Endogenous Portfolio Choice in Incomplete Markets

Benoît Clerc **Yucheng Yang** (PKU '15) Jan Žemlička

University of Zurich and SFI

PKU Conference on Machine Learning in Economics

July 1st, 2025

Introduction

- Rise of ML as new global solution methods for heterogeneous agent (HA) models.
- While now mostly focus on challenging new questions in various fields (e.g., climate economics, asset pricing, monetary/fiscal policy, macro labor), this literature starts with many "proof of concept" papers on **classical models**, mostly Krusell and Smith (1998).
- Good **benchmark problems**: crucial for method development (e.g. ImageNet for DNN).
- Krusell and Smith (1998) as a **benchmark model**:
    1. Advantages: simplest HA models with aggregate shocks. Everybody knows it.
    2. Limitations: miss important dimensions of modern models, e.g. non-trivial market clearing, portfolio choice. Also, can be easily solved with conventional methods.
- Goal of this paper: use deep learning to solve the Krusell and Smith (1997) model with both risky capital and bond. Potentially, a new **benchmark problem**.

Some History according to Maliar and Maliar (2020)

The HANC class of models received a considerable attention in the literature, in particular, den Haan, Judd and Juliard (2008) edited a special issue of JEDC that summarize the computational approaches proposed for analyzing such models. The participants of the special issue were invited to solve two HANC models: one with savings through capital and the other with savings through bonds. There were many methods that worked accurately and reliably for the former model but the participants did not succeed in producing accurate solutions to the later model. In particular, they tried out higher moments and other statistics of the bond distribution such as histogram or fraction of agents facing the borrowing constraint but those statistics did not have sufficient power for predicting the future aggregates. Eventually, the HANC model with savings through bonds was eventually removed from the original JEDC call and it was not studied in the literature to the best of our knowledge.

Krusell and Smith (1997): Model

Household's Problem

- Production economy with a continuum of households: household i chooses consumption c_{it}, saving in bond $b_{i,t+1}$, saving in risky asset $a_{i,t+1}$ to solve:

$$\max_{c_{it}, a_{i,t+1}, b_{i,t+1}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta_t U(c_{it})$$

subject to:

$$c_{it} + a_{i,t+1} + q_t b_{i,t+1} = a_{it}(1 + r_t) + b_{it} + e_{it}\bar{l} w_t$$

$$a_{i,t+1} \geq \underline{a}, \quad b_{i,t+1} \geq \underline{b}$$

- Idiosyncratic shocks on labor endowment e_{it} follows i.i.d. three-state Markov process.

Firm's Problem and Asset Market

Firm's Problem

- Representative firm produces output using a Cobb-Douglas function:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

- Aggregate productivity $Z_t \in \{z_l, z_m, z_h\}$ follows a three-state Markov process.
- Factor prices set by firm FOCs:

$$w_t = Z_t(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}, \quad r_t = Z_t\alpha\left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta$$

Asset Market

- Agents can invest in two assets to hedge against unemployment risk
- Risky capital whose return fluctuates with aggregate shocks
- Riskless bond, with its price q_t, is set to clear the market

Recursive Equilibrium

- Individual state: e_{it}, total wealth $\omega_{it} \equiv a_{it}(1+r_t) + b_{it}$.
- Aggregate state: distribution μ_t and Z_t
- Recursive equilibrium: value & functions $V(\omega_t, e_t, Z_t, \mu_t)$, $c(\omega_t, e_t, Z_t, \mu_t)$, $a'(\omega_t, e_t, Z_t, \mu_t)$, $b'(\omega_t, e_t, Z_t, \mu_t)$, prices $q(\mu_t, Z_t), r(\mu_t, Z_t), w(\mu_t, Z_t)$, law of motion $\mu' = H(\mu, Z, Z')$. Value and policy functions solve

$$V(\omega, e, Z, \mu) = \max_{c,a',b'} \{u(c) + \beta EV\left(\omega', e', Z', \mu' | e, Z\right)\}$$

  subject to (1) household budget constraint (2) borrowing constraint (3) exogenous shock processes (4) law of motion H.
- Factor prices competitive; market clear; law of motion consistent with policy functions.
- **Main difficulty** compared to KS (1998): bond price pinned down by market clearing.

