Electrodynamics

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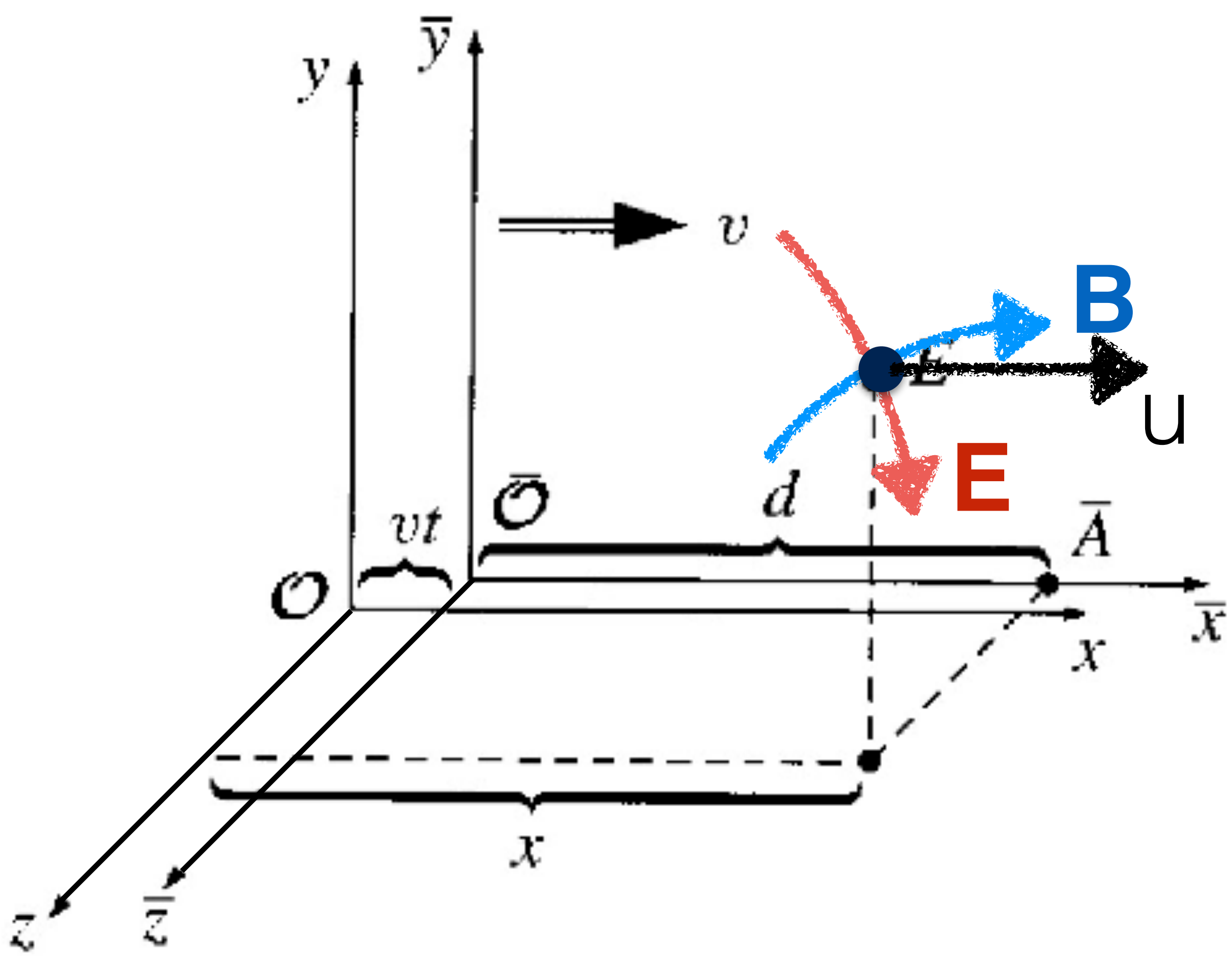
CHAPTER 12 Electrodynamics and Relativity: Griffiths

(12.3) Questions and Discussions

① If a charged particle moves at speed u along a wire with current I, the particle is suffering an electric force or a magnetic force?

② How do the electric and magnetic fields look like, if a charged particle moves with the speed of light?

12.3 Relativistic Electrodynamics

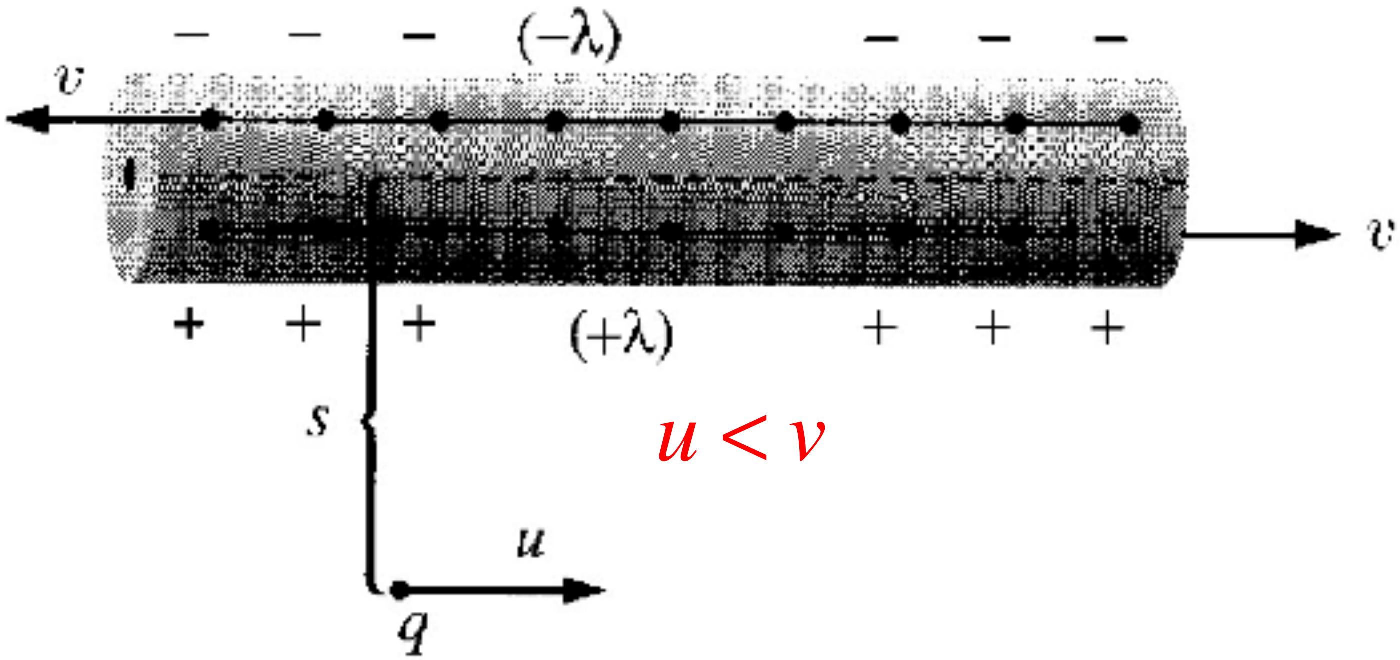


A complete and consistent formulation of relativistic electrodynamics; a deeper understanding of the structure of the electrodynamics: the coherence and inevitablity of the laws .

u: the velocity of a special object.
v: the velocity of a moving system.

12.3.1 Magnetism as a Relativistic Phenomenon

Why had there to be magnetism? With electrostatics and relativity, we can calculate the magnetic force between a current carrying wire and a moving charge without invoking the laws of magnetism.

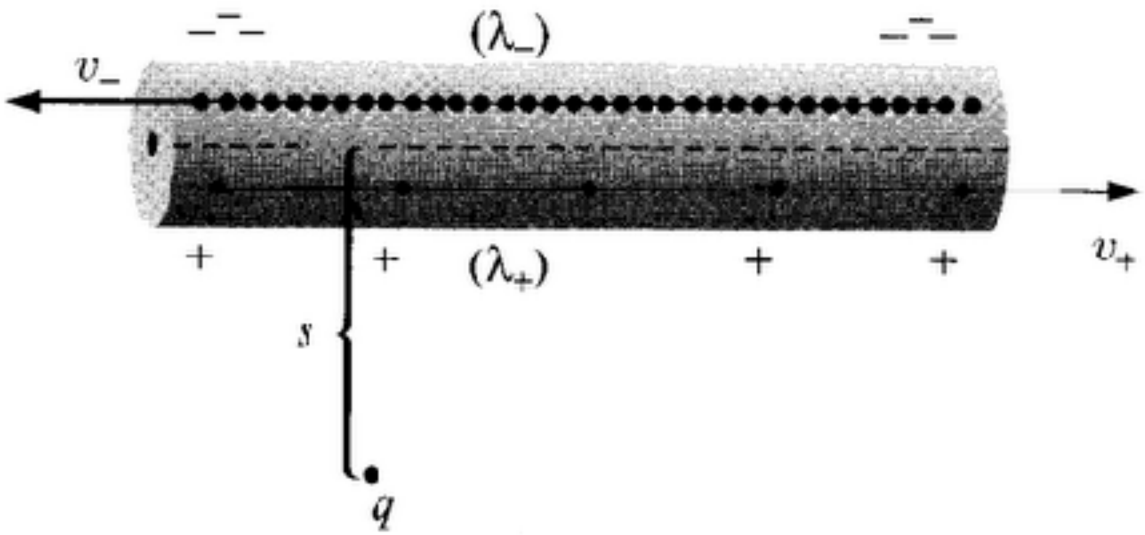


$u < v$

$I = 2\lambda v$

current in system S: the laboratory frame

There is no net charge in the wire, and therefore there is no electric force on the charge q in system S.



