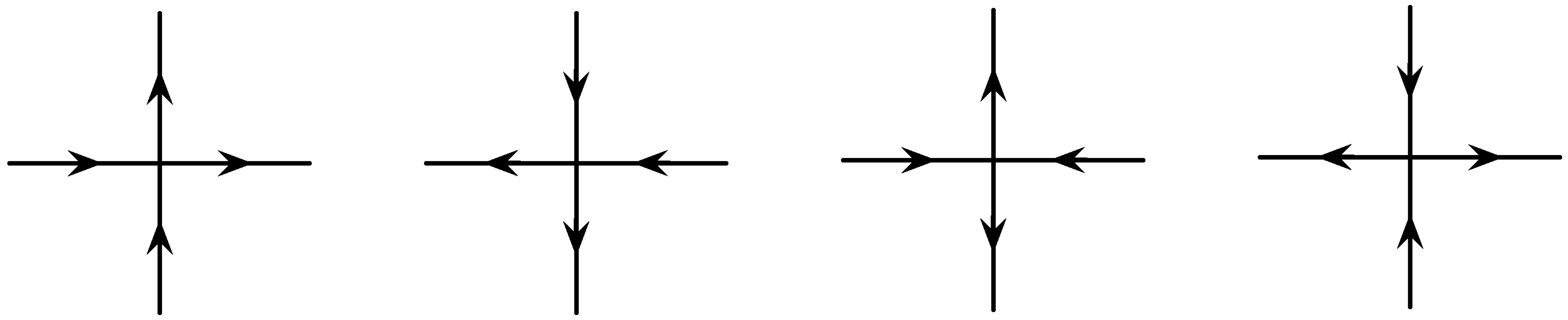

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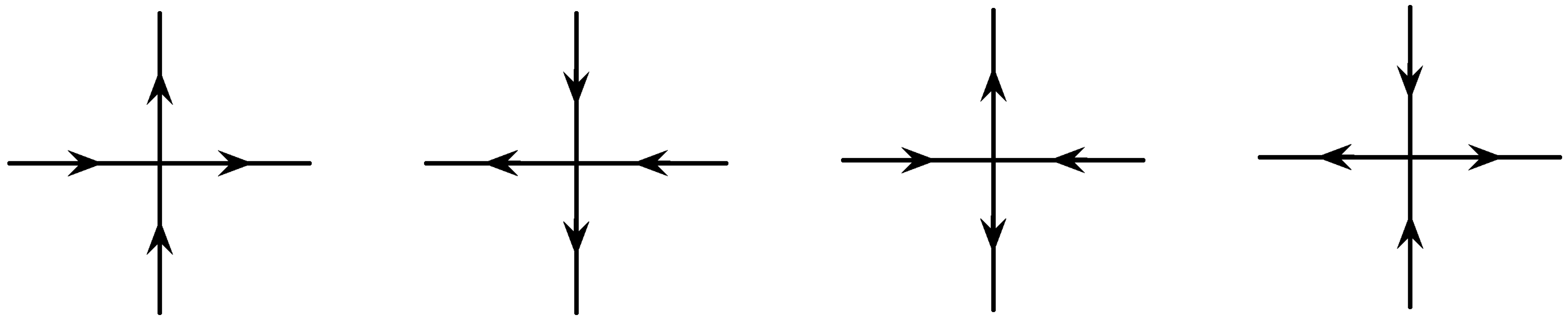

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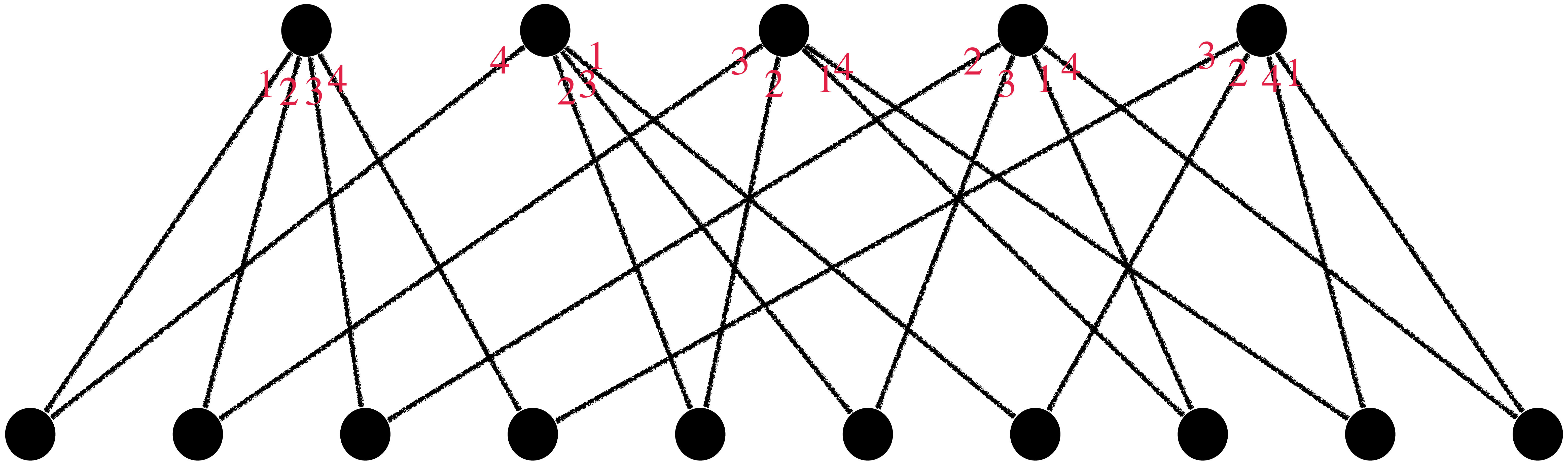

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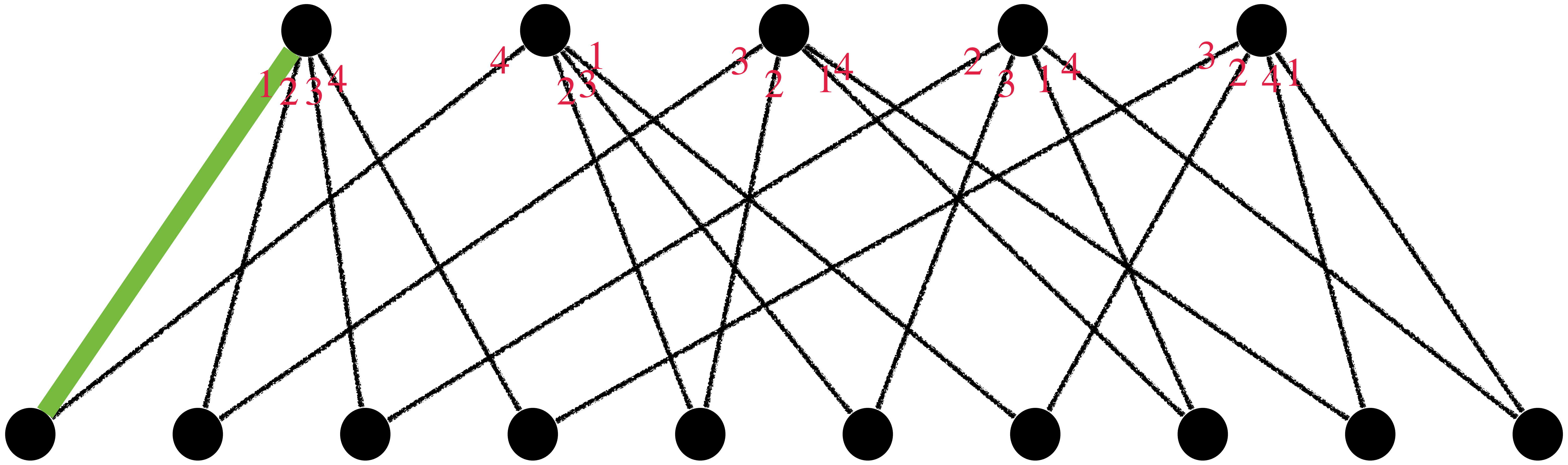

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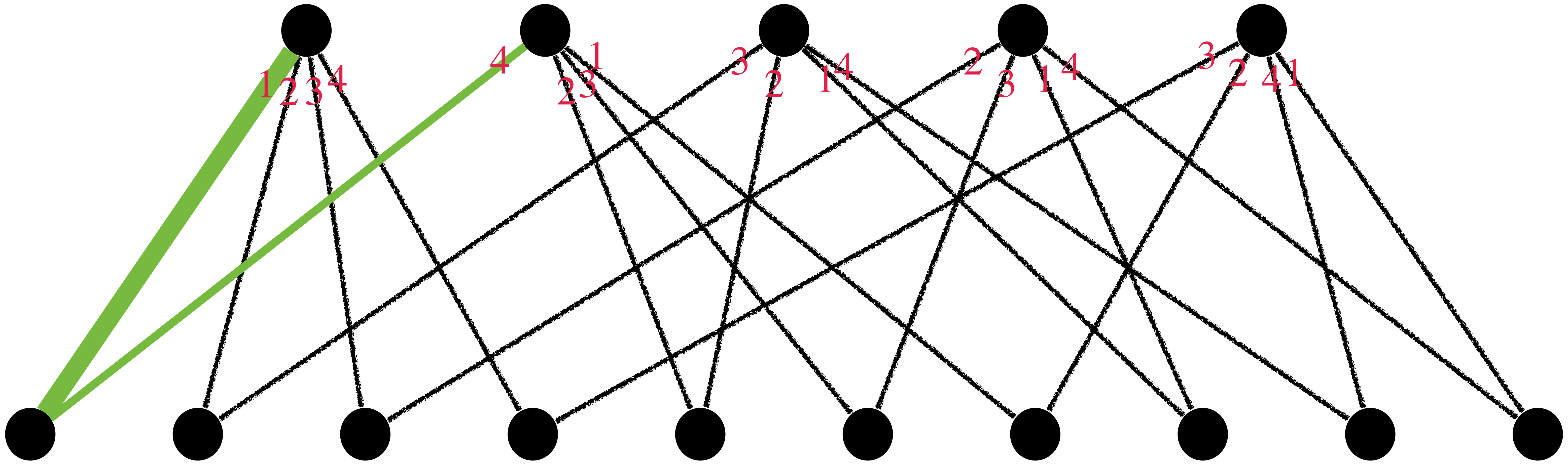

