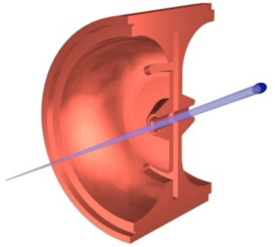

Electrodynamics

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1. Coulomb's Law

➢ Coulomb's Law

- *All of electrostatics stems from the quantitative statement of Coulomb's law!*

$$\boldsymbol{F} = kq_1q_2 \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\left|\boldsymbol{x}_1 - \boldsymbol{x}_2\right|^3}$$

- $\boldsymbol{F}$ is the force on a point charge q_1, located at $\boldsymbol{x}_1$, due to another point charge q_2, located at $\boldsymbol{x}_2$.

- k is the constant of proportionality which depends on the system of units used. In the SI system: $k = 1/\left(4\pi\varepsilon_0\right)$

- Furthermore it was shown experimentally that the total force produced on one small charged body by a number of the other small charged bodies placed around is the vector sum of the individual two-body forces of Coulomb.

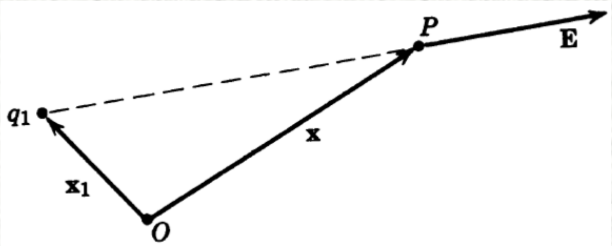
2. Electric Field

➢ Electric Field

- *The electric field can be defined as the force per unit charge acting at a given point.*
- Measurement: Must using a limiting process whereby the ratio of the force on the small test body to the charge on it is measured for smaller and smaller amounts of charge.

$$\boldsymbol{F} = q\boldsymbol{E}$$

- The electric field at the point $\boldsymbol{x}$ due to a point charge q_1 at the point $\boldsymbol{x}_1$:



$$\boldsymbol{E} = kq_1 \frac{\boldsymbol{x} - \boldsymbol{x}_1}{\left|\boldsymbol{x} - \boldsymbol{x}_1\right|^3}$$

One coulomb produces an electric field of 8.987GV/m at a distance of 1 meter!

2. Electric Field

- Linear Superposition

$$\boldsymbol{E}(\boldsymbol{x})=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}q_i\frac{\boldsymbol{x}-\boldsymbol{x}_i}{\left|\boldsymbol{x}-\boldsymbol{x}_i\right|^3}$$

$$\boldsymbol{E}(\boldsymbol{x})=\frac{1}{4\pi\varepsilon_0}\int_l \rho_l\left(\boldsymbol{x}'\right)\frac{\boldsymbol{x}-\boldsymbol{x}'}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|^3}\mathrm{d}l$$

$$\boldsymbol{E}(\boldsymbol{x})=\frac{1}{4\pi\varepsilon_0}\int_{S'} \sigma_S\left(\boldsymbol{x}'\right)\frac{\boldsymbol{x}-\boldsymbol{x}'}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|^3}\mathrm{d}S'$$

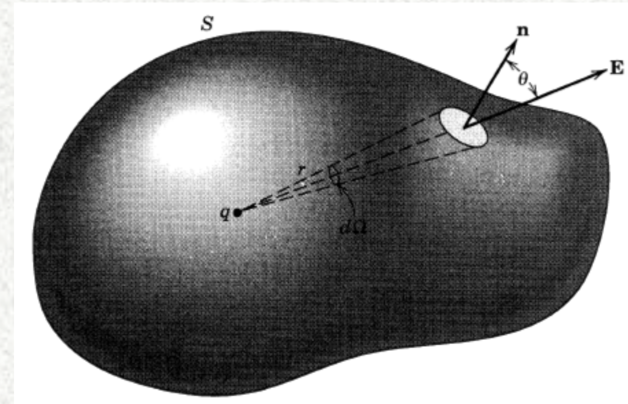
$$\boldsymbol{E}(\boldsymbol{x})=\frac{1}{4\pi\varepsilon_0}\int_{V'} \rho\left(\boldsymbol{x}'\right)\frac{\boldsymbol{x}-\boldsymbol{x}'}{\left|\boldsymbol{x}-\boldsymbol{x}'\right|^3}\mathrm{d}^3x' \quad (1.5)$$

