# DeepHAM: A Global Solution Method for Heterogeneous Agent Models with Aggregate Shocks

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### Abstract

We propose an efficient, reliable, and interpretable global solution method, the *Deep learning-based algorithm for Heterogeneous Agent Models (DeepHAM)*, for solving high dimensional heterogeneous agent models with aggregate shocks. The state distribution is approximately represented by a set of optimal generalized moments. Deep neural networks are used to approximate the value and policy functions, and the objective is optimized over directly simulated paths. In addition to being an accurate global solver, this method has three additional features. First, it is computationally efficient in solving complex heterogeneous agent models, and it does not suffer from the curse of dimensionality. Second, it provides a general and interpretable representation of the distribution over individual states, which is crucial in addressing the classical question of whether and how heterogeneity matters in macroeconomics. Third, it solves the constrained efficiency problem as easily as it solves the competitive equilibrium, which opens up new possibilities for studying optimal monetary and fiscal policies in heterogeneous agent models with aggregate shocks.

**Keywords:** Heterogeneous agent models, aggregate shocks, global solution, deep learning, generalized moments, constrained efficiency.

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# 1 Introduction

The incorporation of both explicit heterogeneity and aggregate fluctuations into quantitative models has been one of the most important recent developments in macroeconomics. It has become an important agenda for a number of reasons. First, uneven dynamics across sectors and population groups after major economic fluctuations and policy shocks suggest that heterogeneity and aggregate shocks are necessary considerations when studying fundamental macroeconomic problems. Second, the recent development of the heterogeneous agent New Keynesian (HANK) models suggests that heterogeneity gives rise to key channels in the transmission of aggregate shocks. These channels are crucial in determining the correct aggregate implications of monetary and fiscal policies. Third, the advancement of computational methods and growth in computing power have allowed economists to build models more realistic than the representative agent models that have long been dominant in both academic and policy research.

Despite the attention this agenda has received, its workhorse models, heterogeneous agent (HA) models with aggregate shocks, still present severe computational challenges. Ideally, one would like to develop solution methods that fulfill the following basic requirements:

- **Efficiency**: The method should be computationally efficient, especially for complex HA models with multiple state variables. This is necessary in order to use the method for calibration, estimation, and further quantitative analysis.

- **Reliability**: The method should produce accurate solutions for all practical situations that HA models are intended for. In particular, it should be applicable beyond the local perturbation regime if nonlinear and nonlocal effects of aggregate shocks are important.

- **Interpretability**: We are not only interested in the numbers that come out of an algorithm, but also in understanding the mechanisms underlying the results. For that, the major components of the algorithm should be interpretable. In particular, solutions to HA models with aggregate shocks usually involve mappings from the distribution over all individual states to the agent's welfare or decision outcomes. An ideal solution should provide interpretability of these mappings through an interpretable representation of the state distribution. An interpretable representation of the distribution is also necessary to derive reduced dynamic models at the aggregate level from the original HA model.

- **Generality**: The method should in principle be applicable to a wide variety of different HA models (simple or complex), and to different notions of equilibrium (e.g.

competitive equilibrium or constrained efficiency problem).

Currently, there are two main approaches for solving HA models with aggregate shocks, and they satisfy only a subset of the requirements listed above. The first is the Krusell-Smith (KS) method, a global solution method proposed in Krusell and Smith (1998). The KS method approximates agent distribution with a small number of moments (e.g., the first moment), which are interpretable. It is efficient when solving simple HA models but becomes less effective when solving complex HA models with multiple shocks, or multiple endogenous states, or when solving the estimation problem. This is due to the large number of variables (such as the possibly large number of moments needed) introduced and the resulting curse of dimensionality problem. That is, the computational cost increases exponentially with the number of variables.

The second approach is the local perturbation method proposed in Reiter (2009). This method allows one to study or estimate complex HA models, but is not reliable for models where aggregate shocks bring significant nonlinear or nonlocal effects. Nonlinear effects are common in models with zero lower bounds (ZLB). Nonlocal effects appear in models with large aggregate shocks, or in models (e.g., macro-finance models) where explicit consideration of aggregate uncertainty plays an important role in shaping agents' behavior, resulting in the deviation of the risky steady state from the deterministic steady state. Table 1 summarizes the advantages and limitations of these two methods.

| Model features | KS method | Perturbation method | DeepHAM |
|---|---|---|---|
| Multiple shocks | No | Yes | Yes |
| Multiple endogenous states | No | Yes | Yes |
| Large shocks | Yes | No | Yes |
| Risky steady state | Yes | No | Yes |
| Nonlinearity (e.g., ZLB) | Yes | No | Yes |

Table 1: Model features that different solution methods for HA models with aggregate shocks can handle.

In this paper, we propose a new solution method, the *Deep learning-based algorithm for Heterogeneous Agent Models (DeepHAM)*, which satisfies all the requirements listed above. We formulate a HA model with $N$ agents, where $N$ is large when we aim to solve a problem with a continuum of agents. To solve HA models, the fundamental objects of interest are agents' value and policy functions. A complication arises from the fact that these functions depend not only on the agent's own state, but also the distribution of all agents' states in the economy. To address this issue, we represent the value function and policy function with deep neural networks, and present an algorithm to update the value and policy functions iteratively. Deep neural networks are a class of functions in deep learning, which have strong representational capability for high dimensional functions and can be efficiently optimized with stochastic gradient descent algorithms. Here we take advantage of deep learning not only as an efficient solution method for high dimensional problems, but also as a flexible representation of high dimensional distributions. In contrast to existing literature that uses deep learning to represent high dimensional policy and value functions directly, we introduce *generalized moments* to represent the state distribution efficiently, and solve for the value and policy functions as functions of the generalized moments. Generalized moments extract useful information from the state distribution, similarly to classical moments, but are automatically determined by the algorithm. The introduction of generalized moments also ensures that the agent's optimal policy and value functions are invariant under permutations of the ordering of the agents. Conceptually, the generalized moments reduce state dimension while remaining readily interpretable and flexible enough to encode, through algorithmically-determined moments of the state distribution, the typical forms of interaction between the agents and the market. As we will see below, a single generalized moment not only leads to more accurate solutions than using only the first moment, but also extracts key information from the agent distribution with first order implications for aggregate welfare and dynamics. Thus, it provides a general and interpretable way to study a key question in macroeconomics: whether, why, and how inequality matters for the macroeconomy.

As we will demonstrate later, DeepHAM meets all the requirements listed above. First, it shows better global accuracy compared with existing methods. In the baseline model we study, DeepHAM with only the first moment in the state vector reduces the Bellman equation error by 37.5% compared to the KS method. DeepHAM with one generalized moment reduces the error by 54.2%. Second, the computational cost of DeepHAM is quite low in solving complex HA models, and it does not suffer from the curse of dimensionality. DeepHAM can efficiently solve HA models with endogenous labor supply, or with a Brunnermeier and Sannikov (2014) type of financial sector. Third, the use of generalized moments allows us to revisit classical questions in macroeconomics of whether and how heterogeneity matters to aggregate welfare and dynamics. Krusell and Smith (1998) famously argued that, in their setup, individual welfare is affected by other agents only through the mean of wealth distribution. With the generalized moments, we find that an unanticipated redistributional policy shock would have a non-zero welfare impact on those households who are not in the policy program, even when the mean of the wealth distribution is not affected. Finally, as a demonstration of the generality of DeepHAM, we show that it can be used to solve the constrained efficiency problem in HA models, which is regarded as a challenging problem in the literature, as easily as solving the competitive equilibrium. This allows us to study optimal fiscal and monetary policy in HA models with aggregate shocks.

DeepHAM should be applicable to a large class of economic problems with heterogeneity and aggregate shocks, and here we list some such examples. First, since DeepHAM does not suffer from the curse of dimensionality with more endogenous states or shocks, we can introduce more realistic portfolio options like housing and mortgage choices (Kaplan, Mitman, and Violante, 2020; Boar, Gorea, and Midrigan, 2021). We can efficiently handle models with *ex ante* heterogeneous agents, such as households and financial experts (Brunnermeier and Sannikov, 2014), rational and bounded-rational agents (Woodford and Xie, 2021), among others. We can study models with rich firm heterogeneity and aggregate shocks (Khan and Thomas, 2013). We can also study HA models with multiple shocks, where the shocks take forms that appear commonly in the DSGE literature (McKay and Reis, 2016). Second, we can use DeepHAM to study models with large shocks such as the COVID-19 shock, or large endogenous fluctuations such as those discussed in Petrosky-Nadeau, Zhang, and Kuehn (2018). We can also use DeepHAM to study asset pricing and the wealth effects of monetary and fiscal policy in a HA model. The empirical literature has shown these factors to be important (Andersen et al., 2021) but, due to computational challenges, they have only been studied in models with limited heterogeneity (Kekre and Lenel, 2020; Caramp and Silva, 2021). We can also study the interaction of asset pricing and wealth inequality (Cioffi, 2021). Third, we can study optimal policy problems with heterogeneous agents using the Ramsey approach (Davila et al., 2012; Nuño and Moll, 2018), such as optimal monetary and fiscal policy (Bhandari, Evans, Golosov, and Sargent, 2021; Dyrda and Pedroni, 2021; Le Grand, Martin-Baillon, and Ragot, 2021), or optimal macroprudential policy (Bianchi and Mendoza, 2018). Such study has heretofore been limited by computational challenges. Last but not least, methodologically, we can extend DeepHAM to do model calibration by introducing a calibration target in the objective function so that we can solve and calibrate HA models in the same algorithmic framework.

**Related Literature.** Our work builds on an extensive literature on solving HA models with aggregate shocks. As discussed, there are two main approaches in the literature (Algan et al., 2014): the global Krusell-Smith (KS) method (Krusell and Smith, 1998; Den Haan, 2010; Fernández-Villaverde et al., 2019; Schaab, 2020), and the local perturbation method (Reiter, 2009; Winberry, 2018; Ahn et al., 2018; Boppart et al., 2018; Auclert et al., 2021). Due to the curse of dimensionality, the KS method cannot handle complex HA models with multiple assets and multiple shocks. The perturbation method has been applied to complex HA models, solving for local dynamics around the deterministic stationary equilibrium in the absence of aggregate shocks, but it is inapplicable to problems with nonlinear dynamics induced by aggregate shocks, or problems that are not close to the deterministic stationary equilibrium. DeepHAM can handle complex HA models with aggregate shocks and provide a global solution.

This paper proposes a general methodology that extracts the key information of the distribution that matters for aggregate welfare and dynamics. This is an important issue in studying the role of heterogeneity in macroeconomics (Kaplan et al., 2018; Auclert, 2019). For an overview of this literature, see Kaplan and Violante (2018). Most papers in this literature study the role of heterogeneity with quantitative decomposition after solving the model, while we propose a general numerical method that extracts key moments of the distribution as part of the numerical solution process. Our idea of the permutation invariant *generalized moments* coincides with the independent and contemporaneous work of Kahou et al. (2021), while we further explore the interpretation of the generalized moments and heterogeneity, and use them to study the impact of an unanticipated redistributional policy shock.

