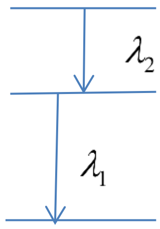
Homework for General PhysicsII-set8

1.

23. The longest wavelength in the Lyman series is 121.5 nm and the shortest wavelength in the Balmer series is 364.6 nm. Use the figures to find the longest wavelength of light that could ionize hydrogen.

Answer:

The longest wavelength λ_1 (smallest in energy) is transition from n=2 to n=1 in Lyman series; and the shortest wavelength λ_2 (highest in energy) is from n=infinity to n=2 in Balmer series.



The total energy is additive (wavelength is not)
The lowest ionization energy is (energy from n=infinity to n=1):

$$\frac{hc}{\lambda} = \frac{hc}{\lambda_1} + \frac{hc}{\lambda_2}$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

$$\lambda = 91.1nm$$

This is the answer for the question.
Also we can check that this wavelength corresponding to the energy of n=1:
The energy corresponding to wavelength is:

$$\frac{hc}{\lambda}$$

$hc = 6.63 \times 10^{-34} \times 3 \times 10^{8} = 1.986 \times 10^{-25} J \cdot m = 1239.8 eV \cdot nm$ (check this combination yourself)

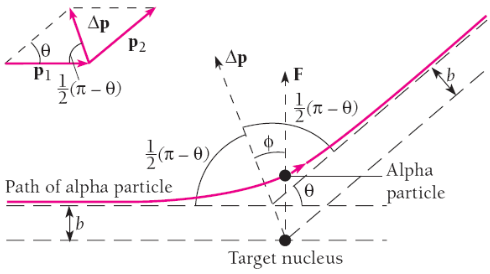
$$\frac{hc}{\lambda} = \frac{1239.8}{91.1} = 13.6eV$$

2. AB's 4-46

46. A beam of 8.3-MeV alpha particles is directed at an aluminum foil. It is found that the Rutherford scattering formula ceases to be obeyed at scattering angles exceeding about 60°. If the alpha-particle radius is assumed small enough to neglect here, find the radius of the aluminum nucleus.

Answer: This is a central field problem we had in last semester. The Rutherford model is based on Coulomb expulsion and it ceases to be obeyed if the alpha particle is too close to the nuclei and nuclear force needs to be taken into consideration.

From deflection angle we can calculate b, the impact parameter: (I copied figure, only need b and θ here,



Using formula 4.29 in the book pg.155 (we worked that out in last semester)

$$\cot\frac{\theta}{2} = \frac{4\pi\varepsilon_0 KE}{Ze^2} b$$

$$b = \frac{Ze^2}{4\pi\varepsilon_0 KE}\cot\frac{\theta}{2}$$

With $\frac{1}{4\pi\varepsilon_0} = 9\times10^9 SI$, Z=13 for Al;

$$KE = 8.3\times10^6 eV = 8.3\times10^6\times1.6\times10^{-19} j$$

$$b = 3.9\times10^{-15} m$$

This b gives the approximate size of Al nuclei.

If you want to be more precise (not necessary, because this is just an estimate of nuclei size at best), you can calculate the r_{min} of the hyberbola (the closest of approach for hyperbolic curve under repulsion), which is **c+*a*** (*a* and c are parameters for hyperbola). Since we know b and deflection angle θ, we can calculate c and a (another approach would be like we discussed in central field, given b and initial E (which is just KE here), everything about hyperbola can be calculated):

$$c = \frac{b}{\sin\frac{1}{2}(\pi-\theta)}, a = \frac{b}{\tan\frac{1}{2}(\pi-\theta)}$$

3. 8. Find the kinetic energy of an electron whose de Broglie wavelength is the same as that of a 100-keV x-ray.

Answer:

From energy of x-ray (light), we can get its wavelength (or momentum p), then we can find momentum of electron, then its kinetic energy. Since energy seems on the order of 100keV which is about same order of electron rest mass (500keV), so I will use relativistic formula for energy and momentum:

De-Broglie $p = \frac{h}{\lambda}$,means if same wavelength, then same momentum.

