Deep Learning for Search And Matching Models

(a.k.a. "DeepSAM")

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Introduction

- Heterogeneity and aggregate shocks are important in markets with search frictions (e.g. labor and financial markets).
- Most search and matching (SAM) models with heterogeneous agents study:
  1. Deterministic steady state (e.g. Shimer-Smith '00),
  2. Aggregate fluctuations, but make assumptions to eliminate distribution from state space (e.g. "block recursivity" in Menzio-Shi '11, Lise-Robin '17; Lagos-Rocheteau '09).
- We present SAM models as high-dim. PDEs with distribution & agg. shocks as states

  ...and develop a new deep learning method, **DeepSAM**, to solve them globally.
- We also extend **DeepSAM** for SMM estimation within efficient computational time.

This Paper

- Develop DeepSAM and apply to canonical search models with aggregate shocks:
  1. Shimer-Smith/Mortensen-Pissarides model with two-sided heterogeneity.
  2. Lise-Robin on-the-job search (OJS) model with endogenous separation & worker bargain.
  3. Duffie-Garleanu-Pederson OTC model with asset and investor heterogeneity (in paper).
- High accuracy in "global" state space (including distribution); efficient compute time for both solution and estimation.
- We can study non-block recursive unemployment dynamics and wage dynamics:
  1. Large impact of distribution on aggregates when aggregate shocks affect agents unevenly.
  2. A search-theoretical explanation for Okun's hypothesis.
  3. Low-type worker wages more procyclical.
  4. Lise-Robin style block recursive equilibria over-predict unemployment & vacancy IRF.

Literature

- Deep learning in macro; for incomplete market heterogeneous agent models (HAM) (e.g. Maliar et al '21, Azinovic et al '22, Kahou et al '21, Han-Yang-E '21 "DeepHAM"; Fernández-Villaverde et al '20, Huang '22, Gu-Laurière-Merkel-Payne '23, among others)
  - This paper: search and matching (SAM) models.

	Distribution	Distribution impact on decisions
HAM	Asset wealth and income	Via aggregate prices
SAM	Type (productivity) of agents in two sides of matching	Via matching process with other types

- Search model with business cycle (e.g. Shimer '05, Menzio-Shi '11, Lise-Robin '17.)
  - This paper: keep distribution in the state vector.

- Integrate deep learning based solution methods with calibration and estimation (e.g., Chen et al '23, Kase et al '23, Friedl et al '23, Duarte & Fonseca '24)
  - This paper: standard internal calibration practice for quantitative macro.

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Shimer-Smith/Mortensen-Pissarides with Two-sided Heterogeneity

- Continuous time, infinite horizon environment.
- Workers $x \in [0,1]$ with exog density $g_t^w(x)$; Firms $y \in [0,1]$ with $g_t^f(y)$ by free entry:
  - Unmatched: unemployed workers get benefit b; vacant firms pay vacancy cost c.
  - Matched: type x worker and type y firm produce output $z_t f(x,y)$.
  - z_t: follows two-state continuous time Markov Chain (can be generalized).
  - Firms make entry decision and then draw a type y from uniform distribution $[0,1]$. More
- Meet randomly at rate $m(\mathcal{U}_t, \mathcal{V}_t)$, $\mathcal{U}_t$ is total unemployment, $\mathcal{V}_t$ is total vacancies.
- Upon meeting, agents choose whether to accept the match:
  - Match surplus $S_t(x,y)$ divided by generalized Nash bargaining: worker get fraction β.
  - Match acceptance decision $\alpha_t(x,y) = \mathbb{1}\{S_t(x,y) > 0\}$. Exogenous dissolve rate $\delta(x,y,z)$.
- Equilibrium object: $g_t(x,y)$ distribution of match $\Rightarrow$ unemployed $g_t^u(x)$, vacant $g_t^v(y)$.

Recursive Equilibrium Part I: Unemployed Workers & KFE

- Idiosyncratic state $= x$, Aggregate states $= (z, g(x, y))$.
- Hamilton-Jacobi-Bellman equation for an unemployed worker's value $V^u(x, z, g)$:

$$\rho V^u(x,z,g) = b + \frac{m(z,g)}{\mathcal{U}(z,g)} \int \underbrace{\overbrace{\alpha(x,\tilde{y},z,g)}^{\text{acceptance decision}} \left(\overbrace{V^e(x,\tilde{y},z,g)}^{\text{employed value}} - V^u(x,z,g)\right)}_{\text{change of value conditional on match}} \frac{g^v(\tilde{y})}{\mathcal{V}(z,g)} d\tilde{y}$$

$$+ \lambda_{z\tilde{z}}(V^u(x,\tilde{z},g) - V^u(x,z,g)) + \underbrace{D_g V^u(x,z,g)}_{\substack{\text{Frechet derivative:} \\ \text{how change of } g \text{ affects } V}} \cdot \mu^g$$

- Dynamics of $g(x, y)$ is given by Kolmogorov forward equation (KFE):

$$\mu^g(x,y,z,g) := \frac{dg_t(x,y)}{dt} = -\delta(x,y,z)g(x,y) + \frac{m(z,g)}{\mathcal{U}(z,g)\mathcal{V}(z,g)} \alpha(x,y,z,g) g^v(y) g^u(x)$$

HJB for employed worker, vacant firm, producing firm

Recursive Characterization For Equilibrium Surplus

- Surplus from match $S(x, y, z, g) := V^p(x, y, z, g) - V^v(y, z, g) + V^e(x, y) - V^u(x, z, g)$.
- Characterize equilibrium with master equation for surplus: Free entry condition

$$\begin{aligned}
\rho S(x, y, z, g) = {} & zf(x, y) - \delta(x, y, z) S(x, y, z, g) \\
& + c - (1-\beta) \frac{m(z, g)}{\mathcal{V}(z, g; S)} \int \alpha(\tilde{x}, y, z, g) S(\tilde{x}, y, z, g) \frac{g^u(\tilde{x})}{\mathcal{U}(z, g)} d\tilde{x} \\
& - b - \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \int \alpha(x, \tilde{y}, z, g) S(x, \tilde{y}, z, g) \frac{g^v(\tilde{y})}{\mathcal{V}(z, g; S)} d\tilde{y} \\
& + \lambda(z)(S(x, y, \tilde{z}, g) - S(x, y, z, g)) + D_g S(x, y, z, g) \cdot \mu^g(z, g)
\end{aligned}$$

- Kolmogorov forward equation (KFE):

$$\frac{dg_t(x, y)}{dt} := \mu^g(x, y, z, g) = -\delta(x, y, z) g(x, y) + \frac{m(z, g)}{\mathcal{U}(z, g)\mathcal{V}(z, g)} \alpha(x, y, z, g) g^v(y) g^u(x)$$

- High-dim PDEs with distribution in state: hard to solve with conventional methods.