system $\bar{S}$: moving with speed u to the right, the frame of q rest

From the Einstein velocity addition rule:

$$v_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

The positive and negative charge densities are different:

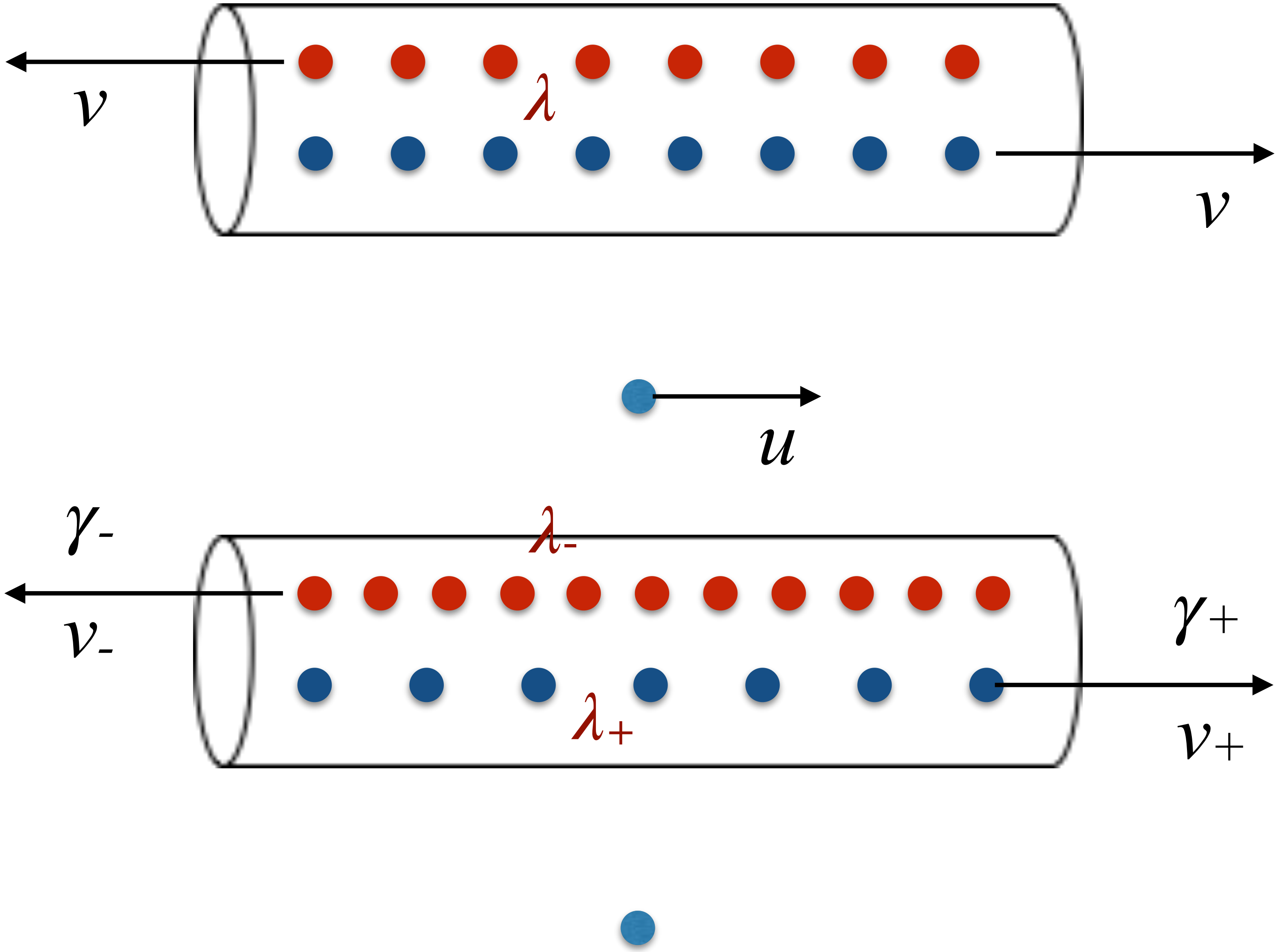
$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0$$

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}$$

And λ_0 is the charge density in its own moving frame.

$$\lambda = \gamma\lambda_0$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$



$$\gamma_{\pm} = \frac{1}{\sqrt{1-\frac{1}{c^2}(v\mp u)^2\left(1\mp vu/c^2\right)^{-2}}}$$

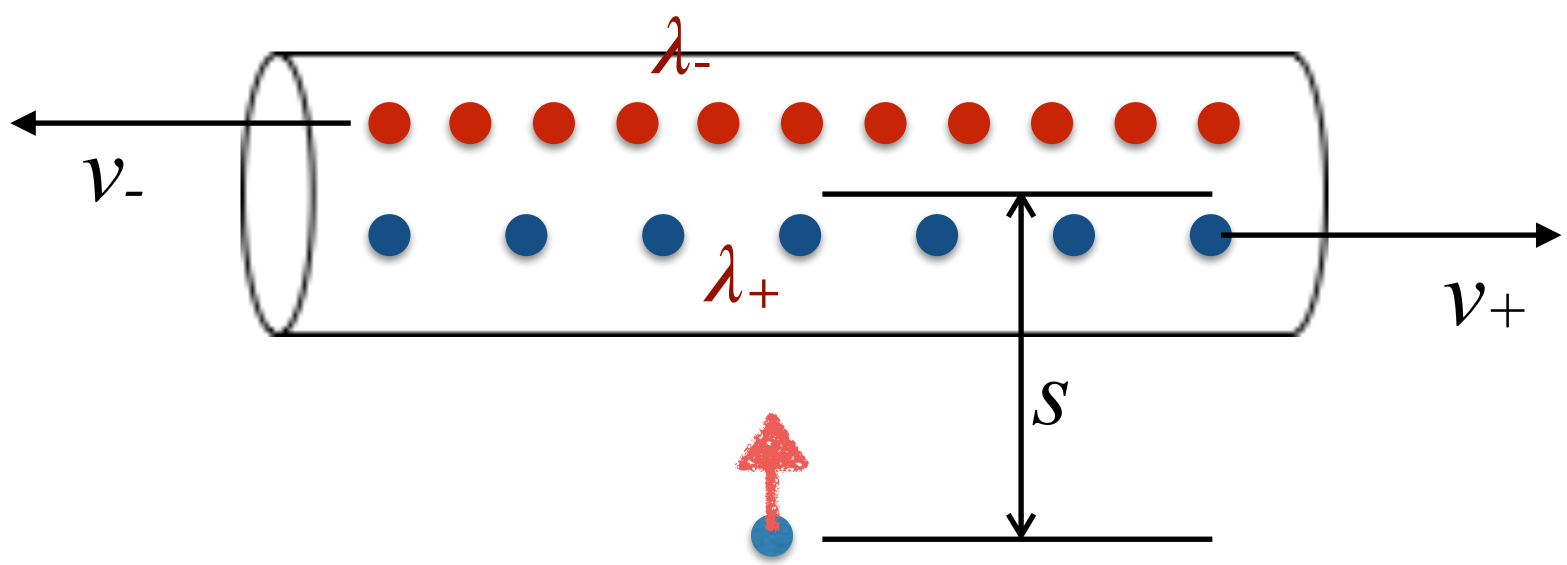
$$= \frac{c^2 \mp uv}{\sqrt{\left(c^2\mp uv\right)^2 - c^2\left(v\mp u\right)^2}}$$

$$= \frac{c^2 \mp uv}{\sqrt{\left(c^2 - v^2\right)\left(c^2-u^2\right)}}$$

$$= \gamma\frac{1\mp uv/c^2}{\sqrt{1-u^2/c^2}}$$

$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0\left(\gamma_+ - \gamma_-\right) = \frac{-2\lambda uv}{c^2\sqrt{1-u^2/c^2}}$$

In the frame of q, the wire has a net negative charge. And positive charge q is suffered an electric force from the wire.