This work is also of relevance to the literature on machine learning-based algorithms for solving high dimensional dynamic programming problems in scientific computing (Han and E, 2016; Han et al., 2018; Fernández-Villaverde et al., 2020) and in macroeconomics (Duarte, 2018; Fernández-Villaverde, Hurtado, and Nuno, 2019; Azinovic, Gaegauf, and Scheidegger, 2019; Maliar, Maliar, and Winant, 2021). A notable recent contribution by Maliar et al. (2021) also uses deep learning to solve HA models with aggregate shocks. DeepHAM differs from that work in the following aspects. First, in the recursive formulation, DeepHAM addresses the optimization problem with directly simulated paths, while Maliar et al. (2021) optimizes over an objective function constructed as a weighted sum of the Bellman residual and the first-order condition. Their setup requires a good approximation not only of the high dimensional value function itself, but also of partial derivatives of the value function, which are challenging to accurately obtain using neural networks. Second, we introduce generalized moments to make the high dimensional policy and value functions permutation invariant to the ordering of agents, which also improves interpretability. Third, this paper provides thorough discussions on both model performance and computational efficiency analysis, which are absent in their work. In addition, this paper presents the first example of the use of machine learning-based algorithms to solve constrained efficiency problems in HA models with aggregate shocks.

The rest of this paper is organized as follows. Section 2 presents the DeepHAM method for solving a general HA model with aggregate shocks. Section 3 illustrates the use of DeepHAM on the classic Krusell-Smith model, and highlights the main features of the current approach. Sections 4 and 5 apply DeepHAM to more complex HA models and the constrained efficiency problem in HA models with aggregate shocks. Section 6 concludes the paper with some perspectives.

# 2 DeepHAM: A New Solution Method

## 2.1 General Setup of HA Models

In this subsection, we discuss the competitive equilibrium in a general HA model. In Section 2.4, we extend our setup to the constrained efficiency problem. Consider a discrete time and infinite horizon economy consisting of $N$ agents. $N$ is large when we aim to solve a problem with a continuum of agents, and smaller when we aim to solve for the strategic equilibrium with finite agents.[1]

For agent $i$, her state dynamics (i.e. law of motion for individual states) which usually come from the agent's budget constraint, are given by:

$$s_{t+1}^i = f(s_t^i, c_t^i, X_t, \bar{\mathbb{S}}_t; z_{t+1}^i), \quad i = 1, \dots, N. \tag{1}$$

Here $s_t^i \in \mathbb{R}^{d_s}$ and $c_t^i \in \mathbb{R}^{d_c}$ denote the state and decision (control) of agent $i$ at period $t$. $z_t^i \in \mathbb{R}^{d_z}$ denotes the idiosyncratic shock at period $t$, and is a subvector of $s_t^i$. The set $\bar{\mathbb{S}}_t = \{(s_t^1, c_t^1), (s_t^2, c_t^2) \dots, (s_t^N, c_t^N)\}$ denotes the (unordered) set of all agents' state-control pairs. Similarly, we use $\mathbb{S}_t = \{s_t^1, s_t^2, \dots, s_t^N\}$ to denote the set of agent states. We will also frequently use $S_t = (s_t^1, s_t^2, \dots, s_t^N)$ to denote the (ordered) vector of agent states. $X_t \in \mathbb{R}^{d_X}$ denotes aggregate state variables excluding $\mathbb{S}_t$. Here, the law of motion of agent states (1) combines the agent's budget constraint, together with other optimization and market clearing conditions that characterize the aggregate prices in the budget constraint. For example, in Krusell and Smith (1998), the household's wealth state in the next period depends on current wealth and consumption, as well as current prices of labor and capital. $\bar{\mathbb{S}}_t$ is included as an input to the law of motion $f$ for the individual state, because prices are functions of (the mean of) the wealth distribution according to the representative firm's optimization conditions and market clearing conditions, and the wealth distribution is a subset of $\bar{\mathbb{S}}_t$.

The control variables are subject to inequality constraints,

$$h_l(s_t^i, X_t, \mathbb{S}_t) \le c_t^i \le h_u(s_t^i, X_t, \mathbb{S}_t), \quad i = 1, \dots, N, \tag{2}$$

[1] Although our model assumes a finite number of agents, numerical results suggest that a choice of $N = 50$ can approximate the solution to HA models with a continuum of *ex ante* homogeneous agents as in Krusell and Smith (1998) quite well. We have also tested values of $N = 100, 200, ...$, with similar results.

where $h_l, h_u$ are vector functions with $d^c$-dimensional output, and the dynamics of $X_t$ are modeled by

$$X_{t+1} = g(X_t, \bar{\mathbb{S}}_t; Z_{t+1}), \tag{3}$$

where $Z_t \in \mathbb{R}^{d_Z}$ denotes the aggregate shock, and is usually a subvector of $X_t$. The aggregate state variable $X_t$ also includes other aggregate quantities that affect the law of motion for individual states (1), but cannot be written only as functions of the collection of state-control pairs $\bar{\mathbb{S}}_t$. An example of $X_t$ that contains more information than $Z_t$ is presented in Section 4.

Here we have indicated that $f, g, h_l, h_u$ depend only on the sets $\bar{\mathbb{S}}_t$ or $\mathbb{S}_t$, not the ordering of the agents, i.e., the state dynamics (1) are invariant to the ordering of the agents in the economy. This is a consequence of the "mean-field" character of the interaction between agents. "Mean-field" is a concept that originated in physics, and describes the situation when the agents, or particles, interact with each other not directly, but through an empirical distribution that all the agents contribute to equally. A typical interaction form of mean-field type is through endogenous aggregate variables $O_t \in \mathbb{R}^{d_O}$ defined by

$$O_t = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(s_t^i, c_t^i).$$

Here $O_t$ can be considered a function of $\bar{\mathbb{S}}_t$ as it is permutation invariant to the ordering of agents. Examples of $O_t$ include the first or other moments of individual states. In this paper, we assume the dependence of $f, g, h_l, h_u$ on $\bar{\mathbb{S}}_t$ or $\mathbb{S}_t$ can be written in explicit functional forms. The Krusell-Smith model mentioned above is such an example. This point will be more clear in our presentation of the concrete examples in Sections 3 to 5.

According to the above description, $(X_t, \mathbb{S}_t)$ completely characterizes the state of the whole economy. Mathematically, we are interested in how agents should make decisions $c_t^i = \mathcal{C}(s_t^i, X_t, \mathbb{S}_t)$ through the decision rule $\mathcal{C}$ to achieve optimality.[2]

In the setting of competitive equilibrium, each agent $i$ seeks to maximize her discounted lifetime utility:

$$\mathbb{E}_\mu \sum_{t=0}^{\infty} \beta^t u(c_t^i).$$

Here $u(\cdot)$ is the utility function and $\beta \in (0, 1)$ is the discount factor. $\mu$ is the full state distribution $(X_0, \mathbb{S}_0)$ at the initial time. We use $\mu(\mathcal{C})$ to denote the stationary distribution of

[2] Note here $\mathcal{C}$ is common across agents. This naturally applies for HA models with *ex ante* homogeneous agents like Krusell and Smith (1998). Similarly, the value function $V$ defined later is common across agents as well. For models with *ex ante* heterogeneous agents, we may introduce additional individual state variables such that $\mathcal{C}$ is common across agents.

$(X_t, \mathbb{S}_t)$ when every agent employs the decision rule $\mathcal{C}$ and we assume that such a stationary distribution always exists. The expectation is taken with respect to both the idiosyncratic and aggregate shocks over all time.

We say that $\mathcal{C}^*$ is an optimal policy in the competitive equilibrium if, for $\forall i \in \{1, \ldots, N\}$, $\mathcal{C}^*$ solves agent $i$'s problem,

$$\begin{aligned}
&\max_{\mathcal{C}} \ \mathbb{E}_{\mu(\mathcal{C}^*)} \sum_{t=0}^{\infty} \beta^t u(s_t^i, c_t^i), \\
&\text{s.t. } (1)(2)(3) \text{ hold, } c_t^i = \mathcal{C}(s_t^i, X_t, \mathbb{S}_t), \\
&\text{given } c_t^j = \mathcal{C}^*(s_t^j, X_t, \mathbb{S}_t), j = 1, \ldots, N, j \neq i.
\end{aligned}$$

Note that in contrast to the perturbation method in Reiter (2009), which only computes the solution around the stationary equilibrium in the absence of aggregate shocks, a global solution method seeks to find the solution according to the stationary distribution $\mu(\mathcal{C}^*)$ of the economy, which may significantly differ from the stationary equilibrium without aggregate shocks (Kekre and Lenel, 2020; Bhandari et al., 2021).

We now make two remarks about the general setup we present above.

*Remark* 1 (Discrete time setup)*.* Throughout this paper, we formulate HA models in discrete time. Founded upon the methodological framework of Achdou, Han, Lasry, Lions, and Moll (2017), the study of HA models in continuous time has received a great deal of interest in recent years. Compared to discrete time, a continuous time setup admits explicit expectation integral formulas (with respect to specific forms of idiosyncratic shocks) and efficient numerical algorithms for solving the corresponding PDE systems. However, when there are aggregate shocks, the stochastic PDE systems (Carmona and Delarue, 2018) must be derived and solved in order to obtain the global solution, which is much more challenging. Here we use a discrete time setup so that the problem is easier to solve and more general forms of shocks can be included.

*Remark* 2 (Infinite horizon)*.* We restrict the setup to the infinite horizon for the sake of concision when introducing the algorithm. As seen in Section 2.3, our method can also handle finite horizon problems such as life cycle models with only minor modification.

## 2.2 Representation of the Agent Distribution and Generalized Moments

The exposition of DeepHAM comprises two steps. The first is the introduction of generalized moments to replace the full agent distribution. This can be considered a model reduction step. The second is the introduction of an algorithm to solve the reduced model. We remark that the first step is of independent interest: The reduced model itself is a reliable and interpretable model that can be used as a general starting point for performing economic analysis. We discuss the idea of the first step in this subsection and present the detailed algorithm in the next subsection.

In HA models, a key question is what variables should be used to represent the whole economy. In algorithmic terms, these are what should be fed into the policy and value function approximators as input. Clearly, the individual state $s_t^i$ and the aggregate state variable $X_t$ should be taken into account. The main question is therefore how to represent the empirical distribution $\mathbb{S}_t = \{s_t^1, s_t^2, \ldots, s_t^N\}$, which also affects the dynamics of $s_t^i$ and $X_t$. On this point, the existing literature generally uses one of the following two representations.

1. The full vector $S_t$ (see, e.g., Maliar et al., 2021). This approach uses the full information of the distribution to characterize the optimal policy and value functions. However, there are two caveats. First, the dimension of $S_t$ is proportional to $N$. It is thus extremely expensive to deal with economies with a large number of agents. Second, the agent's optimal policy and value function should be invariant to the ordering of other agents' states. A function approximation form taking $S_t$ as the direct input cannot straightforwardly impose this restriction and must inefficiently process the irrelevant information of the agents' ordering.

2. Finite moments, usually the first moment of $\mathbb{S}_t$. This is the approach adopted by the KS method. It overcomes the two caveats discussed above of using the full vector $S_t$. However, the moments chosen may not carry the full information necessary for an agent to evaluate her current environment and act optimally. Under this simplification, the solution may deviate from the ground truth, especially in complex HA models.