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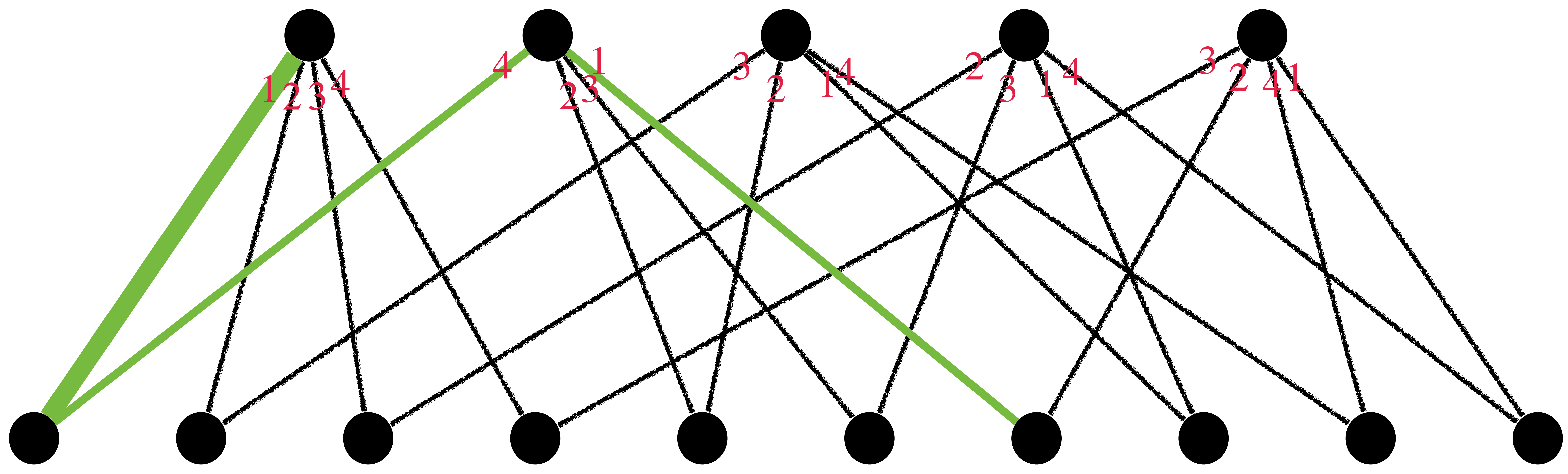

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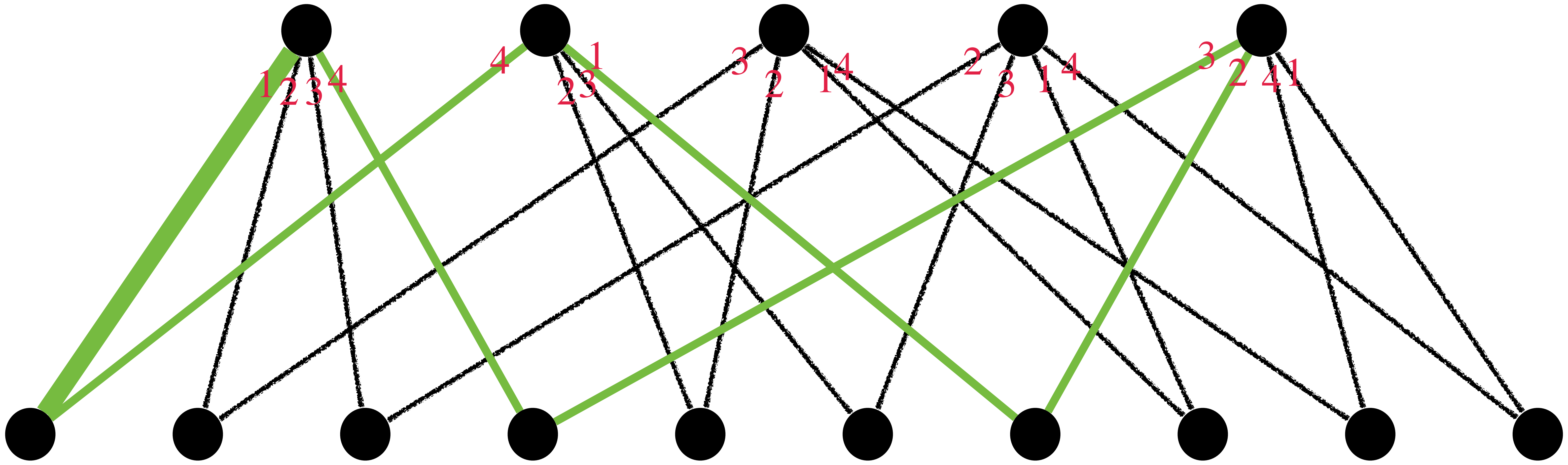

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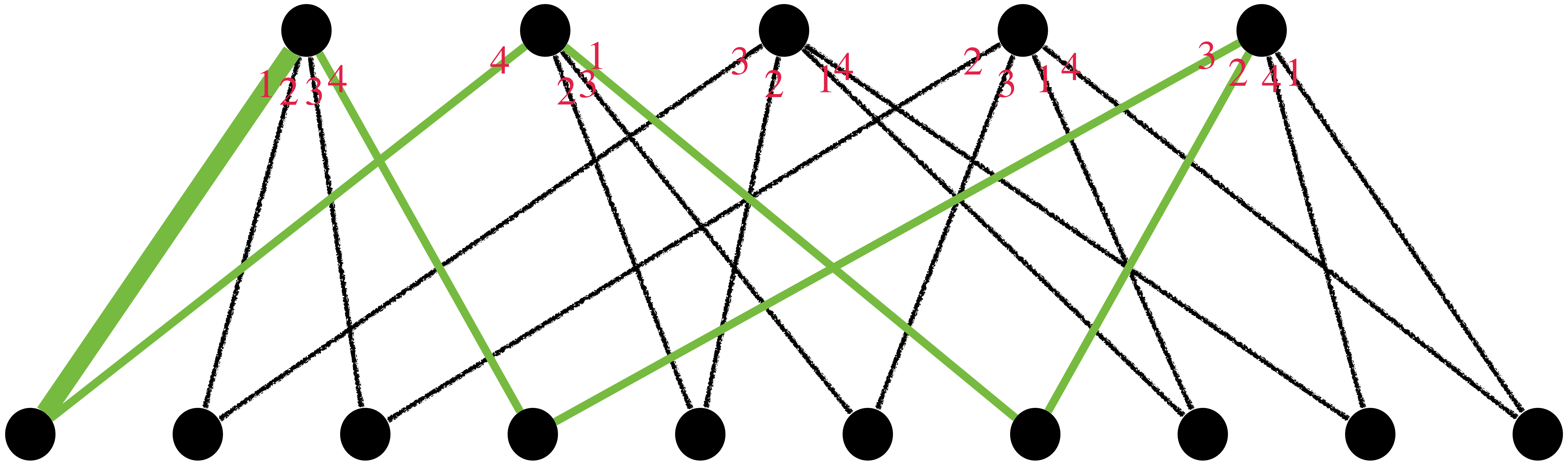

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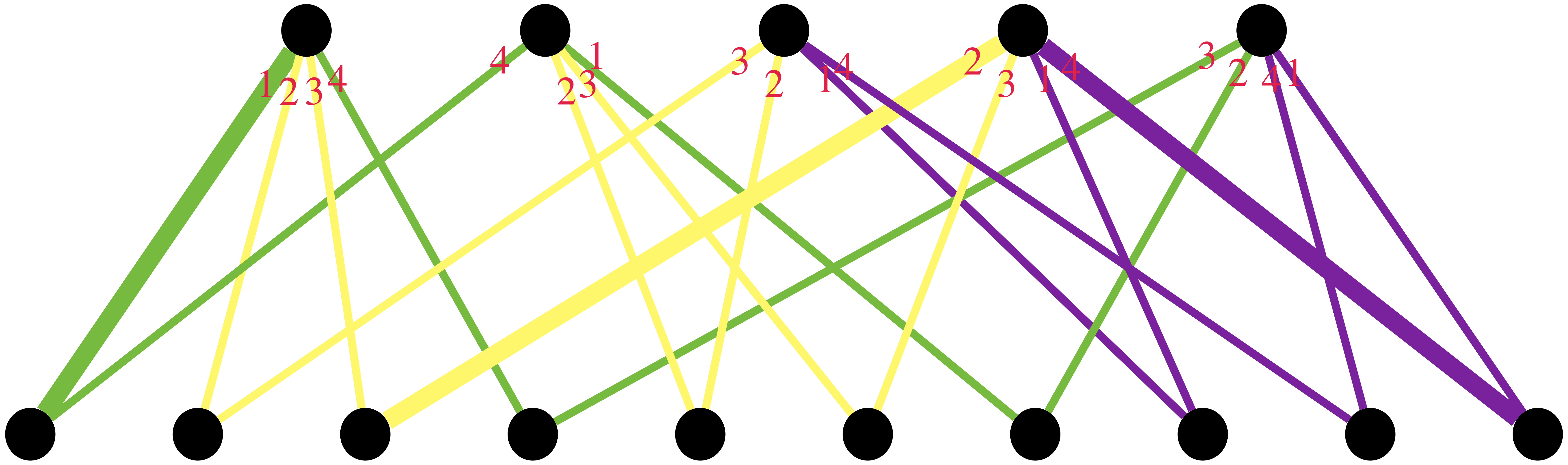

