微积分 A (2)

姚家燕

第 11 讲

在听课过程中,

严禁使用与教学无关的电子产品!

重要通知

- 希望大家认真温习第 1 章!
- 希望大家能重温上学期所学的广义积分!

第 11 讲

例 16. 设 $f\in\mathscr{C}^{(2)}(\mathbb{R}^2)$ 且 $\forall(x,y)\in\mathbb{R}^2$, $f(x,y)>0$, $f''_{xy}(x,y)f(x,y)=f'_x(x,y)f'_y(x,y)$. 求证:

(1) $\forall(x,y)\in\mathbb{R}^2$, 均有 $\frac{\partial}{\partial y}(\frac{f'_x}{f})(x,y)=0$.

(2) $\exists\varphi,\psi\in\mathscr{C}^{(2)}(\mathbb{R})$ 使得 $\forall(x,y)\in\mathbb{R}^2$, 均有

$$f(x,y)=\varphi(x)\psi(y).$$

证明: (1) $\forall(x,y)\in\mathbb{R}^2$, 由题设立刻可得

$$\frac{\partial}{\partial y}\Big(\frac{f'_x}{f}\Big)(x,y)=\frac{f''_{yx}(x,y)f(x,y)-f'_x(x,y)f'_y(x,y)}{(f(x,y))^2}=0.$$

(2) 由 (1) 以及单变量的 Lagrange 中值定理可知, $\forall (x,y)\in\mathbb{R}^2$, 我们有 $\frac{f'_x(x,y)}{f(x,y)}=\frac{f'_x(x,0)}{f(x,0)}$, 也即

$$\frac{\partial(\log f)}{\partial x}(x,y)=\frac{f'_x(x,0)}{f(x,0)},$$

从而 $\exists \varphi_1,\psi_1\in\mathscr{C}(\mathbb{R})$ 使得 $\forall (x,y)\in\mathbb{R}^2$, 我们有

$$\log f(x,y)=\varphi_1(x)+\psi_1(y).$$

$\forall (x,y)\in\mathbb{R}^2$, 定义 $\varphi(x)=e^{\varphi_1(x)}$, $\psi(y)=e^{\psi_1(y)}$, 则我们有 $f(x,y)=\varphi(x)\psi(y)$. 但 $f\in\mathscr{C}^{(2)}(\mathbb{R}^2)$, 于是我们有 $\varphi,\psi\in\mathscr{C}^{(2)}(\mathbb{R})$.

例 17. 求 $z=\frac{e^x}{1-y}$ 在原点的二阶 Taylor 多项式.

解: 当 $(x,y)\to(0,0)$ 时, 我们有

$$\begin{aligned} z &= \left(1+x+\frac{1}{2}x^2+x^2o(1)\right)\left(1+y+y^2+y^2o(1)\right) \\ &= 1+x+y+\frac{1}{2}x^2+xy+y^2+x^2o(1)+y^2o(1) \\ &= 1+x+y+\frac{1}{2}x^2+xy+y^2+o(x^2+y^2). \end{aligned}$$

故所求多项式为 $1+x+y+\frac{1}{2}x^2+xy+y^2$.

例 18. 如果 f 可微, 求证: 曲面 $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) = 0$ 上任意点处的切平面过一定点, 并求该定点.

证明: 定义 $F(x, y, z) = f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)$, 则

$$\operatorname{grad} F = \begin{pmatrix} \partial_1 f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) \frac{1}{z-c} \\ \partial_2 f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) \frac{1}{z-c} \\ \partial_1 f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) \frac{a-x}{(z-c)^2} + \partial_2 f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right) \frac{b-y}{(z-c)^2} \end{pmatrix}.$$

曲面在任意点 $P_0(x_0, y_0, z_0)$ 处的切平面方程为

$$0 = \partial_1 f\left(\frac{x_0-a}{z_0-c}, \frac{y_0-b}{z_0-c}\right) \frac{x-x_0}{z_0-c} + \partial_2 f\left(\frac{x_0-a}{z_0-c}, \frac{y_0-b}{z_0-c}\right) \frac{y-y_0}{z_0-c}$$

$$+\left(\partial_1 f\left(\frac{x_0-a}{z_0-c}, \frac{y_0-b}{z_0-c}\right) \frac{a-x_0}{(z_0-c)^2} + \partial_2 f\left(\frac{x_0-a}{z_0-c}, \frac{y_0-b}{z_0-c}\right) \frac{b-y_0}{(z_0-c)^2}\right)(z-z_0).$$

于是我们有

$$\begin{aligned}
&\partial_1 f\Big(\frac{x_0-a}{z_0-c},\frac{y_0-b}{z_0-c}\Big)(x-x_0)(z_0-c)\\
&+\partial_2 f\Big(\frac{x_0-a}{z_0-c},\frac{y_0-b}{z_0-c}\Big)(y-y_0)(z_0-c)\\
&-\partial_1 f\Big(\frac{x_0-a}{z_0-c},\frac{y_0-b}{z_0-c}\Big)(a-x_0)(z_0-z)\\
&-\partial_2 f\Big(\frac{x_0-a}{z_0-c},\frac{y_0-b}{z_0-c}\Big)(b-y_0)(z_0-z)=0.
\end{aligned}$$

由于点 (a,b,c) 恰好满足上述方程, 故题设曲面任意点处的切平面均过定点 (a,b,c).

