Answers for Hw 12 GP II

By SJ

1. For a free particle with mass m, its initial wavefunction is in form of:

$$\psi(x,t=0)=\begin{cases} A & \text{if } -a<x<a \\ 0 & \text{elsewhere} \end{cases}$$ (a square function with width of 2a), A, a are real, positive constants. Find form of $\psi(x,t)$ (express it in integral form would be ok, it does not have analytical formula as Gaussian wave packet)

(problem 1 is example 2.6 in Griffiths')

Answer: Please refer to his book, I just copy-paste the relevant part below:

Solution: First we need to normalize $\Psi(x,0)$:

$$1=\int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx=|A|^2\int_{-a}^{a}dx=2a|A|^2 \Rightarrow A=\frac{1}{\sqrt{2a}}.$$

Next we calculate $\phi(k)$, using Equation 2.103:

$$\phi(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2a}}\int_{-a}^{a}e^{-ikx}\,dx=\frac{1}{2\sqrt{\pi a}}\frac{e^{-ikx}}{-ik}\bigg|_{-a}^{a}$$
$$=\frac{1}{k\sqrt{\pi a}}\left(\frac{e^{ika}-e^{-ika}}{2i}\right)=\frac{1}{\sqrt{\pi a}}\frac{\sin(ka)}{k}.$$

Finally, we plug this back into Equation 2.100:

$$\Psi(x,t)=\frac{1}{\pi\sqrt{2a}}\int_{-\infty}^{\infty}\frac{\sin(ka)}{k}e^{i(kx-\frac{\hbar k^2}{2m}t)}\,dk. \qquad [2.104]$$

Unfortunately, this integral cannot be solved in terms of elementary functions, though it can of course be evaluated numerically (Figure 2.8). (There are, in fact, precious few cases in which the integral for $\Psi(x,t)$ (Equation 2.100) *can* be calculated explicitly; see Problem 2.22 for a particularly beautiful example.)

2. (Griffiths' problem 2.21) A free particle has initial wavefunction:

$\psi(x,t=0)=Ae^{-a|x|}$, A, a are real positive constants.

(a) Normalize $\psi(x,0)$

(b)Find $\phi(k)$ (I used g(k) in my lecture)

(c)Construct $\psi(x,t)$, in the form of the integral.

(d)Discuss the limiting case (*a* very small and *a* very large)

You may also read his problem 2.22 (which is the Gaussian wave packet I solved in lecture)

Answer: (From his solution manual)

Problem 2.21

(a)

$$1=\int_{-\infty}^{\infty}|\Psi(x,0)|^2dx=2|A|^2\int_0^{\infty}e^{-2ax}dx=2|A|^2\left.\frac{e^{-2ax}}{-2a}\right|_0^{\infty}=\frac{|A|^2}{a}\Rightarrow A=\boxed{\sqrt{a}.}$$

(b)

$$\phi(k)=\frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}e^{-ikx}\,dx=\frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}(\cos kx-i\sin kx)dx.$$

The cosine integrand is even, and the sine is odd, so the latter vanishes and

$$\begin{aligned}\phi(k)&=2\frac{A}{\sqrt{2\pi}}\int_0^{\infty}e^{-ax}\cos kx\,dx=\frac{A}{\sqrt{2\pi}}\int_0^{\infty}e^{-ax}\left(e^{ikx}+e^{-ikx}\right)dx\\&=\frac{A}{\sqrt{2\pi}}\int_0^{\infty}\left(e^{(ik-a)x}+e^{-(ik+a)x}\right)dx=\frac{A}{\sqrt{2\pi}}\left[\frac{e^{(ik-a)x}}{ik-a}+\frac{e^{-(ik+a)x}}{-(ik+a)}\right]\Bigg|_0^{\infty}\\&=\frac{A}{\sqrt{2\pi}}\left(\frac{-1}{ik-a}+\frac{1}{ik+a}\right)=\frac{A}{\sqrt{2\pi}}\frac{-ik-a+ik-a}{-k^2-a^2}=\boxed{\sqrt{\frac{a}{2\pi}}\frac{2a}{k^2+a^2}.}\end{aligned}$$

(c)

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}2\sqrt{\frac{a^3}{2\pi}}\int_{-\infty}^{\infty}\frac{1}{k^2+a^2}e^{i(kx-\frac{\hbar k^2}{2m}t)}dk=\boxed{\frac{a^{3/2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2+a^2}e^{i(kx-\frac{\hbar k^2}{2m}t)}dk.}$$

(d) For *large* a, $\Psi(x,0)$ is a sharp narrow spike whereas $\phi(k)\cong\sqrt{2/\pi a}$ is broad and flat; position is well-defined but momentum is ill-defined. For *small* a, $\Psi(x,0)$ is a broad and flat whereas $\phi(k)\cong(\sqrt{2a^3/\pi})/k^2$ is a sharp narrow spike; position is ill-defined but momentum is well-defined.

