

Deep Learning for Search And Matching Models

(a.k.a. “DeepSAM”)

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Introduction

- ▶ **Heterogeneity** and **aggregate shocks** are important in markets with **search frictions** (e.g. labor and financial markets).
- ▶ Most search and matching (SAM) models with heterogeneous agents study:
 1. Deterministic steady state (e.g. Shimer-Smith '00),
 2. Aggregate fluctuations, but make assumptions to eliminate distribution from state space (e.g. “block recursivity” in Menzio-Shi '11, Lise-Robin '17; Lagos-Rocheteau '09).
- ▶ We present SAM models as **high-dim. PDEs** with **distribution** & **agg. shocks** as states ...and develop a new deep learning method, **DeepSAM**, to solve them globally.
- ▶ We also extend **DeepSAM** for calibration within efficient computational time.

This Paper

- ▶ Develop DeepSAM and apply to canonical search models with aggregate shocks:
 1. Shimer-Smith/Mortensen-Pissarides model with two-sided heterogeneity (today's focus).
 2. Lise-Robin on-the-job search model with worker bargaining.
 3. Duffie-Garleanu-Pederson OTC model with asset and investor heterogeneity (at end).
- ▶ High accuracy in “global” state space (including distribution); efficient compute time for both solution and calibration.
- ▶ We can study non-block recursive unemployment dynamics and wage dynamics:
 1. Large impact of distribution on aggregates when aggregate shocks affect agents unevenly.
 2. A search-theoretical explanation for Okun's hypothesis.
 3. Low-type worker wages more procyclical, especially those in high-type firms.
 4. Lise-Robin style block recursive equilibria over-predict unemployment & vacancy IRF.

Literature

- ▶ Deep learning in macro; for incomplete market heterogeneous agent models (HAM) (e.g. Maliar et al '21, Azinovic et al '22, Kahou et al '21, Han-Yang-E '21 “DeepHAM”; Fernández-Villaverde et al '20, Huang '22, Gu-Laurière-Merkel-Payne '23, among others)
 - ▶ This paper: [search and matching \(SAM\) models](#).

	Distribution	Distribution impact on decisions
HAM	Asset wealth and income	Via aggregate prices
SAM	Type (productivity) of agents in two sides of matching	Via matching probability with other types

- ▶ Search model with business cycle (e.g. Shimer '05, Menzio-Shi '11, Lise-Robin '17.)
 - ▶ This paper: [keep distribution in the state vector](#).
- ▶ Integrate deep learning based solution methods with calibration and estimation (e.g., Chen et al '23, Kase et al '23, Friedl et al '23, Duarte & Fonseca '24)
 - ▶ This paper: [standard calibration practice for quantitative macro](#).

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Shimer-Smith/Mortensen-Pissarides with Two-sided Heterogeneity

- ▶ Continuous time, infinite horizon environment.
- ▶ **Workers** $x \in [0, 1]$ with exog density $g_t^w(x)$; **Firms** $y \in [0, 1]$ with $g_t^f(y)$ by free entry:
 - ▶ Unmatched: unemployed workers get benefit b ; vacant firms produce nothing.
 - ▶ Matched: type x worker and type y firm produce output $z_t f(x, y)$.
 - ▶ z_t : follows two-state continuous time Markov Chain (can be generalized).
 - ▶ Firms can pay entry cost c and draw a firm type y from uniform distribution $[0, 1]$. [More](#)
- ▶ **Meet randomly** at rate $m(\mathcal{U}_t, \mathcal{V}_t)$, \mathcal{U}_t is total unemployment, \mathcal{V}_t is total vacancies.
- ▶ Upon meeting, agents choose whether to accept the match:
 - ▶ Match surplus $S_t(x, y)$ divided by **generalized Nash bargaining**: worker get fraction β .
 - ▶ Match acceptance decision $\alpha_t(x, y) = \mathbb{1}\{S_t(x, y) > 0\}$. Match dissolve rate $\delta(x, y, z)$.
- ▶ Equilibrium object: **$g_t(x, y)$ mass** of match $(x, y) \Rightarrow$ unemployed $g_t^u(x)$, vacant $g_t^v(y)$.

Recursive Equilibrium Part I: Unemployed Workers & KFE

- ▶ Idiosyncratic state = x , Aggregate states = $(z, g(x, y))$.
- ▶ Hamilton-Jacobi-Bellman equation for an unemployed worker's value $V^u(x, z, g)$:

$$\begin{aligned} \rho V^u(x, z, g) = & b + \frac{m(z, g)}{\mathcal{U}(z, g)} \int \underbrace{\overbrace{\alpha(x, \tilde{y}, z, g)}^{\text{acceptance decision}} \overbrace{(V^e(x, \tilde{y}, z, g) - V^u(x, z, g))}^{\text{employed value}}}_{\text{change of value conditional on match}} \frac{g^v(\tilde{y})}{\mathcal{V}(z, g)} d\tilde{y} \\ & + \lambda_{z\tilde{z}}(V^u(x, \tilde{z}, g) - V^u(x, z, g)) + \underbrace{D_g V^u(x, z, g)}_{\text{Frechet derivative: how change of } g \text{ affects } V} \cdot \mu^g \end{aligned}$$

- ▶ Dynamics of $g(x, y)$ is given by Kolmogorov forward equation (KFE):

$$\mu^g(x, y, z, g) := \frac{dg_t(x, y)}{dt} = -\delta(x, y, z)g(x, y) + \frac{m(z, g)}{\mathcal{U}(z, g)\mathcal{V}(z, g)}\alpha(x, y, z, g)g^v(y)g^u(x)$$

Recursive Characterization For Equilibrium Surplus

- ▶ Surplus from match $S(x, y, z, g) := V^p(x, y, z, g) - V^v(y, z, g) + V^e(x, y) - V^u(x, z, g)$.
- ▶ Characterize equilibrium with master equation for surplus: Free entry condition

$$\begin{aligned}\rho S(x, y, z, g) &= z f(x, y) - \delta(x, y, z) S(x, y, z, g) \\ &\quad - (1 - \beta) \frac{m(z, g)}{\mathcal{V}(z, g; S)} \int \alpha(\tilde{x}, y, z, g) S(\tilde{x}, y, z, g) \frac{g^u(\tilde{x})}{\mathcal{U}(z, g)} d\tilde{x} \\ &\quad - b - \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \int \alpha(x, \tilde{y}, z, g) S(x, \tilde{y}, z, g) \frac{g^v(\tilde{y})}{\mathcal{V}(z, g; S)} d\tilde{y} \\ &\quad + \lambda(z)(S(x, y, \tilde{z}, g) - S(x, y, z, g)) + D_g S(x, y, z, g) \cdot \mu^g(z, g)\end{aligned}$$

- ▶ Kolmogorov forward equation (KFE):

$$\frac{dg_t(x, y)}{dt} := \mu^g(x, y, z, g) = -\delta(x, y, z) g(x, y) + \frac{m(z, g)}{\mathcal{U}(z, g) \mathcal{V}(z, g)} \alpha(x, y, z, g) g^v(y) g^u(x)$$

- ▶ High-dim PDEs with distribution in state: hard to solve with conventional methods.