The electric force in system $\bar{S}$:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\varepsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

The force in system S: the laboratory frame

$$F = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\frac{qu}{s}$$

$$I = 2\lambda v$$

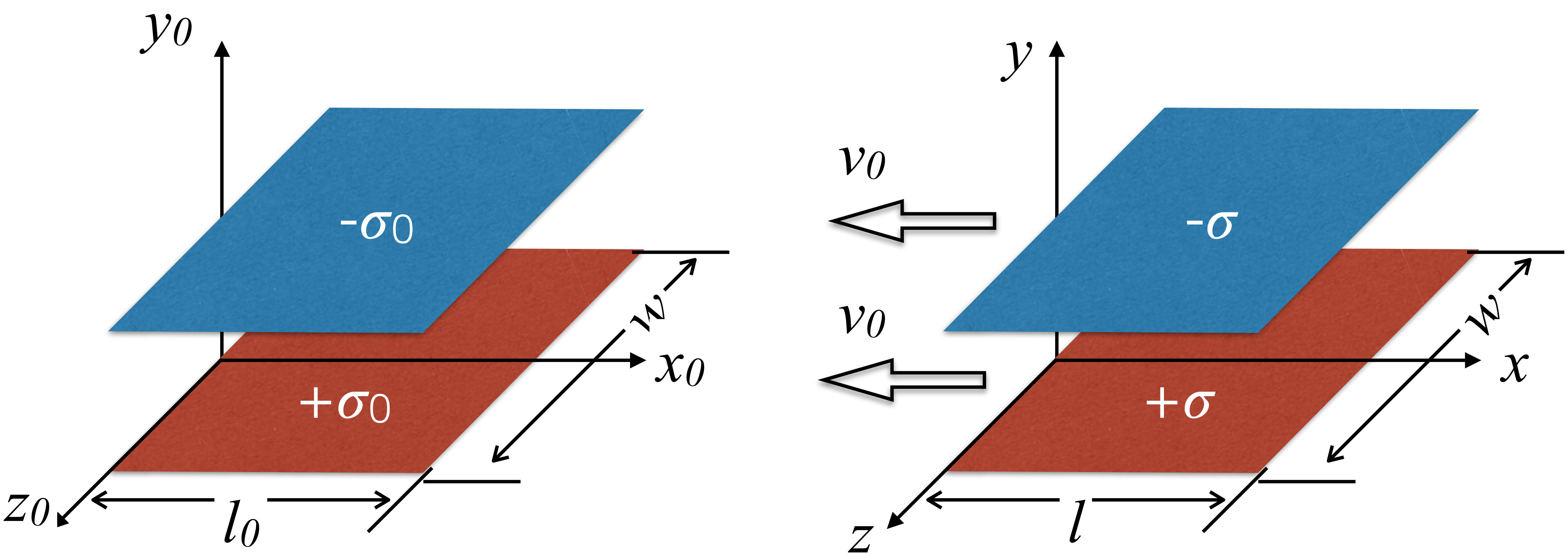
$$c^2 = 1/\varepsilon_0\mu_0$$

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

The force in S must not be electric force.
What?
Magnetic Force!

12.3.2 How the Fields Transform

- Two assumption:
  - ☐ Charge is invariant.
  - ☐ The transformation rules are the same no matter how the fields were produced.



$$\boldsymbol{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{\boldsymbol{y}} \qquad \boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{y}}$$

The total charge on each plate is invariant, and the width (w) is unchanged, but the length (l) is Lorentz-contracted by a factor:

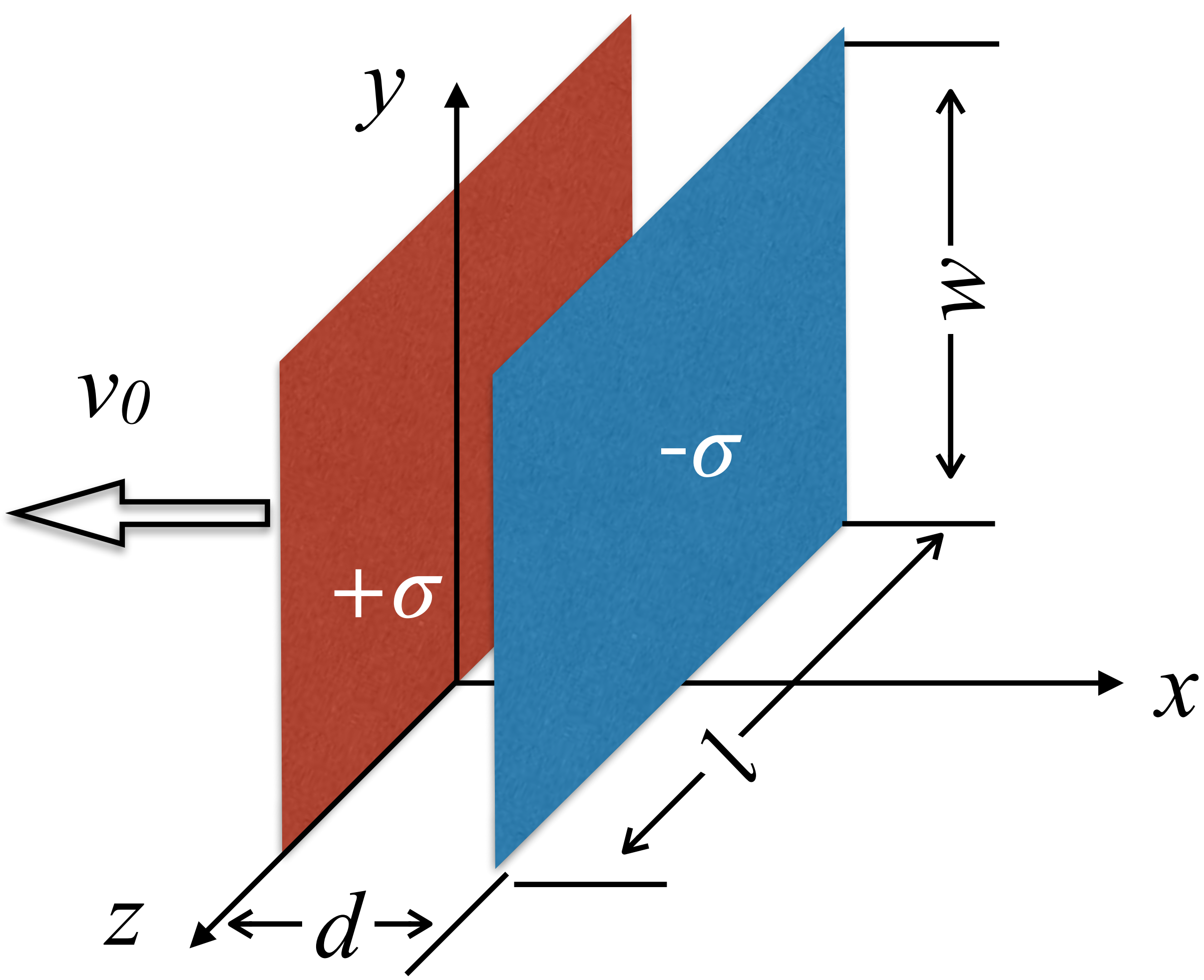
$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}$$

The surface charge density:

$$\sigma = \gamma_0 \sigma_0$$

The transverse electric fields:

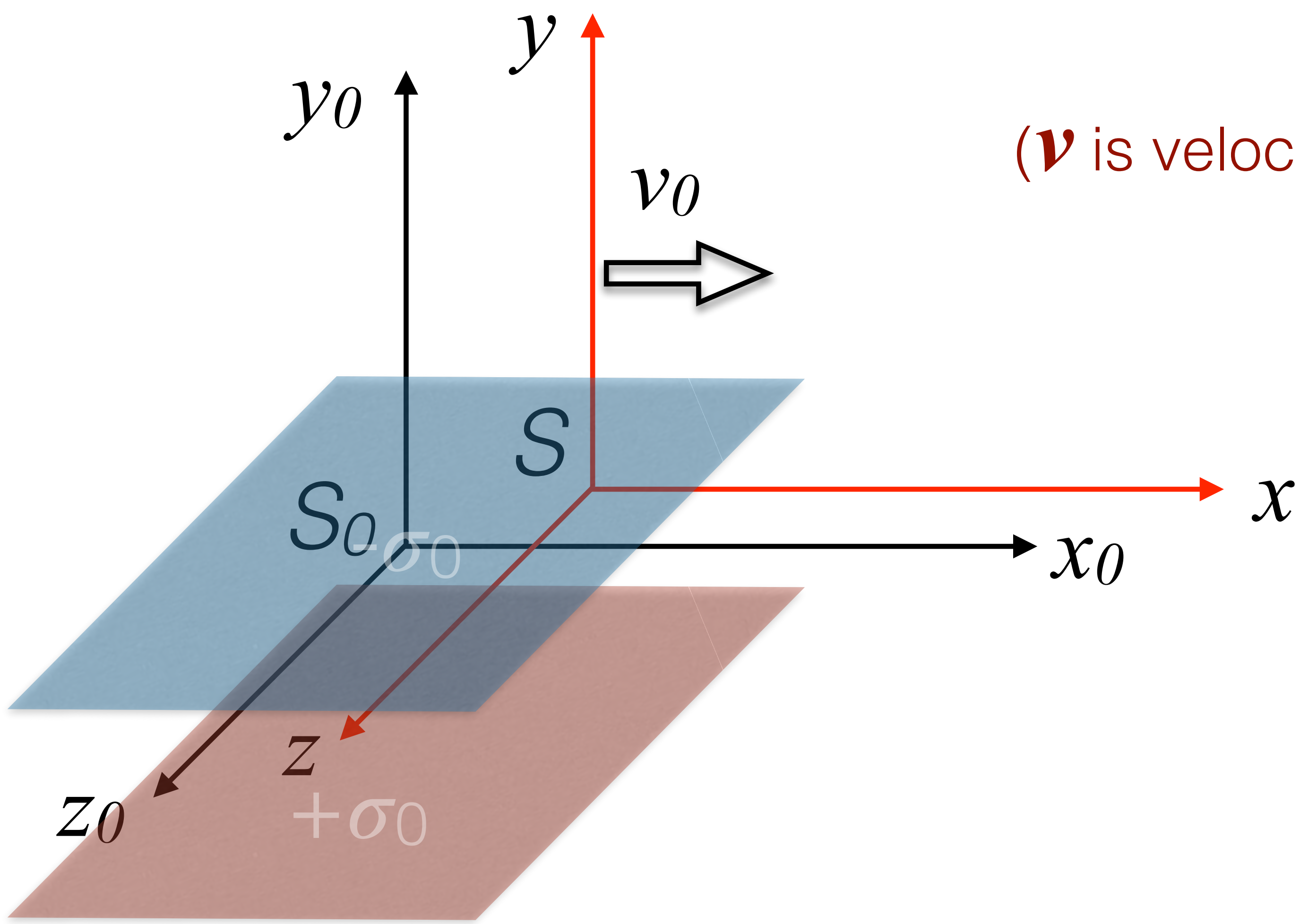
$$\boldsymbol{E}^{\perp} = \gamma_0 \boldsymbol{E}_0^{\perp}$$