The momentum of photon (x-ray) is:

$E = pc$ (I will adopt c=1 in the following):

$$p = E = 100keV$$

Total energy of the electron then:

$$E_e^2 = p^2 + m_e^2 \rightarrow E_e = 510keV$$

$$KE = E_e - m_e = 10keV$$

(Kinetic energy of electron is much less than rest mass, so if you use classical energy formula, the answer would be close enough)

4. AB 3-10:

10. Show that the de Broglie wavelength of a particle of mass m and kinetic energy KE is given by

$$\lambda = \frac{hc}{\sqrt{KE(KE + 2mc^2)}}$$

Answer:

I use $m_0 = m$ specifies rest mass. (In the following I still use c=1, and finally add c from dimension analysis):

$E = KE + m_0$ Also:

$$E^2 = p^2 + m_0^2$$

Then:

$$KE^2 + m_0^2 + 2m_0 KE = p^2 + m_0^2$$

$$p^2 = KE^2 + 2m_0 KE$$

$$p = \sqrt{KE^2 + 2m_0 KE}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{KE^2 + 2m_0 KE}}$$

Add c's to make dimension correct:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{KE^2 + 2m_0 c^2 KE}}$$

5. For a "particle" which has duality(a matter wave), its particle behavior can be described as a momentum p and velocity u, prove the following:

a) The associated matter wave generally has relation between phase and group velocity, the group velocity is just u. i.e. the group velocity of the matter wave corresponds to classical u

b) $v_\varphi v_g = c^2$

Answer:

a) Since $E^2 = p^2c^2 + m_0^2c^4$

$$v_g = \frac{dE}{dp} = c^2 \frac{p}{E} = u$$

b) $E = \gamma_u m_0 c^2, p = \gamma_u m_0 u$

So: $v_\varphi = E / p = \dfrac{c^2}{u}$

Then: $v_\varphi v_g = c^2$

6. AB's 3-13

13. An electron and a proton have the same velocity. Compare the wavelengths and the phase and group velocities of their de Broglie waves.

Answer:

Since they have same velocity, they have same γ factor.

Their momentum ratio: $p_e / p_{proton} = \dfrac{m_e}{m_p}$, so will be their energy ratio:

$$E_e / E_{proton} = \frac{m_e}{m_p}$$

$$v_\varphi = \frac{E}{p}$$

$$\frac{v_{\varphi e}}{v_{\varphi p}} = \frac{E_e / p_e}{E_p / p_p} = 1$$

$v_g = u$ also same for electron and proton since they have same u.

7. AB's 3-19

19. Find the phase and group velocities of the de Broglie waves of an electron whose kinetic energy is 500 keV.

Answer: need to find E, P for phase velocity and use c^2 to get group v:

$$E = m_0 + KE = 500keV + 500keV = 1000keV$$

$$p = \sqrt{E^2 - m_0^2} = 866keV$$

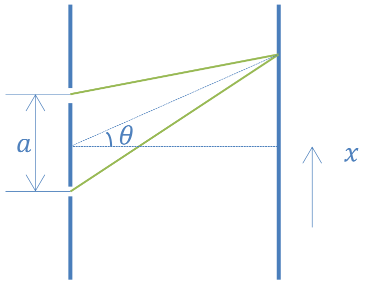
$v_\varphi = E / p = 1.15c$ (add c here to make dimension correct)

Don't be alarmed that phase velocity is larger than c, phase velocity doesnot corresponds to travelling speed of "particle", it is the group velocity as we see in 5 above:

$$v_g v_\varphi = c^2 \rightarrow v_g = 0.87c$$

8. **An electron microscope using electrons to study the micro-structure, for the electron at 1eV, using them to study the structure, what is the spatial resolution limit?**
   (You may wonder whether you can directly use the results learned from light diffraction, the answer is you can. The reasoning is I only used superposition principle of waves in the treatment of light interference and diffraction, the same superposition principle also holds in Q.M, though the physical meaning of the wave itself is different from that of classical wave theory)
   Answer:
   $$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m\varepsilon}} = \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 1.6 \times 10^{-19}}} = 1.23 \times 10^{-9} m$$
   Spatial resolution limit:
   $1.22\lambda = 1.5 \times 10^{-9} m$

9. **Using the uncertainly relation between position and momentum to derived the Fraunhofer single slit diffraction patter, i.e. $\boldsymbol{a \cdot \Delta\theta \sim \lambda}$, where $\boldsymbol{a}$ is the width of slit and $\boldsymbol{\Delta\theta}$ is the angular width between the maximum and first zero in the diffraction pattern.**
   Answer:



Consider the uncertainty in the x direction:

$\Delta x \sim a$

$$\Delta p_x = \frac{h}{\lambda} sin\Delta\theta \approx \frac{h}{\lambda}\Delta\theta$$

$\because \Delta x \Delta p_x \sim h$

$\therefore a \cdot \Delta\theta \sim \lambda$

10. **Use the uncertainty relation to explain the stability of electron in hydrogen atom. (You may find discussion outlined in my notes, please work out the detail)**

Answer:

As described in my note, taking the consideration of uncertainty, as radius decreases, the average kinetic energy increase:

$$assume\ \Delta r \sim r\ ;\ \Delta p \sim \frac{h}{r}$$

$K = \frac{<p^2>}{2m} = \frac{(\Delta p)^2}{2m}$ (This is because $(\Delta p)^2 = <p^2> - <p>^2, <p>=0$)

$$E = K + U = \frac{h^2}{2mr^2} - \frac{e^2}{r}$$

Here as well as in my note I realized I used esu unit of electric, in the SI unit, e^2 here and below would be replaced by $\frac{e^2}{4\pi\varepsilon_0}$.

The lowest energy now would not be at r=0, but be calculated by taking:

$$\frac{dE}{dr} = 0 \rightarrow r_0 = \frac{h^2}{m_e e^2}$$

Another way to argue is by:

If the electron is constrained in the nuclei of the atom:

$\Delta x \approx 10^{-14} m$

$\because \Delta x \Delta p \sim h$

$$\therefore \Delta v \approx \frac{\Delta p}{m} \approx \frac{h}{m\Delta x} \approx \frac{6 \times 10^{-34}}{9 \times 10^{-31} \times 10^{-14}} \sim 10^{11} m/s \gg c$$

It's impossible.

So the electron won't drop into the nuclei.

11. **A problem of particle in an infinite deep potential. Of course you may solve this problem with Schrödinger equation and the answers are in all textbooks. Here I urge you to use general properties of wave (please recall the discussion on standing wave, it is a**

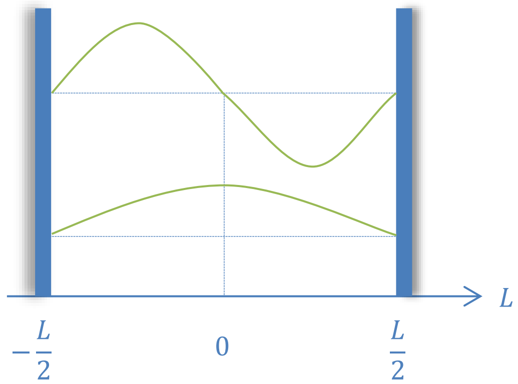
superposition of two traveling wave against each other) and simple quantum relation ($p = \frac{h}{\lambda}$) to solve it.

The potential is defined as:

$$V(x) = \begin{cases} 0, |x| < L/2 \\ \infty, otherwise \end{cases}$$

A particle of mass m is inside such potential well.

(1) Plot this one dimension potential, and also plot the *two* waveforms that can exist inside such potential with longest wavelengths. (One is the longest and the other is next to it.)



(2) What are the energies of the particle associated with this two waveform?
(You are calculating lowest two energy levels in this case. The energy is still K+V (kinetic+potential); potential is 0 everywhere inside the well)
Answer:

$$\lambda_1 = 2\text{L} \Longrightarrow p_1 = \frac{h}{\lambda_1} = \frac{h}{2L} \Longrightarrow \varepsilon_1 = \frac{p_1^2}{2m} = \frac{h^2}{8mL^2}$$

$$\lambda_2 = \text{L} \Longrightarrow p_2 = \frac{h}{\lambda_2} = \frac{h}{L} \Longrightarrow \varepsilon_2 = \frac{p_2^2}{2m} = \frac{h^2}{2mL^2}$$

(3) You should find that the lowest energy level for the system is actually not zero, and if you guess that is due to uncertainty relations, it's great.
What is the energy of the system you estimated from uncertainty relations?
(May not exactly as you get in question (2), but would be same order)
Answer:
Uncertainty principle:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x \sim L$$

$$\Delta p \sim p$$

For the lowest energy level:

$$\Delta x \Delta p = \frac{\hbar}{2}$$

$$\Longrightarrow p = \frac{\hbar}{2L} \Longrightarrow \varepsilon = \frac{p^2}{2m} = \frac{\hbar^2}{8mL^2}$$

(4) The uncertainty of the momentum is defined as $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$ (<>means average) which equals to $\langle p^2 \rangle - \langle p \rangle^2$.