Equilibrium Conditions

1. **Household's KKT Conditions:**

   - Capital:

$$u'(c_t) = \beta \mathbb{E}_t[(1 + r_{t+1})u'(c_{t+1})] + \lambda_t$$

$$\lambda_t(a_t - \underline{a}) = 0, \quad \lambda_t \geq 0, \quad a_t \geq \underline{a}$$

   - Bonds:

$$q_t u'(c_t) = \beta \mathbb{E}_t[u'(c_{t+1})] + \nu_t$$

$$\nu_t(b_t - \underline{b}) = 0, \quad \nu_t \geq 0, \quad b_t \geq \underline{b}$$

2. **Markets clearing conditions:**

$$\int b_{i,t}\, d\mu = \bar{B} \quad \int a_{i,t}\, d\mu = K_t$$

Deep Learning Method

Neural Networks Representation

- $x^{agg} = (\mu, Z, K, r, w)$: state vector for price function (extra prices for stability), $x^{id} = (x^{agg}, \omega, e)$: state vector for policy functions.
- Distribution representation μ: histogram on wealth & income grids $N_{grid} \times N_e$.
- **Aggregate Network** for bond price

$$\mathcal{N}_{\Theta^{agg}} : \mathbb{R}^{N_e \times N_{grid}+4} \to \mathbb{R}^1, \quad \hat{q}_t = \mathcal{N}_{\Theta^{agg}}(x_t^{agg})$$

- **Policy Network:**

$$\mathcal{N}_{\Theta^{pol}} : \mathbb{R}^{N_e \times N_{grid}+6} \to \mathbb{R}^2$$

  Outputs: (1) saving rate (s^{rate}), (2) fraction of savings allocated to risky assets (ρ):

$$(s_t^{rate}, \rho_t) = \mathcal{N}_{\Theta^{pol}}(x_t^{agg}, \omega_t, e_t)$$

$$\text{cash on hand: } coh_t = \omega_t + w_t(K_t, Z_t, e_t) - \underline{a} - \hat{q}_t \underline{b}$$

$$\hat{c}_t = (1 - s_t^{rate}) \times coh_t, \quad \hat{a}_{t+1} = \underline{a} + s_t^{rate} \times \rho_t \times coh_t, \quad \hat{b}_{t+1} = \underline{b} + \frac{s_t^{rate} \times (1-\rho_t) \times coh_t}{\hat{q}_t}$$

Solution Algorithm

- We solve the problem with the Deep Equilibrium Nets (DEQN) framework proposed by Azinovic et al. (2022), and "homotopy" ideas from Azinovic and Zemlicka (2023).
- General steps:
  1. Generate a training sample $\mathcal{D}_{\text{train}} = (x_1, ..., x_N)$ using the most recent policy function NN. Histogram-based simulation following Young (2010)'s non-stochastic method
  2. Evaluate the loss function $\mathcal{L}$ on the sample
  3. Update the NN parameters $\Theta = (\Theta^{pol}, \Theta^{agg})$ using gradient descent:

$$\Theta_{t+1} = \Theta_t - \alpha^{\text{rate}} \times \frac{\partial \mathcal{L}}{\partial \Theta_t}$$

  4. Repeat the process until the error converges to an acceptable level.

Loss Function

Errors in KKT conditions are encoded with Fischer-Burmeister equation (Maliar, Maliar, Winant, 2021): $\psi(x, y) \equiv x + y - \sqrt{x^2 + y^2}$

$$\text{err}_\theta^{a,i}(\mathbf{x_t^{id}}) = \psi\left(\frac{(u')^{-1}\left(\beta\mathbb{E}_t\left[(1 + r_{t+1})u'(\hat{c}_{t+1}^i)\right]\right)}{\hat{c}_t^i} - 1, \frac{\hat{a}_t^i - \underline{a}}{\hat{c}_t^i}\right),$$