3. Griffiths P2.5.

A particle in the infinite square well has its initial wave function in form of mixture of first two stationary states: $\psi(x,0) = A(\psi_1(x) + \psi_2(x))$

(a) Normalize the $\psi(x,0)$.

(b) Find the $\psi(x,t)$ and $|\psi(x,t)|^2$. Express the latter as a sinusoidal function of time. To simplify the result, let $\omega \equiv \pi^2\hbar / 2ma^2$

(c) Compute <x>. Notice that it oscillates in time. What is the angular frequency and amplitude of the oscillation?

(d) Compute <p> and better do it the quick way.

(e) If you measure the energy of this state, what values you might get and what is the probability of getting each of them? Find the expectation value of H.

Answer:

(a) Since ψ_1, ψ_2 are orthogonal and normalized, the normalized state is:

$$\psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

(b) $E_1/\hbar = \omega = \pi^2\hbar / 2ma^2, E_2/\hbar = 4\omega$

$$\psi(x,t) = \frac{1}{\sqrt{2}}[e^{-i\omega t}\psi_1(x) + e^{-i4\omega t}\psi_2(x)]$$

$$|\psi(x,t)|^2 = \psi^*\psi = \frac{1}{2}[e^{i\omega t}\psi_1(x) + e^{i4\omega t}\psi_2(x)][e^{-i\omega t}\psi_1(x) + e^{-i4\omega t}\psi_2(x)]$$

$$= \frac{1}{2}(\psi_1^2 + \psi_2^2 + 2\cos(3\omega t)\psi_1\psi_2)$$

$$= \frac{1}{a}[\sin^2(\frac{\pi x}{a}) + \sin^2(\frac{2\pi x}{a}) + 2\cos(3\omega t)\sin(\frac{\pi x}{a})(\frac{2\pi x}{a})]$$

(c)

$$< x >= \int_0^a x\,|\psi(x,t)|^2$$

$$= \int_0^a x\frac{1}{a}[\sin^2(\frac{\pi x}{a}) + \sin^2(\frac{2\pi x}{a}) + 2\cos(3\omega t)\sin(\frac{\pi x}{a})(\frac{2\pi x}{a})]dx$$

$$= \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t)$$

(d) $< p >= m\dfrac{d < x >}{dt}$ (using the Ehrenfest theorem for d<x>/dt)

$$< p >= 3m\omega\frac{16}{9\pi^2}\sin(3\omega t)$$

(e) The energies are either $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$ or $E_2 = 4\dfrac{\pi^2\hbar^2}{2ma^2}$, each with probability of 1/2. The average is:

$< H >= \dfrac{E_1 + E_2}{2}$ and does not change with time.

4. Griffiths P2.4.

Calculate <x>, <x^2>, <p>, <p^2>, σ_x and σ_p ($\sigma_x^2 =< x^2 > - < x >^2$), for the nth stationary state of the infinite well. Check that the uncertainty principle is satisfied and what state comes closest to the uncertainty limit?

Answer: The energy and stationary values are given by:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\pi^2\hbar^2}{2ma^2}n^2, \psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$$

$$<x>=\frac{2}{a}\int_0^a x\sin^2(\frac{n\pi x}{a})dx=\frac{2}{a}\int_0^a x(\frac{1-\cos(2\frac{n\pi x}{a})}{2})dx=\frac{a}{2}$$

$$<x^2>=\frac{2}{a}\int_0^a x^2(\frac{1-\cos(2\frac{n\pi x}{a})}{2})dx=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}$$

$$\sigma_x=(<x^2<-<x>^2)^{1/2}=\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}a$$

$$<p>=\frac{2}{a}(-i\hbar)\int_0^a \sin(\frac{n\pi x}{a})\frac{d}{dx}[\sin(\frac{n\pi x}{a})]dx=0$$

$$<p^2>=2m<H>=2mE_n=\frac{n^2\pi^2\hbar^2}{a^2}$$

$$\sigma_p=\frac{n\pi\hbar}{a}$$

$$\sigma_x\sigma_p=n\pi\hbar\sqrt{\frac{1}{12}-\frac{1}{2n^2\pi^2}}=\hbar\sqrt{\frac{n^2\pi^2}{12}-\frac{1}{2}}$$

