微积分 A (2)

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第 17 讲

在听课过程中,

严禁使用与教学无关的电子产品!

期中考试评讲

例 1. 设 $K \subseteq \mathbb{R}^k$ 为有界闭集, $f \in \mathscr{C}(\mathbb{R}^m \times K)$. $\forall X \in \mathbb{R}^m$, 令 $g(X) = \min\limits_{Y \in K} f(X, Y)$. 求证: $g \in \mathscr{C}(\mathbb{R}^m)$.

证明: 固定 $X_0 \in \mathbb{R}^m$, 并令 $E = \overline{B(X_0, 1)} \times K$, 则 E为有界闭集. 由于 $f \in \mathscr{C}(\mathbb{R}^m \times K)$, 则 f 在 E 上一致连续, 从而 $\forall \varepsilon > 0$, $\exists \delta \in (0, 1)$ 使得 $\forall (X_1, Y_1), (X_2, Y_2) \in E$, 若 $\|(X_1, Y_1) - (X_2, Y_2)\| < \delta$, 则我们有 $|f(X_1, Y_1) - f(X_2, Y_2)| < \varepsilon$. 特别地, $\forall X \in B(X_0, \delta)$ 以及 $Y \in K$, 我们有

$$|f(X, Y) - f(X_0, Y)| < \varepsilon,$$

也即我们有

$$f(X_0,Y)-\varepsilon < f(X,Y) < f(X_0,Y)| + \varepsilon.$$

将上式对 $Y \in K$ 取下确界可得

$$g(X_0)-\varepsilon \leqslant g(X) \leqslant g(X_0)+\varepsilon.$$

故 g 在点 X_0 处连续, 进而可知 $g \in \mathscr{C}(\mathbb{R}^m)$.

例 2. 设 $\Omega \subseteq \mathbb{R}^2$ 为有界闭区域, 而 $f \in \mathscr{C}(\Omega)$. 求证: 至多存在一个 $u \in \mathscr{C}(\Omega)$ 使 $u \in \mathscr{C}^{(2)}(\mathring{\Omega})$,

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = e^{u(x,y)}, & \forall (x,y) \in \mathring{\Omega}, \\ u(x,y) = f(x,y), & \forall (x,y) \in \partial\Omega. \end{cases}$$

证明: 用反证法, 假设 $\exists u, v \in \mathscr{C}(\Omega)$ 满足题设条件且 $u \neq v$. 定义 $g = u - v$. 由于 $g \in \mathscr{C}(\Omega)$ 且 Ω 为有界闭集, 则由最值定理、$g \neq 0$ 以及 $g\big|_{\partial\Omega} = 0$ 知, 我们在 Ω 上可找到 g 的最值点 P_0 使 $g(P_0) \neq 0$, 则 $P_0 \in \mathring{\Omega}$. 不失一般性, 设 $g(P_0) < 0$,

否则考虑 $-g$. 于是 P_0 为 g 的最小值点, 从而海赛矩阵 $H_g(P_0)$ 为正定或半正定, 于是

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2}(P_0) &= (1,0)H_g(P_0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \geqslant 0, \\ \frac{\partial^2 g}{\partial y^2}(P_0) &= (0,1)H_g(P_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \geqslant 0. \end{aligned}$$

但由题设又可得知

$$0 \leqslant \frac{\partial^2 g}{\partial x^2}(P_0) + \frac{\partial^2 g}{\partial y^2}(P_0) = e^{u(P_0)} - e^{v(P_0)} < 0,$$

矛盾! 故所证结论成立.

例 3. $\forall y \in \mathbb{R}$, 令 $I(y) = \int_0^{+\infty} e^{-x^2} \sin(2xy)\,\mathrm{d}x$.
求证: $I(y) = e^{-y^2} \int_0^y e^{x^2}\,\mathrm{d}x$.

证明: $\forall y \in \mathbb{R}, \forall x \geqslant 0$, $|e^{-x^2} \sin(2xy)| \leqslant e^{-x^2}$, 且

$$\left|\frac{\partial(e^{-x^2} \sin(2xy))}{\partial y}\right| = |2xe^{-x^2} \cos(2xy)| \leqslant xe^{-x^2},$$

又 $\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x$, $\int_0^{+\infty} xe^{-x^2}\,\mathrm{d}x$ 收敛, 由 Weierstrass 判别准则, 含参广义积分 $\int_0^{+\infty} e^{-x^2} \sin(2xy))\,\mathrm{d}x$, $\int_0^{+\infty} 2xe^{-x^2} \cos(2xy))\,\mathrm{d}x$ 关于 $y \in \mathbb{R}$ 一致收敛,