$$\rho(\boldsymbol{x}')=\sum_{i=1}^{n}q_i\delta\left(\boldsymbol{x}'-\boldsymbol{x}_i\right)$$

(1.5) is not always the most suitable form for the evaluation of electric fields.

3. Gauss's Law

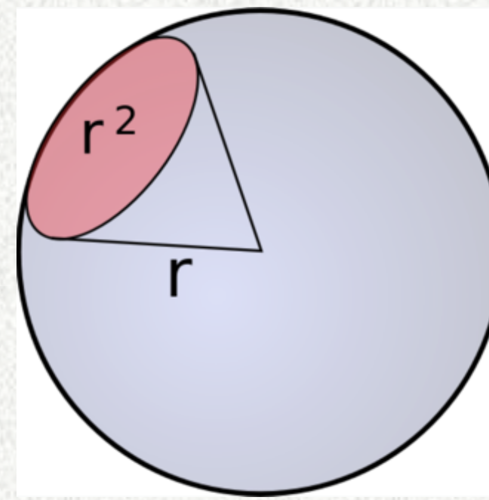
- Gauss's law for a single point charge



$$\boldsymbol{E}\bullet\boldsymbol{n}\mathrm{d}a=\frac{q}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}\mathrm{d}a=\frac{q}{4\pi\varepsilon_0}\mathrm{d}\Omega$$

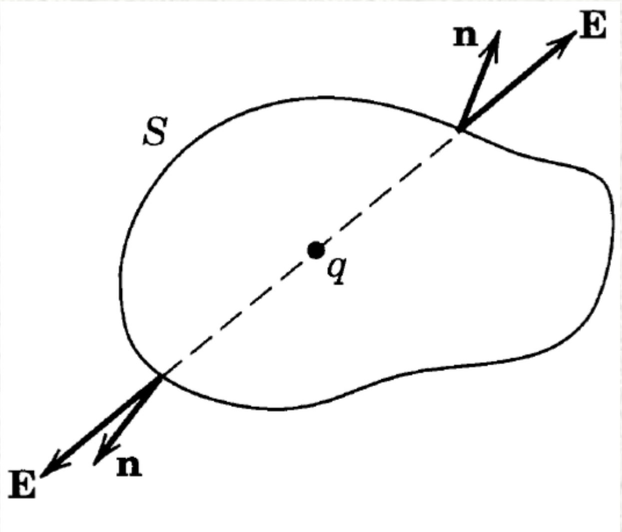
$$\left(\cos\theta\mathrm{d}a=r^2\mathrm{d}\Omega\right)$$

An object's **solid angle**(in steradians): equal to the area of the segment of a unit sphere, centered at the angle's vertex, that the object covers.

