# Deep Learning for Search And Matching Models

# (a.k.a. "DeepSAM")

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# Introduction

- Heterogeneity and aggregate shocks are important in markets with search frictions.
- Most search and matching (SAM) models with heterogeneous agents study:
  1. Deterministic steady state (e.g. Shimer-Smith '00),
  2. Aggregate fluctuations, but impose restrictions to eliminate distribution from state space (e.g. "block recursivity" in Menzio-Shi '11, Lise-Robin '17; Lagos-Rocheteau '09).
- We present SAM models as high-dim. PDEs with distribution & agg. shocks as states

  …and develop a new deep learning method, DeepSAM, to solve them globally.
- We extend the method for internal calibration using SMM in efficient compute time.

# This Paper

- Develop method and apply to "canonical" search models with aggregate shocks:
  1. Shimer-Smith/Mortensen-Pissarides model with two-sided heterogeneity.
  2. Lise-Robin on-the-job search (OJS) model with worker bargaining power.
  3. Duffie-Garleanu-Pederson OTC model with asset and investor heterogeneity.
- High accuracy + efficient compute time for both solution and estimation.
- This talk: study unemployment and wage dynamics during business cycles and crises:
  1. Distribution feedback most important when aggregate shocks are asymmetric.
  2. Low-type worker wages are more procyclical.
  3. Low-type workers benefit more from longer expansions ("Okun's hypothesis").
  4. Block recursive model assumptions amplify IRFs.

# Literature

- Deep learning in macro; for incomplete market heterogeneous agent models (HAM) (e.g. Maliar et al '21, Azinovic et al '22, Kahou et al '21, Han-Yang-E '21; Fernández-Villaverde et al '19, Huang '22, Gu-Laurière-Merkel-Payne '23, among others)
    - This paper: search and matching (SAM) models.

| | Distribution | Distribution impact on decisions |
|---|---|---|
| HAM | Asset wealth and income | Via aggregate prices |
| SAM | Type (productivity) of agents in two sides of matching | Via matching process with other types |

- Search model with business cycle (e.g. Shimer '05, Menzio-Shi '11, Lise-Robin '17.)
    - This paper: keep distribution in the state vector.

- Integrate deep learning based solution methods with calibration and estimation (e.g. F-V et al '19, Chen et al '23, Kase et al '23, Friedl et al '23)
    - This paper: standard internal calibration practice for quantitative macro.

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# Shimer-Smith/Mortensen-Pissarides with Two-sided Heterogeneity

- Continuous time, infinite horizon environment.
- Workers $x \in [0,1]$ with exog density $g_t^w(x)$; Firms $y \in [0,1]$ with $g_t^f(y)$ by free entry:
  - Unmatched: unemployed workers get benefit $b$; vacant firms pay vacancy cost $c$.
  - Matched: type $x$ worker and type $y$ firm produce output $z_t f(x,y)$.
  - $z_t$: follows continuous time Markov Chain.
  - Firms make entry decision and then draw a type y from uniform distribution $[0,1]$. More
- Meet randomly at rate $m(\mathcal{U}_t, \mathcal{V}_t)$, $\mathcal{U}_t$ is total unemployment, $\mathcal{V}_t$ is total vacancies.
- Upon meeting, agents choose whether to accept the match:
  - Match surplus $S_t(x,y)$ divided by Nash bargaining: worker gets fraction $\beta$.
  - Match acceptance decision indicator $\alpha_t(x,y) \in \{0,1\}$. Exogenous dissolve rate $\delta(x,y,z)$.
- Equilibrium object: $g_t(x,y)$ "density" of matches $\Rightarrow$ unemployed $g_t^u(x)$, vacant $g_t^v(y)$.

# Recursive Equilibrium: Agent Problems

- E.g. worker idiosyncratic state $= x$, Aggregate states $= (z, g(x, y))$.
  Let $dg/dt = \breve{\mu}^g(x, y, z, g)$ denote agents' belief about the evolution of $g$.

- Hamilton-Jacobi-Bellman equation (HJBE) for unemployed worker's value $V^u(x, z, g)$:

$$\rho V^u(x,z,g) = b + \frac{m(z,g)}{\mathcal{U}(z,g)} \int \underbrace{\overbrace{\alpha(x,\tilde{y},z,g)}^{\text{acceptance decision}} \left(\overbrace{V^e(x,\tilde{y},z,g)}^{\text{employed value}} - V^u(x,z,g)\right)}_{\text{change of value conditional on match}} \frac{g^v(\tilde{y})}{\mathcal{V}(z,g)} d\tilde{y}$$

$$+ \lambda(z)(V^u(x,\tilde{z},g) - V^u(x,z,g)) + \underbrace{D_g V^u(x,z,g)}_{\substack{\text{Frechet derivative:} \\ \text{how change of } g \text{ affects } V}} \cdot \underbrace{\breve{\mu}^g}_{\substack{\text{Belief about} \\ g \text{ evolution}}}$$

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$+$ HJBEs for employed workers $V^e$, vacant firms $V^v$, and producing firms $V^p$ More.

$\Rightarrow$ Match surplus $S(x,y,z,g) := V^p(x,y,z,g) - V^v(y,z,g) + V^e(x,y) - V^u(x,z,g)$

$\Rightarrow \alpha(x,y,z,g) = \mathbb{1}\{S(x,y,z,g) > 0\}$

# Recursive Equilibrium: Agent Problems and Distribution Evolution

- E.g. worker idiosyncratic state $= x$, Aggregate states $= (z, g(x,y))$.
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  + HJBEs for employed workers $V^e$, vacant firms $V^v$, and producing firms $V^p$ More.

- Given $\alpha$ choices, evolution of $g_t(x,y)$ is given by Kolmogorov forward equation (KFE):

$$\mu^g(x,y,z,g) := \frac{dg_t(x,y)}{dt} = \underbrace{-\delta(x,y,z) g_t(x,y)}_{\substack{\text{Outflow:} \\ \text{Breakup of matches}}} + \underbrace{\frac{m(z,g)}{\mathcal{U}(z,g)\mathcal{V}(z,g)} \alpha(x,y,z,g) g_t^v(y) g_t^u(x)}_{\text{Inflow: Creation of new matches}}$$

# Recursive Equilibrium: Distribution Evolution

A **(recursive) equilibrium** is a collection of functions $\{V^u, V^e, V^v, V^p,\ w, \alpha, g^f\}$ of the state variables $(z, g)$ such that:

1. Given beliefs about the evolution of $g_t$, $(V^u, V^e, V^v, V^p, \alpha)$ solve the HJB equations,
2. The division of surplus satisfies Nash bargaining,
3. $g^f$ satisfies the free entry condition details, and
4. Agent beliefs about the evolution of $g_t$ are consistent: $\breve{\mu}^g = \mu^g$.

# Recursive Characterization For Equilibrium Surplus

- We characterize equilibrium with master equation for surplus: C.f. block recursive

$$
\begin{aligned}
\rho S(x,y,z,g) &= zf(x,y) - \delta(x,y,z)S(x,y,z,g) \\
&+ c - (1-\beta)\frac{m(z,g)}{\mathcal{V}(z,g;S)}\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)\frac{g^u(\tilde{x})}{\mathcal{U}(z,g)}d\tilde{x} \\
&- b - \beta\frac{m(z,g)}{\mathcal{U}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)\frac{g^v(\tilde{y})}{\mathcal{V}(z,g;S)}d\tilde{y} \\
&+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + D_g S(x,y,z,g)\cdot\mu^g(z,g)
\end{aligned}
$$

- High-dim PDEs with distribution in state: hard to solve with conventional methods.

# Finite Type Approximation

- Approximate $g(x, y)$ on finite types: $x \in \mathcal{X} = \{x_1, \dots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \dots, y_{n_y}\}$. (Other approaches of discretization: projection and finite population More)
- Finite state approximation $\Rightarrow$ analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \le n_x, j \le n_y}$
- Approximated master equation for surplus:

$$
\begin{aligned}
0 = \mathcal{L}^S S(x,y,z,g) &= -(\rho+\delta) S(x,y,z,g) + z f(x,y) + c - b \\
&- (1-\beta) \frac{m(z,g)}{\mathcal{V}(z,g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z,g)} \\
&- \beta \frac{m(z,g)}{\mathcal{U}(z,g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z,g)} \\
&+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x,y,z,\{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g)
\end{aligned}
$$

where the acceptance decision approximated by $\alpha(x,y,z,g) = (1 + e^{-\xi S(x,y,z,g)})^{-1}$

# Algorithm for Solving the Master Equation

1. Approximate surplus by neural network $S(x, y, z, g) \approx \widehat{S}(x, y, z, g; \boldsymbol{\Theta})$. Function form
2. Start with initial parameter guess $\boldsymbol{\Theta}^0$. At iteration $n$ with $\boldsymbol{\Theta}^n$:
   2.1 Generate $K$ sample points, $Q^n = \left\{ \left( x_k, y_k, z_k, \{g_{ij,k}\}_{i \le n_x, j \le n_y} \right) \right\}_{k \le K}$.
   2.2 Calculate the average mean squared error of surplus master equation on sample points:
   $$L\left(\boldsymbol{\Theta}^n, Q^n\right) := \frac{1}{K} \sum_{k \le K} \left| \mathcal{L}^S \widehat{S}\left( x_k, y_k, z_k, \{g_{ij,k}\}_{i \le n_x, j \le n_y} \right) \right|^2$$
   2.3 Update NN parameters with stochastic gradient descent (SGD) or variants:
   $$\boldsymbol{\Theta}^{n+1} = \boldsymbol{\Theta}^n - \zeta^n \nabla_{\boldsymbol{\Theta}} L\left(\boldsymbol{\Theta}^n, Q^n\right)$$
   2.4 Repeat until $L\left(\boldsymbol{\Theta}^n, Q^n\right) \le \epsilon$ with precision threshold $\epsilon$.
3. Once $S$ is solved, we have $\alpha$ and can solve for worker and firm value functions.

