Tidal Theory with Newtonian Cooling

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Summary – Processes representable by Newtonian cooling models are found to be of importance for tides in various planetary atmospheres. The equations of tidal theory are, therefore, rederived to include a rather general Newtonian cooling, and some of the effects of this inclusion are investigated.

1. Introduction

The classical theory of linearized, adiabatic tides has been developed with fair thoroughness (see, for example SIEBERT [1],[3]) LINDZEN [2]). Moreover, some attention has been devoted to the breakdown of this theory due to the growing importance of neglected nonlinear terms (PEKERIS and ALTERMAN [3]) and molecular diffusion (YANOWITCH [4]). Little attention has been given, however, to the influence of radiative processes—apart from their role in establishing thermal drives for tides (BUTLER and SMALL [5], LINDZEN [2], SIEBERT [1]). It turns out, however, that radiative cooling is in several cases sufficiently rapid to play a discernible role in motions with tidal periods. One such case occurs in the vicinity of the mesopeak of the Earth's atmosphere. Here the combined effects of CO_2 infrared cooling and O_3 photochemistry lead to cooling rates whose time scales are on the order of one day (LINDZEN and GOODY [6]). Another case concerns the Martian atmosphere which is composed largely of CO_2 (PRABHAKARA and HOGAN [7]). Here too, radiative responses can be very rapid. Moreover, numerical experiments suggest a very large diurnal tide on Mars (LEOVY and MINTZ [8]). In both these cases it appears possible to describe the effect of radiation (combined in the first case with photochemistry) upon deviations of temperature away from some equilibrium approximately with Newtonian cooling models; i.e., the radiative cooling is proportional to some function of space times the temperature deviation, δT. When the function of space depends only on altitude, it is a simple matter to extend classical tidal theory to allow for the inclusion of Newtonian cooling. This paper describes that extension. Apart from the potential practical importance of this extension in the above mentioned cases, it also displays how a dissipative process may affect the gross structure of tidal waves.

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[3]) Numbers in brackets refer to References, page 96.

2. *Mathematical development*

The basic dynamic and thermodynamic equations used are those described by SIEBERT [1]. The inclusion of Newtonian cooling alters only the expression of the first law of thermodynamics (equation (3.17) of SIEBERT [1]) which now becomes

$$\frac{DT}{Dt} = \frac{g\,H}{\varrho_0}\,\frac{M(\gamma-1)}{R}\,\frac{D\varrho}{Dt} + \frac{M(\gamma-1)}{R}\,J - a(z)\,\delta T\,, \tag{1}$$

where

T = temperature,

ϱ = density,

δT = tidal temperature perturbation from basic temperature, T_0,

J = heating per unit mass per unit time,

γ = $C_p/C_v = 1.40$,

M = molecular weight,

R = gas constant,

ϱ_0 = basic density,

$a(z)$ = rate coefficient for Newtonian cooling,

H = $R\,T_0/M\,g$,

g = acceleration of gravity.

Also,

K = $(\gamma-1)/\gamma$.

The remaining equations are

$$i\,\omega\,u - 2\,\Omega\,v\cos\theta = -\frac{1}{a}\,\frac{\partial}{\partial\theta}\left(\frac{\delta p}{\varrho_0}\right), \tag{2}$$

$$i\,\omega\,v + 2\,\Omega\,u\cos\theta = -\frac{1}{a\sin\theta}\,\frac{\partial}{\partial\varphi}\left(\frac{\delta p}{\varrho_0}\right), \tag{3}$$

$$\frac{\partial\delta p}{\partial z} = -\,g\,\delta\varrho\,, \tag{4}$$

$$i\,\omega\,\delta\varrho + w\,\frac{d\varrho_0}{dz} + \varrho_0\,\chi = 0\,, \tag{5}$$

and

$$\frac{\delta p}{p_0} = \frac{\delta\varrho}{\varrho_0} + \frac{\delta T}{T_0}\,, \tag{6}$$

where a time dependence of the form $e^{i\omega t}$ has been assumed and

u = westerly velocity,

v = northerly velocity,

w = vertical velocity,

δp = tidal pressure perturbation,

P_0 = basic pressure,

$\delta\varrho$ = tidal density perturbation,

Θ = colatitude,

φ = longitude,

Ω = rotation rate = $2\pi/1$ day,

a = radius of planet,

χ = velocity divergence.

Gravitational excitation has been neglected.