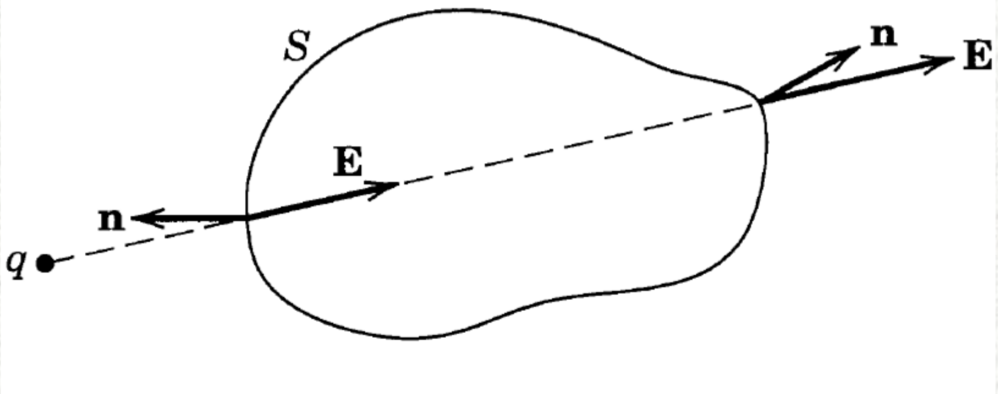


Any area on a sphere which is equal in area to the square of its radius, when observed from its center, subtends precisely **one steradian**.

3. Gauss's Law



q inside S



q outside S

$$\oint_S \boldsymbol{E} \bullet \boldsymbol{n} \, \mathrm{d}a = \begin{cases} q/\varepsilon_0 & \text{if } q \text{ lies inside S} \\ 0 & \text{if } q \text{ lies outside S} \end{cases}$$

3. Gauss's Law

- Gauss's law for a discrete set of charges and continuous distribution of charges

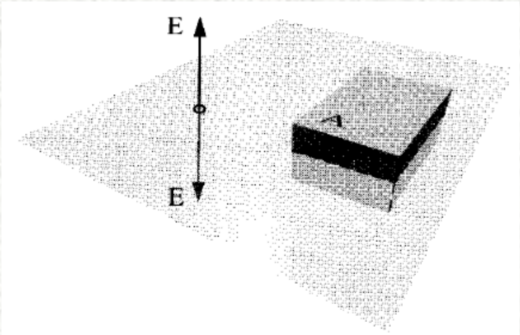
$$\oint_S \boldsymbol{E} \bullet \boldsymbol{n} \,\mathrm{d}a = \frac{1}{\varepsilon_0} \sum_i q_i$$

$$\oint_S \boldsymbol{E} \bullet \boldsymbol{n} \,\mathrm{d}a = \frac{1}{\varepsilon_0} \int_V \rho(\boldsymbol{x}) \mathrm{d}^3 x$$

3. Gauss's Law

- Example 1: An infinite plane carries a uniform surface charge σ. Find its electric field.

Draw a "Gaussian pillbox", Extending equal distances above and below the plane. Apply Gauss's law to this surface:



$$\oint_S \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{a} = \frac{\sigma A}{\varepsilon_0}$$

$$\oint_S \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{a} = 2A|\boldsymbol{E}|$$

$$2A|\boldsymbol{E}| = \frac{\sigma A}{\varepsilon_0}$$

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$\hat{n}$ is a unit vector pointing away from the surface.

