Homework 16 for GP1

By SJ

1. KK 13.9

   Answer: a) Radiation Pressure

   $p = \frac{E}{c}$ Assuming Absorbing Surface.

   Unit area absorbs light per unit time is just $I = 1.4 \times 10^3 W / m^2$, so $1m^2$ area, momentum change is $\Delta p = \frac{I}{c} = F \Delta t (\Delta t = 1s)$

   To put unit all correct: $\Delta p = \frac{I Area \Delta t}{c} = F \Delta t$

   $\frac{I}{c} = \frac{F}{Area} = \mathrm{Pr}essure$

   $\frac{1.4 \times 10^3}{3 \times 10^8} = 0.47 \times 10^{-5} Pascal$

   You probably guess that this radiation pressure is much smaller than gravitational force, which I bet will be correct, otherwise, the earth will not orbit the sun. But do the arithmatic anyway, need distance to sun, their mass, and radius of earth. Total radiation force would be approximation by a Disk with earth radius (From Wiki: $M_{earth} = 6 \times 10^{24} kg$, $M_{sun} = 2 \times 10^{30} kg$, $R = 1.5 \times 10^{11} m$, $G = 6.7 \times 10^{-11}$, $r_{earth} = 6.4 \times 10^6 m$)

   $F = G \frac{M_{sun} M_{earth}}{R^2} \approx 3.6 \times 10^{22} N$

   $F_{pressure} = 0.47 \times 10^{-5} \pi r^2 \approx 6 \times 10^8$ only 10^{-14} of gravity as expected.

   b) Specific gravity is just density at earth (weight instead of mass), specific gravity 5 means the density is 5000kg/m^3 (5 times that of water).

   Consider the particle at earth (for we know the Sun intensity here), this is not so special as it appears. Since both gravity & light intensity changes as $\frac{1}{R^2}$, so if the particle experience larger radiation force on earth than gravity, it will be anywhere. Particle has radius r:

   $Volume = \frac{4}{3} \pi r^3 \quad m = \frac{4}{3} \pi r^3 \cdot 5 \times 10^3$

   $F_G = 6.7 \times 10^{-11} M_{sun} \frac{20}{3} \pi r^3 \Big/ 2.25 \times 10^{22} \approx 4 \times 10 \pi r^3$

   $F_R = \pi r^2 \cdot pressure = 4.7 \times 10^{-6} \pi r^2$

   $F_R > F_G \quad r < 10^{-7} m$

2. KK 13.11

Answer: $p_1 = p_1'\cos\theta + p_2'\cos\varphi \qquad p_2'\sin\varphi = p_1'\sin\theta$

$$\cot\varphi = \frac{p_1 - p_1'\cos\theta}{p_1'\sin\theta}$$

(P_1 is the initial photon, p'_1 is scattered photon, p'_2 is scattered particle) We know the initial energy and final energy of the photon (see next problem for proof or directly use the result of Compton's effect) as well as photon's scattered angle, so the above equation is good enough to calculate the particle's angle, but I shall further simplify the equation further by using the result of Compton Effect to get KK's answer:

$$E' = \frac{E_0}{1 + \frac{E_0}{m_0 c^2}(1-\cos\theta)}$$

$$p_1' = \frac{p_1}{1 + \frac{E_0}{m_0 c^2}(1-\cos\theta)}$$

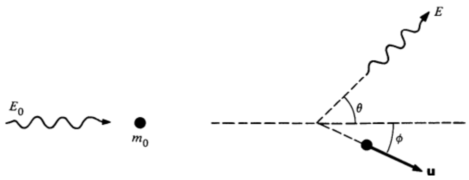
$$p_1 = p_1' + \frac{E_0}{m_0 c^2}(1-\cos\theta)\, p_1'$$

$$p_1 - p_1'\cos\theta = p_1'(1+\frac{E_0}{m_0c^2}) - p_1'\cos\theta(1+\frac{E_0}{m_0c^2}) = (1+\frac{E_0}{m_0c^2})(1-\cos\theta)\,p_1'$$

$$\cot\varphi = (1+\frac{E_0}{m_0c^2})(1-\cos\theta)\Big/\sin\theta = (1+\frac{E_0}{m_0c^2})2\sin^2\frac{\theta}{2}\Big/2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = (1+\frac{E_0}{m_0c^2})\tan\frac{\theta}{2}$$

