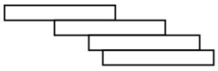
7. The length of stacking blocks:



N identical blocks with uniform density are stacked together, each with length of 2m, and mass M. What is the maximum distance that these N blocks can reach without falling? If we want to use these blocks to built a span of 20m, approximately how many blocks are needed?

Hint: Much easier starting from the top (say set the top block left end as 0) and use property of center of mass (though I may not have formally introduced it, it is not difficult to grasp). The position of center of mass between two objects of masses M_1 and M_2 is: $X_{CM}=(M_1X_1+M_2X_2)/(M_1+M_2)$. The equilibrium (statics) requires that the center of mass of the top blocks has to be supported by the bottom one. You shall find the length goes as a famous series 1+1/2+1/3+... You may need calculus to estimate this series: $\int \frac{1}{x}dx = \ln x$.

$N=2$  block 1 重心 $x=1$

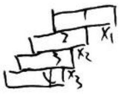
$x_1 \le 1$

$L_{max} = 2 + x_1 \le 2+1$

$N=3$  block 1+2 重心 $\frac{1+(x_1+1)}{2}$

$x_1+x_2 \le \frac{1+(x_1+1)}{2}$

$L_{max} = 2+x_1+x_2 \le 2+1+\frac{x_1}{2} \le 2+1+\frac{1}{2}$

$N=4$  block 1+2+3 重心 $\frac{1+(1+x_1)+(1+x_1+x_2)}{3}$

$x_1+x_2+x_3 \le \frac{1+(1+x_1)+(1+x_1+x_2)}{3} = 1+\frac{2x_1}{3}+\frac{x_2}{3}$

$L_{max} = 2+x_1+x_2+x_3 \le 2+1+\frac{2x_1}{3}+\frac{x_2}{3} \le 2+1+\frac{x_1}{3}+\frac{2+x_1}{6} \le 2+1+\frac{1}{2}+\frac{1}{3}$

$\therefore N$: $L_{max} = 2+1+\frac{1}{2}+\cdots+\frac{1}{N-1}$

$= 2+\sum_{n=1}^{N-1}\frac{1}{n} \approx 2+\int_1^N \frac{1}{x}dx$

$= 2+\ln N$

$2+\ln N = 20 \quad N = e^{18}$

2.1. **Hanging rope**

A rope with length L and mass density per unit length ρ is suspended vertically from one end. Find the tension as a function of height along the rope.

Let $T(y)$ be the tension as a function of height. Consider a small piece of the rope between y and $y + dy$ ($0 \le y \le L$). The forces on this piece are $T(y + dy)$ upward, $T(y)$ downward, and the weight $\rho g\, dy$ downward. Since the rope is at rest, we have $T(y + dy) = T(y) + \rho g\, dy$. Expanding this to first order in dy gives $T'(y) = \rho g$. The tension in the bottom of the rope is zero, so integrating from $y = 0$ up to a position y gives

$$T(y) = \rho g y. \tag{2.16}$$

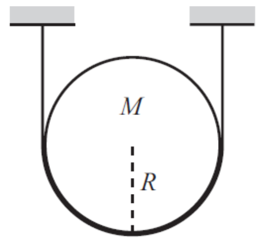
As a double-check, at the top end we have $T(L) = \rho g L$, which is the weight of the entire rope, as it should be.

Alternatively, you can simply write down the answer, $T(y) = \rho g y$, by noting that the tension at a given point in the rope is what supports the weight of all the rope below it.

2.6. **Supporting a disk** **

(a) A disk of mass M and radius R is held up by a massless string, as shown in Fig. 2.12. The surface of the disk is frictionless. What is the tension in the string? What is the normal force per unit length that the string applies to the disk?

(b) Let there now be friction between the disk and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point?



(a) The gravitational force downward on the disk is Mg, and the force upward is $2T$. These forces must balance, so

$$T = \frac{Mg}{2}. \tag{2.22}$$

We can find the normal force per unit length that the string applies to the disk in two ways.