To go beyond the above limitations, we introduce a class of generalized aggregate variables $Q_t \in \mathbb{R}^{d_Q}$ into the state vector:

$$Q_t = \frac{1}{N}\sum_{i=1}^{N} \mathcal{Q}(s_t^i).$$

Here the basis function $\mathcal{Q}$ may take a pre-specified functional form, or it may be a general basis function with variational parameters. For pre-specified functional forms, for example, it could be the identity function, making $Q_t$ the first moment. $\mathcal{Q}$ might also be an indicator function of whether an agent is at the borrowing constraint, so that $Q_t$ captures the share of hand-to-mouth agents (Kaplan, Violante, and Weidner, 2014) in the economy. A $Q_t$ based on pre-specified $\mathcal{Q}$ nests the moment representation we discuss above. A general basis function

$\mathcal{Q}$ can be parameterized with neural networks (see, e.g., Han et al., 2019) and the optimal representation solved for in the algorithm. When $\mathcal{Q}$ is a general basis function, we call the components of the resulting $Q_t$ *"generalized moments"*.[3]

With $Q_t$ as the representation of the distribution, we use $(s_t^i, X_t, Q_t)$ as the state vector taken as input by the policy and value function approximations. Conceptually, one can think of this in the following two ways.

1. Instead of parameterizing the mapping $[(s_t^i, X_t, \mathbb{S}_t) \mapsto \text{ output}]$ with neural network models, we decompose it into $[(s_t^i, X_t, \mathbb{S}_t) \mapsto (s_t^i, X_t, Q_t) \mapsto \text{ output}]$ and parameterize the two components by two neural networks, the first step is an encoding network and the second step is a fitting network. In this sense, by specifying the dimension $d_Q$, we ensure that the complexity of the neural networks does not increase rapidly as $N$ increases. Furthermore, the final policy functions are permutation invariant by design. In this way, both shortcomings mentioned above are overcome.

2. $Q_t$ shares the same formula with $O_t$, which can be interpreted as a vector of generalized moments. If $\mathcal{Q}$ is parameterized by a set of specific functional forms informed by structural details, $Q_t$ are selected from a large set of interpretable aggregate moments. If $\mathcal{Q}$ is directly parameterized by neural networks, optimizing $\mathcal{Q}$ can guide the agent to finding the aggregate variables most relevant to their decision making. Compared to the KS method, generalized moments have the flexibility to more closely capture those features of the whole economy most relevant to the optimal decision rules, obtaining a more accurate macroeconomic model without sacrificing interpretability.

In this paper, we use two separate sets of generalized moments, $Q_t^c$ and $Q_t^V$, to extract the distribution information for the policy and value functions, respectively. This choice simplifies their updating rule. Algorithmically, when we fed $Q_t^c/Q_t^V$ into the policy/value functions, the variational parameters of the basis and policy/value parameters can be trained jointly end-to-end through the corresponding objective function. Using shared generalized moments for both the policy and value functions will be investigated in future work.

## 2.3 Solution Method for Competitive Equilibrium

We first describe the algorithm for solving for the competitive equilibrium of economies of the form described in Section 2.1. The algorithm for solving the constrained efficiency problem is quite similar and will be discussed in Section 2.4. To solve the model, we rewrite agents' objective function based on the dynamic programming principle over $T$ periods. In

[3] A brief mathematical introduction to neural networks is presented in Appendix A.

the competitive equilibrium, for $\forall i \in \{1, \ldots, N\}$, the policy function $\mathcal{C}^*$ solves agent $i$'s problem,

$$\begin{aligned} \max_{\mathcal{C}} \ & \mathbb{E}_{\mu(\mathcal{C}^*)} \left[ \sum_{t=0}^{T} \beta^t u\left(s_t^i, c_t^i\right) + \beta^T V(s_T^i, X_T, \mathbb{S}_T) \right], \\ & \text{s.t. (1)(2)(3) hold, } c_t^i = \mathcal{C}(s_t^i, X_t, \mathbb{S}_t), \\ & \text{given } c_t^j = \mathcal{C}^*(s_t^j, X_t, \mathbb{S}_t), j = 1, \ldots, N, j \neq i, \end{aligned} \tag{4}$$

where the value function $V$ is defined by

$$\begin{aligned} V(s_0^i, X_0, \mathbb{S}_0) = \ & \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u\left(s_t^i, c_t^i\right) \middle| X_0, \mathbb{S}_0 \right], \\ & \text{s.t. (1)(2)(3) hold, } c_t^i = \mathcal{C}^*(s_t^i, X_t, \mathbb{S}_t), i = 1, \ldots, N. \end{aligned} \tag{5}$$

The overall idea of DeepHAM is an iterative procedure starting from the initial guess of the policy $\mathcal{C}_0$, as presented in Algorithm 1. Each iteration includes three steps that we will discuss in detail: (a) prepare the stationary distribution $\mu(\mathcal{C})$; (b) update the value function according to (5); (c) optimize the policy function according to (4). We refer to one such high-level iteration step as a *round* and repeat $N_k$ rounds until convergence. It is similar to the conventional value function iteration algorithm (Ljungqvist and Sargent, 2018, Chapter 3), except that:

1. We parameterize both value and policy functions with neural networks, each of which nests two sub-networks with a feedforward architecture: one approximates the basis function $\mathcal{Q}$, the other approximates the mapping from $(s_t^i, X_t, Q_t)$ to policy or value function values.[4]

2. We solve for optimal policy to maximize the total utility in (4) over Monte Carlo simulations for $T$ periods, instead of one period, which is typically used in the conventional value function iteration algorithm. When the state vector is high dimensional, it is computationally expensive or even infeasible to update the policy with one period calculation on a grid. Instead, we update the parameters of the policy function neural network to maximize the expected total utility in (4) over simulated paths. We use $T$ periods to reduce the variance in Monte Carlo simulation.

Now we further explain the details of the three main steps of DeepHAM in the $k$-th round.

[4] If we choose to use some pre-specified basis to define $Q$, such as the first moment, we will only have the second sub-network in both the policy and value functions.

**Algorithm 1** DeepHAM for solving the competitive equilibrium

```
Require: Input: the initial policy $\mathcal{C}_0$, the initial value and policy neural networks with
    parameters $\Theta^V$ and $\Theta^C$, respectively
for $k = 1, 2, \dots, N_k$ do
    prepare the stationary distribution $\mu(\mathcal{C}_{k-1})$ according to the policy $\mathcal{C}_{k-1}$
    for $m = 1, 2, \dots, N_{m_1}$ do                          ▷ update the value function
        sample $N_{b_1}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$
        compute the realized total utility in (8) through a single simulated path
        use the empirical version of (9) to compute the gradient $\nabla_{\Theta_V}$
        update $\Theta^V$ with $\nabla_{\Theta_V}$
    end for
    for $m = 1, 2, \dots, N_{m_2}$ do                          ▷ optimize the policy function
        sample $N_{b_2}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$
        use the empirical version of (10) to compute the gradient $\nabla_{\Theta_C}$
        update $\Theta^C$ with $\nabla_{\Theta_C}$
    end for
    define $\mathcal{C}_k$ according to (11)
end for
```

**Preparing the stationary distribution.** This is done by simulating the economy (1)(3) forward for sufficiently many periods, under the policy $\mathcal{C}_{k-1}$.

**Updating the value function.** Given the policy $\mathcal{C}_{k-1}$, updating the value function can be formulated as a supervised learning problem. Denote the parameters in the value function neural network by $\Theta^V = (\Theta^{VQ}, \Theta^{VO})$, where $\Theta^{VQ}$ are the parameters in the general basis function defining $Q_t^V$ and $\Theta^{VO}$ are the parameters in the function that maps $(s_t^i, X_t, Q_t^V)$ to value outcomes. Our approximation to the value function can thus be written,

$$V_{\text{NN}}(s_t^i, X_t, \mathbb{S}_t; \Theta^V) := \widetilde{V}_{\text{NN}}(s_t^i, X_t, Q_t^V; \Theta^{VO}) = \widetilde{V}_{\text{NN}}(s_t^i, X_t, \frac{1}{N}\sum_{i=1}^{N} \mathcal{Q}_{\text{NN}}(s_t^i; \Theta^{VQ})); \Theta^{VO}). \quad (6)$$

We want to use $V_{\text{NN}}$ to approximate the agents' expected lifetime utility under the policy $\mathcal{C}_{k-1}$, i.e.,

$$V_{\text{NN}}(s_t^i, X_t, \mathbb{S}_t; \Theta^V) \approx \mathbb{E}\left[\sum_{\tau=0}^{\infty} \beta^\tau u(s_{t+\tau}^i, c_{t+\tau}^i) \,\middle|\, s_t^i, X_t, \mathbb{S}_t\right]. \quad (7)$$

However, evaluating the expectation in (7) is still computationally expensive. To reduce the computational cost, given each sample $(s_0^i, X_0, \mathbb{S}_0)$ from the stationary distribution $\mu(\mathcal{C}_{k-1})$, we only simulate a single path for $T_{\text{simul}}$ periods (with $T_{\text{simul}}$ sufficiently large) under the policy $\mathcal{C}_{k-1}$ to get the truncated realized total utility

$$\widehat{V}^i = \sum_{\tau=0}^{T_{\text{simul}}} \beta^\tau u(s_\tau^i, c_\tau^i). \tag{8}$$

Note that $\widehat{V}_t^i$ is a random variable influenced by the realization of the idiosyncratic and aggregate shocks. Still, we know that the true value function minimizes the expected difference with the realized total utility. Thus, we only need to solve the following regression problem to update the value function:

$$\min_{\Theta^V} \mathbb{E}_{\mu(\mathcal{C}_{k-1})} \left[ V_{\text{NN}}(s_0^i, X_0, \mathbb{S}_0; \Theta^V) - \widehat{V}^i \right]^2. \tag{9}$$

We use the stochastic gradient descent algorithm to solve (9). Specifically, in each update step, we sample $N_{b_1}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$, use the empirical version of (9) to compute the gradient with respect to $\Theta^V$ by backpropagation, and update $\Theta^V$ accordingly. We repeat $N_{m_1}$ steps to achieve convergence. As $\Theta^V = (\Theta^{VQ}, \Theta^{VO})$, we also obtain, at the end of the update, the updated basis function $\mathcal{Q}_{\text{NN}}(\cdot; \Theta^{VQ})$ and the generalized moments $Q_t$ at the same time.[5]

**Optimizing the policy function.** In the competitive equilibrium, the policy function is updated iteratively following the spirit of fictitious play (Brown, 1951). A similar idea has been used in Han and Hu (2020) and Hu (2021) to solve stochastic differential games based on neural networks. Similar to our approach to the value function, we will update the parameters associated with the policy function neural network through stochastic gradient descent. We call each update a "play". In each "play", we fix everyone but agent $i = 1$'s policy as that from the last play, and consider agent $i = 1$'s utility maximization problem to update the neural network parameters, to get the new policy in this "play". All agents then adopt the new policy in this "play". We repeat the "plays" until convergence.[6]

For the utility maximization problem of agent $i = 1$, the algorithm essentially builds upon the one proposed in Han and E (2016): optimizing the parameters of the policy function neural network over simulated paths. Given the updated value function $V_{\text{NN}}(s_t^i, X_t, \mathbb{S}_t; \Theta^V)$ in the same round and other agents' policy function from the last "play", agent $i = 1$ aims

[5] Backpropagation is an algorithm allowing an efficient computation of all partial derivatives of the neural network (composition of a series of functions) with respect to its parameters.