3. Gauss's Law

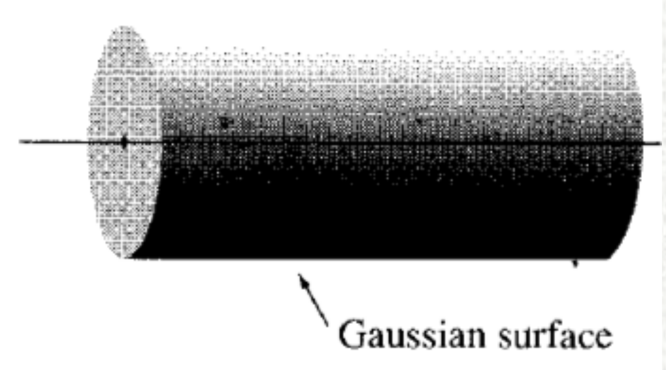
➢ Discussions

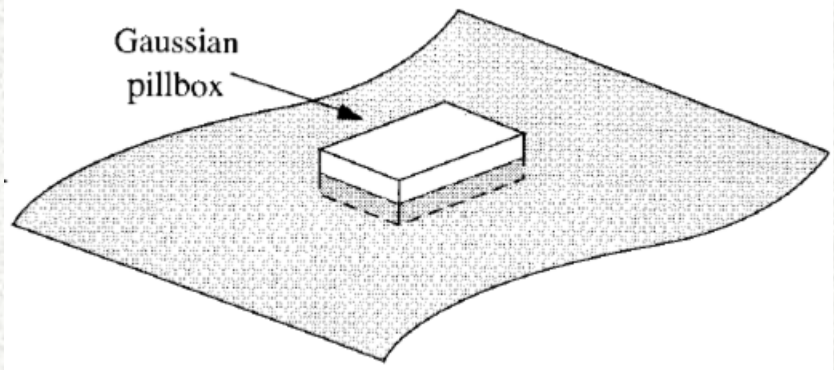
- Gauss's law is always true, but it is not always *useful*.
- *Symmetry* is crucial to this application of Gauss's law.
- There are only three kinds of symmetry that work:

(1) Spherical symmetry. Make your Gaussian surface a concentric sphere.

(2) Cylindrical symmetry. Make your Gaussian surface a coaxial cylinder.

(3) Plane symmetry. Use a Gaussian "pillbox", which straddles the surface.





4. Differential Form of Gauss's Law

- Differential Form of Gauss's Law

From divergence theorem:

$$\oint_S \boldsymbol{A} \bullet \boldsymbol{n} \mathrm{d}a = \int_V (\nabla \bullet \boldsymbol{A}) \mathrm{d}^3 x$$

and the Gauss's Law in the integral form:

$$\oint_S \boldsymbol{E} \bullet \boldsymbol{n} \mathrm{d}a = \frac{1}{\varepsilon_0} \int_V \rho(\boldsymbol{x}) \mathrm{d}^3 x$$

we can obtain $\int_V (\nabla \bullet \boldsymbol{E} - \rho/\varepsilon_0) \mathrm{d}^3 x = 0$ for an arbitrary volume V.

The differential form of Gauss's Law:

$$\nabla \bullet \boldsymbol{E} = \rho/\varepsilon_0$$

5. Another equation of electrostatic and the scalar potential

- Another Equation of Electrostatic

$$\nabla \times \boldsymbol{E} = 0$$

- Scalar Potential

$$\boldsymbol{E} = -\nabla \Phi$$

From (1.15) :

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\mathrm{d}^3x'$$

$$\nabla\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) = -\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}$$

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\left[-\nabla\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\right]\mathrm{d}^3x'$$

$$= -\frac{1}{4\pi\varepsilon_0}\nabla\int \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x' \qquad (1.15)$$

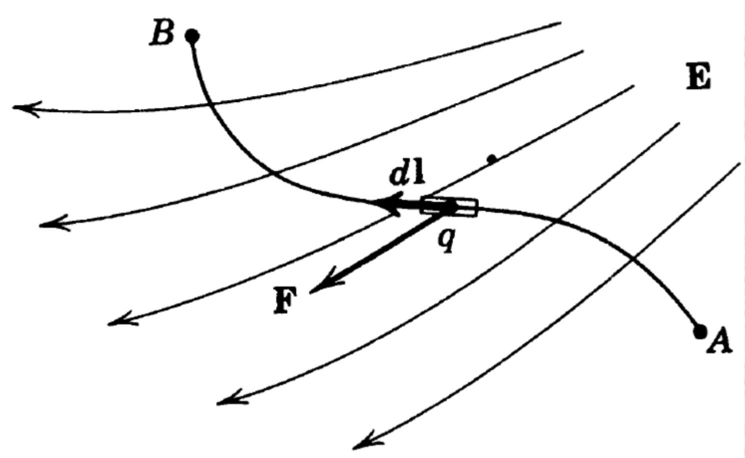
5. Another equation of electrostatic and the scalar potential