Finite Type Approximation

- ▶ Approximate $g(x, y)$ on finite types: $x \in \mathcal{X} = \{x_1, \dots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \dots, y_{n_y}\}$.
- ▶ Finite state approximation \Rightarrow analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \leq n_x, j \leq n_y}$
- ▶ Approximated master equation for surplus:

$$\begin{aligned} 0 = \mathcal{L}^S S(x, y, z, g) = & -(\rho + \delta)S(x, y, z, g) + zf(x, y) - b \\ & - (1 - \beta) \frac{m(z, g)}{\mathcal{V}(z, g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z, g)} \\ & - \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z, g)} \\ & + \lambda(z)(S(x, y, \tilde{z}, g) - S(x, y, z, g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x, y, z, \{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g) \end{aligned}$$

DeepSAM Algorithm for Solving the Model

- ▶ Approximate surplus by neural network $S(x, y, z, g) \approx \hat{S}(x, y, z, g; \Theta)$. Function form
- ▶ Start with initial parameter guess Θ^0 . At iteration n with Θ^n :
 1. Generate K sample points, $Q^n = \{(x_k, y_k, z_k, \{g_{ij,k}\}_{i \leq n_x, j \leq n_y})\}_{k \leq K}$.
 2. Calculate the average mean squared error of surplus master equation on sample points:

$$L(\Theta^n, Q^n) := \frac{1}{K} \sum_{k \leq K} \left| \mathcal{L}^S \hat{S}(x_k, y_k, z_k, \{g_{ij,k}\}_{i \leq n_x, j \leq n_y}) \right|^2$$

3. Update NN parameters with stochastic gradient descent (SGD) method:

$$\Theta^{n+1} = \Theta^n - \zeta^n \nabla_{\Theta} L(\Theta^n, Q^n)$$

4. Repeat until $L(\Theta^n, Q^n) \leq \epsilon$ with precision threshold ϵ .

- ▶ Once S is solved, we have α and can solve for worker and firm value functions.

DeepSAM for Calibration and Estimation

- ▶ DeepSAM for **solving** the model (e.g. 59 dimension PDE):

$$\mathcal{L}^S S(x, y, z, g) = 0 \quad (1)$$

- ▶ Include structural parameters directly in state space: DeepSAM for **calibrating** the model, solve (e.g. $59 + \dim(\Omega)$ dimension PDE):

$$\mathcal{L}^{\tilde{S}} \tilde{S}(x, y, z, g, \Omega) = 0 \quad (2)$$

Ω : structural parameters for internal calibration.

- ▶ Dimension of (2) is only marginally higher than (1). Solving (2), we obtain the model solution over a range of parameter space, enabling calibration through simulation.
 - ▶ We use simulation data to build a surrogate model mapping parameters to moments.
- ▶ Calibration only takes a marginally longer time than solving the model.

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Calibration of Shimer-Smith Model with Aggregate Shocks

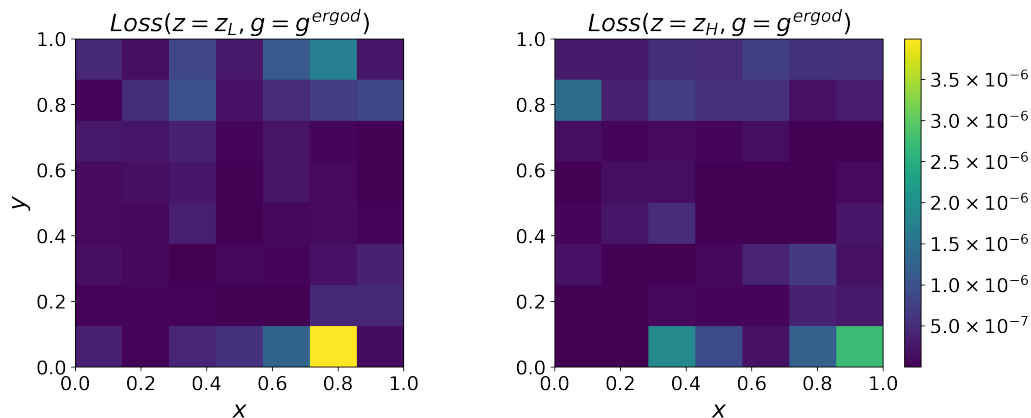
Frequency: annual.

Parameter	Interpretation	Value	Target/Source
ρ	Discount rate	0.05	Kaplan, Moll, Violante '18
δ	Job destruction rate	0.2	BLS job tenure 5 years
ξ	Extreme value distribution for α choice	2.0	
$f(x, y)$	Production function for match (x, y)	$0.6 + 0.4 (\sqrt{x} + \sqrt{y})^2$	Hagedorn et al '17
β	Surplus division factor	0.72	Shimer '05
c	Entry cost	4.86	Steady state $\mathcal{V}/\mathcal{U} = 1$
z, \tilde{z}	TFP shocks	1 ± 0.015	Lise Robin '17
$\lambda_z, \lambda_{\tilde{z}}$	Poisson transition probability	0.08	Shimer '05
$\delta, \tilde{\delta}$	Separation shocks	0.2 ± 0.02	Shimer '05
$\lambda_{\delta}, \lambda_{\tilde{\delta}}$	Poisson transition probability	0.08	Shimer '05
$m(\mathcal{U}, \mathcal{V})$	Matching function	$\kappa \mathcal{U}^{\nu} \mathcal{V}^{1-\nu}$	Hagedorn et al '17
ν	Elasticity parameter for meeting function	0.5	Hagedorn et al '17
κ	Scale parameter for meeting function	5.4	Unemployment rate 5.9%
b	Worker unemployment benefit	0.5	Shimer '05
n_x	Discretization of worker types	7	
n_y	Discretization of firm types	8	

Numerical Performance: Accuracy I

Calibration

- Mean squared loss as a function of type in the master equations of S (at ergodic g).