# Comment: Some Technical Details

- ***Q.*** *How do we choose where to sample?*
  - We start by drawing distributions "between" steady states for different fixed $z$.
  - Can move to simulation based sampling once the error is small.
  - Additional discussion of sampling approaches. More
- ***Q.*** *Why global solution?* Non-linear approximation across broad state space.
- ***Q.*** *Which equilibrium concept?*
  - We solve for a rational expectations equilibrium
    $\Rightarrow$ requires solving the master equation where $\mu^g = \breve{\mu}^g$ on and off equilibrium path.
  - Different to "reinforcement learning" where agents optimize in response to simulations
    $\Rightarrow$ $\mu^g \neq \breve{\mu}^g$ does not necessarily hold, especially off equilibrium path. More

Neural networks do not train in seconds.

So how can we choose structural parameters to match data?

# Internal Calibration with Simulated Method of Moments

- So far, we have solved:

$$0 = \mathcal{L}^S S(x, y, z, \underline{g})$$

- For internal calibration, we include economic parameters $\boldsymbol{\Psi}$ as pseudo states and solve extended master equation:

$$0 = \mathcal{L}^S S(x, y, z, \underline{g}, \boldsymbol{\Psi})$$

- Simulate the model under different structural parameter vectors $\{\boldsymbol{\Psi}_l\}_l$ and ... fit a surrogate function mapping structural parameters to simulated moments.

- Use surrogate function to find the parameters that match data moments.

More

# Extensions

- On-the-job search: More
  - Bertrand competition between firms (Postel-Vinay and Robin (2002)).
  - Nash bargaining over incremental surplus from moving.
- Match specific idiosyncratic productivity shocks. More
- Endogenous separation. More
- Variations on free entry and exit. More
- Agent type switching. More
- Asset trade rather than worker-firm production (e.g., OTC search markets). More
- Non-transferable utility. More

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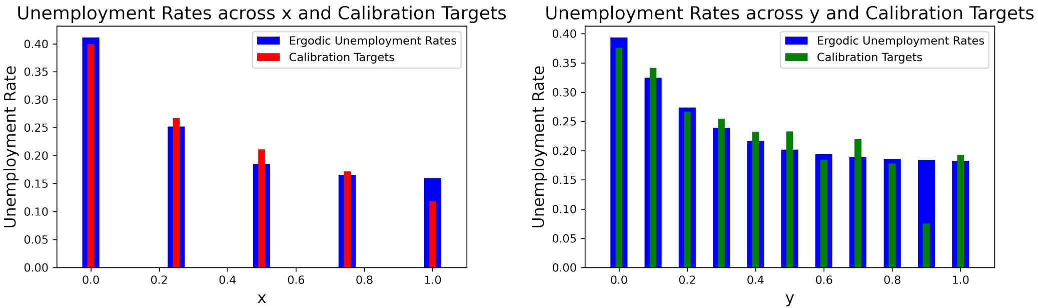
# Crisis Shock in Shimer-Smith (2000)

- We test the solution method for the baseline model
  ...with a recession where the job separation rate significantly increases.

- We calibrate the separation rate $\delta(x, y, z)$ to match the heterogeneous employment declines during COVID-19 for different workers/firms (Cajner et al., 2020). More




- Worker and firm types: $(n_x, n_y) = (5, 11) \Rightarrow 58$ dimensional PDE for $S(x, y, z, g)$

# Numerical Accuracy of Solution Technique

Technically, our method achieves high numerical accuracy using a number of measures:

- Small numerical error. Use DeepSAM to solve the problem (58 dimensional PDE), and compute loss across long simulation. MSE: $10^{-7} \sim 10^{-6}$. More

- Verification on models with known solution. Use DeepSAM to solve model without aggregate shocks (57 dimensional PDE) and obtain steady state solution. Compare to steady state solution from conventional methods. MSE Difference: $10^{-6} \sim 10^{-5}$. More

# Q1. Does Distribution Feedback Matter? Evidence From COVID-19

- Aggregate dynamics with and without distribution feedback to agent decisions:

$$\text{Full dynamics: } \frac{dg_t(x,y)}{dt} = -\delta(x,y,z_t)g_t(x,y) + \frac{m_t(z,g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x,y,z_t,g_t)g_t^u(x)g_t^v(y)$$

$$\text{No distribution feedback: } \frac{dg_t(x,y)}{dt} = -\delta(x,y,z_t)g_t(x,y) + \frac{m_t(z,g_t)}{\mathcal{U}_t(g_t)\mathcal{V}_t(g_t)}\alpha(x,y,z_t,g^{\text{ergodic}})g_t^u(x)g_t^v(y)$$

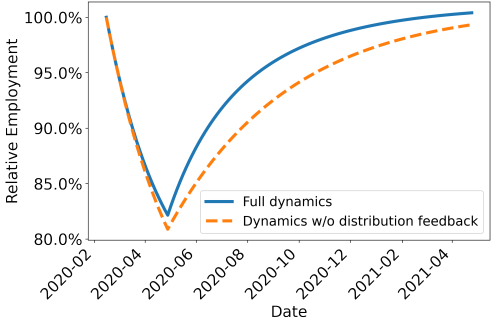
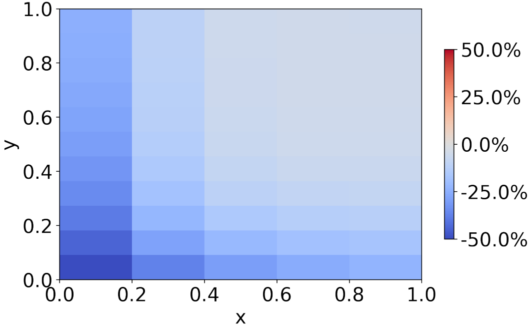


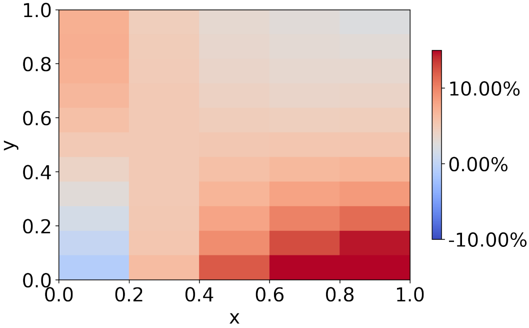

Figure: Employment drop after the COVID-19 shock

# A1. Mechanism: Asymmetric Shock Makes Agents "Less Picky"

Intuition: COVID-19 lead to relatively more low-type unemployed workers and firms
⇒ workers and firms are are less willing to wait for a good match.




(a) Distribution difference: after COVID-19 shock compared to ergodic SS.




(b) Acceptance difference: after COVID-19 shock compared to ergodic SS.

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# Solving and Estimating OJS Model

- Lise-Robin (2017) with worker bargaining power $\beta > 0$ Model details
- Discretization: $(n_x, n_y) = (7, 8) \Rightarrow S(x, y, z, g)$ is 59-dimensional.

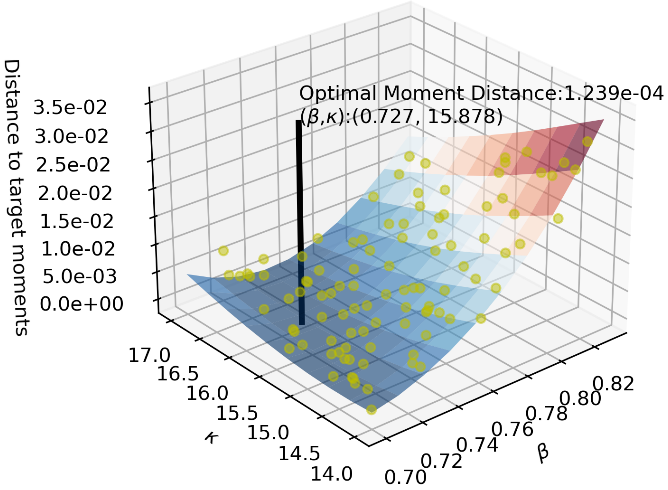
# Solving and Estimating OJS Model

- Lise-Robin (2017) with worker bargaining power $\beta > 0$ Model details
- Discretization: $(n_x, n_y) = (7, 8) \Rightarrow S(x, y, z, g)$ is 59-dimensional.
- Internal calibration for $\{\beta, \kappa, c, b, \delta\}$: solve the model over economic parameter space, and simulate across 10,000 parameter combinations for simulated method of moments.