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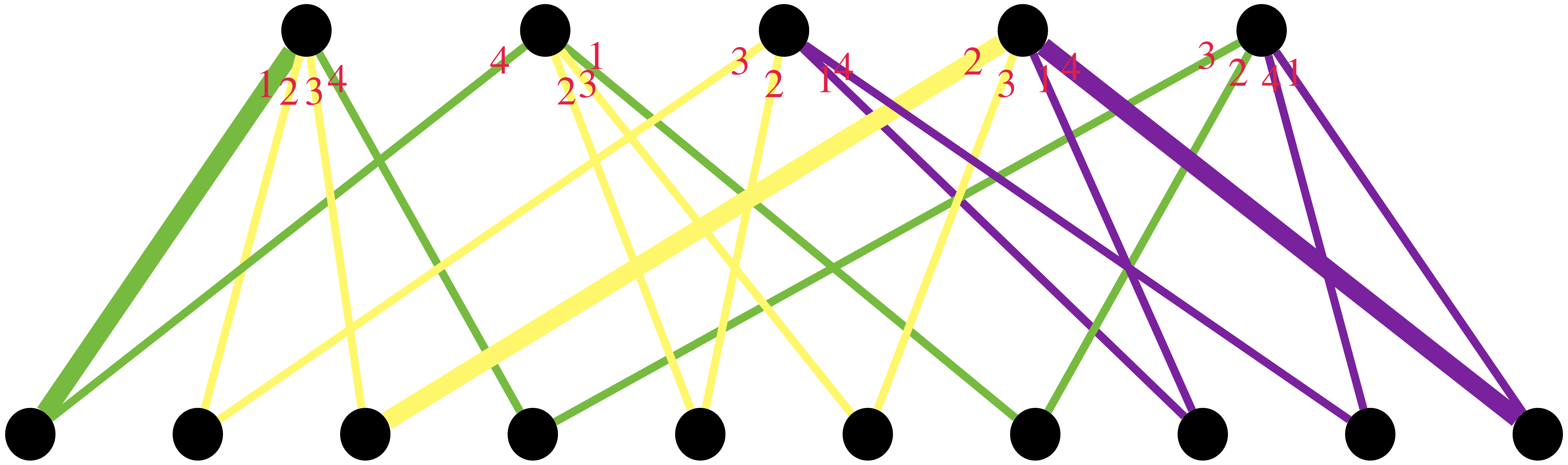

Graph of circuits

$$G_C = (\mathscr{C}, E_C)$$

$\mathscr{C} = \{C_1, C_2 \cdots\}$: set of circuits

$(C_i, C_j) \in E_C$ iff C_i and C_j have common vertex

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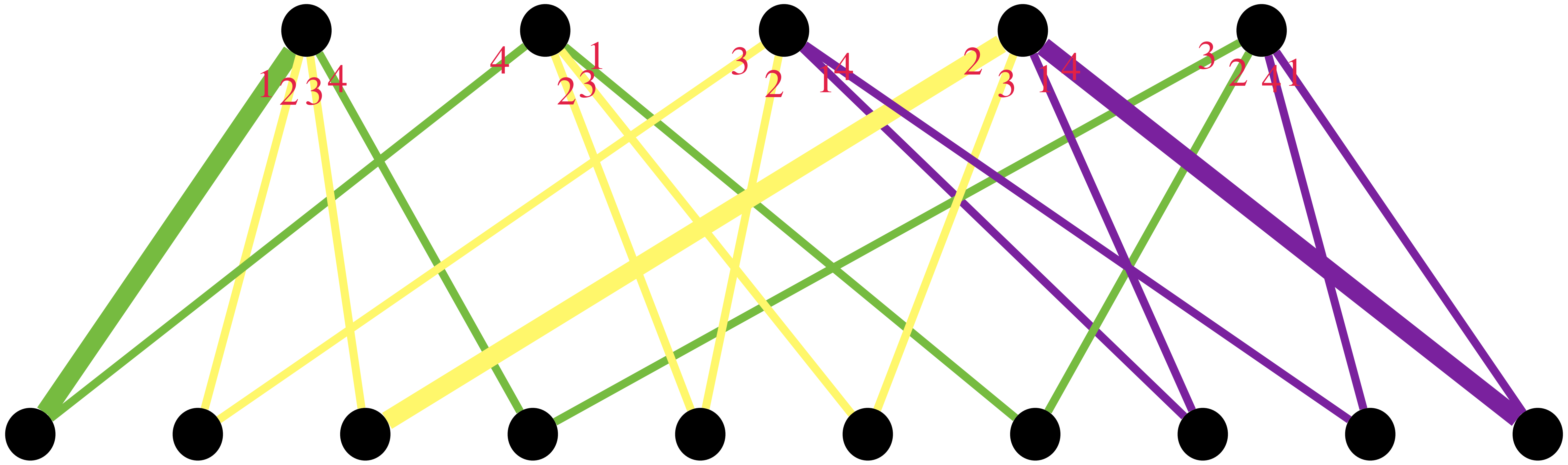

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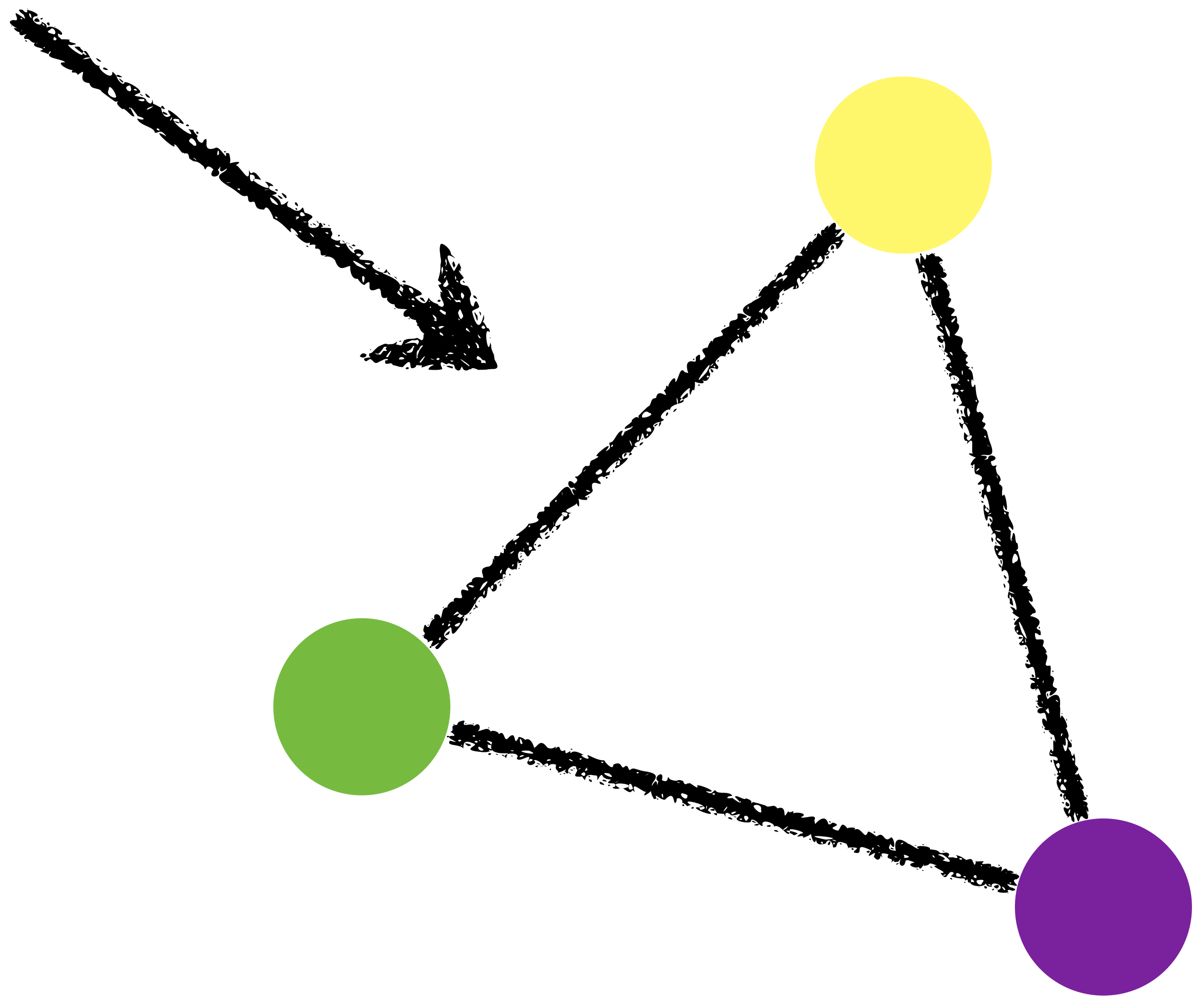
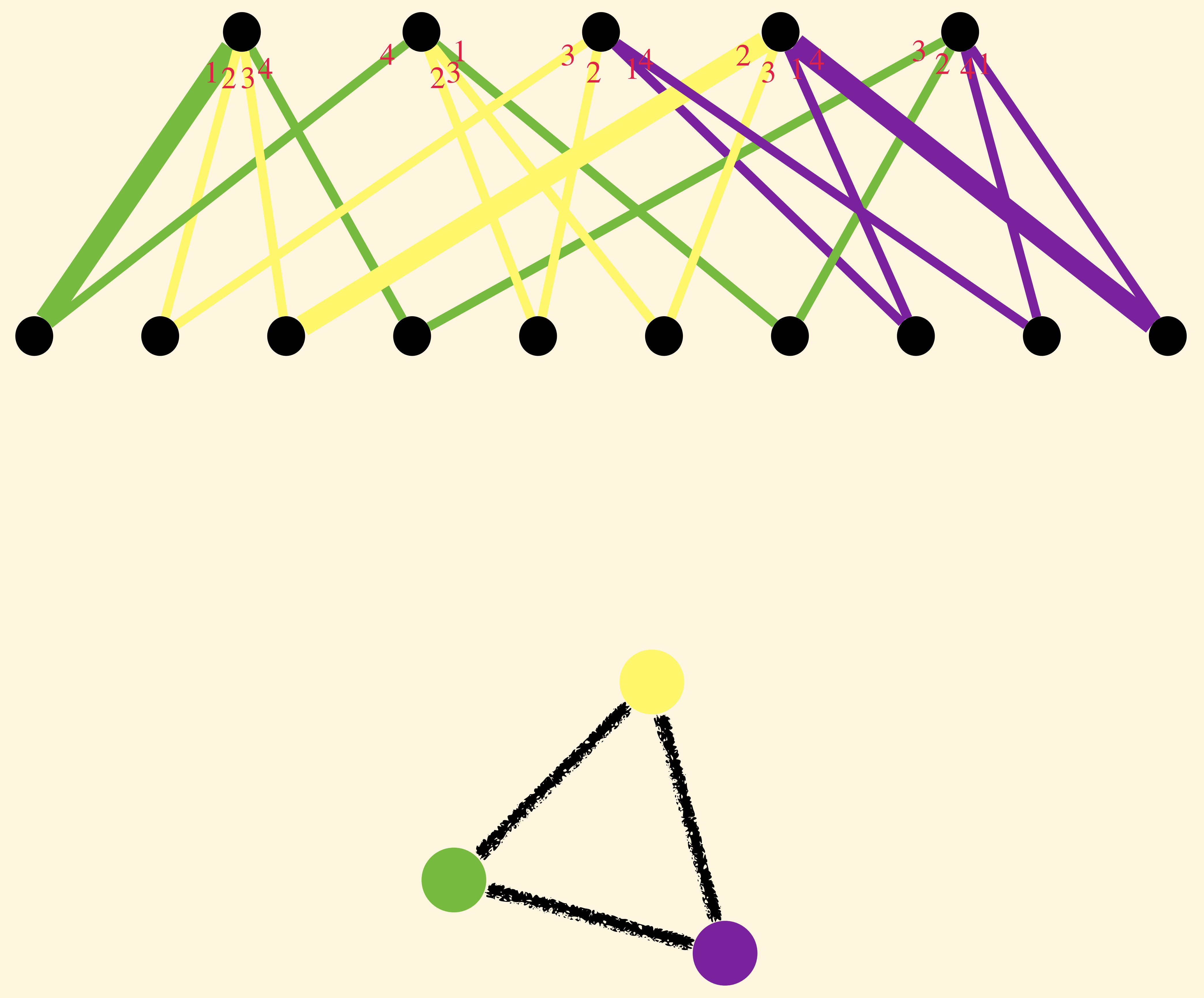