于是由求导与积分可交换性知 I 连续可导, 且

$$
\begin{aligned}
I'(y) &= \int_0^{+\infty} 2xe^{-x^2}\cos(2xy)\,\mathrm{d}x \\
&= -e^{-x^2}\cos(2xy)\Big|_0^{+\infty} - 2y\int_0^{+\infty} e^{-x^2}\sin(2xy)\,\mathrm{d}x \\
&= 1 - 2yI(y).
\end{aligned}
$$

注意到 $I(0)=0$, 则我们有

$$
\begin{aligned}
I(y) &= e^{-\int_0^y 2x\,\mathrm{d}x}\left(I(0) + \int_0^y e^{\int_0^x 2t\,\mathrm{d}t}\,\mathrm{d}x\right) \\
&= e^{-y^2}\int_0^y e^{x^2}\,\mathrm{d}x.
\end{aligned}
$$

期中考试到此结束

综合练习

例 1. 假设 $D \subset \mathbb{R}^n$ 为有界闭区域, $f: D \to \mathbb{R}$ 为连续函数, 而 $\varphi: D \to \mathbb{R}$ 为非负可积函数. 求证:

$$\exists X_0 \in D \text{ 使得 } \int_D f(X)\varphi(X)\,\mathrm{d}X = f(X_0)\int_D \varphi(X)\,\mathrm{d}X.$$

证明: 因函数 f 连续, 而 D 为有界闭集, 从而 f 在 D 上有最大值 M 和最小值 m, 于是我们有

$$m\int_D \varphi(X)\,\mathrm{d}X \leqslant \int_D f(X)\varphi(X)\,\mathrm{d}X \leqslant M\int_D \varphi(X)\,\mathrm{d}X.$$

如果 $\int_D \varphi(X)\,\mathrm{d}X = 0$, 则 $\int_D f(X)\varphi(X)\,\mathrm{d}X = 0$, 此时所证等式对任意 $X_0 \in D$ 均成立.

若 $\int_D \varphi(X)\,\mathrm{d}X \neq 0$, 则 $m \leqslant \dfrac{\int_D f(X)\varphi(X)\,\mathrm{d}X}{\int_D \varphi(X)\,\mathrm{d}X} \leqslant M$, 从而由连续函数介值定理可知 $\exists X_0 \in D$ 使得

$$f(X_0) = \frac{\int_D f(X)\varphi(X)\,\mathrm{d}X}{\int_D \varphi(X)\,\mathrm{d}X},$$

由此立刻可得所证结论成立.

例 2. 交换下述累次积分次序

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \mathrm{d}y \int_{\frac{1}{2}}^{\sqrt{y}} f(x,y)\,\mathrm{d}x + \int_{\frac{1}{2}}^{1} \mathrm{d}y \int_{y}^{\sqrt{y}} f(x,y)\,\mathrm{d}x.$$

解: 上述累次积分等于

$$= \iint\limits_{\substack{\frac{1}{2}\leqslant x\leqslant\sqrt{y} \\ \frac{1}{4}\leqslant y\leqslant\frac{1}{2}}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{y\leqslant x\leqslant\sqrt{y} \\ \frac{1}{2}\leqslant y\leqslant 1}} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{\substack{\frac{1}{2}\leqslant x\leqslant 1 \\ x^2\leqslant y\leqslant x}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\frac{1}{2}}^{1} \mathrm{d}x \int_{x^2}^{x} f(x,y)\,\mathrm{d}y.$$

例 3. 交换下述累次积分的次序

$$\int_0^1 \int_0^{x^2} f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^1 f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_2^3 \int_0^{3-x} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

解: 上述累次积分等于

$$\begin{aligned}
&\iint\limits_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant x^2}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{1\leqslant x\leqslant 2\\ 0\leqslant y\leqslant 1}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{2\leqslant x\leqslant 3\\ 0\leqslant y\leqslant 3-x}} f(x,y)\,\mathrm{d}x\mathrm{d}y\\
=&\iint\limits_{\substack{0\leqslant y\leqslant 1\\ \sqrt{y}\leqslant x\leqslant 1}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{0\leqslant y\leqslant 1\\ 1\leqslant x\leqslant 2}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{0\leqslant y\leqslant 1\\ 2\leqslant x\leqslant 3-y}} f(x,y)\,\mathrm{d}x\mathrm{d}y\\
=&\int_0^1 \int_{\sqrt{y}}^1 f(x,y)\,\mathrm{d}x\mathrm{d}y + \int_0^1 \int_1^2 f(x,y)\,\mathrm{d}x\mathrm{d}y + \int_0^1 \int_2^{3-y} f(x,y)\,\mathrm{d}x\mathrm{d}y\\
=&\int_0^1 \int_{\sqrt{y}}^{3-y} f(x,y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