| | Solution Given the Value of Structural Parameters | Solution with Structural Parameters as Pseudo-states | Simulation & Training Surrogate Model | Simulated Method of Moments | Entire Estimation |
|---|---|---|---|---|---|
| MSE Loss | $1.97 \times 10^{-6}$ | $4.8 \times 10^{-6}$ | $6.13 \times 10^{-7}$ | $1.24 \times 10^{-4}$ | - |
| Time | 55min | 4h 1min | 1h 3min | 1.4min | **5h 5min** |

| Moments | $\mathbb{E}[U]$ | $\mathbb{E}[V]$ | $\mathbb{E}[E2E]$ | $\mathbb{E}[U2E]$ | $\mathbb{E}[E2U]$ |
|---|---|---|---|---|---|
| Data | 0.058 | 0.037 | 0.025 | 0.468 | 0.025 |
| Model | 0.058 | 0.037 | 0.026 | 0.431 | 0.026 |

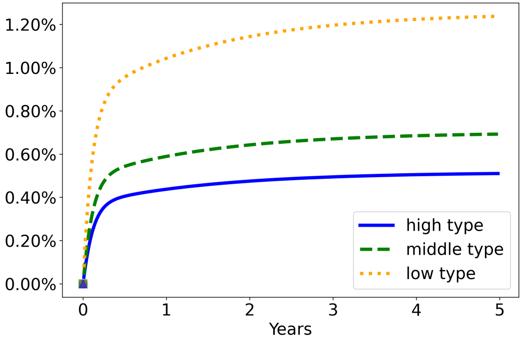
# Estimation of OJS Model: Visualization in 2D



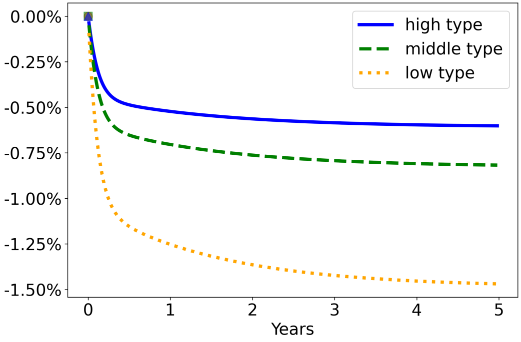

Target moment: $\mathbb{E}[U], \mathbb{E}[V]$. Parameter: matching efficiency $\kappa$, worker bargaining power $\beta$.

# Q2. Are wage dynamics heterogeneous across distribution? A2. Yes

- In Lise-Robin: "wages cannot be solved for exactly... need to solve worker values where the distribution of workers across jobs is a state variable."
- DeepSAM can solve wage dynamics with rich heterogeneity.
- Low-type worker wages more procyclical.




(a) IRF to positive shocks




(b) IRF to negative shocks

# Q3. Who benefits more over a longer expansion? A3. Low Types

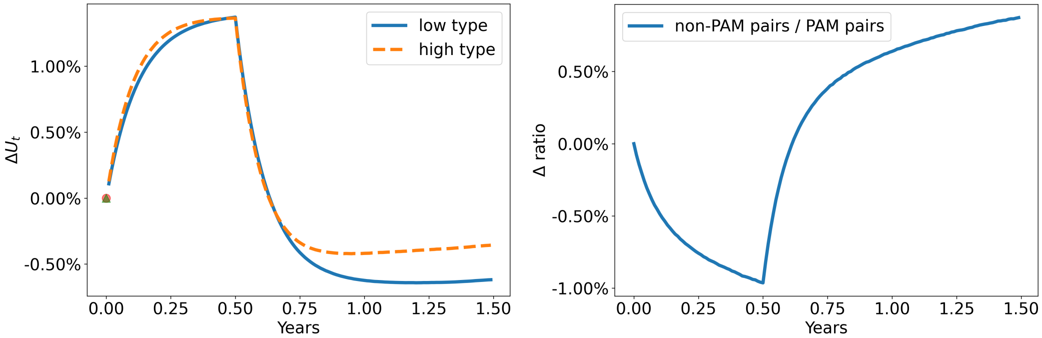



Figure: Left: $\Delta U_t$ for different workers. Right: expansion $\Rightarrow$ positive assortative matching $\downarrow$.

- Mechanism: sorting weakens over time in expansions, high-type firms more inclined to hire low-type workers during longer expansions.
- Important that workers & firms understand the distribution of matches over time.

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# Conclusion

- We develop an integrated global solution and estimation method, DeepSAM, to search and matching models with heterogeneity and aggregate shocks.
- We apply DeepSAM to three general setups in labor and financial search models (without simplification assumptions). OTC model details
- The method is accurate, solves new variables (e.g. wage), and generates novel economic insights.
- A foundational tool for a large literature with more applications:
  - Richer models in labor, financial, and money search, combined with rich micro data.
  - Spatial and network models with aggregate uncertainty (similar math structure).

Thank You!

# Approximate $S$ by Neural Network (Feed Forward, Fully Connected)

- Let $\boldsymbol{\omega} = (x, y, z, g)$. We approximate surplus $S(\boldsymbol{\omega})$ by neural network with form:

$$\begin{aligned}
\boldsymbol{h}^{(1)} &= \phi^{(1)}(W^{(1)}\boldsymbol{\omega} + \boldsymbol{b}^{(1)}) && \ldots\text{Hidden layer 1} \\
\boldsymbol{h}^{(2)} &= \phi^{(2)}(W^{(2)}\boldsymbol{h}^{(1)} + \boldsymbol{b}^{(2)}) && \ldots\text{Hidden layer 2} \\
&\vdots \\
\boldsymbol{h}^{(H)} &= \phi^{(H)}(W^{(H)}\boldsymbol{h}^{(H-1)} + \boldsymbol{b}^{(H)}) && \ldots\text{Hidden layer H} \\
S &= \sigma(\boldsymbol{h}^{(H)}) && \ldots\text{Surplus}
\end{aligned}$$

- Terminology (our parameter choices are in blue):
  - $H$: is the number of *hidden layers*, $(H = 5)$
  - Length of vector $\boldsymbol{h}^{(i)}$: number of *neurons* in hidden layer $i$, $(Length = 50)$
  - $\phi^{(i)}$: is the *activation function* for hidden layer $i$, $(\phi^{i} = tanh)$
  - $\sigma$: is the *activation function* for the final layer, $(\sigma = tanh)$
  - $\boldsymbol{\Theta} = (W^{1}, \ldots W^{(H)}, b^{(1)}, \ldots, b^{(H)})$ are the *parameters*,

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# Free Entry Condition

- Firms make entry decision and then draw type y from uniform distribution $[0,1]$:

$$0 = \mathbb{E}[V_t^v] = \int V^v(\tilde{y}, z, g)d\tilde{y}. \tag{1}$$

- As the matching function is homothetic $\frac{m(z_t,g_t)}{\mathcal{V}_t} = \widehat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right)$, combining free entry condition with HJB equation for $V^v$ gives:

$$\widehat{m}\left(\frac{\mathcal{V}_t}{\mathcal{U}_t}\right) = \frac{\rho c}{\int\int \alpha(\tilde{x},\tilde{y})\frac{g_t^u(\tilde{x})}{\mathcal{U}_t}(1-\beta)S_t(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}} \Rightarrow \quad \mathcal{V}_t = \mathcal{U}_t\widehat{m}^{-1}(\cdots) \tag{2}$$

  where $g_t^u = g_t^w - \int g_t^m(x,y)dy$ and so the RHS can be computed from $g_t^m$ and $S_t$.
- $g_t^f = \mathcal{V}_t + \mathcal{P}_t$, where $\mathcal{V}_t$ and $\mathcal{P}_t$ can be expressed in terms of $g$ and $S$.
- With free entry condition, the master equation expression for surplus takes the same form as without free entry, but with different expressions of $g^f(y)$.

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# Recursive Equilibrium Part II: Other Equations

- Hamilton-Jacobi-Bellman equation (HJBE) for employed worker's value $V^e(x,y,z,g)$:

$$\rho V^e(x,y,z,g) = w(x,y,z,g) + \delta(x,y,z)\left(V^u(x,z,g) - V^e(x,y,z,g)\right) + \lambda_{z\tilde{z}}(V^e(x,y,\tilde{z},g) - V^e(x,y,z,g)) + D_g V^e(x,y,z,g) \cdot \breve{\mu}^g$$

- HJBE for a vacant firm's value $V^v(y,z,g)$:

$$\rho V^v(y,z,g) = -c + \frac{m(z,g)}{\mathcal{V}(z,g)} \int \alpha(\tilde{x},y,z,g)(V^p(\tilde{x},y,z,g) - V^v(y,z,g)) \frac{g^u(\tilde{x})}{\mathcal{U}(z,g)} d\tilde{x} + \lambda_{z\tilde{z}}(V^v(x,\tilde{z},g) - V^v(x,z,g)) + D_g V^v(y,z,g) \cdot \breve{\mu}^g$$

- HJBE for a producing firm's value $V^p(x,y,g)$:

$$\rho V^p(x,y,z,g) = zf(x,y) - w(x,y,z,g) + \delta(x,y,z)(V^v(y,z,g) - V^p(x,y,z,g)) + \lambda_{z\tilde{z}}(V^p(x,y,\tilde{z},g) - V^p(x,y,z,g)) + D_g V^p(x,y,z,g) \cdot \breve{\mu}^g$$

back

# Comparison to Other Heterogeneous Agent Search Models

- Lise-Robin ’17: sets $\beta = 0$ (and other conditions, including Postal-Vinay Robin style Bertrand competition for workers searching on-the-job)

$$S(x, y, z, g) = S(x, y, z), \quad \alpha(x, y, z, g) = \alpha(x, y, z)$$

- Menzio-Shi ’11: competitive search (directed across a collection of sub-markets):

$$S(x, y, z, g) = S(x, y, z)$$

- We look for a solution for $S$ and $\alpha$ in terms of the distribution $g$.