例 19. 设曲面 S 由 $ax+by+cz=G(x^2+y^2+z^2)$ 确定, 其中 G 可导并且 a,b,c 不全为零. 求证: S 上任意点处的法线与某定直线相交或平行.

证明: 令 $F(x,y,z)=G(x^2+y^2+z^2)-(ax+by+cz)$, 则

$$\mathrm{grad}F=\begin{pmatrix}2xG'(x^2+y^2+z^2)-a\\2yG'(x^2+y^2+z^2)-b\\2zG'(x^2+y^2+z^2)-c\end{pmatrix}.$$

设所求定直线的方程为 $\frac{x-x_1}{\alpha}=\frac{y-y_1}{\beta}=\frac{z-z_1}{\gamma}$.

上述直线与曲面 S 上任意点 $P_0(x_0, y_0, z_0)$ 处的法线相交或平行当且仅当下述三个向量

$$\mathrm{grad}F(P_0), \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix}$$

线性相关, 也即我们有

$$\begin{vmatrix} 2x_0 G'(x_0^2 + y_0^2 + z_0^2) - a & \alpha & x_1 - x_0 \\ 2y_0 G'(x_0^2 + y_0^2 + z_0^2) - b & \beta & y_1 - y_0 \\ 2z_0 G'(x_0^2 + y_0^2 + z_0^2) - c & \gamma & z_1 - z_0 \end{vmatrix} = 0.$$

根据行列式的性质, 该式等价于

$$2G'(x_0^2+y_0^2+z_0^2)\begin{vmatrix} x_0 & \alpha & x_1-x_0 \\ y_0 & \beta & y_1-y_0 \\ z_0 & \gamma & z_1-z_0 \end{vmatrix} = \begin{vmatrix} a & \alpha & x_1-x_0 \\ b & \beta & y_1-y_0 \\ c & \gamma & z_1-z_0 \end{vmatrix}.$$

而当 $\alpha=a$, $\beta=b$, $\gamma=c$, $x_1=y_1=z_1=0$ 时,

上式成立, 也即曲面任意点处的法线与定直线

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

平行或相交.

例 20. 设 $f:(0,+\infty)\to\mathbb{R}$ 为二阶连续可导函数且 $u(x,y,z)=f(\sqrt{x^2+y^2+z^2})$ 满足

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

求函数 f 的表达式.

解: 由题设可知 $\frac{\partial u}{\partial x}=\frac{xf'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}$, 则

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+\frac{x^2f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}\\ &\quad -\frac{x^2f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}}.\end{aligned}$$

由对称性可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}} + \frac{y^2 f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2} - \frac{y^2 f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}} + \frac{z^2 f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2} - \frac{z^2 f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

于是我们有

$$0=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial^2 z}=\frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+\frac{x^2f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}$$

$$-\frac{x^2f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+\frac{y^2f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}$$

$$-\frac{y^2f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+\frac{z^2f''(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}$$

$$-\frac{z^2f'(\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2f'(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+f''(\sqrt{x^2+y^2+z^2}).$$

从而 $\forall r>0$, $f''(r)+\frac{2f'(r)}{r}=0$, 故 $f'(r)=-\frac{C_1}{r^2}$,
进而 $f(r)=\frac{C_1}{r}+C_2$, 其中 C_1, C_2 为常数.

例 21. 设 $u = x^2 + y^2 + z^2$, 其中 $z = z(x,y)$ 是由方程 $z = x - ye^z$ 确定的隐函数, 求 $\mathrm{d}u$ 和 $\frac{\partial^2 u}{\partial x \partial y}$.

解: 由题设知 $\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$, $\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$.

注意到 $z = x - ye^z$, 两边分别对 x, y 求偏导数可得 $\frac{\partial z}{\partial x} = \frac{1}{1+ye^z}$, $\frac{\partial z}{\partial y} = -\frac{e^z}{1+ye^z}$. 于是

$$\mathrm{d}u = \left(2x + \frac{2z}{1+ye^z}\right)\mathrm{d}x + \left(2y - \frac{2ze^z}{1+ye^z}\right)\mathrm{d}y.$$

与此同时, 我们也有

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(2y + 2z\frac{\partial z}{\partial y}\right) = 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + 2z\frac{\partial^2 z}{\partial x \partial y} \\
&= -\frac{2e^z}{(1+ye^z)^2} + 2z\frac{\partial}{\partial x}\left(-\frac{e^z}{1+ye^z}\right) \\
&= -\frac{2e^z}{(1+ye^z)^2} - 2z \cdot \frac{e^z(1+ye^z)\frac{\partial z}{\partial x} - e^z \cdot ye^z\frac{\partial z}{\partial x}}{(1+ye^z)^2} \\
&= -\frac{2e^z}{(1+ye^z)^2} - \cdot\frac{2ze^z}{(1+ye^z)^3} \\
&= -\frac{2e^z(1+ye^z+z)}{(1+ye^z)^3} = -\frac{2(1+x)e^z}{(1+ye^z)^3}.
\end{aligned}
$$

例 22. 设隐函数 $z=z(x,y)$ 由方程 $x=u+v$, $y=u^2+v^2$, $z=u^3+v^3$ 确定. 求 $\frac{\partial^2 z}{\partial x^2}$.