It is smallest when n=1, and it is larger than $\hbar/2$

5. A particle of mass m is in a 1-D infinite square well where V(x)=0 when $0\le x\le a$, and infinity elsewhere. At t=0 its normalized wave function is:

$$\psi(x,t=0)=\sqrt{8/5a}[1+\cos(\frac{\pi x}{a})]\sin(\frac{\pi x}{a})$$

(a) What is the wave function at a later time t=t_0?

(b) What is the average energy of the system at t=0 and t= t_0?

(c) What is the probability of finding the particle at the left half of the box (i.e. [0, *a*/2]) at t= t_0?

Answer:

$$\psi(x,t=0)=\sqrt{8/5a}[1+\cos(\frac{\pi x}{a})]\sin(\frac{\pi x}{a})$$
$$=\sqrt{8/5a}\sin(\frac{\pi x}{a})+\sqrt{8/5a}\cos(\frac{\pi x}{a})\sin(\frac{\pi x}{a})=\sqrt{8/5a}\sin(\frac{\pi x}{a})+\sqrt{2/5a}\sin(\frac{2\pi x}{a})$$
$$=\sqrt{4/5}\psi_1+\sqrt{1/5}\psi_2$$

$\psi_{1,2}$ are the first two stationary states for infinity well with width a.

(a)
$$\psi(x,t)=\sqrt{4/5}\psi_1 e^{-i\omega t}+\sqrt{1/5}\psi_2 e^{-i4\omega t}$$
$$\omega=E_1/\hbar=\frac{\pi^2\hbar}{2ma^2}$$

(b) The <H> does not change over time, and it is:

$$<H>=\frac{4}{5}E_1+\frac{1}{5}E_2=\frac{8}{5}E_1=\frac{4\pi^2\hbar^2}{5ma^2}$$

(c)

$$P(0<x<a/2)=\int_0^{a/2}\psi^*\psi dx=\frac{4}{5}\int_0^{a/2}\psi_1^2 dx+\frac{1}{5}\int_0^{a/2}\psi_2^2 dx+\frac{2}{5}\int_0^{a/2}2\cos(3\omega t)\psi_1\psi_2 dx$$
$$=\frac{1}{2}+16/15\pi\cos(3\omega t)$$

6. For a general 3-D problem, let’s consider one particular general as well as simple case, that is the potential V(r) can be written as: $V(x,y,z)=V_x(x)+V_y(y)+V_z(z)$, then prove that the energy would be in form of: Solving the 3-D problem will become solving three 1-D equations, with $E=E_x+E_y+E_z$, and the stationary state wave function is in form of: $\psi(r)=\psi(x,y,z)=\psi_x(x)\psi_y(y)\psi_z(z)$

Answer:

We can write the Hamiltonian of the system for this special case

$$\frac{P^2}{2m} = -\frac{\hbar^2}{2m}\Delta = -\frac{\hbar^2}{2m}(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}) = \frac{P_x^2 + P_y^2 + P_z^2}{2m}$$

$$H = H_1 + H_2 + H_3$$

$$H_1 = \frac{P_x^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_x(x)$$

Similar for H_2,H_3 with y and z respectively. Time-independent S-equation becomes:

$$H\psi = E\psi$$

We shall try separation of variables, guessing the solution in form of:

$\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z)$ and try to solve $\psi_{x,y,z}$:

$$H\psi(x,y,z) = -\frac{\hbar^2}{2m}[\psi_y\psi_z\frac{d^2\psi_x}{dx^2} + \psi_x\psi_z\frac{d^2\psi_y}{dy^2} + \psi_x\psi_y\frac{d^2\psi_z}{dz^2}] + (V_x + V_y + V_z)\psi_x\psi_y\psi_z$$
$$= E\psi_x\psi_y\psi_z$$

divide above by $\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z)$ (1):

$$E = \frac{1}{\psi_x}H_1\psi_x + \frac{1}{\psi_y}H_2\psi_y + \frac{1}{\psi_z}H_3\psi_z \quad (2)$$

Each term on the left is a function of x only, y only and z only respectively. The right is a constant independent of x, y, z. This means the three terms have to be equal to some constants respectively, we name these constants E_x, E_y, E_z , i.e.