$$\text{err}_\theta^{b,i}(\mathbf{x_t^{id}}) = \psi\left(\frac{(u')^{-1}\left(\frac{\beta}{\hat{q}_t}\mathbb{E}_t\left[u'(\hat{c}_{t+1}^i)\right]\right)}{\hat{c}_t^i} - 1, \frac{\hat{b}_t^i - \underline{b}}{\hat{c}_t^i}\right),$$

$$\text{err}_\theta^{mc}(\mathbf{x}_t^{\text{agg}}) = B^{\text{supply}} - B^{\text{demand}}, \qquad B^{\text{demand}} = \sum_{i=1}^{N_{\text{grid}}}\sum_e \mu_t^{i,e} b_t^{i,e}$$

$$\Rightarrow \mathcal{L} = \frac{1}{N_{\text{size}}}\sum_{b=1}^{N_{\text{size}}}\left((\text{err}_\theta^{\text{mc}})^2 + \frac{1}{N^{\text{id}}}\sum_{i=1}^{N^{\text{id}}}\left((\text{err}_\theta^{\text{a},i})^2 + (\text{err}_\theta^{\text{b},i})^2\right)\right)$$

Training procedure

- Portfolio choice problems are difficult to solve because assets are close substitutes.
- High accuracy is required to differentiate assets, especially problematic at the beginning of the training
- We use homotopy (or "step-wise") approach following Azinovic and Zemlicka (2023): begin with a simpler model and gradually transit to more complex ones

Homotopy Steps

1. Solve a nested one-asset (capital only) model and calibrate the aggregate function NN.
2. Introduce bonds into the economy:
    1. Relax the parameter $\underline{b}$ and increase the relative weight of bonds in the loss term.
    2. Set equal weights for bonds and capital errors, and relax $\underline{b}$ until the limit is reached.

Homotopy Step 1: Solve a Nested One-asset Model

Set $\underline{b} = 0$, $B^{supply} = 0$. Add an error term to calibrate the aggregate network

$$q_t^{\text{implied}} = \max\left(\frac{\beta \mathbb{E}_t\left[u'\left(\hat{c}_{t+1}^i\right)\right]}{u'\left(c_t\right)}\right), \quad \text{err}_\Theta^q = \hat{q}_t - q_t^{\text{implied}}$$

$$\mathcal{L} = \frac{1}{N_{\text{size}}} \sum_{b=1}^{N_{\text{size}}} \left((\text{err}_\theta^{\text{mc}})^2 + (\text{err}_\theta^{\text{q}})^2 + \frac{1}{N^{\text{id}}} \sum_{i=1}^{N^{\text{id}}} \left((\text{err}_\theta^{\text{a},i})^2 + 0 \times (\text{err}_\theta^{\text{b},i})^2 \right) \right)$$

Homotopy Step 2: Introduce the Bond Gradually to the Economy

Step 2.1

Introduce a weighting term w^b into the loss function, where w^b increases iteratively from 0 to 1. Simultaneously, let $\underline{b}$ decrease from 0 to -1.

$$\mathcal{L} = \frac{1}{N_{\text{size}}} \sum_{b=1}^{N_{\text{size}}} \left((\text{err}_\theta^{\text{mc}})^2 + \frac{1}{N^{\text{id}}} \sum_{i=1}^{N^{\text{id}}} \left((\text{err}_\theta^{\text{a},i})^2 + {\color{red} w^b} (\text{err}_\theta^{\text{b},i})^2 \right) \right)$$

Step 2.2

Fix $w^b = 1$ and allow $\underline{b}$ to decrease from -1 until it reaches its target value.

