# 微积分 A (2)

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第 6 讲

在听课过程中，

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# 第 5 讲回顾: 方向导数

- 方向导数的定义, 方向导数存在并不意味着偏导数存在.
- 若沿某一个坐标轴的偏导数存在, 则沿该轴正、反两方向的方向导数存在且互负.
- 函数在一点处沿任意的方向均有方向导数, 并不意味着函数在该点可微.
- 方向导数的表达式 (借助微分或偏导数).

# 回顾：数量场的梯度

- 梯度的定义及其意义.
- 当函数为可微时，其梯度可由偏导数构成的列向量表示，而方向导数则可为梯度与指示方向的单位向量的内积.
- 常值函数的梯度等于零；梯度满足与单变量函数求导类似的四则运算及复合法则.
- 典型问题：求函数在一点处的梯度与最大的方向导数，以及沿某一向量的方向导数.

# 回顾: 高阶偏导数

- 二阶偏导数: $\frac{\partial^2 f}{\partial x_j \partial x_i} = \partial_{ji} f = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$, $\frac{\partial^2 f}{\partial x_i^2}$.
- $k$ 阶偏导数: $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$.
- 求偏导数一般不能交换次序.
- 设 $\Omega \subset \mathbb{R}^n$ 为开集. 若 $f : \Omega \to \mathbb{R}$ 在 $\Omega$ 上有二阶偏导函数 $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, 并且当中一个在点 $X_0 \in \Omega$ 连续, 则 $\frac{\partial^2 f}{\partial x_j \partial x_i}(X_0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(X_0)$.

# 回顾：函数空间 $\mathscr{C}^{(k)}(\Omega)$

- 空间 $\mathscr{C}^{(k)}(\Omega)$ $(k \geqslant 0$ 为整数$)$.
- 若 $f \in \mathscr{C}^{(k)}(\Omega)$, 则称之在 $\Omega$ 上为 $k$ 阶连续可导或 $k$ 阶连续可微.
- 设 $k \geqslant 2$ 为整数. 若 $f \in \mathscr{C}^{(k)}(\Omega)$, 则对任意整数 $1 \leqslant r \leqslant k$, 均有 $f \in \mathscr{C}^{(r)}(\Omega)$ 并且 $f$ 的任意 $r$ 阶偏导数均与求偏导的次序无关.

# 第 6 讲

## §5. 向量值函数的微分

回顾: 设 $A:\mathbb{R}^n\to\mathbb{R}^m$ 为线性映射, $\vec{e}_1,\ldots,\vec{e}_n$ 为 $\mathbb{R}^n$ 的自然基底, 而 $\vec{f}_1,\ldots,\vec{f}_m$ 为 $\mathbb{R}^m$ 的自然基底. 令 $a_{ij}=A\vec{e}_j\cdot\vec{f}_i$ $(1\leqslant i\leqslant m,\,1\leqslant j\leqslant n)$, 则 $A\vec{e}_j=\displaystyle\sum_{i=1}^m a_{ij}\vec{f}_i$. 若 $X=\displaystyle\sum_{j=1}^n x_j\vec{e}_j$, 那么

$$AX=\sum_{j=1}^n x_jA\vec{e}_j=\sum_{j=1}^n x_j\sum_{i=1}^m a_{ij}\vec{f}_i=\sum_{i=1}^m\Big(\sum_{j=1}^n a_{ij}x_j\Big)\vec{f}_i.$$

故 $(AX)_i=(AX)\cdot\vec{f}_i=\sum_{j=1}^n a_{ij}x_j$. 于是 $A$ 与矩阵 $(a_{ij})_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}$ 对应起来, 可将之视为同一.

**定义 1.** 设 $X_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in \mathbb{R}^n$, $r > 0$, 而 $\vec{f} : B(X_0, r) \subset \mathbb{R}^n \to \mathbb{R}^m$, $g : B(X_0, r) \to \mathbb{R}$ 为映射. 若 $\lim\limits_{X \to X_0} \frac{\|\vec{f}(X)\|}{|g(X)|} = 0$, 则记

$$\vec{f}(X) = \vec{o}(|g(X)|) = |g(X)|\vec{o}(1) \ (X \to X_0).$$

如果记 $\vec{f} = (f_1, \ldots, f_m)^T$, 则上式成立当且仅当对任意的整数 $1 \leqslant i \leqslant m$, 我们均有

$$f_i(X) = o(|g(X)|) \ (X \to X_0).$$

定义 2. 假设 $X_0 = (x_1^{(0)}, \ldots, x_n^{(0)}) \in \mathbb{R}^n$, $r > 0$, $\vec{f}: B(X_0, r) \subset \mathbb{R}^n \to \mathbb{R}^m$ 为向量值函数. 如果存在线性映射 $A: \mathbb{R}^n \to \mathbb{R}^m$ 使得 $X \to X_0$ 时,

$$\vec{f}(X) - \vec{f}(X_0) = A(X - X_0) + \vec{o}(\|X - X_0\|),$$

则称 $\vec{f}$ 在点 $X_0$ 可微并将映射 $A$ 记作 $\mathrm{d}\vec{f}(X_0)$, 称为 $\vec{f}$ 在点 $X_0$ 的全微分或微分. 线性映射 $A$ 所对应的矩阵记作 $J\vec{f}(X_0)$, 也被记作 $J_{\vec{f}}(X_0)$, 称为 $\vec{f}$ 在点 $X_0$ 处的 Jacobi 矩阵.

