第 6 次作业题解答

1. 比较 $\displaystyle\iint\limits_D (x+y)^2\,\mathrm{d}x\mathrm{d}y$ 与 $\displaystyle\iint\limits_D (x+y)^3\,\mathrm{d}x\mathrm{d}y$ 的大小, 其中

$$D=\big\{(x,y)\mid (x-2)^2+(y-2)^2\leqslant 2\big\}.$$

解: $\forall (x,y)\in D$, 我们有 $(x+y-4)^2\leqslant 2\big((x-2)^2+(y-2)^2\big)\leqslant 4$, 故

$$x+y=4+(x+y-4)\geqslant 4-|x+y-4|\geqslant 2,$$

于是 $(x+y)^3\geqslant 2(x+y)^2$, 从而由积分严格保序性可知

$$\iint\limits_D (x+y)^2\,\mathrm{d}x\mathrm{d}y<\iint\limits_D (x+y)^3\,\mathrm{d}x\mathrm{d}y.$$

2. 设函数 f 在原点 $(0,0)$ 的某个邻域内连续, 计算极限

$$\lim_{r\to 0^+}\frac{1}{r^2}\iint\limits_{x^2+y^2\leqslant r^2} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

解: 由题设可知 $\exists R>0$ 使得 f 在 $B((0,0);R)$ 上连续. $\forall r\in(0,R)$, 由积分中值定理可知, $\exists(\xi_1(r),\xi_2(r))\in\bar{B}((0,0);r)$ 使得

$$\frac{1}{r^2}\iint\limits_{x^2+y^2\leqslant r^2} f(x,y)\,\mathrm{d}x\mathrm{d}y=\pi f(\xi_1(r),\xi_2(r)).$$

由夹逼原理可知

$$\lim_{r\to 0^+}\xi_1(r)=\lim_{r\to 0^+}\xi_2(r)=0,$$

而 f 在原点 $(0,0)$ 连续, 于是由复合函数极限法则立刻可知

$$\lim_{r\to 0^+}\frac{1}{r^2}\iint\limits_{x^2+y^2\leqslant r^2} f(x,y)\,\mathrm{d}x\mathrm{d}y=\pi f(0,0).$$

3. 将二重积分 $\displaystyle\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y$ 化成累次积分, 其中

$$D=\big\{(x,y)\mid y\geqslant x-2,\ x\geqslant y^2\big\}.$$

解: 由题设知 D 可化成 $D=\big\{(x,y)\mid -1\leqslant y\leqslant 2,\ y^2\leqslant x\leqslant y+2\big\}$, 则

$$\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y=\int_{-1}^{2}\Big(\int_{y^2}^{y+2} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y.$$

4. 改变 $\int_0^1\Big(\int_{2\sqrt{1-x}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x+\int_1^2\Big(\int_0^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x$ 的积分次序.

解: 由题设可知

$$
\begin{aligned}
& \int_0^1\Big(\int_{2\sqrt{1-x}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x+\int_1^2\Big(\int_0^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
= & \iint\limits_{\substack{0\leqslant x\leqslant 1\\ 2\sqrt{1-x}\leqslant y\leqslant\sqrt{4-x^2}}} f(x,y)\,\mathrm{d}x\mathrm{d}y+\iint\limits_{\substack{1\leqslant x\leqslant 2\\ 0\leqslant y\leqslant\sqrt{4-x^2}}} f(x,y)\,\mathrm{d}x\mathrm{d}y \\
= & \iint\limits_{\substack{0\leqslant y\leqslant 2\\ 1-\frac{y^2}{4}\leqslant x\leqslant\sqrt{4-y^2}}} f(x,y)\,\mathrm{d}x\mathrm{d}y \\
= & \int_0^2\Big(\int_{1-\frac{y^2}{4}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y.
\end{aligned}
$$

5. 计算下列二重积分:

(1) $\iint\limits_D |xy|\,\mathrm{d}x\mathrm{d}y$, 其中 $D=\{(x,y)\mid x^2+y^2\leqslant R^2\}$ 且 $R>0$;

(2) $\iint\limits_D (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$, 其中 D 是以 $y=x$, $y=x+1$, $y=1$, $y=4$ 为其边的平行四边形.

解: (1) 由对称性可得

$$
\begin{aligned}
\iint\limits_D |xy|\,\mathrm{d}x\mathrm{d}y &= 4\iint\limits_{\substack{x^2+y^2\leqslant R^2\\ x,y\geqslant 0}} xy\,\mathrm{d}x\mathrm{d}y \\
&= 4\int_0^R\Big(\int_0^{\sqrt{R^2-x^2}} xy\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&= 2\int_0^R x(R^2-x^2)\,\mathrm{d}x \\
&= (R^2x^2-\frac{1}{2}x^4)\Big|_0^R=\frac{1}{2}R^4.
\end{aligned}
$$

(2) 由题设可知 $D=\big\{(x,y)\mid 1\leqslant y\leqslant 4,\ y-1\leqslant x\leqslant y\big\}$, 于是

$$
\begin{aligned}
\iint\limits_D (x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \int_1^4\Big(\int_{y-1}^{y}(x^2+y^2)\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&= \int_1^4\Big(\frac{1}{3}y^3-\frac{1}{3}(y-1)^3+y^2\Big)\,\mathrm{d}y \\
&= \Big(\frac{1}{12}y^4-\frac{1}{12}(y-1)^4+\frac{1}{3}y^3\Big)\Big|_1^4 \\
&= \frac{71}{2}.
\end{aligned}
$$

6. 分别求出由平面 $z=x-y$, $z=0$ 与圆柱面 $x^2+y^2=2x$ 所围成的两个空间几何体的体积.