Reduction to Ising Model



Reduction to Ising Model

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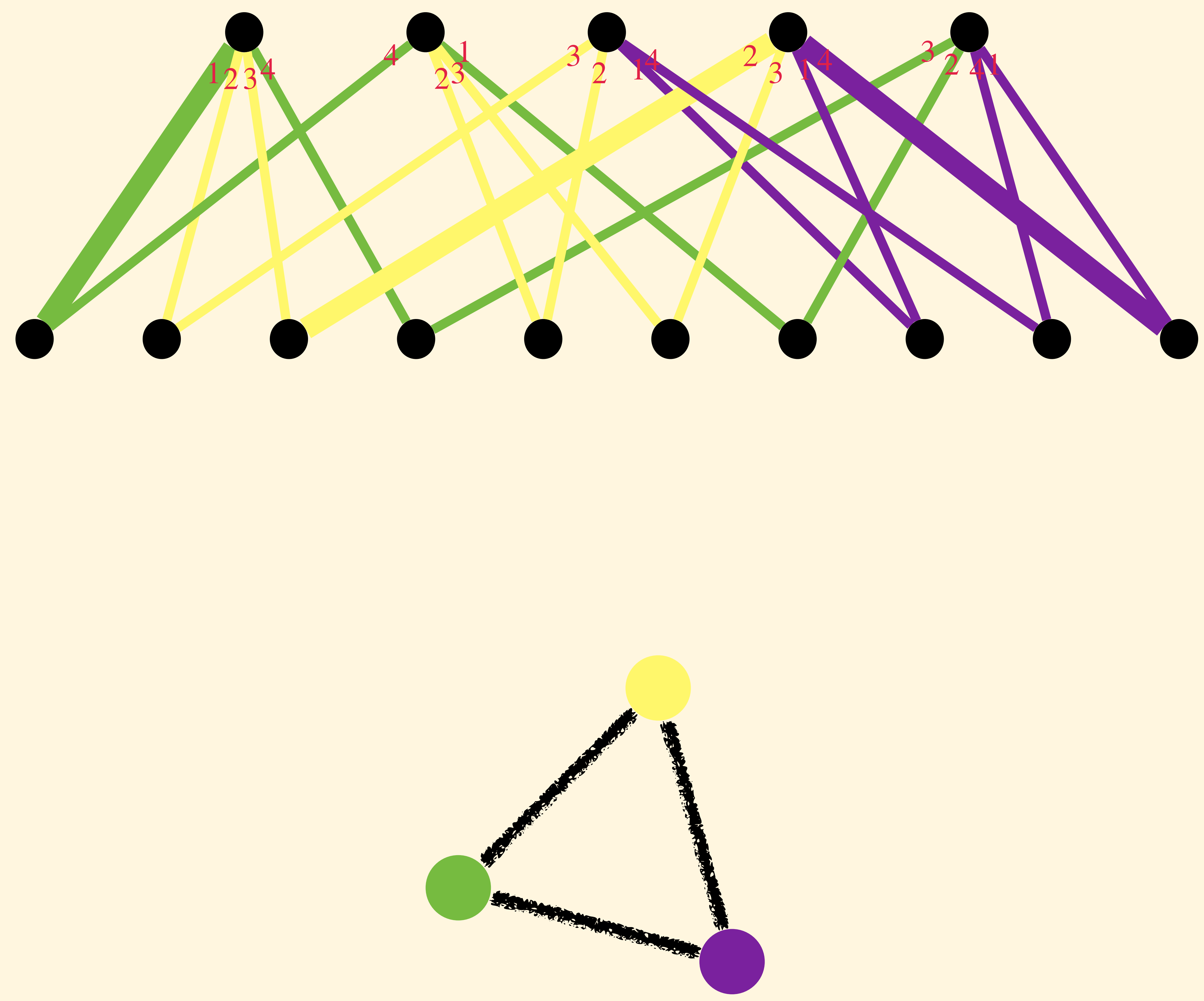

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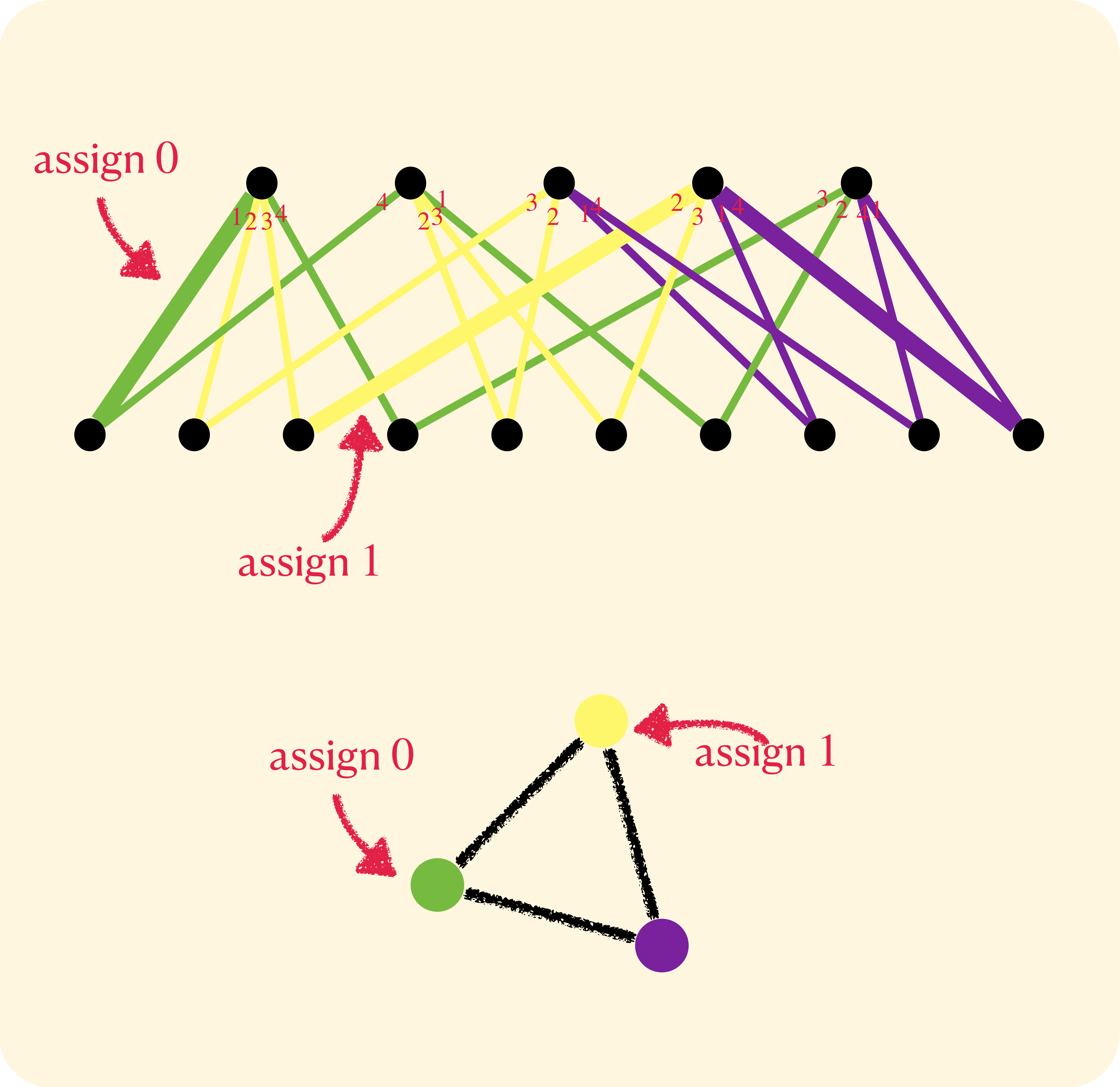