From (2) and (3) one obtains

$$\chi - \frac{\partial w}{\partial z} = \frac{i\,\omega}{4\,a^2\,\Omega^2}\,F\left(\frac{\delta p}{\varrho_0}\right), \tag{7}$$

where F is the following operator

$$\left.\begin{aligned} F = {} & \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{f^2-\cos^2\theta}\,\frac{\partial}{\partial\theta}\right) \\ & + \frac{1}{f^2-\cos^2\theta}\left[\frac{i}{f}\,\frac{f^2+\cos^2\theta}{f^2-\cos^2\theta}\,\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right], \end{aligned}\right\} \tag{8}$$

and

$$f = \frac{\omega}{2\,\Omega}.$$

In general, J, χ, w, δp, $\delta\varrho$ and δT may be expanded in terms of the eigenfunctions of the equation

$$F(\psi_n) + \frac{4\,a^2\,\omega^2}{g\,h_n}\,\psi_n = 0, \tag{9}$$

where h_n = equivalent depth.

(9) is Laplace's Tidal Equation. If we restrict ourselves to a given n, we must obtain an equation describing the vertical (z) structure of that mode. In adiabatic tidal theory the equation has its simplest form for the dependent variable

$$Y_n = \chi_n - \frac{K\,J_n}{g\,H}, \tag{10}$$

where χ_n, J_n, etc. are the coefficients in the expansions

$$\chi = \sum_n \chi_n(z)\,\psi_n(\theta,\varphi)\,e^{iwt},$$

$$J = \sum_n J_n(z)\,\psi_n(\theta,\varphi)\,e^{iwt}, \quad \text{etc.}$$

Moreover, it is especially easy to relate the various fields to Y_n (see SIEBERT [1]).

We should like to determine such a simplifying dependent variable for the present case as well. The procedure is implicitly contained in LINDZEN [9]. There, it was found that for the solar diurnal tide, there is a mode with a horizontal dependence of the form $\sin\theta\,\cos\theta$, and for this mode the quantity $\chi - (\kappa\,J/g\,H)$ is identically equal to zero. Redoing the analysis of that paper, using equation (1) of this paper, one finds that for the same horizontal dependence, the quantity

$$\left(1 + \frac{a(z)}{i\,\omega\,\gamma}\right)\chi - \frac{\kappa\,J}{g\,H} - \frac{a(z)}{i\,\omega\,\gamma}\,\frac{w}{H}\,\frac{dH}{dz}$$

is identically equal to zero. This same quantity proves to be a simple dependent variable. However, it proves even simpler to use the above quantity multiplied by $(1 + [a(z)/i\,\omega])^{-1}$. Thus, as our dependent variable we will take

$$L_n = \left(1 + \frac{a}{i\,\omega}\right)^{-1}\left\{\left(1 + \frac{a}{i\,\omega\,\gamma}\right)\chi_n - \frac{\kappa\,J_n}{g\,H} - \frac{a}{i\,\omega\,\gamma}\,\frac{w}{H}\,\frac{dH}{dz}\right\}. \tag{11}$$

Using (11), together with (1), (4), (5), (6) and (9) one obtains after some manipulation

$$\left.\begin{aligned} &\frac{d^2L_n}{dx^2} - \left\{1 + \frac{a}{i\,\omega\,\gamma}\left(1 + \frac{a}{i\,\omega\,\gamma}\right)\frac{1}{H}\,\frac{dH}{dx}\right\}\frac{dL_n}{dx} \\ &\qquad + \left\{\frac{1}{h_n}\left(\frac{dH}{dx} + K\,H\right)\frac{a}{i\,\omega\,\gamma}\,\frac{1}{H}\,\frac{dH}{dx}\right\}\left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1} L_n \\ &\qquad = \left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1}\frac{\kappa}{\gamma\,g\,h_n}\,J_n\,, \end{aligned}\right\} \tag{12}$$

where

$$x = \int_0^z \frac{dz'}{H(z')}\,. \tag{13}$$

From equations (1) to (7), and (9) we obtain the following equations relating u_n, v_n, w_n, δp_n and δT_n to $L_n(z)$ and $\Psi_n(\theta, \varphi)$:

$$u_n = \frac{\gamma\,g\,h_n}{4\,a\,\Omega^2(f^2 - \cos^2\theta)}\left(\frac{dL_n}{dx} - L_n\right)\left(\frac{\partial}{\partial\theta} - \frac{i}{f}\,\omega\,t\,\theta\,\frac{\partial}{\partial\psi}\right)\psi_n\,, \tag{14}$$

$$v_n = \frac{i\,\gamma\,g\,h_n}{4\,a\,\Omega^2(f^2 - \cos^2\theta)}\left(\frac{dL_n}{dx} - L_n\right)\left(\frac{\cos\theta}{f}\,\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\,\frac{\partial}{\partial\varphi}\right)\psi_n\,, \tag{15}$$