[6] In a more general setup, we can also fix the other agents' policies for several "plays" and then update from the agent $i = 1$'s policy.

to solve

$$\max_{\Theta^C} \mathbb{E}_{\mu(\mathcal{C}_{k-1})} \left[ \sum_{t=0}^{T} \beta^t u(s_t^i, c_t^i) + \beta^T V_{\text{NN}}(s_T^i, X_T, \mathbb{S}_T; \Theta^V) \right], \tag{10}$$

with her policy parameterized by neural networks in the form of

$$\begin{aligned} c_t^i &= \mathcal{C}(s_t^i, X_t, \mathbb{S}_t; \Theta^C) \\ &= \big(h_u(s_t^i, X_t, \mathbb{S}_t) - h_l(s_t^i, X_t, \mathbb{S}_t)\big) \odot c_{\text{NN}}\left(s_t^i, X_t, \mathbb{S}_t; \Theta^C\right) + h_l(s_t^i, X_t, \mathbb{S}_t). \end{aligned} \tag{11}$$

Here the outputs of $h_u(\cdot), h_l(\cdot)$, and $c_{\text{NN}}(\cdot)$ are $d^c$ dimensional, and $\odot$ denotes element-wise multiplication. We use the sigmoid function $\frac{1}{1+e^{-x}} \in [0,1]$ as the last composed function of $c_{\text{NN}}(\cdot)$, so that the inequality constraints (2) are always satisfied. For a mathematical introduction to the composition structure of neural networks, see Appendix A. Similar to the value function, the parameters in the policy function neural network $\Theta^C = (\Theta^{CQ}, \Theta^{CO})$, where $\Theta^{CQ}$ are parameters in the general basis function and $\Theta^{CO}$ are parameters in the function that maps $(s_t^i, X_t, Q_t)$ to policy outcomes. So we have

$$c_{\text{NN}}(s_t^i, X_t, \mathbb{S}_t; \Theta^C) = \widetilde{c}_{\text{NN}}(s_t^i, X_t, Q_t^C; \Theta^{CO}) = \widetilde{c}_{\text{NN}}(s_t^i, X_t, \frac{1}{N}\sum_{i=1}^{N} \mathcal{Q}_{\text{NN}}(s_t^i; \Theta^{CQ})); \Theta^{CO}). \tag{12}$$

Once more, we use the stochastic gradient descent algorithm corresponding to (10). We sample $N_{b_2}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$ as the initial conditions, use the empirical version of (9) to compute the gradient with respect to $\Theta^C$, and update $\Theta^C$ accordingly. The gradient with respect to $\Theta^C$ can be obtained by backpropagation as well, because all the component functions in (10)(11)(12) are explicit and differentiable; see Figure 1 for the computational graph corresponding to (10).

From the above description, we can see one merit of DeepHAM: its ready ability to handle models with aggregate shocks. The algorithm remains almost the same when solving models with aggregate shocks or without. In contrast, the continuous time PDE approach (Achdou et al., 2017) can solve models without aggregate shocks efficiently (in the low-dimensional case), but faces challenges in the presence of aggregate shocks.

## 2.4 Extension to Constrained Efficiency Problem

In the constrained efficiency problem, a benevolent social planner seeks to find a policy rule $\mathcal{C}$, determining each agent's decision variable $c_t^i$, in order to maximize the discounted sum of social welfare $\Omega(\bar{\mathbb{S}}_t)$. The social welfare depends on the collection of all agents' state-control

Figure 1: Computational Graph to Solve HA Models using DeepHAM

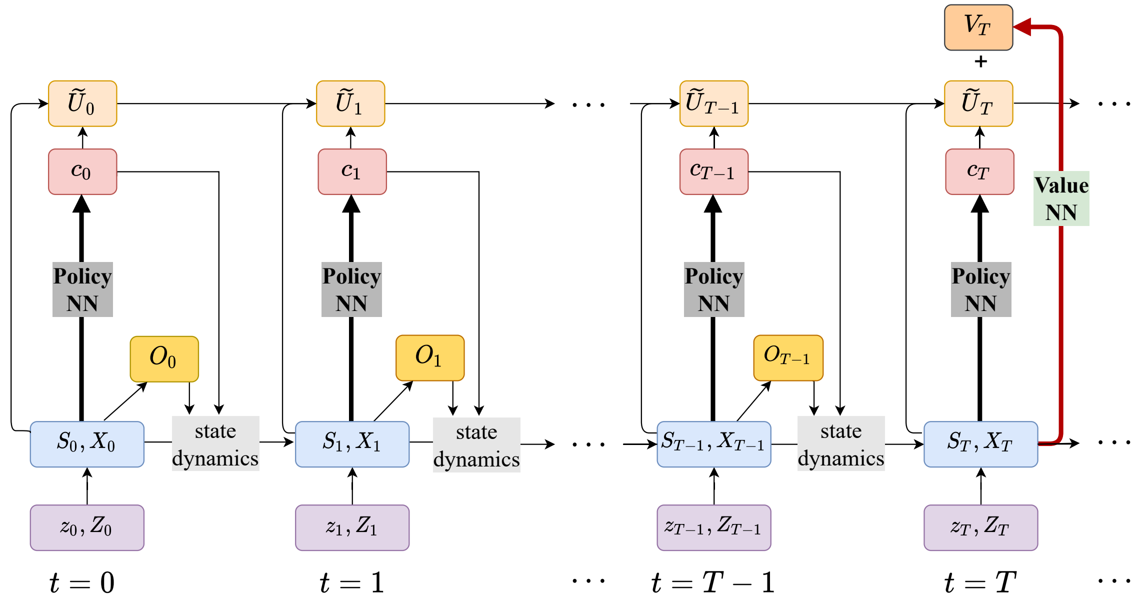


Figure note: $S_t$, $z_t$, and $c_t$ denote the collection of all agents' states, idiosyncratic shocks, and decisions at time $t$, respectively. $Z_t$ denotes aggregate shocks at time $t$. $\widetilde{U}_t$ denotes the collection of all agents' cumulative utilities up to period $t$, i.e., $\widetilde{U}_t^i = \sum_{\tau=0}^{t} \beta^\tau u\left(s_\tau^i, c_\tau^i\right)$.

pairs:

$$\max_{\mathcal{C}} \ \mathbb{E}_{\mu(\mathcal{C})} \sum_{t=0}^{\infty} \beta^t \Omega(\bar{\mathbb{S}}_t),$$

$$\text{s.t. } (1)(2)(3) \text{ hold, } c_t^i = \mathcal{C}(s_t^i, X_t, \mathbb{S}_t), i = 1, \dots, N.$$

Here the social welfare function can take the utilitarian form $\Omega(\bar{\mathbb{S}}_t) = \frac{1}{N}\sum_{i=1}^{N} u(s_t^i, c_t^i)$, or $\Omega(\bar{\mathbb{S}}_t) = \sum_{i=1}^{N} \omega_i u(s_t^i, c_t^i)$ with Negishi weights $\omega_i = \frac{u_c(s_t^i, c_t^i)}{\sum_{i=1}^{N} u_c(s_t^i, c_t^i)}$ (Bhandari et al., 2021), or other general forms.

The overall procedure for solving the constrained efficiency problem is the same as that for solving the competitive equilibrium, as presented in Algorithm 2: each round consists of (a) preparing the stationary distribution, (b) updating the value function, and (c) optimizing the policy function. Below, we explain the three steps in the $k$-th round in detail and mainly highlight the differences between these and the procedure for solving for the value and policy objectives in competitive equilibrium.

**Preparing the stationary distribution.** This is done exactly the same as for the competitive equilibrium problem. That is, by simulating the economy (1)(3) for sufficiently many periods under the policy $\mathcal{C}_{k-1}$.

**Algorithm 2** DeepHAM for solving the constrained efficiency problem

```
Require: Input: the initial policy $\mathcal{C}_0$, the initial value and policy neural networks with parameters $\Theta^V$ and $\Theta^C$, respectively
for $k = 1, 2, \dots, N_k$ do
    prepare the stationary distribution $\mu(\mathcal{C}_{k-1})$ according to the policy $\mathcal{C}_{k-1}$
    for $m = 1, 2, \dots, N_{m_1}$ do                              ▷ update the value function
        sample $N_{b_1}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$
        compute the realized total social welfare in (14) through a single simulated path
        use the empirical version of (15) to compute the gradient $\nabla_{\Theta_V}$
        update $\Theta^V$ with $\nabla_{\Theta_V}$
    end for
    for $m = 1, 2, \dots, N_{m_2}$ do                              ▷ optimize the policy function
        sample $N_{b_2}$ samples of $(X_0, \mathbb{S}_0)$ from $\mu(\mathcal{C}_{k-1})$
        use the empirical version of (16) to compute the gradient $\nabla_{\Theta_C}$
        update $\Theta^C$ with $\nabla_{\Theta_C}$
    end for
    define $\mathcal{C}_k$ according to (11)
end for
```

**Updating the value function.** Given the policy $\mathcal{C}_{k-1}$, we want to use $V_{\text{NN}}$ to approximate the expected total social welfare under the policy $\mathcal{C}_{k-1}$, i.e.,

$$V_{\text{NN}}(X_t, \mathbb{S}_t; \Theta^V) \approx \mathbb{E}\left[ \sum_{\tau=0}^{\infty} \beta^{\tau} \Omega(\bar{\mathbb{S}}_{t+\tau}) \,\middle|\, X_t, \mathbb{S}_t \right], \tag{13}$$

where

$$V_{\text{NN}}(X_t, \mathbb{S}_t; \Theta^V) = V_{\text{NN}}(X_t, Q_t^V; \Theta^{VO}) = V_{\text{NN}}(X_t, \frac{1}{N}\sum_{i=1}^{N} \mathcal{Q}_{\text{NN}}(s_t^i; \Theta^{VQ}); \Theta^{VO}).$$

Similarly, to avoid the computational cost of evaluating the expectation in (13) given each sample $(X_0, \mathbb{S}_0)$ from the stationary distribution $\mu(\mathcal{C}_{k-1})$, we only simulate a single path for $T_{\text{simul}}$ periods (with $T_{\text{simul}}$ sufficiently large) under the policy $\mathcal{C}_{k-1}$ to obtain the truncated realized total social welfare

$$\widehat{V} = \sum_{\tau=0}^{T_{\text{simul}}} \beta^{\tau} \Omega(\bar{\mathbb{S}}_{\tau}). \tag{14}$$

Then we only need to solve the following regression problem to update the value function:

$$\min_{\Theta^V} \mathbb{E}_{\mu(\mathcal{C}_{k-1})} \left[ V_{\text{NN}}(X_0, \mathbb{S}_0; \Theta^V) - \widehat{V} \right]^2. \tag{15}$$

This can be done using stochastic gradient descent in the same way as for (9).