# 评注

- 若 $\vec{f}$ 在点 $X_0$ 处可微, 则其微分唯一.
- 可微性蕴含连续性.
- 若记 $\vec{f} = (f_1, \ldots, f_m)^T$, 则 $A = (a_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}$ 为 $\vec{f}$ 在点 $X_0$ 处的微分当且仅当 $X \to X_0$ 时, 对任意的整数 $1 \leqslant i \leqslant m$, 我们均有

$$f_i(X) - f_i(X_0) = \sum_{j=1}^{n} a_{ij}(x_j - x_j^{(0)}) + o(\|X - X_0\|).$$

也即 $f_i$ 在点 $X_0$ 处可微, 并且有 $a_{ij}=\frac{\partial f_i}{\partial x_j}(X_0)$. 故 $\mathrm{d}\vec{f}(X_0)$ 所对应的矩阵的第 $i$ 个行向量正好对应于 $\mathrm{d}f_i(X_0)$ 所对应的矩阵. 由此可知 $\vec{f}$ 在点 $X_0$ 可微当且仅当 $f_1,\ldots,f_m$ 在该点可微且

$$\mathrm{d}\vec{f}(X_0)=\begin{pmatrix}\mathrm{d}f_1(X_0)\\ \vdots\\ \mathrm{d}f_m(X_0)\end{pmatrix}=\begin{pmatrix}\sum\limits_{j=1}^{n}\frac{\partial f_1}{\partial x_j}(X_0)\,\mathrm{d}x_j\\ \vdots\\ \sum\limits_{j=1}^{n}\frac{\partial f_m}{\partial x_j}(X_0)\,\mathrm{d}x_j\end{pmatrix},$$

进而我们就有

$$\mathrm{d}\vec{f}(X_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X_0) & \cdots & \frac{\partial f_1}{\partial x_n}(X_0) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(X_0) & \cdots & \frac{\partial f_m}{\partial x_n}(X_0) \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix},$$

也即 $J_{\vec{f}}(X_0) = \left(\frac{\partial f_i}{\partial x_j}(X_0)\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$. 若将最右边那个列向量记作 $\mathrm{d}X$, 则 $\mathrm{d}\vec{f}(X_0) = J_{\vec{f}}(X_0)\,\mathrm{d}X$. 通常也将 $J_{\vec{f}}(X_0)$ 记作 $\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(X_0)$ 或 $\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}\Big|_{X_0}$.

当 $m = n$ 时, 相应行列式被称为 Jacobi 行列式, 记作 $\frac{D(f_1,\ldots,f_m)}{D(x_1,\ldots,x_n)}(X_0)$ 或 $\frac{D(f_1,\ldots,f_m)}{D(x_1,\ldots,x_n)}\Big|_{X_0}$.

例 1. $\forall (r,\varphi)\in D=(0,+\infty)\times(-\pi,\pi)$, 定义

$$\vec{f}(r,\varphi)=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}.$$

求 $\vec{f}$ 在点 $(r,\varphi)$ 处的微分及其 Jacobi 行列式.

解: 由于 $\vec{f}$ 的分量均为初等函数, 故可微且

$$J_{\vec{f}}(r,\varphi)=\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}.$$

则所求 Jacobi 行列式 $\frac{D(x,y)}{D(r,\varphi)} = r$, 而所求微分为

$$
\begin{aligned}
\mathrm{d}\vec{f}(r,\varphi) &= \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\varphi \end{pmatrix} \\
&= \begin{pmatrix} \cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi \\ \sin\varphi\,\mathrm{d}r + r\cos\varphi\,\mathrm{d}\varphi \end{pmatrix}.
\end{aligned}
$$

作业题: 第 1.5 节第 54 页第 2 题并求其微分.

# 可微复合向量值函数的微分

回顾: 矩阵的范数. 令 $A=(a_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}$. 定义

$$\|A\|=\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{\frac{1}{2}},$$

称为矩阵 $A$ 的范数. $\forall X=(x_1,\ldots,x_n)^T\in\mathbb{R}^n$, 令 $Y=AX=(y_1,\ldots,y_m)^T$, 则我们有

$$y_i=\sum_{j=1}^{n}a_{ij}x_j,$$

由此可立刻导出

$$\|Y\|_m^2 = \sum_{i=1}^{m} |y_i|^2 = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}x_j\Big)^2$$

$$\leqslant \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}||x_j|\Big)^2 \leqslant \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}|^2\Big)\Big(\sum_{j=1}^{n} |x_j|^2\Big)$$

$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} |a_{ij}|^2\Big)\|X\|_n^2 = \|A\|^2\|X\|_n^2,$$

从而我们有 $\|AX\|_m = \|Y\|_m \leqslant \|A\| \cdot \|X\|_n$.