Numerical Performance: Accuracy II Calibration

- Compare **steady state solution without aggregate shocks** to solution using conventional methods.

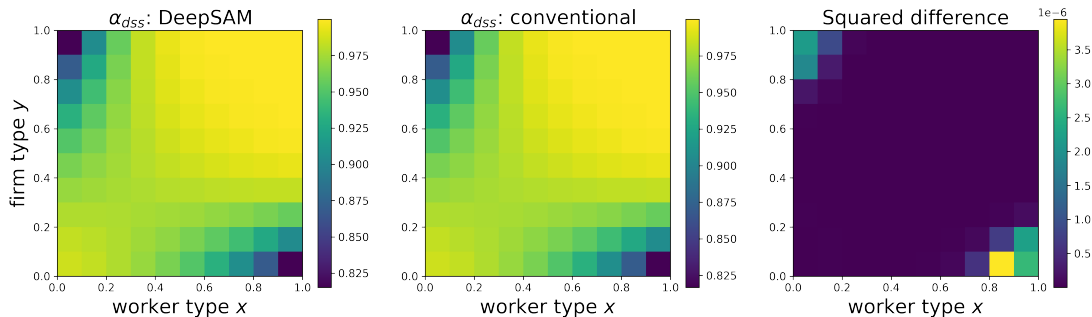


Figure: Comparison with steady-state solution

Comparison for discrete α

Computational Speed for Solving and Calibrating OJS Model

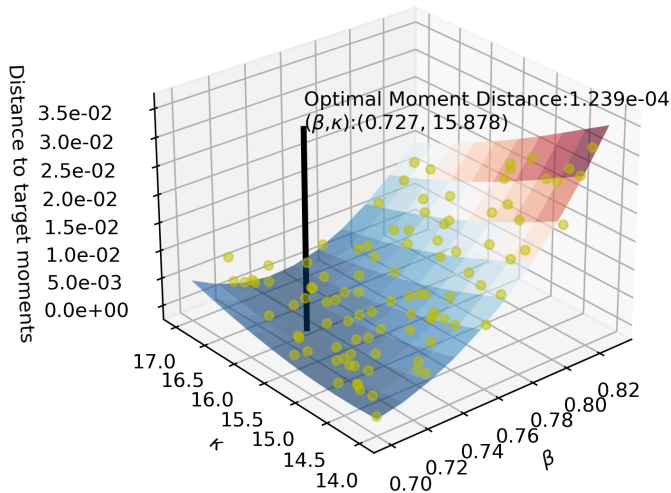
	Solution Given Structural Parameters	Solution with Structural Parameters as Pseudo-states	Simulation & Training Surrogate Model	Simulated Method of Moments	Full Calibration
MSE Loss	1.97×10^{-6}	4.8×10^{-6}	6.13×10^{-7}	1.24×10^{-4}	-
Time	55min	4h 1min	1h 3min	1.4min	5h 5min

- Calibration: solve the model over economic parameter space, and simulate across 10,000 parameter combinations for simulated method of moments.

Moments	$\mathbb{E}[U]$	$\mathbb{E}[V]$	$\mathbb{E}[E2E]$	$\mathbb{E}[U2E]$	$\mathbb{E}[E2U]$
Data	0.058	0.037	0.025	0.468	0.025
Model	0.058	0.037	0.026	0.431	0.026

Table: Calibration Results

Calibration of OJS Model: Visualization in 2D



Target moment: $\mathbb{E}[U], \mathbb{E}[V]$. Parameter: matching efficiency κ , worker bargaining power β .

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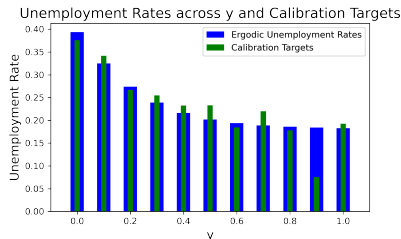
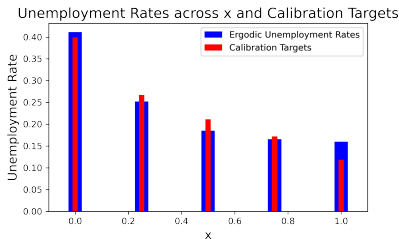
Algorithm Performance

Distribution and Business Cycle Dynamics

More Applications: OJS and OTC Search

Q1. Is the feedback from g to α important? Evidence from COVID

- ▶ Workers and firms have heterogeneous exposure to aggregate shocks.
- ▶ We calibrate separation rate $\delta(x, y, z)$ to match the heterogeneous employment effect of COVID on different workers/firms (Cajner et al., 2020).



- ▶ Study aggregate dynamics **with** and **without** distribution feedback to agent decision:

Full dynamics:
$$\frac{dg_t(x, y)}{dt} = -\delta(x, y, z_t)g_t(x, y) + \frac{m_t(z, g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x, y, z_t, g_t)g_t^u(x)g_t^v(y)$$

No distribution feedback:
$$\frac{dg_t(x, y)}{dt} = -\delta(x, y, z_t)g_t(x, y) + \frac{m_t(z, g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x, y, z_t, g^{\text{ergodic}})g_t^u(x)g_t^v(y)$$

A1. Feedback from g to α matter for asymmetric shocks.

Full dynamics:
$$\frac{dg_t(x, y)}{dt} = -\delta(x, y, z_t)g_t(x, y) + \frac{m_t(z, g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x, y, z_t, \mathbf{g}_t)g_t^u(x)g_t^v(y)$$

No distribution feedback:
$$\frac{dg_t(x, y)}{dt} = -\delta(x, y, z_t)g_t(x, y) + \frac{m_t(z, g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x, y, z_t, \mathbf{g}^{\text{ergodic}})g_t^u(x)g_t^v(y)$$

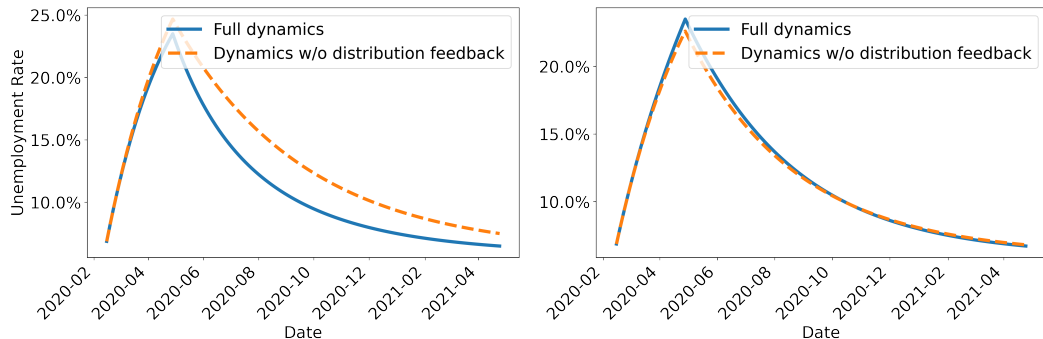


Figure: Unemployment U_t after (left) true COVID shock, (right) counterfactual “symmetric” shock.