- Work on a test charge q in transporting it from one point A to another point B in the presence of an electric field ***E***(***x***): *W*

$$W = -\int_A^B \boldsymbol{F}_e \bullet \mathrm{d}\boldsymbol{l} = -q\int_A^B \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l} = q\int_A^B \nabla\boldsymbol{\Phi} \bullet \mathrm{d}\boldsymbol{l} = q\int_A^B \mathrm{d}\boldsymbol{\Phi} = q\left(\boldsymbol{\Phi}_B - \boldsymbol{\Phi}_A\right)$$

$$\int_A^B \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l} = -\left(\boldsymbol{\Phi}_B - \boldsymbol{\Phi}_A\right) \qquad \int_S \left(\nabla \times \boldsymbol{A}\right) \bullet \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{A} \bullet d\boldsymbol{l}$$

$$\oint \boldsymbol{E} \bullet \mathrm{d}\boldsymbol{l} = 0 \quad \Longrightarrow \quad \nabla \times \boldsymbol{E} = 0$$

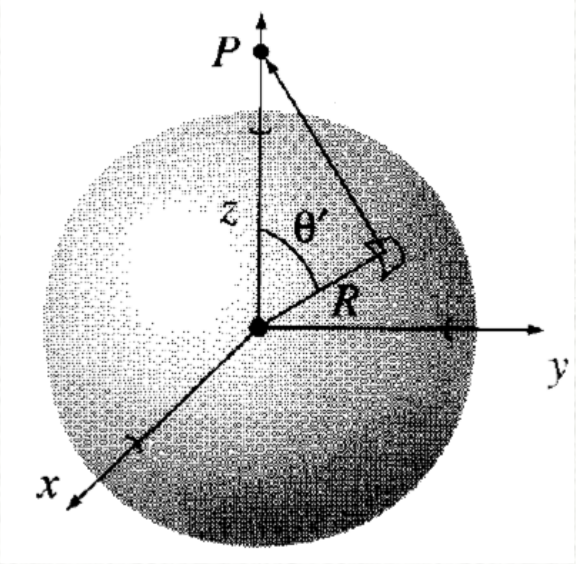


5. Another equation of electrostatic and the scalar potential

- Example 2: Find the potential inside and outside a spherical shell of radius *R*, which carries a uniform surface charge (total charge on the shell is *q*). Set the reference point at infinity.

From Gauss's law, the field outside is $\boldsymbol{E}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r}$.

The field inside is zero.



For points outside the sphere (*r*>*R*),

$$\Phi(r)=-\int_{\infty}^{r}\boldsymbol{E}\bullet \mathrm{d}\boldsymbol{l}=-\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r^2}\mathrm{d}r=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\bigg|_{\infty}^{r}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

For points inside the shell (*r*<*R*),

$$\Phi(r)=-\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{R}\frac{q}{r^2}\mathrm{d}r-\int_{R}^{r}(0)\mathrm{d}r=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\bigg|_{\infty}^{R}=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

Can we solve it with $\Phi(\boldsymbol{x})=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int\frac{\sigma(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}a'$?

6. Poisson and Laplace Equations

- Poisson Equation

$$\left.\begin{aligned}\nabla\cdot\boldsymbol{E}&=\rho/\varepsilon_0\\ \nabla\times\boldsymbol{E}=0\quad\rightarrow\quad\boldsymbol{E}&=-\nabla\Phi\end{aligned}\right\}\Rightarrow\nabla^2\Phi=-\rho/\varepsilon_0$$