First method: Let $N\,d\theta$ be the normal force on an arc of the disk that subtends an angle $d\theta$. Such an arc has length $R\,d\theta$, so N/R is the desired normal force per unit arclength. The tension in the string is the same throughout it, because the string is massless. So all points are equivalent, and hence N is constant, independent of θ. The upward component of the normal force is $N\,d\theta\cos\theta$, where θ is measured from the vertical (that is, $-\pi/2 \le \theta \le \pi/2$ here). Since the total upward force is Mg, we must have

$$\int_{-\pi/2}^{\pi/2} N\cos\theta\,d\theta = Mg. \tag{2.23}$$

The integral equals $2N$, so we have $N = Mg/2$. The normal force per unit length, N/R, therefore equals $Mg/2R$.

Second method: Consider the normal force, $N\,d\theta$, on a small arc of the disk that subtends an angle $d\theta$. The tension forces on each end of the corresponding small piece of string almost cancel, but they don't exactly, because they point in slightly different directions. Their nonzero sum is what produces the normal force on the disk. From Fig. 2.43, we see that the two forces have a sum of $2T\sin(d\theta/2)$, directed "inward". Since $d\theta$ is small, we can use $\sin x \approx x$ to approximate this as $T\,d\theta$. Therefore, $N\,d\theta = T\,d\theta$, and so $N = T$. The normal force per unit arclength, N/R, therefore equals T/R. Using $T = Mg/2$ from Eq. (2.22), we arrive at $N/R = Mg/2R$.

(b) Let $T(\theta)$ be the tension, as a function of θ, for $-\pi/2 \le \theta \le \pi/2$. T now depends on θ, because there is a tangential friction force. Most of the work for this problem was already done in the "Rope wrapped around a pole" example in Section 2.1. We'll simply invoke Eq. (2.7), which in the present language says[9]

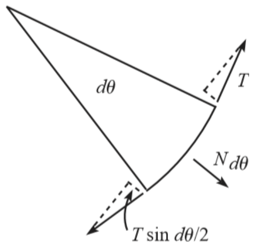


Fig. 2.3

Solution: Consider a small piece of the rope that subtends an angle $d\theta$. Let the tension in this piece be T (which varies slightly over the small length). As shown in Fig. 2.3, the pole exerts a small outward normal force, $N_{d\theta}$, on the piece. This normal force exists to balance the "inward" components of the tensions at the ends. These inward components have magnitude $T\sin(d\theta/2)$.[1] Therefore, $N_{d\theta} = 2T\sin(d\theta/2)$. The small-angle approximation, $\sin x \approx x$, allows us to write this as $N_{d\theta} = T\,d\theta$.

The friction force on the little piece of rope satisfies $F_{d\theta} \le \mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the piece. In other words, the tension, as a function of θ, satisfies

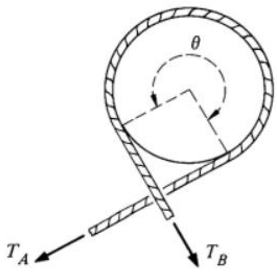
$$
\begin{aligned}
T(\theta + d\theta) &\le T(\theta) + \mu T\,d\theta \\
\Longrightarrow \quad dT &\le \mu T\,d\theta \\
\Longrightarrow \quad \int \frac{dT}{T} &\le \int \mu\,d\theta \\
\Longrightarrow \quad \ln T &\le \mu\theta + C \\
\Longrightarrow \quad T &\le T_0 e^{\mu\theta},
\end{aligned}
\tag{2.7}
$$

Letting $\theta = \pi/2$, and using $T(\pi/2) = Mg/2$, gives $Mg/2 \le T(0)e^{\mu\pi/2}$. We therefore see that the tension at the bottom point must satisfy

$$T(0) \ge \frac{Mg}{2} e^{-\mu\pi/2}. \tag{2.25}$$

Remark: This minimum value of $T(0)$ goes to $Mg/2$ as $\mu \to 0$, as it should. And it goes to zero as $\mu \to \infty$, as it should (imagine a very rough surface, so that the friction force from the rope near $\theta = \pi/2$ accounts for essentially all the weight). But interestingly, the tension at the bottom doesn't exactly equal zero, no matter now large μ is. Basically, the smaller T is, the smaller N is. But the smaller N is, the smaller the change in T is (because N determines the friction force). So T doesn't decrease much when it's small, and this results in it never being able to reach zero. ♣

