第 9 次作业题

1. 比较下列积分的大小:
(1) $\int_0^1 x\,\mathrm{d}x$ 和 $\int_0^1 x^2\,\mathrm{d}x$, (2) $\int_0^{\frac{\pi}{2}} x\,\mathrm{d}x$ 和 $\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x$.

2. 求证: $\frac{1}{2} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\,\mathrm{d}x < \frac{\sqrt{2}}{2}$.

3. 求证: 若 $f, g \in \mathscr{R}[a,b]$, 则 $\min(f,g), \max(f,g) \in \mathscr{R}[a,b]$.

4. 若 $f \in \mathscr{C}[a,b]$ 且 $\forall x \in [a,b]$, 均有 $f(x) > 0$, 求证:

$$\Big(\int_a^b f(x)\,\mathrm{d}x\Big)\Big(\int_a^b \frac{\mathrm{d}x}{f(x)}\Big) \geqslant (b-a)^2.$$

5. 求证: $\lim\limits_{n\to\infty} \int_{n^2}^{n^2+n} \frac{\mathrm{d}x}{\sqrt{x}e^{\frac{1}{x}}} = 1$.

6. 求下列函数的导函数:
(1) $F(x) = \int_{\sqrt{x}}^{x^2} e^{-t^2}\,\mathrm{d}t$, (2) $F(x) = \int_0^{\arctan x} \tan t\,\mathrm{d}t$.

7. 函数 $y = y(x)$ 由方程 $\int_0^y e^{-t^2}\,\mathrm{d}t + \int_0^x \cos t^2\,\mathrm{d}t = 0$ 确定, 求 $y'(x)$.

8. 设曲线 $y = y(x)$ 由方程 $x = \int_1^t \frac{\cos u}{u}\mathrm{d}u$, $y = \int_1^t \frac{\sin u}{u}\mathrm{d}u$ 来确定, 求该曲线在 $t = \frac{\pi}{4}$ 时的斜率.

9. 若 $f \in \mathscr{C}[0,+\infty)$ 使得 $\forall x \geqslant 0$, 均有 $\int_0^{\sqrt{x}} f(t)\,\mathrm{d}t = x + \sin x$, 求 $f(x)$.

10. $\forall x \in \mathbb{R}$, 定义 $F(x) = \int_0^x te^{-t^2}\,\mathrm{d}t$ 的极值点与拐点的横坐标.

11. 求下列极限:
(1) $\lim\limits_{x\to+\infty} \frac{\int_0^x \arctan t^2\,\mathrm{d}t}{\sqrt{1+x^2}}$, (2) $\lim\limits_{x\to 0} \frac{\int_{\sin x}^x \sqrt{1-t^2}\,\mathrm{d}t}{x^3}$.

12. 设 $f(x) = \begin{cases} x+1, & \text{若 } x \in [-1,0) \\ x, & \text{若 } x \in [0,1] \end{cases}$. $\forall x \in [-1,1]$, 令 $F(x) = \int_{-1}^x f(t)\,\mathrm{d}t$. 讨论函数 F 的连续性与可导性.

13. 若 $f \in \mathscr{C}^{(2)}[a,b]$, 求证: $\exists \xi \in [a,b]$ 使得

$$\int_a^b f(x)\,\mathrm{d}x = f\big(\frac{a+b}{2}\big)(b-a) + \frac{(b-a)^3}{24} f''(\xi).$$

14. 若 $f \in \mathscr{R}[a,b]$ 在 (a,b) 内连续, 求证: $\exists \xi \in (a,b)$ 使得

$$\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b-a).$$

15. 求证: $\lim\limits_{n\to\infty}\int_0^1 \frac{\mathrm{d}x}{1+x^n}=1$.

16. 问下列函数在 $(-\infty,+\infty)$ 上是否有原函数? 若有, 求出原函数, 若没有, 请说明理由.

$$(1)\ f(x)=\begin{cases} x^2+1, & \text{若 } x\leqslant 0 \\ \cos x, & \text{若 } x>0 \end{cases},\quad (2)\ f(x)=\begin{cases} x^2+1, & \text{若 } x\leqslant 0 \\ \cos x+\frac{\pi}{4}, & \text{若 } x>0 \end{cases}.$$

17. 求下列不定积分:

(1) $\int (x-x^{-2})\sqrt{x\sqrt{x}}\,\mathrm{d}x$,
(2) $\int (1-2\cot^2 x)\,\mathrm{d}x$,
(3) $\int \left(\frac{4}{\sqrt{1-x^2}}+\sin x\right)\mathrm{d}x$,
(4) $\int |(x-1)(3x-2)|\,\mathrm{d}x$,
(5) $\int \frac{\mathrm{d}x}{(1+x^2)\arctan x}$,
(6) $\int \frac{1}{x^2}\mathrm{sh}\,\frac{1}{x}\,\mathrm{d}x$,
(7) $\int \frac{x}{\sqrt{1+x^2}}\sin\sqrt{1+x^2}\,\mathrm{d}x$,
(8) $\int \frac{\mathrm{d}x}{e^x+e^{-x}}$,
(9) $\int \sec x\,\mathrm{d}x$,
(10) $\int \frac{x^2}{\sqrt{a^2+x^2}}\mathrm{d}x\ (a>0)$,
(11) $\int \frac{\sqrt{x^2-4}}{x}\mathrm{d}x$,
(12) $\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}}$,
(13) $\int \frac{2x-1}{\sqrt{4x^2+4x+5}}\mathrm{d}x$.