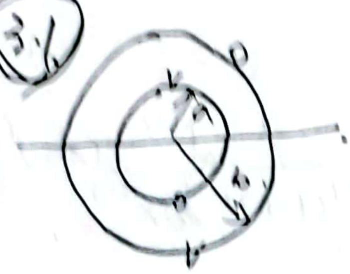
3.1



$$\phi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta)$$

$$\phi(a,\theta)=\begin{cases}V, & 0<\theta<\frac{\pi}{2}\\ 0, & \frac{\pi}{2}<\theta<\pi\end{cases}$$

$$\phi(b,\theta)=\begin{cases}0, & 0<\theta<\frac{\pi}{2}\\ V, & \frac{\pi}{2}<\theta<\pi\end{cases}$$

$$V\cdot\frac{2l+1}{2}\int_0^{\frac{\pi}{2}}P_l(\cos\theta)\,d(\cos\theta) = A_l a^l + B_l a^{-l-1}$$

$$V\cdot\frac{2l+1}{2}\int_{\frac{\pi}{2}}^{\pi}P_l(\cos\theta)\,d\theta = A_l b^l + B_l b^{-l-1}$$

$$\rightarrow\begin{cases}A_l=\dfrac{(2l+1)V\left[b^{l+1}\int_0^1\cdots - a^{-l-1}\int_{-1}^0\cdots\right]}{2\left(a^l b^{-l-1}-b^l a^{-l-1}\right)}\\ B_l=\dfrac{a^{l+1}(2l+1)V}{2}\int_0^1\cdots dx - A_l a\cdots\end{cases}$$

$$\int_0^1 P_l(x)\,dx=\int_0^1\frac{1}{2^l l!}\frac{d^l(x^2-1)^l}{dx^l}dx=\begin{cases}(-1)^{\frac{l+1}{2}}\dfrac{(l-2)!!}{(l+1)!!}, & l=2n+1\\ 0, & l=2n\end{cases}$$

同理 $$\int_{-1}^0 P_l(x)\,dx=\begin{cases}(-1)^{\frac{l+1}{2}}\dfrac{-(l-2)!!}{(l+1)!!}, & l=2n+1\\ 0, & l=2n\end{cases}$$

$$\therefore\ \phi(r,\theta)=\frac{V}{2}+\left[\frac{3V(a^2+b^2)}{4(a^3-b^3)}r+\frac{3Va^2(-b^3-b^2a)}{4(a^3-b^3)}r^{-2}\right]P_1(\cos\theta)$$

$$+\left(\frac{7V(a^4+b^4)}{16(a^7-b^7)}r^3+\frac{7Va^4(-b^7-b^4a^3)}{16(a^7-b^7)}r^{-4}\right)P_3(\cos\theta)+\cdots$$

$b\to\infty,\ a\to 0$

$$\phi(r,\theta)=\frac{V}{2}-\frac{3V}{4b}rP_1(\cos\theta)+\frac{7V}{16b^3}r^3P_3(\cos\theta)-\cdots$$

1. $\frac{dP_{l+1}}{dx} = \frac{d}{dx}\left(\frac{1}{2^{l+1}(l+1)!}\frac{d^{l+1}}{dx^{l+1}}(x^2-1)^{l+1}\right)$

$= \frac{1}{2^{l+1}(l+1)!}\frac{d^{l+2}}{d^{l+2}}(x^2-1)^{l+1} = \frac{1}{2^l l!}\frac{d^{l+1}}{dx^{l+1}}\left(x\cdot(x^2-1)^l\right)$

$= \cancel{\frac{2^{l+1}(l+1)!}{2^{l+1}(l+1)!}}\ (2l+1)P_l = (2l+1)\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

$(2l+1)P_l + \frac{dP_{l-1}}{dx} = \frac{1}{2^l(l-1)!}\frac{d^l}{dx^l}(x^2-1)^l + \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l + \frac{1}{2^{l-1}(l-1)!}\frac{d^{l-1}}{dx^{l-1}}(x^2-1)^{l-1}$

$\frac{dP_{l+1}}{dx} = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left((x^2-1)^l + 2l\,x^2(x^2-1)^{l-1}\right)$

$= \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l + \frac{1}{2^{l-1}(l-1)!}\frac{d^l}{dx^l}\left(x^2\cdot(x^2-1)^{l-1}\right)$

$\hookrightarrow (x^2-1)(x^2-1)^{l-1} + (x^2-1)^{l-1}$
$= (x^2-1)^l + (x^2-1)^{l-1}$

$= P_l + 2lP_l + \frac{dP_{l-1}}{dx}$

$= \frac{dP_{l-1}}{dx} + (2l+1)P_l$.

3.9

$\varphi(r,\phi,z) = R(r)X(\phi)Y(z)$.

$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$

$-\frac{1}{r^3}\frac{\partial R}{\partial r}XY + \frac{1}{r^2}\frac{\partial^2 R}{\partial r^2}XY + \frac{1}{r^2}\frac{\partial^2 X}{\partial\phi}RY + \frac{\partial^2 Y}{\partial z^2}RX = 0$

1. $\frac{\partial^2 X}{X\partial^2\phi^2} = \lambda^2 \rightarrow X = C_1\sin(\lambda\phi) + C_2\cos(\lambda\phi)$

2. $\frac{\partial^2 Y}{Y\partial^2 z^2} = \eta^2 \rightarrow Y = C_3\sin(\eta z) + C_4\sin(\eta z)$ $\quad Y(0)=0$.

$= C_3\sin(\eta z) = C_3\sin\frac{n\pi}{L}z$. $\quad Y(L)=0$

$\frac{\partial^2 R}{\partial r} - \frac{1}{r}\frac{\partial R}{\partial r} + \lambda^2 R + \eta^2 r^2 R = 0$

设解为 $C_4(r)$.

$$\phi = \sum_{\lambda,n=0}^{\infty} C_4(r)\sin\frac{n\pi}{L}z\left(C_{1\lambda n}\sin\lambda\phi + C_{2\lambda n}\cos\lambda\phi\right)$$

其中 $C_{\lambda n} = \frac{2}{L\pi}\int_0^L \phi\,\sin\left(\frac{n\pi}{L}z\right)\sin\lambda\phi\,d\phi.$

$D_{\lambda n} = \frac{2}{L\pi}\int_0^L \phi\,\sin\frac{n\pi}{L}z\,\cos\lambda\phi\,d\phi.$

3.6

$$\phi(r,\theta,\phi) = \sum_{l=0}^{m}\sum_{m=-1}^{1}\left[A_{lm}r^{l} + B_{lm}r^{-(l+1)}\right]Y_{lm}(\theta,\phi)$$

$$\phi(r,0,0) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|r-a|} - \frac{1}{4\pi\varepsilon_0}\frac{q}{|r+a|}$$???