The longitudinal electric fields:

$$E^{\parallel} = E_0^{\parallel}$$

General Situation: for a system with both electric and magnetic fields



($\boldsymbol{v}$ is velocity relative to S)

In the system S: In addition to the electric field,

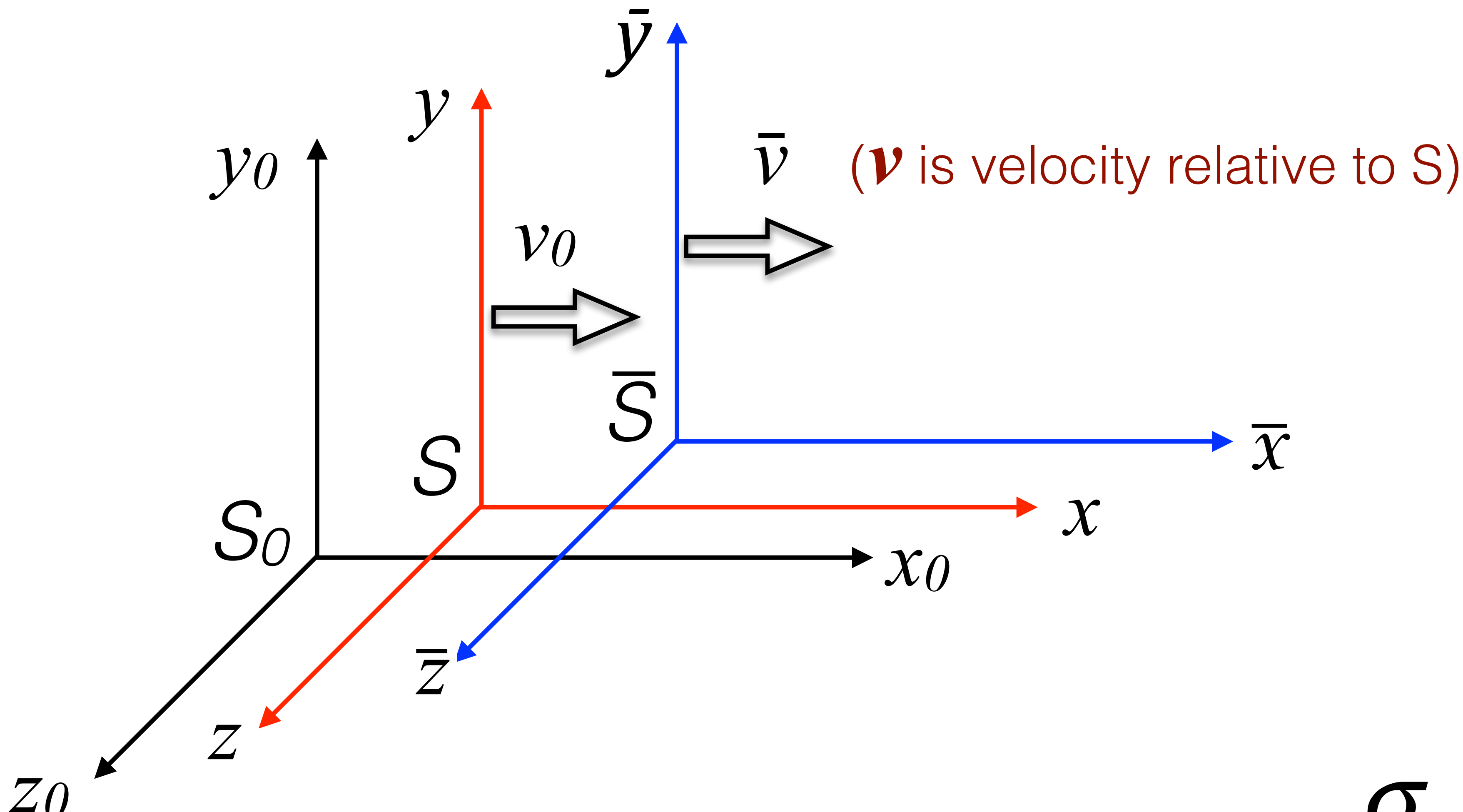
$$E_y = \frac{\sigma}{\varepsilon_0}$$

there is magnetic fields because of the surface current:

$$\boldsymbol{K}_{\pm} = \mp \sigma v_0 \hat{\boldsymbol{x}}$$

$$B_z = -\mu_0 \sigma v_0$$

In the system $\bar{S}$ with speed $\boldsymbol{v}$ relative to S



In the system S: In addition to the electric field, $E_y = \dfrac{\sigma}{\varepsilon_0}$

there is magnetic fields because of the surface current: $\boldsymbol{K}_{\pm} = \mp\sigma v_0 \hat{\boldsymbol{x}}$

$$B_z = -\mu_0 \sigma v_0$$

In the system $\bar{S}$ $\quad \bar{E} = ? \quad \bar{B} = ?$

In the system $\bar{S}$ with speed v relative to S

The fields will be: $$\bar{E}_y = \frac{\bar{\sigma}}{\varepsilon_0},\ \bar{B}_z = -\mu_0 \bar{\sigma}\bar{v}$$

$$\bar{v} = \frac{v + v_0}{1 + v v_0 / c^2},\ \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2 / c^2}}$$

$$\bar{\sigma} = \bar{\gamma}\sigma_0$$

$$\sigma = \gamma_0 \sigma_0 \quad \frac{1}{\gamma_0} = \sqrt{1 - v_0^2 / c^2}$$

$$\bar{\sigma} = \frac{\bar{\gamma}}{\gamma_0}\sigma$$

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0},\ \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0 \sigma \bar{v}$$

γ_0 is system S to S_0, and

$\bar{\gamma}$ is system $\bar{S}$ to S_0:

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \bar{v}^2/c^2}} = \frac{1 + vv_0/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(1 + \frac{vv_0}{c^2}\right)$$

$$\bar{v} = \frac{v + v_0}{1 + vv_0/c^2}$$

γ is system $\bar{S}$ to S:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0}$$