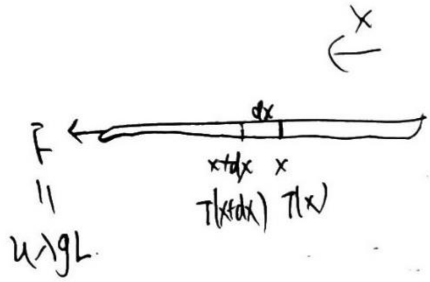
2.24 A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing shows about three-fourths turn). The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B. Can you show that $T_B = T_A e^{-\mu\theta}$, where μ is the coefficient of friction and θ is the total angle subtended by the rope on the drum?



与上题类似 不过摩擦力的方向延θ增大的方向

$$T(\theta + d\theta) = T(\theta) - \mu T d\theta$$
$$dT = -\mu T d\theta$$
$$\ln T = -\mu\theta + C$$
$$T_B = T_A e^{-\mu\theta}$$

2－12. 拖着一根重绳在粗糙的水平面上匀速前进，它各点的张力一样吗？

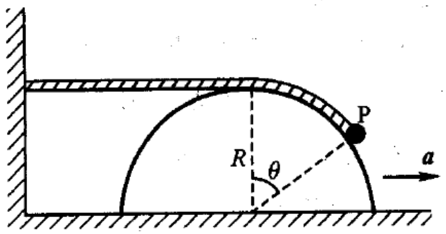


设质量密度λ 摩擦系数u
总长L

$$T(x+dx) = T(x) + ug\lambda dx$$
$$dT = ug\lambda dx$$
$$T = T_0 + ug\lambda x$$
$$\cancel{x=L} \quad \cancel{F} \quad x=L \quad T=F=u\lambda gL$$
$$\therefore T_0 = 0$$
$$T = ug\lambda x$$

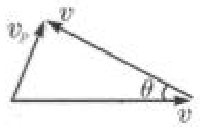
各点张力不一样

1-31 一半径为 R 的半圆柱面在水平面上向右作加速度为 a 的匀加速运动．在柱面上有一系在绳子一端的小球 P，绳子的另一端水平地连在墙上，如图所示．当小球相对半柱面的角位置为 θ 时，半柱面的速度为 v，求此时小球的速度与加速度．



假设半圆柱体不动,则小球P在圆柱体上的运动相当于绳子的水平段向左以加速度为a在匀加速地拉动绳子,因此,当小球相对于半圆柱面的角位置为θ时,小球P相对圆柱面的速度为υ,切向加速度为$a_\tau = a$.

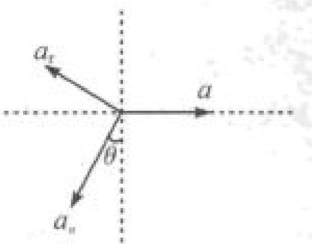
小球P的速度υ_p由半圆柱体的速度(牵连速度)与P相对于圆柱体的速度合成,其矢量关系如图所示。



由此可得$\upsilon_p = 2\upsilon\sin\dfrac{\theta}{2}$。

小球P相对于半圆柱体除了切向加速度外,还存在相对圆柱体做圆周运动的向心加速度$a_n = \dfrac{\upsilon^2}{R}$

小球P的加速度为a_τ、a_n、a三个加速度的合成,其矢量关系如图所示。

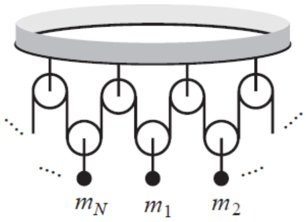


所以,小球P的加速度为

$$a_P = \sqrt{(a\cos\theta - a_\tau)^2 + (a\sin\theta - a_n)^2} = \sqrt{a^2(\cos\theta - 1)^2 + \left(a\sin\theta - \frac{\upsilon^2}{R}\right)^2}\text{。}$$

3.5. **Ring of pulleys** **

Consider the system of pulleys shown in Fig. 3.15. The string (which is a loop with no ends) hangs over N fixed pulleys that circle around the underside of a ring. N masses, $m_1, m_2, \ldots, m_N$, are attached to N pulleys that hang on the string. What are the accelerations of all the masses?



