**I.**

以$\hat{p_x}$为例，

$$\int_{-\infty}^{\infty}\psi^*(\hat{p_x}\phi)dx = -i\hbar\int_{-\infty}^{\infty}\psi^*\frac{\partial\phi}{\partial x}dx$$
$$= -i\hbar\int_{-\infty}^{\infty}d(\psi^*\phi) + i\hbar\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\phi dx$$

不妨令$\psi(x) = C_1e^{ik_1x}$，$\phi(x) = C_2e^{ik_2x}$，

$$\int_{-\infty}^{\infty}d(\psi^*\phi) = \int_{-\infty}^{\infty}iC_1^*C_2(k_2-k_1)e^{i(k_2-k_1)x}dx$$
$$= 2i\pi C_1^*C_2(k_2-k_1)\delta(k_2-k_1)$$

对$\forall k_1, k_2$，$(k_2-k_1)\delta(k_2-k_1) = 0$，所以对$\forall\psi(x), \phi(x)$，

$$\int_{-\infty}^{\infty}\psi^*(\hat{p_x}\phi)dx = i\hbar\int_{-\infty}^{\infty}\frac{\partial\psi^*}{\partial x}\phi dx = \int_{-\infty}^{\infty}(\hat{p_x}\psi)^*\phi dx$$

所以$\hat{p_x}$是厄密算符，同理，$\hat{p_y}$和$\hat{p_z}$也是厄密算符

**II.**

转置算符定义为：

$$(\psi, \hat{A}^T\phi) = (\phi^*, \hat{A}\psi^*)$$

则

$$(\psi, \hat{A}^{T*}\phi) = (\phi^*, \hat{A}^*\psi^*)$$

厄密共轭算符定义为：

$$(\psi, \hat{A}^+\phi) = (\hat{A}\psi, \phi)$$

因此

$$(\psi, \hat{A}^+\phi) = (\phi, \hat{A}\psi)^* = (\phi^*, \hat{A}^*\psi^*)$$

即

$$(\psi, \hat{A}^{T*}\phi) = (\psi, \hat{A}^+\phi)$$
$$\hat{A}^+ = \hat{A}^{T*}$$

**III.**

$$\vec{r} = r\cdot\vec{e_r} = rsin\theta cos\varphi\vec{e_x} + rsin\theta sin\varphi\vec{e_y} + rcos\theta\vec{e_z}$$

$$\begin{cases} \vec{e_r} = sin\theta cos\varphi\vec{e_x} + sin\theta sin\varphi\vec{e_y} + cos\theta\vec{e_z} \\ \vec{e_\theta} = cos\theta cos\varphi\vec{e_x} + cos\theta sin\varphi\vec{e_y} - sin\theta\vec{e_z} \\ \vec{e_\varphi} = -sin\varphi\vec{e_x} + cos\varphi\vec{e_y} \end{cases}$$

代入$\vec{L}=-i\hbar(\vec{e_\varphi}\frac{\partial}{\partial\theta}-\vec{e_\theta}\frac{1}{sin\theta}\frac{\partial}{\partial\varphi})$，有：

$$\vec{L}=-i\hbar[(-sin\varphi\frac{\partial}{\partial\theta}-cot\theta cos\varphi\frac{\partial}{\partial\varphi})\vec{e_x}+(cos\varphi\frac{\partial}{\partial\theta}-cot\theta sin\varphi\frac{\partial}{\partial\varphi})\vec{e_y}-(\frac{\partial}{\partial\varphi})\vec{e_z}]$$

所以

$$\hat{L_z}=-i\hbar\frac{\partial}{\partial\varphi}$$

**IV.**

因为$A^+=A,B^+=B$，所以

$$\frac{1}{2}(AB+BA)^+=\frac{1}{2}(B^+A^++A^+B^+)=\frac{1}{2}(BA+AB)=\frac{1}{2}(AB+BA)$$

$$[\frac{1}{2i}(AB-BA)]^+=-\frac{1}{2i}(B^+A^+-A^+B^+)=-\frac{1}{2i}(BA-AB)=\frac{1}{2i}(AB-BA)$$