**定理 1.** 假设 $\Omega_1 \subseteq \mathbb{R}^n$, $\Omega_2 \subseteq \mathbb{R}^m$ 均为非空开集, $X_0 \in \Omega_1$, 而映射 $\vec{g} : \Omega_1 \to \Omega_2$ 在点 $X_0$ 处可微, $\vec{f} : \Omega_2 \to \mathbb{R}^k$ 在点 $Y_0 = \vec{g}(X_0)$ 处可微, 则 $\vec{f} \circ \vec{g}$ 在点 $X_0$ 处可微, 并且

$$\mathrm{d}(\vec{f} \circ \vec{g})(X_0) = \mathrm{d}\vec{f}(Y_0) \circ \mathrm{d}\vec{g}(X_0).$$

**证明:** 令 $A = \mathrm{d}\vec{g}(X_0)$, $B = \mathrm{d}\vec{f}(Y_0)$, 则我们有

$$\vec{g}(X) - \vec{g}(X_0) = A(X - X_0) + \vec{o}(\|X - X_0\|_n) \ (X \to X_0),$$
$$\vec{f}(Y) - \vec{f}(Y_0) = B(Y - Y_0) + \vec{o}(\|Y - Y_0\|_m) \ (Y \to Y_0).$$

于是当 $X \to X_0$ 时, 我们有

$$
\begin{aligned}
&\|\vec{g}(X) - \vec{g}(X_0)\|_m \\
&= \|A(X - X_0) + \vec{o}(\|X - X_0\|_n)\|_m \\
&\leqslant \|A(X - X_0)\|_m + \|\|X - X_0\|_n \vec{o}(1)\|_m \\
&\leqslant \|A\| \cdot \|X - X_0\|_n + \|X - X_0\|_n o(1) \\
&= \|X - X_0\|_n O(1).
\end{aligned}
$$

$$
\|B\big(\vec{o}(\|X - X_0\|_n)\big)\|_k \leqslant \|B\| \cdot \|\|X - X_0\|_n \vec{o}(1)\|_m
$$

$$
\leqslant \|B\| \cdot \|X - X_0\|_n o(1) = \|X - X_0\|_n o(1).
$$

从而当 $X \to X_0$ 时, 我们有

$$\begin{aligned}\vec{f}\circ\vec{g}(X)-\vec{f}\circ\vec{g}(X_0) &= B(\vec{g}(X)-\vec{g}(X_0)) \\ &\qquad +\vec{o}(\|\vec{g}(X)-\vec{g}(X_0)\|_m) \\ &= B\big(A(X-X_0)+\vec{o}(\|X-X_0\|_n)\big) \\ &\qquad +\|\vec{g}(X)-\vec{g}(X_0)\|_m\vec{o}(1) \\ &= B\circ A(X-X_0)+\|X-X_0\|_n\vec{o}(1) \\ &\qquad +\|X-X_0\|_nO(1)\vec{o}(1) \\ &= B\circ A(X-X_0)+\|X-X_0\|_n\vec{o}(1).\end{aligned}$$

由微分的定义可知 $\vec{f}\circ\vec{g}$ 在点 $X_0$ 可微且其微分为 $B\circ A$, 即 $\mathrm{d}(\vec{f}\circ\vec{g})(X_0)=\mathrm{d}\vec{f}(Y_0)\circ\mathrm{d}\vec{g}(X_0)$.

# 可微复合向量值函数微分的矩阵表示

- $J_{\vec{f}\circ\vec{g}}(X_0) = J_{\vec{f}}(\vec{g}(X_0)) \cdot J_{\vec{g}}(X_0)$.
- 记 $\vec{g} = (g_1, \ldots, g_m)^T$, $\vec{f} = (f_1, \ldots, f_k)^T$, 则

$$\frac{\partial(f_1 \circ \vec{g}, \ldots, f_k \circ \vec{g})}{\partial(x_1, \ldots, x_n)}\Big|_{X_0} = \frac{\partial(f_1, \ldots, f_k)}{\partial(y_1, \ldots, y_m)}\Big|_{\vec{g}(X_0)} \cdot \frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_n)}\Big|_{X_0}.$$

- 当 $k = 1$ 时, 我们有

$$\frac{\partial(f \circ \vec{g})}{\partial(x_1, \ldots, x_n)} = \left(\frac{\partial f \circ \vec{g}}{\partial x_1}, \ldots, \frac{\partial f \circ \vec{g}}{\partial x_n}\right),$$

$$\frac{\partial(f)}{\partial(y_1, \ldots, y_m)} = \left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_m}\right),$$