Finite Type Approximation

- Approximate $g(x,y)$ on finite types: $x \in \mathcal{X} = \{x_1, \ldots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \ldots, y_{n_y}\}$.
- Finite state approximation $\Rightarrow$ analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \leq n_x, j \leq n_y}$
- Approximated master equation for surplus:

$$
\begin{aligned}
0 = \mathcal{L}^S S(x,y,z,g) &= -(\rho+\delta) S(x,y,z,g) + z f(x,y) + c - b \\
&- (1-\beta) \frac{m(z,g)}{\mathcal{V}(z,g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z,g)} \\
&- \beta \frac{m(z,g)}{\mathcal{U}(z,g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z,g)} \\
&+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x,y,z,\{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g)
\end{aligned}
$$

DeepSAM Algorithm for Solving the Model

- Approximate surplus by neural network $S(x, y, z, g) \approx \widehat{S}(x, y, z, g; \boldsymbol{\Theta})$. Function form
- Start with initial parameter guess $\boldsymbol{\Theta}^0$. At iteration n with $\boldsymbol{\Theta}^n$:
  1. Generate K sample points, $Q^n = \left\{\left(x_k, y_k, z_k, \{g_{ij,k}\}_{i \leq n_x, j \leq n_y}\right)\right\}_{k \leq K}$.
  2. Calculate the average mean squared error of surplus master equation on sample points:
  $$L\left(\boldsymbol{\Theta}^n, Q^n\right) := \frac{1}{K} \sum_{k \leq K} \left| \mathcal{L}^S \widehat{S}\left(x_k, y_k, z_k, \{g_{ij,k}\}_{i \leq n_x, j \leq n_y}\right) \right|^2$$
  3. Update NN parameters with stochastic gradient descent (SGD) method:
  $$\boldsymbol{\Theta}^{n+1} = \boldsymbol{\Theta}^n - \zeta^n \nabla_{\boldsymbol{\Theta}} L\left(\boldsymbol{\Theta}^n, Q^n\right)$$
  4. Repeat until $L\left(\boldsymbol{\Theta}^n, Q^n\right) \leq \epsilon$ with precision threshold ϵ.
- Once S is solved, we have α and can solve for worker and firm value functions.

DeepSAM for Estimation with Simulated Method of Moment

- DeepSAM for solving the model (e.g. 59 dimension PDE):

$$\mathcal{L}^S S(x,y,z,g) = 0 \tag{1}$$

- Include structural parameters directly in state space: DeepSAM for estimating the model, solve (e.g. $59 + dim(\Omega)$ dimension PDE):

$$\mathcal{L}^{\widetilde{S}} \widetilde{S}(x,y,z,g,\Omega) = 0 \tag{2}$$

  Ω: structural parameters for estimation.

- Dimension of (2) is only marginally higher than (1). Solving (2), we obtain the model solution over a range of parameter space, enabling estimation through simulation.
  - We use simulation data to build a surrogate model mapping parameters to moments.
- Estimation only takes a marginally longer time than solving the model.

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Distribution and Business Cycle Dynamics

More Applications: OJS and OTC Search

Numerical Accuracy

We confirm DeepSAM achieves high numerical accuracy using two measures:

- Small numerical error. Use DeepSAM to solve the problem (59 dimensional PDE), and compute loss everywhere in high-dimensional state space. Loss: $10^{-7} \sim 10^{-6}$.
- Verification on models with known solution. Use DeepSAM to solve model without aggregate shocks (58 dimensional PDE) and obtain solution at steady state. Compare with steady state solution from conventional methods. Difference: $10^{-7} \sim 10^{-6}$.

Details

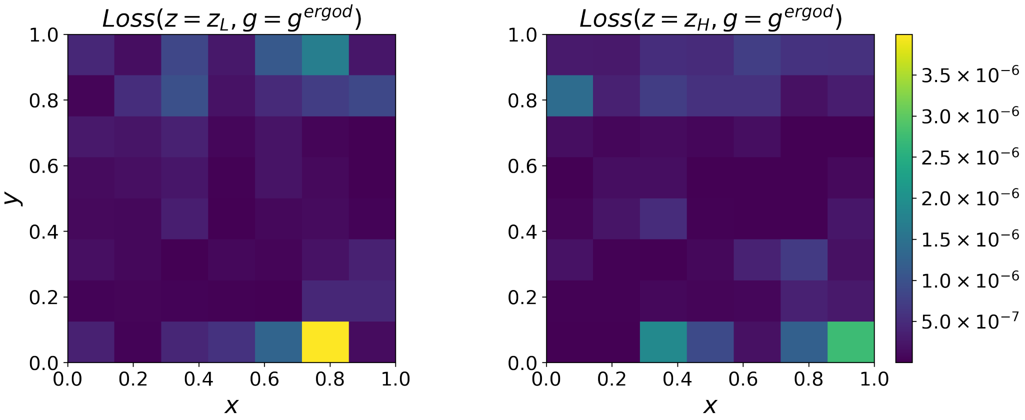
Calibration of Shimer-Smith Model with Aggregate Shocks

Frequency: annual.

Parameter	Interpretation	Value	Target/Source
ρ	Discount rate	0.05	Kaplan, Moll, Violante '18
δ	Job destruction rate	0.2	BLS job tenure 5 years
ξ	Extreme value distribution for α choice	2.0	
$f(x,y)$	Production function for match (x,y)	$0.6+0.4\left(\sqrt{x}+\sqrt{y}\right)^2$	Hagedorn et al '17
β	Surplus division factor	0.72	Shimer '05
c	Entry cost	4.86	Steady state $\mathcal{V}/\mathcal{U}=1$
$z, \tilde{z}$	TFP shocks	1 ± 0.015	Lise Robin '17
$\lambda_z, \lambda_{\tilde{z}}$	Poisson transition probability	0.08	Shimer '05
$\delta, \tilde{\delta}$	Separation shocks	0.2 ± 0.02	Shimer '05
$\lambda_\delta, \lambda_{\tilde{\delta}}$	Poisson transition probability	0.08	Shimer '05
$m(\mathcal{U},\mathcal{V})$	Matching function	$\kappa\mathcal{U}^\nu\mathcal{V}^{1-\nu}$	Hagedorn et al '17
ν	Elasticity parameter for meeting function	0.5	Hagedorn et al '17
κ	Scale parameter for meeting function	5.4	Unemployment rate 5.9%
b	Worker unemployment benefit	0.5	Shimer '05
n_x	Discretization of worker types	7	
n_y	Discretization of firm types	8	

Numerical Performance: Accuracy I Calibration

- Mean squared loss as a function of type in the master equations of S (at ergodic g).



Numerical Performance: Accuracy II Calibration

- Compare steady state solution without aggregate shocks to solution using conventional methods.