例 4. 交换下述累次积分的次序

$$\int_{-1}^{0} \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x + \int_{0}^{1} \mathrm{d}y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,\mathrm{d}x.$$

解: 上述累次积分等于

$$\iint\limits_{\substack{-\sqrt{1-y^2}\leqslant x\leqslant\sqrt{1-y^2}\\ -1\leqslant y\leqslant 0}} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint\limits_{\substack{-\sqrt{1-y}\leqslant x\leqslant\sqrt{1-y}\\ 0\leqslant y\leqslant 1}} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{\substack{-1\leqslant x\leqslant 1\\ -\sqrt{1-x^2}\leqslant y\leqslant 1-x^2}} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$= \int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,\mathrm{d}y.$$

例 5. 设 $D \subset \mathbb{R}^n$ 为有界闭区域, 而 $f: D \to \mathbb{R}$ 非负连续. 求证: 若 $\int_D f(X)\,\mathrm{d}X = 0$, 则 $f \equiv 0$.

证明: 用反证法, 设函数 f 在 D 上不恒等于 0. 则由连续性可知 f 在 $\mathrm{int}D$ 上也不恒为零, 于是 $\exists X_0 \in \mathrm{int}D$ 使得 $f(X_0) > 0$. 又由连续性可知, $\exists r > 0$ 使得 $\forall X \in B(X_0, r) \subset \mathrm{int}D$, 我们均有 $f(X) > \frac{1}{2} f(X_0)$, 从而我们有

$$\int_D f(X)\,\mathrm{d}X \geqslant \int_{B(X_0,r)} f(X)\,\mathrm{d}X \geqslant \frac{1}{2} f(X_0)|B(X_0,r)| > 0,$$

矛盾! 故所证结论成立.

例 6. $\forall f \in \mathscr{C}[a,b]$, 求证:

$$\Big(\int_a^b f(x)\,\mathrm{d}x\Big)^2 \leqslant (b-a)\int_a^b (f(x))^2\,\mathrm{d}x,$$

其中等号成立当且仅当 f 为常值函数.

证明: 由题设可知

$$\begin{aligned}\Big(\int_a^b f(x)\,\mathrm{d}x\Big)^2 &= \Big(\int_a^b f(x)\,\mathrm{d}x\Big)\Big(\int_a^b f(y)\,\mathrm{d}y\Big)\\ &= \int_a^b \Big(\int_a^b f(x)f(y)\,\mathrm{d}x\Big)\mathrm{d}y.\end{aligned}$$

由此我们立刻可得

$$
\begin{aligned}
\Big(\int_a^b f(x)\,\mathrm{d}x\Big)^2 &\leqslant \int_a^b \Big(\int_a^b \frac{1}{2}\big((f(x))^2+(f(y))^2\big)\,\mathrm{d}x\Big)\mathrm{d}y \\
&= \frac{1}{2}\int_a^b \Big(\int_a^b (f(x))^2\,\mathrm{d}x\Big)\mathrm{d}y + \frac{1}{2}\int_a^b \Big(\int_a^b (f(y))^2\,\mathrm{d}x\Big)\mathrm{d}y \\
&= (b-a)\int_a^b (f(x))^2\,\mathrm{d}x,
\end{aligned}
$$

其中等号成立当且仅当 $\forall x, y \in [a,b]$, 均有

$$
f(x)f(y) = \tfrac{1}{2}\big((f(x))^2+(f(y))^2\big),
$$

也即 $f(x)=f(y)$, 这等价于说 f 为常值函数.

例 7. 求平面 $3x-y-z=\pm1$, $-x+3y-z=\pm1$, $-x-y+3z=\pm1$ 所围成的立体的体积.

解: 考虑坐标变换

$$\begin{cases} u = 3x - y - z, \\ v = -x + 3y - z, \\ w = -x - y + 3z. \end{cases}$$

所围立体在此变换下变为:

$$-1 \leqslant u \leqslant 1,\ -1 \leqslant v \leqslant 1,\ -1 \leqslant w \leqslant 1.$$

此外我们还有

$$\frac{D(u,v,w)}{D(x,y,z)} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16,$$

故 $\frac{D(x,y,z)}{D(u,v,w)} = \frac{1}{16}$. 由此可知所求体积为

$$\begin{aligned} V &= \iiint\limits_{\substack{-1\leqslant u\leqslant 1\\ -1\leqslant v\leqslant 1\\ -1\leqslant w\leqslant 1}} \left|\frac{D(x,y,z)}{D(u,v,w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= \frac{1}{16}\int_{-1}^{1}\left(\int_{-1}^{1}\left(\int_{-1}^{1}\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}w = \frac{1}{2}. \end{aligned}$$

例 8. 求曲线 $(\frac{x^2}{a^2}+\frac{y^2}{b^2})^2=x^2+y^2$ $(a,b>0)$ 所围平面图形的面积.