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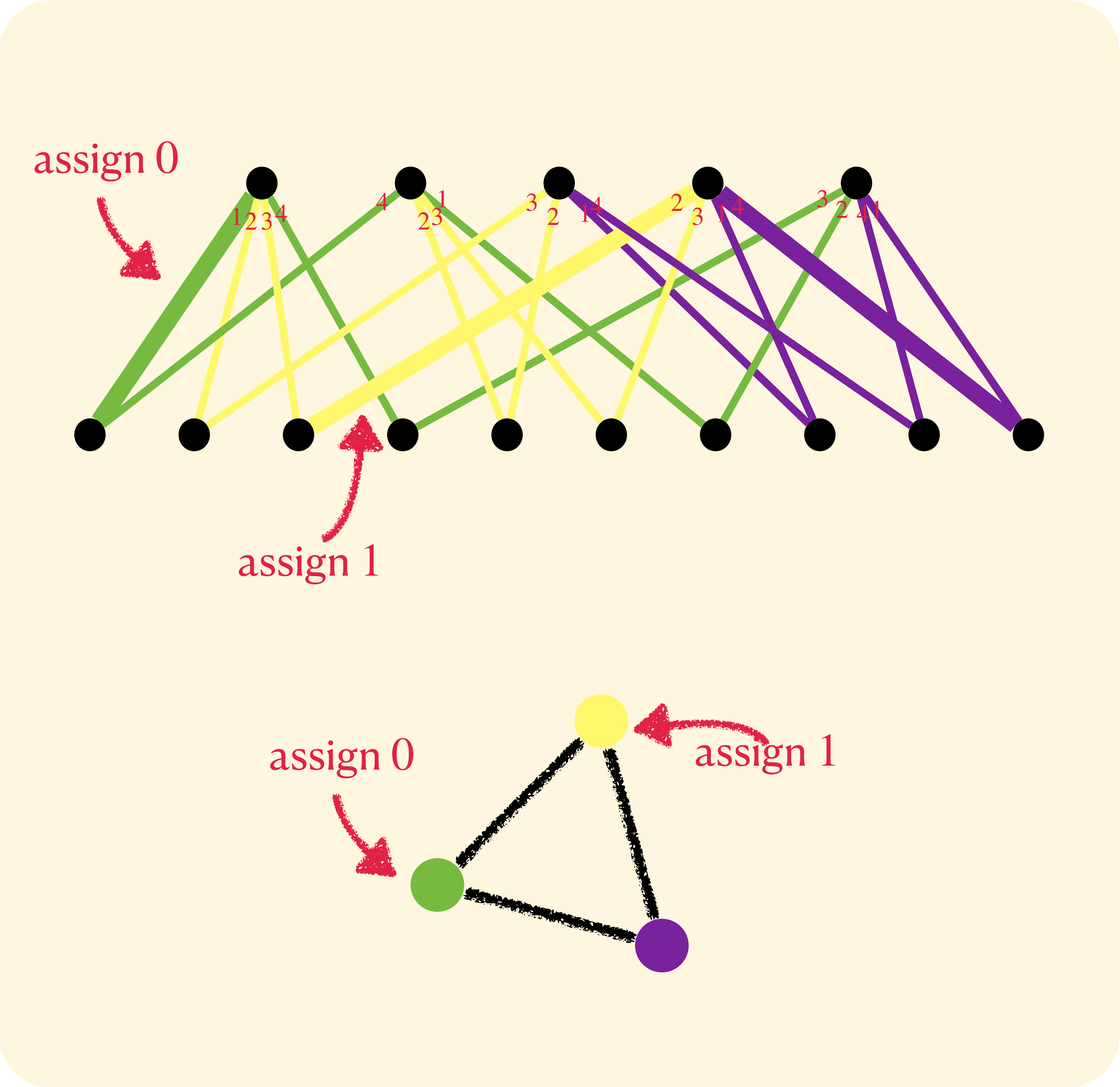

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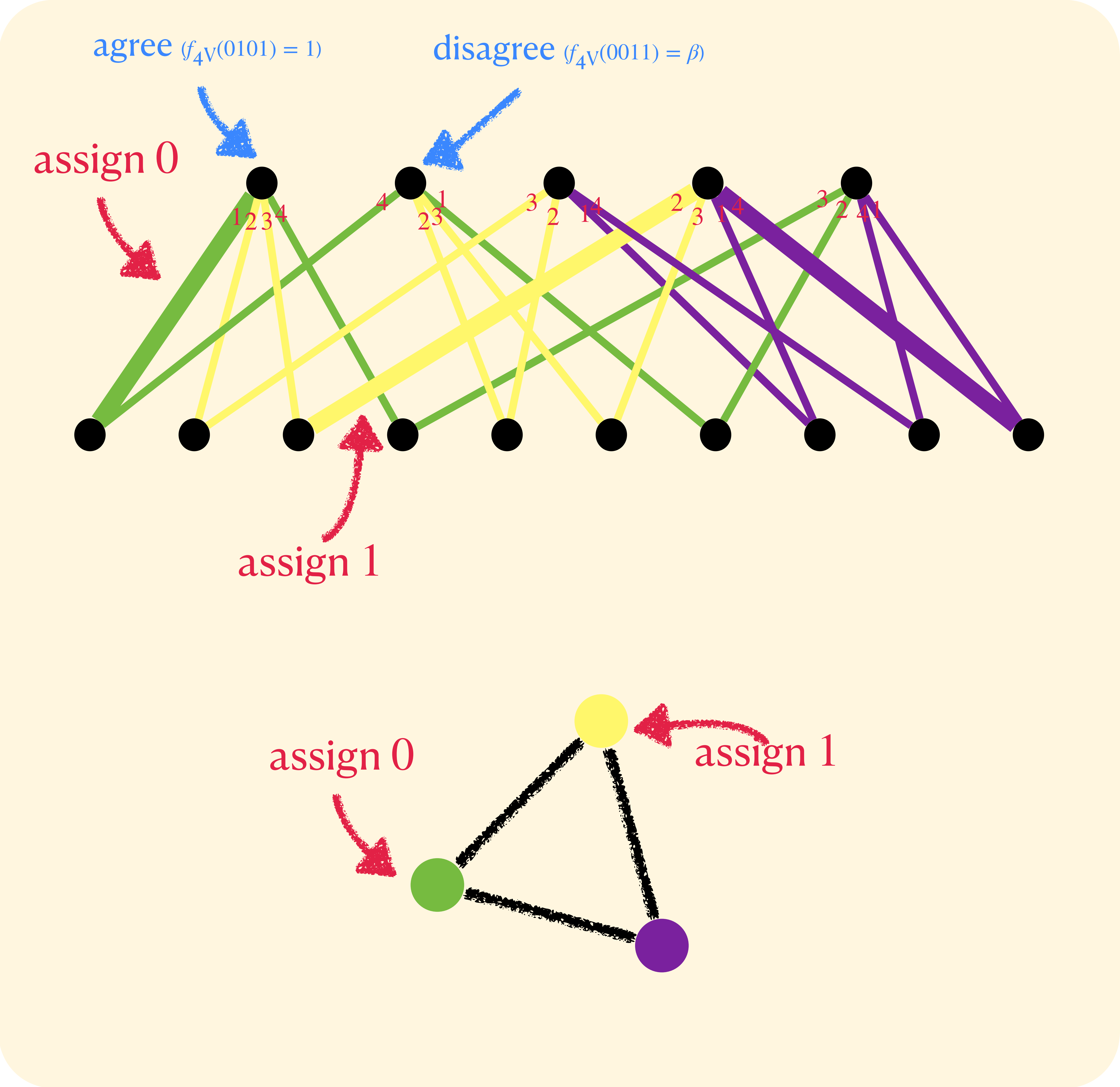

Reduction to Ising Model

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The value of a circuit is the value of its initial edge

$$\sigma : E' \to \{0,1\} \Longrightarrow \tau : \mathscr{C} \to \{0,1\}$$

Reduce signatures

Classify common vertex between $u, v \in \mathscr{C}$

agree vertex: take value β if $\tau(u) = \tau(v)$

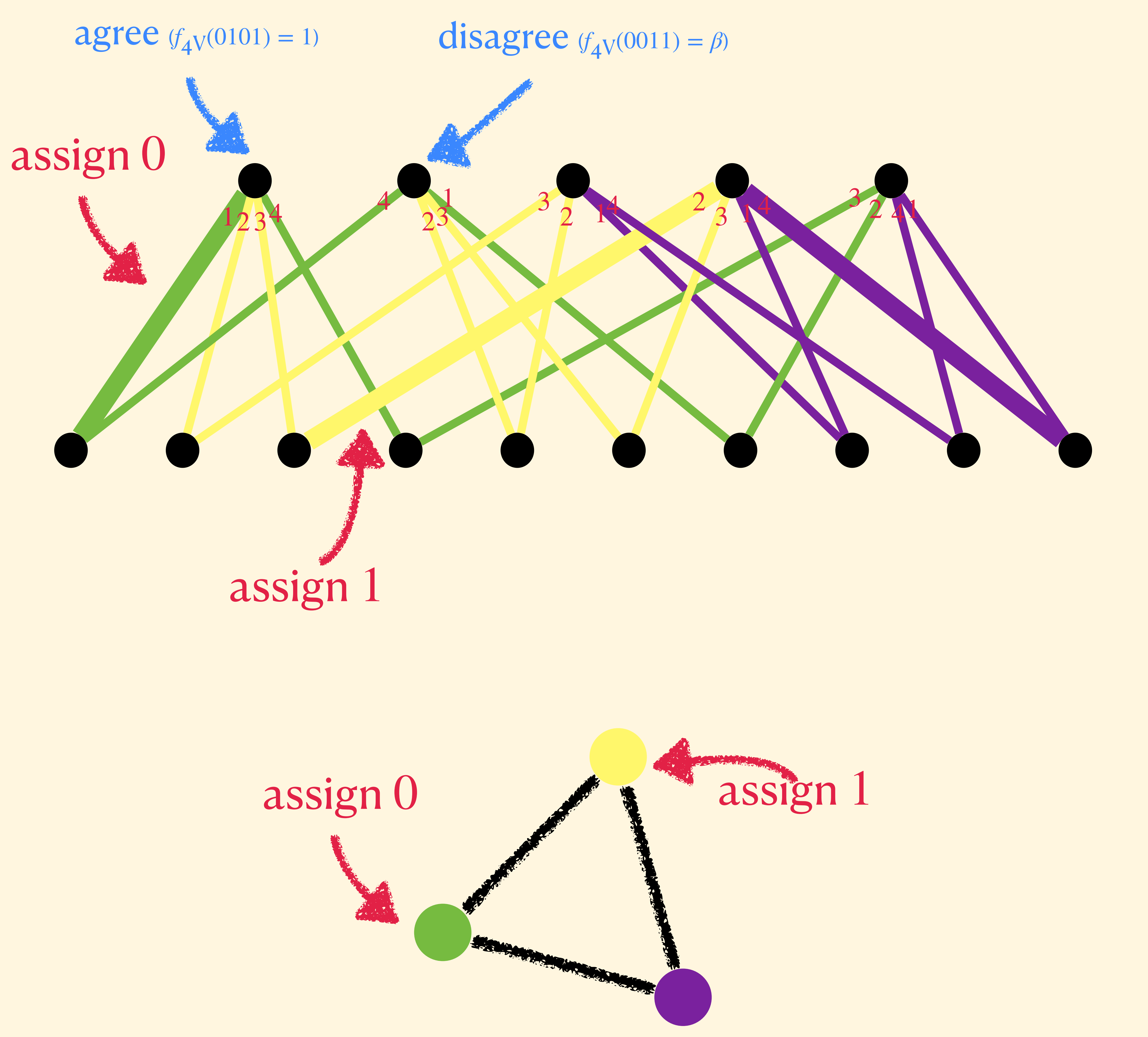
take value 1 if $\tau(u) \neq \tau(v)$

disagree vertex is the opposite

$$f_{4\mathrm{V}}\left(\begin{bmatrix} 0000 & 0010 & 0001 & 0011 \\ 0100 & 0110 & 0101 & 0111 \\ 1000 & 1010 & 1001 & 1011 \\ 1100 & 1110 & 1101 & 1111 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow f_{uv}\left(\begin{bmatrix} 00 & 01 \\ 10 & 11 \end{bmatrix}\right) = \begin{bmatrix} \beta^{A_{uv}} & \beta^{D_{uv}} \\ \beta^{D_{uv}} & \beta^{A_{uv}} \end{bmatrix} = \beta^{D_{uv}} \begin{bmatrix} \beta^{A_{uv}-D_{uv}} & 1 \\ 1 & \beta^{A_{uv}-D_{uv}} \end{bmatrix}$$

