微积分 A (2)

姚家燕

第 16 讲

在听课过程中,

严禁使用与教学无关的电子产品!

期中考试时间与地点

时间: 2021 年 4 月 17 日星期六 13:30–15:30

地点: 二教 401 (工物系, 车辆学院)-86,

二教 402 (其余)-73

请大家务必提前 30 分钟到场!

重要提示: 考试时需且只许带学生证和考试用具!

答疑: 4 月 16 日 18:00–21:00 (数学系 A 216)

期中考试内容

- 多元微分学 (第 1 章)
- 含参积分以及广义含参积分 (第 2 章)

第 15 讲回顾: 极坐标下二重积分的累次积分

1. 假设

$$\begin{aligned} D_1 &= \{(\rho,\varphi) \mid \alpha \leqslant \varphi \leqslant \beta,\ \rho_1(\varphi) \leqslant \rho \leqslant \rho_2(\varphi)\}, \\ D_2 &= \{(x,y) \mid x = \rho\cos\varphi,\ y = \rho\sin\varphi, (\rho,\varphi) \in D_1\}, \end{aligned}$$

其中 $\rho_2 \geqslant \rho_1 \geqslant 0$ 连续. 若 $f \in \mathscr{C}(D_2)$, 则

$$\begin{aligned} &\iint\limits_{D_2} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint\limits_{D_1} f(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= \int_\alpha^\beta \left(\int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho \right) \mathrm{d}\varphi. \end{aligned}$$

2. 假设

$$
\begin{aligned}
D_1 &= \{(\rho,\varphi) \mid \rho_1 \leqslant \rho \leqslant \rho_2,\ \alpha(\rho) \leqslant \varphi \leqslant \beta(\rho)\}, \\
D_2 &= \{(x,y) \mid x = \rho\cos\varphi,\ y = \rho\sin\varphi, (\rho,\varphi) \in D_1)\},
\end{aligned}
$$

其中 $\beta \geqslant \alpha$ 连续. 若 $f \in \mathscr{C}(D_2)$, 则

$$
\begin{aligned}
&\iint_{D_2} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{D_1} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\
&= \int_{\rho_1}^{\rho_2} \left(\int_{\alpha(\rho)}^{\beta(\rho)} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\varphi \right) \mathrm{d}\rho.
\end{aligned}
$$

回顾：三重积分在直角坐标系下的累次积分

命题 1. 设 $D\subset\mathbb{R}^2$ 为 Jordan 可测集，$f_1, f_2\in\mathscr{C}(D)$ 使得 $\forall(x,y)\in D$，均有 $f_1(x,y)\leqslant f_2(x,y)$. 令

$$\Omega=\big\{(x,y,z)\mid f_1(x,y)\leqslant z\leqslant f_2(x,y),\ (x,y)\in D\big\}.$$

则 Ω 为 Jordan 可测集且 $\forall f\in\mathscr{C}(\Omega)$，我们均有

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint\limits_{D}\left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y.$$

评注

- Jordan 可测集 Ω 的体积为

$$V(\Omega) = \iint\limits_{D} \big(f_2(x,y) - f_1(x,y)\big)\,\mathrm{d}x\mathrm{d}y.$$

- 若 $D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$, 则

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{D} \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}x\mathrm{d}y$$

$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x.$$

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例 4. 计算 $I=\iiint\limits_{\Omega} xyz\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$\Omega=\left\{(x,y,z)\mid x^2+y^2+z^2\leqslant 1,\ x,y,z\geqslant 0\right\}$.

解: 由题设可知 $(x,y,z)\in\Omega$ 当且仅当

$0\leqslant z\leqslant\sqrt{1-x^2-y^2},\ x^2+y^2\leqslant 1,\ x,y\geqslant 0.$

而这又等价于说

$$0\leqslant z\leqslant\sqrt{1-x^2-y^2},$$
$$0\leqslant y\leqslant\sqrt{1-x^2},\ 0\leqslant x\leqslant 1,$$

由此立刻可得

$$
\begin{aligned}
I &= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz \,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \frac{1}{2}(1-x^2-y^2)xy \,\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{2}(1-x^2)y^2x - \frac{1}{4}y^4x \right)\Big|_0^{\sqrt{1-x^2}} \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{2}(1-x^2)^2x - \frac{1}{4}(1-x^2)^2x \right) \mathrm{d}x \\
&= -\frac{1}{48}(1-x^2)^3\Big|_0^1 = \frac{1}{48}.
\end{aligned}
$$

例 5. 计算 $I = \iiint\limits_{\Omega} (y+z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega = \left\{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1,\ \ z \geqslant 0\right\}.$$

解: 由线性性与对称性可知

$$\begin{aligned} I &= \iiint\limits_{\Omega} y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} \left(\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} z\,\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} \left(\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} z \,\mathrm{d}z \right) \mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1} \frac{c^2}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \mathrm{d}x\mathrm{d}y \\
&= 4 \iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1 \\ x,y\geqslant 0}} \frac{c^2}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= 2c^2 \iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\\ x,y\geqslant 0}} \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \mathrm{d}x\mathrm{d}y \\
&\overset{\substack{x=a\rho\cos\varphi\\ y=b\rho\sin\varphi}}{=} 2c^2 \int_0^{\frac{\pi}{2}} \left(\int_0^1 (1-\rho^2)ab\rho\,\mathrm{d}\rho\right) \mathrm{d}\varphi \\
&= 2abc^2 \cdot \frac{\pi}{2} \cdot \left(\frac{\rho^2}{2}-\frac{\rho^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{4}abc^2.
\end{aligned}
$$

作业题: 第 3.4 节第 161 页第 5 题第 (1) 小题.