解: 方法 1. 由题设可知

$$1=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x},\quad 0=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}.$$

由此我们可立刻导出

$$\frac{\partial u}{\partial x}=\frac{v}{v-u},\qquad \frac{\partial v}{\partial x}=\frac{u}{u-v},$$

于是 $\frac{\partial z}{\partial x}=3u^2\frac{\partial u}{\partial x}+3v^2\frac{\partial v}{\partial x}=-3uv$, 进而可得

$$\frac{\partial^2 z}{\partial x^2}=-3v\frac{\partial u}{\partial x}-3u\frac{\partial v}{\partial x}=-3(u+v)=-3x.$$

方法 2. 由题设可知 $2uv = x^2 - y$, 从而

$$
\begin{aligned}
z &= (u+v)(u^2 - uv + v^2) \\
&= x\left(y - \frac{1}{2}(x^2 - y)\right) \\
&= \frac{3}{2}xy - \frac{1}{2}x^3,
\end{aligned}
$$

由此我们立刻可得

$$
\frac{\partial z}{\partial x} = \frac{3}{2}y - \frac{3}{2}x^2, \ \frac{\partial^2 z}{\partial x^2} = -3x.
$$

例 23. 设 $x = f(y,z)$, $y = g(x,z)$, $z = h(x,y)$, 其中 f, g, h 为可微函数. 求证:

$$\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial z} \cdot \frac{\partial h}{\partial x} = -1.$$

证明: 由题设可知

$$x = f(g(x,z),z), \; z = h(x, g(x,z)).$$

由此立刻可得

$$1 = \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x}, \quad 0 = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial g}{\partial x}, \quad 1 = \frac{\partial h}{\partial y} \cdot \frac{\partial g}{\partial z}.$$

于是我们就有

$$\frac{\partial f}{\partial y}=\frac{1}{\frac{\partial g}{\partial x}},\ \frac{\partial h}{\partial x}=-\frac{\partial h}{\partial y}\cdot\frac{\partial g}{\partial x},\ \frac{\partial g}{\partial z}\cdot\frac{\partial h}{\partial y}=1.$$

进而立刻可得知

$$\begin{aligned}\frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial z}\cdot\frac{\partial h}{\partial x} &= \frac{1}{\frac{\partial g}{\partial x}}\cdot\frac{\partial g}{\partial z}\cdot\left(-\frac{\partial h}{\partial y}\frac{\partial g}{\partial x}\right)\\ &= -\frac{\partial g}{\partial z}\cdot\frac{\partial h}{\partial y}=-1.\end{aligned}$$

例 24. $\forall (x,y,z) \in \mathbb{R}^3$, 定义

$$f(x,y,z) = x^2 + 2y - xyz.$$

设 f 在点 $(1,1,0)$ 处的梯度方向为 $\vec{\ell}$, 求 $\frac{\partial f}{\partial \vec{\ell}}(1,1,0)$.

解: 由题设可知

$$\operatorname{grad} f(1,1,0) = \begin{pmatrix} 2x - yz \\ 2 - xz \\ -xy \end{pmatrix} \Bigg|_{(1,1,0)} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},$$

于是我们有 $\frac{\partial f}{\partial \vec{\ell}}(1,1,0) = \|\operatorname{grad} f(1,1,0)\| = 3.$

例 25. 设隐函数 $z=z(x,y)$ 由方程

$$z=f(x+y+z)$$

确定, 其中 f 为 $\mathscr{C}^{(2)}$ 类函数且 $f'\neq 1$, 求 $\frac{\partial^2 z}{\partial x^2}$.

解: 将方程 $z=f(x+y+z)$ 对 x 求偏导数, 则

$$\begin{aligned}\frac{\partial z}{\partial x} &= f'(x+y+z)\frac{\partial(x+y+z)}{\partial x}\\ &= f'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right),\end{aligned}$$

由此立刻可得 $\frac{\partial z}{\partial x}=\frac{f'(x+y+z)}{1-f'(x+y+z)}$.

进而我们就有

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{f'(x+y+z)}{1-f'(x+y+z)}\Big) \\
&= \frac{\partial}{\partial x}\Big(-1+\frac{1}{1-f'(x+y+z)}\Big) \\
&= -\frac{1}{(1-f'(x+y+z))^2}\cdot\frac{\partial(1-f'(x+y+z))}{\partial x} \\
&= \frac{f''(x+y+z)}{(1-f'(x+y+z))^2}\cdot\frac{\partial(x+y+z)}{\partial x} \\
&= \frac{f''(x+y+z)}{(1-f'(x+y+z))^2}\Big(1+\frac{\partial z}{\partial x}\Big) \\
&= \frac{f''(x+y+z)}{(1-f'(x+y+z))^3}.
\end{aligned}
$$

例 26. 设 p 为实数. $\forall (x,y) \in \mathbb{R}^2$, 令

$$f(x,y) = \begin{cases} (x^2+y^2)^p \sin \frac{1}{\sqrt{x^2+y^2}}, & \text{若 } (x,y) \neq (0,0), \\ 0, & \text{若 } (x,y) = (0,0). \end{cases}$$

请分析 p 取何值时, 函数 f 在原点处:

(1) 连续; (2) 可导; (3) 可微.

解: (1) 若 $p > 0$, 则 $\forall (x,y) \in \mathbb{R}^2$, 我们有

$$|f(x,y)| \leqslant (x^2+y^2)^p.$$

由夹逼原理可得 $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

因此函数 f 在原点处连续.