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# Modification 1: Finite Type Approximation

- Approximate $g(x, y)$ on finite types: $x \in \mathcal{X} = \{x_1, \ldots, x_{n_x}\}$, $y \in \mathcal{Y} = \{y_1, \ldots, y_{n_y}\}$.
- Finite state approximation $\Rightarrow$ analytical (approximate) KFE: $g \approx \{g_{ij}\}_{i \leq n_x, j \leq n_y}$
- Approximated master equation for surplus:

$$
\begin{aligned}
0 = \mathcal{L}^S S(x, y, z, g) &= -(\rho + \delta) S(x, y, z, g) + z f(x, y) - b \\
&- (1 - \beta) \frac{m(z, g)}{\mathcal{V}(z, g)} \frac{1}{n_x} \sum_{i=1}^{n_x} \alpha(\tilde{x}_i, y, z, g) S(\tilde{x}_i, y, z, g) \frac{g^u(\tilde{x}_i)}{\mathcal{U}(z, g)} \\
&- \beta \frac{m(z, g)}{\mathcal{U}(z, g)} \frac{1}{n_y} \sum_{j=1}^{n_y} \alpha(x, \tilde{y}_j, z, g) S(x, \tilde{y}_j, z, g) \frac{g^v(\tilde{y}_j)}{\mathcal{V}(z, g)} \\
&+ \lambda(z) (S(x, y, \tilde{z}, g) - S(x, y, z, g)) + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \partial_{g_{ij}} S(x, y, z, \{g_{ij}\}_{i,j}) \mu^g(\tilde{x}_i, \tilde{y}_j, z, g)
\end{aligned}
$$

# Modification 2: Approximate Discrete Choice

- In the original model,

$$\alpha(x,y,z,g) = \mathbb{1}\{S(x,y,z,g) > 0\}$$

- Discrete choice $\alpha \Rightarrow$ discontinuity of $S(x,y,z,g)$ at some $g$.

- To ensure master equation well defined & NN algorithm works, we approximate with

$$\alpha(x,y,z,g) = \frac{1}{1+e^{-\xi S(x,y,z,g)}}$$

- Interpretation: logit choice model with utility shocks $\sim$ extreme value distribution. ($\xi \to \infty \Rightarrow$ discrete choice $\alpha$.)

# Finite Dimensional "Distribution" Approximation ($a =$ idio. state)

| | Finite Population | Discrete State | Projection |
|---|---|---|---|
| Params $\hat{\varphi}$ | Agent states $\hat{\varphi}_t = \{(a_t^i)\}_{i \leq N}$ | Masses on grid $\hat{\varphi}_{i,t}, \ \forall (a^i)_{i \leq N}$ | Basis coefficients $\hat{\varphi}_{i,t}, \ \forall b_i(a)\|_{i \leq N}$ |
| Dist. approx. | $\frac{1}{N} \sum_{i=1}^{N} \delta_{(a_t^i)}$ | $\sum_{i=1}^{N} \hat{\varphi}_{i,t} \delta_{(a^i)}$ | $\sum_{i=0}^{N} \hat{\varphi}_{i,t} b_i(a) \approx \mu_g\left(a, z, \sum_{i=1}^{N} \hat{\varphi}_{i,t} b_i\right)$ |
| KFE approx. ($\mu^{\hat{\varphi}}$) | Evolution of other agents' states | Evolution of mass between grid points (e.g. finite diff.) | Evolution of projection coefficients (least squares) |

Finite population works well for Walrasian markets; less well for search Back

# Sampling Approaches

- Sampling $(a, z, \zeta, K)$: draw from uniform distribution, then add draws where error high.

- Sampling the parameters in the distribution approximation $(\hat{\varphi}^i)_{i \leq N}$:
    - *Moment sampling:*
        1. Draw samples for selected moments of the distribution (that are important for $\hat{Q}(z, \hat{\varphi})$).
        2. Sample $\hat{\varphi}$ from a distribution that satisfies the moments drawn in the first step.
    - *Mixed steady state sampling:*
        1. Solve for the steady state for a collection of fixed aggregate states $z$.
        2. Draw random, perturbed mixtures of this collection of steady state distributions.
    - *Ergodic sampling:*
        1. Simulate economy using current value function approximation.
        2. Use simulated distributions as training points.

Back

# Comment 2: Different NN Techniques $\Leftrightarrow$ Different Eqm Concepts

Borrowing from Sargent (2008): Back

- **Rational expectations eqm** (this paper; $\mu^g = \breve{\mu}^g$):
  - Agent "perceived law of motion (PLM)" is the *true law of motion* if agents follow PLM
  - True *on and off the equilibrium path* $\Rightarrow$ requires solving the master equation.
- **Self-confirming eqm** (some model-based reinforcement learning, e.g. DeepHAM)
  - PLM is *best statistical fit to data generated by economy* where agents follow that PLM.
  - So, agents would never reject a misspecification test
  - but may have off-equilibrium beliefs that differ from rational expectations ($\mu^g \neq \breve{\mu}^g$)
  - $\Rightarrow$ can be solved by updating reaction functions.
- **Misspecified belief eqm**
  - Agents have a "perceived law of motion" for the equilibrium variables in the economy
  - ...that *satisfies a fixed parametric form* but doesn't satisfy a misspecification test.

# Calibration and Estimation Challenges

- Let $\boldsymbol{\Psi} \in \Omega^{\Psi}$ be the structural parameters to be calibrated internally.
- Let $\breve{\varphi} = (\breve{\varphi}_1, \ldots, \breve{\varphi}_N)$ be the $N \times 1$ data moments that we want to match.
- Let $\phi(\boldsymbol{\Psi}) = (\phi_1(\boldsymbol{\Psi}), \ldots, \phi_N(\boldsymbol{\Psi}))$ be the corresponding simulated model moments.
- **Computation challenge**: need to solve $\phi(\boldsymbol{\Psi})$ for many structural parameters $\boldsymbol{\Psi}$.
- **Solution**: include $\boldsymbol{\Psi}$ as a pseudo state vector and solve extended master equation:

$$0 = \mathcal{L}^S S(x, y, z, \underline{g}, \boldsymbol{\Psi})$$

  - Approximate extended surplus function by NN: $S(x, y, z, g, \boldsymbol{\Psi}) \approx \widehat{S}(x, y, z, g, \boldsymbol{\Psi}; \boldsymbol{\Theta})$.
  - Fit $\widehat{S}$ using same approach as before but sampling over states & structural parameters.

Back

# Simulated Method of Moments

- Fit surrogate neural network $\hat{\mathbf{\Phi}}$ mapping structural parameters to simulated moments:

$$\mathbf{\Psi} \mapsto \hat{\mathbf{\Phi}}(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi}) = \left(\hat{\phi}_1(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi}), \ldots, \hat{\phi}_N(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi})\right), \qquad \mathbf{\Theta}^{\Phi} \text{ are NN parameters}$$

  - Simulate the model under many different structural parameter vectors $\{\mathbf{\Psi}_l\}_l$
  - Compute the resulting model moments $\{(\phi_1(\mathbf{\Psi}_l), \ldots, \phi_N(\mathbf{\Psi}_l))\}_l$
  - Fit neural network parameters $\mathbf{\Theta}^{\Phi}$ to approx relationship $\hat{\mathbf{\Phi}}(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi})$ using simulations.

- **Think**: surrogate NN has incorporated the economic structure implicitly into $\mathbf{\Theta}^{\Phi}$

- With $\hat{\mathbf{\Phi}}(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi})$, we can find the parameters to match data moments:
  (i.e. perform simulated method of moments)

$$\mathbf{\Psi}^* = \arg\min_{\mathbf{\Psi}} \sum_{i=1}^{N} \omega_i \left( \frac{\breve{\varphi}_i - \hat{\phi}_i(\mathbf{\Psi}; \mathbf{\Theta}^{\Phi})}{\breve{\varphi}_i} \right)^2.$$

Back

# Calibration and Estimation

1. Fit extended surplus function by NN: $S(x, y, z, g, \boldsymbol{\Psi}) \approx \widehat{S}(x, y, z, g, \boldsymbol{\Psi}; \boldsymbol{\Theta})$.
   Start with initial parameter guess $\boldsymbol{\Theta}^0$. At iteration $n$ with $\boldsymbol{\Theta}^n$:
   1.1 Generate $K$ sample points, $Q^n = \left\{ \left( x_k, y_k, z_k, \{g_{ij,k}\}_{i \le n_x, j \le n_y}, \boldsymbol{\Psi}_k \right) \right\}_{k \le K}$.
   1.2 Calculate the average mean squared error of surplus master equation on sample points:
   $$L\left(\boldsymbol{\Theta}^n, Q^n\right) := \frac{1}{K} \sum_{k \le K} \left| \mathcal{L}^S \widehat{S}\left( x_k, y_k, z_k, \{g_{ij,k}\}_{i \le n_x, j \le n_y}, \boldsymbol{\Psi}_k \right) \right|^2$$
   1.3 Update NN parameters with stochastic gradient descent (SGD) method.
2. Fit surrogate NN mapping structural parameters to moments: $\hat{\boldsymbol{\Phi}}(\boldsymbol{\Psi}; \boldsymbol{\Theta}^{\Phi})$.
3. Use surrogate NN to find parameters that match data moments.