Mechanism: low-type worker/firms accept faster given distribution shift

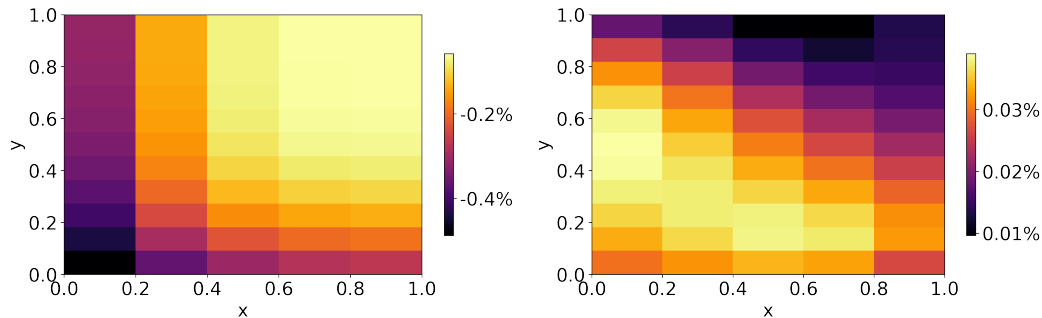


Figure: Difference of distribution and acceptance after the asymmetric COVID-19 shock, compared to the ergodic steady state.

Q2. How do block recursive models restrict aggregate dynamics?
(IRF to negative TFP shock for block recursive vs other calibrations)

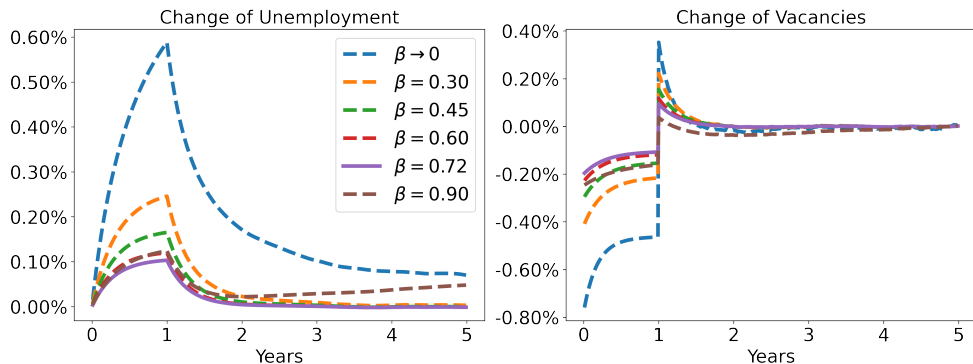


Figure: IRF with different β 's vs. block-recursive model with $\beta = 0$

- By assuming firms get all surplus, block recursive models predict high U_t response (because firms' vacancy posting is very elastic to aggregate shocks).

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More Applications: OJS and OTC Search

More Applications in the Paper

1. SAM model with on-the-job search and endogenous separation. [details](#)
 - ▶ Similar to Lise-Robin '17, but allow for $\beta \in (0, 1)$.
 - ▶ We also do not assume that vacancies are destroyed if not filled. Vacancy is a stock variable.
2. OTC financial market with heterogeneous investors, different bond maturities, and aggregate default risk. [details](#)

Q3. Who benefits more over a longer expansion?

- ▶ Okun's (1973) hypothesis: longer expansion benefits low-income workers more.
- ▶ We find U_t for low-income workers drops more than high-income in longer expansion.

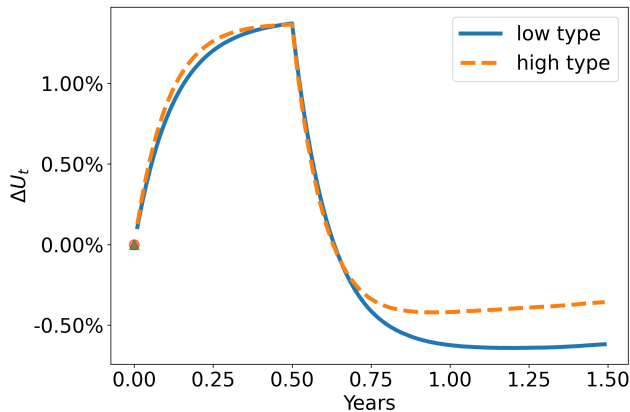


Figure: ΔU_t for workers in different groups

A Search-Theoretical Explanation for Okun's Hypothesis

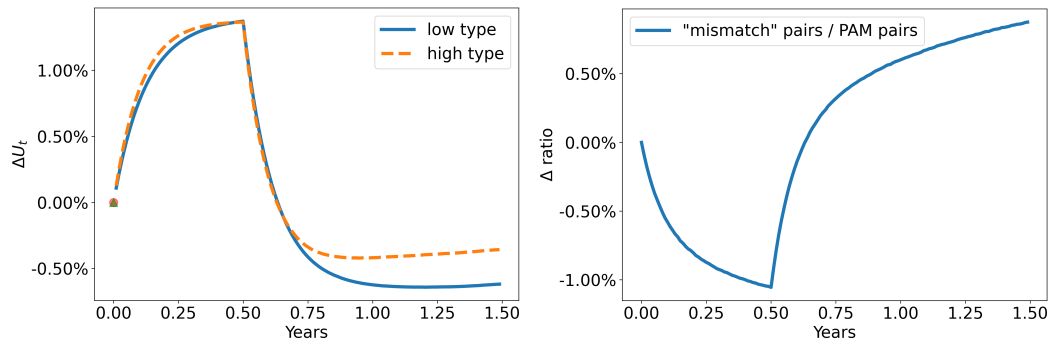


Figure: Left: ΔU_t for different workers. Right: expansion \Rightarrow positive assortative matching \downarrow .