**Optimizing the policy function.** In order to find the constrained optimum, we need to update the policy function by solving

$$\max_{\Theta^C} \mathbb{E}_{\mu(\mathcal{C}_{k-1})} \left[ \sum_{t=0}^{T} \beta^t \Omega(\bar{\mathbb{S}}_t) + \beta^T V_{\mathrm{NN}}(X_T, \mathbb{S}_T; \Theta^V) \right]. \tag{16}$$

Compared to the policy function optimization in the competitive equilibrium problem, here we get rid of the fictitious play step and instead optimize all the agents' policies simultaneously to maximize the total social welfare. The optimization problem (16) can be solved in the same way as for the problem (10) with the stochastic gradient descent algorithm. The gradient with respect to $\Theta^C$ can be obtained by backpropagation in the same computational graph as in Figure 1.

The above description demonstrates that the DeepHAM Algorithm 2 for solving the constrained efficiency problem is identical to Algorithm 1 for the competitive equilibrium, except for the two differences in lines 6 and 11, in which we use different objectives for the value function and policy function corresponding to the different setting. Meanwhile, the pipeline of data sampling and optimization methods for these different objectives is exactly the same. In this sense, DeepHAM can solve the constrained efficiency problem as easily as the competitive equilibrium problem.

# 3 DeepHAM for the Krusell-Smith Model

In this section, we illustrate DeepHAM on the classic Krusell-Smith model, and highlight the advantages of this method.

## 3.1 Model Setup

The setup follows Den Haan (2010). Household $i$'s state $s_t^i = (a_t^i, z_t^i) \in \mathbb{R}^2$, with beginning-of-period wealth $a_t^i$, employment status $z_t^i \in \{0, 1\}$. The consumption $c_t^i \in \mathbb{R}$ is the control variable. Households have log utility over consumption. $Z_t \in \{Z^h, Z^l\}$ denotes aggregate productivity. The process $z_t^i, Z_t$ follows a first-order Markov process. The aggregate state variable $X_t = Z_t$, so its dynamics (3) are trivial. The state dynamics of the household comes from the household budget constraint,

$$\begin{aligned} &a_{t+1}^i = (1 + r_t - \delta) a_t^i + [(1 - \tau_t)\bar{l} z_t^i + b(1 - z_t^i)] w_t - c_t^i, \\ &a_{t+1}^i \geq 0, \quad c_t^i \geq 0, \end{aligned}$$

where the net rate of return of capital is $r_t - \delta$, with depreciation rate $\delta$. The factor prices $r_t, w_t$ are determined by the first order condition (FOC) of the representative firm, which produces with a Cobb-Douglas technology $Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$, in the competitive factor market,

$$w_t = Z_t(1-\alpha)(K_t/L_t)^{\alpha}, \quad r_t = Z_t \alpha (K_t/L_t)^{\alpha-1},$$

with aggregate capital $K_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i$ and labor supply $L_t = \bar{l}(L^h \mathbb{1}_{Z_t=Z^h} + L^l \mathbb{1}_{Z_t=Z^l})$, in which $\bar{l}$ is the time endowment of each agent. Unemployed agents ($z_t^i = 0$) receive unemployment benefits $bw_t$ where $b$ is the unemployment benefit rate. Employed agents ($z_t^i = 1$) earn after-tax labor income $(1-\tau_t)\bar{l}w_t$, with tax rate $\tau_t = b(1-L_t)/\bar{l}L_t$, such that government budget constraint always holds (total tax income equals unemployment benefits). This completes the specification of (1). The borrowing constraint and non-negative consumption constraint specifies (2). The calibration of the model follows Den Haan (2010) and is presented in Appendix B.1.

## 3.2 Results

We solve the Krusell-Smith model described above in the case of $N = 50$ using DeepHAM. We have tried other choices of $N = 100, 200, ...$, and we find $N = 50$ is large enough to approximate the solution to the Krusell-Smith model. The computational graph for this problem is shown in Figure 2.

### 3.2.1 Solution Accuracy

In Table 2, we compare the Bellman equation errors (defined in Appendix C) of DeepHAM to the same error for the KS method implemented in Maliar et al. (2010). For DeepHAM, we present accuracy measures for the cases where we include (1) the first moment of the household wealth distribution and (2) one generalized moment of the household wealth distribution in the state variable. We present the standard deviation of the Bellman errors from multiple runs of the numerical algorithm in the last column of Table 2. We see that all results are statistically significant.[7]

As can be seen in Table 2, the solutions obtained using DeepHAM are highly accurate. Compared to the KS method, DeepHAM with the first moment in the state vector reduces the Bellman equation error by 27.2%. DeepHAM with one generalized moment reduces the error by 40.3%. Generalized moments play an important role in improving solution accuracy because they provide a more concise representation of the household distribution and extract

[7] Following the moment construction in Krusell and Smith (1998), here the generalized moments are constructed on the wealth distribution, rather than the joint distribution of wealth and employment status.

Figure 2: Computational Graph to Solve Krusell and Smith (1998) using DeepHAM

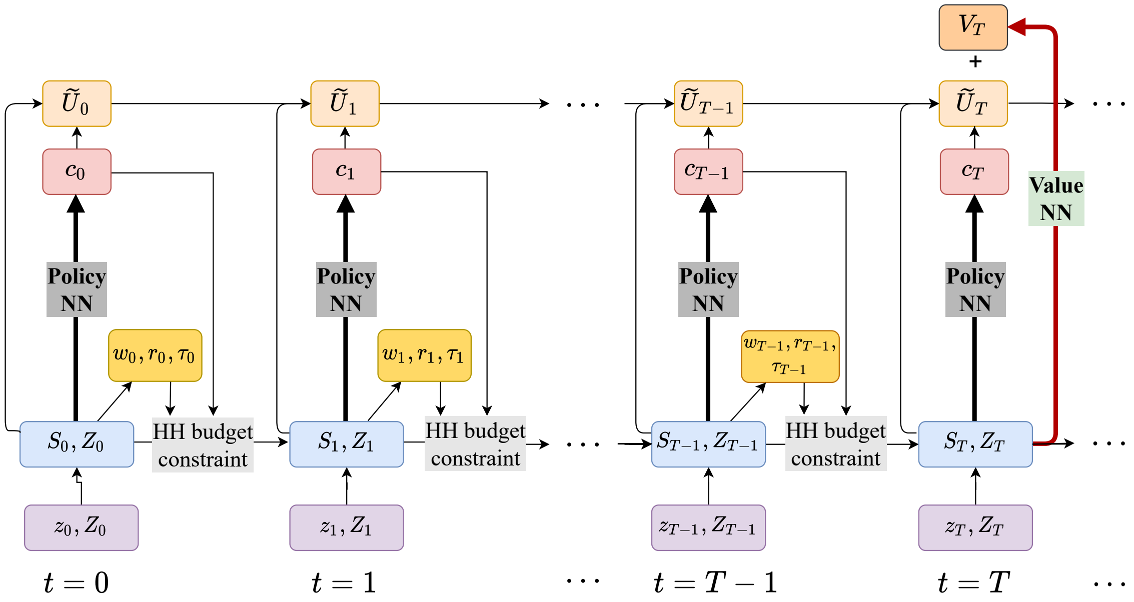


Figure note: $S_t$, $z_t$, $c_t$, and $\widetilde{U}_t$ denote the collection of all agents' states, idiosyncratic shocks, decisions, and cumulative utilities at time $t$, respectively. $Z_t$ denotes aggregate shocks at time $t$. Aggregate prices $w_t, r_t$ are determined by FOCs of the representative firm in the competitive factor market. Income tax rate $\tau_t$ depends on the aggregate shock $Z_t$ and is pinned down in the government budget constraint.

| Method and Moment Choice | Bellman error | Std of error |
|---|---|---|
| KS Method (Maliar et al., 2010) | 0.0253 | 0.0002 |
| DeepHAM with 1st moment | 0.0184 | 0.0023 |
| DeepHAM with 1 generalized moment | 0.0151 | 0.0015 |

Table 2: Comparison of solution accuracy for Krusell-Smith problem

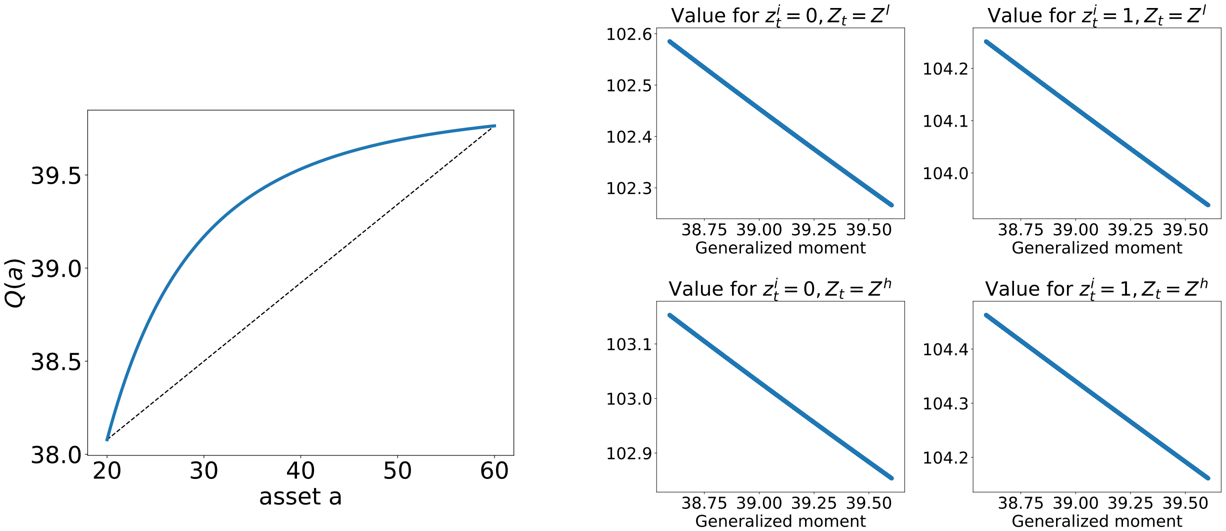
more relevant information than the first moment. We discuss and interpret the generalized moment we obtain in the Krusell-Smith problem in the next subsection.[8]

### 3.2.2 Generalized moments and redistributional effect

Among the three results in Table 2, DeepHAM with one generalized moment yields the most accurate solution. To better understand the improvement, we visualize the mapping from individual asset holdings $a_t^i$ to the basis function $\mathcal{Q}(a_t^i)$, and the mapping from the

[8] The KS method with the first moment is known to solve the Krusell-Smith model reasonably well. This is confirmed by the small Bellman error for the KS method in Table 2. Though DeepHAM can further improve the solution accuracy, the simulated economy based on the DeepHAM solution is highly consistent with the simulation based on the KS method. We present this comparison in Appendix D. This further confirms the accuracy of the DeepHAM solution for the Krusell-Smith model.

generalized moment $\frac{1}{N}\sum_i \mathcal{Q}(a_t^i)$ to the value function in Figure 3.