3. Compton Scattering: The initial photon with wavelength λ_0 collides with a stationary free electron m_0, show that the scattered photon will have a wavelength shift (Compton effect):

$\lambda = \lambda_0 + \dfrac{h}{m_0 c}(1-\cos\theta)$ (the KK has solution, you should try yourself)



Answer: See the book for one solution.
Here I shall use 4-vector approach. (set c=1)

$$\underset{\rightarrow}{P}_{ip} = (E_0, E_0, 0, 0);\ \underset{\rightarrow}{P}_{ie} = (m_0, 0, 0, 0)$$

$$\underset{\to}{P}_{fp} = (E, E\cos\theta, E\sin\theta, 0); \underset{\to}{P}_{fe} = (E', P'\cos\phi, -P'\sin\phi, 0)$$

$$\underset{\to}{P}_{ip} + \underset{\to}{P}_{ie} = \underset{\to}{P}_{fp} + \underset{\to}{P}_{fe}$$

The final electron is not our concern, so that I shall use:

$$\underset{\to}{P}_{ip} + \underset{\to}{P}_{ie} - \underset{\to}{P}_{fp} = \underset{\to}{P}_{fe}$$

$$\underset{\to}{P}_{fe}{}^2 = \underset{\to}{P}_{ip}{}^2 + \underset{\to}{P}_{ie}{}^2 + \underset{\to}{P}_{fp}{}^2 + 2\underset{\to}{P}_{ip} \cdot \underset{\to}{P}_{ie} - 2\underset{\to}{P}_{ie} \cdot \underset{\to}{P}_{fp} - 2\underset{\to}{P}_{ip} \cdot \underset{\to}{P}_{fp}$$

Each term (especially the P^2 terms) can be easily evaluated.

$$m_0^2 = 0 + m_0^2 + 0 + 2E_0 m_0 - 2Em_0 - 2(E_0 E - E_0 E\cos\theta)$$

$$E_0 m_0 = E(m_0 + E_0 - E_0\cos\theta)$$

$$E = \frac{m_0 E_0}{m_0 + E_0 - E_0\cos\theta} \xrightarrow{put\ back\ c} E = \frac{m_0 c^2 E_0}{m_0 c^2 + E_0 - E_0\cos\theta}$$

$$\frac{1}{E} = \frac{1}{E_0} + \frac{1}{m_0 c^2}(1 - \cos\theta) \qquad (1)$$

$$E = h\upsilon = \frac{hc}{\lambda} \to \lambda = \frac{hc}{E}$$

Insert this relation into (1) and we get the Compton shift.

4. A particle of mass m and energy E collide with an identical stationary particle. What is the threshold energy (the minimum value of E) for a final state that containing N particles with mass m?

   Answer: (I will just use energy, momentum conservation and will not resort to 4-vector, but this can equally solved using 4-vector)

   As discussed in the note, the threshold energy is achieved when the products of N particles "stick" together and do not move with respect to each other, but travel together as a cluster. (this has the minimum possible final energy and in turn means minimum initial energy, and since one of the initial particle is stationary whose energy is fixed, then this will give the minimum energy of the moving particle m)

   For particle A: E_A, P_A

   Particle B: $E_B, P_B; where\ E_B = mc^2 = m, P_B = 0$ in lab frame with c set to1.