再注意到

$$\frac{\partial(g_1,\ldots,g_m)}{\partial(x_1,\ldots,x_n)} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(X_0) & \cdots & \frac{\partial g_1}{\partial x_n}(X_0) \\ \vdots & \cdots & \vdots \\ \frac{\partial g_m}{\partial x_1}(X_0) & \cdots & \frac{\partial g_m}{\partial x_n}(X_0) \end{pmatrix},$$

于是对任意整数 $1 \leqslant i \leqslant n$, 我们有

$$\frac{\partial f\circ\vec{g}}{\partial x_i}(X_0) = \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(\vec{g}(X_0))\frac{\partial g_j}{\partial x_i}(X_0).$$

也即我们有

$$\frac{\partial f(g_1,\ldots,g_m)}{\partial x_i}(X_0)=\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}(Y_0)\frac{\partial g_j}{\partial x_i}(X_0)$$

$$=\frac{\partial f}{\partial y_1}(Y_0)\frac{\partial g_1}{\partial x_i}(X_0)+\frac{\partial f}{\partial y_2}(Y_0)\frac{\partial g_2}{\partial x_i}(X_0)+\cdots+\frac{\partial f}{\partial y_m}(Y_0)\frac{\partial g_m}{\partial x_i}(X_0),$$

$$\frac{\partial f(g_1,\ldots,g_m)}{\partial x_i}=\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}\frac{\partial g_j}{\partial x_i}$$

$$=\frac{\partial f}{\partial y_1}\frac{\partial g_1}{\partial x_i}+\frac{\partial f}{\partial y_2}\frac{\partial g_2}{\partial x_i}+\cdots+\frac{\partial f}{\partial y_m}\frac{\partial g_m}{\partial x_i}.$$

例 2. 假设 $z = f(u,v) = u^2v - uv^2$, $u = x\sin y$, $v = x\cos y$. 求 $\frac{\partial z}{\partial x}$.

解: 由题设可得

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial(x\sin y)}{\partial x} + \frac{\partial f}{\partial(x\cos y)}\frac{\partial v}{\partial x} \quad \text{(严重错误!)}$$

$$= \frac{\partial f}{\partial u}\frac{\partial(x\sin y)}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial(x\cos y)}{\partial x}$$

$$= (2uv - v^2)\sin y + (u^2 - 2uv)\cos y$$

$$= (2x^2\sin y\cos y - x^2\cos^2 y)\sin y$$
$$+(x^2\sin^2 y - 2x^2\sin y\cos y)\cos y$$

$$= \frac{3}{2}x^2(\sin y - \cos y)\sin(2y).$$

例 3. 设 $z=f(xy,x^2-y^2)$, $f$ 可微. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解: 由题设可知

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \partial_1 f(xy,x^2-y^2)\frac{\partial(xy)}{\partial x} \\
&\quad +\partial_2 f(xy,x^2-y^2)\frac{\partial(x^2-y^2)}{\partial x} \\
&= y\partial_1 f(xy,x^2-y^2)+2x\partial_2 f(xy,x^2-y^2). \\
\frac{\partial z}{\partial y} &= \partial_1 f(xy,x^2-y^2)\frac{\partial(xy)}{\partial y} \\
&\quad +\partial_2 f(xy,x^2-y^2)\frac{\partial(x^2-y^2)}{\partial y} \\
&= x\partial_1 f(xy,x^2-y^2)-2y\partial_2 f(xy,x^2-y^2).
\end{aligned}
$$

例 4. 设 $z=\frac{y}{x}+xyf(\frac{y}{x})$, $f$ 可微, 求 $\frac{\partial z}{\partial x}$.

解: 由题设可得

$$\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{y}{x^2}+yf(\frac{y}{x})+xy\cdot f'(\frac{y}{x})\cdot(-\frac{y}{x^2})\\ &= -\frac{y}{x^2}+yf(\frac{y}{x})-\frac{y^2}{x}f'(\frac{y}{x}).\end{aligned}$$

**例 5.** 设 $z=f(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n))$, $f,g_1,\ldots,g_m$ 二阶可微, 求 $\frac{\partial^2 z}{\partial x_i \partial x_j}$ $(1\leqslant i,j\leqslant n)$.