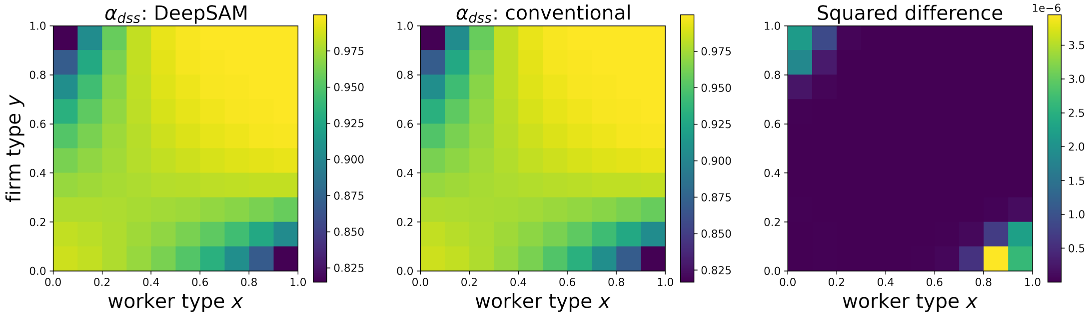


Figure: Comparison with steady-state solution

Comparison for discrete α

back

Computational Speed for Solving and Estimating OJS Model

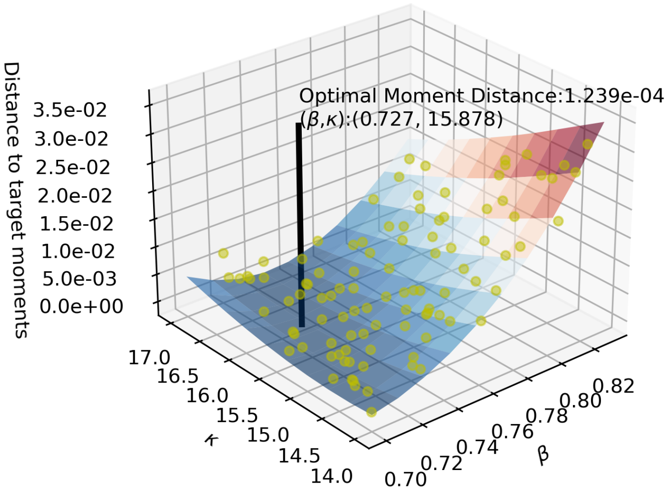
	Solution Given the Value of Structural Parameters	Solution with Structural Parameters as Pseudo-states	Simulation & Training Surrogate Model	Simulated Method of Moments	Entire Estimation
MSE Loss	1.97×10^{-6}	4.8×10^{-6}	6.13×10^{-7}	1.24×10^{-4}	-
Time	55min	4h 1min	1h 3min	1.4min	**5h 5min**

- Solution: 59-dimension PDE.
- Estimation: solve the model over economic parameter space, and simulate across 10,000 parameter combinations for simulated method of moments.

Moments	$\mathbb{E}[U]$	$\mathbb{E}[V]$	$\mathbb{E}[E2E]$	$\mathbb{E}[U2E]$	$\mathbb{E}[E2U]$
Data	0.058	0.037	0.025	0.468	0.025
Model	0.058	0.037	0.026	0.431	0.026

Table: Estimation Results

Estimation of OJS Model: Visualization in 2D



Target moment: $\mathbb{E}[U], \mathbb{E}[V]$. Parameter: matching efficiency κ, worker bargaining power β.

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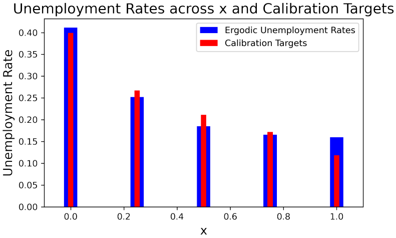
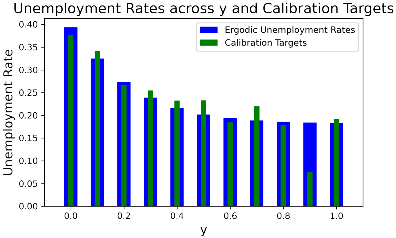
Q1. Does distribution feedback matter? Evidence from COVID-19

- We first apply our method to study labor market dynamics after COVID-19 shock.
- Similar to most recessions, COVID-19 hits workers and firms in a heterogeneous way, which shifts the distribution of matches.
- We calibrate separation rate $\delta(x, y, z)$ to match the heterogeneous employment effect of COVID-19 on different workers/firms (Cajner et al., 2020).





- Study aggregate dynamics with and without distribution feedback to agent decision.

A1. Distribution feedback matters after asymmetric shocks.

- Aggregate dynamics with and without distribution feedback to agent decision:

$$\text{Full dynamics: } \frac{dg_t(x,y)}{dt} = -\delta(x,y,z_t)g_t(x,y) + \frac{m_t(z,g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x,y,z_t,g_t)g_t^u(x)g_t^v(y)$$

$$\text{No distribution feedback: } \frac{dg_t(x,y)}{dt} = -\delta(x,y,z_t)g_t(x,y) + \frac{m_t(z,g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x,y,z_t,g^{\text{ergodic}})g_t^u(x)g_t^v(y)$$

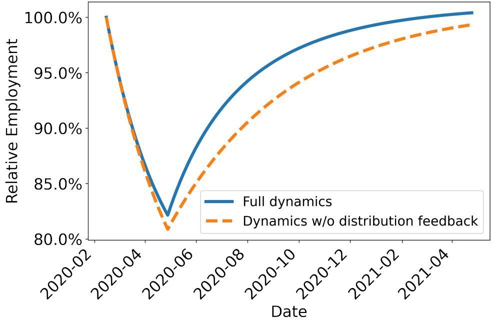


Figure: Employment drop after the COVID-19 shock

Mechanism: high-low matches are more likely given distribution shift

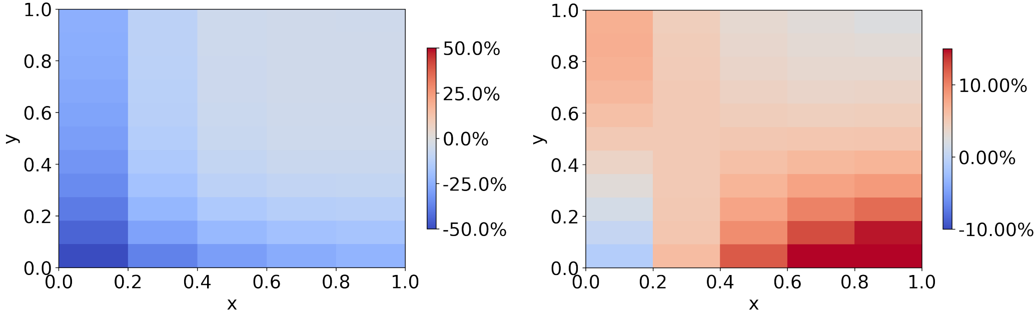


Figure: Difference of distribution and acceptance after the asymmetric COVID-19 shock, compared to the ergodic steady state.

- For high-type firms: relatively more low-type unemployed workers are available, so accept more of them.

