量子力学作业-第五周

题目 1. (教材 2.2) 设粒子处于一维无限深方势阱中,

$$V(x)=\begin{cases}0, & 0<x<a\\ \infty, & x<0,x>a\end{cases}$$

证明处于能量本征态 $\psi_n(x)$ 的粒子, $\bar{x}=a/2$

$$\overline{(x-\bar{x})^2}=\frac{a^2}{12}\left(1-\frac{6}{n^2\pi^2}\right)$$

讨论 $n\to\infty$ 的情况, 并与经典力学计算结果比较.

解答. 易得题设势阱中的本征态波函数为

$$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right),n=1,2,3\cdots$$

因此

$$\begin{aligned}\bar{x}&=\int_{-\infty}^{\infty}\psi^*\left(x\right)x\psi\left(x\right)dx=\frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx\\&=\frac{2}{a}\int_0^a x\frac{1-\cos\left(\frac{2n\pi}{a}x\right)}{2}dx=\frac{2}{a}\frac{a^2}{4}-\frac{1}{2}\int_0^a x\cos\left(\frac{2n\pi}{a}x\right)dx\\&=\frac{a}{2}-\frac{1}{2}\int_0^a\frac{a}{2n\pi}xd\sin\left(\frac{2n\pi}{a}x\right)\\&=\frac{a}{2}-\frac{1}{2}\frac{a}{2n\pi}x\sin\left(\frac{2n\pi}{a}x\right)|_0^a+\frac{a}{4n\pi}\int_0^a\sin\left(\frac{2n\pi}{a}x\right)dx\\&=\frac{a}{2}-\frac{a}{4n\pi}\frac{a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right)|_0^a\\&=\frac{a}{2}\end{aligned}$$

$$
\begin{aligned}
\overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2 &= \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right)dx - \frac{a^2}{4} \\
&= \frac{2}{a}\int_0^a x^2 \frac{1-\cos\left(\frac{2n\pi}{a}x\right)}{2}dx - \frac{a^2}{4} \\
&= \frac{a^2}{12} - \frac{1}{a}\int_0^a x^2 \cos\left(\frac{2n\pi}{a}x\right)dx \\
&= \frac{a^2}{12} - \frac{1}{a}\frac{a}{2n\pi}x^2 \sin\left(\frac{2n\pi}{a}x\right)\Big|_0^a + \frac{1}{a}\frac{a}{2n\pi}\int_0^a 2x\sin\left(\frac{2n\pi}{a}x\right)dx \\
&= \frac{a^2}{12} - \frac{1}{n\pi}\frac{a}{2n\pi}x\cos\left(\frac{2n\pi}{a}x\right)\Big|_0^a + \frac{1}{n\pi}\frac{a}{2n\pi}\int_0^a \cos\left(\frac{2n\pi}{a}x\right)dx \\
&= \frac{a^2}{12}\left(1-\frac{6}{n^2\pi^2}\right)
\end{aligned}
$$

当 $n \to \infty$ 时，$\bar{x} \to \frac{a}{2}$,$\overline{(x-\bar{x})^2} \to \frac{a^2}{12}$，对于经典情形，粒子出现在阱内各处的概率相同，有

$$\bar{x} = \frac{1}{a}\int_0^a x dx = \frac{a}{2}$$

$$\overline{(x-\bar{x})^2} = \overline{x^2} - \bar{x}^2 = \frac{1}{a}\int_0^a x^2 dx - \frac{a^2}{4} = \frac{a^2}{12}$$

可见 $n \to \infty$ 时，经典力学与量子力学的结果相同.

题目 2. (教材 2.4) 设粒子处于无限深方势阱

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & x < 0, x > a \end{cases}$$

中, 粒子波函数为 $\psi(x) = Ax(x-a)$, A 为归一化常数.

(a) 求 A;

(b) 求测得粒子处于能量本征态 $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ 的概率 P_n, 特别是 P_1. 提示: 用 $\psi_n(x)$ 展开, $\psi(x) = \sum_n C_n\psi_n(x), P_n = |C_n|^2$

(c) 作图, 比较 $\psi(x)$ 与 $\psi_1(x)$ 曲线. 从 $P_1 \gg P_n(n \neq 1)$ 来说明两条曲线非常相似, 即 $\psi(x)$ 几乎与基态 $\psi_n(x)$ 完全相同.