**解:**

$$\frac{\partial^2 z}{\partial x_i \partial x_j}=\frac{\partial}{\partial x_i}\Big(\frac{\partial z}{\partial x_j}\Big)=\frac{\partial}{\partial x_i}\Big(\sum_{k=1}^m \frac{\partial f}{\partial y_k}(*)\frac{\partial g_k}{\partial x_j}\Big)$$

$$=\sum_{k=1}^m\Big[\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial y_k}(*)\Big)\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial}{\partial x_i}\Big(\frac{\partial g_k}{\partial x_j}\Big)\Big]$$

$$=\sum_{k=1}^m\Big[\Big[\sum_{l=1}^m\frac{\partial}{\partial y_l}\Big(\frac{\partial f}{\partial y_k}\Big)(*)\frac{\partial g_l}{\partial x_i}\Big]\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial^2 g_k}{\partial x_i\partial x_j}\Big]$$

$$=\sum_{k=1}^m\Big[\sum_{l=1}^m\frac{\partial^2 f}{\partial y_l\partial y_k}(*)\frac{\partial g_l}{\partial x_i}\frac{\partial g_k}{\partial x_j}+\frac{\partial f}{\partial y_k}(*)\frac{\partial^2 g_k}{\partial x_i\partial x_j}\Big].$$

例 6. (Laplace 方程) 定义 $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $r = \sqrt{x^2 + y^2 + z^2}$. 求证: 在 $\mathbb{R}^3 \setminus \{(0,0,0)\}$ 上,

$$\Delta\left(\frac{1}{r}\right) = \frac{\partial^2(\frac{1}{r})}{\partial x^2} + \frac{\partial^2(\frac{1}{r})}{\partial y^2} + \frac{\partial^2(\frac{1}{r})}{\partial z^2} = 0.$$

证明: 在 $\mathbb{R}^3 \setminus \{(0,0,0)\}$ 上, 我们有

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{x}{r^3}.$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x}{r^4}\frac{\partial r}{\partial x} = -\frac{1}{r^3} + \frac{3x^2}{r^5}.$$

于是由对称性可得

$$\begin{aligned}\Delta\left(\frac{1}{r}\right) &= \left(-\frac{1}{r^3}+\frac{3x^2}{r^5}\right)+\left(-\frac{1}{r^3}+\frac{3y^2}{r^5}\right)\\&\quad+\left(-\frac{1}{r^3}+\frac{3z^2}{r^5}\right)\\&= -\frac{3}{r^3}+\frac{3(x^2+y^2+z^2)}{r^5}=0.\end{aligned}$$

作业题: 第 1.5 节第 54 页第 3 题第 (1) 小题, 第 5 题, 第 7 题, 第 9 题第 (1) 小题.

## §6. 隐 (向量值) 函数、反 (向量值) 函数的存在性及其微分

问题: 如何解方程 $F(x, y) = 0$? 具体来说, 如何从方程 $F(x, y) = 0$ 出发来求解 $y = y(x)$?

线性的情形: 假设 $F(x, y) = ax + by + c$. 此时可从 $F(x, y) = 0$ 解出 $y$ 当且仅当 $\frac{\partial F}{\partial y} = b \neq 0$, 这时我们有 $y = -\frac{1}{b}(ax + c)$.

圆周: 现在考虑方程 $F(x,y) := x^2 + y^2 - 1 = 0$. 此时我们有 $y = \pm\sqrt{1-x^2}$.

- 当 $y > 0$ 时, $y = \sqrt{1-x^2}$, $\frac{\partial F}{\partial y} = 2y > 0$.
- 当 $y < 0$ 时, $y = -\sqrt{1-x^2}$, $\frac{\partial F}{\partial y} = 2y < 0$.
- 在 $(1,0)$ 的附近, 无法求 $y$, 而 $\frac{\partial F}{\partial y}(1,0) = 0$.

启示: 方程 $F(x,y) = 0$ 有解 $y = y(x)$ 与 $\frac{\partial F}{\partial y}$ 是否等于零有关?

# 隐函数定理

**定理 1.** 设 $X_0 = (x_0, y_0) \in \mathbb{R}^2$, $r > 0$, 而数量值函数 $F : B(X_0, r) \to \mathbb{R}$ 为 $\mathscr{C}^{(1)}$ 类的函数使得 $F(x_0, y_0) = 0$, $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$. 则 $\exists \delta, \eta > 0$ 使得 $B(x_0, \delta) \times B(y_0, \eta) \subset B(X_0, r)$ 且 $\forall x \in B(x_0, \delta)$, $\exists! y \in B(y_0, \eta)$ 使得 $F(x, y) = 0$. 定义 $f(x) = y$. 则 $f : B(x_0, \delta) \to B(y_0, \eta)$ 为 $\mathscr{C}^{(1)}$ 类函数, 并且 $\forall x \in B(x_0, \delta)$, 均有 $f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}$.

证明: 不失一般性, 我们可假设 $\frac{\partial F}{\partial y}(x_0, y_0) > 0$. 否则考虑函数 $-F$.

存在性: 由题设可知 $\frac{\partial F}{\partial y}$ 连续, 则 $\exists \eta > 0$ 使得 $\forall (x, y) \in B(X_0, \sqrt{2}\eta) \subsetneq B(X_0, r)$, $\frac{\partial F}{\partial y}(x, y) > 0$.