$\frac{1}{\psi_x}H_1\psi_x = E_x \to H_1\psi_x = E_x\psi_x$, which is just 1-D S-equation. Similarly:

$$H_2\psi_y = E_y\psi_y, H_3\psi_z = E_z\psi_z$$

(2) requires:

$$E = E_x + E_y + E_z$$

We see that we reduce a particle in 3-D problem to three 1-D equations.

7. We already learned for the 1-D free (V(x)=0) particle, the wave function with particular momentum is (expressed with k_x instead of p_x): $\psi(x)=\frac{1}{\sqrt{2\pi}}e^{ik_x x}$, with momentum $p_x=k_x\hbar, E=\frac{\hbar^2k_x^2}{2m}$. Now consider a 3-D free particle, what is the general expression of wave function with particular momentum and what is the expression for energy? (You will see the function is really like a 3-D plane wave)

Answer: The potential here are just $V_x=V_y=V_z=0$ for the isotropic 3-D free space and $V=V_x+V_y+V_z=0$. We see that we can apply what we derived in the last problem. The three 1-D equations are identical except variable change, and the solutions are (from those provided already for x-dimension):

$$\psi(x)=\frac{1}{\sqrt{2\pi}}e^{ik_x x}, p_x=k_x\hbar, E_x=\frac{\hbar^2k_x^2}{2m}$$

$$\psi(y)=\frac{1}{\sqrt{2\pi}}e^{ik_y y}, p_y=k_y\hbar, E_x=\frac{\hbar^2k_y^2}{2m}$$

$$\psi(z)=\frac{1}{\sqrt{2\pi}}e^{ik_z z}, p_x=k_x\hbar, E=\frac{\hbar^2k_x^2}{2m}$$

There is no restriction (quantization on the possible value of $k_{x,y,z}$)

Then the total energy is:

$$E=E_x+E_y+E_z=\frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2)=\frac{\hbar^2}{2m}k^2$$

$$\psi(x,y,z)=(\frac{1}{2\pi})^{3/2}e^{ik_x x}e^{ik_y y}e^{ik_z z}=(\frac{1}{2\pi})^{3/2}e^{i(\vec{k}\cdot\vec{r})}$$

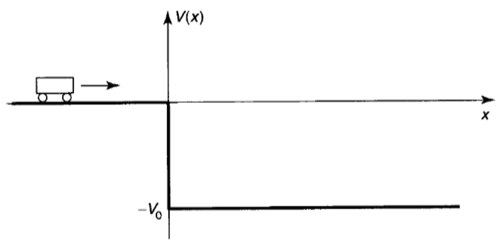
$$\psi(x,y,z,t)=(\frac{1}{2\pi})^{3/2}e^{i(\vec{k}\cdot\vec{r})}e^{-i\omega t},\omega=E/\hbar=\frac{\hbar k^2}{2m}$$

This is just a plane wave.

Note here the wave function is highly degenerated given the energy, i.e. knowing the energy we cannot fix the wave function. This is in contrary to 1-D, where the degeneracy for plane wave is at most 2-fold (+k, -k). The increase of degeneracy is the result of increased symmetry (from line to spherical).

8. Griffiths P2.35

A particle of mass m and kinetic energy E>0 approaches an abrupt potential drop V_0:



(a) What is the probability that it will "reflect" back, if $E=V_0/3$?

(b) The figure is drawn as a approaching a cliff, obviously the probability of bouncing back from the edge of cliff is far smaller than what you got in (a). Explain why this potential does not correctly represent a cliff.

(c) When a free neutron enters a nucleus, it experiences a sudden drop in potential energy, from V=0 outside to around V=-12MeV inside.

Suppose a neutron emitted with kinetic energy 4 MeV by a fission event, strikes such nucleus. What is the probability it will be absorbed? Hint: You may use T=1-R to get the transmission ratio.