(a) Plot of $\mathcal{Q}_1(a)$

(b) Map $\frac{1}{N}\sum_i \mathcal{Q}_1(a_i)$ to the value function

Figure 3: Generalized moments for the Krusell-Smith problem. Left panel (solid blue line): a concave mapping from the individual asset to the basis function of the generalized moment. Right panel: mapping from the generalized moment to the value function, assuming households' individual assets fixed at the average level in the stationary equilibrium. Each figure in the right panel corresponds to one realization of idiosyncratic and aggregate shocks.

We find that the basis function is concave in the individual asset, while the value function is linear with regard to the generalized moment. That is, households with different levels of wealth will have heterogeneous contributions to the generalized moment: giving an additional unit of assets to poor households increases the generalized moment more than giving the same assets to rich households. This phenomenon means that a purely redistributional policy would affect aggregate welfare and dynamics even in the simple setup of Krusell and Smith (1998). Consider an unanticipated one-time policy shock (MIT shock): if one unit of the asset is redistributed from the richest households to the poorest households, the welfare of "middle" households who are not in the redistribution program would decrease on impact since the generalized moment increases. Such an unanticipated policy shock will lead to higher aggregate savings in the future, since there are fewer people on the borrowing constraint. The increase in savings would lead to a higher future wage and a lower future asset return. Under the calibration of this model, the "middle" households who are not in the redistribution program receive more capital income than labor income, so the unanticipated policy shock would make them worse off. This sensible logic differs from the implications of the solution of the KS method. According to the KS method, households' welfare only depends on the first moment and individual states. So the redistributional policy shock would have no instantaneous welfare impact on those "middle" households who are not in the redistribution program, since the first moment of individual wealth distribution would not change.

# 4 DeepHAM for More Complex HA Models

In this section, we use DeepHAM to solve a HA model with a financial sector and aggregate shocks, as proposed in Fernández-Villaverde, Hurtado, and Nuno (2019). Compared to the Krusell-Smith model with the two-state aggregate shocks in Section 3, here the aggregate shocks take values in a continuous range, which makes the problem more costly to solve.

## 4.1 Fernández-Villaverde et al. (2019): Model Setup

The setup is the discrete time version of Fernández-Villaverde et al. (2019). We use the subscript $t$ and $t+\Delta t$ to highlight that the model comes from the discretization of a continuous time model, but should be interpreted as representing the dynamics between $t$ and $t+1$ in the general setup in Section 2. In this economy, there are $N$ households who save in risk-free bonds and consume. Their labor supply is exogenous and exposed to idiosyncratic shocks. There is a representative financial expert who issues risk-free bonds to households and invests in productive capital. A representative firm produces with capital from the financial expert and with labor supplied by the households. The growth rate of productive capital is exposed to aggregate shocks.

**Household's problem.** For household $i$, her state is $s_t^i = (a_t^i, z_t^i) \in \mathbb{R}^2$, with beginning-of-period risk-free asset $a_t^i$, and the idiosyncratic shocks on labor supply $z_t^i \in \{z_1, z_2\}$ with $0 < z_1 < z_2$. The process $z_t^i$ follows a first-order Markov process with ergodic mean 1 such that the aggregate labor supply $L_t = 1$. Household $i$ has constant relative risk aversion (CRRA) utility from consumption $c_t^i$ with parameter $\gamma > 0$ and discount factor $e^{-\rho\Delta t}$.

The household budget and borrowing constraints determines the state dynamics (1) of household $i$:

$$\begin{aligned} a_{t+\Delta t}^i &= a_t^i + (w_t z_t^i + r_t a_t^i - c_t^i)\Delta t, \\ a_{t+\Delta t}^i &\geq 0, \quad c_t^i \geq 0, \end{aligned} \tag{17}$$

where the aggregate prices are characterized below. Aggregate risk-free asset demand $B_t = \frac{1}{N}\sum_{i=1}^N a_t^i$.

**Representative firm's problem.** The firm produces with Cobb-Douglas technology $Y_t = K_t^\alpha L_t^{1-\alpha}$. It hires labor $L_t$ from households at wage $w_t$, and rents capital $K_t$ from the financial expert at rental rate $rc_t$, both in the competitive factor market:

$$w_t = (1-\alpha)(K_t/L_t)^\alpha, \quad rc_t = \alpha(K_t/L_t)^{\alpha-1}. \tag{18}$$

**Financial expert's problem.** The representative financial expert issues a risk-free bond $B_t$ at rate $r_t$ to households, and rents capital $K_t$ at rate $rc_t$ to the representative firm. Her net worth $W_t = K_t - B_t$. For the financial expert, the instantaneous return rate on capital is exposed to aggregate shocks $Z_t$:

$$\frac{K_{t+\Delta t} - K_t}{K_t} = (rc_t - \delta)\Delta t + \sigma Z_t \sqrt{\Delta t},$$

where $\delta$ is the depreciation rate of capital, $\sigma$ is the volatility of aggregate shocks, and $Z_t$ follows an i.i.d. standard normal distribution.[9]

The financial expert has log utility with discount rate $\widehat{\rho} < \rho$ over consumption $\widehat{C}_t$, so she consumes a constant share of her net worth: $\widehat{C}_t = \widehat{\rho} W_t$, and chooses a leverage ratio proportional to excess return of risky capital $\frac{K_t}{W_t} = \frac{1}{\sigma^2}(rc_t - \delta - r_t)$. So the risk-free return is

$$r_t = \alpha(K_t/L_t)^{\alpha-1} - \delta - \sigma^2 \frac{K_t}{W_t}. \tag{19}$$

The budget constraint of the financial expert $W_{t+\Delta t} = W_t + (rc_t - \delta)K_t\Delta t + \sigma K_t Z_t \sqrt{\Delta t} - B_t r_t \Delta t - \widehat{C}_t \Delta t$ implies the following dynamics of net worth $W_t$:

$$W_{t+\Delta t} = W_t + \left(\alpha K_t^{\alpha-1} - \delta - \widehat{\rho} - \sigma^2 \left(1 - \frac{K_t}{W_t}\right)\frac{K_t}{W_t}\right) W_t \Delta t + \sigma K_t Z_t \sqrt{\Delta t}. \tag{20}$$

Using the general descriptive variables in Section 2, the aggregate state $X_t = W_t$. Since $K_t = B_t + W_t$, the evolution of $W_t$ only depends on $W_t, B_t = \frac{1}{N}\sum_{i=1}^N a_t^i$, and $Z_t$. The aggregate state dynamics (3) are specified as (20). Equations (17)(18)(19), together with the stochastic process of $z_t^i$, complete the specification of (1) and (2). The calibration of the model follows Fernández-Villaverde et al. (2019) and is presented in Appendix B.2.

[9] In the continuous time model, the aggregate shock is a white noise process with volatility $\sigma$. There would be a slight numerical difference to the discretized model, but the difference is tiny as we choose a tiny $\Delta t$.

## 4.2 Solution Accuracy and Efficiency

We use DeepHAM to obtain the global solution to the problem described above with $N = 50$. We compare the Bellman equation errors (see the definition in Appendix C) of DeepHAM to the generalized KS method with the nonlinear perceived law of motion implemented in Fernández-Villaverde et al. (2019) in Table 3. For DeepHAM, we present accuracy measures for the cases where we include in the state variable (1) only the first moment or (2) one generalized moment of household asset distribution.

| Method and Moment Choice | Bellman error | Std of error |
|---|---|---|
| KS Method (Fernández-Villaverde et al., 2019) | 0.00417 | 0.00011 |
| DeepHAM with 1st moment | 0.00405 | 0.00059 |
| DeepHAM with 1 generalized moment | 0.00422 | 0.00086 |

Table 3: Comparison of solution accuracy on a HA model with a financial sector and aggregate shocks. The KS method refers to the solution method implemented by (Fernández-Villaverde et al., 2019).

As we see in Table 3, the solutions obtained using DeepHAM are highly accurate. Compared to the generalized KS method with the nonlinear law of motion implemented by Fernández-Villaverde et al. (2019), DeepHAM with either the first moment or a generalized moment can obtain global solutions with the same level of accuracy. We present the standard deviation of the Bellman errors from multiple runs of the numerical algorithm in the last column of Table 3.[10]

Solving this HA model, with a financial sector and aggregate shocks which take values in a continuous range, takes DeepHAM 12% longer than solving the simple Krusell-Smith model in Section 3. This result demonstrates the efficiency of DeepHAM in studying complex HA models with aggregate shocks: unlike the grid-based method, the computational cost of DeepHAM does not increase quickly when the number of state variables or grid points increases.

[10] Note that Fernández-Villaverde et al. (2019) builds and solves the model in continuous time. To compare it with DeepHAM, which solves the discrete time version of the model, we evaluate their solution also on the discrete time Bellman equation error defined in Appendix C. Since we chose a small $\Delta t$, the continuous time solution should give a meaningful approximate Bellman error in discrete time. We find that DeepHAM obtains an accurate solution which is comparable to Fernández-Villaverde et al. (2019).

# 5 DeepHAM for Constrained Efficiency Problem in HA Models

In this section, we solve the constrained efficiency problem in HA models using DeepHAM. In contrast to the competitive equilibrium, the constrained optimum of an HA model, defined as the allocation decided by a benevolent social planner who maximizes social welfare, is much harder to solve. Existing literature only handles constrained optima of HA models without aggregate shocks (Davila, Hong, Krusell, and Ríos-Rull, 2012; Nuño and Moll, 2018), and at a much higher computational cost than for the competitive equilibrium of the same model.

In contrast, as presented in Section 2.4, DeepHAM can solve the constrained efficiency problem as easily as it can solve the competitive equilibrium. We illustrate this advantage by using DeepHAM to solve the constrained efficiency problem in an Aiyagari model as in Davila et al. (2012), and in a HA model with aggregate shocks.

## 5.1 Model Setup

**Baseline setup without aggregate shocks.** The baseline setup is an Aiyagari model that follows the "high wealth dispersion" calibration in Davila et al. (2012) to match empirical US wealth inequality. There are $N$ *ex ante* homogeneous households in this economy. Household $i$'s labor supply is subject to idiosyncratic shocks $z_t^i \in \{e_0, e_1, e_2\}$, which are i.i.d. across agents and follow a Markov process. Household $i$ accumulates asset $a_t^i$ in the form of real capital. Household $i$'s state $s_t^i = (a_t^i, z_t^i)$ follows

$$\begin{aligned} a_{t+1}^i &= (1 + r_t - \delta)a_t^i + w_t z_t^i - c_t^i, \\ a_{t+1}^i &\geq 0, \quad c_t^i \geq 0, \end{aligned} \tag{21}$$

where consumption $c_t^i$ is the only control variable. The representative firm produces with a Cobb-Douglas technology $Y_t = K_t^\alpha L_t^{1-\alpha}$ and rents capital and hires labor in a competitive factor market. So the wage $w_t$ and capital rental rate $r_t$ are:

$$w_t = (1-\alpha)(K_t/L_t)^\alpha, \quad r_t = \alpha(K_t/L_t)^{\alpha-1}, \tag{22}$$

where aggregate saving $K_t = \frac{1}{N}\sum_{i=1}^N a_t^i$ and labor supply $L_t = \bar{L}$ is constant. This completes the specification of (1) and (2). Since there is no aggregate shock in the baseline setup, there is no aggregate state variable $X_t$ nor are there dynamics as in (3).