- ▶ Mechanism: sorting weakens over time in expansions, high-type firms more inclined to hire low-type workers during longer expansions.
- ▶ Important that workers&firms understand the distribution shift over time.

Q4. Are wage dynamics heterogeneous across distribution?

- ▶ In Lise-Robin: “wages cannot be solved for exactly... need to solve worker values where the distribution of workers across jobs is a state variable.”
- ▶ DeepSAM can solve wage dynamics with rich heterogeneity.
- ▶ Low-type worker wages more procyclical, especially those in high-type firms.

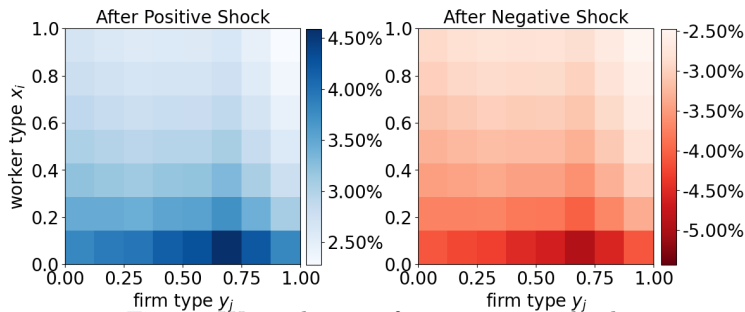


Figure: Wage change after aggregate shocks

Conclusion and Future Work

- ▶ We develop an integrated global solution and calibration method, DeepSAM, to search and matching models with heterogeneity and aggregate shocks.
- ▶ We apply DeepSAM to canonical labor search models, and find important interaction between heterogeneity and aggregate shocks that we cannot study before.
- ▶ A powerful tool to be combined with rich data of heterogeneous workers, firms, investors, & assets over business cycles!
- ▶ More applications:
 - ▶ Richer models in labor, financial, and money search.
 - ▶ Spatial models with aggregate uncertainty.
 - ▶ Network models with aggregate uncertainty.

Thank You!

Deep Learning for Economic Models

- ▶ Deep learning has been successful in high-dimensional scientific computing problems.
- ▶ We can use deep learning to solve high-dim value & policy functions in economics:

1. Use deep neural networks to approximate value function $V : \mathbb{R}^N \rightarrow \mathbb{R}$

$$V(\mathbf{x}) \approx \mathcal{L}^P \circ \dots \circ \mathcal{L}^p \circ \dots \circ \mathcal{L}^1(\mathbf{x}), \quad \mathbf{x}: \text{high-dim state vector},$$
$$\mathbf{h}_p = \mathcal{L}^p(\mathbf{h}_{p-1}) = \sigma(\mathbf{W}_p \mathbf{h}_{p-1} + \mathbf{b}_p), \quad \mathbf{h}_0 = \mathbf{x},$$

σ : element-wise nonlinear fn, e.g. $\text{Tanh}(\cdot)$. Want to solve unknown parameters $\Theta = \{\mathbf{W}_p, \mathbf{b}_p\}_p$.

2. Cast high-dim function into a loss function, e.g. Bellman equation residual.
 3. Optimize unknown parameters, Θ , to minimize average loss on a “global” state space, using stochastic gradient descent (SGD) method.
- ▶ Similar procedure to polynomial “projection”, but more efficient in practice. [back](#)

Methodology Q & A

► Q. What about dimension reduction?

- Krusell-Smith '98 suggest approximating distribution by mean.
- For random search, **not clear what moment enables approximation** of:

$$\int \alpha(\tilde{x}, y, z, g) S(\tilde{x}, y, z, g) \frac{g^u(\tilde{x})}{\mathcal{U}(z, g)} d\tilde{x}, \quad \text{and} \quad \int \alpha(x, \tilde{y}, z, g) S(x, \tilde{y}, z, g) \frac{g^v(\tilde{y})}{\mathcal{V}(z, g)} d\tilde{y}$$

► Q. How do we choose where to sample?

- We start by drawing distributions **“between” steady states** for **different fixed z** .
- Can move to **ergodic** sampling once error is small.
- Can increase sampling in regions of the state space **where errors are high**.

► Q. Why are SAM models hard to solve?

- Compared to PINNs, we have feedback between agent optimization and distribution.
- Difficult when feedback is strong & $\hat{S}(x, y, z, g; \Theta)$ has sharp curvature. Use “homotopy”.

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Labor Search Model

On-The-Job Search Model

OTC Market

Appendix A

Comparison to Other Heterogeneous Agent Search Models

- Lise-Robin '17: sets $\beta = 0$ (and other conditions, including Postal-Vinay Robin style Bertrand competition for workers searching on-the-job)

$$S(x, y, z, \textcolor{red}{g}) = S(x, y, z), \quad \alpha(x, y, z, \textcolor{red}{g}) = \alpha(x, y, z)$$

- Menzio-Shi '11: competitive search (directed across a collection of sub-markets):

$$S(x, y, z, \textcolor{red}{g}) = S(x, y, z)$$

- We look for a solution for S and α in terms of the distribution g .

Modification 1: Finite Type Approximation

- ▶ Approximate $g(x, y)$ on finite types: $x \in \mathcal{X} = \{x_1, \dots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \dots, y_{n_y}\}$.
- ▶ Finite state approximation \Rightarrow analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \leq n_x, j \leq n_y}$
- ▶ Approximated master equation for surplus:

$$\begin{aligned} 0 = \mathcal{L}^S S(x, y, z, g) = & -(\rho + \delta)S(x, y, z, g) + zf(x, y) - b \\ & - (1 - \beta) \frac{m(z, g)}{\mathcal{V}(z, g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z, g)} \\ & - \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z, g)} \\ & + \lambda(z)(S(x, y, \tilde{z}, g) - S(x, y, z, g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x, y, z, \{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g) \end{aligned}$$