- Laplace Equation

$$\nabla^2\Phi=0\qquad\text{when }\rho=0$$

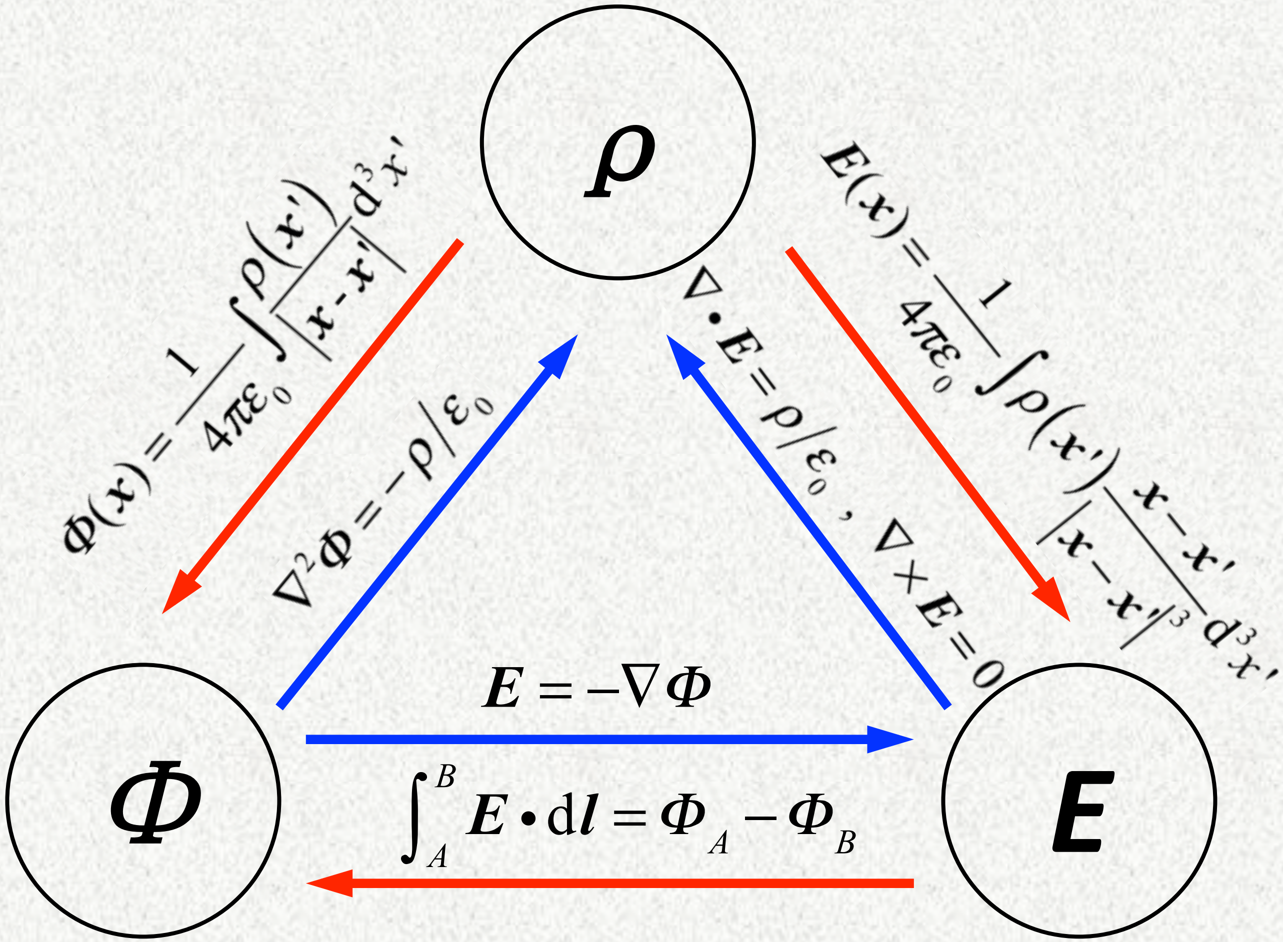
- Example 3: Show that the following scalar potential satisfy the Poisson equation:

$$\Phi(\boldsymbol{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\begin{aligned}\nabla^2\Phi&=\frac{1}{4\pi\varepsilon_0}\nabla^2\int\frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'\\&=\frac{1}{4\pi\varepsilon_0}\int\rho(\boldsymbol{x}')\nabla^2\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\mathrm{d}^3x'\\&=-\frac{1}{\varepsilon_0}\int\rho(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}^3x'\\&=-\rho(\boldsymbol{x})/\varepsilon_0\end{aligned}$$

$$\nabla^2\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)=-4\pi\delta(\boldsymbol{x}-\boldsymbol{x}')$$