2. 三重积分在柱坐标系下的累次积分法:

考虑广义柱坐标变换 (其中 $a, b > 0$)

$$\begin{cases} x = a\rho\cos\varphi, \\ y = b\rho\sin\varphi, \quad (\rho \geqslant 0,\ 0 \leqslant \varphi < 2\pi,\ z \in \mathbb{R}). \\ z = z, \end{cases}$$

该变换为连续可导且我们有

$$\frac{D(x,y,z)}{D(\rho,\varphi,z)} = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi & 0 \\ b\sin\varphi & b\rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = ab\rho.$$

假设 $\Omega \subset \mathbb{R}^3$ 为 Jordan 可测集, 并且它在广义柱坐标系下变为 Ω_1, 则我们有

$$
\begin{aligned}
&\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
= &\iiint\limits_{\Omega_1} f(a\rho\cos\varphi, b\rho\sin\varphi, z)ab\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z.
\end{aligned}
$$

当 $a = b = 1$ 时, 我们就得到了标准 (通常的) 柱坐标变换.

例 6. 将 $\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ 在柱坐标系下化成累次积分, 其中

$$\Omega=\left\{(x,y,z)\mid(x-R)^2+y^2\leqslant R^2,\ 0\leqslant z\leqslant H\right\}.$$

解: 在柱坐标下 Ω 变为

$$\Omega_1=\left\{(\rho,\varphi,z)\mid-\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2},\ 0\leqslant\rho\leqslant 2R\cos\varphi,\ 0\leqslant z\leqslant H\right\}.$$

由此立刻可得

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{2R\cos\varphi}\left(\int_0^H f(\rho\cos\varphi,\rho\sin\varphi,z)\rho\,\mathrm{d}z\right)\mathrm{d}\rho\right)\mathrm{d}\varphi.$$

例 7. 求积分 $\iiint\limits_{\Omega}(x^2+y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中立体 Ω 为平面曲线 $\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$ 绕 z 轴旋转一周形成的旋转面与平面 $z=8$ 所围成的空间区域.

解: 在柱坐标系下Ω 变为

$$\Omega_1 = \Big\{(\rho,\varphi,z) \mid 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \rho \leqslant 4,\ \frac{1}{2}\rho^2 \leqslant z \leqslant 8\Big\}.$$

由此立刻可得

$$\iint\limits_{\Omega}(x^2+y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^{2\pi}\Big(\int_0^4\Big(\int_{\frac{1}{2}\rho^2}^8 \rho^3\,\mathrm{d}z\Big)\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi = \frac{1024}{3}\pi.$$

例 8. 计算 $\iiint\limits_{\Omega} x^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega = \left\{(x, y, z) \mid \sqrt{x^2+y^2} \leqslant z \leqslant \sqrt{R^2-x^2-y^2}\right\}.$$

解: 在柱坐标系下 Ω 变为

$$\Omega_1 = \left\{(\rho, \varphi, z) \mid \rho \leqslant z \leqslant \sqrt{R^2-\rho^2},\ 0 \leqslant \rho \leqslant \frac{\sqrt{2}R}{2},\ 0 \leqslant \varphi \leqslant 2\pi\right\}.$$

由此立刻可得

$$\iiint\limits_{\Omega} x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega_1} (\rho\cos\varphi)^2\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z = \int_0^{2\pi}\Big(\int_0^{\frac{\sqrt{2}R}{2}}\Big(\int_\rho^{\sqrt{R^2-\rho^2}} \rho^3\cos^2\varphi\mathrm{d}z\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi$$

$$= \pi\int_0^{\frac{\sqrt{2}R}{2}} \rho^3\big(\sqrt{R^2-\rho^2}-\rho\big)\,\mathrm{d}\rho = \frac{\pi R^5}{5}\Big(\frac{2}{3}-\frac{5\sqrt{2}}{12}\Big).$$

例 9. 计算 $I=\iiint\limits_{\Omega}\frac{xy}{\sqrt{z}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中 Ω 为锥面 $(\frac{z}{c})^2=(\frac{x}{a})^2+(\frac{y}{b})^2$ 与平面 $z=c$ 所围成的区域在第一卦限的部分.

