1. $\begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix}\begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}$

$= \begin{pmatrix} 5\times3+8\times4-4\times9 & 5\times2-8\times1-4\times6 & 5\times5+8\times3-4\times5 \\ 6\times3+9\times4-5\times9 & 6\times2-9\times1-5\times6 & 5\times6+9\times3-5\times5 \\ 4\times3+7\times4-3\times9 & 4\times2-7\times1-3\times6 & 4\times5+7\times3-3\times5 \end{pmatrix}$

$= \begin{pmatrix} 11 & -22 & 29 \\ 9 & -27 & 32 \\ 13 & -17 & 26 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

& $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$

$\Downarrow$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

3. 略

4. $H_1 H_2 H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$H_2 H_1 H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

5. $$A^3 - 6A^2 + 10A - 4I = (A-2I)^3 - 2A + 4I$$
$$= (A-2I)^3 - 2(A-2I)$$
$$= ((A-2I)^2 - 2I)(A-2I)$$
$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= 0$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

注意顺序不要写错

$$A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(A-2I)^2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(A-2I)^2 - 2I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

6. 1) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Generally, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$

$$\begin{cases} a^2+bc=1 \\ d^2+bc=1 \\ b(a+d)=0 \\ c(a+d)=0 \end{cases}$$

$\begin{pmatrix} 6 & 5 \\ -5 & -6 \end{pmatrix}\begin{pmatrix} 6 & 5 \\ -5 & -6 \end{pmatrix} = \begin{pmatrix} 36-25 & 0 \\ 0 & - \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

2) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

7. $\begin{pmatrix} M(0) & M(\alpha) & M(\frac{\pi}{2}) \\ M(\beta) & M(0) & S \end{pmatrix}\begin{pmatrix} M(\gamma) \\ S \\ I \end{pmatrix}$

$= \begin{pmatrix} M(\gamma)+M(\alpha)S+M(\frac{\pi}{2}) \\ M(\beta+\gamma)+S+S \end{pmatrix}$

$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} \cos\alpha & \cos\alpha+\sin\alpha \\ -\sin\alpha & -\sin\alpha+\cos\alpha \end{pmatrix}$

$= \begin{pmatrix} \cos\gamma+\cos\alpha & \sin\gamma+\cos\alpha+\sin\alpha+1 \\ -\sin\gamma-\sin\alpha-1 & \cos\gamma-\sin\alpha+\cos\alpha \\ \cos(\beta+\gamma)+2 & \sin(\beta+\gamma)+2 \\ -\sin(\beta+\gamma) & \cos(\beta+\gamma)+2 \end{pmatrix}$

8. $h_1 = \begin{pmatrix} H_1 & 0 \\ 0 & 1 \end{pmatrix}$, $h_2 = \begin{pmatrix} H_2 & 0 \\ 0 & 1 \end{pmatrix}$, $h_3 = \begin{pmatrix} 1 & 0 \\ 0 & H_{\cancel{3}\,2} \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & H_1 \end{pmatrix}$

$$h_1h_2h_1 = \begin{pmatrix} H_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} H_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} H_1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} H_1H_2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} H_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H_1H_2H_1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h_2h_1h_2 = \begin{pmatrix} H_2H_1H_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h_2h_3h_2 = \begin{pmatrix} 1 & 0 \\ 0 & H_1H_2H_1 \end{pmatrix}, \quad h_3h_2h_3 = \begin{pmatrix} 1 & 0 \\ 0 & H_2H_1H_2 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad h_3 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

$$h_1h_3 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$h_3h_1 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

9. $\because \forall B \in Mat_{n\times n}(\mathbb{R})$ s.t. $[A, B] = 0$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{array}{c} \\ t \\ \\ \end{array} \begin{pmatrix} 0 & \cdots & \overset{s}{0} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

$B = E_{st}$

$$\begin{array}{c} t \\ \\ \end{array} \begin{pmatrix} 0 & \cdots & \overset{s}{0} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & a_{1t} & \cdots & 0 \\ 0 & \cdots & a_{2t} & \cdots & 0 \\ \vdots & & \vdots & & \\ 0 & \cdots & a_{nt} & \cdots & 0 \end{pmatrix}$$

$$BA = \begin{array}{c|c} & \quad s \\ t & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} = \begin{array}{cc} t & \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ a_{s1} & \cdots a_{st} & \cdots & a_{sn} \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \end{array}$$

$$[A, E_{st}] = \begin{array}{cc} & s \\ t & \begin{pmatrix} 0 & \cdots & a_{1t} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_{s1} & \cdots & -a_{ss}+a_{tt} & \cdots & -a_{sn} \\ \vdots & & s \neq t & & \vdots \\ 0 & \cdots & a_{nt} & \cdots & 0 \end{pmatrix} \end{array} = 0$$

And all the other elements are zero except:

$$\Rightarrow a_{ss} = a_{tt} \Longrightarrow a_{11} = a_{22} = \cdots = a_{nn}$$

$\Downarrow$ 因为 s, t 是我们任意取的

$$\Rightarrow A = cI_{n \times n}$$

10. Since $A^n = 0 \Longrightarrow (I+A)^{-1} = \sum_{k=0}^{n-1} (-1)^k A^k$

You can check that $(I+A) \sum_{k=0}^{n-1} (-1)^k A^k = I$.

$$A_{11}+\cdots+A_{n1}=A_{11} \qquad A_{21}=0$$

$$A_{12}+\cdots+A_{n2}=A_{11} \qquad \vdots$$

$$\vdots \qquad A_{n1}=0$$

$$A_{1n}+\cdots+A_{n1}=A_{11}$$

$$\begin{pmatrix} A_{11} & \cdots & A_{11} \\ A_{21} & \cdots & A_{21} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{n1} \end{pmatrix}$$

$$\begin{pmatrix} \sum A_{i1} & \sum A_{i2} & \cdots & \sum A_{in} \\ 0 & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} \sum A \\ \\ \\ \end{pmatrix} \qquad \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \\ \\ \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$(H_1)_{ij} = \delta_{w+i-k_1}$$

$$(H_1H_2)_{ik} = \sum_j (H_1)_{ij}(H_2)_{jk}$$

$$=\sum_j \delta(j+i-k_1)\delta(k+j-k_2)$$

$j=k_1-i \quad k_1-i=k_2-k$

$j=k_2-k \quad k=k_2+k_1-i$

$$=\delta(k-i-k_2+k_1) \quad k_2-k+i-k_1$$

$$(H_2H_1)_{ik}=\sum_j \delta(j+i-k_2)\delta(k+j-k_1)$$