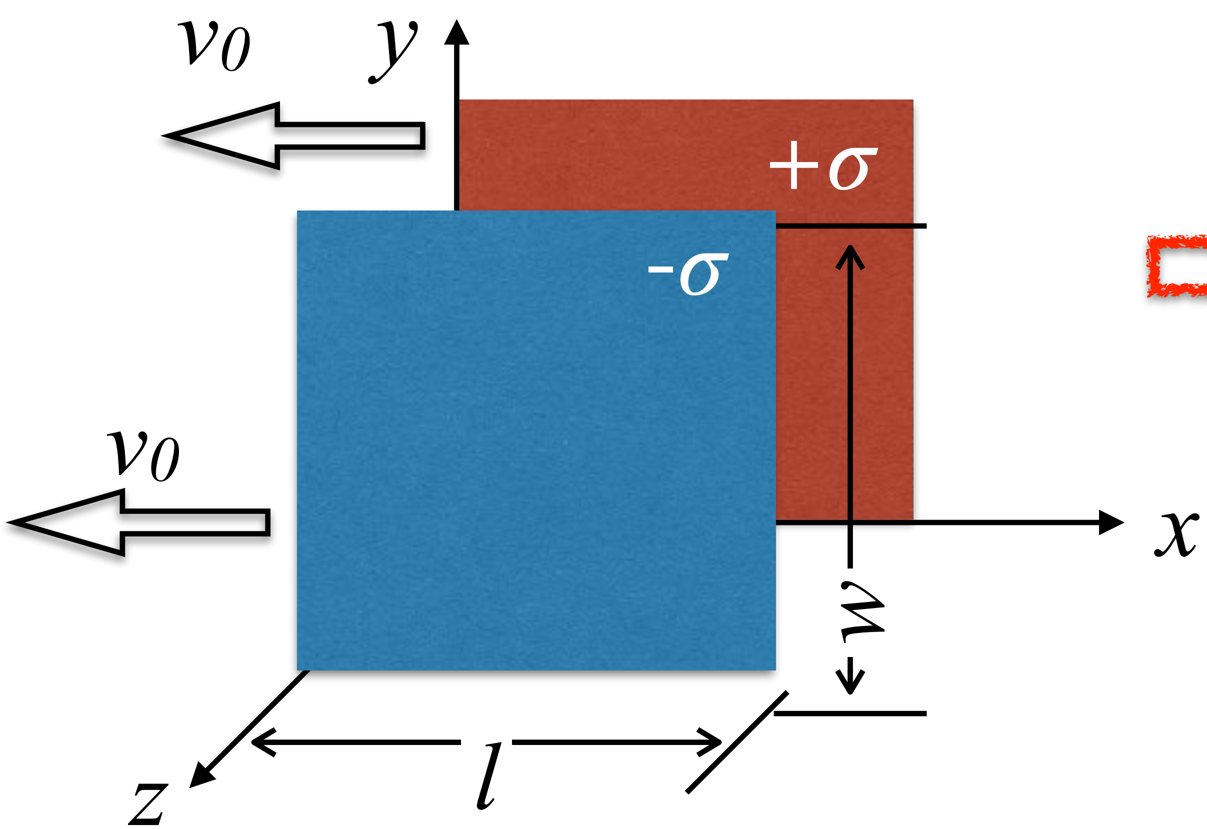
$$E_y = \frac{\sigma}{\varepsilon_0} \qquad B_z = -\mu_0 \sigma v_0$$

$$\bar{E}_y = \gamma\left(1 + \frac{vv_0}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(E_y - \frac{v}{c^2 \varepsilon_0 \mu_0} B_z\right)$$

$$\bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0 \sigma \bar{v} = -\gamma\left(1 + \frac{vv_0}{c^2}\right)\mu_0 \sigma \left(\frac{v + v_0}{1 + vv_0/c^2}\right) = \gamma\left(B_z - \varepsilon_0 \mu_0 v E_y\right)$$

With $\dfrac{1}{\varepsilon_0\mu_0} = c^2$

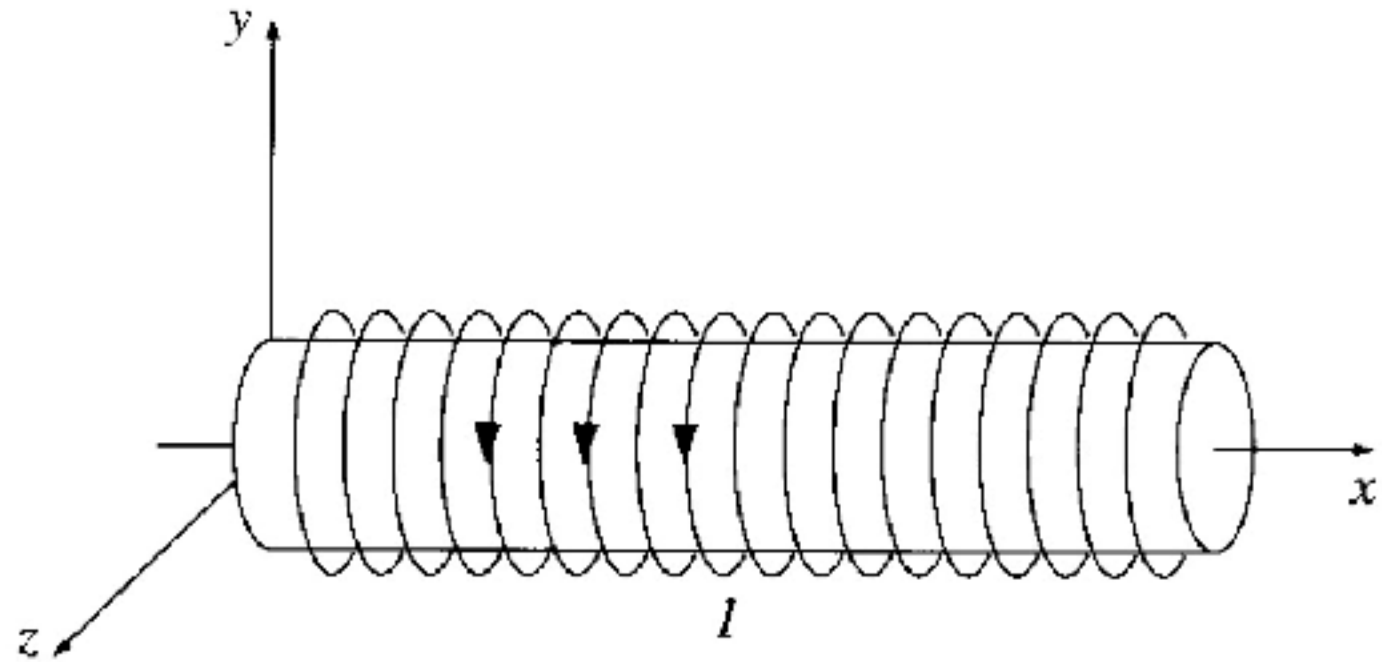
$$\left.\begin{aligned}\bar{E}_y &= \gamma\left(E_y - vB_z\right)\\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)\end{aligned}\right\}$$



$$\left.\begin{aligned}\bar{E}_z &= \gamma\left(E_z + vB_y\right)\\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right)\end{aligned}\right\}$$

Longitudinal component $\bar{E}_x = E_x$

Transformation of Bx:



$$B_x = \mu_0 n I$$

where, n is the number of turns per unit length, and I is the current.

In the system $\bar{S}$ Lorentz-contracted in x direction $\bar{n} = \gamma n$

Time dilates: $\bar{I} = \frac{1}{\gamma} I$

$$\Rightarrow \bar{B}_x = B_x$$

The complete set of transformation rules:

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - vB_z\right), \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

***B**=0* in system S:

$$\bar{\mathbf{B}} = \gamma\frac{v}{c^2}\left(E_z\hat{\mathbf{y}} - E_y\hat{\mathbf{z}}\right) = \frac{v}{c^2}\left(\bar{E}_z\hat{\mathbf{y}} - \bar{E}_y\hat{\mathbf{z}}\right)$$

$$\Downarrow \quad \mathbf{v} = v\hat{\mathbf{x}}$$

$$\bar{\mathbf{B}} = -\frac{1}{c^2}\left(\mathbf{v}\times\bar{\mathbf{E}}\right)$$

***E**=0* in system S:

$$\bar{\mathbf{E}} = -\gamma v\left(B_z\hat{\mathbf{y}} - B_y\hat{\mathbf{z}}\right) = -v\left(\bar{B}_z\hat{\mathbf{y}} - \bar{B}_y\hat{\mathbf{z}}\right)$$