解: 由题设可知 $(x,y,z)\in\Omega$ 当且仅当

$$x,y,z\geqslant 0,\ \left(\frac{z}{c}\right)^2\geqslant\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2,\ z\leqslant c.$$

考虑广义柱坐标变换

$$\begin{cases} x=a\rho\cos\varphi, \\ y=b\rho\sin\varphi, \quad (\rho\geqslant 0,\ 0\leqslant\varphi<2\pi,\ z\in\mathbb{R}). \\ z=z, \end{cases}$$

在柱坐标系下, 积分区域 Ω 变为

$$\Omega_1=\left\{(\rho,\varphi,z)\mid 0\leqslant\rho\leqslant 1,\ 0\leqslant\varphi\leqslant\frac{\pi}{2},\ c\rho\leqslant z\leqslant c\right\}.$$

由此立刻可得

$$\begin{aligned}
I &= \iiint\limits_{\Omega}\frac{xy}{\sqrt{z}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= \iint\limits_{\substack{0\leqslant\rho\leqslant 1\\ 0\leqslant\varphi\leqslant\frac{\pi}{2}}}\left(\int_{c\rho}^{c}\frac{ab\rho^2\sin\varphi\cos\varphi}{\sqrt{z}}ab\rho\,\mathrm{d}z\right)\mathrm{d}\rho\mathrm{d}\varphi\\
&= a^2b^2\left(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\varphi\,\mathrm{d}\varphi\right)\int_0^1\left(\rho^3 2\sqrt{z}\Big|_{c\rho}^{c}\right)\mathrm{d}\rho=\frac{a^2b^2\sqrt{c}}{36}.
\end{aligned}$$

例 10. 求 $I=\iiint\limits_{\Omega}(1+x^2+y^2)z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega=\left\{(x,y,z)\mid\sqrt{x^2+y^2}\leqslant z\leqslant H\right\}.$$

解: 在柱坐标系下, 积分区域 Ω 变为

$$\Omega_1=\left\{(\rho,\varphi,z)\mid\rho\leqslant z\leqslant H,\ 0\leqslant\rho\leqslant H,\ 0\leqslant\varphi\leqslant 2\pi\right\}.$$

由此立刻可得

$$I=\iiint\limits_{\Omega_1}(1+\rho^2)z\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z=\int_0^{2\pi}\Big(\int_0^H\Big(\int_\rho^H(1+\rho^2)z\rho\,\mathrm{d}z\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi$$

$$=2\pi\int_0^H\frac{1}{2}\rho(1+\rho^2)(H^2-\rho^2)\mathrm{d}\rho=\pi\Big(\frac{H^4}{4}+\frac{H^6}{12}\Big).$$

作业题: 第 3.4 节第 161 页第 7 题第 (1) 小题.

3. 三重积分在球坐标系下的累次积分法:

考虑球坐标变换

$$\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta, \end{cases} \quad (r \geqslant 0,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi < 2\pi).$$

该变换为连续可导且我们有

$$\begin{aligned} \frac{D(x,y,z)}{D(r,\theta,\varphi)} &= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\ &= r^2\sin\theta. \end{aligned}$$

下面假设 $\Omega \subset \mathbb{R}^3$ 为 Jordan 可测集, 并且它在球坐标系下变为 Ω_1, 则我们有

$$
\begin{aligned}
&\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
=&\iiint\limits_{\Omega_1} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\left|\frac{D(x,y,z)}{D(r,\varphi,\theta)}\right|\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
=&\iiint\limits_{\Omega_1} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi.
\end{aligned}
$$

例 11. 计算 $I = \iiint\limits_{\Omega} (x^2 + z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega = \{(x, y, z) \mid x^2 + y^2 + (z - R)^2 \leqslant R^2\}.$$

解: 由题设可知 $(x, y, z) \in \Omega$ 当且仅当

$$x^2 + y^2 + (z - R)^2 \leqslant R^2,$$

也即 $r^2 \sin^2\theta + (r\cos\theta - R)^2 \leqslant R^2$, 这又等价于

$r \leqslant 2R\cos\theta$. 特别地, 我们有 $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

由此立刻可得

$$
\begin{aligned}
I &= \iiint\limits_{\Omega} (x^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \iiint\limits_{\substack{r\leqslant 2R\cos\theta \\ 0\leqslant\varphi\leqslant 2\pi,\ 0\leqslant\theta\leqslant\frac{\pi}{2}}} (r^2\sin^2\theta\cos^2\varphi + r^2\cos^2\theta) r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{2}}\left(\int_0^{2\pi}\left(\int_0^{2R\cos\theta} r^4(\sin^2\theta\cos^2\varphi+\cos^2\theta)\sin\theta\,\mathrm{d}r\right)\mathrm{d}\varphi\right)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\left(\int_0^{2\pi}\frac{(2R\cos\theta)^5}{5}(\sin^2\theta\cos^2\varphi+\cos^2\theta)\sin\theta\,\mathrm{d}\varphi\right)\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \Big(\int_0^{2\pi} \frac{(2R\cos\theta)^5}{5} (\sin^2\theta\cos^2\varphi + \cos^2\theta)\sin\theta\,\mathrm{d}\varphi\Big)\mathrm{d}\theta \\
&= \frac{32R^5}{5} \int_0^{\frac{\pi}{2}} \Big(\cos^5\theta\Big(\big(\frac{\varphi}{2} + \frac{1}{4}\sin 2\varphi\big)\sin^3\theta + (\cos^2\theta\sin\theta)\varphi\Big)\Big|_0^{2\pi}\Big)\mathrm{d}\theta \\
&= \frac{32R^5}{5} \int_0^{\frac{\pi}{2}} \Big(\pi\sin^3\theta + 2\pi\cos^2\theta\sin\theta\Big)\cos^5\theta\,\mathrm{d}\theta \\
&= \frac{32\pi R^5}{5} \int_0^{\frac{\pi}{2}} \Big(1 - \cos^2\theta + 2\cos^2\theta\Big)\cos^5\theta\,\mathrm{d}(-\cos\theta) \\
&= -\frac{32\pi R^5}{5}\Big(\frac{1}{6}\cos^6\theta + \frac{1}{8}\cos^8\theta\Big)\Big|_0^{\frac{\pi}{2}} = \frac{28}{15}\pi R^5.
\end{aligned}
$$