   Product particle C(the N particles stick like one snow ball with mass Nm): E_C, P_C

   $$P_A + P_B = P_C \to P_A = P_C$$

   $E_A + E_B = E_C \to E_A^2 + E_B^2 + 2E_A E_B = E_c^2 = P_c^2 + (Nm)^2$ (I set c=1 and will put it back after calculation through dimension analysis)

$2E_A E_B = (Nm)^2 + (P_c^2 - E_A^2) - E_b^2 = (Nm)^2 - m^2 - m^2$ (use $P_A = P_C$ and energy-momentum relation for particle A), and since $E_B = m$:

$E_A = \frac{(N^2-2)m}{2}$ and you can put c back (of course c^2 would make the unit as energy),

so $E_A = \frac{(N^2-2)m}{2}c^2$

This is just like the example I gave in the note and the 4-vector solution is also provided in the note. (Here I purposely use the method set c=1, while avoiding doing these in the notes most of time)

5. For a photon collide with a particle m head on and create one final new particle M. If the total energy of the system is E , how should it divided between the photon and mass m so that the final particle has largest M?
   Answer: With in the system the total energy is fixed. From energy-mass equivalence and conservation of energy, the final energy is $E = \gamma Mc^2$ for the new particle. For maximum M means smallest gamma, and gamma should be 1, i.e. the final product is at rest. (This makes perfect sense as we showed in the last problem and in the examples in notes, there you want to create certain mass with minimum energy, here you want to create maximum mass with certain energy) So we'd better divide the energy between photon and initial particle m so that their total momentum is 0.
   Photon: $E_1, P_1 = \frac{E_1}{c}$
   For particle m: $E_2, P_2 = -P_1 = \frac{-E_1}{c}$
   The relations are: $E_1 + E_2 = E$; $E_2^2 - P_2^2c^2 = m^2c^4$, combined with above, straightforward to get:
   $E_2^2 - E_1^2 = m^2c^4$ $E_2 - E_1 = m^2c^4 / E$, then $E_2 = \frac{1}{2}(E + \frac{m^2c^4}{E}), E_1 = \frac{1}{2}(E - \frac{m^2c^4}{E})$
   (I did not set c=1 though I'd like to, please try practice that yourself, do not ever forget put c back in the final answer!!)

6. To create Higgs boson (whose rest energy is over 100GeV), we collide a proton with an antiproton (their rest energy is 1 GeV).How much energy is required to create Higgs if:
   (a) A moving proton to collide with a stationary antiproton.
   (b) The proton and antiproton have equal and opposite momentum (that is the energy needed at least in the LHC (Large Hardron Collider)).

Answer:

(a) This is just like problem 4 with N=100, and the answer is:

$$E_{proton} = \frac{100^2 - 2}{2} \approx 5000GeV$$

(b) When we arrange head-to-head, the energy-momentum conservation give us(since the proton and antiproton have same mass, and if same magnitude of momentum means they are same energy): 2E=100Gev, E=50GeV
Strictly speaking, above is total energy needed and the supplied kinetic energy by accelerator should be above subtracting the rest energies of proton and anti-proton. But since the rest energies of the reactants are usually only small percent of total energy needed, 50GeV would be a good estimate of energy needed by the accelerator.

7. The 4-vector of frequency-wave vector: Considering a general plane wave form of $\sin(\omega t - \vec{k}\cdot\vec{r}) = \sin(\omega t - k_x x - k_y y - k_z z)$, the phase part should be invariant upon LT, otherwise the peak or valley of the wave observed in one frame would not be in another. This means $(\omega/c, \vec{k})\cdot(ct, \vec{r})$, the scalar product in 4-vector form is invariant upon LT, so we can construct a 4-vector $(\omega/c, \vec{k})$:

   (a) Using the transform property of 4-vector, prove the Doppler effect for a **light** wave whose k is in x-y plane: $\vec{k} = |k|(\cos\theta, \sin\theta, 0)$, find relation between ω', ω.

   (b) Prove that for the light wave, the E/ω is a constant independent of frames.

Answer:

(a) The 4-vector of angular frequency-wave vector is $(\omega/c, |k|\cos\theta, |k|\sin\theta, 0)$, for a light wave we have relation: $\omega/|k| = c$

In another frame S', according to Lorentz Transform, the expression of the 4-vector will be:

$$\frac{\omega'}{c} = \gamma(\frac{\omega}{c} - \beta k_x) = \gamma(\frac{\omega}{c} - \beta|k|\cos\theta)$$

$$k'_x = |k'|\cos\theta' = \gamma(|k|\cos\theta - \beta\frac{\omega}{c})$$

$$k'_y = |k'|\sin\theta' = |k|\sin\theta$$

We only need 1st relation to get Doppler shift, while the rest actually gives us angle relations between S and S'.