解: 作变换

$$x = a\rho\cos\varphi,\ \ y = b\rho\sin\varphi.$$

在此变换下, 积分区域变为

$$D'=\left\{(\rho,\varphi)\mid 0\leqslant\varphi\leqslant 2\pi,\ \ 0\leqslant\rho\leqslant\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right\}.$$

与此同时, 我们还有 $\frac{D(x,y)}{D(\rho,\varphi)} = ab\rho.$

从而所求面积为

$$\begin{aligned}
S &= \iint\limits_{(\frac{x^2}{a^2}+\frac{y^2}{b^2})^2\leqslant x^2+y^2} \mathrm{d}x\mathrm{d}y = \iint\limits_{D'} ab\rho\,\mathrm{d}\rho\mathrm{d}\varphi \\
&= ab\int_0^{2\pi}\left(\int_0^{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}\rho\,\mathrm{d}\rho\right)\mathrm{d}\varphi \\
&= \frac{1}{2}ab\int_0^{2\pi}(a^2\cos^2\varphi+b^2\sin^2\varphi)\,\mathrm{d}\varphi \\
&= \frac{1}{2}ab\left(a^2\frac{\varphi+\frac{1}{2}\sin 2\varphi}{2}+b^2\frac{\varphi-\frac{1}{2}\sin 2\varphi}{2}\right)\Bigg|_0^{2\pi} \\
&= \frac{1}{2}ab\pi(a^2+b^2).
\end{aligned}$$

例 9. 设 $h=\sqrt{a^2+b^2+c^2}>0$. 若 $f\in\mathscr{C}[-h,h]$,
求证: $\iiint\limits_V f(ax+by+cz)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\pi\int_{-1}^{1}(1-t^2)f(ht)\,\mathrm{d}t$,
其中 $V=\{(x,y,z)\mid x^2+y^2+z^2\leqslant 1\}$.

解: 定义 $\vec{e}_1=\left(\frac{a}{h},\frac{b}{h},\frac{c}{h}\right)$, 则 $\|\vec{e}_1\|=1$, 由此可构造两单位行向量 $\vec{e}_2$, $\vec{e}_3$ 使得 $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ 为两两正交.
令 $U=\begin{pmatrix}\vec{e}_2\\ \vec{e}_3\\ \vec{e}_1\end{pmatrix}$, 则 U 为正交矩阵.

作变换

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = U \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

由正交矩阵的性质 $U^T = U^{-1}$ 可知

$$x^2 + y^2 + z^2 = (x, y, z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (X, Y, Z)(U^{-1})^T U^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = X^2 + Y^2 + Z^2,$$

由此立刻可得

$$
\begin{aligned}
&\iiint\limits_{V} f(ax+by+cz)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
=& \iiint\limits_{x^2+y^2+z^2\leqslant 1} f(ax+by+cz)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
=& \iiint\limits_{X^2+Y^2+Z^2\leqslant 1} f(hZ)\left|\frac{D(x,y,z)}{D(X,Y,Z)}\right|\mathrm{d}X\mathrm{d}Y\mathrm{d}Z \\
=& \iiint\limits_{X^2+Y^2+Z^2\leqslant 1} f(hZ)|\det U^{-1}|\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z
\end{aligned}
$$

$$
\begin{aligned}
&= \iiint\limits_{X^2+Y^2+Z^2\leqslant 1} f(hZ)|\det U^{-1}|\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z \\
&= \iiint\limits_{X^2+Y^2+Z^2\leqslant 1} f(hZ)\,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z \\
&\overset{\substack{X=\rho\cos\varphi \\ Y=\rho\sin\varphi \\ Z=t}}{=} \int_{-1}^{1}\left(\int_0^{2\pi}\left(\int_0^{\sqrt{1-t^2}} f(ht)\rho\,\mathrm{d}\rho\right)\mathrm{d}\varphi\right)\mathrm{d}t \\
&= 2\pi\int_{-1}^{1}\left(\frac{1}{2}f(ht)\rho^2\right)\Big|_0^{\sqrt{1-t^2}}\,\mathrm{d}t \\
&= \pi\int_{-1}^{1}(1-t^2)f(ht)\,\mathrm{d}t.
\end{aligned}
$$

例 10. 设 $f:[0,+\infty)\to\mathbb{R}$ 可微. $\forall t>0$, 定义

$$F(t)=\iiint\limits_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

计算 $F'(t)$.

