1、 刚开始 $v_0=\frac{3}{2}\sqrt{gr_0}$

$$L=mv_0r_0=\frac{3m\sqrt{g}}{2}r_0^{\frac{3}{2}} \qquad E=\frac{mv_0^2}{2}=\frac{9mg}{4}r_0$$

放手后，设在 r、v 时~~绳~~半径最大，此时下端 m 速度为 0.

$$L=mvr=\frac{3m\sqrt{g}}{2}r_0^{\frac{3}{2}} \Rightarrow v=\frac{3\sqrt{g}}{2}\frac{r_0^{\frac{3}{2}}}{r}$$

$$E'=\frac{mv^2}{2}+mg(r-r_0)$$

$$E'=E \Rightarrow \frac{9mg}{4}\frac{r_0^3}{r^2}+mg(r-r_0)=\frac{9mg}{4}r_0.$$

$$\frac{9r_0}{4}\left(\frac{r^2-r_0^2}{r^2}\right)=r-r_0$$

$$\frac{(r-r_0)(r-3r_0)(4r+3r_0)}{4r^2}=0$$

$\therefore$ $r=3r_0$ 最长. $r=r_0$ 最短.

2. 刚开始： $L_1=mva$ $\quad L_2=3mva$ $\quad T=mv^2$

过程中 L_1、L_2、T 守恒

最短： $v_{||}=0$

$\therefore$ $L_1'=mv_1r_1$ $\qquad L_2'=mv_2(4a-r_1)$

$$L_1'=L_1 \quad L_2'=L_2 \Rightarrow v_1=\frac{va}{r_1} \quad v_2=\frac{3va}{4a-r_1}$$

$$T'=\frac{1}{2}m(v_1^2+v_2^2)$$

$$T'=T \Rightarrow \left(\frac{a^2}{r_1^2}+\frac{9a^2}{(4a-r_1)^2}\right)=2.$$

$r_1=6.13515a$. $\quad r_1=a$ $\quad r_1=-0.785715a$ $\quad r_1=1.65317a$.

$\therefore$ $r_{max}\approx1.65a$.

(2) 对1 $\quad -T = ma_{r_1} = m\left(\ddot{r}_1 - \frac{v_1^2}{r_1}\right)$ ①

对2 $\quad -T = ma_{r_2} = m\left(\ddot{r}_2 - \frac{v_2^2}{r_2}\right)$ ②

∵ $r_1 + r_2 = 4a$ $\quad$ ∴ $\ddot{r}_1 = -\ddot{r}_2$

①+② ⇒ ∴ $T = \frac{m}{2}\left(\frac{v_1^2}{r_1} + \frac{v_2^2}{4a - r_1}\right)$

∵ $v_1 = \frac{va}{r_1}$ $\quad$ $v_2 = \frac{3va}{4a - r_1}$

∴ $T = \frac{mv^2a^2}{2}\left(\frac{1}{r_1^3} + \frac{9}{(4a - r_1)^3}\right)$

$\frac{\partial T}{\partial r_1} = 0 \Rightarrow r_1 = 2(\sqrt{3} - 1)a.$

代入T得 $T_{min} \approx 0.435\frac{mv^2}{a}.$

7.2. **Cross section**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{C}{3r^3}. \tag{7.57}$$

Setting the derivative equal to zero gives $r = mC/L^2$. Plugging this into $V_{\text{eff}}(r)$ gives

$$V_{\text{eff}}^{\text{max}} = \frac{L^6}{6m^3C^2}. \tag{7.58}$$

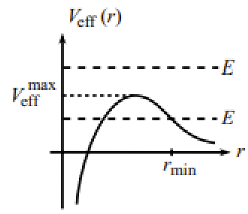


Fig. 7.11

(b) If the energy E of the particle is less than $V_{\text{eff}}^{\text{max}}$, then the particle will reach a minimum value of r, and then head back out to infinity (see Fig. 7.11). If E is greater than $V_{\text{eff}}^{\text{max}}$, then the particle will head all the way in to $r = 0$, never to return. The condition for capture is therefore $V_{\text{eff}}^{\text{max}} < E$. Using $L = mv_0b$ and $E = E_\infty = mv_0^2/2$, this condition becomes

$$\frac{(mv_0b)^6}{6m^3C^2} < \frac{mv_0^2}{2} \quad \Longrightarrow \quad b < \left(\frac{3C^2}{m^2v_0^4}\right)^{1/6} \equiv b_{\text{max}}. \tag{7.59}$$

The cross section for capture is therefore

$$\sigma = \pi b_{\text{max}}^2 = \pi\left(\frac{3C^2}{m^2v_0^4}\right)^{1/3}. \tag{7.60}$$

It makes sense that this should increase with C and decrease with m and v_0.

7.5. Spring ellipse

With $V(r) = \beta r^2$, Eq. (7.16) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2m\beta r^2}{L^2}. \tag{7.71}$$

As stated in Section 7.4.1, we could take a square root, separate variables, integrate to find $\theta(r)$, and then invert to find $r(\theta)$. But let's solve for $r(\theta)$ in a slick way, as we did for the gravitational case, where we made the change of variables, $y \equiv 1/r$. Since there are lots of r^2 terms floating around in Eq. (7.71), it's reasonable to try the change of variables, $y \equiv r^2$ or $y \equiv 1/r^2$. The latter turns out to be the better choice. So, using $y \equiv 1/r^2$ and $dy/d\theta = -2(dr/d\theta)/r^3$, and multiplying Eq. (7.71) through by $1/r^2$, we obtain

$$\begin{aligned}\left(\frac{1}{2}\frac{dy}{d\theta}\right)^2 &= \frac{2mEy}{L^2} - y^2 - \frac{2m\beta}{L^2}. \\ &= -\left(y - \frac{mE}{L^2}\right)^2 - \frac{2m\beta}{L^2} + \left(\frac{mE}{L^2}\right)^2. \end{aligned} \tag{7.72}$$