$$w_n = \gamma\,h_n\left(\frac{dL_n}{dx} + \left(\frac{H}{h_n} - 1\right)L_n\right)\psi_n\,, \tag{16}$$

$$\delta p_n = \frac{\varrho_0\,\gamma\,g\,h_n}{i\,\omega}\left(\frac{dL_n}{dx} - L_n\right)\psi_n\,, \tag{17}$$

$$\left.\begin{aligned} \delta T_n = \frac{M}{i\,\omega\,R}\left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1}\Big[\gamma\,g\Big(\frac{dH}{dx}\left(\frac{h_n}{H} - 1\right) - K\,H\Big)\,L_n& \\ - \frac{\gamma\,g\,h_n}{H}\,\frac{dH}{dx}\,\frac{dL_n}{dx} + K\,J_n\Big]\,\psi_n\,.& \end{aligned}\right\} \tag{18}$$

Reference to SIEBERT [1] will show that (14) to (18) in $L_n(x)$ are identical to the equations for the adiabatic case in terms of Y_n [see (10)]—with the exception of the factor $[1 + (a/i\,\omega\,\gamma)]^{-1}$ in the equation for δT_n.

We see from the equations, that the effects of Newtonian cooling fall into two categories:

1. To the extent that L_n, as determined from equation (12), differs from Y_n, its adiabatic counterpart, the vertical structure of the various fields will differ.

2. Insofar as equations (14) to (17) in L_n are identical to the adiabatic equations in Y_n, while equation (18) differs by the factor $[1 + (a/i\,\omega\,\gamma)]^{-1}$ from the adiabatic

equation for δT_n, the relative phases between the temperature and the velocity fields will be altered by Newtonian cooling.

The last matter may prove particularly important since the relative phase between δT and the velocity fields determines the mean tidal heat advection, and anything affecting this phase relation can result in mean tidal heating. The practical implications of this will be explored in subsequent papers. It should be noted, however, that the effect is simple to deal with since it enters primarily through a multiplicative factor in equation (18).

3. *Asymptotic behavior*

In most potential applications of the above theory, H is not constant, and the ratio $|a/\omega|$ is of order unity. However, taking H a constant vastsy simplifies the problem. Moreover, the consideration of extreme values of $|a/\omega|$ makes clear certain qualitative effects of this particular dissipative process.

In proceeding it proves simpler to transform (12) to the following

$$\left.\begin{aligned}
&\frac{d^2\Phi_n}{dx^2} + \left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1} \left\{\frac{1}{h_n}\frac{dH}{dx} + \frac{\kappa H}{h_n} - \frac{1}{4}\right. \\
&\quad + \frac{1}{2}\frac{a}{i\,\omega\,\gamma}\left[\frac{1}{H}\frac{dH}{dx} - \frac{1}{2} + \frac{1}{H}\frac{dH}{dx}\frac{1}{a}\frac{da}{dx} + \frac{1}{H}\frac{d^2H}{dx^2}\right. \\
&\quad \left.\left. - \frac{1}{H^2}\left(\frac{dH}{dx}\right)^2 - \frac{a}{i\,\omega\,\gamma}\left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1}\frac{1}{H}\frac{dH}{dx}\left(\frac{1}{a}\frac{da}{dx} + \frac{1}{2}\frac{1}{H}\frac{dH}{dx}\right)\right]\right\}\Phi_n \\
&= \frac{\kappa J_n}{\gamma\, g\, h_n\left(1 + \dfrac{a}{i\,\omega\,\gamma}\right)}\, e^{-x/2} \exp\left\{-\frac{1}{2}\int_0^x \left[\frac{\dfrac{a}{i\,\omega\,\gamma}\dfrac{1}{H}\dfrac{dH}{dx}}{1 + \dfrac{a}{i\,\omega\,\gamma}}\right] dx\right\},
\end{aligned}\right\} \tag{19}$$

where

$$L_n = \exp\left[\frac{x}{2} + \frac{1}{2}\int_0^x \left\{\frac{\dfrac{a}{i\,\omega\,\gamma}\dfrac{1}{H}\dfrac{dH}{dx}}{1 + \dfrac{a}{i\,\omega\,\gamma}}\right\} dx\right]\Phi_n\,. \tag{20}$$

Assuming $H = \text{constant}$ (i.e., an isothermal atmosphere) (19) becomes

$$\frac{d^2\Phi_n}{dx^2} + \left(1 + \frac{a}{i\,\omega\,\gamma}\right)^{-1}\left\{\frac{K H}{h_n} - \frac{1}{4} - \frac{1}{4}\frac{a}{i\,\omega\,\gamma}\right\}\Phi_n = \frac{\kappa J_n}{\gamma\, g\, h_n\left(1 + \dfrac{a}{i\,\omega\,\gamma}\right)}\, e^{-x/2}, \tag{21}$$

where

$$L_n = e^{x/2}\,\Phi_n\,. \tag{22}$$

Consider now the case where $|a(z)/\omega\,\gamma| \gg 1$ for all z, and where, moreover, $|a/\omega\,\gamma| \gg |\kappa H/h_n|$.