解: 由题设可知

$$\begin{aligned}F(t) &= \iiint\limits_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{t} f(r^2)r^2\sin\theta\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}\varphi\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{t} r^2 f(r^2) \sin\theta \,\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}\varphi \\
&= 2\pi \left(\int_0^{\pi} \sin\theta \,\mathrm{d}\theta \right) \left(\int_0^{t} r^2 f(r^2) \,\mathrm{d}r \right) \\
&= 4\pi \int_0^{t} r^2 f(r^2) \,\mathrm{d}r,
\end{aligned}
$$

由此我们立刻可得

$$
F'(t) = 4\pi t^2 f(t^2).
$$

例 11. 计算

$$I=\iiint\limits_{x^2+y^2+z^2\leqslant 1}\left(x\sin(x^2+z^2)+ye^{x^2+z^2}+z(y^2+z^2)+1\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

解: 由对称性可知

$$\begin{aligned}I &= \iiint\limits_{x^2+y^2+z^2\leqslant 1} x\sin(x^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z+\iiint\limits_{x^2+y^2+z^2\leqslant 1} ye^{x^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad+\iiint\limits_{x^2+y^2+z^2\leqslant 1} z(y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z+\iiint\limits_{x^2+y^2+z^2\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &= \iiint\limits_{x^2+y^2+z^2\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{4}{3}\pi.\end{aligned}$$

例 12. 求由下述方程定义的两个球体

$$x^2+y^2+z^2 \leqslant 1,\ x^2+y^2+(z-2)^2 \leqslant 4$$

相交部分的体积.

解: 将两球体相交部分记作 Ω, 则

$$\Omega=\left\{(x,y,z) \mid x^2+y^2 \leqslant \frac{15}{16},\right.$$
$$\left. 2-\sqrt{4-x^2-y^2} \leqslant z \leqslant \sqrt{1-x^2-y^2}\right\},$$

由此可知所求体积为

$$|\Omega| = \iint\limits_{x^2+y^2\leqslant\frac{15}{16}} \left(\sqrt{1-x^2-y^2} - (2-\sqrt{4-x^2-y^2})\right) \mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{x^2+y^2\leqslant\frac{15}{16}} (\sqrt{1-x^2-y^2} + \sqrt{4-x^2-y^2}) \,\mathrm{d}x\mathrm{d}y - 2\pi\cdot\frac{15}{16}$$

$$\overset{\substack{x=\rho\cos\varphi\\ y=\rho\sin\varphi}}{=} \int_0^{2\pi} \left(\int_0^{\frac{\sqrt{15}}{4}} (\sqrt{1-\rho^2} + \sqrt{4-\rho^2})\rho \,\mathrm{d}\rho \right) \mathrm{d}\varphi - \frac{15}{8}\pi$$

$$= -2\pi\left(\frac{1}{3}(1-\rho^2)^{\frac{3}{2}} + \frac{1}{3}(4-\rho^2)^{\frac{3}{2}}\right)\Bigg|_0^{\frac{\sqrt{15}}{4}} - \frac{15}{8}\pi = \frac{13}{24}\pi.$$

例 13. 求曲面 $z=\sqrt{x^2-y^2}$ 在由方程

$$(x^2+y^2)^2=a^2(x^2-y^2)$$

定义的柱面内的部分的面积, 其中 $a>0$.

解: 将曲面 $z=\sqrt{x^2-y^2}$ 位于柱面内的那部分记作 Σ, 则其满足的方程为

$$z=\sqrt{x^2-y^2},\quad (x,y)\in D,$$

其中 $D=\{(x,y)\mid (x^2+y^2)^2\leqslant a^2(x^2-y^2)\}$.

注意到 D 关于 x, y 轴对称且在极坐标下变为

$$D_1=\left\{(\rho,\varphi)\mid 0\leqslant\rho\leqslant a\sqrt{\cos 2\varphi},\ -\frac{\pi}{4}\leqslant\varphi\leqslant\frac{\pi}{4}\text{ 或 }\frac{3\pi}{4}\leqslant\varphi\leqslant\frac{5\pi}{4}\right\},$$