Defining $z \equiv y - mE/L^2$ for convenience, we have

$$\begin{aligned}\left(\frac{dz}{d\theta}\right)^2 &= -4z^2 + 4\left(\frac{mE}{L^2}\right)^2\left(1 - \frac{2\beta L^2}{mE^2}\right) \\ &\equiv -4z^2 + 4B^2. \end{aligned} \tag{7.73}$$

As in Section 7.4.1, we can just look at this equation and observe that

$$z = B\cos 2(\theta - \theta_0) \tag{7.74}$$

is the solution. We can rotate the axes so that $\theta_0 = 0$, so we'll drop the θ_0 from here on. Recalling our definition $z \equiv 1/r^2 - mE/L^2$ and also the definition of B from Eq. (7.73), Eq. (7.74) becomes

$$\frac{1}{r^2} = \frac{mE}{L^2}(1 + \epsilon\cos 2\theta), \tag{7.75}$$

where

$$\epsilon \equiv \sqrt{1 - \frac{2\beta L^2}{mE^2}}. \tag{7.76}$$

It turns out, as we'll see below, that ϵ is *not* the eccentricity of the ellipse, as it was in the gravitational case.

We will now use the procedure in Section 7.4.3 to show that Eq. (7.76) represents an ellipse. For convenience, let

$$k \equiv \frac{L^2}{mE}. \tag{7.77}$$

Multiplying Eq. (7.75) through by kr^2, and using

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{x^2}{r^2} - \frac{y^2}{r^2}, \tag{7.78}$$

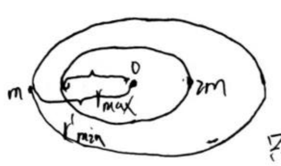
and also $r^2 = x^2 + y^2$, we obtain $k = (x^2 + y^2) + \epsilon(x^2 - y^2)$. This can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad a = \sqrt{\frac{k}{1+\epsilon}}, \quad \text{and} \quad b = \sqrt{\frac{k}{1-\epsilon}}. \tag{7.79}$$

This is the equation for an ellipse with its center located at the origin (as opposed to a focus located at the origin, as in the gravitational case). In Fig. 7.16, the semi-major and semi-minor axes are b and a, respectively, and the focal length is $c = \sqrt{b^2 - a^2} = \sqrt{2k\epsilon/(1-\epsilon^2)}$. The eccentricity is $c/b = \sqrt{2\epsilon/(1+\epsilon)}$.

REMARK: If $\epsilon = 0$, then $a = b$, which means that the ellipse is actually a circle. Let's see if this makes sense. Looking at Eq. (7.76), we see that we want to show that circular motion implies $2\beta L^2 = mE^2$. For circular motion, the radial $F = ma$ equation is $mv^2/r = 2\beta r \Longrightarrow v^2 = 2\beta r^2/m$. The energy is therefore $E = mv^2/2 + \beta r^2 = 2\beta r^2$. Also, the square of the angular momentum is $L^2 = m^2v^2r^2 = 2m\beta r^4$. Therefore, $2\beta L^2 = 2\beta(2m\beta r^4) = m(2\beta r^2)^2 = mE^2$, as we wanted to show. ♣

5、



O为质心.

(ε相同)

质心系下：两者轨道形状完全相同.
只是 m的轨道比2m的大一倍.

($r_0'=2r_0$).

$2m$： $r=\dfrac{r_0}{1+\varepsilon\cos\theta}$ m： $r'=\dfrac{r_0'}{1+\varepsilon\cos\theta}=\dfrac{2r_0}{1+\varepsilon\cos\theta}$

轨道相交：

$r_{max}=r'_{min}$.

即 $\dfrac{r_0}{1-\varepsilon}=\dfrac{2r_0}{1+\varepsilon}\Rightarrow\varepsilon=\dfrac{1}{3}$

6、 $r=\dfrac{r_0}{1-\varepsilon\cos\theta}$ $r_0=\dfrac{L^2}{mc}$ $\varepsilon=\sqrt{1+\dfrac{2EL^2}{mc^2}}$

初始圆轨道. $\varepsilon=0\Rightarrow E=-\dfrac{mc^2}{2L^2}$

(对圆周运动,有 $\dfrac{mv^2}{r}=\dfrac{c}{r^2}\Rightarrow mv^2=\dfrac{c}{r}$

$T=\dfrac{mv^2}{2}=\dfrac{c}{2r}$ $V=-\dfrac{c}{r}$ $\therefore V=-2T$

$\therefore E=V+T=-2T+T=-T$)

$\therefore T=-E=\dfrac{mc^2}{2L^2}$ $V=-2T=-\dfrac{mc^2}{L^2}$

抛物线： $\varepsilon=1\Rightarrow E'=0$

$E'=T'+V'=T'+V=T'-\dfrac{mc^2}{L^2}=0$

$\therefore T'=\dfrac{mc^2}{L^2}=2T$

$\therefore v'=\sqrt{2}v_0$ $f=\sqrt{2}$.

与方向无关.

$r=\dfrac{r_0'}{1-\cos\theta}$ $\therefore r_{min}=\dfrac{r_0'}{2}$

$\because r_0'=\dfrac{L'^2}{mc}$ 径向加速,角动量不变.

$\therefore L'=L$ $\therefore r_0'=r_0$

$r_{min}=\dfrac{r_0}{2}$