第 11 次作业题

1. 验证 $\frac{1}{x^2}(yz\,\mathrm{d}x - zx\,\mathrm{d}y - xy\,\mathrm{d}z)$ 为某个三元函数 u 的全微分并求该函数.

解: 方法 1. 由题设立刻可得

$$\begin{aligned}
\frac{\partial}{\partial y}\Big(-\frac{y}{x}\Big) - \frac{\partial}{\partial z}\Big(-\frac{z}{x}\Big) &= \Big(-\frac{1}{x}\Big) - \Big(-\frac{1}{x}\Big) = 0,\\
\frac{\partial}{\partial z}\Big(\frac{yz}{x^2}\Big) - \frac{\partial}{\partial x}\Big(-\frac{y}{x}\Big) &= \Big(\frac{y}{x^2}\Big) - \Big(\frac{y}{x^2}\Big) = 0,\\
\frac{\partial}{\partial x}\Big(-\frac{z}{x}\Big) - \frac{\partial}{\partial y}\Big(\frac{yz}{x^2}\Big) &= \Big(\frac{z}{x^2}\Big) - \Big(\frac{z}{x^2}\Big) = 0,
\end{aligned}$$

因此 $\frac{1}{x^2}(yz\,\mathrm{d}x - zx\,\mathrm{d}y - xy\,\mathrm{d}z)$ 为某个三元函数 u 的全微分. 另外我们有

$$\begin{aligned}
\mathrm{d}u &= \frac{1}{x^2}(yz\,\mathrm{d}x - zx\,\mathrm{d}y - xy\,\mathrm{d}z) = -yz\,\mathrm{d}\big(\frac{1}{x}\big) - \frac{z}{x}\,\mathrm{d}y - \frac{y}{x}\,\mathrm{d}z\\
&= -\Big(yz\,\mathrm{d}\big(\frac{1}{x}\big) + \frac{1}{x}\,\mathrm{d}\big(yz\big)\Big) + \frac{1}{x}\,\mathrm{d}\big(yz\big) - \frac{z}{x}\,\mathrm{d}y - \frac{y}{x}\,\mathrm{d}z\\
&= -\mathrm{d}\Big(\frac{yz}{x}\Big),
\end{aligned}$$

于是所求原函数为 $u(x,y,z) = -\frac{yz}{x} + C$, 其中 C 为任意常数.

方法 2. 由题设可知

$$\begin{aligned}
&\frac{1}{x^2}(yz\,\mathrm{d}x - zx\,\mathrm{d}y - xy\,\mathrm{d}z) = -yz\,\mathrm{d}\big(\frac{1}{x}\big) - \frac{z}{x}\,\mathrm{d}y - \frac{y}{x}\,\mathrm{d}z\\
= &-\Big(yz\,\mathrm{d}\big(\frac{1}{x}\big) + \frac{1}{x}\,\mathrm{d}\big(yz\big)\Big) + \frac{1}{x}\,\mathrm{d}\big(yz\big) - \frac{z}{x}\,\mathrm{d}y - \frac{y}{x}\,\mathrm{d}z\\
= &-\mathrm{d}\Big(\frac{yz}{x}\Big),
\end{aligned}$$

故题设微分形式为 u 的全微分, 其中 $u(x,y,z) = -\frac{yz}{x} + C$, 而 C 为任意常数.

2. 证明曲线积分 $\displaystyle\int_{(0,0,0)}^{(1,2,1)} (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z$ 与路径无关, 并求积分值.

解: 由于 $(y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z = \mathrm{d}(xy+yz+zx)$, 故题设积分与路径无关, 且其积分值为 $(xy+yz+zx)\big|_{(0,0,0)}^{(1,2,1)} = 5$.

3. 已知标量函数 u, 向量值函数 $\vec{V}, \vec{A}, \vec{B}$ 为 $\mathbb{R}^3$ 中的光滑函数, 证明:

(1) $\mathrm{div}(u\vec{V}) = u\,\mathrm{div}\vec{V} + \mathrm{grad}u \cdot \vec{V}$;

(2) $\mathrm{rot}(u\vec{A}) = u\,\mathrm{rot}\vec{A} + \mathrm{grad}u \times \vec{A}$;

(3) $\mathrm{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\mathrm{rot}\vec{A} - \vec{A}\cdot\mathrm{rot}\vec{B}$;

(4) $\mathrm{rot}(\mathrm{grad}u) = \vec{0}$;

(5) $\mathrm{div}(\mathrm{rot}\vec{A}) = 0$.

证明： (1) 设 $\vec{V}=(V_1,V_2,V_3)^T$, 则我们有

$$\begin{aligned}
\mathrm{div}(u\vec{V}) &= \frac{\partial(uV_1)}{\partial x}+\frac{\partial(uV_2)}{\partial y}+\frac{\partial(uV_3)}{\partial z}\\
&= u\frac{\partial V_1}{\partial x}+\frac{\partial u}{\partial x}V_1+u\frac{\partial V_2}{\partial y}+\frac{\partial u}{\partial y}V_2+u\frac{\partial V_3}{\partial z}+\frac{\partial u}{\partial z}V_3\\
&= u\Big(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\Big)+\Big(\frac{\partial u}{\partial x}V_1+\frac{\partial u}{\partial y}V_2+\frac{\partial u}{\partial z}V_3\Big)=u\,\mathrm{div}\vec{V}+\mathrm{grad}u\cdot\vec{V}.
\end{aligned}$$