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More Applications in the Paper

1. SAM model with on-the-job search and endogenous separation. details
   - Similar to Lise-Robin '17, but allow for $\beta \in (0, 1)$.
   - We also do not assume that vacancies are destroyed if not filled.
2. OTC financial market with heterogeneous investors, different bond maturities, and aggregate default risk. details
   - Introduce idiosyncratic type switching and asset trade compared to our labor model.
   - Offers a search-theoretic rationale for the volatility of the term structure.

Q2. Are wage dynamics heterogeneous across distribution?

- In Lise-Robin: "wages cannot be solved for exactly... need to solve worker values where the distribution of workers across jobs is a state variable."
- DeepSAM can solve wage dynamics with rich heterogeneity.
- Low-type worker wages more procyclical.

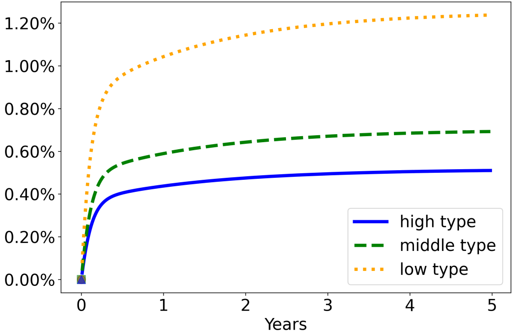


Figure: Wage change after positive aggregate shocks.

Q3. Who benefits more over a longer expansion?

- Okun's (1973) hypothesis: longer expansion benefits low-income workers more.
- We find U_t for low-income workers drops more than high-income in longer expansion.

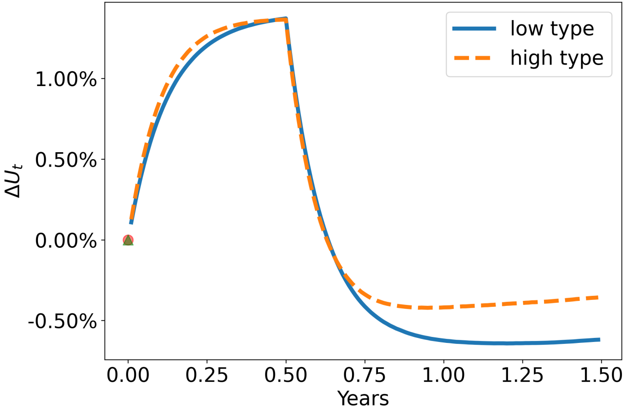
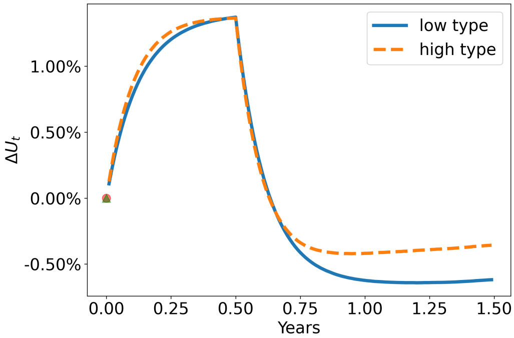


Figure: ΔU_t for workers in different groups

A Search-Theoretical Explanation for Okun's Hypothesis



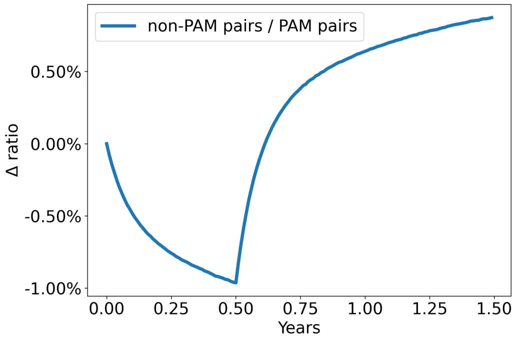


Figure: Left: ΔU_t for different workers. Right: expansion $\Rightarrow$ positive assortative matching $\downarrow$.

- Mechanism: sorting weakens over time in expansions, high-type firms more inclined to hire low-type workers during longer expansions.
- Important that workers&firms understand the distribution of matches over time.

Conclusion and Future Work

- We develop an integrated global solution and estimation method, DeepSAM, to search and matching models with heterogeneity and aggregate shocks.
- We apply DeepSAM to three general setups in labor and financial search models (without simplification assumptions). The method works well, solves new variables (e.g. wage), and generates novel economic insights.
- A foundational tool for a large literature with more applications:
  - Richer models in labor, financial, and money search, combined with rich micro data.
  - Spatial and network models with aggregate uncertainty (similar math structure).

Thank You!

Deep Learning for Economic Models

- Deep learning has been successful in high-dimensional scientific computing problems.
- We can use deep learning to solve high-dim value & policy functions in economics:
  1. Use deep neural networks to approximate value function $V : \mathbb{R}^N \to \mathbb{R}$

$$V(\mathbf{x}) \approx \mathcal{L}^P \circ \cdots \circ \mathcal{L}^p \circ \cdots \circ \mathcal{L}^1(\mathbf{x}), \quad \mathbf{x}\text{: high-dim state vector,}$$
$$\mathbf{h}_p = \mathcal{L}^p(\mathbf{h}_{p-1}) = \sigma\big(\mathbf{W}_p \mathbf{h}_{p-1} + \mathbf{b}_p\big), \quad \mathbf{h}_0 = \mathbf{x},$$

σ : element-wise nonlinear fn, e.g. Tanh$(\cdot)$. Want to solve unknown parameters $\boldsymbol{\Theta} = \{\mathbf{W}_p, \mathbf{b}_p\}_p$.

  2. Cast high-dim function into a loss function, e.g. Bellman equation residual.
  3. Optimize unknown parameters, $\boldsymbol{\Theta}$, to minimize average loss on a "global" state space, using stochastic gradient descent (SGD) method.

- Similar procedure to polynomial "projection", but more efficient in practice. back

Methodology Q & A

- ***Q.*** *What about dimension reduction?*
  - Krusell-Smith '98 suggest approximating distribution by mean.
  - For random search, not clear what moment enables approximation of:

$$\int \alpha(\tilde{x}, y, z, g) S(\tilde{x}, y, z, g) \frac{g^u(\tilde{x})}{\mathcal{U}(z, g)} d\tilde{x}, \quad \text{and} \quad \int \alpha(x, \tilde{y}, z, g) S(x, \tilde{y}, z, g) \frac{g^v(\tilde{y})}{\mathcal{V}(z, g)} d\tilde{y}$$

- ***Q.*** *How do we choose where to sample?*
  - We start by drawing distributions "between" steady states for different fixed z.
  - Can move to ergodic sampling once error is small.
  - Can increase sampling in regions of the state space where errors are high.
- ***Q.*** *Why are SAM models hard to solve?*
  - Compared to PINNs, we have feedback between agent optimization and distribution.
  - Difficult when feedback is strong & $\widehat{S}(x, y, z, g; \boldsymbol{\Theta})$ has sharp curvature. Use "homotopy".