解答. (1) 易得

$$
\begin{aligned}
\int_0^a A^2x^2(x-a)^2\,dx &\xlongequal{t=x-\frac{a}{2}} A^2\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(t+\frac{a}{2}\right)^2\left(t-\frac{a}{2}\right)^2dt\\
&=A^2\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(t^4-\frac{a^2}{2}t^2+\frac{a^4}{16}\right)dt\\
&=A^2\left(\frac{t^5}{5}-\frac{a^2}{6}t^3+\frac{a^4}{16}t\right)|_{-\frac{a}{2}}^{\frac{a}{2}}\\
&=2A^2\left(\frac{a^5}{5\cdot 32}-\frac{a^5}{6\cdot 8}+\frac{a^5}{16\cdot 2}\right)\\
&=\frac{A^2}{30}a^5=1
\end{aligned}
$$

因此

$$A=\sqrt{\frac{30}{a^5}}$$

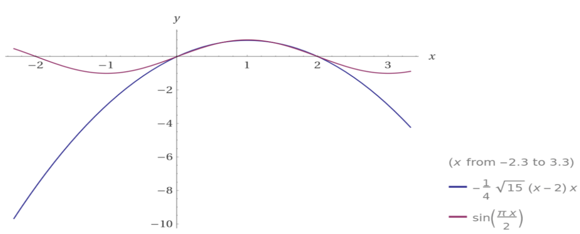
(2) 系数

$$
\begin{aligned}
C_n=\int_0^a\psi_n^*(x)\,\psi(x)\,dx&=A\int_0^a\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\cdot x(x-a)\,dx\\
&=A\sqrt{\frac{2}{a}}\int_0^a\sin\left(\frac{n\pi x}{a}\right)\cdot x(x-a)\,dx\\
&=-A\sqrt{\frac{2}{a}}\frac{a}{n\pi}\left[x(x-a)\,|_0^a-\int_0^a\cos\left(\frac{n\pi x}{a}\right)dx\,(x-a)\right]\\
&=A\sqrt{\frac{2}{a}}\frac{a}{n\pi}\left[2\int_0^a x\cos\left(\frac{n\pi x}{a}\right)dx-a\int_0^a\cos\left(\frac{n\pi x}{a}\right)dx\right]\\
&=2A\sqrt{\frac{2}{a}}\left(\frac{a}{n\pi}\right)^2\left[x\sin\left(\frac{n\pi x}{a}\right)|_0^a-\int_0^a\sin\left(\frac{n\pi x}{a}\right)dx\right]\\
&=2A\sqrt{\frac{2}{a}}\left(\frac{a}{n\pi}\right)^3\cos\left(\frac{n\pi x}{a}\right)|_0^a=2\sqrt{\frac{60}{a^6}}\frac{a^3}{n^3\pi^3}\left[(-1)^n-1\right]
\end{aligned}
$$

得粒子处于能量本征态 $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$ 的概率

$$P_n=|C_n|^2=\frac{480}{a^6}\frac{a^6}{n^6\pi^6}\left[1-(-1)^n\right]=\frac{480}{n^6\pi^6}\left[1-(-1)^n\right]$$

(3) 不妨取 $a=2$，此时二者图像为



可以看出，两个图像几乎完全重合，这是因为 $P_n \sim \frac{1}{n^6}$，故 $P_1 \gg P_n(n \neq 1)$ ，即 $\psi(x)$ 几乎与基态 $\psi_n(x)$ 完全相同.

题目 3. (教材 2.5) 同上题. 设粒子处于基态 $(n=1), E_1 = \pi^2\hbar^2/2ma^2$. 设 $t=0$ 时刻阱宽突然变为 $2a$, 粒子波函数来不及改变, 即

$$\psi(x,0) = \psi_1(x) = \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}$$

试问: 对于加宽了的无限深方势阱

$$V(x) = \begin{cases} 0, & 0 < x < 2a \\ \infty, & x < 0, x > 2a \end{cases}$$

$\psi(x,0)$ 是否还是能量本征态? 求测得粒子能量仍为 E_1 的概率.