- Gradually relax the borrowing constraint to prevent abrupt shifts in the distribution.
- The evolution of financial wealth, $\omega_{t+1} = a_{t+1}(1 + r_{t+1}) + b_{t+1}$, depends on agents' policies but also on q_t, r_{t+1}, and $\underline{b}$.
- A large shock in any of these variables can cause the economy to deviate from its ergodic states ▸ Example

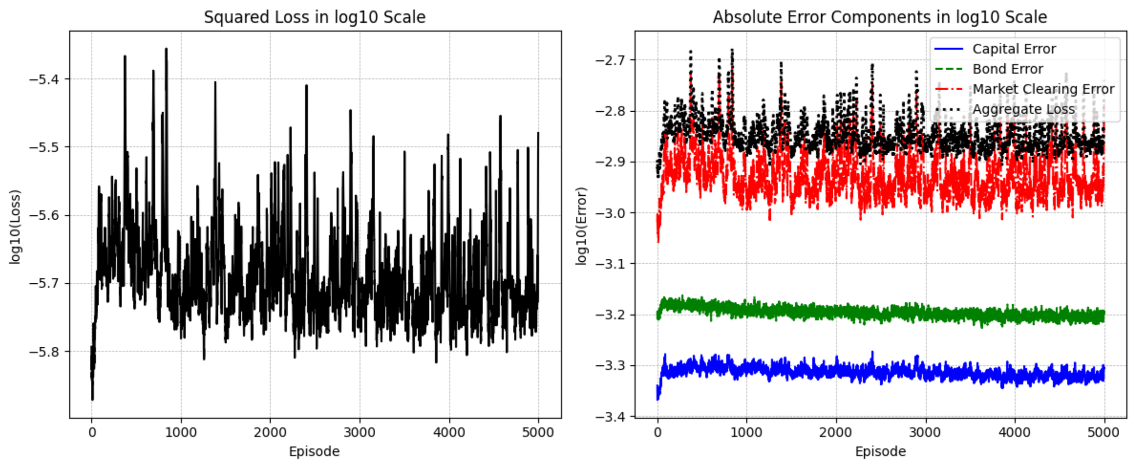
Results

Calibration

- Adapt the quarterly calibration from Krusell and Smith (1997) to a yearly calibration. Coutercyclical idiosyncratic shock following Storesletten et al (2004).

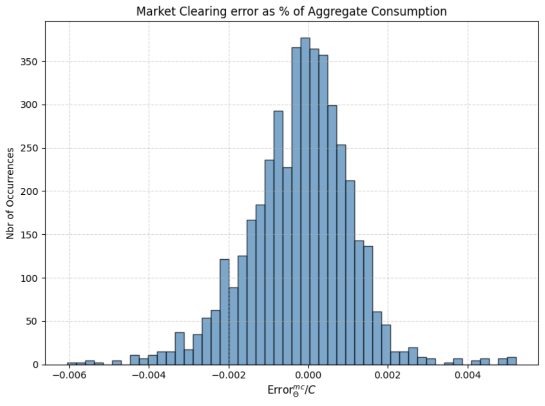
Parameter	**Description**	**Value**
α	Capital share in production	0.36
β	Discount factor	0.95
δ	Depreciation rate of capital	10%
γ	Risk aversion coefficient	5
Z_l	TFP in low state	0.95
Z_m	TFP in middle state	1.00
Z_h	TFP in high state	1.05
$\underline{a}$	Risky asset constraint	0
$\underline{b}$	Riskless bond constraint	-2.4