现假设 $p \leqslant 0$. $\forall n \geqslant 1$, 令 $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$,

则 $f(x_n, 0) = 0$, $f(y_n, 0) = y_n^{2p} \geqslant 1$. 故 $\{x_n\}$, $\{y_n\}$

收敛到 0, 但 $\{f(x_n, 0)\}$, $\{f(y_n, 0)\}$ 却不收敛到

同一个极限. 这表明 f 在原点间断.

综上所述可知 f 在原点连续当且仅当 $p > 0$.

(2) 若 $p > \frac{1}{2}$, 由夹逼原理可知

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}|x|^{2p-1}(\operatorname{sgn} x)\sin\frac{1}{|x|} = 0,$$

$$\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}|y|^{2p-1}(\operatorname{sgn} y)\sin\frac{1}{|y|} = 0,$$

故 f 在原点可导且 $f'_x(0,0) = f'_y(0,0) = 0$.

若 $p \leqslant \frac{1}{2}$, 由 $\lim\limits_{x\to 0}(\operatorname{sgn} x)\sin\frac{1}{|x|}$ 不存在可知 f 在原点不可导. 故 f 在原点可导当且仅当 $p > \frac{1}{2}$.

(3) 若 f 在原点可微, 则它在该点可导, 故 $p > \frac{1}{2}$.

现假设 $p > \frac{1}{2}$, 此时我们有

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f'_x(0,0)x - f'_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} (x^2+y^2)^{p-\frac{1}{2}} \sin\frac{1}{\sqrt{x^2+y^2}} = 0,$$

于是由可微的定义可知 f 在原点可微.

综上所述可知 f 在原点可微当且仅当 $p > \frac{1}{2}$.

例 27. 固定 $k>0$. $\forall(x,y)\in\mathbb{R}^2$, 定义

$$f(x,y)=\begin{cases}\frac{|xy|^k}{x^2+y^2}, & 若\ (x,y)\neq(0,0),\\ 0, & 若\ (x,y)=(0,0).\end{cases}$$

问 k 为何值时函数 f 在原点连续, 可导, 可微或连续可导?

解: 连续性. 当 $k>1$ 时, $\forall(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$,

$$\frac{|xy|^k}{x^2+y^2}\leqslant\frac{(\frac{1}{2}(x^2+y^2))^k}{x^2+y^2}=\frac{1}{2^k}(x^2+y^2)^{k-1}.$$

于是由夹逼原理可知此时函数 f 在原点连续.

当 $k=1$ 时, 由于 $\lim\limits_{x\to 0} f(x,x)=\frac{1}{2}$, 由复合极限法则可知此时函数 f 在原点不连续.

当 $k<1$ 时, 因 $\lim\limits_{x\to 0} f(x,x)=\lim\limits_{x\to 0} \frac{1}{2}|x|^{2k-2}=+\infty$, 则由复合极限法则可知 f 在原点不连续.

综上所述可知 f 在原点连续当且仅当 $k>1$.

可导性. $\forall x,y\in\mathbb{R}\setminus\{0\}$, 因 $f(x,0)=f(0,y)=0$, 于是由偏导数的定义可知 $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$, 故函数 f 在原点处可导.

可微性. 因 $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$, 则由微分的定义立刻可知 f 在原点可微当且仅当

$$\lim_{(x,y)\to(0,0)} \frac{|xy|^k}{(x^2+y^2)^{\frac{3}{2}}}=0,$$

而这又等价于说

$$\lim_{(x,y)\to(0,0)} \frac{|xy|^{\frac{2k}{3}}}{x^2+y^2}=0,$$

由此可知 f 在原点处可微当且仅当 $k>\frac{3}{2}$.

连续可导性. 若 f 在原点连续可导, 则由前面讨论可知 $k > \frac{3}{2}$. 此时 $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\frac{\partial f}{\partial x}(x,y) = \frac{k|x|^{k-1}|y|^k \operatorname{sgn} x}{x^2+y^2} - \frac{2x|xy|^k}{(x^2+y^2)^2},$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{k|x|^{k}|y|^{k-1} \operatorname{sgn} y}{x^2+y^2} - \frac{2y|xy|^k}{(x^2+y^2)^2}.$$

由此立刻可得

$$\left|\frac{\partial f}{\partial x}(x,y)\right| \leqslant \frac{k|x|^{k-1}|y|^k}{x^2+y^2} + \frac{2|x|^{k+1}|y|^k}{(x^2+y^2)^2}$$

$$\begin{aligned} &\leqslant \frac{k}{2^{k-1}}(x^2+y^2)^{k-2}|y| + \frac{1}{2^{k-1}}(x^2+y^2)^{k-2}|x| \\ &\leqslant \frac{k}{2^{k-1}}(x^2+y^2)^{k-\frac{3}{2}} + \frac{1}{2^{k-1}}(x^2+y^2)^{k-\frac{3}{2}} \\ &= \frac{k+1}{2^{k-1}}(x^2+y^2)^{k-\frac{3}{2}}, \end{aligned}$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \leqslant \frac{k|x|^k|y|^{k-1}}{x^2+y^2} + \frac{2|x|^k|y|^{k+1}}{(x^2+y^2)^2} \leqslant \frac{k+1}{2^{k-1}}(x^2+y^2)^{k-\frac{3}{2}},$$

由夹逼原理可知 $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ 在原点连续.