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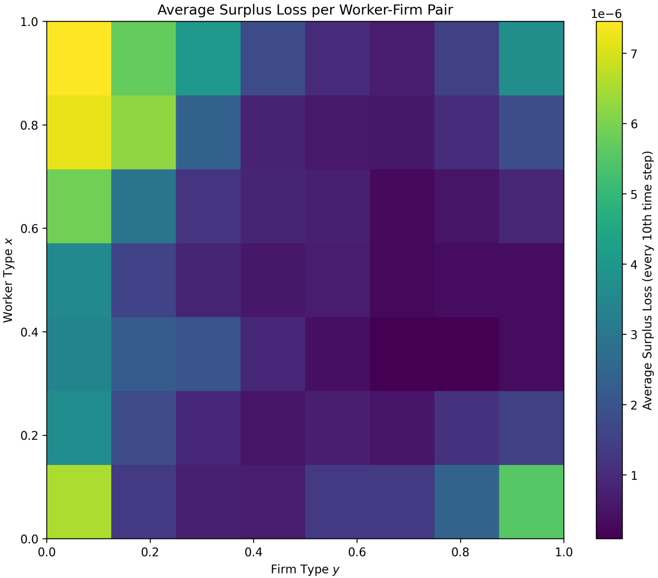
# Calibration of Shimer-Smith Model with Aggregate Shocks back

Frequency: annual.

| Parameter | Interpretation | Value | Target/Source |
|---|---|---|---|
| $\rho$ | Discount rate | 0.05 | Kaplan, Moll, Violante '18 |
| $\delta$ | Job destruction rate | 0.2 | BLS job tenure 5 years |
| $\xi$ | Extreme value distribution for $\alpha$ choice | 2.0 | |
| $f(x, y)$ | Production function for match $(x, y)$ | $0.6 + 0.4\left(\sqrt{x} + \sqrt{y}\right)^2$ | Hagedorn et al '17 |
| $\beta$ | Surplus division factor | 0.72 | Shimer '05 |
| c | Entry cost | 4.86 | Steady state $\mathcal{V}/\mathcal{U} = 1$ |
| $z, \tilde{z}$ | TFP shocks | $1 \pm 0.015$ | Lise Robin '17 |
| $\lambda_z, \lambda_{\tilde{z}}$ | Poisson transition probability | 0.08 | Shimer '05 |
| $\delta, \tilde{\delta}$ | Separation shocks | $0.2 \pm 0.02$ | Shimer '05 |
| $\lambda_\delta, \lambda_{\tilde{\delta}}$ | Poisson transition probability | 0.08 | Shimer '05 |
| $m(\mathcal{U}, \mathcal{V})$ | Matching function | $\kappa \mathcal{U}^\nu \mathcal{V}^{1-\nu}$ | Hagedorn et al '17 |
| $\nu$ | Elasticity parameter for meeting function | 0.5 | Hagedorn et al '17 |
| $\kappa$ | Scale parameter for meeting function | 5.4 | Unemployment rate 5.9% |
| $b$ | Worker unemployment benefit | 0.5 | Shimer '05 |
| $n_x$ | Discretization of worker types | 7 | |
| $n_y$ | Discretization of firm types | 8 | |

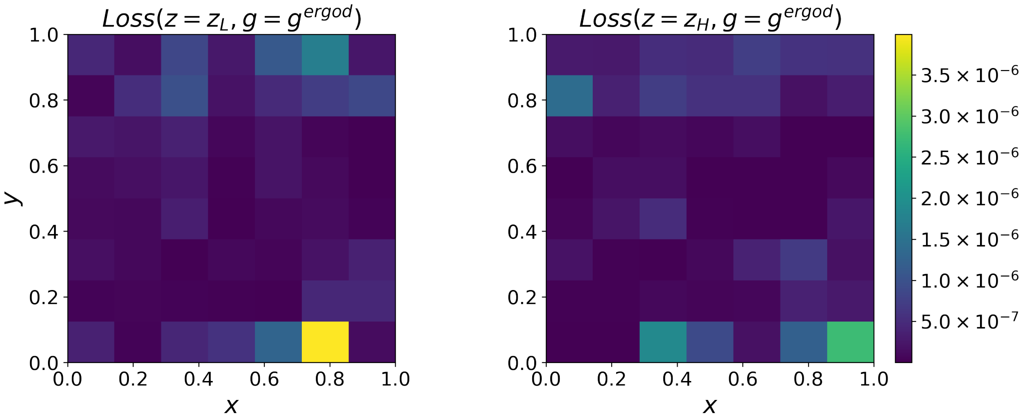
# Numerical Performance: Accuracy I back




- Average master equation MSE along a simulated path. For the optimal parameters found during the estimation process, 100 economy paths are generated of length 3001 time steps, all starting at the deterministic steady state.
- For every 10th timestep of the simulation, calculate the surplus loss for each firm-worker pair, then average the losses over the timesteps.

# Numerical Performance: Accuracy I back

- Mean squared loss as a function of type in the master equations of $S$ (at ergodic g).



# Numerical Performance: Accuracy II

- Compare steady state solution without aggregate shocks to solution using conventional methods.

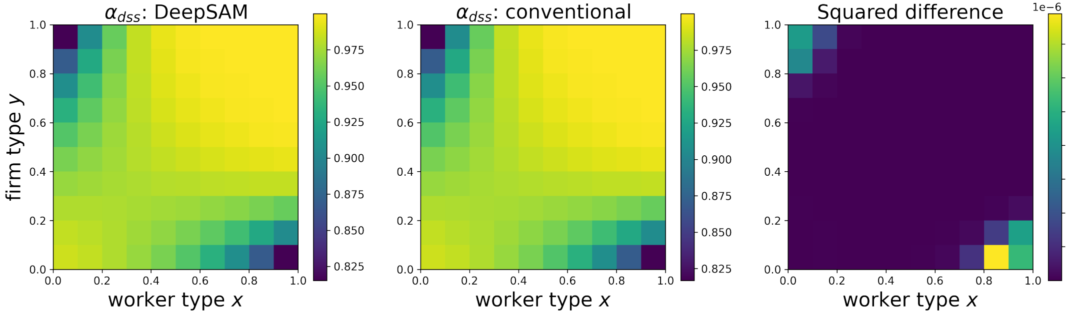
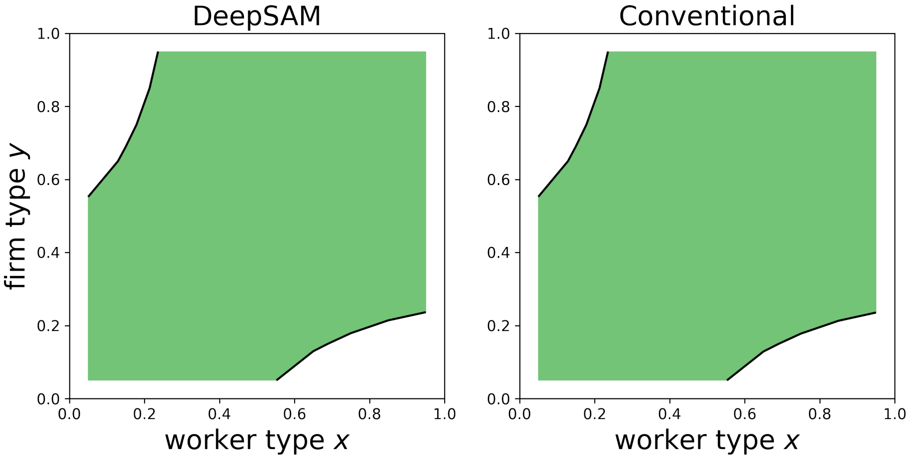