而 $\dfrac{\partial z}{\partial x}=\dfrac{x}{\sqrt{x^2-y^2}}$, $\dfrac{\partial z}{\partial y}=-\dfrac{y}{\sqrt{x^2-y^2}}$, 则所求面积为

$$S=\iint\limits_D\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,\mathrm{d}x\mathrm{d}y=\iint\limits_D\sqrt{\frac{2x^2}{x^2-y^2}}\,\mathrm{d}x\mathrm{d}y$$

$$=4\int_0^{\frac{\pi}{4}}\Big(\int_0^{a\sqrt{\cos 2\varphi}}\sqrt{\frac{2\cos^2\varphi}{\cos 2\varphi}}\rho\,\mathrm{d}\rho\Big)\mathrm{d}\varphi=2\sqrt{2}a^2\int_0^{\frac{\pi}{4}}\cos\varphi\sqrt{\cos 2\varphi}\mathrm{d}\varphi$$

$$=2\sqrt{2}a^2\int_0^{\frac{\pi}{4}}\sqrt{1-2\sin^2\varphi}\,\mathrm{d}(\sin\varphi)=\frac{1}{2}\pi a^2.$$

例 14. 设 $f \in \mathscr{C}[0,+\infty)$. $\forall t \geqslant 0$, 定义

$$F(t) = \iiint\limits_{\Omega_t} \left(z^2 + f(x^2+y^2)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

其中 $\Omega_t = \{(x,y,z) \mid 0 \leqslant z \leqslant h,\ x^2+y^2 \leqslant t^2\}$, 计算 $\lim\limits_{t\to 0^+} \dfrac{F(t)}{t^2}$.

解: 在柱坐标系下, 积分区域 Ω_t 变为

$$\Omega_t' = \{(\rho,\varphi,z) \mid 0 \leqslant z \leqslant h,\ 0 \leqslant \rho \leqslant t,\ 0 \leqslant \varphi \leqslant 2\pi\}.$$

由此我们立刻可得

$$
\begin{aligned}
F(t) &= \iiint\limits_{\Omega_t} \left(z^2 + f(x^2+y^2)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{\Omega_t'} \left(z^2 + f(\rho^2)\right)\rho \,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z \\
&= \int_0^h \left(\int_0^{2\pi} \left(\int_0^t \left(z^2 + f(\rho^2)\right)\rho \,\mathrm{d}\rho \right)\mathrm{d}\varphi \right)\mathrm{d}z \\
&= 2\pi \int_0^h \left(\int_0^t z^2\rho \,\mathrm{d}\rho \right)\mathrm{d}z + 2\pi \int_0^h \left(\int_0^t f(\rho^2)\rho \,\mathrm{d}\rho \right)\mathrm{d}z \\
&= \frac{\pi}{3} t^2 h^3 + \pi h \int_0^{t^2} f(u)\,\mathrm{d}u.
\end{aligned}
$$

于是由 f 的连续性以及 L'Hospital 法则可知

$$\begin{aligned}
\lim_{t\to 0^+}\frac{F(t)}{t^2} &= \frac{\pi}{3}h^3+\lim_{t\to 0^+}\frac{\pi h}{t^2}\int_0^{t^2} f(u)\,\mathrm{d}u \\
&= \frac{\pi}{3}h^3+\lim_{r\to 0^+}\frac{\pi h}{r}\int_0^{r} f(u)\,\mathrm{d}u \\
&= \frac{\pi h^3}{3}+\lim_{r\to 0^+}\pi h f(r) \\
&= \frac{\pi h^3}{3}+\pi h f(0).
\end{aligned}$$

例 15. 计算 $\displaystyle\iiint\limits_{x^2+y^2+z^2\leqslant 2z} (ax+by+cz)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

解: 由对称性可得

$$
\begin{aligned}
\iiint\limits_{x^2+y^2+z^2\leqslant 2z} (ax+by+cz)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \iiint\limits_{x^2+y^2+(z-1)^2\leqslant 1} ax\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \iiint\limits_{x^2+y^2+(z-1)^2\leqslant 1} by\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_{x^2+y^2+(z-1)^2\leqslant 1} c(z-1)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \iiint\limits_{x^2+y^2+(z-1)^2\leqslant 1} c\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{x^2+y^2+(z-1)^2\leqslant 1} c\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{4}{3}\pi c.
\end{aligned}
$$

例 16. 求证:

$$\int_0^1 \left(\int_x^1 \left(\int_x^y f(x)f(y)f(z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x = \frac{1}{6} \left(\int_0^1 f(x)\,\mathrm{d}x \right)^3.$$

证明: 由题设立刻可知

$$\begin{aligned} I &:= \int_0^1 \left(\int_x^1 \left(\int_x^y f(x)f(y)f(z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\ &= \iiint\limits_{\substack{0\leqslant x\leqslant 1, x\leqslant y\leqslant 1 \\ x\leqslant z\leqslant y}} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iiint\limits_{0\leqslant x\leqslant z\leqslant y\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned}$$