Since $|k| = \frac{\omega}{c}$ for light wave, from the 1st equation in the LT above:

$$\omega' = \gamma\omega(1 - \beta\cos\theta)$$

Noticed here the angle theta is referred to frame S, so let's choose this S to be moving with respect to the source and the measured frequency of the light wave in this is ω. The frame S' is then the frame in which the source is stationary and the ω' is the measured frequency here, which is $2\pi\upsilon_0$. Thus from above relation we recovered $\upsilon_0 = \gamma(1-\beta\cos\theta)\upsilon$, which is the Doppler relation we derived a while ago. (Here you may also set $\omega' = 2\pi\upsilon$ and $\omega = 2\pi\upsilon_0$. i.e. just switch the assignments of S and S' I used above. This is perfectly ok, since calling which frame S, S' is arbitrary. However, the angle theta's meaning in the above equation will be the angle viewed by the source frame, and the result will still be consistent with our earlier ones)

(b) It is attempting to use products among 4-vectors, and use the fact that dot product between 4-vectors is a scalar, but I have not found a suitable combination this way. So I shall just use the LT on 4-vectors:

I have shown that the $\omega' = \gamma\omega(1-\beta\cos\theta)$

And for the energy, I can assume that the momentum is also along the same direction as wave propagates, i.e. the direction of the photon travels is along same theta in S. For the photon |p|=E/c, and the transformation gives the energy is S' is:

$E' = \gamma E(1-\beta\cos\theta)$

Thus: $\frac{E'}{\omega'} = \frac{E}{\omega}$ this ratio will be same in S and S', i.e. this ratio does not depend on frame. (of course the ratio is Planck constant $\hbar$)

8. For a number of material particles (rest mass m_i>0), each has (E_i/c,P_i) is S frame:
   (a) Prove that we can find one frame in which the total momentum P=0. (Hint: 1) You can prove that the energy-momentum 4-vector is time-like for one material particle, and for two particles, the total-energy-momentum can also be proved time-like: so for any number particles the total energy-momentum is also time-like. 2) Minkowski diagram can be used to easily show that you can find a frame in which total momentum is zero)
   (b) Prove the zero-momentum frame moves relative to S by: $\beta = \sum \vec{p}_i c \big/ \sum E_i$
   (c) Use the conclusion in this problem to show that it is impossible for a single photon to create a positron + electron pair.
   (d) If someone claims that he could create two identical particles each with mass m out of a single particle whose M<2m, because he can make arbitrary large energy by accelerating the M, is the claim correct or wrong and why?

   Answer:

   (a) For each particle, its energy-momentum 4-vector is $(E_i / c, \vec{p}_i)$, the magnitude of this 4-vector is: $(E_i^2 / c^2) - \vec{p}_i \cdot \vec{p}_i = E_i^2 / c^2 - |p_i|^2 = m_i^2 c^2 > 0$ (true for material particle m>0)This is analogous to the space-time interval $s^2 \equiv (ct)^2 - |r|^2 > 0$ which we call time-like.

For two particles, their total 4-vectors is (set c=1 for simple typing):

$(E_1 + E_2, \vec{p}_1 + \vec{p}_2)$, and its magnitude is:

$$E_1^2 - |p_1|^2 + E_2^2 - |p_2|^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2$$

The first 4 terms are $m_1^2 + m_2^2 > 0$ (this is true because it is for the material particle m>0), and because $E_i > |p_i|, \vec{p}_1 \cdot \vec{p}_2 < |p_1 p_2|, E_1E_2 > \vec{p}_1 \cdot \vec{p}_2$, so

$$E_1^2 - |p_1|^2 + E_2^2 - |p_2|^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 > 0$$

The total E-p 4-vector is also time like.