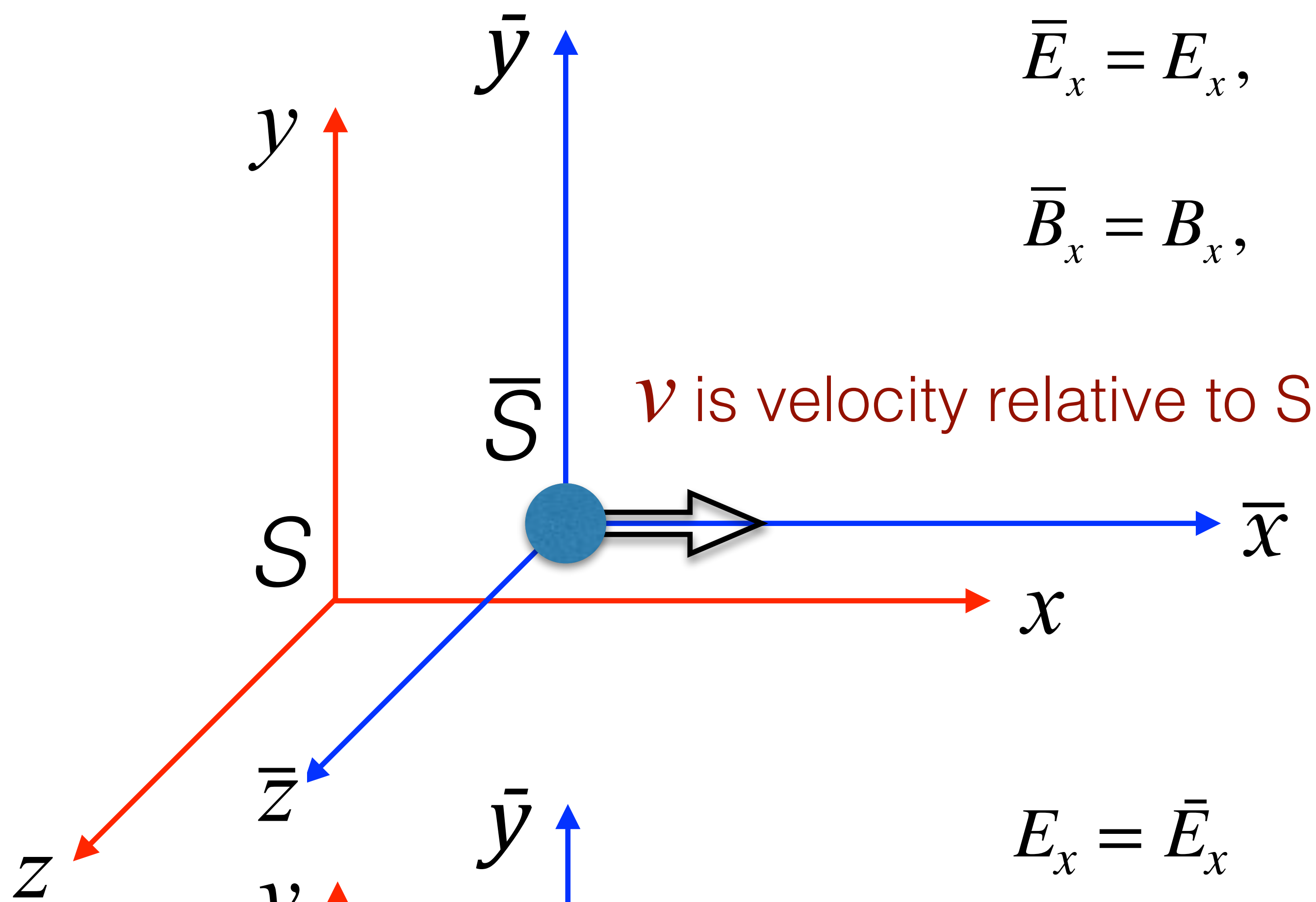
$$\Downarrow \quad \mathbf{v} = v\hat{\mathbf{x}}$$

$$\bar{\mathbf{E}} = \mathbf{v}\times\bar{\mathbf{B}}$$

Example: The electric and magnetic fields of a point charge moving with constant velocity v.

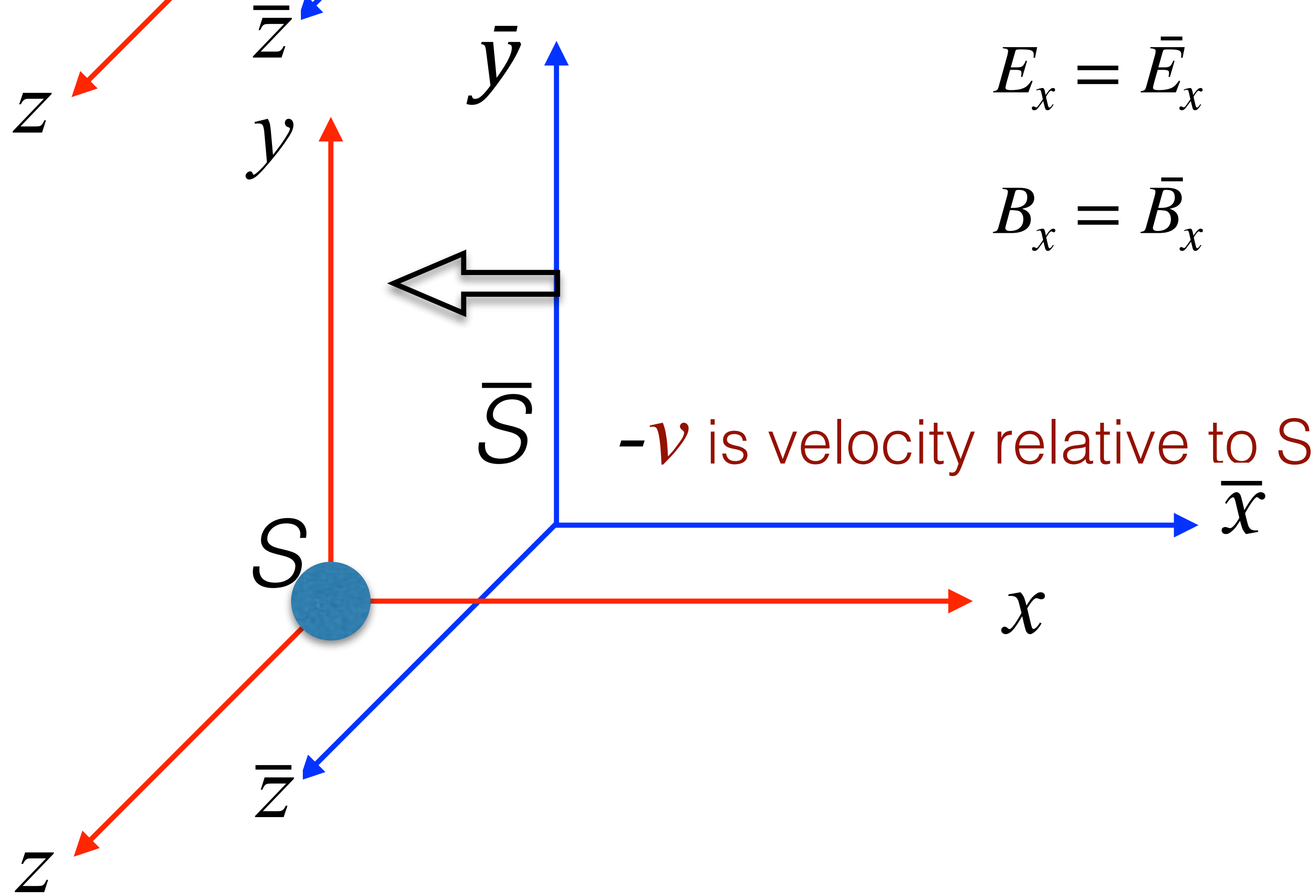
$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - vB_z\right), \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$



$$E_x = \bar{E}_x \qquad E_y = \gamma(\bar{E}_y + v\bar{B}_z) \qquad E_z = \gamma(\bar{E}_z - v\bar{B}_y)$$

$$B_x = \bar{B}_x \qquad B_y = \gamma(\bar{B}_y - \frac{v}{c^2}\bar{E}_z) \qquad B_z = \gamma(\bar{B}_z + \frac{v}{c^2}\bar{E}_y)$$



B=0 in system $\bar{S}$:

Electric fields in S:

$$E_x = \bar{E}_x, \quad E_y = \gamma \bar{E}_y, \quad \bar{E}_z = \gamma \bar{E}_z$$

In the rest frame: $\bar{\boldsymbol{E}} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r_0^2} \hat{\boldsymbol{r}}_0$

$$\begin{cases} \bar{E}_x = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\bar{x}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \\ \bar{E}_y = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\bar{y}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \\ \bar{E}_z = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\bar{z}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \end{cases}$$

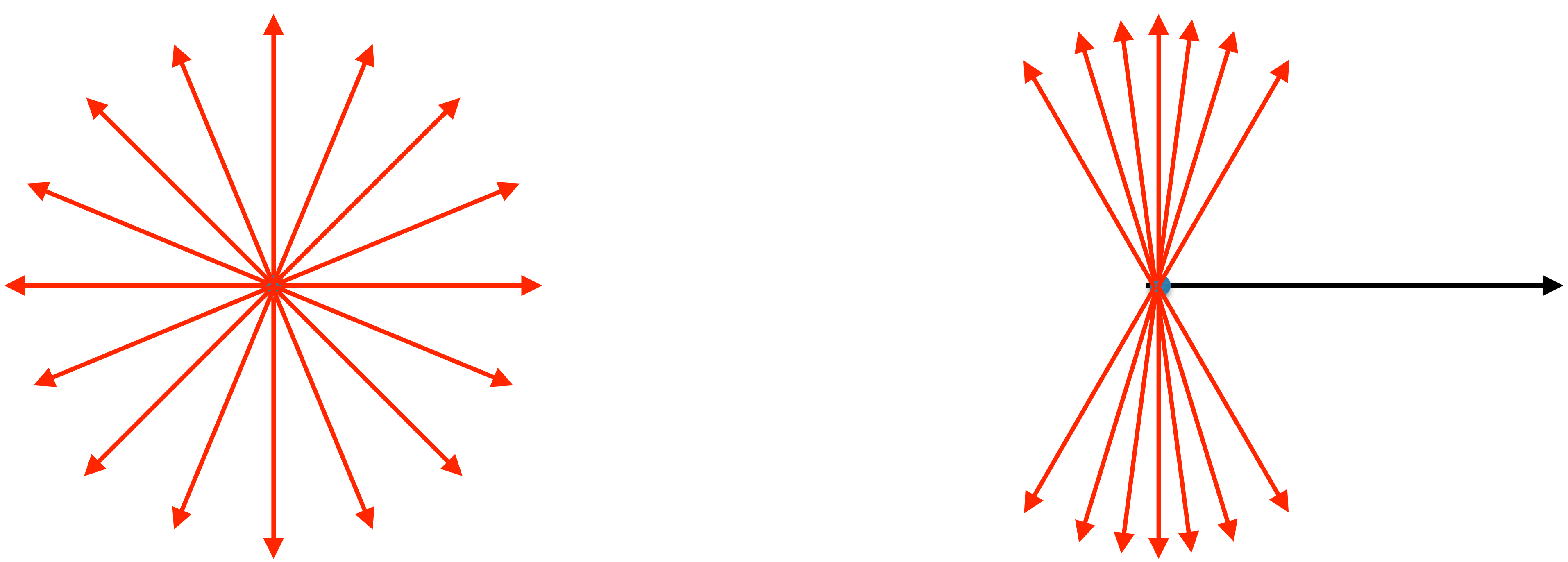
In frame S:

$$\begin{cases} E_x = \bar{E}_x = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\bar{x}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \\ E_y = \gamma\bar{E}_y = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\gamma q\bar{y}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \\ E_z = \gamma\bar{E}_z = \dfrac{1}{4\pi\varepsilon_0} \dfrac{\gamma q\bar{z}}{(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{3/2}} \end{cases}$$

$$\begin{cases} E_x = \bar{E}_x = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q\bar{x}}{(\bar{x}^2+\bar{y}^2+\bar{z}^2)^{3/2}} \\ E_y = \gamma\bar{E}_y = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma q\bar{y}}{(\bar{x}^2+\bar{y}^2+\bar{z}^2)^{3/2}} \\ E_z = \gamma\bar{E}_z = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\gamma q\bar{z}}{(\bar{x}^2+\bar{y}^2+\bar{z}^2)^{3/2}} \end{cases}$$

$$\begin{cases} \bar{x} = \gamma(x+vt) = \gamma R_x \\ \bar{y} = y = R_y \\ \bar{z} = z = R_z \end{cases}$$

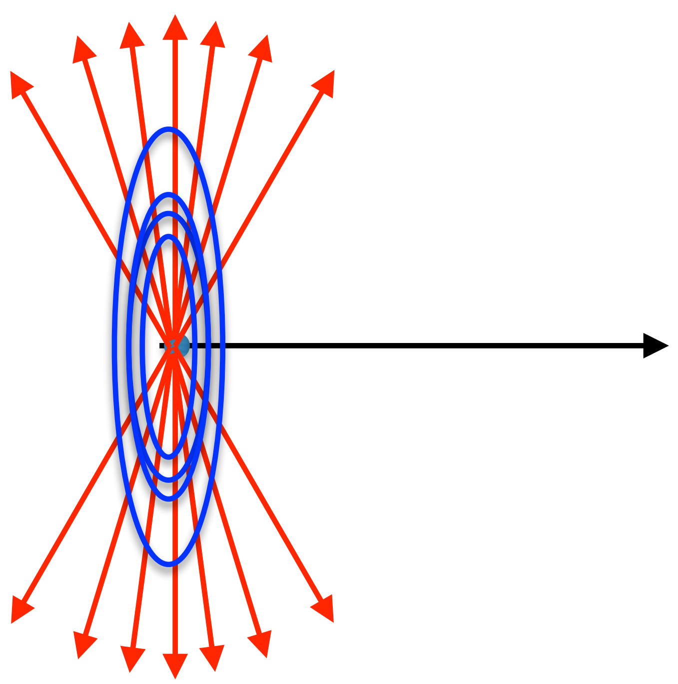
$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q\boldsymbol{R}}{\left(\gamma^2R^2\cos^2\theta + R^2\sin^2\theta\right)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{q\left(1-v^2/c^2\right)}{\left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{3/2}}\frac{\hat{\boldsymbol{R}}}{R^2}$$



Magnetic fields in S:

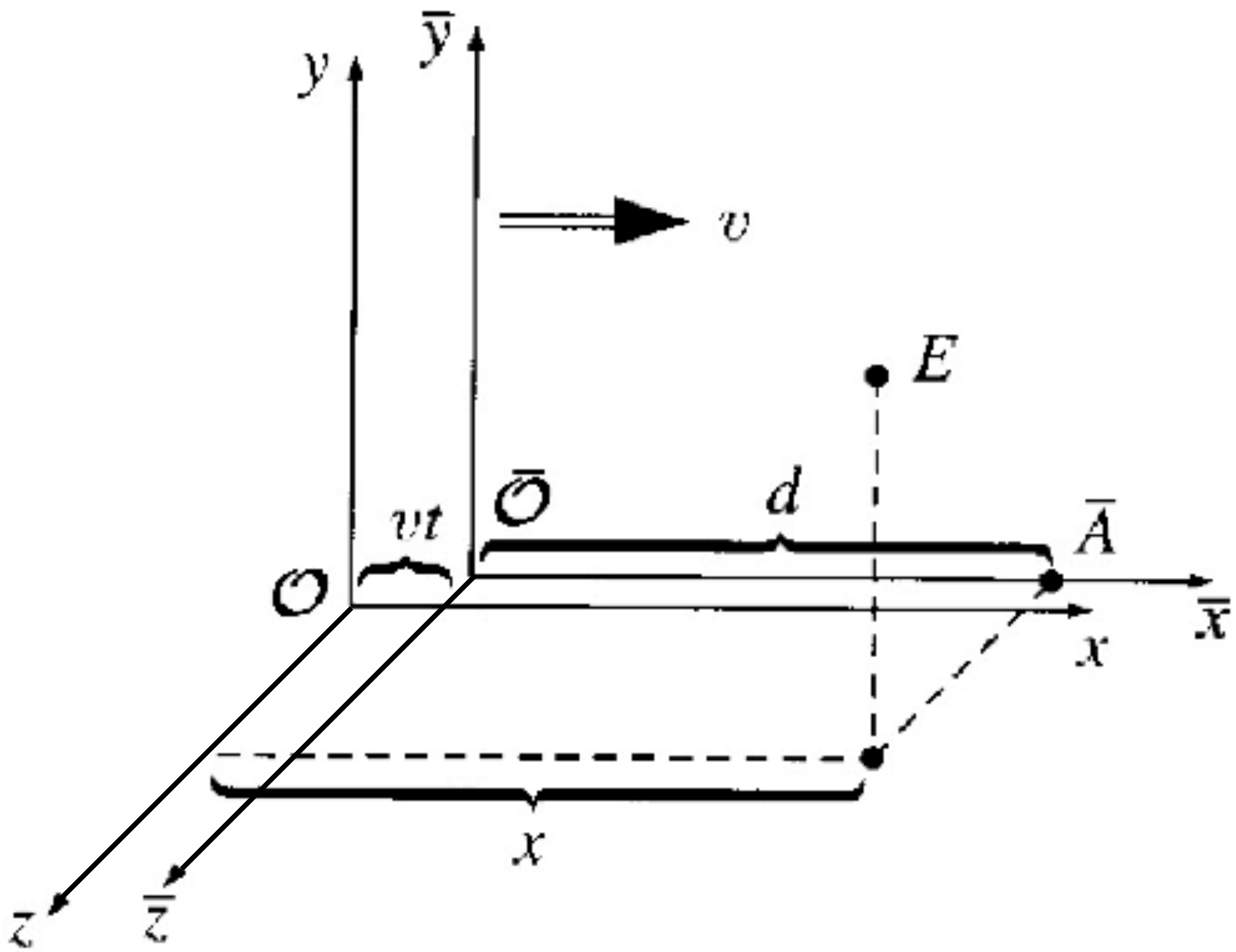
$$\boldsymbol{B} = \frac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E})$$

$$B = \frac{\mu_0}{4\pi} \frac{qv\left(1 - v^2/c^2\right)\sin\theta}{\left[1 - \left(v^2/c^2\right)\sin^2\theta\right]^{3/2}} \frac{\hat{\phi}}{R^2}$$



12.3.3 The Field Tensor

- Lorentz Transformations



$$\bar{a}^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

second-rank tensor.

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

for a 4 dimension tensor

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}$$

$$t^{\mu\nu} = t^{\nu\mu} \quad \text{(symmetric tensor)}$$

$$t^{\mu\nu} = -t^{\nu\mu} \quad \text{(antisymmetric tensor)}$$

The transformation of electromagnetic fields is connected by an **antisymmetric, second-rank tensor**.

$$t^{\mu\nu} = \begin{bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{bmatrix}$$

How to transfer?