Let T be the tension in the string. Then $F = ma$ for m_i gives

$$2T - m_i g = m_i a_i, \tag{3.72}$$

with upward taken to be positive. The a_i's are related by the fact that the string has fixed length, which implies that the sum of the displacements of all the masses is zero. In other words,

$$a_1 + a_2 + \cdots + a_N = 0. \tag{3.73}$$

If we divide Eq. (3.72) by m_i, and then add the N such equations together and use Eq. (3.73), we find that T is given by

$$2T\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right) - Ng = 0. \tag{3.74}$$

Therefore,

$$T = \frac{NMg}{2}, \quad \text{where} \quad \frac{1}{M} \equiv \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N} \tag{3.75}$$

is the so-called *reduced mass* of the system. Substituting this value for T into (3.72) gives

$$a_i = g\left(\frac{NM}{m_i} - 1\right). \tag{3.76}$$

REMARK: A few special cases are: If all the masses are equal, then all the $a_i = 0$. If $m_k = 0$ (and all the others are not zero), then $a_k = (N-1)g$, and all the other $a_i = -g$. If $N-1$ of the masses are equal and much smaller than the remaining one, m_k, then $m_k \approx -g$, and all the other $a_i \approx g/(N-1)$. ♣

3.12. **Throwing a beach ball** ***

A beach ball is thrown upward with initial speed v_0. Assume that the drag force from the air is $F_\text{d} = -m\alpha v$. What is the speed of the ball, v_f, right before it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

On both the way up and the way down, the total force on the ball is

$$F = -mg - m\alpha v. \tag{3.96}$$

On the way up, v is positive, so the drag force points downward, as it should. And on the way down, v is negative, so the drag force points upward. Our strategy for finding v_f will be to produce two different expressions for the maximum height h, and then equate them. We'll find these two expressions by considering the upward and then the downward motion of the ball. In doing so, we will need to write the acceleration of the ball as $a = v\,dv/dy$. For the upward motion, $F = ma$ gives

$$-mg - m\alpha v = mv\frac{dv}{dy} \quad \Longrightarrow \quad \int_0^h dy = -\int_{v_0}^0 \frac{v\,dv}{g+\alpha v}\,. \tag{3.97}$$

where we have taken advantage of the fact that the speed of the ball at the top is zero. Writing $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$, the integral yields

$$h = \frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{3.98}$$

Now consider the downward motion. Let v_f be the final speed, which is a positive quantity. The final velocity is then the negative quantity, $-v_\text{f}$. Using $F = ma$, we obtain

$$\int_h^0 dy = -\int_0^{-v_\text{f}} \frac{v\,dv}{g+\alpha v}\,. \tag{3.99}$$

Performing the integration (or just replacing the v_0 in Eq. (3.98) with $-v_\text{f}$) gives

$$h = -\frac{v_\text{f}}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_\text{f}}{g}\right). \tag{3.100}$$

Equating the expressions for h in Eqs. (3.98) and (3.100) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_\text{f} = \frac{g}{\alpha}\ln\left(\frac{g+\alpha v_0}{g-\alpha v_\text{f}}\right). \tag{3.101}$$

Remarks: In the limit of small α (more precisely, in the limit $\alpha v_0/g \ll 1$), we can use $\ln(1+x) = x - x^2/2 + \cdots$ to obtain approximate values for h in Eqs. (3.98) and (3.100). The results are, as expected,

$$h \approx \frac{v_0^2}{2g}, \quad \text{and} \quad h \approx \frac{v_\text{f}^2}{2g}\,. \tag{3.102}$$

We can also make approximations for large α (or large $\alpha v_0/g$). In this limit, the log term in Eq. (3.98) is negligible, so we obtain $h \approx v_0/\alpha$. And Eq. (3.100) gives $v_f \approx g/\alpha$, because the argument of the log must be very small in order to give a very large negative number, which is needed to produce a positive h on the left-hand side. There is no way to relate v_f and h in this limit, because the ball quickly reaches the terminal velocity of $-g/\alpha$ (which is the velocity that makes the net force equal to zero), independent of h. ♣

Let's now find the times it takes for the ball to go up and to go down. We'll present two methods for doing this.