The benevolent social planner seeks to find a policy rule $\mathcal{C}$ determining $c_t^i$ for all the households $i = 1, ..., N$ and $t = 0, 1, ..., \infty$ to maximize the utilitarian objective,

$$\max_{\mathcal{C}} \frac{1}{N}\mathbb{E}_{\mu(\mathcal{C})}\sum_{i=1}^{N}\sum_{t=0}^{\infty}\beta^t u(c_t^i),$$

subject to the constraints in (21).[11]

**Setup with aggregate shocks.** We also solve the constrained efficiency problem of a HA model with aggregate shocks. On top of the baseline model above, we introduce aggregate productivity shocks $Z_t \in \{Z^l, Z^h\}$, that follow a Markov process, on the production technology of the representative firm $Y_t = Z_t K_t^\alpha L_t^{1-\alpha}$, such that the factor prices are,

$$w_t = Z_t(1-\alpha)(K_t/L_t)^\alpha, \quad r_t = Z_t\alpha(K_t/L_t)^{\alpha-1}, \tag{23}$$

where aggregate saving $K_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i$ and labor supply $L_t = (L^h \mathbb{1}_{Z_t=Z^h} + L^l \mathbb{1}_{Z_t=Z^l})$. Using the general descriptive variables in Section 2, the aggregate state variable $X_t = Z_t$. We also introduce countercyclical idiosyncratic risk to the model, so that the probability that households enter the low income state $z_t^i = e_0$ becomes larger in the bad aggregate state, and smaller in the good aggregate state. Our setup follows the "integration principle" proposed by Krusell et al. (2009), so that when aggregate shocks are eliminated, the model will exactly reduce to the baseline setup. Equations (21) and (22) (or (23)), together with the stochastic process of $z_t^i$, complete the specifications of (1) and (2). The calibration of both models are presented in Appendix B.3.

## 5.2 Results

We solve the constrained planner's problems with $N = 50$ in both the baseline model without aggregate shocks, and in the model with aggregate shocks and countercyclical idiosyncratic shocks. The equilibrium statistics of these problems are presented in Table 4 and 5. In comparison, we also present equilibrium statistics under the competitive equilibrium of the same models.[12]

The main findings are as follows. First, in both models (with or without aggregate

[11] Given the form of the utilitarian objective, we have an alternative approach to approximate the expected social welfare besides the one presented in Section 2.4. We can learn the individual value function defined in (7) with the approximation form (6) and use that to approximate the expected total social welfare by taking the average of the individual value function. We have used the latter approach in this work.

[12] When solving the constrained efficiency problem, we usually find two local maxima of the problem. Since we are solving the constrained planner's problem, we only take the local optimum with higher expected total welfare. We leave the study of the "second best" constrained optimum for future research.

|  | No aggregate shock |  | Aggregate shock |  |
|---|---|---|---|---|
|  | Market | Constrained Opt. | Market | Constrained Opt. |
| Average assets | 30.635 | 119.741 | 34.296 | 95.811 |
| Output | 10.294 | 16.816 | 12.159 | 17.592 |
| Capital-output ratio | 2.976 | 7.120 | 2.821 | 5.446 |
| Interest rate | 4.097% | -2.944% | 4.678% | -1.433% |
| Coefficient of variation of wealth | 2.621 | 2.483 | 2.574 | 2.924 |
| Wealth Gini | 0.864 | 0.862 | 0.812 | 0.878 |
| Coefficient of variation of consumption | 1.548 | 0.710 | 1.699 | 0.736 |
| Consumption Gini | 0.615 | 0.386 | 0.578 | 0.388 |

Table 4: Equilibrium statistics in the market outcome (competitive equilibrium) and constrained optimum for models without or with aggregate shocks.

|  | Positive aggregate shock |  | Negative aggregate shock |  |
|---|---|---|---|---|
|  | Market | Constrained Opt. | Market | Constrained Opt. |
| Average assets | 36.316 | 99.793 | 32.260 | 91.826 |
| Output | 13.925 | 20.038 | 10.393 | 15.146 |
| Capital-output ratio | 2.608 | 4.980 | 3.104 | 6.063 |
| Interest rate | 5.147% | -1.116% | 4.208% | -1.750% |
| Coefficient of variation of wealth | 2.533 | 2.894 | 2.614 | 2.953 |
| Wealth Gini | 0.815 | 0.877 | 0.805 | 0.877 |
| Coefficient of variation of consumption | 1.693 | 0.756 | 1.697 | 0.713 |
| Consumption Gini | 0.599 | 0.407 | 0.542 | 0.345 |

Table 5: Equilibrium statistics in the market outcome (competitive equilibrium) and constrained optimum for the HA model with aggregate shocks, conditional on the realization of aggregate shock.

shocks), the constrained optimum requires a much higher level of capital than the competitive equilibrium. In the absence of aggregate shocks, the planner chooses a capital level 3.90 times that of the laissez-faire equilibrium, which is consistent with the finding of Davila et al. (2012). This is because the planner with the utilitarian objective aims to redistribute from rich households to poor households in order to improve social welfare. Since poor households have a higher labor income share, the planner would raise the aggregate capital level, so that the wage rate increases and capital return decreases according to equations (22) and (23). By raising the aggregate capital level, such a redistribution leaves poor households better off. Meanwhile, in both models, the constrained efficiency problem features a similar level of wealth inequality and a lower level of consumption inequality relative to the market outcome.

Second, compared to the constrained optimum in the absence of aggregate shocks, the model with aggregate shocks features a lower level of aggregate capital stock. With aggregate shocks and countercyclical unemployment risks, the planner chooses a capital level 2.79 times that of the laissez-faire equilibrium, which is lower than the 3.90 times for the model without aggregate shocks. This is because with aggregate shocks, households, especially poor households, have a stronger precautionary saving motive, and their labor income share is thus lower than in the model without aggregate shocks. So the planner would still raise aggregate capital to redistribute through price changes, but not as much as in the economy without aggregate shocks. We provide further validation of this explanation in Section 5.3.

Third, according to Table 5 presenting equilibrium statistics conditional on the realization of aggregate shocks, the planner intends for households to increase their savings in a higher ratio (2.84) in the bad aggregate state, compared to (2.74) in the good aggregate state.

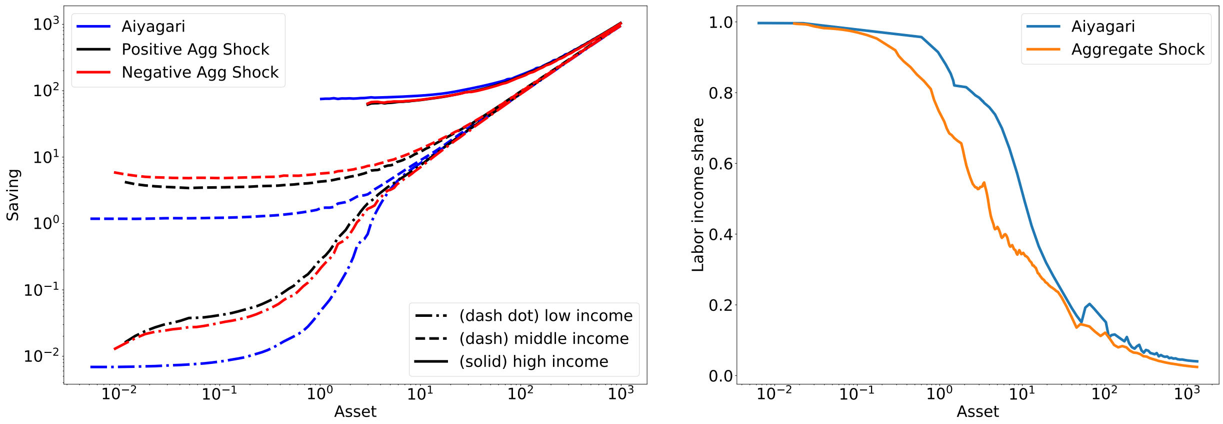
## 5.3 Impact of Aggregate Shocks on Constrained Optimum

To further understand the impact of aggregate shocks on the constrained efficiency problem, we compare households' saving policy and labor share across the asset distribution in the constrained optimum with aggregate shocks and without aggregate shocks ("Aiyagari economy") in Figure 4. Figure 4a shows that in the presence of aggregate shocks, households, especially wealth-poor households, save more than they do in the Aiyagari economy due to precautionary motives. As a result, households, especially wealth-poor households, have a lower labor income share compared to the Aiyagari economy, which is shown in Figure 4b. As a result, in order to redistribute towards wealth-poor households, the constrained planner does not need to raise the aggregate capital level as much in the presence of aggregate shocks as in the economy without aggregate shocks.

## 5.4 Computational Efficiency for Constrained Efficiency Problem

In this section, we highlight the computational efficiency of DeepHAM in solving the constrained planner's problem. In Table 6, we report the computational cost for DeepHAM and for the classical method (Davila et al., 2012) when solving the constrained efficiency problem in the baseline Aiyagari model and in the model with aggregate shocks.

According to Table 6, it is very costly to solve the constrained efficiency problem using the classical method, even for the baseline Aiyagari model without aggregate shocks. To our knowledge, a global solution of the constrained efficiency problem in HA models with



(a) Saving policy

(b) Labor share

Figure 4: Household's saving policy and labor share along asset distribution in the constrained optimum. In the model with aggregate shocks, households, especially wealth-poor households, have more precautionary saving and a lower labor income share, compared to the economy without aggregate shocks.

| | Aiyagari model | With aggregate shocks |
|---|---|---|
| Classical method | 15 hours | not solved in the literature |
| DeepHAM | 20 minutes | 32 minutes |

Table 6: Comparison of the computational speed for the constrained efficiency problem. The conventional method (Davila et al., 2012) is implemented on a laptop with a 2.3Ghz Dual-Core Intel Core i5 processor. DeepHAM is implemented on a small cluster with a NVIDIA Tesla P100 GPU.

aggregate shocks has not been presented in the literature. In contrast, DeepHAM can handle this class of problems quite efficiently.

# 6 Conclusion

We present DeepHAM, an efficient, reliable, and interpretable deep learning-based method for globally solving HA models with aggregate shocks. DeepHAM achieves highly accurate results, and can be applied to complex HA models without suffering from the curse of dimensionality. The algorithm automatically generates a flexible and interpretable representation of the agent distribution through generalized moments. The generalized moments extract key information of the distribution that are most relevant to individual decision rules and thus, through aggregation, to welfare and the evolution of the aggregate economy. They furthermore help us better understand whether and how heterogeneity matters in macroeconomics. Moreover, DeepHAM can solve the constrained efficiency problem as fast as solving the competitive equilibrium, a significant advantage over existing methods. The results demonstrate that DeepHAM is a powerful tool for studying global patterns of complex HA models with aggregate shocks, opening up many exciting possibilities for future research.