即$\frac{1}{2}(AB+BA)$和$-\frac{1}{2i}(AB-BA)$均为厄密算符

$$F=\frac{F}{2}+\frac{F}{2}=\frac{1}{2}(F+F^+)+\frac{1}{2}(F-F^+)=F_++iF_-$$

而$F_+$和$F_-$显然均为厄密算符

**V.**

对于$\vec{l}=\vec{r}\times\vec{p}$，有

$$l_x=yp_z-zp_y$$

$$l_x^+=(yp_z-zp_y)^+=(p_z^+y^+-p_y^+z^+)=(p_zy-p_yz)=yp_z-zp_y=l_x$$

同理，

$$l_y^+=l_y,l_z^+=l_z$$

所以，$\vec{l}=\vec{r}\times\vec{p}=\vec{l}^+$是厄密算符

对于$\vec{r}\cdot\vec{p}$，有

$$(\vec{r}\cdot\vec{p})^+=(xp_x+yp_y+zp_z)^+=p_x^+x^++p_y^+y^++p_z^+z^+$$

$$=p_xx+p_yy+p_zz\neq xp_x+yp_y+zp_z$$

所以 $\vec{r}\cdot\vec{p}$ 不是厄密算符，而

$$(\vec{r}\cdot\vec{p})^+=\vec{p}\cdot\vec{r}=p_xx+p_yy+p_zz=(xp_x-i\hbar)+(yp_y-i\hbar)+(zp_z-i\hbar)=\vec{r}\cdot\vec{p}-3i\hbar$$

可构造相应的厄密算符为$\frac{1}{2}[\vec{r}\cdot\vec{p}+\vec{p}\cdot\vec{r}]=\vec{r}\cdot\vec{p}-3i\hbar/2$

对于$\vec{p}\times\vec{l}$，

$$(\vec{p}\times\vec{l})_x^+=(p_yl_z-p_zl_y)^+=l_zp_y-l_yp_z=(xp_y-yp_x)p_y-(zp_x-xp_z)p_z=xp_y^2+xp_z^2-yp_xp_y-zp_xp_z$$

$$(\vec{p}\times\vec{l})_x = p_y l_z - p_z l_y = p_y(xp_y - yp_x) - p_z(zp_x - xp_z) = xp_y^2 + xp_z^2 - p_y y p_x + p_z z p_x$$

利用$[x,p]=i\hbar$，得到

$$(\vec{p}\times\vec{l})_x = xp_y^2 + xp_z^2 - (yp_y - i\hbar)p_x + (zp_z - i\hbar)p_x$$

$$= xp_y^2 + xp_z^2 - yp_x p_y - zp_x p_z + 2i\hbar p_x = (\vec{p}\times\vec{l})_x^+ + 2i\hbar p_x$$

同理，

$$(\vec{p}\times\vec{l})_y = (\vec{p}\times\vec{l})_y^+ + 2i\hbar p_y,$$

$$(\vec{p}\times\vec{l})_z = (\vec{p}\times\vec{l})_z^+ + 2i\hbar p_z$$

所以

$$(\vec{p}\times\vec{l})^+ = \vec{p}\times\vec{l} - 2i\hbar\vec{p}$$

故 $\vec{p}\times\vec{l}$ 不是厄密算符，但可构造相应的厄密算符为

$$\frac{1}{2}[(\vec{p}\times\vec{l}) + (\vec{p}\times\vec{l})^+] = \vec{p}\times\vec{l} - i\hbar\vec{p}$$

对于$\vec{r}\times\vec{l}$，类似的可证明：

$$(\vec{r}\times\vec{l})^+ = \vec{r}\times\vec{l} - 2i\hbar\vec{r}$$

即$\vec{r}\times\vec{l}$也不是厄密算符，但可构造相应的厄密算符为

$$\frac{1}{2}[(\vec{r}\times\vec{l}) + (\vec{r}\times\vec{l})^+] = \vec{r}\times\vec{l} - i\hbar\vec{r}$$