$\forall (x, y) \in B(X_0, \sqrt{2}\eta)$, 我们令 $g_x(y) = F(x, y)$. 则对于每个固定的 $x \in [x_0 - \eta, x_0 + \eta]$, 函数 $g_x$ 在 $[y_0 - \eta, y_0 + \eta]$ 上可导且 $g'_{x_0}(y) = \frac{\partial F}{\partial y}(x_0, y) > 0$, 从而 $g_{x_0}$ 为严格递增函数. 又 $g_{x_0}(y_0) = 0$, 故

$$\begin{aligned} F(x_0, y_0 - \eta) &= g_{x_0}(y_0 - \eta) < g_{x_0}(y_0) = 0 \\ &< g_{x_0}(y_0 + \eta) = F(x_0, y_0 + \eta). \end{aligned}$$

注意到 $F$ 连续, 于是由连续函数的保号性知, $\exists \delta \in (0,\eta)$ 使得 $\forall x \in (x_0-\delta, x_0+\delta)$, 均有

$$\begin{aligned} g_x(y_0-\eta) &= F(x,y_0-\eta) < 0, \\ g_x(y_0+\eta) &= F(x,y_0+\eta) > 0. \end{aligned}$$

又 $\forall y \in [y_0-\eta, y_0+\eta]$, 均有 $g_x'(y) = \frac{\partial F}{\partial y}(x,y) > 0$, 因此 $g_x$ 在 $[y_0-\eta, y_0+\eta]$ 上严格递增且连续, 由连续函数介值定理, $\exists! y \in (y_0-\eta, y_0+\eta)$ 使得 $F(x,y) = g_x(y) = 0$. 令 $f(x) = y$. 则 $f$ 为所求.

连续性: 由前面讨论知, $\forall \varepsilon \in (0,\eta)$, $\exists \delta' \in (0,\varepsilon)$ 使 $\forall x \in B(x_0,\delta')$, $\exists ! y \in B(y_0,\varepsilon)$ 使 $F(x,y)=0$, 此时 $y=f(x)$, 也即当 $|x-x_0|<\delta'$ 时, 我们有 $|f(x)-f(x_0)|<\varepsilon$. 故函数 $f$ 在点 $x_0$ 处连续.

取 $x_1 \in B(x_0,\delta)$, $y_1=f(x_1)$, 则 $F(x_1,y_1)=0$ 且 $(x_1,y_1)\in B((x_0,y_0),\sqrt{2}\eta)$, 故 $\frac{\partial F}{\partial y}(x_1,y_1)>0$. 由前面的讨论可知, 存在 $\delta_1 \in (0,\delta), \eta_1 \in (0,\eta)$ 以及在 $x_1$ 连续的函数 $g: B(x_1,\delta_1) \to B(y_1,\eta_1)$ 使 $F(x,g(x))=0$. 另外可设 $B(x_1,\delta_1) \subset B(x_0,\delta)$.

由唯一性知 $\forall x \in B(x_1, \delta_1)$, 均有 $f(x) = g(x)$, 故 $f$ 在点 $x_1$ 处连续.

可导性: 取$x \in B(x_0, \delta)$, $h \in \mathbb{R}$ 使$x+h \in B(x_0, \delta)$. 令$y = f(x)$, $\Delta y = f(x+h) - f(x)$. 由Lagrange 中值定理可知, $\exists \theta_1, \theta_2 \in (0,1)$ 使得

$$\begin{aligned} 0 &= F(x+h, y+\Delta y) - F(x, y) \\ &= \big(F(x+h, y+\Delta y) - F(x, y+\Delta y)\big) \\ &\quad + \big(F(x, y+\Delta y) - F(x, y)\big) \\ &= \frac{\partial F}{\partial x}(x+\theta_1 h, y+\Delta y)h + \frac{\partial F}{\partial y}(x, y+\theta_2 \Delta y)\Delta y. \end{aligned}$$

由于 $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ 均连续, 于是由夹逼原理以及复合函数极限法则可知

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= -\lim_{h \to 0} \frac{\frac{\partial F}{\partial x}(x + \theta_1 h, y + \Delta y)}{\frac{\partial F}{\partial y}(x, y + \theta_2 \Delta y)} \\
&= -\frac{\frac{\partial F}{\partial x}(x, y)}{\frac{\partial F}{\partial y}(x, y)} = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}.
\end{aligned}
$$

上式同时表明 $f'$ 为连续函数, 故 $f$ 连续可导.

定理 2. 设 $X_0\in\mathbb{R}^n$, $y_0\in\mathbb{R}$, $r>0$, 而数量值函数 $F:B((X_0,y_0),r)\to\mathbb{R}$ 为 $\mathscr{C}^{(1)}$ 类使 $F(X_0,y_0)=0$, $\frac{\partial F}{\partial y}(X_0,y_0)\neq 0$. 则 $\exists\delta,\eta>0$ 使得我们有

$$B(X_0,\delta)\times B(y_0,\eta)\subset B((X_0,y_0);r),$$

且 $\forall X\in B(X_0,\delta)$, $\exists! y\in B(y_0,\eta)$ 使 $F(X,y)=0$. 令 $f(X)=y$. 则 $f:B(X_0,\delta)\to B(y_0,\eta)$ 为 $\mathscr{C}^{(1)}$ 类且 $\forall X\in B(X_0,\delta)$ 与任意整数 $1\leqslant i\leqslant n$, 均有

$$\frac{\partial f}{\partial x_i}(X)=-\frac{\frac{\partial F}{\partial x_i}(X,f(X))}{\frac{\partial F}{\partial y}(X,f(X))}.$$