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Labor Search Model

On-The-Job Search Model

OTC Market

Comparison to Other Heterogeneous Agent Search Models

- Lise-Robin '17: sets $\beta = 0$ (and other conditions, including Postal-Vinay Robin style Bertrand competition for workers searching on-the-job)

$$S(x, y, z, g) = S(x, y, z), \quad \alpha(x, y, z, g) = \alpha(x, y, z)$$

- Menzio-Shi '11: competitive search (directed across a collection of sub-markets):

$$S(x, y, z, g) = S(x, y, z)$$

- We look for a solution for S and α in terms of the distribution g.

Modification 1: Finite Type Approximation

- Approximate $g(x, y)$ on finite types: $x \in \mathcal{X} = \{x_1, \dots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \dots, y_{n_y}\}$.
- Finite state approximation $\Rightarrow$ analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \le n_x, j \le n_y}$
- Approximated master equation for surplus:

$$\begin{aligned}
0 = \mathcal{L}^S S(x, y, z, g) &= -(\rho + \delta) S(x, y, z, g) + z f(x, y) - b \\
&- (1 - \beta) \frac{m(z, g)}{\mathcal{V}(z, g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z, g)} \\
&- \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z, g)} \\
&+ \lambda(z) (S(x, y, \tilde{z}, g) - S(x, y, z, g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x, y, z, \{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g)
\end{aligned}$$

Modification 2: Approximate Discrete Choice

- In the original model,

$$\alpha(x, y, z, g) = \mathbb{1}\{S(x, y, z, g) > 0\}$$

- Discrete choice $\alpha \Rightarrow$ discontinuity of $S(x, y, z, g)$ at some g.

- To ensure master equation well defined & NN algorithm works, we approximate with

$$\alpha(x, y, z, g) = \frac{1}{1 + e^{-\xi S(x,y,z,g)}}$$

- Interpretation: logit choice model with utility shocks $\sim$ extreme value distribution.
  ($\xi \to \infty \Rightarrow$ discrete choice α.)

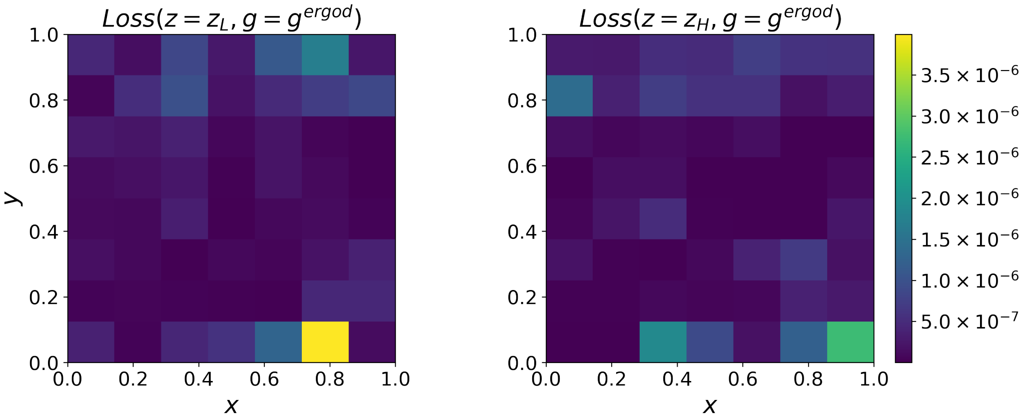
Calibration of Shimer-Smith Model with Aggregate Shocks

Frequency: annual.

Parameter	Interpretation	Value	Target/Source
ρ	Discount rate	0.05	Kaplan, Moll, Violante '18
δ	Job destruction rate	0.2	BLS job tenure 5 years
ξ	Extreme value distribution for α choice	2.0	
$f(x, y)$	Production function for match (x, y)	$0.6 + 0.4 \left(\sqrt{x} + \sqrt{y}\right)^2$	Hagedorn et al '17
β	Surplus division factor	0.72	Shimer '05
c	Entry cost	4.86	Steady state $\mathcal{V}/\mathcal{U} = 1$
$z, \tilde{z}$	TFP shocks	1 ± 0.015	Lise Robin '17
$\lambda_z, \lambda_{\tilde{z}}$	Poisson transition probability	0.08	Shimer '05
$\delta, \tilde{\delta}$	Separation shocks	0.2 ± 0.02	Shimer '05
$\lambda_\delta, \lambda_{\tilde{\delta}}$	Poisson transition probability	0.08	Shimer '05
$m(\mathcal{U}, \mathcal{V})$	Matching function	$\kappa \mathcal{U}^\nu \mathcal{V}^{1-\nu}$	Hagedorn et al '17
ν	Elasticity parameter for meeting function	0.5	Hagedorn et al '17
κ	Scale parameter for meeting function	5.4	Unemployment rate 5.9%
b	Worker unemployment benefit	0.5	Shimer '05
n_x	Discretization of worker types	7	
n_y	Discretization of firm types	8	

Numerical Performance: Accuracy I Calibration

- Mean squared loss as a function of type in the master equations of S (at ergodic g).



Numerical Performance: Accuracy II Calibration

- Compare steady state solution without aggregate shocks to solution using conventional methods.

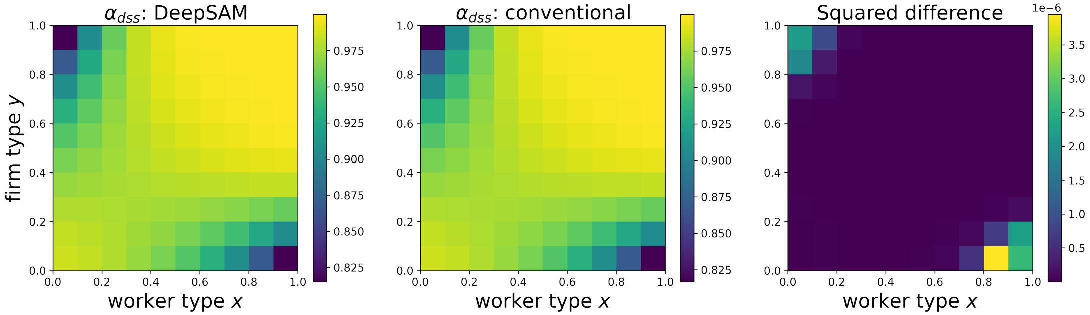


Figure: Comparison with steady-state solution

Comparison for discrete α

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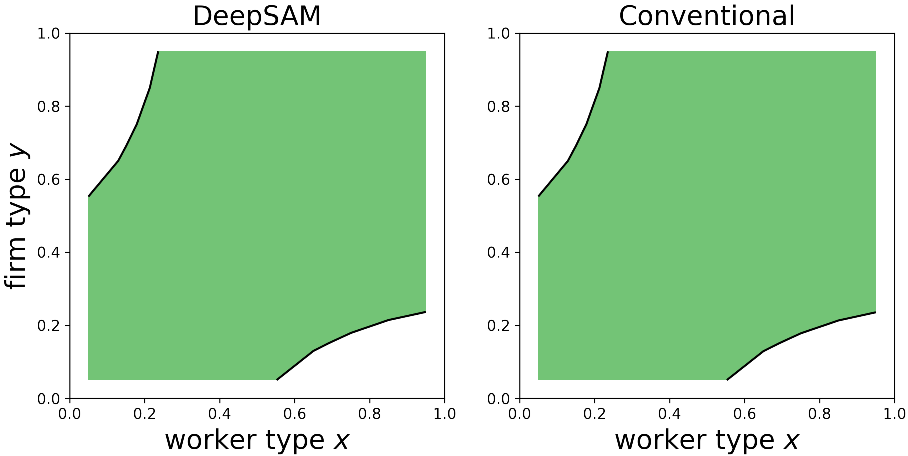
DeepSAM vs Conventional method at DSS: discrete case