提示: 阱宽为 $2a$ 的无限深方势阱中的粒子能量本征值 $\varepsilon_n = n^2\pi^2\hbar^2/8ma^2$, 本征函数 $\psi_n(x) = \sqrt{\frac{1}{a}}\sin\frac{n\pi x}{2a}, \varepsilon_2 = E_1, \psi(x,0)$ 用 $\psi_n(x)$ 展开, $\psi(x,0) = \sum_n C_n\psi_n(x)$, 求出 $|C_2|^2$.

解答. 变化前，能量本征值和本征波函数为

$$E_n = \frac{\pi^2 n^2\hbar^2}{2ma^2}, \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

变化后，能量本征值和本征波函数为

$$\varepsilon_n = \frac{\pi^2 n^2\hbar^2}{8ma^2}, \varphi_n(x) = \sqrt{\frac{1}{a}}\sin\left(\frac{n\pi x}{2a}\right)$$

变化瞬间，能量与波函数来不及改变，因此

$$E = E_1 = \frac{\pi^2 n^2\hbar^2}{2ma^2} = \varepsilon_2$$

$$\psi(x,0) = \psi_1(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$$

变化瞬间的波函数可以展开为变化后的本征波函数叠加

$$\psi(x,0) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) = \sum_{n=1}^{\infty} C_n\varphi_n(x)$$

有

$$C_2 = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\sqrt{\frac{1}{a}}\sin\left(\frac{\pi x}{a}\right)dx = \frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx = \frac{1}{\sqrt{2}}$$

因此所求概率

$$P_2 = |C_2|^2 = \frac{1}{2}$$

题目 4. (教材 2.7) 利用 Hermite 多项式的递推关系 (附录 A3, 式 (13)), 证明谐振子波函数满足下列关系

$$\begin{aligned}
x\psi_n(x) =& \frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x)+\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right] \\
x^2\psi_n(x) =& \frac{1}{2\alpha^2}\left[\sqrt{n(n-1)}\psi_{n-2}(x)+(2n+1)\psi_n(x)\right. \\
& \left.+\sqrt{(n+1)(n+2)}\psi_{n+2}(x)\right]
\end{aligned}$$

并由此证明, 在 ψ_n 态下, $\bar{x}=0, \bar{V}=E_n/2$.

解答. 谐振子的波函数

$$\psi_n(x)=N_nH_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}=\sqrt{\frac{\alpha}{\sqrt{\pi}2^nn!}}H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}$$

Hermite 多项式的递推公式为

$$H_{n+1}\left(x\right)-2xH_n\left(x\right)+2nH_{n-1}\left(x\right)=0$$

因此

$$\begin{aligned}
x\psi_n(x) &= x\sqrt{\frac{\alpha}{\sqrt{\pi}2^nn!}}H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2} \\
&= \frac{1}{2\alpha}e^{-\frac{1}{2}\alpha^2x^2}\sqrt{\frac{\alpha}{\sqrt{\pi}2^nn!}}\left[H_{n+1}\left(\alpha x\right)+2nH_{n-1}\left(\alpha x\right)\right] \\
&= \frac{1}{2\alpha}\sqrt{\frac{\alpha}{\sqrt{\pi}2^nn!}}\left[\frac{\psi_{n+1}(x)}{\sqrt{\frac{\alpha}{\sqrt{\pi}2^{n+1}(n+1)!}}}+2n\frac{\psi_{n-1}(x)}{\sqrt{\frac{\alpha}{\sqrt{\pi}2^{n-1}(n-1)!}}}\right] \\
&= \frac{1}{2\alpha}\left[\sqrt{2\left(n+1\right)}\psi_{n+1}(x)+2n\sqrt{\frac{1}{2n}}\psi_{n-1}(x)\right] \\
&= \frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x)+\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right] \\
x^2\psi_n(x) &= \frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}x\psi_{n-1}(x)+\sqrt{\frac{n+1}{2}}x\psi_{n+1}(x)\right] \\
&= \frac{1}{\alpha^2}\left[\sqrt{\frac{n}{2}}\left(\sqrt{\frac{n-1}{2}}\psi_{n-2}(x)+\sqrt{\frac{n}{2}}\psi_n(x)\right)+\sqrt{\frac{n+1}{2}}\left(\sqrt{\frac{n+1}{2}}\psi_n(x)+\sqrt{\frac{n+2}{2}}\psi_{n+2}(x)\right)\right] \\
&= \frac{1}{2\alpha^2}\left[\sqrt{n\left(n-1\right)}\psi_{n-2}(x)+\left(2n+1\right)\psi_n(x)+\sqrt{n\left(n+1\right)}\psi_{n+2}(x)\right]
\end{aligned}$$