$$A_{uv} = \#\mathrm{AgreeVertex}, \quad D_{uv} = \#\mathrm{DisagreeVertex}$$



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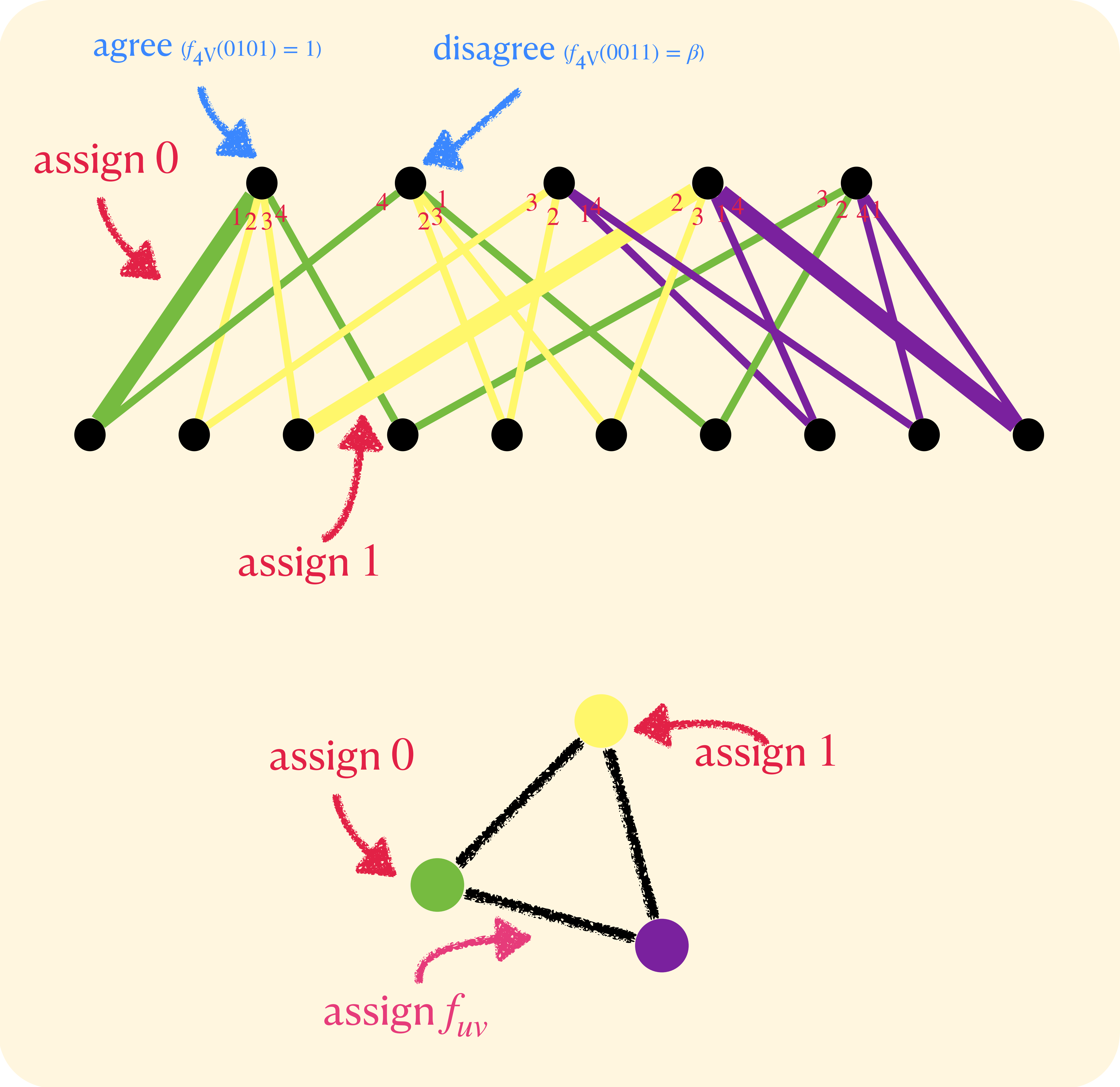

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Ising model [Lenz, 1920]

graph $G = (V, E)$ and local interaction $\beta = \{\beta_e\}_{e \in E}$

$$Z_{\text{Ising}}(G; \beta) = \sum_{\tau: V \to \{0,1\}} \prod_{e=(u,v) \in E} \beta_e^{\mathbf{I}(\tau(u)=\tau(v))}$$

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Approximability of Ising model [Jerrum and Sinclair, 1993]

If $\beta_e > 1$ for all $e \in E$, the system is *ferromagnetic*

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we use

- Jerrum-Sinclair chain
- **Worm process** [Prokof'ev and Svistunov, 2001]
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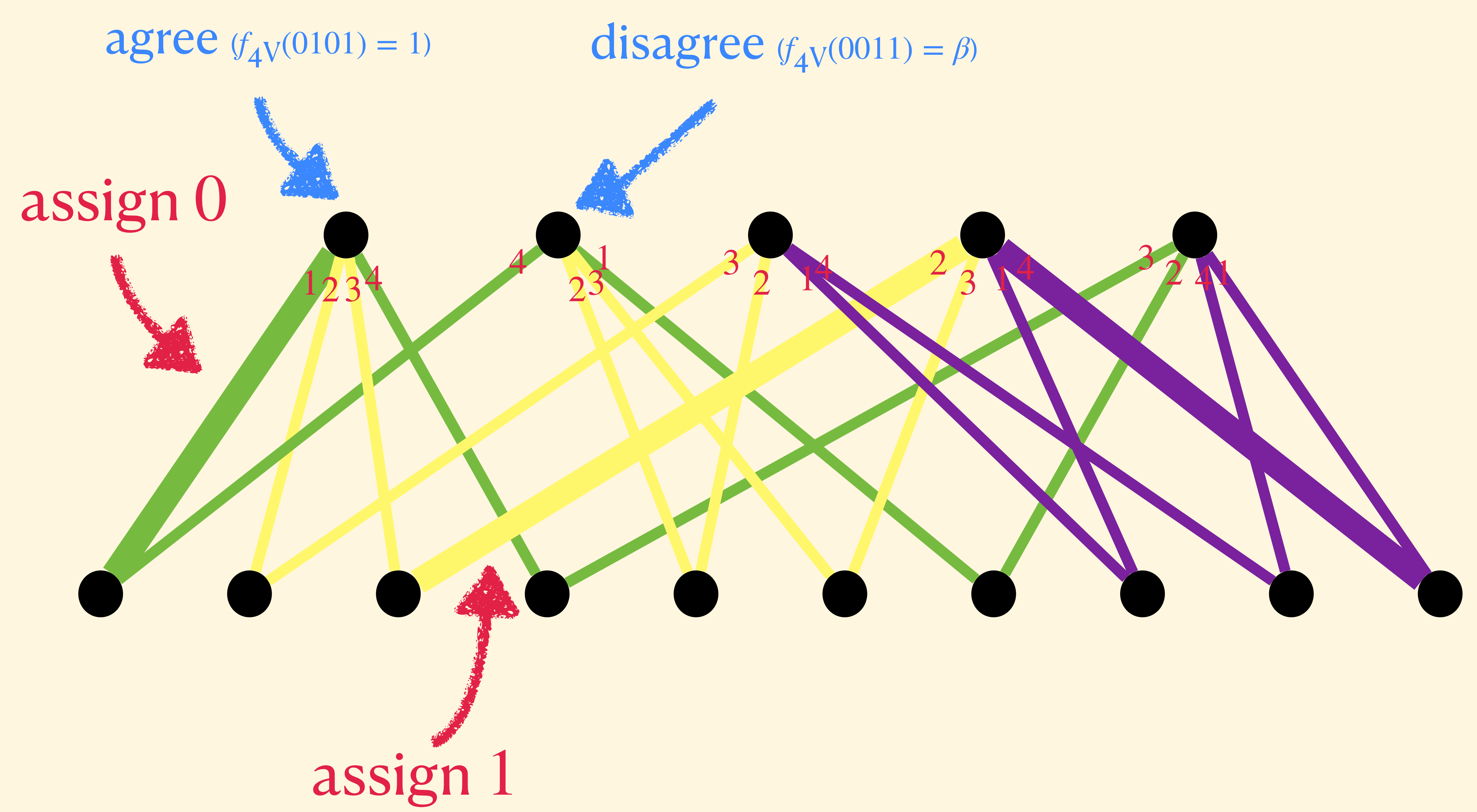
Choose initial edges “wisely”

Observation

Changing the initial edge of a circuit will swap the agree and disagree vertices between it and its neighbors