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Free Entry Condition

- Firms make entry decision and then draw type y from uniform distribution $[0,1]$:

$$0 = \mathbb{E}[V_t^v] = \int V^v(\tilde{y}, z, g) d\tilde{y}. \tag{3}$$

- As the matching function is homothetic $\frac{m(z_t, g_t)}{\mathcal{V}_t} = \widehat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right)$, combining free entry condition with HJB equation for V^v gives:

$$\widehat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right) = \frac{\rho c}{\int\int \alpha(\tilde{x}, \tilde{y}) \frac{g_t^u(\tilde{x})}{\mathcal{U}_t} (1-\beta) S_t(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}} \Rightarrow \quad \mathcal{V}_t = \mathcal{U}_t \widehat{m}^{-1}(\cdots) \tag{4}$$

  where $g_t^u = g_t^w - \int g_t^m(x, y) dy$ and so the RHS can be computed from g_t^m and S_t.
- $g_t^f = \mathcal{V}_t + \mathcal{P}_t$, where $\mathcal{V}_t$ and $\mathcal{P}_t$ can be expressed in terms of g and S.
- With free entry condition, the master equation expression for surplus takes the same form as without free entry, but with different expressions of $g^f(y)$.

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Recursive Equilibrium Part II: Other Equations

- Hamilton-Jacobi-Bellman equation (HJBE) for employed worker's value $V^e(x,y,z,g)$:

$$\begin{aligned}\rho V^e(x,y,z,g) = w(x,y,z,g) + \delta(x,y,z)\left(V^u(x,z,g) - V^e(x,y,z,g)\right) \\ + \lambda_{z\tilde{z}}(V^e(x,y,\tilde{z},g) - V^e(x,y,z,g)) + D_g V^e(x,y,z,g)\cdot\mu^g\end{aligned}$$

- HJBE for a vacant firm's value $V^v(y,z,g)$:

$$\begin{aligned}\rho V^v(y,z,g) = & -c + \frac{m(z,g)}{\mathcal{V}(z,g)}\int \alpha(\tilde{x},y,z,g)(V^p(\tilde{x},y,z,g) - V^v(y,z,g))\frac{g^u(\tilde{x})}{\mathcal{U}(z,g)}d\tilde{x} \\ & + \lambda_{z\tilde{z}}(V^v(x,\tilde{z},g) - V^v(x,z,g)) + D_g V^v(y,z,g)\cdot\mu^g\end{aligned}$$

- HJBE for a producing firm's value $V^p(x,y,g)$:

$$\begin{aligned}\rho V^p(x,y,z,g) = zf(x,y) - w(x,y,z,g) + \delta(x,y,z)(V^v(y,z,g) - V^p(x,y,z,g)) \\ + \lambda_{z\tilde{z}}(V^p(x,y,\tilde{z},g) - V^p(x,y,z,g)) + D_g V^p(x,y,z,g)\cdot\mu^g\end{aligned}$$

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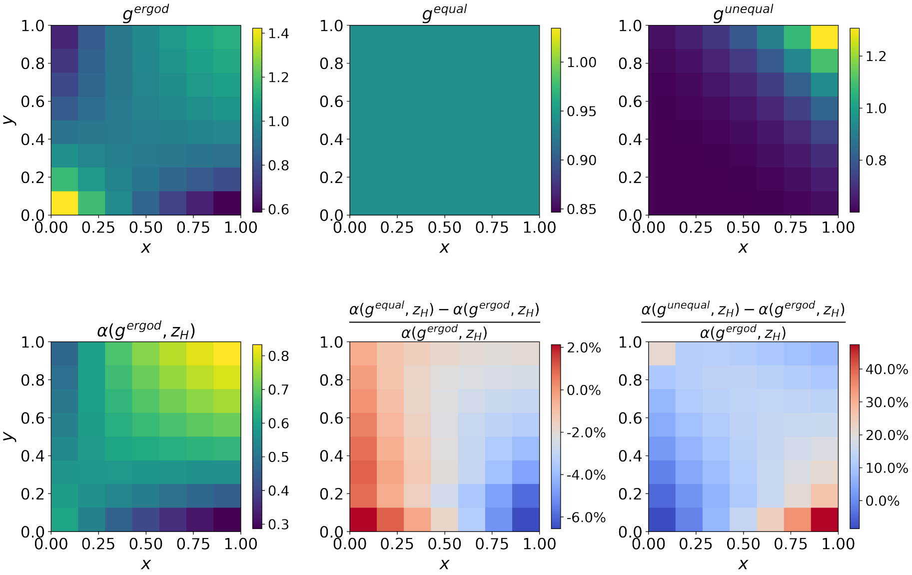
Variation in α as the Distribution Varies



Q2. How do block recursive models restrict aggregate dynamics?

(IRF to negative TFP shock for block recursive vs other calibrations)

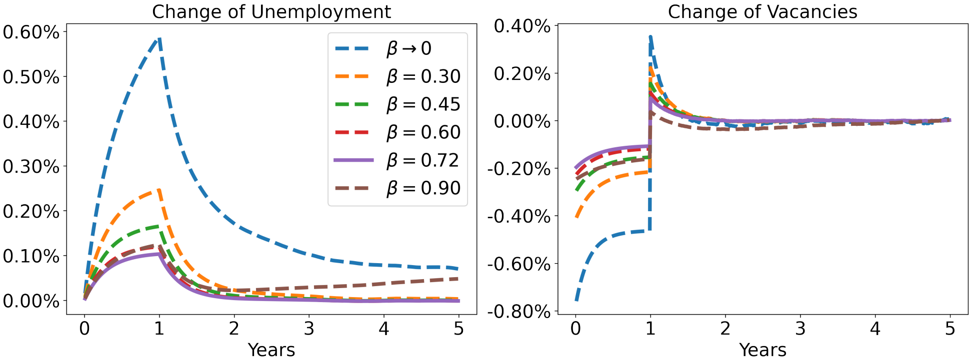


Figure: IRF with different β's vs. block-recursive model with $\beta = 0$

- By assuming firms get all surplus, block recursive models predict high U_t response (because firms' vacancy posting is very elastic to aggregate shocks).