Modification 2: Approximate Discrete Choice

- ▶ In the original model,

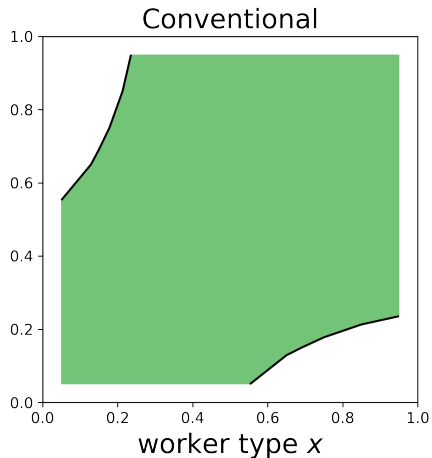
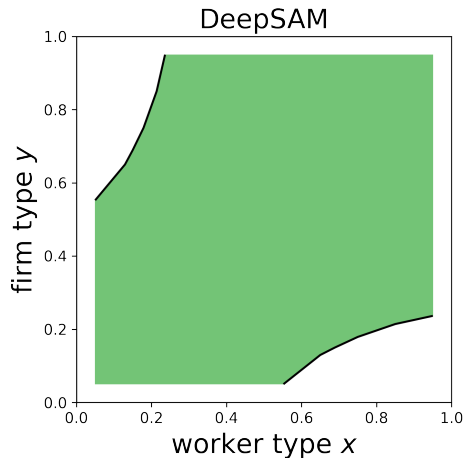
$$\alpha(x, y, z, g) = \mathbb{1}\{S(x, y, z, g) > 0\}$$

- ▶ Discrete choice $\alpha \Rightarrow$ discontinuity of $S(x, y, z, g)$ at some g .
- ▶ To ensure master equation well defined & NN algorithm works, we approximate with

$$\alpha(x, y, z, g) = \frac{1}{1 + e^{-\xi S(x, y, z, g)}}$$

- ▶ Interpretation: logit choice model with utility shocks \sim extreme value distribution.
($\xi \rightarrow \infty \Rightarrow$ discrete choice α .)

DeepSAM vs Conventional method at DSS: discrete case



[back](#)

Free Entry Condition

- ▶ Firms can pay entry cost c and draw a firm type y from uniform distribution $[0, 1]$:

$$c = \mathbb{E}[V_t^v] = \int V^v(\tilde{y}, z, g) d\tilde{y}. \quad (3)$$

- ▶ As the matching function is homothetic $\frac{m(z_t, g_t)}{\mathcal{V}_t} = \hat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right)$, combining free entry condition with HJB equation for V^v gives:

$$\hat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right) = \frac{\rho c}{\int \int \alpha(\tilde{x}, \tilde{y}) \frac{g_t^u(\tilde{x})}{\mathcal{U}_t} (1 - \beta) S_t(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}} \Rightarrow \mathcal{V}_t = \mathcal{U}_t \hat{m}^{-1}(\dots) \quad (4)$$

where $g_t^u = g_t^w - \int g_t^m(x, y) dy$ and so the RHS can be computed from g_t^m and S_t .

- ▶ $g_t^f = \mathcal{V}_t + \mathcal{P}_t$, where \mathcal{V}_t and \mathcal{P}_t can be expressed in terms of g and S .
- ▶ With free entry condition, the master equation expression for surplus takes the same form as without free entry, but with different expressions of $g^f(y)$.

Recursive Equilibrium Part II: Other Equations

- ▶ Hamilton-Jacobi-Bellman equation (HJBE) for employed worker's value $V^e(x, y, z, g)$:

$$\begin{aligned}\rho V^e(x, y, z, g) = & w(x, y, z, g) + \delta(x, y, z) (V^u(x, z, g) - V^e(x, y, z, g)) \\ & + \lambda_{z\tilde{z}}(V^e(x, y, \tilde{z}, g) - V^e(x, y, z, g)) + D_g V^e(x, y, z, g) \cdot \mu^g\end{aligned}$$

- ▶ HJBE for a vacant firm's value $V^v(y, z, g)$:

$$\begin{aligned}\rho V^v(y, z, g) = & \frac{m(z, g)}{\mathcal{V}(z, g)} \int \alpha(\tilde{x}, y, z, g) (V^p(\tilde{x}, y, z, g) - V^v(y, z, g)) \frac{g^u(\tilde{x})}{\mathcal{U}(z, g)} d\tilde{x} \\ & + \lambda_{z\tilde{z}}(V^v(y, \tilde{z}, g) - V^v(y, z, g)) + D_g V^v(y, z, g) \cdot \mu^g\end{aligned}$$

- ▶ HJBE for a producing firm's value $V^p(x, y, z, g)$:

$$\begin{aligned}\rho V^p(x, y, z, g) = & z f(x, y) - w(x, y, z, g) + \delta(x, y, z) (V^v(y, z, g) - V^p(x, y, z, g)) \\ & + \lambda_{z\tilde{z}}(V^p(x, y, \tilde{z}, g) - V^p(x, y, z, g)) + D_g V^p(x, y, z, g) \cdot \mu^g\end{aligned}$$

Variation in α as the Distribution Varies

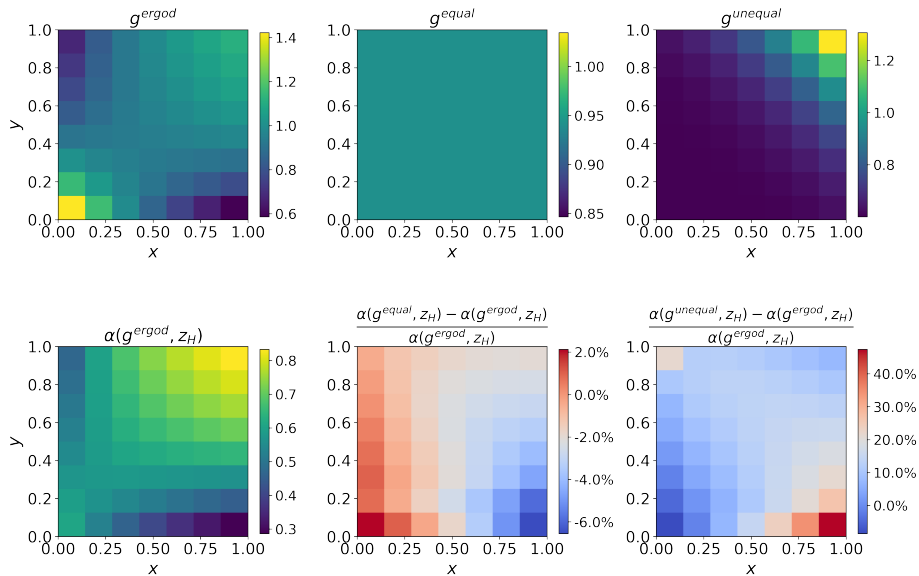


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OTC Market

On-The-Job Search: Environment Features

- ▶ Same worker types, firm types, and production function.
- ▶ Now all workers search; meeting rate is $m(\mathcal{W}_t, \mathcal{V}_t)$; total search effort is $\mathcal{W}_t := \mathcal{U}_t + \phi \mathcal{E}_t$
- ▶ Terms of trade when a vacant \tilde{y} -firm meets:
 - ▶ Unemployed x -worker: Nash bargaining where workers get surplus fraction β ,
 - ▶ Worker in (x, y) match: Nash bargaining over incremental surplus.
If $S_t(x, \tilde{y}) > S_t(x, y)$, worker moves to firm \tilde{y} and gets additional $\beta(S_t(x, \tilde{y}) - S_t(x, y))$.
- ▶ Endogenous separation $\alpha_t^b(x, y) = 1$ when $S_t(x, y) < 0$.