Similar to the (ct,r) case, we can have Minkowski diagram with (E/c,p) where E/c corresponds to ct and p to r. We had proved either using LT or geometrically in the diagram that for time-like case, there is always a frame where different events happened at same place (r=0) but different time from which the name time-like comes. Here similar argument will lead to that we can find one frame that the p=0. This is the total-zero-momentum frame (I also call it CM frame borrowed from classical mechanics)

(b) The 4-vector transforms obeying LT:

In the frame S, to total E-p vector is ($\sum_i E_i, \sum_i \vec{p}_i$), I will choose the direction of total momentum as x, and frame S' will move along x with v relative to S and S' is the zero-momentum frame, thus $P_x = \sum_i \vec{p}_{ix} = |\sum_i \vec{p}_i|$

$P_x' = 0 = \gamma(P_x - \beta E)$, $E = \sum_i E_i$

$\beta = \dfrac{P_x}{E}$, take into the consideration of direction and put back c to make unit correct:

$$\beta = \frac{c\sum_i \vec{p}_i}{\sum_i E_i}$$

(c) In the case of single photon create two material particles (electron and positron), based upon our above proof, we can always find one frame within which the total momentum of the electron+positron is 0. However, in this frame, the initial momentum of the photon is not zero (the photons momentum is E/c in all frames, where E is the energy of the photon which can change depending on frame (remember Doppler?), but it cannot be zero otherwise means no photon at all). So the momentum conservation cannot be satisfied in this case. Though I only pick the special zero-momentum frame for discussion, using the property of 4-vector, this is true in all frames.

(d) For this question you can use the zero momentum frame, say the M frame in which M is stationary, so the energy in such frame for M is just M. The energies for the final particles will be $\gamma_1 m + \gamma_2 m > 2m > M$, this means it is impossible to have energy conservation in this frame, and thus impossible in all frames. The claim is wrong.

9. KK 14.1

Answer: $\vec{p}_1 = (\frac{E_1}{c}, p_1, 0, 0) = (\gamma_v m_0 c, \gamma_v m_0 v, 0, 0)$

$$\vec{p}_2 = (\frac{E_2}{c}, p_2 \cos\theta, p_2 \sin\theta, 0)$$

$$\vec{p}_3 = (\frac{E_3}{c}, p_3 \cos\theta, -p_3 \sin\theta, 0)$$

$$\vec{p}_1 = \vec{p}_2 + \vec{p}_3$$

$$p_1^2 = p_2^2 + p_3^2 + 2\vec{p}_2 \cdot \vec{p}_3$$

$$m_0^2 c^2 = 0 + 0 + 2\frac{E^2}{c^2} - p^2 \cos^2\theta + p^2 \sin^2\theta$$

$$m_0^2 c^4 = 2E^2 - E^2 \cos 2\theta$$

$$\rightarrow \cos 2\theta = 0.1775$$

For β_v, using conservation of energy alone: $\gamma_v \cdot 135 MeV = 200 MeV$

$$\gamma_v = 1.48 \rightarrow \beta_v^2 = 1 - \frac{1}{\gamma^2} \rightarrow \beta_v \square 0.74$$

10. KK 14.3 (working with c=1)

Answer: $p_\gamma^2 + p_1^2 + 2p_\gamma \cdot p_1 = (3 \times 0.51 MeV)^2$

Assume the electron is at rest at beginning, or if you really want to have minimum energy, the electron has to move against the γ photon to have 0 total momentum initially.