# 评注

上述最后一个等式可由对恒等式

$$F(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0$$

求偏导数而得. 事实上, 对 $x_i$ 求偏导数可得

$$\frac{\partial F}{\partial x_i}(X, f(X)) + \frac{\partial F}{\partial y}(X, f(X))\frac{\partial f}{\partial x_i}(X) = 0,$$

由此我们可立刻导出

$$\frac{\partial f}{\partial x_i}(X) = -\frac{\frac{\partial F}{\partial x_i}(X, f(X))}{\frac{\partial F}{\partial y}(X, f(X))}.$$

定理 3. 设 $X_0\in\mathbb{R}^n$, $Y_0\in\mathbb{R}^m$, $r>0$, 向量值函数 $\vec{F}=(F_1,\dots,F_m)^T\colon B((X_0,Y_0),r)\to\mathbb{R}^m$ 为 $\mathscr{C}^{(1)}$ 类使得 $\vec{F}(X_0,Y_0)=\vec{0}$, $\frac{\partial(F_1,\dots,F_m)}{\partial(y_1,\dots,y_m)}(X_0,Y_0)$ 可逆. 那么 $\exists\delta,\eta>0$ 使 $B(X_0,\delta)\times B(Y_0,\eta)\subset B((X_0,Y_0);r)$ 且 $\forall X\in B(X_0,\delta)$, $\exists! Y\in B(Y_0,\eta)$ 使 $\vec{F}(X,Y)=0$. 令 $\vec{f}(X)=Y$. 则 $\vec{f}\colon B(X_0,\delta)\to B(Y_0,\eta)$ 为 $\mathscr{C}^{(1)}$ 类, 并且 $\forall X\in B(X_0,\delta)$, 我们均有

$$J_{\vec{f}}(X)=-\left(\frac{\partial(F_1,\dots,F_m)}{\partial(y_1,\dots,y_m)}(X,\vec{f}(X))\right)^{-1}\cdot\frac{\partial(F_1,\dots,F_m)}{\partial(x_1,\dots,x_n)}(X,\vec{f}(X)).$$

# 评注

- 上述定理也可表述成: $\forall X \in B(X_0, \delta)$ 以及 $\forall Y \in B(Y_0, \eta)$, 等式 $\vec{F}(X, Y) = \vec{0}$ 成立当且仅当我们有 $Y = \vec{f}(X)$.
- 若将 $\mathscr{C}^{(1)}$ 换成 $\mathscr{C}^{(k)}$ $(k \geqslant 1)$, 定理依然成立.
- 将 $F_i(X, \vec{f}(X)) = 0$ 对 $x_j$ 求偏导可得

$$\frac{\partial F_i}{\partial x_j}(X, \vec{f}(X)) + \sum_{l=1}^{m} \frac{\partial F_i}{\partial y_l}(X, \vec{f}(X)) \frac{\partial f_l}{\partial x_j}(X) = 0,$$

进而我们可以导出

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(x_1,\ldots,x_n)}(X,\vec{f}(X)) + \frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X,\vec{f}(X))\cdot\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(X) = \vec{0},$$

于是我们有

$$\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}(X) = -\Big(\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}(X,\vec{f}(X))\Big)^{-1} \cdot\frac{\partial(F_1,\ldots,F_m)}{\partial(x_1,\ldots,x_n)}(X,\vec{f}(X)).$$

例 1. $\forall(x,y,z)\in\mathbb{R}^3$, 定义

$$F(x,y,z)=x(1+yz)+e^{x+y+z}-1.$$

问方程 $F(x,y,z)=0$ 是否能在原点的附近确定一个隐函数 $z=f(x,y)$? 如果能, 求该隐函数在点 $(0,0)$ 处的偏导数.

解: 由题设可知 $F$ 为初等函数, 从而为 $\mathscr{C}^{(1)}$ 类并且我们还有 $F(0,0,0)=0$, $\frac{\partial F}{\partial z}=xy+e^{x+y+z}$. 于是 $\frac{\partial F}{\partial z}(0,0,0)=1\neq 0$, 因此方程 $F(x,y,z)=0$ 能在原点附近确定一个隐函数 $z=f(x,y)$.