Recursive Characterization For Equilibrium Surplus

- Can characterize equilibrium with the master equation for the surplus:

$$\begin{aligned}\rho S(x, y, z, g) &= z f(x, y) - (\delta + \alpha^b(x, y, z, g)) S(x, y, z, g) \\ &\quad - \frac{m(z, g)}{\mathcal{W}(z, g) \mathcal{V}(z, g)} \left[(1 - \beta) \int \alpha(\tilde{x}, y, z, g) S(\tilde{x}, y, z, g) g^u(\tilde{x}) d\tilde{x} \right. \\ &\quad - \phi(1 - \beta) \int \alpha^p(\tilde{x}, y, \tilde{y}, z, g) (S(\tilde{x}, y, z, g) - S(\tilde{x}, \tilde{y}, z, g)) g(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} \\ &\quad \left. + \phi\beta \int \alpha^p(x, \tilde{y}, y, z, g) S(x, y, z, g) g^v(\tilde{y}) d\tilde{y} \right] \\ &\quad - b - \beta \frac{m(z, g)}{\mathcal{W}(z, g) \mathcal{V}(z, g)} \int \alpha(x, \tilde{y}, z, g) S(x, \tilde{y}, z, g) g^v(\tilde{y}) d\tilde{y} \\ &\quad + \lambda(z) (S(x, y, \tilde{z}, g) - S(x, y, z, g)) + D_g S(x, y, z, g) \cdot \mu^g(z, g)\end{aligned}$$

where:

$$\alpha^p(\tilde{x}, y, \tilde{y}, z, g) := \mathbb{1}\{S(\tilde{x}, y, z, g) \geq S_t(\tilde{x}, \tilde{y}, z, g) \geq 0\}$$

KFE

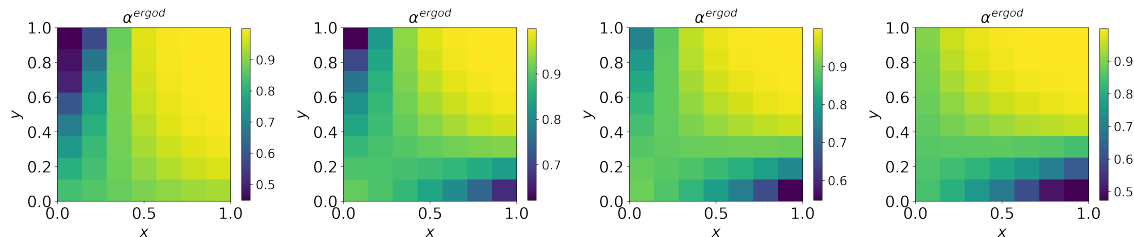
On-the-job-search: KFE

- The KFE becomes:

$$\begin{aligned} dg_t^m(x, y) = & -\delta g_t^m(x, y)dt \\ & -\phi \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} g_t^m(x, y) \int \alpha_t^p(x, y, \tilde{y}) g_t^v(\tilde{y}) d\tilde{y} dt \\ & + \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} \alpha_t(x, y) g_t^u(x) g_t^v(y) dt \\ & + \phi \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} \int \alpha_t^p(\tilde{x}, \tilde{y}, y) g_t^v(y) \frac{g_t^m(\tilde{x}, \tilde{y})}{\mathcal{E}_t} d\tilde{x} d\tilde{y} dt \end{aligned}$$

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Worker Bargaining Power Influences Assortative Matching

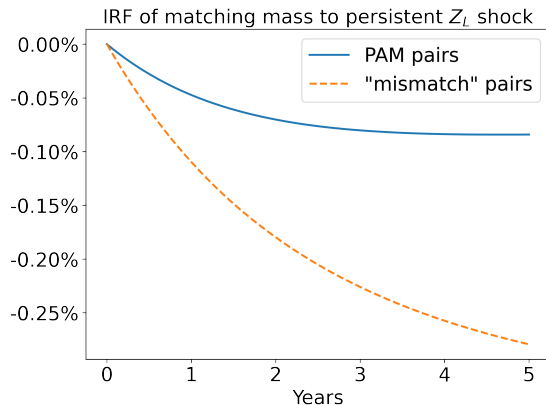
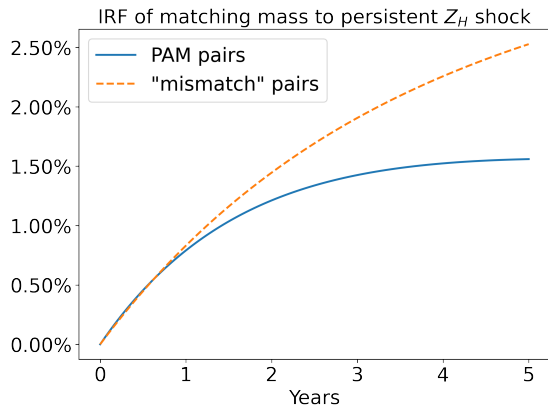


Sorting at the ergodic distribution for different worker bargaining power β . Left to right $\beta = 0$ (Lise-Robin '17), 0.5, 0.72 (benchmark), 1.

Additional parameter calibration: $\phi = 0.2$.

Sorting Over Business Cycles

- Study how “mismatch” changes over the business cycle. [back](#)



“PAM” pairs: pairs where x & y are close. “Mismatch”: pairs where x & y are **not** close.