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Labor Search Model

On-The-Job Search Model

OTC Market

On-The-Job Search: Environment Features

- Same worker types, firm types, and production function.
- Now all workers search; meeting rate is $m(\mathcal{W}_t, \mathcal{V}_t)$; total search effort is $\mathcal{W}_t := \mathcal{U}_t + \phi \mathcal{E}_t$
- Terms of trade when a vacant $\tilde{y}$-firm meets:
  - Unemployed x-worker: Nash bargaining where workers get surplus fraction β,
  - Worker in (x, y) match: Nash bargaining over incremental surplus.

    If $S_t(x, \tilde{y}) > S_t(x, y)$, worker moves to firm $\tilde{y}$ and gets additional $\beta(S_t(x, \tilde{y}) - S_t(x, y))$.
- Endogenous separation $\alpha_t^b(x, y) = 1$ when $S_t(x, y) < 0$.

Recursive Characterization For Equilibrium Surplus

- Can characterize equilibrium with the master equation for the surplus:

$$
\begin{aligned}
\rho S(x,y,z,g) &= zf(x,y) - (\delta + \alpha^b(x,y,z,g))S(x,y,z,g) \\
&- \frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\Bigg[(1-\beta)\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)g^u(\tilde{x})d\tilde{x} \\
&-\phi(1-\beta)\int \alpha^p(\tilde{x},y,\tilde{y},z,g)(S(\tilde{x},y,z,g)-S(\tilde{x},\tilde{y},z,g))g(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} \\
&+\phi\beta\int \alpha^p(x,\tilde{y},y,z,g)S(x,y,z,g)g^v(\tilde{y})d\tilde{y}\Bigg] \\
&- b - \beta\frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)g^v(\tilde{y})d\tilde{y} \\
&+ \lambda(z)(S(x,y,\tilde{z},g)-S(x,y,z,g)) + D_g S(x,y,z,g)\cdot\mu^g(z,g)
\end{aligned}
$$

where:

$$
\alpha^p(\tilde{x},y,\tilde{y},z,g) := \mathbb{1}\{S(\tilde{x},y,z,g) \geq S_t(\tilde{x},\tilde{y},z,g) \geq 0\}
$$

KFE

On-the-job-search: KFE

- The KFE becomes:

$$
\begin{aligned}
dg_t^m(x,y) = & -\delta g_t^m(x,y)dt \\
& -\phi\frac{m(\mathcal{W}_t,\mathcal{V}_t)}{\mathcal{W}_t\mathcal{V}_t}g_t^m(x,y)\int\alpha_t^p(x,y,\tilde{y})g_t^v(\tilde{y})d\tilde{y}dt \\
& +\frac{m(\mathcal{W}_t,\mathcal{V}_t)}{\mathcal{W}_t\mathcal{V}_t}\alpha_t(x,y)g_t^u(x)g_t^v(y)dt \\
& +\phi\frac{m(\mathcal{W}_t,\mathcal{V}_t)}{\mathcal{W}_t\mathcal{V}_t}\int\alpha_t^p(\tilde{x},\tilde{y},y)g_t^v(y)\frac{g_t^m(\tilde{x},\tilde{y})}{\mathcal{E}_t}d\tilde{x}d\tilde{y}dt
\end{aligned}
$$

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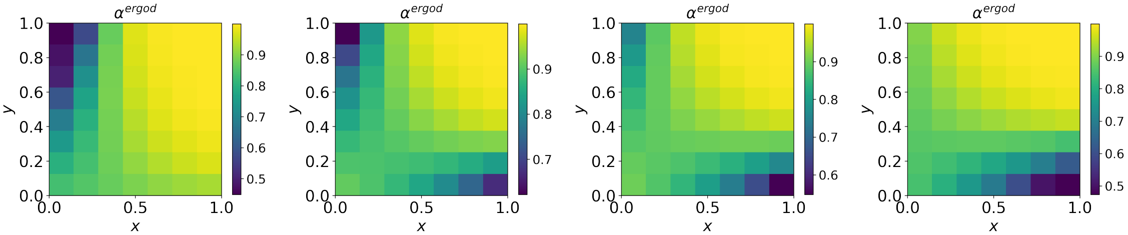
Worker Bargaining Power Influences Assortative Matching

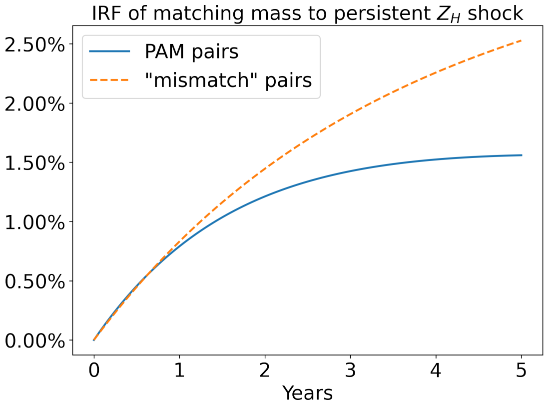


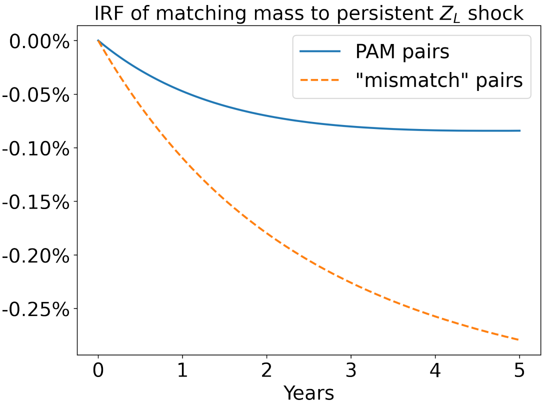
Sorting at the ergodic distribution for different worker bargaining power β. Left to right $\beta = 0$ (Lise-Robin '17), $0.5, 0.72$ (benchmark), 1.

Additional parameter calibration: $\phi = 0.2$.

Sorting Over Business Cycles

- Study how "mismatch" changes over the business cycle. back





"PAM" pairs: pairs where x & y are close. "Mismatch": pairs where x & y are not close.

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Labor Search Model

On-The-Job Search Model

OTC Market

Environment: Setting, Bonds, and Households

- Continuous time, infinite horizon environment.
- There are many bonds, $k \in \{1, \ldots, K\}$, in positive net supply s_k:
  - Every bond pays the same dividend $\delta > 0$.
  - Bond k matures at rate $1/\tau_k$ (so it has average maturity τ_k).
- Populated by a unit-mass continuum of infinitely-lived and risk-neutral investors:
  - An investor can hold either zero or one share of at most one type of asset.
  - Investor type $j \in \{1, \ldots, J\}$ gets flow utility $\delta - \psi(j, k)$ from holding bond k.
  - Agents switch from type i to j at rate $\lambda_{i,j}$.
- Aggregate (default) state $z \in \{z_1, \ldots, z_n\}$, switches at rate $\zeta_{z,z'}$.
  At state z, asset k pays a fraction $\phi(k, z)$ of the coupon and the principal.