例 12. 计算 $\iiint\limits_{\Omega} z^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant R^2\}.$$

解: 方法 1. 在球坐标下 Ω 变为

$$\Omega_1 = \{(r, \theta, \varphi) \mid 0 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi\}.$$

由此立刻可得

$$\iiint\limits_{\Omega} z^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega_1} (r\cos\theta)^2 r^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$
$$= \left(\int_0^{2\pi} \mathrm{d}\varphi\right)\left(\int_0^{\pi} \cos^2\theta \sin\theta \,\mathrm{d}\theta\right)\left(\int_0^{R} r^4 \,\mathrm{d}r\right) = \frac{4}{15}\pi R^5.$$

方法 2. 由对称性立刻可得

$$\iiint\limits_{\Omega} x^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega} y^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega} z^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

进而再利用球坐标变换可得

$$\begin{aligned} \iiint\limits_{\Omega} z^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \frac{1}{3}\iiint\limits_{\Omega} (x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = \ & \frac{1}{3}\int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{R} r^2(r^2\sin\theta)\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}\varphi \\ = \ & \frac{2\pi}{3}\left(\int_0^{R} r^4\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\right) = \frac{4}{15}\pi R^5. \end{aligned}$$

例 13. 计算 $\iiint\limits_{\Omega} x^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega=\left\{(x,y,z)\mid \sqrt{x^2+y^2}\leqslant z\leqslant \sqrt{R^2-x^2-y^2}\right\}.$$

解: 在球坐标系下 Ω 变为

$$\Omega_1=\left\{(r,\theta,\varphi)\mid 0\leqslant r\leqslant R,\ 0\leqslant\theta\leqslant\frac{\pi}{4},\ 0\leqslant\varphi\leqslant 2\pi\right\}.$$

由此立刻可得

$$\iiint\limits_{\Omega} x^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega_1}(r\sin\theta\cos\varphi)^2r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

$$=\left(\int_0^{2\pi}\cos^2\varphi\,\mathrm{d}\varphi\right)\left(\int_0^{\frac{\pi}{4}}\sin^3\theta\,\mathrm{d}\theta\right)\left(\int_0^{R}r^4\,\mathrm{d}r\right)=\frac{\pi R^5}{5}\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right).$$

例 14. 计算心脏线 $r = a(1+\cos\theta)$ $(0 \leqslant \theta \leqslant \pi)$ 与极轴 (这里为 z 轴) 围成的图形绕极轴旋转一周后所得到的旋转体 Ω 的体积, 其中 $a > 0$.

解: 设旋转体 Ω 在球坐标系下变为 Ω_1. 由题设可知 $(r, \theta, \varphi) \in \Omega_1$ 当且仅当我们有

$$0 \leqslant r \leqslant a(1+\cos\theta),\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi.$$

故所求体积为

$$V = \iiint_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_1} r^2 \sin\theta \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

$$
\begin{aligned}
&= \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{a(1+\cos\theta)} r^2\sin\theta\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}\varphi \\
&= 2\pi\int_0^{\pi}\frac{a^3}{3}(1+\cos\theta)^3\sin\theta\,\mathrm{d}\theta \\
&= -\frac{2\pi}{3}a^3\int_0^{\pi}(1+\cos\theta)^3\,\mathrm{d}\cos\theta \\
&= -\frac{2\pi}{3}a^3\left(\frac{1}{4}(1+\cos\theta)^4\right)\Big|_0^{\pi} \\
&= \frac{8}{3}\pi a^3.
\end{aligned}
$$

作业题: 第 3.4 节第 161 页第 7 题第 (2) 小题.