**VI.**

$$\hat{P}_r = \frac{1}{2}(\frac{\vec{r}}{r}\cdot\vec{p} + \vec{p}\cdot\frac{\vec{r}}{r})$$

$$\hat{P_r}^+ = \frac{1}{2}(\frac{\vec{r}}{r}\cdot\vec{p} + \vec{p}\cdot\frac{\vec{r}}{r})^+ = \frac{1}{2}[\hat{p}^+\hat{r}^+(\frac{1}{r})^+ + (\frac{1}{r})^+\hat{r}^+\hat{p}^+]$$

$$= \frac{1}{2}(\vec{p}\cdot\frac{\vec{r}}{r} + \frac{\vec{r}}{r}\cdot\vec{p}) = \hat{P}_r$$

所以$\hat{P}_r$是厄密算符

$$\hat{P}_r = \frac{1}{2}(\frac{\vec{r}}{r}\cdot\vec{p} + \vec{p}\cdot\frac{\vec{r}}{r}) = -\frac{i\hbar}{2}(\vec{\nabla}\cdot\frac{\vec{r}}{r} + \frac{\vec{r}}{r}\cdot\vec{\nabla})$$

对一任意波函数$\psi$，

$$\hat{P}_r\psi = -\frac{i\hbar}{2}[\vec{\nabla}\cdot(\frac{\vec{r}}{r}\psi) + \frac{\vec{r}}{r}\cdot\vec{\nabla}\psi]$$

$$\vec{\nabla}\cdot(\frac{\vec{r}}{r}\psi) = \frac{\partial}{\partial x}(x\frac{\psi}{r}) + \frac{\partial}{\partial y}(y\frac{\psi}{r}) + \frac{\partial}{\partial z}(z\frac{\psi}{r}) \tag{1}$$

又

$$\frac{\partial}{\partial x}(x\frac{\psi}{r}) = \frac{x}{r}\frac{\partial\psi}{\partial x} + \psi\frac{\partial}{\partial x}(\frac{x}{r}) = \frac{x}{r}\frac{\partial\psi}{\partial x} + (\frac{1}{r} - \frac{x^2}{r^3})\psi$$

代入(1)式，

$$\vec{\nabla}\cdot(\frac{\vec{r}}{r}\psi) = \frac{x}{r}\frac{\partial\psi}{\partial x} + \frac{y}{r}\frac{\partial\psi}{\partial y} + \frac{z}{r}\frac{\partial\psi}{\partial z} + [\frac{3}{r} - (\frac{x^2}{r^3} + \frac{y^2}{r^3} + \frac{z^2}{r^3})]\psi$$

$$= \frac{\vec{r}}{r} \cdot \vec{\nabla}\psi + (\frac{3}{r} - \frac{r^2}{r^3})\psi = \frac{\vec{r}}{r} \cdot \vec{\nabla}\psi + \frac{2}{r}\psi$$

所以，

$$\vec{\nabla} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{r} \cdot \vec{\nabla} + \frac{2}{r}$$

在球坐标中：

$$\vec{\nabla} = \vec{e_r}\frac{\partial}{\partial r} + \vec{e_\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e_\varphi}\frac{1}{rsin\theta}\frac{\partial}{\partial \varphi}$$

故

$$\vec{r} \cdot \vec{\nabla} = r\frac{\partial}{\partial r}$$

所以，

$$\hat{P}_r = -\frac{i\hbar}{2}(\vec{\nabla}\frac{\vec{r}}{r} + \frac{\vec{r}}{r} \cdot \vec{\nabla}) = -\frac{i\hbar}{2}(2\frac{\vec{r}}{r} \cdot \vec{\nabla} + \frac{2}{r})$$

$$= -i\hbar(\frac{\partial}{\partial r} + \frac{1}{r})$$