解: 由题设可知所围成的两个空间几何体为

$$\begin{aligned}\Omega_1 &= \big\{(x,y)\mid x^2+y^2\leqslant 2x,\ 0\leqslant z\leqslant x-y\big\},\\ \Omega_2 &= \big\{(x,y)\mid x^2+y^2\leqslant 2x,\ x-y\leqslant z\leqslant 0\big\},\end{aligned}$$

则 $\Omega_1=\Omega_{11}\cup\Omega_{12}$, $\Omega_2=\big\{(x,y)\mid 0\leqslant x\leqslant 1,\ x\leqslant y\leqslant \sqrt{2x-x^2},\ x-y\leqslant z\leqslant 0\big\}$,

$$\begin{aligned}\Omega_{11} &= \big\{(x,y)\mid 0\leqslant x\leqslant 1,\ -\sqrt{2x-x^2}\leqslant y\leqslant x,\ 0\leqslant z\leqslant x-y\big\},\\ \Omega_{12} &= \big\{(x,y)\mid 1\leqslant x\leqslant 2,\ -\sqrt{2x-x^2}\leqslant y\leqslant \sqrt{2x-x^2},\ 0\leqslant z\leqslant x-y\big\},\end{aligned}$$

于是它们的体积分别为

$$\begin{aligned}
|\Omega_1| &= |\Omega_{11}|+|\Omega_{12}|\\
&= \int_0^1\Big(\int_{-\sqrt{2x-x^2}}^{x}(x-y)\,\mathrm{d}y\Big)\,\mathrm{d}x+\int_1^2\Big(\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}(x-y)\,\mathrm{d}y\Big)\,\mathrm{d}x\\
&= \int_0^1\Big(xy-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{2x-x^2}}^{x}\,\mathrm{d}x+\int_1^2\Big(xy-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}\,\mathrm{d}x\\
&= \int_0^1(x\sqrt{2x-x^2}+x)\,\mathrm{d}x+\int_1^2 2x\sqrt{2x-x^2}\,\mathrm{d}x\\
&= \frac{1}{2}+\int_0^1 x\sqrt{2x-x^2}\,\mathrm{d}x+\int_1^2 2x\sqrt{2x-x^2}\,\mathrm{d}x\\
&\overset{u=x-1}{=\!=\!=} \frac{1}{2}+\int_{-1}^0(u+1)\sqrt{1-u^2}\,\mathrm{d}u+\int_0^1 2(u+1)\sqrt{1-u^2}\,\mathrm{d}x\\
&\overset{u=\sin t}{=\!=\!=} \frac{1}{2}+\int_{-\frac{\pi}{2}}^0(\sin t+1)\cos t\,\mathrm{d}(\sin t)+\int_0^{\frac{\pi}{2}}2(\sin t+1)\cos t\,\mathrm{d}(\sin t)\\
&= \frac{1}{2}+\int_{-\frac{\pi}{2}}^0(\sin t\cos^2 t+\cos^2 t)\,\mathrm{d}t+\int_0^{\frac{\pi}{2}}2(\sin t\cos^2 t+\cos^2 t)\,\mathrm{d}t\\
&= \frac{1}{2}+\Big(-\frac{1}{3}\cos^3 t+\frac{\sin 2t}{4}+\frac{t}{2}\Big)\Big|_{-\frac{\pi}{2}}^0+2\Big(-\frac{1}{3}\cos^3 t+\frac{\sin 2t}{4}+\frac{t}{2}\Big)\Big|_0^{\frac{\pi}{2}}\\
&= \frac{1}{2}+\Big(\frac{\pi}{4}-\frac{1}{3}\Big)+2\Big(\frac{\pi}{4}+\frac{1}{3}\Big)=\frac{3}{4}\pi+\frac{5}{6},\\
|\Omega_2| &= \int_0^1\Big(\int_x^{\sqrt{2x-x^2}}(y-x)\,\mathrm{d}y\Big)\,\mathrm{d}x=\int_0^1\Big(\frac{1}{2}y^2-xy\Big)\Big|_x^{\sqrt{2x-x^2}}\,\mathrm{d}x\\
&= \int_0^1\Big(x-x\sqrt{2x-x^2}\Big)\,\mathrm{d}x=\frac{1}{2}-\int_0^1 x\sqrt{2x-x^2}\,\mathrm{d}x\\
&= \frac{1}{2}-\Big(\frac{\pi}{4}-\frac{1}{3}\Big)\\
&= \frac{5}{6}-\frac{\pi}{4}.
\end{aligned}$$