4. 其它类型的坐标变换:

例 15. 求 $\iiint\limits_{\Omega} (x+y+z)\cos(x+y+z)^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z$,

其中 $\Omega = \Big\{(x,y,z) \mid 0 \leqslant x-y \leqslant 1,$

$$0 \leqslant x-z \leqslant 1,\ 0 \leqslant x+y+z \leqslant 1\Big\}.$$

解: 作变量替换

$$\begin{cases} u = x - y, \\ v = x - z, \\ w = x + y + z. \end{cases}$$

在此变换下, 积分区域 Ω 变为

$$\Omega_1 = \{(u, v, w) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1,\ 0 \leqslant w \leqslant 1\}.$$

与此同时, 我们有

$$\frac{D(u, v, w)}{D(x, y, z)} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3,$$

由此我们立刻可得 $\frac{D(x,y,z)}{D(u,v,w)} = \frac{1}{3}$.

于是我们有

$$
\begin{aligned}
&\iiint\limits_{\Omega} (x+y+z)\cos(x+y+z)^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
= \ & \int_0^1 \left(\int_0^1 \left(\int_0^1 (w\cos(w^2)) \cdot \frac{1}{3} \,\mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}w \\
= \ & \frac{1}{3} \int_0^1 \left(\int_0^1 (w\cos(w^2)) \,\mathrm{d}v \right) \mathrm{d}w \\
= \ & \frac{1}{3} \int_0^1 (w\cos(w^2)) \,\mathrm{d}w = \frac{1}{6} \sin(w^2) \Big|_0^1 = \frac{1}{6} \sin 1.
\end{aligned}
$$

例 16. 计算 $\iiint\limits_{\Omega}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中,

$$\Omega=\left\{(x,y,z)\mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\right\},\ a,b,>0.$$

解: 考虑广义球坐标变换

$$\begin{cases} x=ar\sin\theta\cos\varphi, \\ y=br\sin\theta\sin\varphi, \\ z=cr\cos\theta, \end{cases}\quad (r\geqslant 0,\ 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi<2\pi).$$

该变换为连续可导且我们有

$$\frac{D(x,y,z)}{D(r,\theta,\varphi)}=\begin{vmatrix} a\sin\theta\cos\varphi & ar\cos\theta\cos\varphi & -ar\sin\theta\sin\varphi \\ b\sin\theta\sin\varphi & br\cos\theta\sin\varphi & br\sin\theta\cos\varphi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix}$$

$$=abcr^2\sin\theta.$$

在此变换下 Ω 变为

$$\Omega_1 = \{(r, \theta, \varphi) \mid 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi\}.$$

由此立刻可得

$$\begin{aligned} & \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = & \iiint_{\Omega_1} r^2 (abcr^2 \sin\theta) \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\ = & \ abc\left(\int_0^1 r^4 \, \mathrm{d}r\right)\left(\int_0^{\pi} \sin\theta \, \mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\varphi\right) = \frac{4\pi}{5} abc. \end{aligned}$$

例 17. 求曲面 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ 围成的立体的体积.

解: 设所围区域为 Ω. 考虑广义球坐标变换

$$\begin{cases} x = ar\sin\theta\cos\varphi, \\ y = br\sin\theta\sin\varphi, \quad (r \geqslant 0,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi < 2\pi), \\ z = cr\cos\theta, \end{cases}$$

在该变换下 Ω 变为

$$\Omega_1 = \{(r,\theta,\varphi) \mid 0 \leqslant r \leqslant \sin\theta,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi\}.$$

由此立刻可知所求体积为

$$\begin{aligned}
|\Omega| &= \iiint\limits_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{\Omega_1} abcr^2 \sin\theta \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{\sin\theta} abcr^2 \sin\theta \,\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}\varphi \\
&= \frac{2}{3}\pi abc \int_0^{\pi} \sin^4\theta \,\mathrm{d}\theta = \frac{4}{3}\pi abc \int_0^{\frac{\pi}{2}} \cos^4\theta \,\mathrm{d}\theta \\
&= \frac{1}{3}\pi abc \int_0^{\frac{\pi}{2}} (1 + 2\cos 2\theta + \cos^2 2\theta) \,\mathrm{d}\theta = \frac{\pi^2}{4} abc.
\end{aligned}$$

作业题: 第 3.4 节第 162 页第 8 题第 (2) 小题.

5. n 重积分:

例 18. 计算 $\mathbb{R}^n$ 中的单位球

$$\Omega_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 \leqslant 1\right\}$$

的体积 V_n.

解: 由题设可知

$$\Omega_n = \left\{(x_1, \ldots, x_n) \mid -1 \leqslant x_n \leqslant 1, \sum_{j=1}^{n-1} x_j^2 \leqslant 1 - x_n^2\right\}.$$

令 $D_{n-1}(x_n)=\left\{(x_1,\ldots,x_{n-1})\mid\sum\limits_{j=1}^{n-1}x_j^2\leqslant 1-x_n^2\right\}$, 则

$$V_n=\int_{-1}^{1}\Big(\int\cdots\int\limits_{D_{n-1}(x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}\Big)\mathrm{d}x_n.$$

对于 $|x_n|<1$, 考虑坐标变换

$$x_j=\sqrt{1-x_n^2}u_j\ (1\leqslant j\leqslant n-1).$$

该变换为 $\mathscr{C}^{(1)}$ 类, 其逆亦如此. 它将 $D_{n-1}(x_n)$ 变成 $\mathbb{R}^{n-1}$ 中的单位球 Ω_{n-1}, 且我们有

$$\frac{D(x_1,\ldots,x_{n-1})}{D(u_1,\ldots,u_{n-1})}=\left(\sqrt{1-x_n^2}\right)^{n-1}.$$