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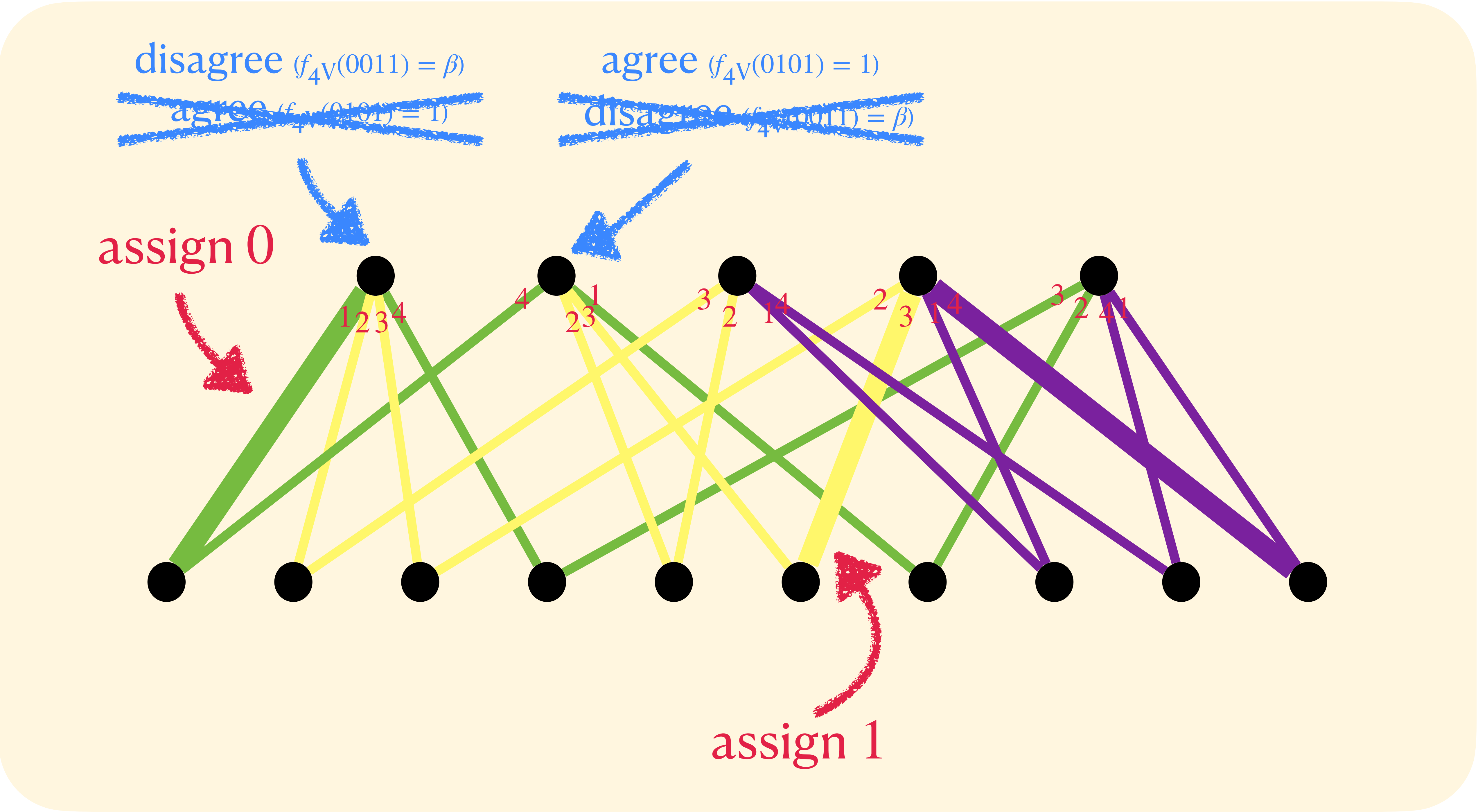

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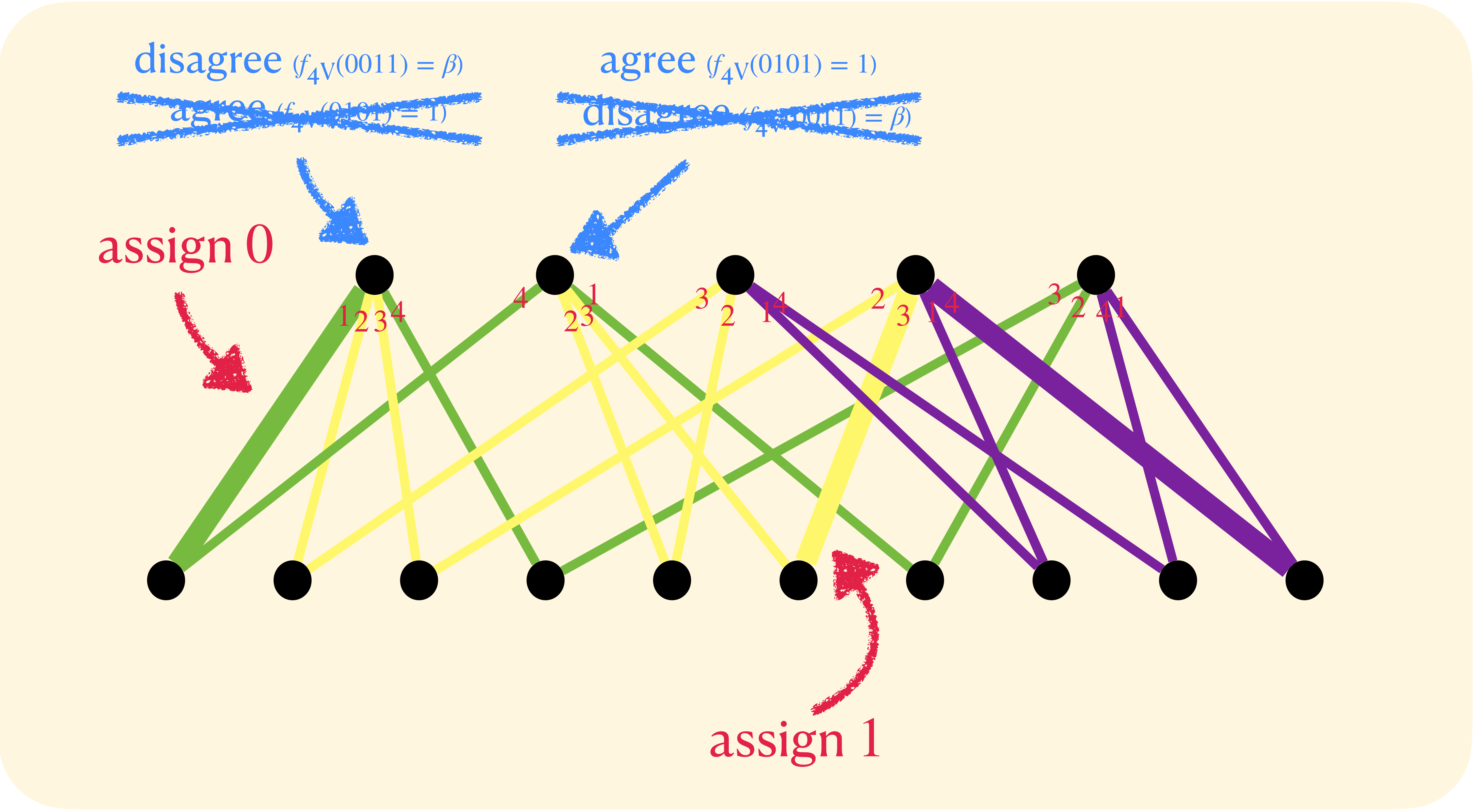
Change some initial edges to make $A_{uv} > D_{uv}$ for all $(u, v) \in E_C$ — obtain a ferromagnetic system



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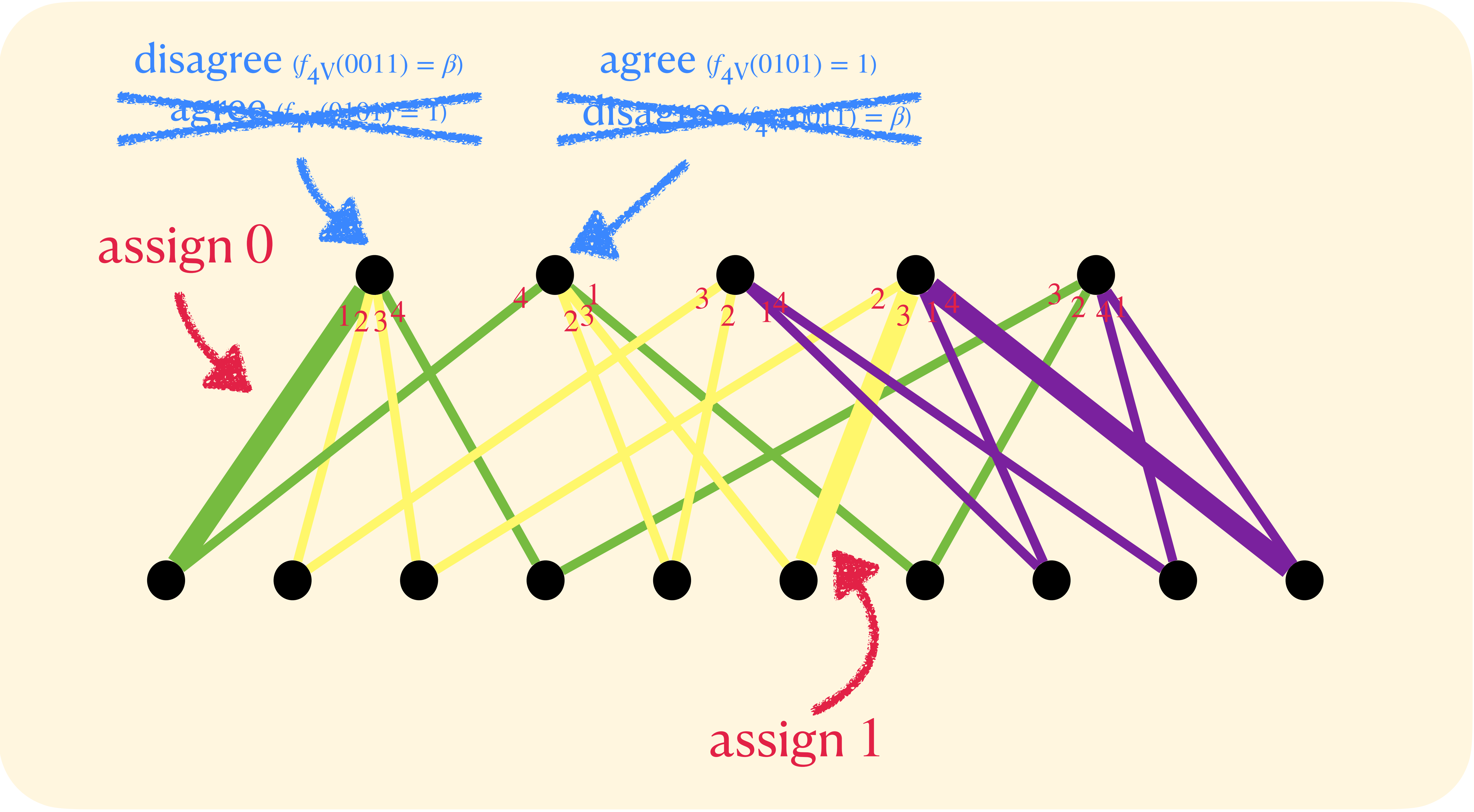