从而 x 的平均值

$$\bar{x}=\int\psi_n^*\left(x\right)x\psi_n\left(x\right)dx=\int\psi_n^*\left(x\right)\frac{1}{\alpha}\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x)+\sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right]dx=0$$

势能平均值

$$
\begin{aligned}
\bar{V} &= \int \psi_n^*(x)\,\frac{1}{2}m\omega^2 x^2 \psi_n(x)\,dx \\
&= \frac{1}{2}m\omega^2 \int \psi_n^*(x)\,\frac{1}{2\alpha^2}\left[\sqrt{n(n-1)}\psi_{n-2}(x) + (2n+1)\,\psi_n(x) + \sqrt{n(n+1)}\psi_{n+2}(x)\right]dx \\
&= \frac{1}{2}m\omega^2 \int \psi_n^*(x)\,\frac{1}{2\alpha^2}(2n+1)\,\psi_n(x)dx = \frac{(2n+1)\,m\omega^2}{4\alpha^2} = \frac{(2n+1)\hbar\omega}{4} = \frac{E_n}{2}
\end{aligned}
$$

题目 5. (教材 2.8) 同上题, 利用 Hermite 多项式的求导公式 (附录 A3, 式 (14)), 证明

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x) =& \alpha\left[\sqrt{\frac{n}{2}}\psi_{n-1} - \sqrt{\frac{n+1}{2}}\psi_{n+1}\right] \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x) =& \frac{\alpha^2}{2}\left[\sqrt{n(n-1)}\psi_{n-2} - (2n+1)\psi_n\right. \\
&\left. + \sqrt{(n+1)(n+2)}\psi_{n+2}\right]
\end{aligned}
$$

并由此证明, 在 ψ_n 态下, $\bar{p}=0, \bar{T}=\bar{p^2}/2m=E_n/2$.

解答. 谐振子的波函数

$$
\psi_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2} = \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}} H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}
$$

Hermite 多项式的求导公式为

$$
H_n'(x) = 2nH_{n-1}(x)
$$

因此

$$
\begin{aligned}
\frac{d}{dx}\psi_n(x) &= \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\left[\frac{dH_n(\alpha x)}{dx}\mathrm{e}^{-\frac{1}{2}\alpha^2x^2} + H_n(\alpha x)\frac{de^{-\frac{1}{2}\alpha^2x^2}}{dx}\right] \\
&= \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\left[\alpha H_n'(\alpha x)\,\mathrm{e}^{-\frac{1}{2}\alpha^2x^2} - \alpha^2 x H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}\right] \\
&= \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\left[2\alpha n H_{n-1}(\alpha x)\,\mathrm{e}^{-\frac{1}{2}\alpha^2x^2} - \alpha^2 x H_n(\alpha x)e^{-\frac{1}{2}\alpha^2x^2}\right] \\
&= 2\sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\alpha n H_{n-1}(\alpha x)\,\mathrm{e}^{-\frac{1}{2}\alpha^2x^2} - \alpha^2 x\psi_n(x) \\
&= 2\sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\alpha n \frac{\psi_{n-1}(x)}{\sqrt{\frac{\alpha}{\sqrt{\pi}2^{n-1}(n-1)!}}} - \alpha^2 x\psi_n(x) \\
&= 2\alpha\sqrt{\frac{n}{2}}\psi_{n-1}(x) - \alpha^2 x\psi_n(x) \\
&= 2\alpha\sqrt{\frac{n}{2}}\psi_{n-1}(x) - \alpha\left[\sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x)\right]
\end{aligned}
$$

