矩阵乘法

1. ① $\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b & a+b+c \\ 0 & a & a+b \\ 0 & 0 & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{pmatrix}$

② $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$= I + A$

$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$A^3 = 0$

$\Rightarrow (I+A)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k$

$= I + nA + \frac{n(n-1)}{2} A^2 = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$

2. $((AB)C)_{ij} = \sum_k (AB)_{ik} C_{kj}$

$= \sum_{k,l} A_{il} B_{lk} C_{kj}$

$$
\begin{aligned}
(A(BC))_{ij} &= \sum_k A_{ik}(BC)_{kj} \\
&= \sum_k A_{ik} \sum_l B_{kl} C_{lj} \\
&= \sum_{k,l} A_{ik} B_{kl} C_{lj}
\end{aligned}
$$

exchange k, l in $(AB)C$ to l, k & we obtain that $(AB)C = A(BC)$

3. For two upper triangular matrix

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{pmatrix}
\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & \cdots & b_{2n} \\ 0 & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12}+a_{12}b_{22} & \cdots & \sum_{i=1}^{n} a_{1i}b_{in} \\ 0 & a_{22}b_{22} & & \sum_{i=2}^{n} a_{2i}b_{in} \\ & & \ddots & \\ 0 & & & a_{nn}b_{nn} \end{pmatrix}
$$

Similar computation for the under triangular matrix

4. Note that $(A^T)^{-1} A^T = I$

& use the formula $(AB)^T = B^T A^T$

$$\Rightarrow (AA^{-1})^T = (A^{-1})^T A^T = I$$

Since the inverse matrix is unique $\Rightarrow (A^T)^{-1} = (A^{-1})^T$

5. Direct computation:

$$(BB^T)_{ij} = \sum_k B_{ik} B^T_{kj} = \sum_k B_{ik} B_{jk}$$

$$\Rightarrow (BB^T)_{ji} = \sum_k B_{jk} B^T_{ki} = \sum_k B_{jk} B_{ik}$$

$$\Rightarrow (BB^T)_{ij} = (BB^T)_{ji}$$

6. $$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} AD_1 & AD_2 \\ BD_3 & BD_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$\Rightarrow D_1 = A^{-1}$$
$$D_4 = B^{-1}$$

Since A & B are invertible & $AD_2 = 0$, $BD_3 = 0$ $\Rightarrow$ $D_2 = 0$, $D_3 = 0$

$$\begin{array}{cc} n\times n & n\times m \\ \end{array}$$
$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} AD_1 & AD_2 + CD_4 \\ BD_3 & BD_4 \end{pmatrix}$$
$$m\times m$$

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} AD_1 & D_1C + D_2B \\ D_3A & D_4B \end{pmatrix}$$

$D_1 = A^{-1}$ $\quad A^{-1}C + D_2B = 0$

$AD_2 + CB^{-1} = 0$

$D_2B = -A^{-1}C \Rightarrow D_2 = -A^{-1}CB^{-1}$

$AD_2 = -CB^{-1} \Rightarrow D_2 = -A^{-1}CB^{-1}$

$$\Rightarrow \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

消元法、行约化

1.
$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 & ① \\ 3x_1 + x_2 = 2 & ② \\ x_1 - 4x_2 - 2x_3 = 3 & ③ \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ -3 \\ \frac{16}{3} \end{pmatrix}$$

① − ③ $\Rightarrow$ $6x_2 + 3x_3 = -2$ ④

3① − ② $\Rightarrow$ $5x_2 + 3x_3 = 1$ ⑤

④ − ⑤ $\Rightarrow$ $x_2 = -3$

$\Rightarrow$ $x_3 = \frac{16}{3}, \; x_1 = \frac{5}{3}$

②

$$\begin{pmatrix}1&3&5&7&9\\6&0&4&2&0\end{pmatrix}$$

$$\Downarrow$$

1) 行阶梯矩阵

$$\begin{pmatrix}6&18&30&42&54\\6&0&4&2&0\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}6&18&30&42&54\\0&-18&-26&-40&-54\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}1&3&5&7&9\\0&-18&-26&-40&-54\end{pmatrix} \Rightarrow \begin{pmatrix}1&3&5&7&9\\0&1&\frac{13}{9}&\frac{20}{9}&3\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}1&3&5&7&9\\0&3&\frac{13}{3}&\frac{20}{3}&9\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}1&0&\frac{2}{3}&\frac{1}{3}&0\\0&1&\frac{13}{9}&\frac{20}{9}&3\end{pmatrix} \Leftarrow \begin{pmatrix}1&0&\frac{2}{3}&\frac{1}{3}&0\\0&3&\frac{13}{3}&\frac{20}{3}&9\end{pmatrix}$$

$$\begin{pmatrix}1&2&1&1\\3&0&0&4\\1&-4&-2&2\end{pmatrix} \Rightarrow \begin{pmatrix}1&2&1&1\\0&-6&-3&1\\1&-4&-2&2\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}1&2&1&1\\0&-6&-3&1\\0&-6&-3&1\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longleftarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -6 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.3

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.4

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} = A$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

5. $Ax=b, \quad Bx=c$

Since $A \sim B \quad \Rightarrow \quad E_1 E_2 \cdots E_n A = B$

$\Rightarrow E_1 E_2 \cdots E_n A x = c$

$\Rightarrow Ax = E_n^{-1} \cdots E_1^{-1} c = b$ QED.

6. a) $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ A & B \end{pmatrix} \quad E = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C+PA & D+PB \end{pmatrix} \quad E = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$

$\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} PA & PB \\ C & D \end{pmatrix} \quad E = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$

b) $C + PA = 0 \Rightarrow P = -CA^{-1}$

$F = D + PB = D - CA^{-1}B$