于是由积分变元记号的对称性可得

$$
\begin{aligned}
I &= \iiint\limits_{0\leqslant x\leqslant y\leqslant z\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{0\leqslant y\leqslant x\leqslant z\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{0\leqslant y\leqslant z\leqslant x\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{0\leqslant z\leqslant x\leqslant y\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{0\leqslant z\leqslant y\leqslant x\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,
\end{aligned}
$$

进而我们有

$$
\begin{aligned}
I &= \frac{1}{6}\Big(\iiint\limits_{0\leqslant x\leqslant z\leqslant y\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_{0\leqslant x\leqslant y\leqslant z\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \iiint\limits_{0\leqslant y\leqslant x\leqslant z\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_{0\leqslant y\leqslant z\leqslant x\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\quad + \iiint\limits_{0\leqslant z\leqslant x\leqslant y\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_{0\leqslant z\leqslant y\leqslant x\leqslant 1} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \Big) \\
&= \frac{1}{6} \iiint\limits_{\substack{0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1, \\ 0\leqslant z\leqslant 1}} f(x)f(y)f(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{6}\Big(\int_0^1 f(x)\,\mathrm{d}x\Big)\Big(\int_0^1 f(y)\,\mathrm{d}y\Big)\Big(\int_0^1 f(z)\,\mathrm{d}z\Big) \\
&= \frac{1}{6}\Big(\int_0^1 f(x)\,\mathrm{d}x\Big)^3.
\end{aligned}
$$

例 17. 设 $A=(a_{ij})_{1\leqslant i,j\leqslant n}$为实对称正定矩阵. 求证: 椭球体 $\sum\limits_{i,j=1}^{n} a_{ij}x_ix_j \leqslant 1$ 的体积 $V_n = \frac{v_n}{\sqrt{\det A}}$, 其中 v_n 为 $\mathbb{R}^n$ 中单位球的体积. 特别地, $V_3 = \frac{4\pi}{3\sqrt{\det A}}$.

证明: 由于 A 为实对称正定矩阵, 设其特征根为 $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_n$, 则存在实正交矩阵 U 使得 $A = U^T\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)U$. 作变换

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

由于 U 为正交变换, 故 $|\det U^{-1}| = 1$. 又

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j = (x_1, x_2, \cdots, x_n)A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= (x_1, x_2, \cdots, x_n)U^T\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)U\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{j=1}^{n} \lambda_j y_j^2,$$

于是所求体积为

$$V_n = \underset{\sum_{j=1}^n \lambda_j y_j^2 \leqslant 1}{\int \cdots \int} \left| \frac{D(x_1, x_2, \cdots, x_n)}{D(y_1, y_2, \cdots, y_n)} \right| \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n$$

$$= \underset{\sum_{j=1}^n \lambda_j y_j^2 \leqslant 1}{\int \cdots \int} |\det U^{-1}| \, \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n = \underset{\sum_{j=1}^n \lambda_j y_j^2 \leqslant 1}{\int \cdots \int} \mathrm{d}y_1 \mathrm{d}y_2 \cdots \mathrm{d}y_n$$

$$\overset{z_j = \sqrt{\lambda_j} y_j \ (1 \leqslant j \leqslant n)}{=} \underset{\sum_{j=1}^n z_j^2 \leqslant 1}{\int \cdots \int} \left| \frac{D(y_1, y_2, \cdots, y_n)}{D(z_1, z_2, \cdots, z_n)} \right| \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n$$

$$= \underset{\sum_{j=1}^n z_j^2 \leqslant 1}{\int \cdots \int} \frac{1}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_n}} \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n = \frac{v_n}{\sqrt{\det A}}.$$

特别地, 当 $n = 3$ 时, $V_3 = \frac{v_3}{\sqrt{\det A}} = \frac{4\pi}{3\sqrt{\det A}}$.

例 18. 求曲面

$$S: \begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \\ z = r\varphi \end{cases} \quad (0 \leqslant r \leqslant R,\ 0 \leqslant \varphi \leqslant 2\pi)$$

的面积.