由此我们立刻可以导出

$$
\begin{aligned}
V_n &= \int_{-1}^{1} \Big(\int_{D_{n-1}(x_n)} \cdots \int \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \Big) \mathrm{d}x_n \\
&= \int_{-1}^{1} \Big(\int_{\Omega_{n-1}} \cdots \int (1-x_n^2)^{\frac{n-1}{2}} \,\mathrm{d}u_1 \cdots \mathrm{d}u_{n-1} \Big) \mathrm{d}x_n \\
&= \int_{-1}^{1} (1-x_n^2)^{\frac{n-1}{2}} V_{n-1} \,\mathrm{d}x_n = 2V_{n-1} \int_0^1 (1-x_n^2)^{\frac{n-1}{2}} \,\mathrm{d}x_n \\
&\overset{t=x_n^2}{=} 2V_{n-1} \int_0^1 (1-t)^{\frac{n-1}{2}} \,\mathrm{d}(\sqrt{t}) = V_{n-1} \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{n-1}{2}} \,\mathrm{d}t \\
&= V_{n-1} B\Big(\frac{1}{2}, \frac{n+1}{2}\Big) = \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+2}{2}\big)} V_{n-1} = \sqrt{\pi} \frac{\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n+2}{2}\big)} V_{n-1}.
\end{aligned}
$$

注意到 $\Omega_1=[-1,1]$, 故 $V_1=2$, 从而我们有

$$
\begin{aligned}
V_n &= \prod_{k=2}^{n}\frac{V_k}{V_{k-1}}\cdot V_1 = 2\prod_{k=2}^{n}\left(\sqrt{\pi}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}\right)\\
&= 2\pi^{\frac{n-1}{2}}\frac{\prod\limits_{k=2}^{n}\Gamma\left(\frac{k+1}{2}\right)}{\prod\limits_{k=2}^{n}\Gamma\left(\frac{k+2}{2}\right)} = 2\pi^{\frac{n-1}{2}}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\\
&= 2\pi^{\frac{n-1}{2}}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.
\end{aligned}
$$

§5. 重积分的应用

1. 物体的重心 (质心或形心) 问题

问题: 假设 $\Omega \subset \mathbb{R}^n$ 中分布有质量, 在点 X 处的密度为 $\rho(X)$, 求其重心.

解: 设其重心为 $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$, 则

$$\bar{x}_j = \frac{1}{M} \int \cdots \int_{\Omega} x_j \rho(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

其中 $M = \int \cdots \int_{\Omega} \rho(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$.

注: 当 $\rho \equiv 1$ 时, 将质心称为形心.

例 1. 假设在 $\Omega=\{(x,y,z)\mid\sqrt{x^2+y^2}\leqslant z\leqslant H\}$ 上分布着密度为 $\rho(x,y,z)=1+x^2+y^2$ 的质量, 求其质心.

解: 由题设可知总质量为

$$\begin{aligned}
M &= \iiint\limits_{\Omega}\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\overset{\substack{x=r\cos\varphi\\ y=r\sin\varphi}}{=} \int_0^{2\pi}\left(\int_0^H\left(\int_r^H(1+r^2)r\,\mathrm{d}z\right)\mathrm{d}r\right)\mathrm{d}\varphi \\
&= 2\pi\int_0^H(1+r^2)r(H-r)\,\mathrm{d}r=\pi\Big(\frac{H^3}{3}+\frac{H^5}{10}\Big).
\end{aligned}$$

由对称性可知, 所求重心 $(\bar{x}, \bar{y}, \bar{z})$ 满足

$$\begin{aligned}
\bar{x} &= \frac{1}{M} \iiint_{\Omega} x\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0, \\
\bar{y} &= \frac{1}{M} \iiint_{\Omega} y\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 0, \\
\bar{z} &= \frac{1}{M} \iiint_{\Omega} z\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M} \int_0^{2\pi} \left(\int_0^H \left(\int_r^H z(1+r^2)r\,\mathrm{d}z \right) \mathrm{d}r \right) \mathrm{d}\varphi \\
&= \frac{2\pi}{M} \int_0^H r(1+r^2) \cdot \frac{1}{2}(H^2 - r^2)\,\mathrm{d}r = \frac{5(H^2+3)H}{2(3H^2+10)}.
\end{aligned}$$

例 2. 求由曲线 $y=x^2$, $x+y=2$ 所围成的平面均匀薄板的质心.