Figure: Comparison with steady-state solution

Comparison for discrete $\alpha$ back

# DeepSAM vs Conventional method at DSS: discrete case

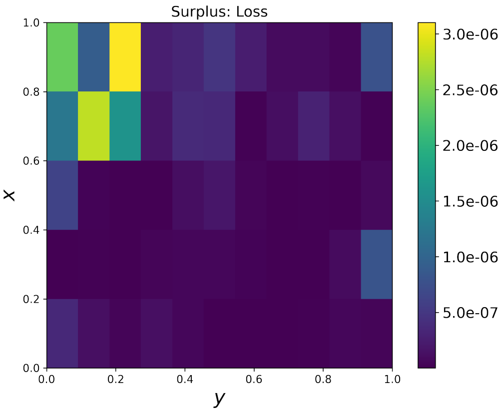



back

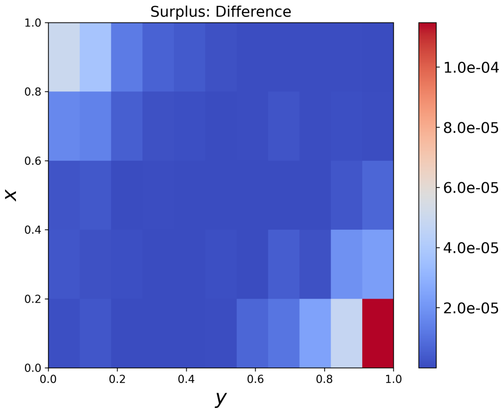
# Calibration of Model with Aggregate Shocks Back

| Parameter | Interpretation | Value | Target/Source |
|---|---|---|---|
| $\rho$ | Discount rate | 0.05 | Interest rate |
| $\xi$ | Extreme value for $\alpha$ choice | 2.0 | |
| $f(x,y)$ | Production for match $(x,y)$ | $0.6+0.4\left(\sqrt{x}+\sqrt{y}\right)^2$ | Hagedorn et al. (2017) |
| $\beta$ | Surplus division factor | 0.72 | Shimer (2005) |
| $m(\mathcal{U},\mathcal{V})$ | Matching function | $\kappa\mathcal{U}^{\nu}\mathcal{V}^{1-\nu}$ | Hagedorn et al. (2017) |
| $\nu$ | Elasticity in meeting function | 0.5 | Hagedorn et al. (2017) |
| $\kappa$ | Scale for meeting function | 5.4 | Unemployment rate |
| $b$ | Worker unemployment benefit | 0.5 | Shimer (2005) |
| c | Entry cost | 4.86 | Steady state $\mathcal{V}/\mathcal{U}=1$ |
| Steady State: | | | |
| $\bar{z}$ | Steady state TFP | 1 | Shimer (2005) |
| $\bar{\delta}$ | Steady state separation rate | 0.2 | BLS job tenure 5 years |
| Exogenous Aggregate Shock Process: | | | |
| $A_D, A_L, A_H$ | TFP levels | 0.985, 0.985, 1.015 | Lise and Robin (2017) |
| $\delta_L, \delta_H$ | Separation rates | 0.18, 0.22 | Shimer (2005) |
| $\delta_D(x,y)$ | TFP and separation at crisis state | 0.6 to 5.2 | Cajner et al. (2020) |
| $\lambda_z$ | Poisson transition probability | 0.4, 0.001 | Shimer (2005) |
| $n_x$ | Discretization of worker types | 5 | Cajner et al. (2020) |
| $n_y$ | Discretization of firm types | 11 | Cajner et al. (2020) |

# Numerical Performance  Back




**(a)** Model with aggregate shock: loss across state space




**(b)** Model without aggregate shock: difference from conventional solution

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# Different Versions of Entry and Exit

1. Free entry or compensated exit of vacant firms *($g^f$ is a jump variable)*.
2. Free entry and exogenous exit for all firms *($g^f$ is a jump variable)*.
3. Free entry and endogenous non-compensated exit of vacant firms *($g^f$ jump variable)*.
4. Gradual entry and exit *($g^f$ is a state variable)*.

Back

# 1. Free entry or compensated exit of vacant firms Back

- KFE matches is undistorted by firm exit because only vacant firms exit:

$$dg_t(x,y) = -\delta(x,y,z)g_t(x,y)dt + \frac{m(z,g)}{\mathcal{U}(z,g)\mathcal{V}(z,g)}\alpha(x,y,z,g)g_t^v(y)g_t^u(x)dt$$

- Master equation is undistorted by firm exit because firms are compensated for exit:

$$\begin{aligned}\rho S(x,y,z,g) =& zf(x,y) - \delta(x,y,z)S(x,y,z,g)\\ &+ c - (1-\beta)\frac{m(z,g)}{\mathcal{V}(z,g;S)}\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)\frac{g^u(\tilde{x})}{\mathcal{U}(z,g)}d\tilde{x}\\ &- b - \beta\frac{m(z,g)}{\mathcal{U}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)\frac{g^v(\tilde{y})}{\mathcal{V}(z,g;S)}d\tilde{y}\\ &+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + D_g S(x,y,z,g)\cdot\mu^g(z,g)\end{aligned}$$

- Firm distribution $g^f$ jumps to ensure free entry condition is satisfied:

$$0 = \mathbb{E}[V_t^v] = \int V^v(\tilde{y},z,g)d\tilde{y}.$$

## 2. Free entry and exogenous exit for all firms Back

- KFE is now directly exposed to the aggregate shock:

$$dg_t(x,y) = -\delta(x,y,z)g_t(x,y)dt + \frac{m(z,g)}{\mathcal{U}\mathcal{V}}\alpha(x,y,z,g)g_t^v(y)g_t^u(x)dt - \sigma(z)g_t(x,y)dZ_t$$

- Master equation incorporates the cost of exogenous exit:

$$\begin{aligned}\rho S(x,y,z,g) &= zf(x,y) - \delta(x,y,z)S(x,y,z,g)\\ &+ c - (1-\beta)\frac{m(z,g)}{\mathcal{V}(z,g;S)}\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)\frac{g^u(\tilde{x})}{\mathcal{U}(z,g)}d\tilde{x}\\ &- b - \beta\frac{m(z,g)}{\mathcal{U}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)\frac{g^v(\tilde{y})}{\mathcal{V}(z,g;S)}d\tilde{y}\\ &+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + \langle D_g S(x,y,z,g), (\mu^g(z,g), \sigma(z))\rangle - \sigma(z)g_t(x,y)dZ_t\end{aligned}$$

- Firm distribution $g^f$ jumps to ensure free entry condition is satisfied:

$$0 = \mathbb{E}[V_t^v] = \int V^v(\tilde{y},z,g)d\tilde{y}.$$

# 3. Free entry and endogenous non-compensated exit of vacant firms

Back

- KFE is undistorted by form exit because only vacant firms exit:

$$dg_t(x,y) = -\delta(x,y,z)g_t(x,y)dt + \frac{m(z,g)}{\mathcal{U}\mathcal{V}}\alpha(x,y,z,g)g_t^v(y)g_t^u(x)dt$$

- Master equation incorporates the cost of exogenous exit:

$$\begin{aligned}
\rho S(x,y,z,g) &= zf(x,y) - \delta(x,y,z)S(x,y,z,g)\\
&+ c - (1-\beta)\frac{m(z,g)}{\mathcal{V}(z,g;S)}\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)\frac{g^u(\tilde{x})}{\mathcal{U}(z,g)}d\tilde{x}\\
&- b - \beta\frac{m(z,g)}{\mathcal{U}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)\frac{g^v(\tilde{y})}{\mathcal{V}(z,g;S)}d\tilde{y} + \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g))\\
&+ \langle D_g S(x,y,z,g), \mu^g(z,g)\rangle - \sigma(z,g)\lambda(z)V^v(x,y,z,g)
\end{aligned}$$

- Firm distribution $g^f$ jumps to ensure free entry condition is satisfied:

$$0 = \mathbb{E}[V_t^v] = \int V^v(\tilde{y},z,g)d\tilde{y}.$$

# 4. Gradual Entry and Exit Back

- KFE for the distribution of vacant firms:

$$dg_t^f(y) = \psi\alpha^f(z,g)dt - \psi\alpha^x(y,z,g)dt$$
$$\alpha^f(z,g) = \begin{cases} 1, & \text{if } \mathbb{E}[V^v(y,z,g)] = \int V^v(\tilde{y},z,g)\pi^e(y)d\tilde{y} > 0 \\ 0, & \text{otherwise} \end{cases}$$

- Surplus function has an additional state (and we need to solve for $V^v$):

$$\begin{aligned}\rho S(x,y,z,g,g^f) &= zf(x,y) - \delta(x,y,z)S(x,y,z,g,g^f) \\ &+ c - (1-\beta)\frac{m(z,g)}{\mathcal{V}(z,g,g^f)}\int \alpha(\tilde{x},y,z,g,g^f)S(\tilde{x},y,z,g,g^f)\frac{g^u(\tilde{x})}{\mathcal{U}(z,g,g^f)}d\tilde{x} \\ &- b - \beta\frac{m(z,g)}{\mathcal{U}(z,g,g^f)}\int \alpha(x,\tilde{y},z,g,g^f)S(x,\tilde{y},z,g,g^f)\frac{g^v(\tilde{y})}{\mathcal{V}(z,g,g^f)}d\tilde{y} \\ &+ \lambda(z)(S(x,y,\tilde{z},g,g^f) - S(x,y,z,g,g^f))\end{aligned}$$

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# On-The-Job Search (OTJS): Additional Environment Features

- Same worker types, firm types, and production function.

- Now all workers search; meeting rate is $m(\mathcal{W}_t, \mathcal{V}_t)$; total search effort is $\mathcal{W}_t := \mathcal{U}_t + \phi\mathcal{E}_t$

- Vacant $\tilde{y}$-firm meets unemployed $x$-worker: Nash bargaining with worker weight $\beta$.

- *OTJS Terms of trade version 1:* Bertrand competition b/n new & incumbent firm (Postel-Vinay and Robin, 2002)

- *OTJS Terms of trade version 2:* Bargaining: when a vacant $\tilde{y}$-firm meets a worker in $(x, y)$ match they Nash bargain over incremental surplus.