6. Poisson and Laplace Equations



- From just two experimental observations, all the formulas can be derived:

  (1) Coulomb's law

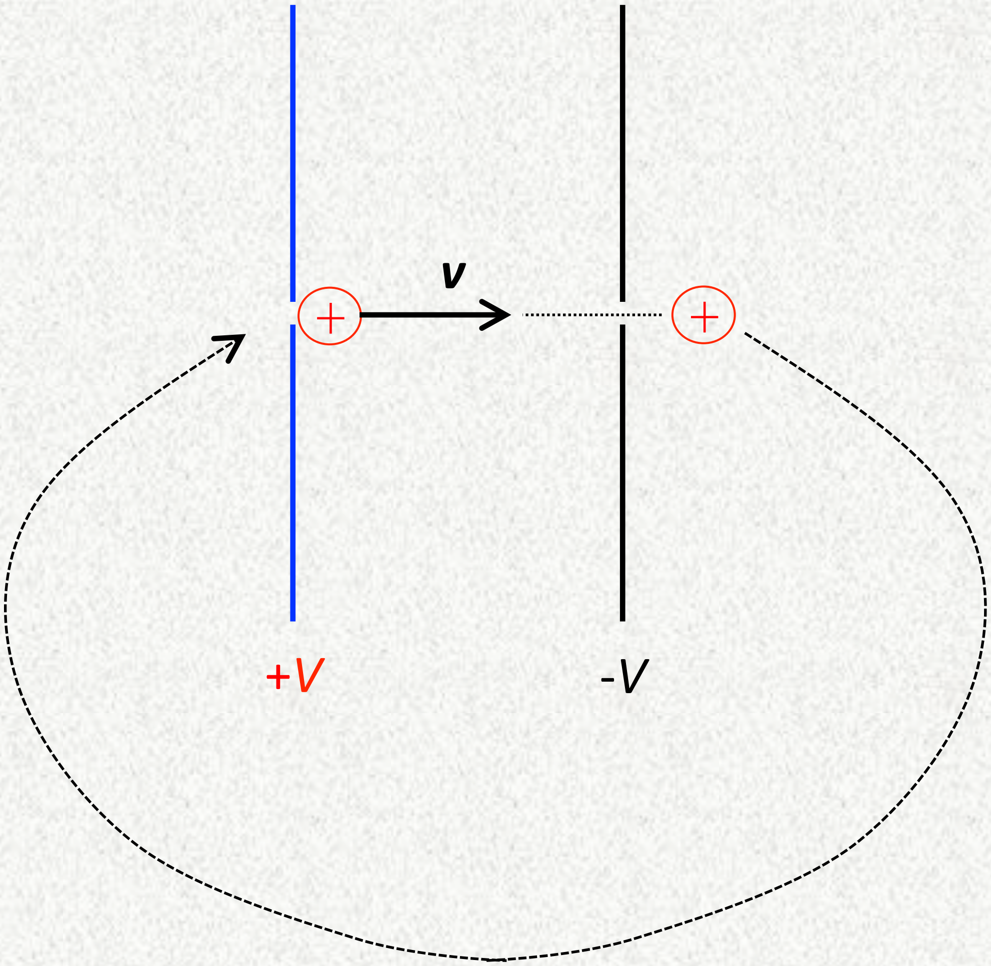
  (2) Principle of superposition

QUESTION-1

The two parallel conductor plates are held at potential of $+V$ and $-V$ respectively. One positively charged particle is accelerated from one plate to another.

Question:

After the particle is accelerated and leaves the plate through the hole, can the particle be accelerated twice, if we can find some way to re-send it to the space between the two plates?