Distribution and Bargaining

- An investor's state is made up of her holding cost $j \in \{1, \ldots, J\}$ and her ownership status, for each asset type $k \in \{1, \ldots, K\}$ (owner o or non-owner n). Hence the set of investor idiosyncratic states is:

$$A = \{1n, 2n, \ldots, Jn, 1o1, \ldots, 1oK, 2o1, \ldots 2oK, Jo1, \ldots, JoK\} \tag{5}$$

- The rate of contact between investors with states a and b is:

$$\mathcal{M}_{a,b} = \kappa_{a,b} g_a g_b \tag{6}$$

- Agents a, b engage in Generalized Nash bargaining with bargaining power $\beta_{a,b}$.

Value Function: Non-Owners

- The value function for non-owner with type i, $V(in, g, z)$, is given by:

$$\begin{aligned}
\rho_i V(in, g, z) = & \sum_a \kappa_{in,a} \alpha(in, a, g, z) \beta_{in,a} S(in, a, z, g) \\
& + \sum_k \xi_{i,k} (V(iok, g, z) - V(in, g, z)) \\
& + \sum_{j \neq i} \lambda_{i,j} (V(jn, g, z) - V(in, g, z)) \\
& + \sum_{z'} \zeta_{z,z'} (V(in, g, z') - V(in, g, z)) + \sum_{a \in A} \partial_{g_a} V(in, g, z) \mu^g(a, z)
\end{aligned}$$

where $\alpha(in, jok, g, z)$ is an indicator for whether the surplus from the trade is positive $S(in, jok, g, z) > 0$ and the trade is accepted upon matching.

Value Function: Owners

- Value function for an investor of type i holding asset k, $V(iok, g, z)$, is given by:

$$\begin{aligned}
\rho_i V(iok, g, z) = \delta\phi(k, z) - \psi(i, k) + \frac{1}{\tau_k}(V(in, g, z) + \pi(k, z) - V(iok, g, z)) \\
+ \sum_a \kappa_{iok,a}\alpha(iok, a, g, z) g_a \beta_{iok,a} S(iok, a, g, z) \\
+ \sum_{j \neq i} \lambda_{i,j}(V(jok, g, z) - V(iok, g, z)) \\
+ \sum_{z'} \zeta_{z,z'}(V(iok, g, z') - V(iok, g, z)) + \sum_{a \in A} \partial_{g_a} V(iok, g, z)\mu^g(a, z).
\end{aligned}$$

Parameter Values: Holding Costs

		Maturity (τ)			
		$\tau_1 = 0.25$	$\tau_2 = 1.0$	$\tau_3 = 5$	$\tau_4 = 10$
Agent Type (i)	A	$\delta\phi(1, z)$	$\delta\phi(2, z)$	$\delta\phi(3, z)$	$\delta\phi(4, z)$
	B	0.02	0.02	0.02	0.02
	C	0.0	0.0	0.0	0.0
	D	0.02	0.02	0.01	0.00

Table: Holding costs: $\psi(i, \tau)$.

Parameter Values: Switching Rates

Parameter Values: Participation in Primary Market

		Maturity (τ)			
		$\tau_1 = 0.25$	$\tau_2 = 1.0$	$\tau_3 = 5$	$\tau_4 = 10$
Agent Type (i)	A	ξ_1	ξ_2	ξ_3	ξ_4
	B	–	–	–	–
	C	–	–	–	–
	D	–	–	–	–

Table: Primary market participation: $\xi(i, \tau)$.

Parameter Values: Mathing Rates and Bargaining

$$\kappa_{a,b} = \begin{cases} 50, & \text{if } (a,b) = (in, jok) \text{ and } i,j \neq A, \\ 50, & \text{if } (a,b) = (iok, jok) \text{ and } i,j \neq A, \\ 75, & \text{if } (a,b) = (in, Aok) \text{ and } i \neq A, \\ 0, & \text{if } (a,b) = (iok, Aol) \text{ and } \forall i, \\ 0, & \text{if } (a,b) = (in, jn) \text{ and } \forall i,j, \end{cases} \tag{7}$$

$$\beta_{a,b} = \begin{cases} 0.5, & \text{if } (a,b) = (in, jok) \text{ and } i,j \neq A, \\ 0.5, & \text{if } (a,b) = (iok, jol) \text{ and } i,j \neq A, \\ 0.05, & \text{if } (a,b) = (in, Aok) \text{ and } i,j \neq A, \end{cases} \tag{8}$$

Parameter Values: Other Values

Parameter	Interpretation	Value	Target/Source
ρ	Discount rate	0.05	Chen at al. (2017)
δ	Bond Coupon Rate	0.01	
Aggregate State: $z \in \{z_L, z_M, z_H\}$			
$\phi(z)$	Coupon haircut	$(0.986, 0.991, 0.997)$	Chen at al. (2017)
$\pi(z)$	Principal haircut	$(0.986, 0.991, 0.997)$	Chen at al. (2017)
$\zeta_{M,L}$, $\zeta_{M,H}$	Rate from 2 to 1 and 2 to 3	0.1	Crisis every 10 years
$\zeta_{L,M}$, $\zeta_{H,M}$	Rate from 1 to 2 and 3 to 2	0.5	Average crisis duration 2 years

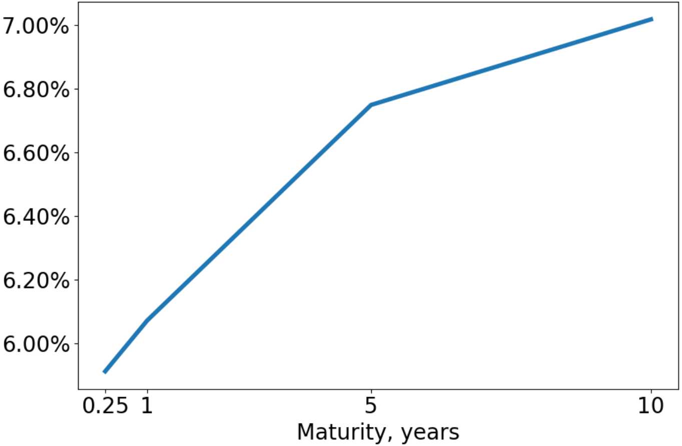
Table: Economic Parameters.

Neural Network Parameter Values

Parameter	Value
Number of layers	8
Neurons per layer	100
Activation function	$\mathrm{GELU}(\cdot)$
Initial learning rate	10^{-4}
Final learning rate	10^{-6}
Initial sample size per epoch	256
Sample size per epoch	1024
Convergence threshold for target calibration	10^{-6}

Table: Neural network parameters

Endogenous Yield Curve back



Yeild IRF back