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

$$\bar{t}^{01} = \Lambda^{0}_{\lambda}\Lambda^{1}_{\sigma}t^{\lambda\sigma}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow$$

$$\begin{aligned} \bar{t}^{01} &= \Lambda^{0}_{0}\Lambda^{1}_{0}t^{00} + \Lambda^{0}_{0}\Lambda^{1}_{1}t^{01} + \Lambda^{0}_{1}\Lambda^{1}_{0}t^{10} + \Lambda^{0}_{1}\Lambda^{1}_{1}t^{11} \\ &= (\Lambda^{0}_{0}\Lambda^{1}_{1} - \Lambda^{0}_{1}\Lambda^{1}_{0})t^{01} \\ &= (\gamma^{2} - \gamma^{2}\beta^{2})t^{01} \\ &= t^{01} \end{aligned}$$

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{t}^{02} = \Lambda^{0}_{\lambda}\Lambda^{2}_{\sigma}t^{\lambda\sigma}$$

$$\bar{t}^{02} = \Lambda^{0}_{0}\Lambda^{2}_{2}t^{02} + \Lambda^{0}_{1}\Lambda^{2}_{2}t^{12}$$

$$\bar{t}^{02} = \gamma t^{02} - \gamma\beta t^{12} = \gamma(t^{02} - \beta t^{12})$$

- The complete set of transformation rules:

$$\left.\begin{array}{l} \bar{t}^{01} = t^{01}, \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right), \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{31}\right) \\ \bar{t}^{23} = t^{23}, \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right), \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right) \end{array}\right\}$$

Compare with the fields transformation:

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - vB_z\right), \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

The Fields Tensor:

$$\bar{t}^{01} = t^{01}, \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right), \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{31}\right)$$

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - vB_z\right), \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)$$

$$\bar{t}^{23} = t^{23}, \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right), \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right)$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

$$\bar{F}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}F^{\lambda\sigma}$$

Another Fields Tensor: **Dual Tensor**

$$\left.\begin{array}{l}\bar{t}^{01} = t^{01}, \bar{t}^{02} = \gamma\left(t^{02} - \beta t^{12}\right), \bar{t}^{03} = \gamma\left(t^{03} + \beta t^{31}\right) \\ \bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\end{array}\right\}$$

$$\left.\begin{array}{l}\bar{t}^{23} = t^{23}, \bar{t}^{31} = \gamma\left(t^{31} + \beta t^{03}\right), \bar{t}^{12} = \gamma\left(t^{12} - \beta t^{02}\right) \\ \bar{E}_x = E_x, \quad \bar{E}_y = \gamma\left(E_y - vB_z\right), \quad \bar{E}_z = \gamma\left(E_z + vB_y\right)\end{array}\right\}$$

$$\Downarrow$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

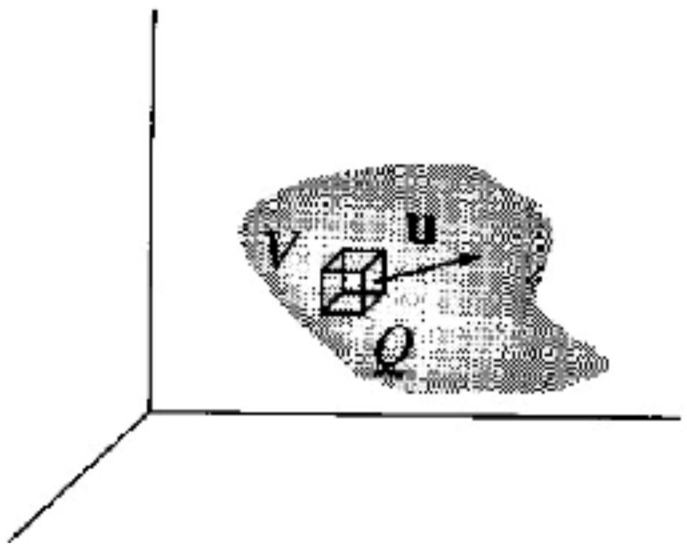
$$\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$$

$$\bar{G}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}G^{\lambda\sigma}$$

12.3.4 Electrodynamics in Tensor Notation

Reformulate the laws of electrodynamics (Maxwell's Equations and the Lorentz Force Law) in the language of tensor.

- The transform of sources of fields:



Charge density: $$\rho = \frac{Q}{V}$$

Current density: $$\boldsymbol{J} = \rho \boldsymbol{u}$$

Proper charge density: $$\rho_0 = \frac{Q}{V_0}$$ the density in the rest system of the charge.

Only one dimension of the volume is Lorentz-contracted: $$V = \sqrt{1-u^2/c^2}\,V_0$$

$$\Rightarrow \quad \rho = \rho_0 \frac{1}{\sqrt{1-u^2/c^2}}, \quad \boldsymbol{J} = \rho_0 \frac{\boldsymbol{u}}{\sqrt{1-u^2/c^2}}$$

Current Density 4-Vector

$$\rho = \rho_0 \frac{1}{\sqrt{1-u^2/c^2}}, \quad \boldsymbol{J} = \rho_0 \frac{\boldsymbol{u}}{\sqrt{1-u^2/c^2}} \qquad \boldsymbol{\eta} = \frac{1}{\sqrt{1-u^2/c^2}}\boldsymbol{u}$$

$$\eta^0 = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{c}{\sqrt{1-u^2/c^2}} \qquad \eta^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

$$J^\mu = \rho_0 \eta^\mu$$

$$\boxed{J^\mu = \left(c\rho, J_x, J_y, J_z\right)}$$

Continuous Equation: $\nabla \bullet \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$

$$\nabla \bullet \boldsymbol{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i}$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$$

$$\boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$$

Four dimension divergence of J^μ

Maxwell's Equations

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \qquad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

μ=0:

$$\begin{aligned} \frac{\partial F^{0\nu}}{\partial x^\nu} &= \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} \\ &= \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \bullet \boldsymbol{E}) \\ &= \mu_0 J^0 = \mu_0 c\rho \end{aligned}$$

$$\Rightarrow \boxed{\nabla \bullet \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho}$$

$$\boxed{\nabla \bullet \boldsymbol{D} = \rho}$$

$$\begin{aligned} \frac{\partial G^{0\nu}}{\partial x^\nu} &= \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} \\ &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \bullet \boldsymbol{B} = 0 \end{aligned}$$

$$\Rightarrow \boxed{\nabla \bullet \boldsymbol{B} = 0}$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \qquad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$\mu=1$:

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \nabla \times \boldsymbol{B}\right)_x$$

$$= \mu_0 J^1 = \mu_0 J_x$$

μ=2,3 give the y and z components

$$\boxed{\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}}$$

$$\boxed{\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}}$$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}\right)_x = 0$$

μ=2,3 give the y and z components

$$\boxed{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}}$$

Lorentz Force Law:

Minkowski Force: $\boldsymbol{K} = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{1}{\sqrt{1-u^2/c^2}}\boldsymbol{F}$ $\quad K^0 = \frac{\mathrm{d}p^0}{\mathrm{d}\tau} = \frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau}$

The Minkowski Force on a charge q: $K^\mu = q\eta_\nu F^{\mu\nu}$

μ=1:

$$K^1 = q\eta_\nu F^{1\nu} = q(\eta_0 F^{10} + \eta_1 F^{11} + \eta_2 F^{12} + \eta_3 F^{13})$$

$$= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}B_z + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right]$$

$$= \frac{q}{\sqrt{1-u^2/c^2}}[\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}]_x$$

μ=2,3 give the y and z components

$$\boxed{\boldsymbol{K} = \frac{q}{\sqrt{1-u^2/c^2}}\left[\boldsymbol{E} + (\boldsymbol{u}\times\boldsymbol{B})\right]} \Rightarrow \boxed{\boldsymbol{F} = q\left[\boldsymbol{E} + (\boldsymbol{u}\times\boldsymbol{B})\right]}$$

12.3.5 Relativistic Potentials

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}, \;\; \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

- Four-Vector Potential:

$$A^{\mu} = \left(\Phi/c, A_x, A_y, A_z\right)$$

- Field Tensor:

$$\boxed{F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$\mu=0, \nu=1$:

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{\partial(\Phi/c)}{\partial x}$$

$$= -\frac{1}{c}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right)_x = \frac{E_x}{c}$$

$\mu=1, \nu=2$:

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \left(\nabla \times \boldsymbol{A}\right)_z = B_z$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^{\nu}}=\mu_0 J^{\mu},\quad \frac{\partial G^{\mu\nu}}{\partial x^{\nu}}=0} \qquad \boxed{F^{\mu\nu}=\frac{\partial A^{\nu}}{\partial x_{\mu}}-\frac{\partial A^{\mu}}{\partial x_{\nu}}}$$

$$\frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right)-\frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right)=\mu_0 J^{\mu}$$

- Gauge invariance:

$$A^{\mu}\rightarrow A^{\mu'}=A^{\mu}+\frac{\partial\lambda}{\partial x_{\mu}}$$

the additional scalar function will not change $F^{\mu\nu}$

The Lorentz gauge condition: $\nabla\cdot A+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0 \Rightarrow \frac{\partial A^{\mu}}{\partial x^{\mu}}=0$

The simplest formula of Maxwell Equations: $\square^2 A^{\mu}=-\mu_0 J^{\mu}$ $\quad x_0=-x^0=-ct$

d'Alembertian

达朗贝尔算符

$$\square^2\equiv\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x^{\nu}}=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Homework

P534: Problem 12.45