Find $\langle p^2\rangle$ *and* $\langle p\rangle$ for the lowest energy level (the longest wavelength).
You really do not need to do calculation if you remember the standing wave. Use this to determine the uncertainty between *x* and *p*, taking $\Delta x = \frac{L}{2}$.
Does it satisfy the limit set by the uncertainty relation?

Answer:

Since it's a standing wave:

$$\langle p\rangle = 0$$

$$\langle p^2\rangle = p_1^2 = \frac{h^2}{4L^2}$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{h^2}{4L^2}$$

$$\Delta p\Delta x = \frac{h}{2L}\cdot\frac{L}{2} = \frac{h}{4} = \frac{\pi\hbar}{2} > \frac{\hbar}{2}, satisfied.$$

(5) **For a small macroscopic object with 1μg (10^{-6}kg) inside a well with L=1μm (10^{-6}m); and an electron (m=10^{-30} kg) trapped inside a well with L=1nm.**
Calculate the energy difference between the lowest two levels for each case.
If you use electron bombardment to excite the particle from the ground (the lowest) to its first excited state, what is the energy (in eV) needed for each case?
If you use light to do the excitation, what are the frequencies of light?
(I hope this convince you that for macroscopic object, the energy spacing is so small that its energy is almost continuous; while for microscopic object, quantization is more obvious)

Answer:

$$\Delta\varepsilon = \varepsilon_2 - \varepsilon_1 = \frac{h^2}{2mL^2} - \frac{h^2}{8mL^2} = \frac{3h^2}{8mL^2}$$

For the macroscopic object:

$$\Delta\varepsilon_m = \frac{3h^2}{8mL^2} = \frac{3\times(7\times10^{-34})^2}{8\times10^{-6}\times(10^{-6})^2} = 1.8\times10^{-49}J = 10^{-30}eV$$

It is the energy needed to excite the particle from ground to the 1st excited state.

If excite it by light, then the frequency:

$$\nu = \frac{\Delta\varepsilon_m}{h} = \frac{1.8\times10^{-49}}{7\times10^{-34}} = 2\times10^{-16}\text{Hz}$$

For the electron:

$$\Delta\varepsilon_e = \frac{3h^2}{8mL^2} = \frac{3\times(7\times10^{-34})^2}{8\times10^{-30}\times(10^{-9})^2} = 1.8\times10^{-19}J = 1eV$$

It is the energy needed to excite the electron from ground to the 1st excited state.

Frequency of the light used to excite:

$$\nu = \frac{\Delta\varepsilon_m}{h} = \frac{1.8\times10^{-19}}{7\times10^{-34}} = 2\times10^{14}\text{Hz}$$

12. Answer;

(a) Single slit diffraction pattern centered around A

(b) Single slit diffraction pattern centered around B

(c) Doubles slit diffraction pattern with interference.

(d) Since we know the pathway of electrons through the slit, the results are summation of intensities of the two diffraction, i.e. summation of intensity from (a) and (b). There is no interference in this case, just like let H polarized light passing A, and V polarized light through B.
(e) There will be interference now, just like that of (c), but the intensity drops to half of the (c).

13. AB's 3-39

39. The frequency of oscillation of a harmonic oscillator of mass m and spring constant C is $\nu = \sqrt{C/m}/2\pi$. The energy of the oscillator is $E = p^2/2m + Cx^2/2$, where p is its momentum when its displacement from the equilibrium position is x. In classical physics the minimum energy of the oscillator is $E_{min} = 0$. Use the uncertainty principle to find an expression for E in terms of x only and show that the minimum energy is actually $E_{min} = h\nu/2$ by setting $dE/dx = 0$ and solving for E_{min}.