$$= \alpha \left[\sqrt{\frac{n}{2}} \psi_{n-1}(x) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(x) \right]$$

$$\begin{aligned}
\frac{d^2\psi_n(x)}{dx^2} &= \alpha \left[\sqrt{\frac{n}{2}} \frac{d\psi_{n-1}(x)}{dx} - \sqrt{\frac{n+1}{2}} \frac{d\psi_{n+1}(x)}{dx} \right] \\
&= \alpha^2 \left[\ \sqrt{\tfrac{n}{2}} \left(\sqrt{\tfrac{n-1}{2}} \psi_{n-2}(x) - \sqrt{\tfrac{n}{2}} \psi_n(x) \right) - \sqrt{\tfrac{n+1}{2}} \left(\sqrt{\tfrac{n+1}{2}} \psi_n(x) - \sqrt{\tfrac{n+2}{2}} \psi_{n+2}(x) \right) \ \right] \\
&= \frac{\alpha^2}{2} \left[\sqrt{n(n-1)} \psi_{n-2} - (2n+1) \psi_n + \sqrt{(n+1)(n+2)} \psi_{n+2} \right]
\end{aligned}$$

从而动量平均值

$$\begin{aligned}
\bar{\boldsymbol{p}}_n &= \int \psi_n^* (x) \, \hat{\boldsymbol{p}} \psi_n (x) \, dx = \int \psi_n^* (x) \left(-i\hbar \frac{d}{dx} \right) \psi_n (x) \, dx \\
&= -i\hbar \int \psi_n^* (x) \, \alpha \left[\sqrt{\frac{n}{2}} \psi_{n-1}(x) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(x) \right] dx = 0
\end{aligned}$$

动能平均值

$$\begin{aligned}
\bar{T} &= \int \psi_n^* (x) \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) \psi_n (x) \, dx \\
&= -\frac{\hbar^2}{2m} \int \psi_n^* (x) \, \frac{\alpha^2}{2} \left[\sqrt{n(n-1)} \psi_{n-2} - (2n+1) \psi_n + \sqrt{(n+1)(n+2)} \psi_{n+2} \right] dx \\
&= \frac{\hbar^2}{2m} \int \psi_n^* (x) \, \frac{\alpha^2}{2} (2n+1) \psi_n dx = \frac{(2n+1)\alpha^2\hbar^2}{4m} = \frac{(2n+1)\hbar\omega}{4} = \frac{E_n}{2}
\end{aligned}$$

题目 6. (教材 2.9) 谐振子处于 ϕ_n 态下, 计算

$$\Delta x = \left[\overline{(x-\bar{x})^2} \right]^{1/2}, \quad \Delta p = \left[\overline{(p-\bar{p})^2} \right]^{1/2}, \quad \Delta x \Delta p = ?$$

解答. 由前两题可知

$$\bar{x} = 0, \bar{p} = 0$$

因此有

$$\begin{aligned}
(\Delta x)^2 = \overline{(x-\bar{x})^2} = \overline{x^2} &= \int \psi_n^* (x) \, x^2 \psi_n (x) \, dx \\
&= \int \psi_n^* (x) \, \frac{1}{2\alpha^2} \left[\sqrt{n\,(n-1)} \psi_{n-2}(x) + (2n+1)\, \psi_n(x) + \sqrt{n\,(n+1)} \psi_{n+2}(x) \right] dx \\
&= \int \psi_n^* (x) \, \frac{1}{2\alpha^2} \, (2n+1) \, \psi_n(x) dx = \frac{(2n+1)}{2\alpha^2}
\end{aligned}$$

$$
\begin{aligned}
(\Delta p)^2 = \overline{(p-\bar{p})^2} = \overline{p^2} &= \int \psi_n^*(x)\left(-\hbar^2\frac{d^2}{dx^2}\right)\psi_n(x)\,dx \\
&= -\hbar^2\int \psi_n^*(x)\,\frac{\alpha^2}{2}\left[\sqrt{n(n-1)}\psi_{n-2}-(2n+1)\psi_n+\sqrt{(n+1)(n+2)}\psi_{n+2}\right]dx \\
&= \hbar^2\int \psi_n^*(x)\,\frac{\alpha^2}{2}(2n+1)\psi_n dx = \frac{(2n+1)\hbar^2\alpha^2}{2}
\end{aligned}
$$