(2) 设 $\vec{A}=(A_1,A_2,A_3)^T$, 而 $\vec{i},\vec{j},\vec{k}$ 为 $\mathbb{R}^3$ 的标准基底, 则我们有

$$\begin{aligned}
\mathrm{rot}(u\vec{A}) &= \Big(\frac{\partial(uA_3)}{\partial y}-\frac{\partial(uA_2)}{\partial z}\Big)\vec{i}+\Big(\frac{\partial(uA_1)}{\partial z}-\frac{\partial(uA_3)}{\partial x}\Big)\vec{j}+\Big(\frac{\partial(uA_2)}{\partial x}-\frac{\partial(uA_1)}{\partial y}\Big)\vec{k}\\
&= \Big(u\frac{\partial A_3}{\partial y}+\frac{\partial u}{\partial y}A_3-u\frac{\partial A_2}{\partial z}-\frac{\partial u}{\partial z}A_2\Big)\vec{i}+\Big(u\frac{\partial A_1}{\partial z}+\frac{\partial u}{\partial z}A_1\\
&\quad -u\frac{\partial A_3}{\partial x}-\frac{\partial u}{\partial x}A_3\Big)\vec{j}+\Big(u\frac{\partial A_2}{\partial x}+\frac{\partial u}{\partial x}A_2-u\frac{\partial A_1}{\partial y}-\frac{\partial u}{\partial y}A_1\Big)\vec{k}\\
&= u\Big(\Big(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\Big)\vec{i}+\Big(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\Big)\vec{j}+\Big(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\Big)\vec{k}\Big)\\
&\quad +\Big(\Big(\frac{\partial u}{\partial y}A_3-\frac{\partial u}{\partial z}A_2\Big)\vec{i}+\Big(\frac{\partial u}{\partial z}A_1-\frac{\partial u}{\partial x}A_3\Big)\vec{j}+\Big(\frac{\partial u}{\partial x}A_2-\frac{\partial u}{\partial y}A_1\Big)\vec{k}\Big)\\
&= u\,\mathrm{rot}\vec{A}+\mathrm{grad}u\times\vec{A}.
\end{aligned}$$

(3) 设 $\vec{B}=(B_1,B_2,B_3)^T$, 则我们有

$$\begin{aligned}
\mathrm{div}(\vec{A}\times\vec{B}) &= \mathrm{div}\Big((A_2B_3-A_3B_2)\vec{i}+(A_3B_1-A_1B_3)\vec{j}+(A_1B_2-A_2B_1)\vec{k}\Big)\\
&= \frac{\partial(A_2B_3-A_3B_2)}{\partial x}+\frac{\partial(A_3B_1-A_1B_3)}{\partial y}+\frac{\partial(A_1B_2-A_2B_1)}{\partial z}\\
&= \Big(B_3\frac{\partial A_2}{\partial x}+A_2\frac{\partial B_3}{\partial x}\Big)-\Big(B_2\frac{\partial A_3}{\partial x}+A_3\frac{\partial B_2}{\partial x}\Big)+\Big(B_1\frac{\partial A_3}{\partial y}+A_3\frac{\partial B_1}{\partial y}\Big)\\
&\quad -\Big(B_3\frac{\partial A_1}{\partial y}+A_1\frac{\partial B_3}{\partial y}\Big)+\Big(B_2\frac{\partial A_1}{\partial z}+A_1\frac{\partial B_2}{\partial z}\Big)-\Big(B_1\frac{\partial A_2}{\partial z}+A_2\frac{\partial B_1}{\partial z}\Big)\\
&= \Big(B_1\Big(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\Big)+B_2\Big(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\Big)+B_3\Big(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\Big)\Big)\\
&\quad -\Big(A_1\Big(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\Big)+A_2\Big(\frac{\partial B_1}{\partial z}-\frac{\partial B_3}{\partial x}\Big)+A_3\Big(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\Big)\Big)\\
&= \vec{B}\cdot\mathrm{rot}\vec{A}-\vec{A}\cdot\mathrm{rot}\vec{B}.
\end{aligned}$$

(4) 因为 $\mathrm{grad}u=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}$, 从而由光滑性可知

$$\mathrm{rot}(\mathrm{grad}u)=\Big(\frac{\partial^2 u}{\partial y\partial z}-\frac{\partial^2 u}{\partial z\partial y}\Big)\vec{i}+\Big(\frac{\partial^2 u}{\partial z\partial x}-\frac{\partial^2 u}{\partial x\partial z}\Big)\vec{j}+\Big(\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y\partial x}\Big)\vec{k}=\vec{0}.$$

(5) 由定义可知

$$\mathrm{rot}\vec{A}=(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z})\vec{i}+(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x})\vec{j}+(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y})\vec{k},$$

进而由光滑性可知

$$\begin{aligned}
\mathrm{div}(\mathrm{rot}\vec{A}) &= \frac{\partial}{\partial x}\Big(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\Big)+\frac{\partial}{\partial y}\Big(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\Big)+\frac{\partial}{\partial z}\Big(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\Big)\\
&= \Big(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}\Big)+\Big(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}\Big)+\Big(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}\Big)=0.
\end{aligned}$$