Training Loss in the Last Iteration

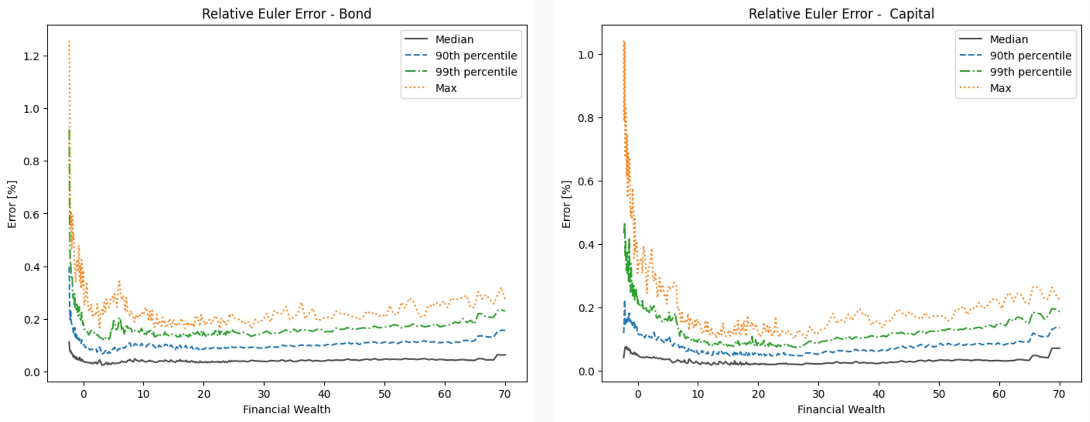


Accuracy on New Test Sample Set: Market Clearing Error

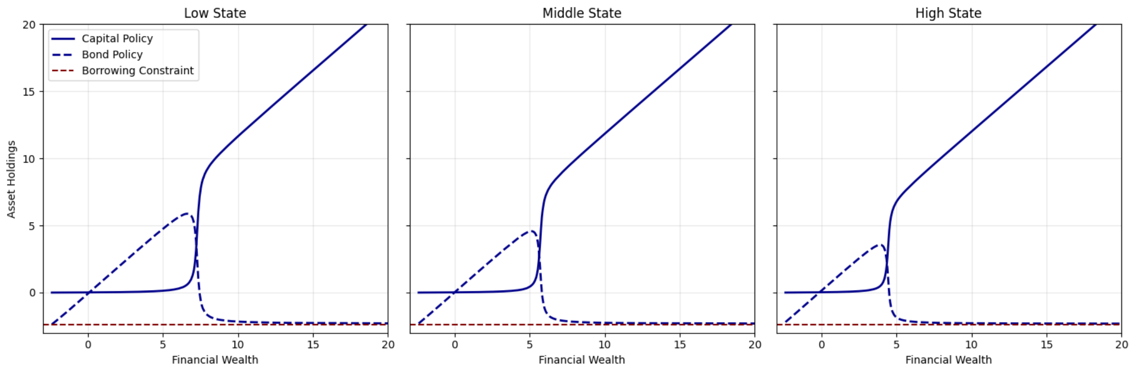
- Simulate for 2^{12} samples without retraining the network
- Market clearing error is distributed around 0
- Error concentration in the lower part of the distribution.



Accuracy on New Test Sample Set: Euler Errors



Results: Savings Policy Functions



Comparison: Policy Function in KS (1997) and Auclert et al (2024)