解: 设曲线所围成的平面区域为 Ω, 则

$$\Omega=\{(x,y)\mid -2\leqslant x\leqslant 1,\ x^2\leqslant y\leqslant 2-x\}.$$

不失一般性, 设其面密度为 $\rho\equiv 1$, 则其质量为

$$M=\iint\limits_{\Omega}\mathrm{d}x\mathrm{d}y=\int_{-2}^{1}\left(\int_{x^2}^{2-x}\mathrm{d}y\right)\mathrm{d}x=\int_{-2}^{1}(2-x-x^2)\,\mathrm{d}x$$

$$=\left(2x-\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-2}^{1}=\frac{9}{2}.$$

设所求质心为 $(\bar{x}, \bar{y})$, 则

$$\begin{aligned}
\bar{x} &= \frac{1}{M} \iint\limits_{\Omega} x \, \mathrm{d}x\mathrm{d}y = \frac{1}{M} \int_{-2}^{1} \left(\int_{x^2}^{2-x} x \, \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{M} \int_{-2}^{1} x(2 - x - x^2) \, \mathrm{d}x = -\frac{1}{2}, \\
\bar{y} &= \frac{1}{M} \iint\limits_{\Omega} y \, \mathrm{d}x\mathrm{d}y = \frac{1}{M} \int_{-2}^{1} \left(\int_{x^2}^{2-x} y \, \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2M} \int_{-2}^{1} \left((2 - x)^2 - x^4 \right) \mathrm{d}x = \frac{8}{5},
\end{aligned}$$

故所求质心为 $(-\frac{1}{2}, \frac{8}{5})$.

例 3. 设曲面 S 在球坐标系下的方程为

$$r = a(1+\cos\theta)\ (a>0).$$

令 Ω 为曲面 S 所围成的有界区域, 求 Ω 在直角坐标系下的形心.

解: 在球坐标系下 Ω 变为

$$\Omega_1 = \left\{(r,\theta,\varphi) \mid 0 \leqslant r \leqslant a(1+\cos\theta),\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi\right\}.$$

由此可得 Ω 的体积为

$$|\Omega| = \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{a(1+\cos\theta)} r^2 \sin\theta \,\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}\varphi = \frac{8}{3}\pi a^3.$$

设所求质心为 $(\bar{x}, \bar{y}, \bar{z})$, 则由对称性可知

$$\begin{aligned}
\bar{x} &= \frac{1}{|\Omega|} \iiint\limits_{\Omega} x \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0, \\
\bar{y} &= \frac{1}{|\Omega|} \iiint\limits_{\Omega} y \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0, \\
\bar{z} &= \frac{1}{|\Omega|} \iiint\limits_{\Omega} z \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{|\Omega|} \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{a(1+\cos\theta)} (r\cos\theta)(r^2\sin\theta)\mathrm{d}r \right) \mathrm{d}\theta \right) \mathrm{d}\varphi = \frac{4}{5}a,
\end{aligned}$$

故所求质心为 $\left(0, 0, \frac{4}{5}a\right)$.

2. 曲面的面积问题

设空间曲面 Σ 的参数方程为

$$\begin{cases} x = x(u,v), \\ y = y(u,v), \quad (u,v) \in D, \\ z = z(u,v), \end{cases}$$

其中 $D \subset \mathbb{R}^2$ 为 Jordan 可测集, 而 x, y, z 为连续可导函数使得 $\frac{\partial(x,y,z)}{\partial(u,v)}$ 的秩为 2.

定义

$$\vec{T}_u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}, \quad \vec{T}_v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix},$$

则我们有

$$\vec{T}_u \times \vec{T}_v = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{D(y,z)}{D(u,v)} \\ \frac{D(z,x)}{D(u,v)} \\ \frac{D(x,y)}{D(u,v)} \end{pmatrix}.$$

由此可得

$$\|\vec{T}_u \times \vec{T}_v\|^2 = \left(\frac{D(y,z)}{D(u,v)}\right)^2 + \left(\frac{D(z,x)}{D(u,v)}\right)^2 + \left(\frac{D(x,y)}{D(u,v)}\right)^2$$

$$=\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)^2$$

$$= \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\left(\frac{\partial x}{\partial v}\right)^2$$

$$+\left(\frac{\partial z}{\partial v}\right)^2\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{\partial y}{\partial u}\right)^2$$

$$-2\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u} - 2\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\frac{\partial z}{\partial v}\frac{\partial x}{\partial u} - 2\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$
\begin{aligned}
&= \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\left(\frac{\partial x}{\partial v}\right)^2 \\
&\quad + \left(\frac{\partial z}{\partial v}\right)^2\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2\left(\frac{\partial y}{\partial u}\right)^2 \\
&\quad - 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} - 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} - 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\
&= \left(\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right) \cdot \left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) \\
&\quad - \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)^2 = \|\vec{T}_u\|^2\|\vec{T}_v\|^2 - \|\vec{T}_u \cdot \vec{T}_v\|^2.
\end{aligned}
$$

定义

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2, \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \end{aligned}$$

则面积微元为

$$\mathrm{d}\sigma = \|\vec{T}_u \times \vec{T}_v\| \,\mathrm{d}u\mathrm{d}v = \sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v,$$

故曲面的面积为 $S = \iint\limits_D \sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v.$

特殊的曲面参数表示:

(1) 设曲面 Σ 的方程为 $z=z(x,y)$, $(x,y)\in D$, 则其参数方程为

$$\begin{cases} x=x, \\ y=y, & \qquad (x,y)\in D, \\ z=z(x,y), \end{cases}$$

故 $E=1+\left(\frac{\partial z}{\partial x}\right)^2$, $G=1+\left(\frac{\partial z}{\partial y}\right)^2$, $F=\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$, 则

$$S=\iint\limits_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y.$$

(2) 设 Σ 的方程为 $x = x(y,z), (y,z) \in D$, 则

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\,\mathrm{d}y\mathrm{d}z.$$

(3) 设 Σ 的方程为 $y = y(x,z), (x,z) \in D$, 则

$$S = \iint\limits_D \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,\mathrm{d}x\mathrm{d}z.$$

例 1. 求半径为 R 的球面的面积.