For rest electron: $p_\gamma = (E, E), p_1 = (m_0, 0)$

$$0 + m_0^2 + 2Em_0 = (3 \times 0.51 MeV)^2$$

$$(0.51)^2 + 2(0.51)E = (3 \times 0.51)^2$$

$$E = 2MeV$$

In case of moving electron to have zero total p (3-vector) initially

$$p_\gamma = (E, E), p_1 = (\gamma_u m, -E)$$

$$\gamma_u m u = E$$

$$2p_\gamma \cdot p_1 = 2\gamma_u m_0 E + E^2$$

$$\rightarrow 2\gamma_u m_0 E + E^2 + m_0^2 = 9m_0^2$$

This calculation will involve evaluate u and gamma, which will complicate the computation, the 'trick' avoiding such is:

$\underset{\rightarrow}{p}_{photon} + \underset{\rightarrow}{p}_e = \underset{\rightarrow}{p}_f \rightarrow \underset{\rightarrow}{p}_e = \underset{\rightarrow}{p}_f - \underset{\rightarrow}{p}_{photon}$, $\underset{\rightarrow}{p}_f = (3m_0, 0)$ here in such zero total momentum arrangement. Square both sides:

$$m_0^2 = 9m_0^2 + 0 - 2(3m_0)E \rightarrow E = \frac{4}{3}m_0 = \frac{2}{3}MeV$$

11. KK 14.4

Answer:
$$p_0 = p_1 + p_2$$

$$p_0{}^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2$$

$$p_1 = (E_1, p_1), p_2 = (E_2, -p_1)$$

$$E_1 + E_2 = M, p_1^2 = E_1^2 - m_1^2, p_2^2 = E_2^2 - m_2^2$$

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 + 2p_1^2$$

$$M^2 = m_1^2 + m_2^2 + 2E_1(M - E_1) + 2(E_1^2 - m_1^2)$$

$$M^2 - m_1^2 - m_2^2 = 2E_1M - 2E_1^2 + 2E_1^2 - 2m_1^2$$

$\frac{M^2 + m_1^2 - m_2^2}{2M} = E_1$ put the dimension correct ($E \sim mc^2$)

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}c^2$$

12. KK 14.6

Answer: a)
$$\underset{\rightarrow}{p}_{photon} = (\frac{E}{c}, \frac{E}{c}) \xrightarrow[c=1]{} (E, E)$$

E is the total energy of the exhausted photon.

$\frac{E}{c} + \vec{p}_{rocket-final} = 0$ Initially the rocket starts from rest

$$\frac{E}{c} = -\gamma m_f v$$

$$\underset{\rightarrow}{p}_{photon} = \gamma m_f v(1, -1)$$

$$\underset{\rightarrow}{p}_{f(rocket)} = (\gamma m_f c^2, \gamma m_f v) = (\gamma m_f, \gamma m_f v)$$

b) Using conservation law (all 4-vectors below):

$$p_{photon} + p_f = p_i$$

$$(p_{photon})^2 + p_f^2 + 2p_{photon} \cdot p_f = p_i^2$$

$$c = 1, 0 + m_f^2 + 2\gamma^2 m_f^2 v + 2\gamma^2 m_f^2 v^2 = M_i^2$$

$$1 + 2\gamma^2 v + 2\gamma^2 v^2 = X^2, X \equiv \frac{M_i}{m_f}$$

$$\frac{2v + 2v^2}{1 - v^2} = X^2 - 1$$

$$\frac{2v}{1 - v} = X^2 - 1$$

$$2v = (X^2 - 1) - v(X^2 - 1)$$

$$v = \frac{X^2 - 1}{X^2 + 1}$$ put the c back to make unit correct

$$v = \frac{X^2 - 1}{X^2 + 1} c$$

13. KK 14.8

Answer: a) This is 4-vector of wave-vector $\vec{k}$ derivation.

$x = \gamma(x' + \beta ct'), ct = \gamma(ct' + \beta x')$

Put this into $\sin(kx - wt)$ $k \equiv \frac{2\pi}{\lambda}$, $w \equiv 2\pi v$, $c = \frac{w}{k} = \lambda v$.

Expressed in x', t':

$$kx - wt = kx - \frac{w}{c}ct = k\gamma x' + k\beta\gamma ct' - \frac{w}{c}\gamma ct' - \frac{w}{c}\gamma\beta x' = (k\gamma - \gamma\beta\frac{w}{c})x' - (\frac{w}{c}\gamma - k\beta\gamma)ct'$$

$k' \equiv k\gamma - \gamma\beta\frac{w}{c} = \gamma(k - \beta\frac{w}{c})$ (1)

$\frac{w'}{c} \equiv \frac{w}{c}\gamma - k\beta\gamma = \gamma(\frac{w}{c} - \beta k)$ (2)

$(1), (2)$ clearly shows $k, \frac{w}{c}$ transform as x, t.

