1. a)

$$\begin{vmatrix} a^2+ab+b^2 & a^2-ab+b^2 \\ a+b & a-b \end{vmatrix} = (a^2+ab+b^2)(a-b)-(a+b)(a^2-ab+b^2)$$

$$= \cancel{a^3}+a^2b+\cancel{ab^2}-a^2b-\cancel{ab^2}-b^3$$

$$-\cancel{a^3}+a^2b-\cancel{ab^2}-a^2b+\cancel{ab^2}-b^3$$

$$=2a^2b-2a^2b-2b^3$$

$$=-2b^3$$

b)

$$\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha+\sin^2\alpha = 1$$

c)

$$\begin{vmatrix} \frac{1-t^2}{1+t^2} & \frac{2t}{1+t^2} \\ -\frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{vmatrix} = \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2} = \frac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2}$$

$$=1$$

2. a)

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1+1=2$$

b)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

c)

$$\begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 1 & 8 & 64 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 25 \\ 1 & 7 & 49 \\ 0 & 1 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 25 \\ 0 & 2 & 24 \\ 0 & 1 & 15 \end{vmatrix} = \begin{vmatrix} 2 & 24 \\ 1 & 15 \end{vmatrix} = 6$$

d) $\begin{vmatrix} 1 & 5 & 6 & 8 \\ 4 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 6 & 8 \\ 4 & 3 & 4 & 6 \\ 0 & -3 & -3 & -4 \\ 0 & -1 & 3 & 5 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 5 & 6 & 8 \\ 0 & -17 & -20 & -26 \\ 0 & -3 & -3 & -4 \\ 0 & -1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} -17 & -20 & -26 \\ -3 & -3 & -4 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= \frac{1}{10} \begin{vmatrix} -17 & -20 & -26 \\ -15 & -15 & -20 \\ -2 & 6 & 10 \end{vmatrix}$$

$$= \frac{1}{10} \begin{vmatrix} 0 & -11 & -16 \\ -15 & -15 & -20 \\ -2 & 6 & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -11 & -16 \\ -3 & -3 & -4 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= \frac{1}{3} \begin{vmatrix} 0 & -11 & -16 \\ -3 & -3 & -4 \\ -3 & 9 & 15 \end{vmatrix} = \begin{vmatrix} 0 & -11 & -16 \\ 0 & -12 & -19 \\ -1 & 3 & 5 \end{vmatrix}$$

$$= - \begin{vmatrix} -11 & -16 \\ -12 & -19 \end{vmatrix} = \begin{vmatrix} -12 & -19 \\ -11 & -16 \end{vmatrix}$$

$$= 12 \times 16 - 11 \times 19$$

$$= -17$$

3. 若不为0，设 $\det \operatorname{rref}(A) \neq 1$ $\Rightarrow$ $\exists$ 倍乘矩阵使得 $\det(A)=1$

$\Rightarrow$ 矛盾 $\Rightarrow$ $\det \operatorname{rref}(A)=1$.

4. $\det(a_1, a_2, \cdots, a_n) = (-1)^{n-1} \det(a_2, a_3, \cdots, a_n, a_1)$

$$= (-1)^{\frac{n(n-1)}{2}} \det(a_n, \cdots, a_1)$$

5.
$$\begin{vmatrix} 0 & \cdots & \cdots & \cdots & a_{1n} \\ 0 & & & a_{2,n-1} & a_{2n} \\ \vdots & & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix} = (-1)^{n} a_{n1} \begin{vmatrix} & 0 & & a_{1n} \\ & & \iddots & \vdots \\ a_{n-1,2} & & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^{\frac{n(n+1)}{2}} a_{n1} a_{n-1,2} \cdots a_{1n}$$

6. 注意到二阶子项 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 只有 $1, 0, -1$ 三种

对于 $a_1 A_1 + a_2 A_2 + a_3 A_3$

若 $A_1 = A_2 = 1$ 即 $A_1, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

则 $A_3 \neq 1$ $\Rightarrow$ $A_3 = 0$ 或 -1

$A_3 = 0 \Rightarrow a_1 + a_2 \Rightarrow \max = 2$

$A_3 = -1 \Rightarrow a_1 + a_2 - a_3 \Rightarrow \max = 2$

$\Rightarrow$ 最大值为 2.

7. a) 直接对 A 的第一列展开，直接计算可得

b) $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ 直接对 A 的第一列展开，然后归纳可得。
(类似 a) 的做法)

c) $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$

$\Rightarrow \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \det \begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}$

$= \det A \det(D-CA^{-1}B)$

10. a) $(AA^*)_{ij} = \sum_k A_{ik}A^*_{kj} = \sum_k A_{ik}(-1)^{k+j}M_{jk}$

$= \sum_k (-1)^{k+j} A_{ik} M_{jk}$

(Laplace公式) $= \delta_{ij} \det A$

$(A^*A)_{ij} = \sum_k A^*_{ik}A_{kj} = \sum_k (-1)^{k+i} M_{ki} A_{kj}$

(Laplace公式) $= \delta_{ij} \det A$

b) 因为 $\det AA^* = \det A \cdot \det A^* = \det(\det A \cdot I_n)$

$= (\det A)^n$

$\Rightarrow \det A^* = (\det A)^{n-1}$

c) 因为 $AA^* = A^*A = \det(A) I_n$, $\det A^* = (\det A)^{n-1}$

$\therefore A^*(A^*)^* = \det(A^*) I_n = (\det A)^{n-1} I_n$

$\Rightarrow (A^*)^* = (A^*)^{-1}(\det A)^{n-1} I_n \quad \because A^* = A^{-1}\det(A)$

$= A(\det A)^{n-2} I_n.$

d) 若 $rk(A)=n-1$，则 A 至少有一个 $(n-1)\times(n-1)$ 子式不为0

$\Rightarrow A^*\neq 0 \Rightarrow rk(A^*)\geq 1$

又 $\because AA^*=0 \Rightarrow rk(A)+rk(A^*)\leq n$

$\Rightarrow rk(A^*)\leq 1 \Rightarrow rk(A^*)=1$

若 $rk(A)<n-1$，所有 $(n-1)\times(n-1)$ 子式均为0 $\Rightarrow rk(A^*)=0$

11. $\det(\lambda I-A)=\begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda-2 & 1 & 0 \\ 0 & 1 & \lambda-2 & 1 \\ 0 & 0 & 1 & \lambda-2 \end{vmatrix}$

$=(\lambda-2)\begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix}-\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-2 \end{vmatrix}$

$=(\lambda-2)^2\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix}-(\lambda-2)\begin{vmatrix} 1 & 0 \\ 1 & \lambda-2 \end{vmatrix}$

$-\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix}$

$=(\lambda-2)^4-(\lambda-2)^2-(\lambda-2)^2-(\lambda-2)^2+1$

$=(\lambda-2)^4-3(\lambda-2)^2+1$

8. a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$A^* = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$, $\det A = -2$

$\therefore A^{-1} = \dfrac{A^*}{\det A} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ $\det A = 6$

$A^* = \begin{pmatrix} 0 & 3 & -6 \\ -2 & 1 & 2 \\ 4 & -2 & 2 \end{pmatrix}$

$\therefore A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

c) $A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ $\det A = -3$

$A^{\tilde{a}} = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 3 & 9 \\ -2 & 3 & 5 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ 1 & -1 & -3 \\ \frac{2}{3} & -1 & -\frac{5}{3} \end{pmatrix}$

9. $\begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$Ax = b \Rightarrow x = A^{-1}b$

$x = \begin{pmatrix} -4 \\ 13 \\ -1 \end{pmatrix}$