即

$$
\Delta x = \frac{1}{\alpha}\sqrt{\frac{2n+1}{2}}，\ \Delta p = \hbar\alpha\sqrt{\frac{2n+1}{2}}
$$

得

$$
\Delta x\Delta p = \left(n+\frac{1}{2}\right)\hbar
$$

题目 7. (教材 3.3) 设 $F(x,p)$ 是 x,p 的整函数, 证明

$$
[p,F] = -\mathrm{i}\hbar\frac{\partial}{\partial x}F, \quad [x,F] = \mathrm{i}\hbar\frac{\partial}{\partial p}F
$$

整函数是指 $F(x,p)$ 可以展开成

$$
F(x,p) = \sum_{m,n=0}^{\infty} C_{mn}x^m p^n
$$

解答. 易知

$$
\begin{aligned}
[p,x^m] &= -i\hbar\frac{\partial}{\partial x}x^m + i\hbar x^m\frac{\partial}{\partial x} = -i\hbar m x^{m-1} \\
[x,p^n] &= (-i\hbar)^n x\frac{\partial^n}{\partial x^n} - (-i\hbar)^n\frac{\partial^n}{\partial x^n}x = (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - \frac{\partial^{n-1}}{\partial x^{n-1}}\frac{\partial}{\partial x}x\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - \frac{\partial^{n-1}}{\partial x^{n-1}}\left(1+x\frac{\partial}{\partial x}\right)\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - \frac{\partial^{n-1}}{\partial x^{n-1}} - \left(\frac{\partial^{n-1}}{\partial x^{n-1}}x\right)\frac{\partial}{\partial x}\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - \frac{\partial^{n-1}}{\partial x^{n-1}} - \left(\frac{\partial^{n-2}}{\partial x^{n-2}}\frac{\partial}{\partial x}x\right)\frac{\partial}{\partial x}\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - \frac{\partial^{n-1}}{\partial x^{n-1}} - \left(\frac{\partial^{n-2}}{\partial x^{n-2}}\left(1+x\frac{\partial}{\partial x}\right)\right)\frac{\partial}{\partial x}\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - k\frac{\partial^{n-1}}{\partial x^{n-1}} - \left(\frac{\partial^{n-k-1}}{\partial x^{n-k-1}}\frac{\partial}{\partial x}x\right)\frac{\partial^k}{\partial x^k}\right] \\
&= (-i\hbar)^n\left[x\frac{\partial^n}{\partial x^n} - n\frac{\partial^{n-1}}{\partial x^{n-1}} - x\frac{\partial^n}{\partial x^n}\right] \\
&= -n(-i\hbar)^n\frac{\partial^{n-1}}{\partial x^{n-1}} = i\hbar n p^{n-1}
\end{aligned}
$$

因此

$$
\begin{aligned}
[p, F] &= \left[p, \sum_{m,n=0}^{\infty} C_{mn} x^m p^n\right] = \sum_{m,n=0}^{\infty} C_{mn} \left[p, x^m\right] p^n \\
&= \sum_{m,n=0}^{\infty} C_{mn} \left(-i\hbar m x^{m-1}\right) p^n = -i\hbar \frac{\partial}{\partial x} \sum_{m,n=0}^{\infty} C_{mn} x^m p^n = -i\hbar \frac{\partial}{\partial x} F \\
[x, F] &= \left[x, \sum_{m,n=0}^{\infty} C_{mn} x^m p^n\right] = \sum_{m,n=0}^{\infty} C_{mn} x^m \left[x, p^n\right] \\
&= \sum_{m,n=0}^{\infty} C_{mn} x^m i\hbar n p^{n-1} = i\hbar n \sum_{m,n=0}^{\infty} C_{mn} x^m p^{n-1} \\
&= i\hbar \frac{\partial}{\partial p} \sum_{m,n=0}^{\infty} C_{mn} x^m p^n = i\hbar \frac{\partial}{\partial p} F
\end{aligned}
$$