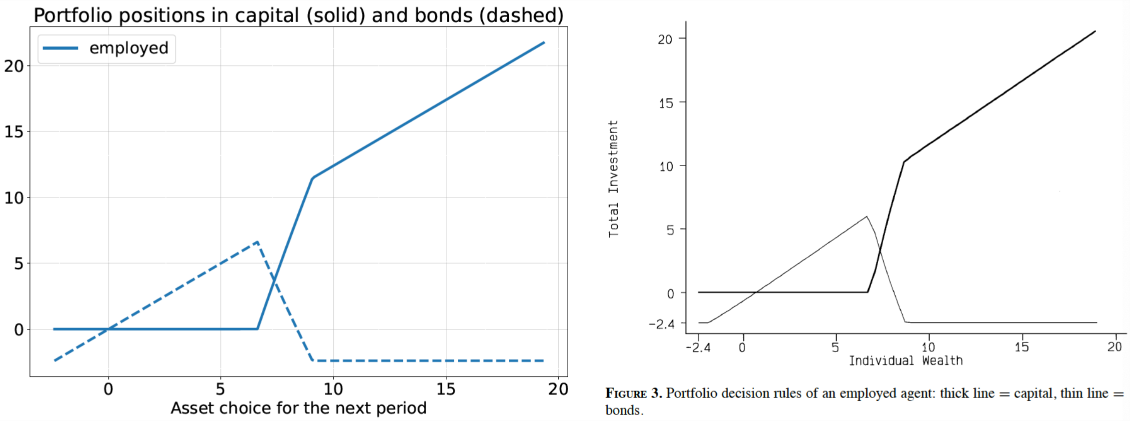
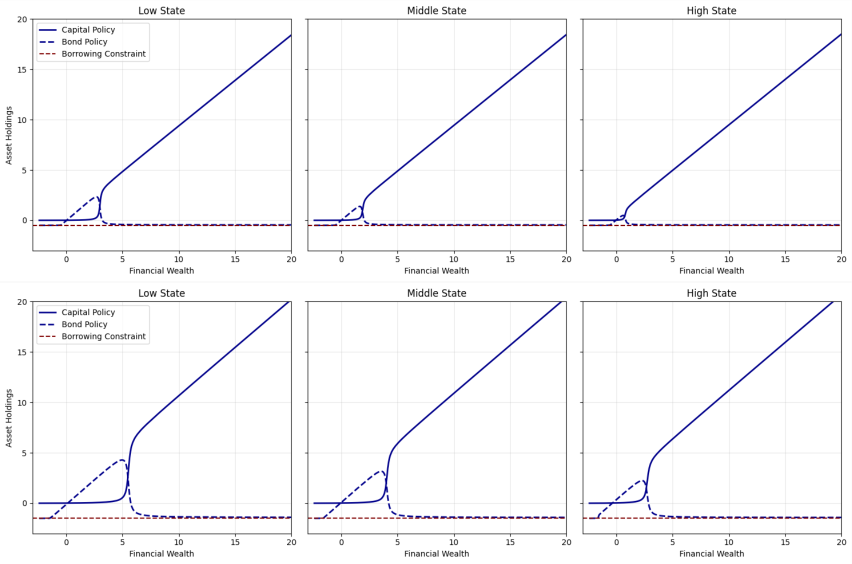


Figure 1: Right: policy function in KS (1997); Left: Auclert et al (2024)'s policy function to solve KS (1997)

Alternative Calibration: Policy functions when $\underline{b} = -0.5, -1.5$

Consumption function



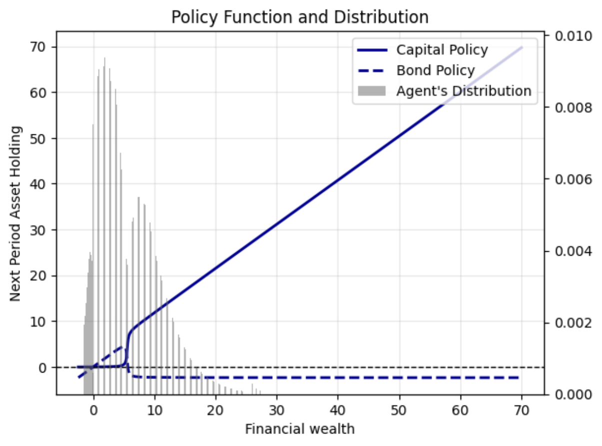
Asset Prices with Different Risk Aversion Parameters

	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$
Aggregate Capital	3.918	4.057	4.327
Return on Capital [%]	5.066	4.763	4.181
Bond Price	0.9516	0.9546	0.9601
Risk-free rate [%]	5.082	4.756	4.152
Equity Premium [%]	-0.01786	0.00056	0.01942

Consistent with KS (1997), we find very low equity premium, about 0.02% on average even when $\gamma = 6$.

Who Price the Assets? Endogenous Market Segmentation

- The majority of agents face borrowing constraints in one asset.
- Capital is priced by the wealthiest agents; Bond is priced by the poorest agent
- Only a small share of agents have an interior portfolio choice



Conclusion

- Method Take-aways
  1. We solve the simplest HA model with endogenous portfolio choice using deep learning.
  2. The model is a potential benchmark problem for future method development.
  3. Specific NN design and training procedure are helpful
- Economic Take-aways:
  1. Equity premium is too low
  2. Portfolio choice results: most agents have binding constraints in one asset: segmented market to price assets.