FIRST METHOD: Let T_1 be the time for the upward path. If we write the acceleration of the ball as $a = dv/dt$, then $F = ma$ gives $-mg - m\alpha v = m\,dv/dt$. Separating variables and integrating yields

$$\int_0^{T_1} dt = -\int_{v_0}^{0} \frac{dv}{g + \alpha v} \quad \Longrightarrow \quad T_1 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{3.103}$$

In a similar manner, we find that the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{3.104}$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right) = \frac{v_0 + v_f}{g}, \tag{3.105}$$

where we have used Eq. (3.101). This result is shorter than the time in vacuum (namely $2v_0/g$) because $v_f < v_0$.

SECOND METHOD: The very simple form of Eq. (3.105) suggests that there is a cleaner way of deriving it. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives Eq. (3.105). Note that this procedure works only because the drag force is proportional to v.

REMARK: The fact that the time here is shorter than the time in vacuum isn't obvious. On one hand, the ball doesn't travel as high in air as it would in vacuum, so you might think that $T_1 + T_2 < 2v_0/g$. But on the other hand, the ball moves slower in air on the way down, so you might think that $T_1 + T_2 > 2v_0/g$. It isn't obvious which effect wins, without doing a calculation.[28] For any α, you can use Eq. (3.103) to show that $T_1 < v_0/g$. But T_2 is harder to get a handle on, because it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f$. Using the results from the previous remark, this becomes $T_2 \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this equals the downward (and upward) time for a ball thrown in vacuum. ♣

3.13. **Balancing a pencil** ***

Consider a pencil that stands upright on its tip and then falls over. Let's idealize the pencil as a mass m sitting at the end of a massless rod of length ℓ.[20]

(a) Assume that the pencil makes an initial (small) angle θ_0 with the vertical, and that its initial angular speed is ω_0. The angle will eventually become large, but while it is small (so that $\sin\theta \approx \theta$), what is θ as a function of time?

(b) You might think that it should be possible (theoretically, at least) to make the pencil balance for an arbitrarily long time, by making the initial θ_0 and ω_0 sufficiently small. However, it turns out that due to Heisenberg's uncertainty principle (which puts a constraint on how well we can know the position and momentum of a particle), it is impossible to balance the pencil for more than a certain amount of time. The point is that you can't be sure that the pencil is initially both at the top *and* at rest. The goal of this problem is to be quantitative about this. The time limit is sure to surprise you.

Without getting into quantum mechanics, let's just say that the uncertainty principle says (up to factors of order 1) that $\Delta x \Delta p \geq \hbar$, where $\hbar = 1.05 \cdot 10^{-34}\,\text{J s}$ is Planck's constant. The implications of this are somewhat vague, but we'll just take it to mean that the initial conditions satisfy $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$. With this constraint, your task is to find the maximum time it can take your $\theta(t)$ solution in part (a) to become of order 1. In other words, determine (roughly) the maximum time the pencil can balance. Assume $m = 0.01$ kg, and $\ell = 0.1$ m.

(a) The component of gravity in the tangential direction is $mg\sin\theta \approx mg\theta$. Therefore, the tangential $F = ma$ equation is $mg\theta = m\ell\ddot{\theta}$, which may be written as $\ddot{\theta} = (g/\ell)\theta$. The general solution to this equation is[29]

$$\theta(t) = Ae^{t/\tau} + Be^{-t/\tau}, \quad \text{where} \quad \tau \equiv \sqrt{\ell/g}. \tag{3.106}$$

The constants A and B are found from the initial conditions,

$$\begin{aligned} \theta(0) = \theta_0 \quad &\Longrightarrow \quad A + B = \theta_0, \\ \dot{\theta}(0) = \omega_0 \quad &\Longrightarrow \quad (A - B)/\tau = \omega_0. \end{aligned} \tag{3.107}$$