另外, 我们还有

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\Bigg|_{(0,0,0)} \\
&= -\frac{1+yz+e^{x+y+z}}{xy+e^{x+y+z}}\Bigg|_{(0,0,0)} = -2.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\Bigg|_{(0,0,0)} \\
&= -\frac{xz+e^{x+y+z}}{xy+e^{x+y+z}}\Bigg|_{(0,0,0)} = -1.
\end{aligned}
$$

例 2. 设 $F$ 为 $\mathscr{C}^{(2)}$ 类, 则由方程 $F(x,y,z)=0$ 确定的隐函数 $z=f(x,y)$ 为 $\mathscr{C}^{(2)}$ 类, 求 $\frac{\partial^2 z}{\partial y\partial x}$.

解: 令 $u=\frac{\partial F}{\partial z}(x,y,z(x,y))\neq 0$. 由题设可得

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(-\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))}\right)\\ &= -\frac{1}{u^2}\Big[\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial x}(x,y,z(x,y))\Big)\frac{\partial F}{\partial z}(x,y,z(x,y))\\ &\quad -\frac{\partial F}{\partial x}(x,y,z(x,y))\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial z}(x,y,z(x,y))\Big)\Big]\end{aligned}$$

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{u^2}\Big[\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial x}(x,y,z(x,y))\Big)\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial z}(x,y,z(x,y))\Big)\Big]$$

$$= -\frac{1}{u^2}\Big[\Big(\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial z \partial x}\frac{\partial z}{\partial y}\Big)\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\Big(\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\Big)\Big]$$

$$= -\frac{1}{u^2}\Big[\Big[\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial z \partial x}\Big(-\frac{\frac{\partial F}{\partial y}}{u}\Big)\Big]\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\Big[\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial^2 F}{\partial z^2}\Big(-\frac{\frac{\partial F}{\partial y}}{u}\Big)\Big]\Big]$$

$$= -\frac{1}{u^3}\Big[\Big(\frac{\partial F}{\partial z}\Big)^2\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial F}{\partial y}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial z^2}\Big]$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial F}{\partial y}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z \partial x} - \frac{\partial F}{\partial x}\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial y \partial z} + \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial z^2}}{\left(\frac{\partial F}{\partial z}\right)^3}.$$

例 3. 求证：下述方程组

$$\begin{cases} F_1(x,y,u,v) = 3x^2 + y^2 + u^2 + v^2 - 1 = 0, \\ F_2(x,y,u,v) = x^2 + 2y^2 - u^2 + v^2 - 1 = 0, \end{cases}$$

在点 $P_0(0, \frac{1}{2}, \sqrt{\frac{1}{8}}, \sqrt{\frac{5}{8}})$ 的某邻域内确定了一个向量值函数 $\begin{pmatrix} u \\ v \end{pmatrix} = \vec{f}(x,y)$，并计算该向量值函数 $\vec{f}$ 在点 $(0, \frac{1}{2})$ 处的 Jacobi 矩阵与微分.

解: 由于 $F_1, F_2$ 均为初等函数, 因此为 $\mathscr{C}^{(1)}$ 类.

又由题设可知 $F_1(P_0) = F_2(P_0) = 0$, 并且

$$\frac{D(F_1, F_2)}{D(u, v)}(P_0) = \left| \begin{vmatrix} 2u & 2v \\ -2u & 2v \end{vmatrix} \right|_{P_0} = 8uv\Big|_{P_0} = \sqrt{5},$$

从而 $\frac{\partial(F_1,F_2)}{\partial(u,v)}(P_0)$ 为可逆矩阵, 于是在点 $P_0$ 的

邻域内, 上述方程组可确定一个向量值函数

$$\begin{pmatrix} u \\ v \end{pmatrix} = \vec{f}(x, y),$$

进而可知所求 Jacobi 矩阵为

$$\frac{\partial(u,v)}{\partial(x,y)}(0,\frac{1}{2}) = -\Big(\frac{\partial(F_1,F_2)}{\partial(u,v)}(P_0)\Big)^{-1}\frac{\partial(F_1,F_2)}{\partial(x,y)}(P_0)$$

$$= -\begin{pmatrix} 2u & 2v \\ -2u & 2v \end{pmatrix}^{-1}\Big|_{P_0}\begin{pmatrix} 6x & 2y \\ 2x & 4y \end{pmatrix}\Big|_{P_0}$$

$$= -\begin{pmatrix} 2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{5}{8}} \\ -2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{5}{8}} \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{5}}\begin{pmatrix} 2\sqrt{\frac{5}{8}} & -2\sqrt{\frac{5}{8}} \\ 2\sqrt{\frac{1}{8}} & 2\sqrt{\frac{1}{8}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{10}}{10} \end{pmatrix}.$$

于是所求微分为

$$\mathrm{d}\vec{f}(0,\frac{1}{2}) = \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}\Big|_{(0,\frac{1}{2})}$$

$$= \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{3\sqrt{10}}{10} \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2}\,\mathrm{d}y \\ -\frac{3\sqrt{10}}{10}\,\mathrm{d}y \end{pmatrix}.$$

作业题: 第 1.6 节第 65 页第 2 题第 (2) 小题, 第 66 页第 6 题.

谢谢大家!