例 28. 假设由方程

$$f(u^2-x^2,u^2-y^2,u^2-z^2)=0$$

可确定 $u=\varphi(x,y,z)$, 其中 f,φ 可微. 记

$$X=(u^2-x^2,u^2-y^2,u^2-z^2).$$

若 $xyzu\neq 0$ 且 $\partial_1 f(X)+\partial_2 f(X)+\partial_3 f(X)\neq 0$,
求证: $\dfrac{u'_x}{x}+\dfrac{u'_y}{y}+\dfrac{u'_z}{z}=\dfrac{1}{u}$.

证明: 由于 $f(u^2-x^2,u^2-y^2,u^2-z^2)=0$, 故

$$\partial_1 f(X)\frac{\partial}{\partial x}(u^2-x^2)+\partial_2 f(X)\frac{\partial}{\partial x}(u^2-y^2)$$
$$+\partial_3 f(X)\frac{\partial}{\partial x}(u^2-z^2)=0,$$

也即我们有

$$\partial_1 f(X)(2uu'_x-2x)+\partial_2 f(X)(2uu'_x)+\partial_3 f(X)(2uu'_x)=0,$$

故 $\frac{u'_x}{x}\big(\partial_1 f(X)+\partial_2 f(X)+\partial_3 f(X)\big)=\frac{1}{u}\partial_1 f(X)$.

由对称性立刻可得

$$\frac{u'_y}{y}\big(\partial_1 f(X)+\partial_2 f(X)+\partial_3 f(X)\big)=\frac{1}{u}\partial_2 f(X),$$
$$\frac{u'_z}{z}\big(\partial_1 f(X)+\partial_2 f(X)+\partial_3 f(X)\big)=\frac{1}{u}\partial_3 f(X).$$

于是我们就有

$$\frac{u'_x}{x}+\frac{u'_y}{y}+\frac{u'_z}{z}=\frac{1}{u}.$$

例 29. $\forall (x,y) \in \mathbb{R}^2$, 定义 $f(x,y) = \sqrt{x^2+y^4}$. 研究 f 在原点的连续性, 可导性以及可微性.

解: 由于 f 为初等函数, 因此在原点连续.

与此同时, 由于极限

$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{|x|}{x}$$

不存在, 故 f 在原点处没有关于第一个变量的偏导数, 进而可知 f 在原点处不可微.

例 30. 假设 $f:\mathbb{R}^2\to\mathbb{R}$ 为二阶连续可导函数, 而隐函数 $z=z(x,y)$ 可由方程 $x+y=f(x,z)$ 确定, 其中 $\partial_2 f(x,z)\neq 0$. 计算 $\frac{\partial^2 z}{\partial x\partial y}$.

解: 将方程两边分别对 x,y 求偏导可得

$$1=\partial_1 f(x,z)+\partial_2 f(x,z)\frac{\partial z}{\partial x},\quad 1=\partial_2 f(x,z)\frac{\partial z}{\partial y},$$

由此我们立刻可知

$$\frac{\partial z}{\partial x}=\frac{1-\partial_1 f(x,z)}{\partial_2 f(x,z)},\quad \frac{\partial z}{\partial y}=\frac{1}{\partial_2 f(x,z)}.$$

于是我们有

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{1}{\partial_2 f(x,z)}\right) = -\frac{1}{(\partial_2 f(x,z))^2}\frac{\partial}{\partial x}\big(\partial_2 f(x,z)\big)$$

$$= -\frac{1}{(\partial_2 f(x,z))^2}\left(\partial_{12} f(x,z) + \partial_{22} f(x,z)\frac{\partial z}{\partial x}\right)$$

$$= -\frac{1}{(\partial_2 f(x,z))^2}\left(\partial_{12} f(x,z) + \partial_{22} f(x,z)\cdot\frac{1-\partial_1 f(x,z)}{\partial_2 f(x,z)}\right)$$

$$= \frac{\partial_{22} f(x,z)\cdot\partial_1 f(x,z) - \partial_{12} f(x,z)\cdot\partial_2 f(x,z) - \partial_{22} f(x,z)}{(\partial_2 f(x,z))^3}.$$

例 31. 求函数 $f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点处的偏导数 $f'_x(0,0)$, $f'_y(0,0)$, 并考察 f 在原点处的连续性和可微性.

解: 由偏导数的定义可知

$$f'_x(0,0)=\lim_{x\to 0}\frac{x}{x}=1,\qquad f'_y(0,0)=\lim_{y\to 0}\frac{-y}{y}=-1.$$

$\forall (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, 我们有

$$0\leqslant |f(x,y)|\leqslant \frac{|x|^3+|y|^3}{x^2+y^2}\leqslant\frac{2(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2}=2\sqrt{x^2+y^2},$$

于是由夹逼原理可知

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

从而 f 在原点处连续. 下证 f 在原点处不可微.

用反证法, 假设 f 在原点处可微, 则

$$\begin{aligned} 0 &= \lim_{(x,y)\to(0,0)} \frac{f(x,y)-(x-y)}{\sqrt{x^2+y^2}} \\ &= \lim_{x\to 0^+} \frac{f(x,-x)-(x+x)}{\sqrt{x^2+x^2}} = \lim_{x\to 0^+} \frac{-x}{\sqrt{2}x} = -\frac{\sqrt{2}}{2}, \end{aligned}$$

矛盾! 故 f 在原点处不可微.