Solving for A and B, and then plugging the results into Eq. (3.106) gives

$$\theta(t) = \frac{1}{2}\left(\theta_0 + \omega_0\tau\right)e^{t/\tau} + \frac{1}{2}\left(\theta_0 - \omega_0\tau\right)e^{-t/\tau}. \tag{3.108}$$

(b) The constants A and B will turn out to be small (they will each be of order $\sqrt{\hbar}$). Therefore, by the time the positive exponential has increased enough to make θ of order 1, the negative exponential will have become negligible. We will therefore ignore this term from here on. In other words,

$$\theta(t) \approx \frac{1}{2}(\theta_0 + \omega_0 \tau)\, e^{t/\tau}. \tag{3.109}$$

The goal is to keep θ small for as long as possible. Hence, we want to minimize the coefficient of the exponential, subject to the uncertainty-principle constraint, $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$. This constraint gives $\omega_0 \geq \hbar/(m\ell^2\theta_0)$. Therefore,

$$\theta(t) \geq \frac{1}{2}\left(\theta_0 + \frac{\hbar\tau}{m\ell^2\theta_0}\right) e^{t/\tau}. \tag{3.110}$$

Taking the derivative with respect to θ_0 to minimize the coefficient, we find that the minimum value occurs at $\theta_0 = \sqrt{\hbar\tau/m\ell^2}$. Substituting this back into Eq. (3.110) gives

$$\theta(t) \geq \sqrt{\frac{\hbar\tau}{m\ell^2}}\; e^{t/\tau}. \tag{3.111}$$

Setting $\theta \approx 1$, and then solving for t gives (using $\tau \equiv \sqrt{\ell/g}$)

$$t \leq \frac{1}{4}\sqrt{\frac{\ell}{g}} \ln\left(\frac{m^2\ell^3 g}{\hbar^2}\right). \tag{3.112}$$

With the given values, $m = 0.01$ kg and $\ell = 0.1$ m, along with $g = 10\,\mathrm{m/s^2}$ and $\hbar = 1.06 \cdot 10^{-34}$ J s, we obtain

$$t \leq \frac{1}{4}(0.1\,\mathrm{s}) \ln(9 \cdot 10^{61}) \approx 3.5\,\mathrm{s}. \tag{3.113}$$

No matter how clever you are, and no matter how much money you spend on the newest cutting-edge pencil balancing equipment, you can never get a pencil to balance for more than about four seconds.

REMARKS:

1. The smallness of this answer is quite amazing. It is remarkable that a quantum effect on a macroscopic object can produce an everyday value for a time scale. Basically, the point is that the fast exponential growth of θ (which gives rise to the log in the final result for t) wins out over the smallness of $\hbar$, and produces a result for t of order 1. When push comes to shove, exponential effects always win.
2. The above value for t depends strongly on ℓ and g, through the $\sqrt{\ell/g}$ term. But the dependence on m, ℓ, and g in the log term is very weak, because $\hbar$ is so small. If m is increased by a factor of 1000, for example, the result for t increases by only about 10%. This implies that any factors of order 1 that we neglected throughout this problem are completely irrelevant. They will appear in the argument of the log term, and will therefore have negligible effect.
3. Note that dimensional analysis, which is generally a very powerful tool, won't get you too far in this problem. The quantity $\sqrt{\ell/g}$ has dimensions of time, and the quantity $\eta \equiv m^2\ell^3 g/\hbar^2$ is dimensionless (it is the only such quantity), so the balancing time must take the form,

$$t \approx \sqrt{\frac{\ell}{g}} f(\eta), \tag{3.114}$$

where f is some function. If the leading term in f were a power (even, for example, a square root), then t would essentially be infinite ($t \approx 10^{30}$ s $\approx 10^{22}$ years for the square root). But f in fact turns out to be a log (which you can't know without solving the problem), which completely cancels out the smallness of $\hbar$, reducing an essentially infinite time down to a few seconds. ♣