Answer:

(a) Using the results in the note:

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

$$k_1 = \sqrt{2mE} / \hbar, k_2 = \sqrt{2m(E+V)} / \hbar$$

(the only difference is –V drop here while in note is +V barrier, and this is clearly E>0 case. Of course you may go through the whole process by solving S-equation using boundary conditions to find the ratio between the transmission and initial wave, using the fact initially only a left to right propagating wave)

$$R = \left(\frac{\sqrt{E+V} - \sqrt{E}}{\sqrt{E+V} + \sqrt{E}}\right)^2 = \left(\frac{\sqrt{1+V/E} - 1}{\sqrt{1+V/E} + 1}\right)^2 = \frac{1}{9} \quad \text{for E=V/3}$$

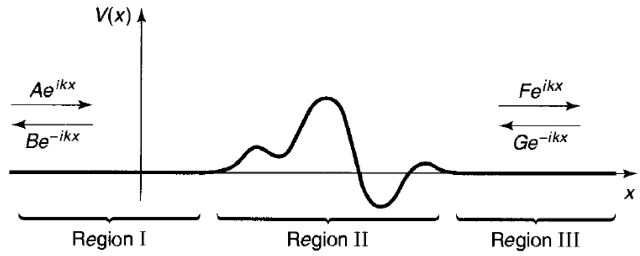
(b) The picture shown is actually not the correct potential felt by the car. Its gravity potential does not abruptly change from mgh to 0, but rather changes continuously depending on initial condition, it varies as: $V(x) \propto -ax - bx^2 \ldots$. The potential shown in the figure is a very "stiff" potential (the "force" is huge at the boundary due to a discontinuity of the V), such as the one shown in (c)

(c) This case is more properly represented by the figure. The initial energy is also 1/3 of the V, so the results in (a) applies well in this case:

R=1/9, so T=1-R=8/9

9. (adapted from Griffiths P2.52,2.53)

We have talked about scattering S-matrix and transfer M-matrix:



For S-matrix, it relates:

$$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$ (Knowing the incoming, we know the outgoing)

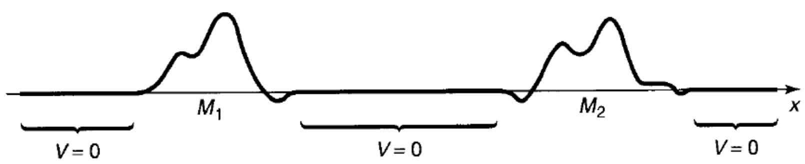
The M-matrix relates:

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$ (Knowing one side we know the other side)

(a) Find the four elements of the M-matrix, in terms of the elements of the S-matrix, and vice versa.

(b) If we have scattering from left, i.e. G=0, express the reflection and transmission coefficients in terms of S and M elements.

(c) For a potential consisting of separated pieces:



Show that the overall M-matrix is the product of the two M-matrices for the each section: $M=M_2M_1$ (note the order).

Answer

(a) $\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$, we have:

$$B = S_{11}A + S_{12}G \qquad (1)$$

$$F = S_{21}A + S_{22}G \qquad (2)$$

We need to rearrange above in formula F to (A,B), and G to (A,B), G to (A,B) is directly from (1):

$$G = -\frac{S_{11}}{S_{12}}A + \frac{1}{S_{12}}B \qquad (3)$$

Put this into (2):

$$F = S_{21}A + S_{22}\left(\frac{B}{S_{12}} - \frac{S_{11}}{S_{12}}A\right) = \frac{S_{21}S_{12} - S_{22}S_{11}}{S_{12}}A + \frac{S_{22}}{S_{12}}B = \frac{-\det S}{S_{12}}A + \frac{S_{22}}{S_{12}}B$$

From above relations, it becomes clear that:

$$M_{11} = -\frac{\det S}{S_{12}}, \quad M_{12} = \frac{S_{22}}{S_{12}}, \quad M_{21} = -\frac{S_{11}}{S_{12}}, M_{22} = \frac{1}{S_{12}} \qquad (4)$$

Exactly same procedure, but staring from Matrix, we can get:

$$S_{11} = -\frac{M_{21}}{M_{22}}, S_{12} = \frac{1}{M_{22}}, S_{21} = \frac{\det M}{M_{22}}, S_{22} = \frac{M_{12}}{M_{22}} \qquad (5)$$

Using (4) or (5) we can straightforward transfer from S to M or vice versa.