例 32. 假设 φ 为二阶连续可微, 而 $z=z(x,y)$ 是由函数方程 $x^3+y^3+z^3=\varphi(z)$ 确定的隐函数, 求 $\frac{\partial^2 z}{\partial x\partial y}$, 并说明隐函数存在的条件.

解: 定义 $F(x,y,z)=x^3+y^3+z^3-\varphi(z)$, 则 F 为二阶连续可微并且 $\frac{\partial F}{\partial z}(x,y,z)=3z^2-\varphi'(z)$, 则当 $3z^2-\varphi'(z)\neq 0$ 时, 由方程 $F(x,y,z)=0$ 可确定隐函数 $z=z(x,y)$. 此时我们还有

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=\frac{3x^2}{\varphi'(z)-3z^2},\quad \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=\frac{3y^2}{\varphi'(z)-3z^2},$$

由此立刻可得

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\Big(\frac{3y^2}{\varphi'(z) - 3z^2}\Big) \\
&= -\frac{3y^2}{(\varphi'(z) - 3z^2)^2}\big(\varphi''(z) - 6z\big)\frac{\partial z}{\partial x} \\
&= \frac{9x^2y^2\big(6z - \varphi''(z)\big)}{(\varphi'(z) - 3z^2)^3}.
\end{aligned}
$$

例 33. 求函数 $z=\frac{\sin x}{1-\sin y}$ 在原点 $(0,0)$ 处带二阶 Peano 余项的 Taylor 展式.

解: 当 $(x,y)\to(0,0)$ 时, 我们有

$$\begin{aligned}\frac{\sin x}{1-\sin y} &= \sin x\,(1+\sin y+o(\sin y)) \\ &= (x+o(x^2))(1+y+o(y)) \\ &= x+xy+xo(y)+(1+y)o(x^2) \\ &= x+xy+o(x^2+y^2).\end{aligned}$$

例 34. 设 $f(x,y)=\begin{cases} x-y+\frac{xy^3}{x^2+y^4}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}$

求证: 函数 f 在原点处连续, 沿任意方向的方向导数都存在, 但不可微.

证明: (1) $\forall (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, 我们有

$$|f(x,y)-f(0,0)|=\left|x-y+\frac{xy^3}{x^2+y^4}\right|$$

$$\leqslant\ |x|+|y|+\frac{|xy^2|\cdot|y|}{x^2+y^4}\leqslant|x|+|y|+\frac{1}{2}|y|.$$

于是由夹逼原理可知函数 f 在原点连续.

(2) 固定 $\vec{\ell}^0 = (\cos\theta, \sin\theta)$. 由定义可知

$$
\begin{aligned}
&\frac{\partial f}{\partial \vec{\ell}^0}(0,0) = \lim_{h\to 0^+} \frac{f(h\vec{\ell}^0) - f(0,0)}{h} \\
= &\lim_{h\to 0^+} \left(\cos\theta - \sin\theta + \frac{h^4 (\cos\theta) \sin^3\theta}{h(h^2\cos^2\theta + h^4\sin^4\theta)} \right) \\
= &\cos\theta - \sin\theta.
\end{aligned}
$$

故 f 在原点处沿任意方向的方向导数存在.

(3) 用反证法, 假设 f 在原点可微. 由定义可得 $\frac{\partial f}{\partial x}(0,0)=1$, $\frac{\partial f}{\partial y}(0,0)=-1$. 由复合函数极限法则,

$$\begin{aligned}
0 &= \lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy^3}{(x^2+y^4)\sqrt{x^2+y^2}} \\
&= \lim_{y\to 0^+} \frac{y^2\cdot y^3}{(y^4+y^4)\sqrt{y^4+y^2}} = \frac{1}{2}.
\end{aligned}$$

矛盾! 故 f 在原点处不可微.

例 35. 假设 $z = z(x, y)$ 为二阶连续可导且满足

$$A\frac{\partial^2 z}{\partial x^2} + 2B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} = 0,$$

其中 $B^2 - AC > 0$ 且 $C \neq 0$. 若令

$$\begin{cases} u = x + \alpha y, \\ v = x + \beta y, \end{cases}$$

试确定 α, β 的值使得原方程等价于

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

解: 由题设可知

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z,\\
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)^2 z,\\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\cdot\alpha+\frac{\partial z}{\partial v}\cdot\beta=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z,\\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)^2 z,\\
\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z\\
&= \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z.
\end{aligned}
$$

带入题设方程可得

$$
\begin{aligned}
0 &= A\frac{\partial^2 z}{\partial x^2} + 2B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} \\
&= A\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^2 z + 2B\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right) z \\
&\quad + C\left(\alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right)^2 z \\
&= (A + 2B\alpha + C\alpha^2)\frac{\partial^2 z}{\partial u^2} \\
&\quad + 2\big(A + B(\alpha+\beta) + C\alpha\beta\big)\frac{\partial^2 z}{\partial u \partial v} \\
&\quad + (A + 2B\beta + C\beta^2)\frac{\partial^2 z}{\partial v^2}.
\end{aligned}
$$

于是要使题设方程等价于 $\frac{\partial^2 z}{\partial u \partial v} = 0$, 只需假设

$$A + 2B\alpha + C\alpha^2 = 0,\ \ A + 2B\beta + C\beta^2 = 0,$$

$$A + B(\alpha + \beta) + C\alpha\beta \neq 0.$$

由于 $B^2 - AC > 0$, 因此我们只需令

$$\begin{aligned} \alpha &= \frac{-B + \sqrt{B^2 - AC}}{C}, \\ \beta &= \frac{-B - \sqrt{B^2 - AC}}{C}. \end{aligned}$$