  ...if $S_t(x,\tilde{y}) > S_t(x,y)$, worker moves to firm $\tilde{y}$ a gets extra $\beta(S_t(x,\tilde{y}) - S_t(x,y))$

- Endogenous separation $\alpha_t^b(x,y) = 1$ when $S_t(x,y) < 0$.

back Application 2

# Recursive Characterization For Equilibrium Surplus (OTJS Version-1)

- Can characterize equilibrium with the master equation for the surplus:

$$
\begin{aligned}
\rho S(x,y,z,g) &= zf(x,y) - (\delta + \alpha^b(x,y,z,g))S(x,y,z,g) \\
&- \frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\Bigg[(1-\beta)\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)g^u(\tilde{x})d\tilde{x} \\
&-\phi(1-\beta)\int \alpha^p(\tilde{x},y,\tilde{y},z,g)(S(\tilde{x},y,z,g) - S(\tilde{x},\tilde{y},z,g))g(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}\Bigg] \\
&- b + c - \beta\frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)g^v(\tilde{y})d\tilde{y} \\
&+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + D_g S(x,y,z,g)\cdot\mu^g(z,g)
\end{aligned}
$$

where:

$$
\alpha^p(\tilde{x},y,\tilde{y},z,g) := \mathbb{1}\{S(\tilde{x},y,z,g) \geq S_t(\tilde{x},\tilde{y},z,g) \geq 0\}
$$

KFE back Application 2

# Recursive Characterization For Equilibrium Surplus (OTJS Version-2)

- Can characterize equilibrium with the master equation for the surplus:

$$
\begin{aligned}
\rho S(x,y,z,g) &= zf(x,y) - (\delta + \alpha^b(x,y,z,g))S(x,y,z,g) \\
&- \frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\Bigg[(1-\beta)\int \alpha(\tilde{x},y,z,g)S(\tilde{x},y,z,g)g^u(\tilde{x})d\tilde{x} \\
&-\phi(1-\beta)\int \alpha^p(\tilde{x},y,\tilde{y},z,g)(S(\tilde{x},y,z,g) - S(\tilde{x},\tilde{y},z,g))g(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} \\
&+\phi\beta\int \alpha^p(x,\tilde{y},y,z,g)S(x,y,z,g)g^v(\tilde{y})d\tilde{y}\Bigg] \\
&- b + c - \beta\frac{m(z,g)}{\mathcal{W}(z,g)\mathcal{V}(z,g)}\int \alpha(x,\tilde{y},z,g)S(x,\tilde{y},z,g)g^v(\tilde{y})d\tilde{y} \\
&+ \lambda(z)(S(x,y,\tilde{z},g) - S(x,y,z,g)) + D_g S(x,y,z,g)\cdot \mu^g(z,g)
\end{aligned}
$$

where:

$$
\alpha^p(\tilde{x},y,\tilde{y},z,g) := \mathbb{1}\{S(\tilde{x},y,z,g) \geq S_t(\tilde{x},\tilde{y},z,g) \geq 0\}
$$

KFE back

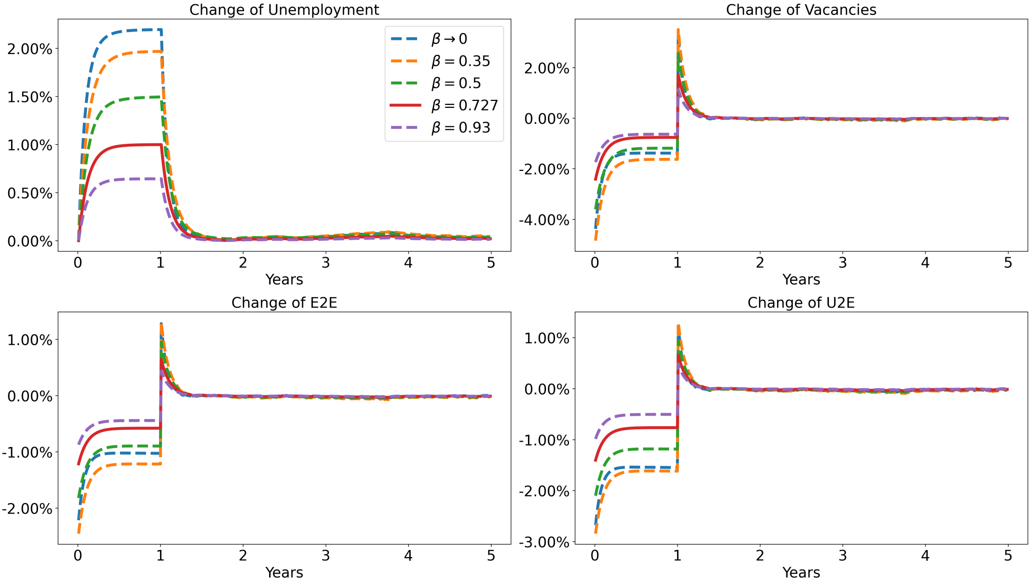
# On-the-job-search: KFE

- The KFE becomes:

$$
\begin{aligned}
dg_t^m(x,y) = & -\delta g_t^m(x,y)dt \\
& - \phi \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} g_t^m(x,y) \int \alpha_t^p(x,y,\tilde{y}) g_t^v(\tilde{y}) d\tilde{y} dt \\
& + \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} \alpha_t(x,y) g_t^u(x) g_t^v(y) dt \\
& + \phi \frac{m(\mathcal{W}_t, \mathcal{V}_t)}{\mathcal{W}_t \mathcal{V}_t} \int \alpha_t^p(\tilde{x},\tilde{y},y) g_t^v(y) \frac{g_t^m(\tilde{x},\tilde{y})}{\mathcal{E}_t} d\tilde{x} d\tilde{y} dt
\end{aligned}
$$

back

# Q4. Block recursive model impact on IRF to negative TFP shock?



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# On-The-Job Search Model with Idiosyncratic Shocks

- Introduce match specific (idiosyncratic) productivity shock $\epsilon$ (Mortensen and Pissarides, 1994) that follows a continuous time Markov chain.
- E.g., production output from matches $F(x, y, z, \epsilon) = z\epsilon f(x, y)$.
- Two sources of endogenous separation: both **idiosyncratic risk** and **aggregate risk**.

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# On-The-Job Search Model with Idiosyncratic Shocks: HJB Equation

$$
\begin{aligned}
&\rho S(x,y,\epsilon,z,g) = \rho(V^p(x,y,\epsilon,z,g) - V^v(y,z,g) + V^e(x,y,\epsilon,z,g) - V^u(x,z,g)) \\
&= F(x,y,z,\epsilon) - (\delta + \eta\alpha^b(x,y,\epsilon,z,g))S(x,y,\epsilon,z,g) - b \\
&- \mathcal{M}^v\mathcal{C}^u \sum_{\tilde{\epsilon}} \pi(\tilde{\epsilon}) \int \alpha(\tilde{x},y,\tilde{\epsilon},z,g)(1-\beta)S(\tilde{x},y,\tilde{\epsilon},z,g)\frac{g^u(\tilde{x})}{\mathcal{U}}d\tilde{x} \\
&- \mathcal{M}^v\mathcal{C}^e \sum_{\tilde{\epsilon},\epsilon} \pi(\epsilon) \int \alpha^p(y,\tilde{x},\epsilon,\tilde{y},\tilde{\epsilon},z,g)\frac{g(\tilde{x},\tilde{y},\tilde{\epsilon})}{\mathcal{E}}(1-\beta)(S(\tilde{x},y,\epsilon,z,g) - S(\tilde{x},\tilde{y},\tilde{\epsilon},z,g))d\tilde{x}d\tilde{y}\Big) \\
&+ \mathcal{M}^e \sum_{\tilde{\epsilon}} \int \alpha^e(x,y,\epsilon,\tilde{y},\tilde{\epsilon},z,g)\beta(S(x,\tilde{y},\tilde{\epsilon},z,g) - S(x,y,\epsilon,z,g))\frac{g^v(\tilde{y})}{\mathcal{V}}d\tilde{y} \\
&- \mathcal{M}^u \int \alpha(x,\tilde{y},\epsilon,z,g)\beta S(x,\tilde{y},\epsilon,z,g)\frac{g^v(\tilde{y})}{\mathcal{V}}d\tilde{y} \\
&+ \lambda(z)(S(x,y,\epsilon,\tilde{z},g) - S(x,y,\epsilon,z,g)) + \lambda(\epsilon,\tilde{\epsilon})(S(x,y,\tilde{\epsilon},z,g) - S(x,y,\epsilon,z,g)) \\
&- (\varsigma + \sigma(z)\lambda(z))(1-\beta)S(y,z,g) + \langle D_g S, \mu^g \rangle
\end{aligned}
$$