(b) For the G=0 case (no wave propagate from right to left initially):

The reflection is given by $|B/A|^2$, and transmission is given by $|F/A|^2$ (for region III and I have same k)

From (1), (2), we see directly:

$$R = |\frac{B}{A}|^2 = |S_{11}|^2 = |\frac{M_{21}}{M_{22}}|^2$$

$$T = |\frac{F}{A}|^2 = |S_{21}|^2 = |\frac{\det M}{M_{22}}|^2$$

So the reflection and transmission is easier to know from S matrix, but also can be calculated from M matrix. However, M matrix has simpler property as next question shows.

(c) Let the wave in the middle be in forms of:

$$\psi_{II} = Ce^{ikx} + De^{-ikx}$$

We have for the first potential barrier, the M matrix M_1 and from Matrix definition, we have:

$$\begin{pmatrix} C \\ D \end{pmatrix} = M_1 \begin{pmatrix} A \\ B \end{pmatrix} \quad (6)$$

For the second potential, (C,D) as input and (F,G) as output, we have:

$$\begin{pmatrix} F \\ G \end{pmatrix} = M_2 \begin{pmatrix} C \\ D \end{pmatrix} \quad (7)$$

Then combine (6) and (7), we have:

$$\begin{pmatrix} F \\ G \end{pmatrix} = M_2 M_1 \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{or} \quad M = M_2 M_1$$

10. An electron with energy E=1eV is incident on a rectangle barrier with height V_0=2eV and width d. What is the value of d so that the transmission probability is 10^{-3}.

Answer:

We shall use formula (12-48) in the note, since T is <<1 here:

$$T = 16\varepsilon(1-\varepsilon)e^{-2k_2'a}$$
$$\varepsilon \equiv E / V_0 = 1/2$$

in our problem

$$k_2' \equiv \sqrt{2m(V-E)} / \hbar \approx \frac{1}{1.96 \times 10^{-10} m} \quad \text{(using 12-50)}$$

$$10^{-3} = 4e^{-2k_2'a} \rightarrow a = -\frac{\ln(10^{-3}/4)}{2k_2'} = 8 \times 10^{-10} m$$

11. (OPTIONAL, you may skip this if you like and it won't relate to exam.

The goal for this problem is to let you know the delta function potential

which is covered in Griffiths' book)

Griffiths P2.27

Consider the double delta-function potential:

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)]$$

Where α, a are positive constants.

(a) Sketch the potential.

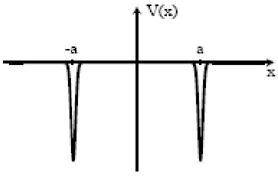
(b) How many bound states does it possess? Find the allowed energies ,

for $\alpha = \hbar^2 / ma$ and for $\alpha = \hbar^2 / 4ma$, and sketch the wave functions.

Answers: (I copied his solution from his solution manual)

Problem 2.27

(a)



(b) From Problem 2.1(c) the solutions are even or odd. Look first for *even solutions*:

$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} + e^{-\kappa x}) & (-a < x < a), \\ Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} + e^{-\kappa a})$, or $A = B(e^{2\kappa a} + 1)$.

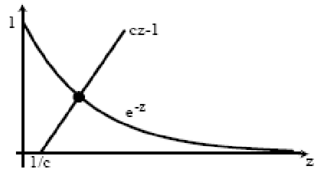
Discontinuous derivative at a, $\Delta \dfrac{d\psi}{dx} = -\dfrac{2m\alpha}{\hbar^2}\psi(a)$:

$$-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} - \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow A + B(e^{2\kappa a} - 1) = \frac{2m\alpha}{\hbar^2\kappa}A;\text{ or}$$

$$B(e^{2\kappa a} - 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = B(e^{2\kappa a} + 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) \Rightarrow e^{2\kappa a} - 1 = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) + \frac{2m\alpha}{\hbar^2\kappa} - 1.$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 + \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a};\ \frac{\hbar^2\kappa}{m\alpha} = 1 + e^{-2\kappa a},\text{ or } \boxed{e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha} - 1.}$$

This is a transcendental equation for κ (and hence for E). I'll solve it graphically: Let $z \equiv 2\kappa a$, $c \equiv \frac{\hbar^2}{2am\alpha}$, so $e^{-z} = cz - 1$. Plot both sides and look for intersections:



From the graph, noting that c and z are both positive, we see that there is one (and only one) solution (for even ψ). If $\alpha = \frac{\hbar^2}{2ma}$, so $c = 1$, the calculator gives $z = 1.278$, so $\kappa^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \Rightarrow E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right) = -0.204\left(\frac{\hbar^2}{ma^2}\right)$.