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Environment: Setting, Bonds, and Households

- ▶ Continuous time, infinite horizon environment.
- ▶ There are many bonds, $k \in \{1, \dots, K\}$, in positive net supply s_k :
 - ▶ Every bond pays the same dividend $\delta > 0$.
 - ▶ Bond k matures at rate $1/\tau_k$ (so it has average maturity τ_k).
- ▶ Populated by a unit-mass continuum of infinitely-lived and risk-neutral investors:
 - ▶ An investor can hold either zero or one share of at most one type of asset.
 - ▶ Investor type $j \in \{1, \dots, J\}$ gets flow utility $\delta - \psi(j, k)$ from holding bond k .
 - ▶ Agents switch from type i to j at rate $\lambda_{i,j}$.
- ▶ Aggregate (default) state $z \in \{z_1, \dots, z_n\}$, switches at rate $\zeta_{z,z'}$.
At state z , asset k pays a fraction $\phi(k, z)$ of the coupon and the principal.

Distribution and Bargaining

- ▶ An investor's state is made up of her holding cost $j \in \{1, \dots, J\}$ and her ownership status, for each asset type $k \in \{1, \dots, K\}$ (owner o or non-owner n). Hence the set of investor idiosyncratic states is:

$$A = \{1n, 2n, \dots, Jn, 1o1, \dots, 1oK, 2o1, \dots, 2oK, Jo1, \dots, JoK\} \quad (5)$$

- ▶ The rate of contact between investors with states a and b is:

$$\mathcal{M}_{a,b} = \kappa_{a,b} g_a g_b \quad (6)$$

- ▶ Agents a, b engage in Generalized Nash bargaining with bargaining power $\beta_{a,b}$.

Value Function: Non-Owners

- The value function for non-owner with type i , $V(in, g, z)$, is given by:

$$\begin{aligned}\rho_i V(in, g, z) = & \sum_a \kappa_{in,a} \alpha(in, a, g, z) \beta_{in,a} S(in, a, z, g) \\ & + \sum_k \xi_{i,k} (V(iok, g, z) - V(in, g, z)) \\ & + \sum_{j \neq i} \lambda_{i,j} (V(jn, g, z) - V(in, g, z)) \\ & + \sum_{z'} \zeta_{z,z'} (V(in, g, z') - V(in, g, z)) + \sum_{a \in A} \partial_{g_a} V(in, g, z) \mu^g(a, z)\end{aligned}$$

where $\alpha(in, jok, g, z)$ is an indicator for whether the surplus from the trade is positive $S(in, jok, g, z) > 0$ and the trade is accepted upon matching.

Value Function: Owners

- Value function for an investor of type i holding asset k , $V(iok, g, z)$, is given by:

$$\begin{aligned}\rho_i V(iok, g, z) = & \delta \phi(k, z) - \psi(i, k) + \frac{1}{\tau_k} (V(in, g, z) + \pi(k, z) - V(iok, g, z)) \\ & + \sum_a \kappa_{iok,a} \alpha(iok, a, g, z) g_a \beta_{iok,a} S(iok, a, g, z) \\ & + \sum_{j \neq i} \lambda_{i,j} (V(jok, g, z) - V(iok, g, z)) \\ & + \sum_{z'} \zeta_{z,z'} (V(iok, g, z') - V(iok, g, z)) + \sum_{a \in A} \partial_{g_a} V(iok, g, z) \mu^g(a, z).\end{aligned}$$

Parameter Values: Holding Costs

Agent Type (i)	Maturity (τ)			
	$\tau_1 = 0.25$	$\tau_2 = 1.0$	$\tau_3 = 5$	$\tau_4 = 10$
A	$\delta\phi(1, z)$	$\delta\phi(2, z)$	$\delta\phi(3, z)$	$\delta\phi(4, z)$
B	0.02	0.02	0.02	0.02
C	0.0	0.0	0.0	0.0
D	0.02	0.02	0.01	0.00

Table: Holding costs: $\psi(i, \tau)$.

Parameter Values: Switching Rates

Parameter Values: Participation in Primary Market

		Maturity (τ)			
		$\tau_1 = 0.25$	$\tau_2 = 1.0$	$\tau_3 = 5$	$\tau_4 = 10$
Agent Type (i)	A	ξ_1	ξ_2	ξ_3	ξ_4
	B	—	—	—	—
	C	—	—	—	—
	D	—	—	—	—

Table: Primary market participation: $\xi(i, \tau)$.

Parameter Values: Mathing Rates and Bargaining

$$\kappa_{a,b} = \begin{cases} 50, & \text{if } (a,b) = (in, jok) \text{ and } i, j \neq A, \\ 50, & \text{if } (a,b) = (iok, jok) \text{ and } i, j \neq A, \\ 75, & \text{if } (a,b) = (in, Aok) \text{ and } i \neq A, \\ 0, & \text{if } (a,b) = (iok, Aol) \text{ and } \forall i, \\ 0, & \text{if } (a,b) = (in, jn) \text{ and } \forall i, j, \end{cases} \quad (7)$$

$$\beta_{a,b} = \begin{cases} 0.5, & \text{if } (a,b) = (in, jok) \text{ and } i, j \neq A, \\ 0.5, & \text{if } (a,b) = (iok, jol) \text{ and } i, j \neq A, \\ 0.05, & \text{if } (a,b) = (in, Aok) \text{ and } i, j \neq A, \end{cases} \quad (8)$$

Parameter Values: Other Values

Parameter	Interpretation	Value	Target/Source
ρ	Discount rate	0.05	Chen at al. (2017)
δ	Bond Coupon Rate	0.01	
Aggregate State: $z \in \{z_L, z_M, z_H\}$			
$\phi(z)$	Coupon haircut	(0.986, 0.991, 0.997)	Chen at al. (2017)
$\pi(z)$	Principal haircut	(0.986, 0.991, 0.997)	Chen at al. (2017)
$\zeta_{M,L}, \zeta_{M,H}$	Rate from 2 to 1 and 2 to 3	0.1	Crisis every 10 years
$\zeta_{L,M}, \zeta_{H,M}$	Rate from 1 to 2 and 3 to 2	0.5	Average crisis duration 2 years

Table: Economic Parameters.

Neural Network Parameter Values

Parameter	Value
Number of layers	8
Neurons per layer	100
Activation function	GELU(\cdot)
Initial learning rate	10^{-4}
Final learning rate	10^{-6}
Initial sample size per epoch	256
Sample size per epoch	1024
Convergence threshold for target calibration	10^{-6}

Table: Neural network parameters

Endogenous Yield Curve [back](#)