解: 方法 1. 以球心为原点建立直角坐标系, 于是上半球面的方程为

$$z = \sqrt{R^2 - x^2 - y^2},$$

其中 $x^2 + y^2 \leqslant R^2$. 由此可知

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}.$$

于是由对称性可知球面的面积为

$$
\begin{aligned}
S &= 2\iint\limits_{x^2+y^2\leqslant R^2}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y \\
&= 2\iint\limits_{x^2+y^2\leqslant R^2}\frac{R}{\sqrt{R^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y \\
&\overset{\substack{x=\rho\cos\varphi\\ y=\rho\sin\varphi}}{=} 2\int_0^{2\pi}\left(\int_0^R\frac{R}{\sqrt{R^2-\rho^2}}\,\rho\,\mathrm{d}\rho\right)\mathrm{d}\varphi \\
&= 4\pi R\cdot\left(-\sqrt{R^2-\rho^2}\right)\Big|_0^R = 4\pi R^2.
\end{aligned}
$$

方法 2. 取球心为原点建立直角坐标系, 从而得

球面的参数方程为

$$\begin{cases} x = R\sin\theta\cos\varphi, \\ y = R\sin\theta\sin\varphi, \quad (0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi), \\ z = R\cos\theta, \end{cases}$$

由此可得 $EG - F^2 = R^4\sin^2\theta$, 故所求面积为

$$S = \int_0^{2\pi}\left(\int_0^{\pi} R^2\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}\varphi = 4\pi R^2.$$

例 2. 求旋转抛物面 $z=x^2+y^2$ 在 $z\leqslant 1$ 那一部分的面积.

解: 令 $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leqslant 1\}$, 则所求曲面的方程为 $z=x^2+y^2$, $(x,y)\in D$, 进而知所求面积为

$$\begin{aligned} S &= \iint\limits_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_D \sqrt{1+4x^2+4y^2}\,\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_D \sqrt{1+4x^2+4y^2}\,\mathrm{d}x\mathrm{d}y \\
&= \int_0^{2\pi} \left(\int_0^1 \sqrt{1+4\rho^2}\rho\,\mathrm{d}\rho \right) \mathrm{d}\varphi \\
&= 2\pi \int_0^1 \sqrt{1+4\rho^2}\rho\,\mathrm{d}\rho = \frac{\pi}{4}\int_0^1 \sqrt{1+4\rho^2}\,\mathrm{d}(1+4\rho^2) \\
&= \frac{\pi}{4}\cdot\frac{2}{3}(1+4\rho^2)^{\frac{3}{2}}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}\pi.
\end{aligned}
$$

作业题: 第 3.5 节第 169 页第 1 题第 (1) 小题 (其中 $a > 0$), 第 170 页第 2 题第 (2) 小题.

第 3 章小结

1. 重积分的概念及其性质:

- $\mathbb{R}^n$ 中的坐标平行体上的积分: $\mathbb{R}^n$ 中的区间或者坐标平行体及其体积, 分割, 步长, 带点分割, Riemann 和, 重积分, Riemann 可积.
- 有界集上的函数的 Riemann 积分: 零延拓成坐标平行体上的函数, 再研究其积分.
- 有界集 Ω 上所有 Riemann 可积函数的全体记作 $\mathscr{R}(\Omega)$, 该集合可能 “非常小” .

- 二重积分的几何意义: 立体的体积.
- Jordan 可测集: 定义, 典型的 Jordan 可测集.
- 如果有界闭集 $\Omega \subset \mathbb{R}^n$ 为 Jordan 可测集, 则我们有 $\mathscr{C}(\Omega) \subset \mathscr{R}(\Omega)$.
- Jordan 可测集上重积分的性质: 有界性, 线性, 区域可加性, (严格) 保号性, (严格) 保序性, 绝对值不等式, 积分的上、下界, 积分中值定理及其应用, 变量替换.

2. 重积分的计算:

- 直角坐标系下二重积分的累次积分法,
- 极坐标坐标系下二重积分的累次积分法,
- 直角坐标系下三重积分的累次积分法,
- 柱坐标系下三重积分的累次积分法,
- 球坐标系下三重积分的累次积分法,
- 一般坐标变换: 目的在于转化成累次积分,
- 对称性在重积分计算当中的应用.

3. 重积分应用: 质心、重心、形心, 曲面面积.

谢谢大家!