Answer: Clearly we need to use uncertainty relation, the key is to express the energy in terms of uncertainties in x and p:
The energy has two parts, kinetic and potential:
For the average energy, the potential part is:

$$PE = \frac{C}{2} < x^2 >$$

From statistics, the uncertainty (standard deviation)is defined as:

$$(\Delta x)^2 =< (x- < x >)^2 >=< x^2 - 2x < x > + < x >^2>$$
$$=< x^2 > -2 < x >< x > + < x >^2 =< x^2 > - < x >^2$$

For the harmonic oscillator, where equilibrium is set at x=0, the average of x:
$$< x >= 0$$

Then: $< x^2 >= (\Delta x)^2$, and if the oscillator is displaced from equilibrium by x, the change of x will be from –x to +x, we take the width ("waist") as:
$\Delta x = x$ (of course here is just one way of approximation, you may argue using 2x as width, then the result will differ, but with same magnitude; remember we never expect any precise answer with uncertainty argument, only a qualitative result)

Thus: $< x^2 >= x^2$

Similarly: $(\Delta p)^2 =< p^2 >$

Now apply uncertainty relation:

$\Delta x \Delta p = \frac{\hbar}{2}$ (here I use the lowest limit 1/2 of h bar; you may use h, or h bar for qualitative estimation)

$$< p^2 >= \frac{\hbar^2}{4x^2}$$

The average energy of oscillator would become:

$$E = \frac{\hbar^2}{8mx^2} + \frac{C}{2}x^2$$

As x (or amplitude) smaller, the kinetic energy becomes big, so lowest energy is not 0 anymore. The lowest energy is at:

$$\frac{dE}{dx} = 0 = -\frac{\hbar^2}{4mx^3} + Cx$$

E is lowest when:

$$x^2 = \sqrt{\frac{\hbar^2}{4mC}} = \frac{\hbar}{2\sqrt{mC}}$$

The lowest energy at this x is:

$$E = \frac{\hbar^2}{8mx^2} + \frac{C}{2}x^2 = \frac{\hbar}{4\sqrt{m/C}} + \frac{\hbar C}{4\sqrt{mC}}$$

$$= \frac{\hbar}{4}\sqrt{\frac{C}{m}} + \frac{\hbar}{4}\sqrt{\frac{C}{m}} = \frac{\hbar}{2}\sqrt{\frac{C}{m}}$$

But $\sqrt{\frac{C}{m}} = \omega$ (the classical oscillator natural angular frequency, C is our old k)

So, the lowest possible energy is:

$$E = \frac{1}{2}\hbar\omega$$

This is exactly the so called zero-point energy for harmonic oscillator using strict but also more complicated quantum method (solving Schrodinger equation). I should stress this kind of estimate using uncertainty which happens to give a precise answer is just a coincidence. Uncertainty is most useful for qualitative estimation.

14. Only answers shown here since this is a long problem:

Answer:

1) *还有两束光在第二个分束器上方,分别是上面光路经分束器的反射和下面光路经分束器的透射*

2) 在探测器能量最低时,能量集中在上问的光路里.即若水平方向干涉抵消,依照能量守恒,则推知能量应该集中在竖直方向上.

3) 沿着光路考虑各个场的位相变化(因为振幅都是一样的,集中考虑位相). E_1,E_2 代表经过上面和下面光路经第二个分束器水平进入探测器 D1（D bottom）的光场. E_1', E_2' 代表经过上面和下面光路经第二个分束器垂直向上的光场进入第二个 D2(Dtop)的光场

$$E_1 = \exp i(\delta_R + \delta_m + \delta_T + kl_{top})$$

$$E_2 = \exp i(\delta_T + \delta_m + \delta_R + kl_{bottom})$$ （只写出位相，振幅一样大）

δ_R, δ_T 是经过分束器带来的位相延迟, δ_m 是镜子反射带来的位相延迟.同样可以写出:

$$E_1' = \exp i(\delta_R + \delta_m + \delta_R + kl_{top})$$

$$E_2' = \exp i(\delta_T + \delta_m + \delta_T + kl_{bottom})$$

当进入探测器 D1 的光场相互抵消时,有 E_1,E_2 的位相差为:

$$kl_{top} - kl_{bottom} = k\Delta l = \pi$$

此时 E_1', E_2'相互增强,位相差为:$2(\delta_R - \delta_T) + k\Delta l = 0$

由此可得:$\delta_R - \delta_T = \phi = \frac{\pi}{2}$

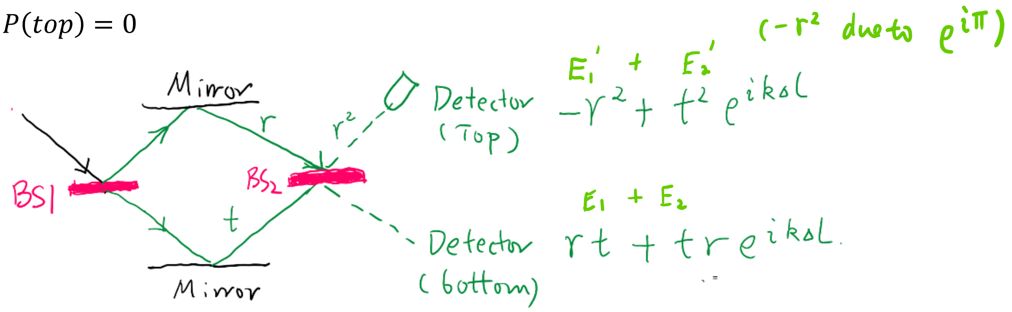
4) 若$\Delta l = 0$, 从 3 问可知： 到达 Bottom 的场(或波) E1, E2 位相差为 0，

$E = E_1 + E_2 = \frac{1}{2} + \frac{1}{2} = 1$

$P(bot) = |E|^2 = 1$

For Top: the phase difference is $2\phi = \pi$ between E_1' and E_2':

$P(top) = 0$



5) $\Delta l = \frac{\lambda}{2}$, then for the bottom there is π phase difference between E_1 andE_2 :

$P(bot) = |E|^2 = 0$; and $P(top) = 1$

6) A) Let the Beam Splitter works on each base vector:

BS will splits top coming light into top (with reflection r) and bottom (with transmission t):

$$BS \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}r\\t\end{pmatrix} = \begin{pmatrix} i\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

Its action on bottom coming light:

$$BS\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}t\\r\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ i\frac{\sqrt{2}}{2}\end{pmatrix}$$

So the matrix for single BS is:

$$BS = \frac{\sqrt{2}}{2}\begin{pmatrix}i & 1\\1 & i\end{pmatrix}$$

B) The input light after two BS will be:

$BS\,BS\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix}$

The output only take the bottom path (the common phase factor i is no importance) as in 4).

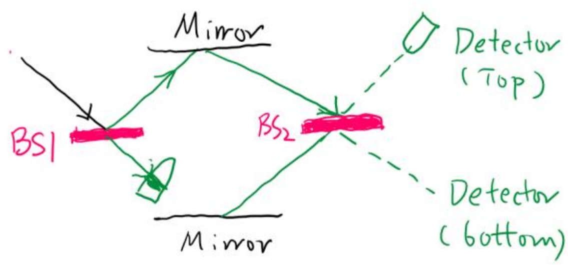
7) If the bad bomb is put at bottom path, since it won't absorb light and appear as air, the situation is exactly as in 4), i.e.

   $P(bot) = |E|^2 = 1;\ P(top) = 0$

   That is the Top detector will never click for the bad bomb

8) For the good bomb at the bottom path: It will absorb light if the single photon takes the lower path, and it will explode: This has 50% of chance.

   However there is another 50% of chance that light will take top path, and the key here is since the bottom path is blocked (which way in interference, and the light reaches the detector must take the top path), so there will be no interference: as shown in the figure:



   $\boldsymbol{P(top)} = 25\% = P(bot)$

   Now the top detector has 25% click and this only happens if the bomb is good!

   So we have **25%** chance successfully detecting good bomb without detonate them!

9) From above; for one measurement on good bomb, 50% chance of detonating it; 25% to be certain it is good; 25% uncertain (that is when bottom detector clicks). Now we can take the 25% uncertain case repeat what we did in 8), then 25% of that (the 25%) will give us certain good; 25% of that (the 25%) will still be uncertain…. This is:

$$P(good) = \frac{1}{4}[1 + \frac{1}{4} + (\frac{1}{4})^2 + .. \left(\frac{1}{4}\right)^n ...] = \frac{1}{4}\frac{1}{1-{}^1/_4} = \frac{1}{3}$$

So 1/3 of the N good bomb will be picked out successfully.