此时我们还有

$$\begin{aligned} & A + B(\alpha+\beta) + C\alpha\beta \\ = \ & A - \frac{2B^2}{C} + A = \frac{2}{C}(AC - B^2) \neq 0. \end{aligned}$$

于是要使题设方程等价于 $\frac{\partial^2 z}{\partial u \partial v} = 0$, 只需假设

$$\alpha = \frac{-B + \sqrt{B^2 - AC}}{C}, \ \beta = \frac{-B - \sqrt{B^2 - AC}}{C}.$$

此时存在两个连续可导函数 f, g 使得

$$z(x, y) = f(u) + g(v) = f(x + \alpha y) + g(x + \beta y).$$

例 36. $\forall x, y, z > 0$, 定义

$$f(x, y, z) = \log x + 2\log y + 3\log z.$$

求 f 在球面 $x^2 + y^2 + z^2 = 6r^2$ $(r > 0)$ 上的最大值, 并证明 $\forall a, b, c > 0$, 均有

$$ab^2c^3 \leqslant 108\Big(\frac{a+b+c}{6}\Big)^6.$$

解: 令 $S = \{(x, y, z) \mid x, y, z > 0,\ x^2 + y^2 + z^2 = 6r^2\}$, 则 S 为二维曲面.

固定 $P^* \in S$. 注意到

$$\lim_{u\to 0^+} \log u = -\infty,$$

而 $\forall (x,y,z) \in S$, 我们有

$$\begin{aligned} f(x,y,z) &= \log x + 2\log y + 3\log z \\ &\leqslant \log x + 2\log(\sqrt{6}r) + 3\log(\sqrt{6}r) \\ &= \log x + 5\log(\sqrt{6}r), \end{aligned}$$

同理, 我们也有

$$\begin{aligned} f(x,y,z) &\leqslant 2\log y + 4\log(\sqrt{6}r), \\ f(x,y,z) &\leqslant 3\log z + 3\log(\sqrt{6}r). \end{aligned}$$

由此知 $\exists\varepsilon > 0$ 使得 $\forall(x, y, z) \in S$, 当 $0 < x < \varepsilon$ 或者 $0 < y < \varepsilon$ 或者 $0 < z < \varepsilon$ 时, 我们总会有 $f(x, y, z) < f(P^*)$. 定义

$$S_\varepsilon = \{(x, y, z) \mid x, y, z \geqslant \varepsilon,\ x^2 + y^2 + z^2 = 6r^2\}.$$

则 S_ε 为有界闭集, 并且 $P^* \in S_\varepsilon$. 而 f 为连续函数, 于是它在 S_ε 上有最大值. 由前面的讨论可知, 该最大值也是 f 在 S 上的最大值. 我们将相应的最大值点记作 (x_0, y_0, z_0).

$\forall x, y, z > 0$ 以及 $\lambda \in \mathbb{R}$, 定义

$$\begin{aligned} L(x, y, z, \lambda) &= \log x + 2\log y + 3\log z \\ &\quad + \lambda(x^2 + y^2 + z^2 - 6r^2). \end{aligned}$$

由 Lagrange 乘数法可知, $\exists \lambda \in \mathbb{R}$ 使得

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x}(x_0, y_0, z_0, \lambda) = \frac{1}{x_0} + 2\lambda x_0, \\ 0 &= \frac{\partial L}{\partial y}(x_0, y_0, z_0, \lambda) = \frac{2}{y_0} + 2\lambda y_0, \\ 0 &= \frac{\partial L}{\partial z}(x_0, y_0, z_0, \lambda) = \frac{3}{z_0} + 2\lambda z_0, \\ 0 &= \frac{\partial L}{\partial \lambda}(x_0, y_0, z_0, \lambda) = x_0^2 + y_0^2 + z_0^2 - 6r^2. \end{aligned}$$

于是我们有

$$x_0 = \frac{1}{\sqrt{-2\lambda}},\ y_0 = \frac{1}{\sqrt{-\lambda}},\ z_0 = \frac{\sqrt{3}}{\sqrt{-2\lambda}},$$
$$-\frac{1}{2\lambda} - \frac{1}{\lambda} - \frac{3}{2\lambda} - 6r^2 = 0,$$

从而 $\lambda = -\frac{1}{2r^2}$, 进而可知 f 在 D 上的最大值点为 $(r, \sqrt{2}r, \sqrt{3}r)$, 相应的最大值为

$$f(r, \sqrt{2}r, \sqrt{3}r) = 6\log r + \log 2 + \frac{3}{2}\log 3.$$

$\forall a,b,c>0$, 我们令

$$r=\sqrt{\frac{1}{6}(a+b+c)},$$

则 $(\sqrt{a})^2+(\sqrt{b})^2+(\sqrt{c})^2=6r^2$, 从而我们有

$$\begin{aligned}&\log\sqrt{a}+2\log\sqrt{b}+3\log\sqrt{c}\\ \leqslant\ &6\log r+\log 2+\frac{3}{2}\log 3.\end{aligned}$$

由此我们立刻可得

$$ab^2c^3\leqslant 108r^{12}=108\left(\frac{a+b+c}{6}\right)^6.$$

例 37. 假设 $x=f(u,v)$, $y=g(u,v)$, $w=h(x,y)$ 均有二阶连续偏导数且满足

$$\frac{\partial f}{\partial u}=\frac{\partial g}{\partial v},\quad \frac{\partial f}{\partial v}=-\frac{\partial g}{\partial u},\quad \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0.$$

证明: $w=h(f(u,v),g(u,v))$ 满足 $\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=0$.