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- Let Boolean variable X_u to indicate whether change u’s initial edge ($X_u = 1$ means “change”)
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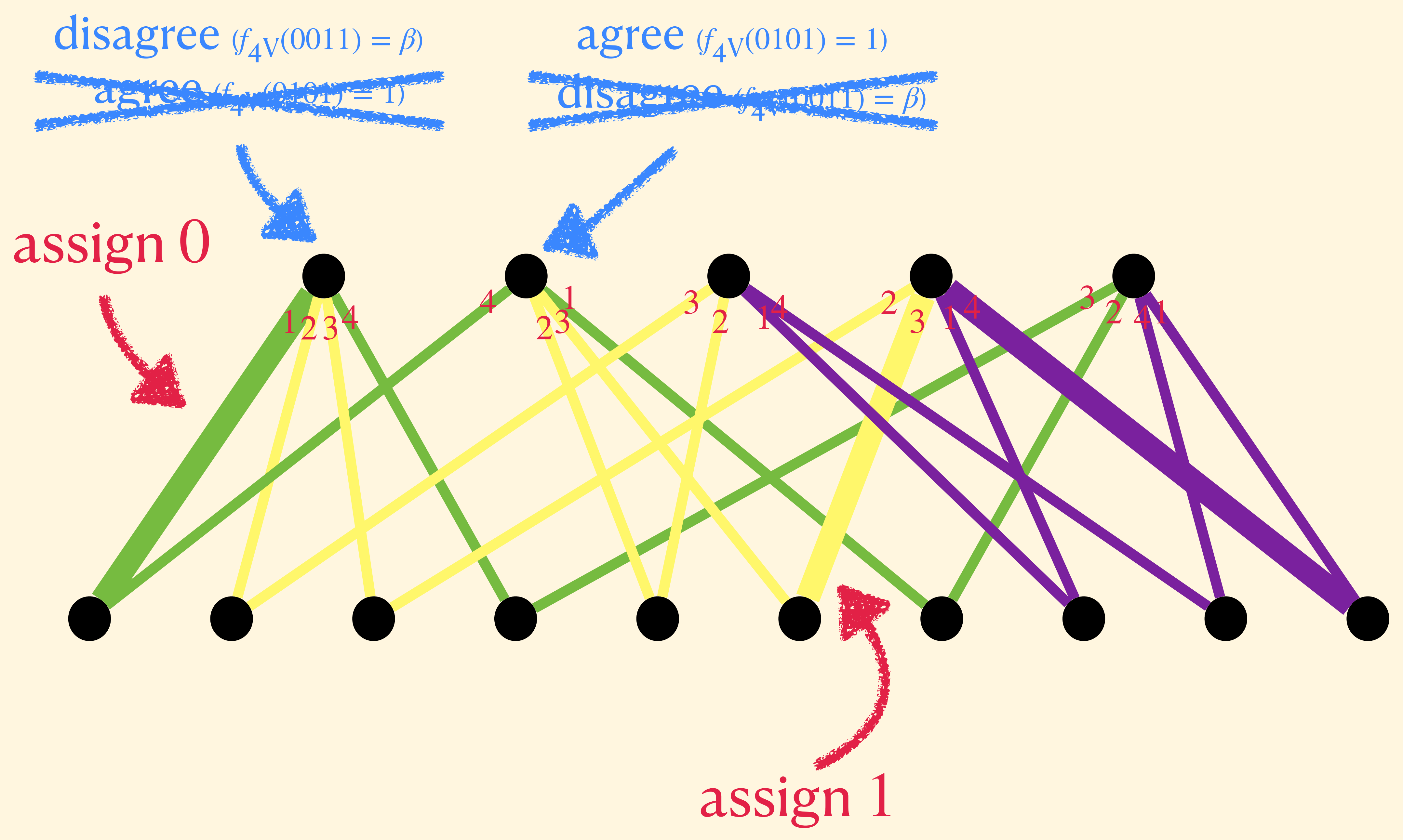

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Algebra criterion to compute Z_{4V}

If this system has a solution, there is ***an FPRAS***.

If this system has no solution, there is ***no FPRAS***.

Planar Graphs and Beyond

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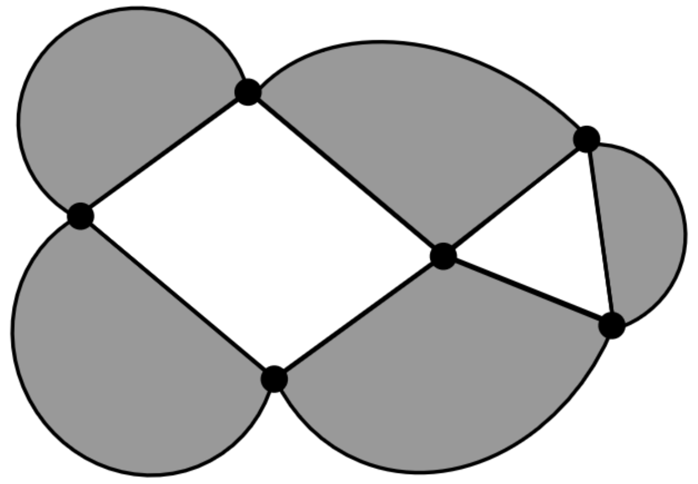
Face 2-coloring

The dual of a planar 4-regular graph is bipartite [Welsh, 1969]



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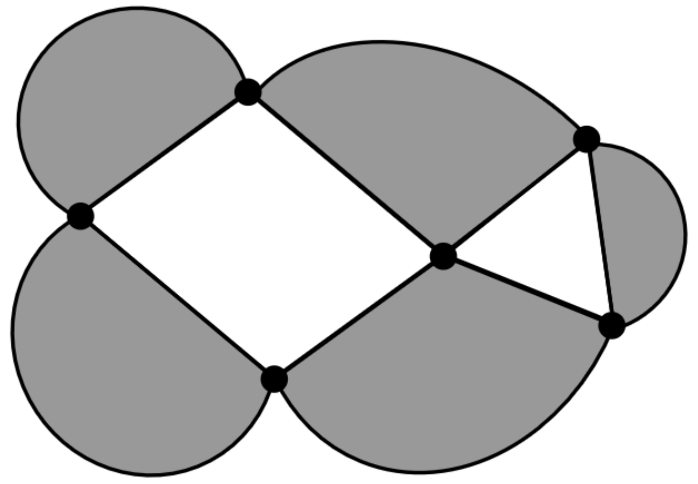
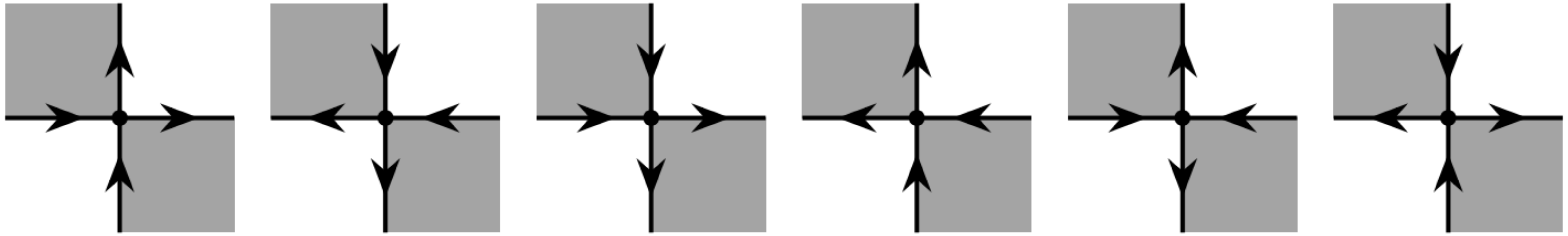


Canonical labeling

Based on face 2-coloring, we design a canonical labeling



1 2 3 4 5 6

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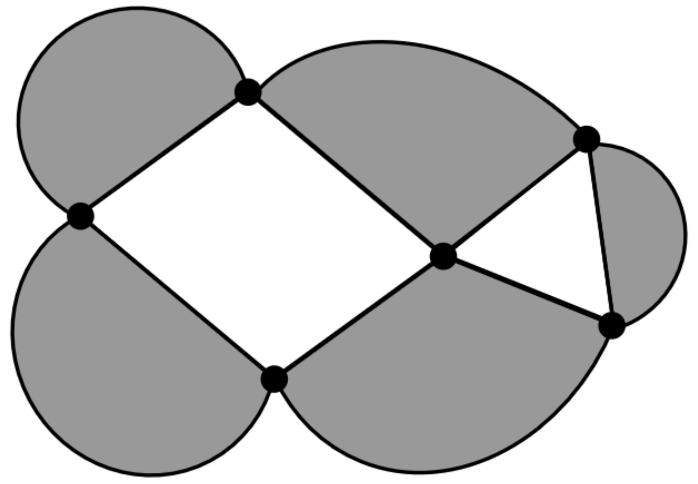

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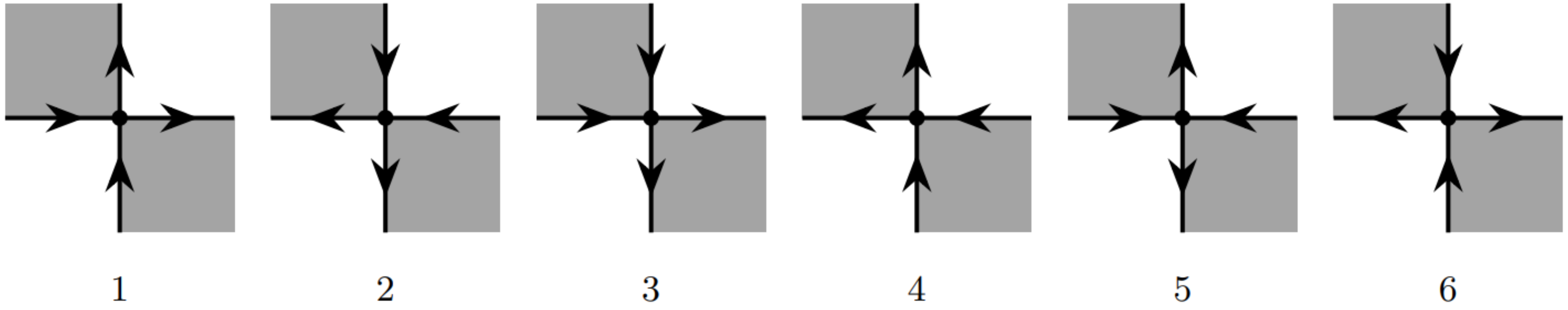

- The canonical labeling already encodes the *staggered ice-type model*
- Under canonical labeling, the XOR equation system is always ***solvable***
- This method also works for *torus grid graphs*.

Thank you!

Theor. Comput. Sci. 114491

Open Problems

- Fully approximability of six-vertex model
- FPTAS for six-vertex model on graphs in the low-temperature regime
- Better or optimal mixing time for six-vertex model
- Spectral independence for six-vertex model