Now look for *odd solutions:*

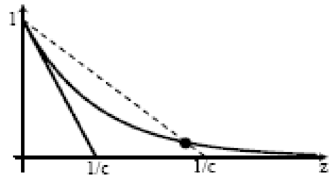
$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x < a), \\ B(e^{\kappa x} - e^{-\kappa x}) & (-a < x < a), \\ -Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a} - e^{-\kappa a})$, or $A = B(e^{2\kappa a} - 1)$.

Discontinuity in ψ': $-\kappa Ae^{-\kappa a} - B(\kappa e^{\kappa a} + \kappa e^{-\kappa a}) = -\dfrac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow B(e^{2\kappa a} + 1) = A\left(\dfrac{2m\alpha}{\hbar^2\kappa} - 1\right)$,

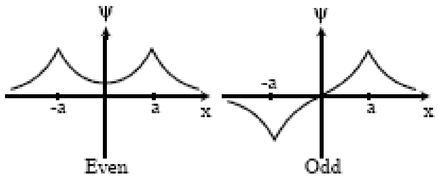
$$e^{2\kappa a} + 1 = (e^{2\kappa a} - 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) - \frac{2m\alpha}{\hbar^2\kappa} + 1,$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 - \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a};\ \frac{\hbar^2\kappa}{m\alpha} = 1 - e^{-2\kappa a},\ \boxed{e^{-2\kappa a} = 1 - \frac{\hbar^2\kappa}{m\alpha},}\text{ or } e^{-z} = 1 - cz.$$



This time there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (solid line), whereas if c is smaller (dashed line) there will be. (Note that $z=0 \Rightarrow \kappa=0$ is *not* a solution, since ψ is then non-normalizable.) The slope of e^{-z} (at $z=0$) is -1; the slope of $(1-cz)$ is $-c$. So there is an *odd* solution $\Leftrightarrow c<1$, or $\alpha > \hbar^2/2ma$.

Conclusion: $\boxed{\textit{One} \text{ bound state if } \alpha \le \hbar^2/2ma;\ \textit{two} \text{ if } \alpha > \hbar^2/2ma.}$



$$\alpha = \frac{\hbar^2}{ma} \Rightarrow c = \frac{1}{2}. \begin{cases} \textit{Even: } e^{-z} = \frac{1}{2}z - 1 \Rightarrow z = 2.21772, \\ \textit{Odd: } \ e^{-z} = 1 - \frac{1}{2}z \Rightarrow z = 1.59362. \end{cases}$$

$$\boxed{E = -0.615(\hbar^2/ma^2);\ E = -0.317(\hbar^2/ma^2).}$$

$$\alpha = \frac{\hbar^2}{4ma} \Rightarrow c = 2.\ \textit{Only even: } e^{-z} = 2z - 1 \Rightarrow z = 0.738835;\ \boxed{E = -0.0682(\hbar^2/ma^2).}$$

5. Griffiths' 2.29

Analyze the odd bound state wave functions for the finite potential well. Derive the transcendental equation for the allowed energies, and solve it graphically. Analyzing the two limiting cases (wide-deep well (meaning Z_0 big) and narrow-shallow well (small Z_0)) and is there always an odd bound state?

Answers from his solution manual:

Problem 2.29

In place of Eq. 2.151, we have: $\psi(x) = \left\{ \begin{array}{ll} Fe^{-\kappa x} & (x > a) \\ D\sin(lx) & (0 < x < a) \\ -\psi(-x) & (x < 0) \end{array} \right\}$.

Continuity of ψ : $Fe^{-\kappa a} = D\sin(la)$; continuity of ψ' : $-F\kappa e^{-\kappa a} = Dl\cos(la)$.

Divide: $-\kappa = l\cot(la)$, or $-\kappa a = la\cot(la) \Rightarrow \sqrt{z_0^2 - z^2} = -z\cot z$, or $\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}.}$

Wide, deep well: Intersections are at $\pi, 2\pi, 3\pi$, etc. Same as Eq. 2.157, but now for n *even*. This fills in the rest of the states for the infinite square well.

Shallow, narrow well: If $z_0 < \pi/2$, there is *no* odd bound state. The corresponding condition on V_0 is $\boxed{V_0 < \frac{\pi^2\hbar^2}{8ma^2} \Rightarrow \textit{no} \text{ odd bound state.}}$