# On-The-Job Search Model with Idiosyncratic Shocks: KFE

$$
\begin{aligned}
dg_t(x,y,\epsilon) = & -(\delta+\eta\alpha_t^b(x,y,\epsilon))g_t(x,y)dt \\
& - \underbrace{\mathcal{M}_t^e}_{\substack{\text{Rate. e-worker}\\ \text{meets}}} \sum_{\tilde{\epsilon}} \pi(\tilde{\epsilon}) \underbrace{\int \underbrace{\alpha_t^e(x,y,\epsilon,\tilde{y},\tilde{\epsilon})}_{\text{Prob. accept}} \underbrace{\frac{g_t^v(\tilde{y})}{\mathcal{V}_t}}_{\text{Prob. meet } \tilde{y}} d\tilde{y} \underbrace{g_t(x,y,\epsilon)}_{\text{Mass at } (x,y)}}_{\text{Mass accepting match}} dt \\
& + \underbrace{\mathcal{M}_t^u}_{\substack{\text{Rate. u-worker}\\ \text{meets}}} \pi(\epsilon)\alpha_t(x,y,\epsilon)\frac{g_t^v(y)}{\mathcal{V}_t} g_t^u(x)dt \\
& + \mathcal{M}_t^e \pi(\epsilon) \sum_{\tilde{\epsilon}} \int \alpha_t^e(x,\tilde{y},\tilde{\epsilon},y,\epsilon)\frac{g_t^v(y)}{\mathcal{V}_t} g_t(x,\tilde{y},\tilde{\epsilon})d\tilde{y}dt \\
& + \sum_{\check{\epsilon}\neq\epsilon} \lambda(\check{\epsilon},\epsilon)g_t(x,y,\check{\epsilon})dt - \sum_{\check{\epsilon}\neq\epsilon} \lambda(\epsilon,\check{\epsilon})g_t(x,y,\epsilon)dt. \\
& - \varsigma g_t(x,y,\epsilon) - \sigma(z)g_t(x,y,\epsilon)dZ_t
\end{aligned}
$$

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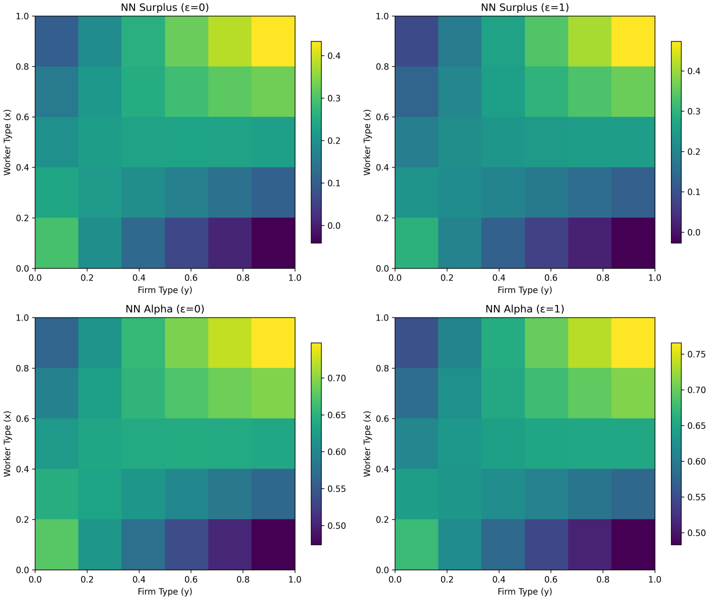
# On-The-Job Search Model with Idiosyncratic Shocks: Solution at DSS



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# Environment: Setting, Bonds, and Households

- Continuous time, infinite horizon environment.

- There are many bonds, $k \in \{1, \dots, K\}$, in positive net supply $s_k$:
  - Every bond pays the same dividend $\delta > 0$.
  - Bond $k$ matures at rate $1/\tau_k$ (average maturity $\tau_k$).

- Populated by a unit-mass continuum of infinitely-lived and risk-neutral investors:
  - An investor can hold either zero or one share of at most one type of asset.
  - Investor type $j \in \{1, \dots, J\}$ gets flow utility $\delta - \psi(j, k)$ from holding bond $k$.
  - Agents **switch type** from $i$ to $j$ at rate $\lambda_{i,j}$.

- Aggregate (default) state $z \in \{z_1, \dots, z_n\}$, switches at rate $\zeta_{z,z'}$.
  At state $z$, asset $k$ pays a fraction $\phi(k, z)$ of the coupon and the principal.

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# Distribution and Bargaining

- An investor's state is made up of her holding cost $j \in \{1, \ldots, J\}$ and her ownership status, for each asset type $k \in \{1, \ldots, K\}$ (owner $o$ or non-owner $n$). Hence the set of investor idiosyncratic states is:

$$A = \{1n, 2n, \ldots, Jn, 1o1, \ldots, 1oK, 2o1, \ldots 2oK, Jo1, \ldots, JoK\} \tag{3}$$

- The rate of contact between investors with states $a$ and $b$ is:

$$\mathcal{M}_{a,b} = \kappa_{a,b} g_a g_b \tag{4}$$

- Agents $a$, $b$ **trade with each other**, engage in generalized Nash bargaining with bargaining power $\beta_{a,b}$.

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# Value Functions for Non-Owners and Owners back

- Value function for non-owner with type $i$ satisfies:

$$\rho_i V(in,g,z) = \sum_a \kappa_{in,a}\alpha(in,a,g,z)\beta_{in,a}S(in,a,z,g)$$
$$+\sum_k \xi_{i,k}(V(iok,g,z)-V(in,g,z))+\sum_{j\neq i}\lambda_{i,j}(V(jn,g,z)-V(in,g,z))$$
$$+\sum_{z'}\zeta_{z,z'}(V(in,g,z')-V(in,g,z))+\sum_{a\in A}\partial_{g_a}V(in,g,z)\mu^g(a,z)$$

- Value function for an investor of type $i$ holding asset $k$ satisfies:

$$\rho_i V(iok,g,z) = \delta\phi(k,z)-\psi(i,k)+\frac{1}{\tau_k}(V(in,g,z)+\pi(k,z)-V(iok,g,z))$$
$$+\sum_a \kappa_{iok,a}\alpha(iok,a,g,z)g_a\beta_{iok,a}S(iok,a,g,z)+\sum_{j\neq i}\lambda_{i,j}(V(jok,g,z)-V(iok,g,z))$$
$$+\sum_{z'}\zeta_{z,z'}(V(iok,g,z')-V(iok,g,z))+\sum_{a\in A}\partial_{g_a}V(iok,g,z)\mu^g(a,z).$$

$\alpha(in,jok,g,z)$: indicator for whether the trade is accepted upon matching.

# Endogenous Yield Curve and Impulse Responses back

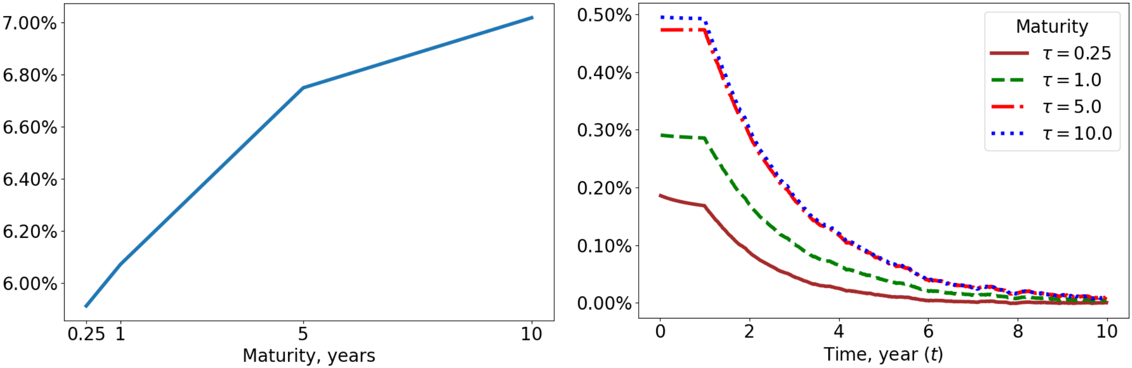


Figure: Yield curve at ergodic steady state and impulse responses. Right figure: proportional bond yield change compared to the ergodic yield at each maturity following a one-year recession.

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# Non-transferable Utility (NTU)

- Main change compared to the benchmark model: for a match $(x, y)$, agent $x$ receives payoff $zf(x, y)$ and agent $y$ receives $zf(y, x)$. No surplus to be split via bargaining.
- Computational implication:
  1. We no longer solve a single master equation for the match surplus.
  2. Instead, we directly approximate the value functions $V^u(x, z, g)$, $V^e(x, y, z, g)$, $V^v(y, z, g)$, and $V^p(x, y, z, g)$ using four neural networks, trained jointly to minimize a weighted sum of residuals from the four master equations.

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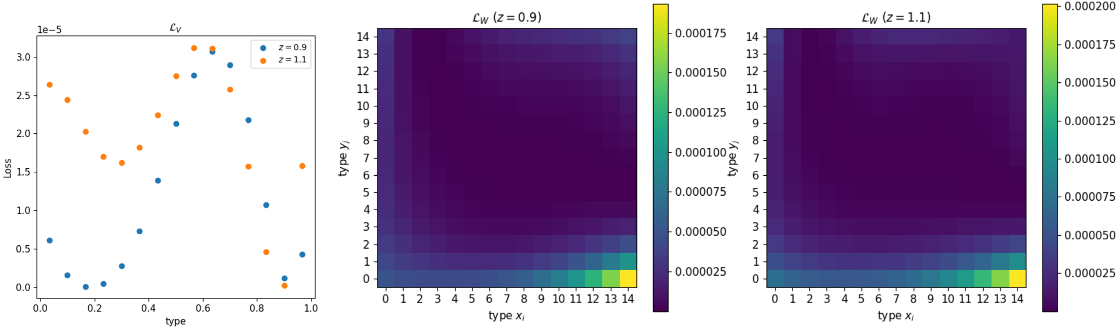
# Non-transferable Utility (NTU, Ct'd)

- Mean squared loss as a function of type in the master equations of $V^u$ (left) and $V^e$ (right)

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