证明: 由复合求导法则可知

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial h}{\partial x}(f(u,v),g(u,v))\frac{\partial f}{\partial u}(u,v)\\ &\quad+\frac{\partial h}{\partial y}(f(u,v),g(u,v))\frac{\partial g}{\partial u}(u,v),\end{aligned}$$

由此我们立刻可以导出

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial h}{\partial x}(f(u,v),g(u,v))\frac{\partial f}{\partial u}(u,v) + \frac{\partial h}{\partial y}(f(u,v),g(u,v))\frac{\partial g}{\partial u}(u,v)\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial h}{\partial x}(f(u,v),g(u,v))\right)\frac{\partial f}{\partial u}(u,v) + \frac{\partial h}{\partial x}(f(u,v),g(u,v))\frac{\partial^2 f}{\partial u^2}(u,v)$$

$$+ \frac{\partial}{\partial u}\left(\frac{\partial h}{\partial y}(f(u,v),g(u,v))\right)\frac{\partial g}{\partial u}(u,v) + \frac{\partial h}{\partial y}(f(u,v),g(u,v))\frac{\partial^2 g}{\partial u^2}(u,v)$$

$$= \left(\frac{\partial^2 h}{\partial x^2}(f(u,v),g(u,v))\frac{\partial f}{\partial u} + \frac{\partial^2 h}{\partial y \partial x}(f(u,v),g(u,v))\frac{\partial g}{\partial u}\right)\frac{\partial f}{\partial u}$$

$$+ \frac{\partial h}{\partial x}(f(u,v),g(u,v))\frac{\partial^2 f}{\partial u^2}$$

$$+ \left(\frac{\partial^2 h}{\partial x \partial y}(f(u,v),g(u,v))\frac{\partial f}{\partial u} + \frac{\partial^2 h}{\partial y^2}(f(u,v),g(u,v))\frac{\partial g}{\partial u}\right)\frac{\partial g}{\partial u}$$

$$+ \frac{\partial h}{\partial y}(f(u,v),g(u,v))\frac{\partial^2 g}{\partial u^2}.$$

为简便记号, 下面省去自变量. 由对称性可得

$$
\begin{aligned}
\frac{\partial^2 w}{\partial v^2} &= \Big(\frac{\partial^2 h}{\partial x^2}\frac{\partial f}{\partial v} + \frac{\partial^2 h}{\partial y \partial x}\frac{\partial g}{\partial v}\Big)\frac{\partial f}{\partial v} + \frac{\partial h}{\partial x}\frac{\partial^2 f}{\partial v^2} \\
&\quad + \Big(\frac{\partial^2 h}{\partial x \partial y}\frac{\partial f}{\partial v} + \frac{\partial^2 h}{\partial y^2}\frac{\partial g}{\partial v}\Big)\frac{\partial g}{\partial v} + \frac{\partial h}{\partial y}\frac{\partial^2 g}{\partial v^2}.
\end{aligned}
$$

由前面讨论立刻可知

$$
\begin{aligned}
\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} &= \frac{\partial^2 h}{\partial x^2}\Big(\Big(\frac{\partial f}{\partial u}\Big)^2 + \Big(\frac{\partial f}{\partial v}\Big)^2\Big) + \frac{\partial^2 h}{\partial y^2}\Big(\Big(\frac{\partial g}{\partial u}\Big)^2 + \Big(\frac{\partial g}{\partial v}\Big)^2\Big) \\
&\quad + 2\frac{\partial^2 h}{\partial x \partial y}\Big(\frac{\partial f}{\partial u}\frac{\partial g}{\partial u} + \frac{\partial f}{\partial v}\frac{\partial g}{\partial v}\Big) + \frac{\partial h}{\partial x}\Big(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\Big) \\
&\quad + \frac{\partial h}{\partial y}\Big(\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\Big).
\end{aligned}
$$

又由于 $\frac{\partial f}{\partial u}=\frac{\partial g}{\partial v}$, $\frac{\partial g}{\partial u}=-\frac{\partial f}{\partial v}$, 于是我们有

$$\begin{aligned}
&\frac{\partial f}{\partial u}\frac{\partial g}{\partial u}+\frac{\partial f}{\partial v}\frac{\partial g}{\partial v}=0,\\
&(\frac{\partial f}{\partial u})^2+(\frac{\partial f}{\partial v})^2=(\frac{\partial g}{\partial v})^2+(\frac{\partial g}{\partial u})^2,\\
&\frac{\partial^2 f}{\partial u^2}=\frac{\partial^2 g}{\partial u\partial v},\ \ \frac{\partial^2 f}{\partial v^2}=-\frac{\partial^2 g}{\partial u\partial v},\\
&\frac{\partial^2 g}{\partial u^2}=-\frac{\partial^2 f}{\partial u\partial v},\ \ \frac{\partial^2 g}{\partial v^2}=\frac{\partial^2 f}{\partial u\partial v}.
\end{aligned}$$

但 $\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0$, 从而最终我们有

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=0.$$

谢谢大家!