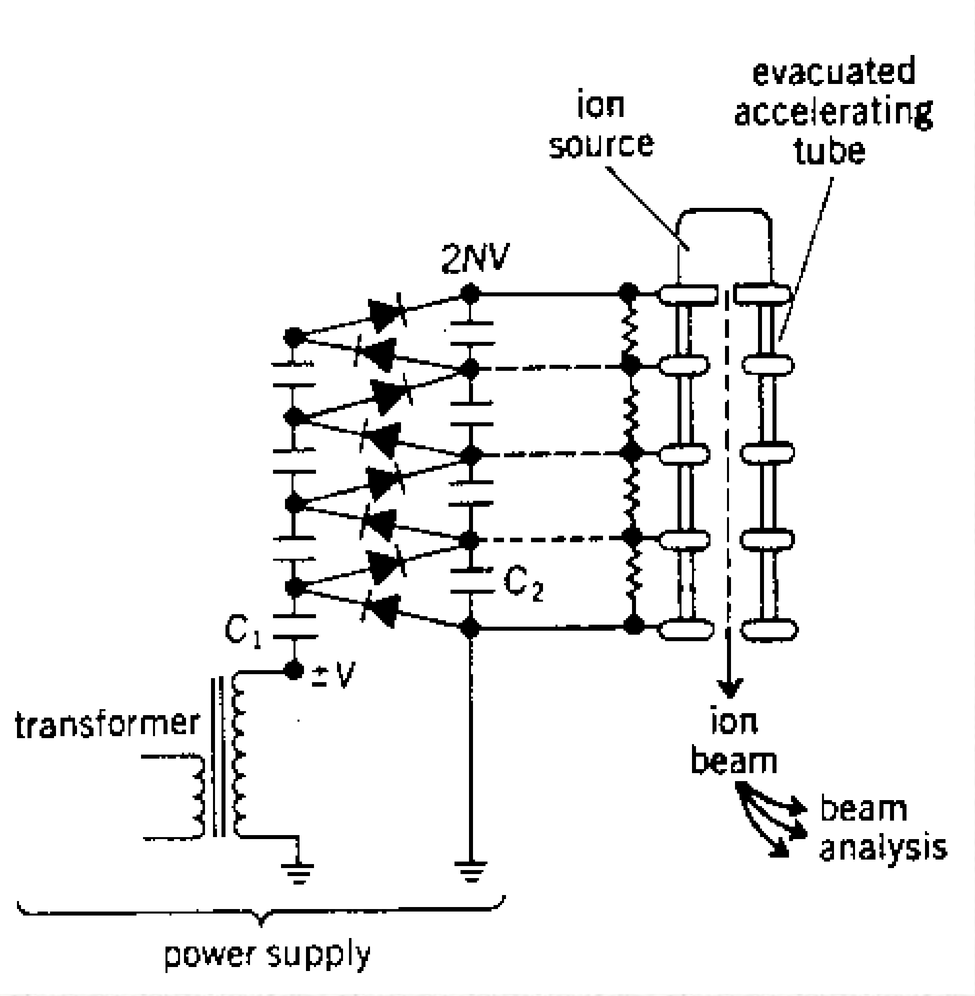
QUESTION-2

In the electrostatic accelerator, the charged particle is accelerated through one tube with several electrodes between which there are static potential across them.

Occassionally people need to insert some conductor plates insulated into the tube to improve the electrostatic field profile.

Question:

Does the energy of the accelerated beam change after the conductor plates are inserted between the electrodes?



Cockcroft-Walton accelerator

Homework

1. If the electric field in some region is given (in spherical coordinates) by the expression

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{A\hat{r} + B\sin\theta\cos\phi\hat{\phi}}{r}$$

where *A* and *B* are constants, what is the charge density?

2. Textbook 1.4

Each of three charged spheres of radius *a*, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n ($n > -3$), has a total charge *Q*. Use Gauss' theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.