解: 由题设可知

$$\frac{\partial(x,y,z)}{\partial(r,\varphi)} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \\ \varphi & r \end{pmatrix}.$$

由此立刻可得

$$
\begin{aligned}
E &= \cos^2\varphi + \sin^2\varphi + \varphi^2 = 1 + \varphi^2, \\
G &= r^2\sin^2\varphi + r^2\cos^2\varphi + r^2 = 2r^2, \\
F &= -r\sin\varphi\cos\varphi + r\sin\varphi\cos\varphi + r\varphi = r\varphi.
\end{aligned}
$$

于是曲面 S 的面积微元为

$$
\begin{aligned}
\mathrm{d}\sigma &= \sqrt{2(1+\varphi^2)r^2 - r^2\varphi^2}\,\mathrm{d}r\mathrm{d}\varphi \\
&= r\sqrt{2+\varphi^2}\,\mathrm{d}r\mathrm{d}\varphi.
\end{aligned}
$$

从而所求曲面的面积为

$$
\begin{aligned}
|S| &= \int_0^{2\pi}\left(\int_0^R r\sqrt{2+\varphi^2}\,\mathrm{d}r\right)\mathrm{d}\varphi \\
&= \frac{R^2}{2}\int_0^{2\pi}\sqrt{2+\varphi^2}\,\mathrm{d}\varphi \\
&= \frac{R^2}{2}\left(\frac{1}{2}\varphi\sqrt{2+\varphi^2}+\log(\varphi+\sqrt{2+\varphi^2})\right)\Bigg|_0^{2\pi} \\
&= \frac{R^2}{2}\left(\pi\sqrt{2+4\pi^2}+\log(\sqrt{2}\pi+\sqrt{1+2\pi^2})\right).
\end{aligned}
$$

例 19. 求球面 $x^2+y^2+z^2=a^2\ (a>0)$ 被平面 $z=\frac{a}{4}$, $z=\frac{a}{2}$ 所夹部分的面积.

解: 曲面的方程为

$$z=\sqrt{a^2-x^2-y^2}\quad \left(\tfrac{3}{4}a^2\leqslant x^2+y^2\leqslant\tfrac{15}{16}a^2\right),$$

则其面积为

$$\begin{aligned} S &= \iint\limits_{\frac{3}{4}a^2\leqslant x^2+y^2\leqslant\frac{15}{16}a^2}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_{\frac{3}{4}a^2\leqslant x^2+y^2\leqslant\frac{15}{16}a^2}\frac{a\,\mathrm{d}x\mathrm{d}y}{\sqrt{a^2-x^2-y^2}} \end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{\frac{3}{4}a^2\leqslant x^2+y^2\leqslant \frac{15}{16}a^2} \frac{a\,\mathrm{d}x\mathrm{d}y}{\sqrt{a^2-x^2-y^2}} \\
&\overset{\substack{x=\rho\cos\varphi \\ y=\rho\sin\varphi}}{=} a\int_0^{2\pi}\left(\int_{\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{15}}{4}a} \frac{\rho}{\sqrt{a^2-\rho^2}}\,\mathrm{d}\rho\right)\mathrm{d}\varphi \\
&= 2\pi a\int_{\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{15}}{4}a} \frac{\rho}{\sqrt{a^2-\rho^2}}\,\mathrm{d}\rho \\
&= -2\pi a\sqrt{a^2-\rho^2}\,\Bigg|_{\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{15}}{4}a} \\
&= \frac{1}{2}\pi a^2.
\end{aligned}
$$

例 20. 求圆柱面 $x^2+y^2=R^2$ 被曲面 $z=R^2-x^2$ 以及平面 $z=0$ 所截部分的侧面积.

解: 由对称性, 只需考虑所求侧面在第一卦限的部分, 其参数方程为

$$y=\sqrt{R^2-x^2}\ (0\leqslant x\leqslant R,\ 0\leqslant z\leqslant R^2-x^2).$$

相应的曲面微元为

$$\begin{aligned}\mathrm{d}\sigma &= \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,\mathrm{d}x\mathrm{d}z\\ &= \sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\,\mathrm{d}x\mathrm{d}z=\frac{R}{\sqrt{R^2-x^2}}\,\mathrm{d}x\mathrm{d}z.\end{aligned}$$

进而可知所求侧面积为

$$
\begin{aligned}
|S| &= 4 \iint\limits_{\substack{0\leqslant x\leqslant R,\\ 0\leqslant z\leqslant R^2-x^2}} \frac{R}{\sqrt{R^2-x^2}}\,\mathrm{d}x\mathrm{d}z \\
&= 4\int_0^R \left(\int_0^{R^2-x^2} \frac{R}{\sqrt{R^2-x^2}}\,\mathrm{d}z\right)\mathrm{d}x \\
&= 4R\int_0^R \sqrt{R^2-x^2}\,\mathrm{d}x \\
&\overset{x=R\sin t}{=} 4R^3\int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t \\
&= 2R^3\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = \pi R^3.
\end{aligned}
$$

谢谢大家!