In this paper, we assume that the dependence of aggregate prices and quantities on individual states could be written in explicit functional forms. This assumption excludes HA models with aggregate variables that are determined recursively as a function of expected future aggregate variables: the inflation rate in a New Keynesian Phillips curve (NKPC), for example, or indirect utility under Epstein-Zin preferences. Handling forward-looking equilibrium conditions in global solution methods is crucial in solving HANK models with aggregate shocks and requires an additional price function be incorporated in the algorithm. We leave this for further discussion in a companion paper. Another avenue for future research is the development of an estimation algorithm based on DeepHAM.

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# Appendix

## A Neural Networks: a Class of Function Approximator

In this paper, we consider deep, fully connected feedforward neural networks. A network $y = u(x; \Theta)$ with $L$ ($L \geq 1$) hidden layers defines a mapping $\mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$, in which $x \in \mathbb{R}^{d_1}$ is the input variable, $y \in \mathbb{R}^{d_2}$ is the output variable, and $\Theta = (W_1, b_1, \ldots, W_{L+1}, b_{L+1})$ is the collection of network parameters. The network's mapping is defined by a series of compositions of linear transformations and nonlinear activation functions:

$$\begin{aligned}
y &= \sigma_{L+1} \circ (W_{L+1} z_L + b_{L+1}), \\
z_L &= \sigma_{L-1} \circ (W_{L-1} z_{L-1} + b_{L-1}), \\
&\vdots \\
z_2 &= \sigma_2 \circ (W_2 z_1 + b_2), \\
z_1 &= \sigma_1 \circ (W_1 x + b_1).
\end{aligned}$$

Here, $W_l \in \mathbb{R}^{m_l \times m_{l-1}}$ is called the weight matrix, and $b_l \in \mathbb{R}^{m_l}$ is called the bias vector, with $l = 1, \ldots, L+1$. We have $m_0 = d_1, m_{L+1} = d_2$, and $m_1, \ldots, m_l$ are set as network hyperparameters. $\sigma_l : \mathbb{R} \to \mathbb{R}$ is a scalar function called an activation function, and $\circ$ denotes element-wise evaluation. The typical choices of $\sigma_l$ include rectified linear units (ReLU) $\sigma(x) = \max\{(0, x)\}$ and the sigmoid function $1/(1 + e^{-x})$, among others. Typically, $\sigma_l$ are the same for all $l = 1, \ldots, L$ and $\sigma_{L+1}$ is chosen as the identity function to ensure the output is unrestricted. In our policy function neural network $c_{\mathrm{NN}}(\cdot)$ in equation (11), we choose $\sigma_{L+1}$ as the sigmoid function such that the inequality constraints on the decision variable can be satisfied.

Hornik et al. (1989); Cybenko (1989) prove that neural networks with one hidden layer neural networks are universal approximators, i.e., they can approximate arbitrary well any unknown Borel measurable function over a compact domain. In recent years, it has been extensively demonstrated empirically and theoretically that deep neural networks with multiple hidden layers have better approximation and optimization efficiency (Goodfellow et al., 2016).

# B Details of the Model Setup

## B.1 Krusell and Smith (1998) Model

**Calibration.** The parameters follow Den Haan (2010) with each period representing one quarter. The capital share $\alpha = 0.36$, depreciation rate of capital $\delta = 0.25$, household discount factor $\beta = 0.99$, labor supply $\bar{l} = 1/0.9$, unemployment benefit rate $b = 0.15$, and households have log utility. Aggregate productivity $Z_t \in \{1.01, 0.99\}$. The joint transition matrix for idiosyncratic and aggregate shock is $\Pi_Z = \begin{bmatrix} 0.525 & 0.35 & 0.03125 & 0.09375 \\ 0.038889 & 0.836111 & 0.002083 & 0.122917 \\ 0.09375 & 0.03125 & 0.291667 & 0.583333 \\ 0.009115 & 0.115885 & 0.024306 & 0.850694 \end{bmatrix}$, where the four rows and columns correspond to $(Z_t, z_t) = (0.99, 0), (0.99, 1), (1.01, 0), (1.01, 1)$ respectively.

## B.2 Fernández-Villaverde et al. (2019) Model

**Calibration.** We follow Fernández-Villaverde et al. (2019) for parameters. The capital share $\alpha = 0.35$, depreciation rate of capital $\delta = 0.1$, household discount rate $\rho = 0.05$, expert discount rate $\widehat{\rho} = 0.04971$, volatility of aggregate shocks $\sigma = 0.014$, and the risk aversion of the households $\gamma = 2$. To solve the problem in discrete time, we choose $\Delta t = 0.2$, which should be a small number so that the solution is comparable to the continuous time solution. Households' discount factor $\beta = e^{-\rho \Delta t}$. The transition matrix of idiosyncratic shocks is $\Pi_e = \begin{bmatrix} 1 - \lambda_1 \Delta t & \lambda_1 \Delta t \\ \lambda_2 \Delta t & 1 - \lambda_2 \Delta t \end{bmatrix}$ where $\lambda_1 = 0.986$, $\lambda_1 = 0.052$.

## B.3 Davila et al. (2012) Model

**Calibration.** The parameter setting in the baseline model follows Davila et al. (2012). The capital share $\alpha = 0.36$, depreciation rate of capital $\delta = 0.08$, discount factor $\beta = 0.887$, and the risk aversion of the households $\gamma = 2$. Labor endowment $z_t^i \in \{e_0 = 1, e_1 = 5.29, e_2 = 46.55\}$, and $\Pi_e = \begin{bmatrix} 0.992 & 0.008 & 0 \\ 0.009 & 0.980 & 0.011 \\ 0 & 0.083 & 0.917 \end{bmatrix}$ with stationary distribution $\{0.498, 0.443, 0.059\}$. The aggregate labor supply $L_t = \bar{L} = 0.498 e_0 + 0.443 e_1 + 0.059 e_3 = 5.574$.

In the model with aggregate shocks, $Z_t \in \{Z^l, Z^h\} = \{0.95, 1.05\}$ with transition matrix $\Pi_Z = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$. The idiosyncratic shocks are countercyclical, and the transition matrix across labor endowment $\Pi_{e,t} = \begin{bmatrix} 0.98 & 0.02 & 0 \\ 0.009 & 0.980 & 0.011 \\ 0 & 0.083 & 0.917 \end{bmatrix}$ when $Z_t > 1$, $\Pi_{e,t} = \begin{bmatrix} 0.6512 & 0.3488 & 0 \\ 0.978 & 0.011 & 0.011 \\ 0 & 0.083 & 0.917 \end{bmatrix}$ when $Z_t < 1$. The aggregate labor supply in good and bad aggregate states are $\{L^h, L^l\} = \{7.525, 3.623\}$, respectively.

# C Accuracy Measures

The main accuracy measure we adopt in this paper is the Bellman equation error defined in this section. We choose it over the Euler equation error since it provides a better measure over the whole state space, especially the region close to the inequality constraints, without the need to introduce the Lagrangian multiplier.

In the general HA model in Section 2, agent $i$'s optimization problem can be characterized recursively:

$$V(s_t^i, X_t, \mathbb{S}_t) = \max_{c_t^i} \left[ u(c_t^i) + \beta \mathbb{E} V(s_{t+1}^i, X_{t+1}, \mathbb{S}_{t+1}) \right].$$

Given the solved value function $V(\cdot)$, we can evaluate the Bellman equation error for each state $(s_t^i, X_t, \mathbb{S}_t)$ in the state space as:

$$\text{err}_\text{B}(s_t^i, X_t, \mathbb{S}_t) = \left| V(s_t^i, X_t, \mathbb{S}_t) - \max_{c_t^i} \left[ u(c_t^i) + \beta \mathbb{E} V(s_{t+1}^i, X_{t+1}, \mathbb{S}_{t+1}) \right] \right|$$

where the expectation operator is approximated by Monte Carlo sampling of aggregate and idiosyncratic shocks, and the consumption choice $c_t^i$ is solved again given the solved value function $V(\cdot)$, rather than directly taken from the optimal policy we solve.

We then average with respect to the stationary distribution over $(X_t, \mathbb{S}_t)$ to calculate the Bellman equation error for the solution we obtain:

$$\text{err}_\text{B} = \mathbb{E}_{\mu(\mathcal{C}^*)} \left| V(s_t^i, X_t, \mathbb{S}_t) - \max_{c_t^i} \left[ u(c_t^i) + \beta \mathbb{E} V(s_{t+1}^i, X_{t+1}, \mathbb{S}_{t+1}) \right] \right|.$$

# D Solution Comparison for Krusell-Smith Model

In this section, we compare the simulated economy based on the DeepHAM solution with the simulation based on the KS method. Under the same realization of idiosyncratic and aggregate shocks, the two economies simulated with 1000 agents based on the two solution methods are presented in Figure 5.[13]

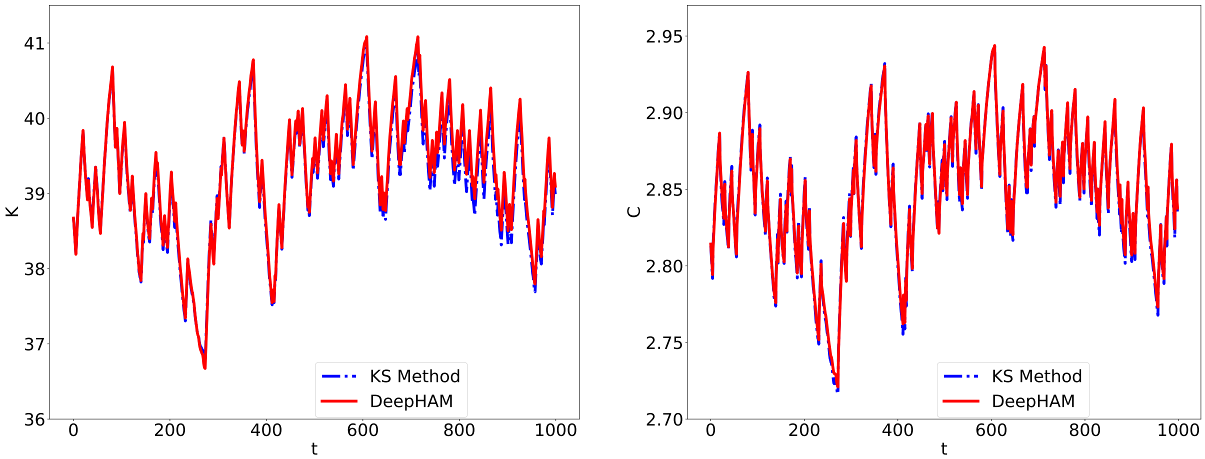

Figure 5: The simulated paths based on solutions obtained from the KS method and DeepHAM, under the same realization of idiosyncratic and aggregate shocks. Left panel: aggregate capital ($K_t$); right panel: aggregate consumption ($C_t$).

The KS method with the first moment can solve the Krusell-Smith model reasonably well, as has been widely validated in the literature. Although DeepHAM can further improve the solution accuracy, as we present in Section 3.2.1, the simulated economy based on the DeepHAM solution is highly consistent with the simulation based on the KS method, as shown in Figure 5. This further confirms the accuracy of the DeepHAM solution for the Krusell-Smith model.

[13] The DeepHAM solution comes from a finite agent model with $N = 50$, while we simulate the two economies with 1000 agents to show the two solutions are consistent with each other in simulations with a relatively large number of agents.