Equation (21) becomes

$$\frac{d^2\Phi_n}{dx^2} - \frac{1}{4}\Phi_n = \frac{i\,\omega}{\gamma\, g\, h_n}\,\kappa\, J_n\, e^{-x/2}\,. \tag{23}$$

Let J_n go to zero above some height. The solution of (23) above that height will be of the form

$$\Phi_n = A\, e^{-x/2} + B\, e^{x/2} . \tag{24}$$

The condition of boundedness requires, moreover, that $B = 0$. Thus, using (22), we have that L_n and consequently u_n, v_n, w_n, and δT_n, are independent of height. This contrasts with the adiabatic case where Y_n would increase with height if $h_n > 0$ and decrease with height if $h_n < 0$. Both these cases lead to constancy with height for sufficiently rapid Newtonian cooling.

Another pedagogically interesting case arises when not only H, but also a is a constant, and where, furthermore $| a/\omega\, \gamma | \ll 1$. (21) becomes

$$\frac{d^2\Phi_n}{dx^2} + \left(\frac{K\,H}{h_n} - \frac{1}{4} - \frac{a}{i\,\omega\,\gamma}\,\frac{K\,H}{h_n}\right)\Phi_n = \frac{K\,J_n}{\gamma\, g\, h_n}\left(1 - \frac{a}{i\,\omega\,\gamma}\right) e^{-x/2} . \tag{25}$$

Again, let J_n go to zero above some height. The solution of (25) above that height will be of the form

$$\Phi_n = A\, e^{\lambda_n X} + B\, e^{-\lambda_n X} , \tag{26}$$

where

$$\lambda_n = \sqrt{-\frac{K\,H}{h_n} + \frac{1}{4} + \frac{a}{i\,\omega\,\gamma}\,\frac{K\,H}{h_n}}$$

or

$$\lambda_n \cong \sqrt{-\frac{K\,H}{h_n} + \frac{1}{4}}\left(1 - \frac{1}{2}\,\frac{\frac{a}{i\,\omega\,\gamma}\,\frac{K\,H}{h_n}}{\frac{K\,H}{h_n} - \frac{1}{4}}\right) . \tag{27}$$

Let us now restrict ourselves to the case where $K\,H/h_n > 1/4$. Then (27) becomes

$$\lambda_n \cong i\,\sqrt{\frac{K\,H}{h_n} - \frac{1}{4}}\left(1 - \frac{1}{2}\,\frac{\frac{a}{i\,\omega\,\gamma}\,\frac{K\,H}{h_n}}{\frac{K\,H}{h_n} - \frac{1}{4}}\right). \tag{28}$$

Boundeness, as $x \to \infty$, now leads to $B = 0$ in (26) which is, in fact, the traditional radiation condition (WILKES [10]).

4. *Concluding remarks*

We have, in this paper, rederived the equations of tidal theory including the effects of Newtonian cooling. It was shown that the inclusion of Newtonian cooling affects 1. the altitude distribution of the amplitude and phase of all tidal fields, and 2. the relative phase between the tidal temperature and velocity fields. The latter may result in net tidal heating.

Certain asymptotic effects of Newtonian cooling are also investigated. Very rapid Newtonian cooling leads to a constancy of wave amplitude with height in cases both where the adiabatic wave amplitude would increase with altitude and where it would decrease with altitude. Very small constant Newtonian cooling leads to the traditional radiation condition. Concerning the last matter, a word of caution is necessary. Consider the following situation: we have a thermal drive concentrated entirely below some height z_1; Newtonian cooling is negligible below some height $z_2 > z_1$; above z_2 Newtonian cooling increases to an appreciable value, but eventually decreases to a small constant value above some height z_3. According to the analysis of section 3, the radiation condition would apply above z_3. However, there is no reason to assume it holds for the region above the drive region between z_1 and z_3. That is to say, the existence of a dissipation mechanism at high levels does not imply that the radiation condition is appropriate directly above some lower level drive.

As a final note, one must point out that in practical situations processes other than those considered here are certainly operative, and the application of the above equations is limited by these omissions.

5. *Acknowledgement*

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