So we can construct a 4-vector

$\vec{k} \equiv (\frac{w}{c}, \vec{k}) = 2\pi(\frac{v}{c}, \frac{1}{\lambda})$

To test $\frac{w}{k} = c$ in one frame

$$\frac{w'}{k'}=\frac{\gamma(\frac{w}{c}-\beta k)}{\gamma(k-\beta\frac{w}{c})}c=\frac{\frac{w}{kc}-\beta}{1-\beta\frac{w}{kc}}c=\frac{1-\beta}{1-\beta}c=c$$

Indeed c is invariant if expressed as $\frac{w}{k}$

Actually, in Quantum picture where $E=h\nu=\hbar w, \bar{p}=\hbar k$.

$\vec{k}$ is nothing but the energy-momentum 4-vector for photon!

$$\bar{p}=(\frac{E}{c},\bar{p})=\hbar(\frac{w}{c},\vec{k})$$

b) $\frac{w'}{c}=\gamma(\frac{w}{c}-\beta k)=\gamma\frac{w}{c}(1-\beta)$

$$w'=\gamma w(1-\beta)=(1-\beta)\frac{1}{\sqrt{(1-\beta)(1+\beta)}}w=\sqrt{\frac{1-\beta}{1+\beta}}w$$

$$v'=\sqrt{\frac{1-\beta}{1+\beta}}v$$

c) If $k=(0,k,0)$ only along y direction

$$\vec{k}=(\frac{w}{c},0,k,0)$$

$$\frac{w'}{c}=\gamma\frac{w}{c} \quad k_y'=k$$

$$w'=\gamma w\Rightarrow v'=\gamma v,\ \frac{1}{\tau}=\gamma\frac{1}{T}$$

Consider 2-D case generally

$$k=(k\cos\theta, k\sin\theta)$$

$$\vec{k}=(\frac{w}{c},k\cos\theta,k\sin\theta)$$

$$\frac{w'}{c}=\gamma(\frac{w}{c}-\beta k\cos\theta)=\gamma\frac{w}{c}(1-\beta\cos\theta)$$

$$v=\frac{v'}{\gamma(1-\beta\cos\theta)}=\frac{v_0}{\gamma(1-\beta\cos\theta)}$$

14. For a particle with charge q moves in an uniform magnetic field B, and the initial velocity is perpendicular to the B, the force is Lorentz force=q|v||B| perpendicular to the motion.

Show that the motion is a circular motion with radius: $r=|P|/q|B|$ even in the relativistic domain.

Answer: Here the rest mass of particle is a constant and we can safely use formula derived in the notes. Since the force is always perpendicular to the motion. i.e. force perpendicular to velocity and thus perpendicular to momentum you may use $\frac{d\vec{p}}{dt}\cdot\vec{p}=0$, thus

$\frac{d(\vec{p}\cdot\vec{p})}{dt}=\frac{d(|p|^2)}{dt}=0, |p|$ is constant. Thus p is a vector with constant magnitude, its change over time could be expressed as: $q|v||B|=d\vec{p}/dt=\vec{\omega}\times\vec{p}=\omega|p|$

$$\omega=\frac{|v|}{R}=\frac{q|v||B|}{|p|}\to R=\frac{|p|}{q|B|}$$

You may argue that I have not proved that the motion is circular. Indeed I did not prove it directly (I did show that the R is a constant given the B field and initial momentum). Let's see from force-acceleration point of view. In this special case that force is perpendicular to velocity, we have a very simple force-acceleration relation, starting from (14-12) in my notes:

$$\vec{a}=\frac{1}{\gamma_u m_0}(\vec{f}-\frac{\vec{f}\cdot\vec{u}}{c^2}\vec{u})$$

$\vec{f}\cdot\vec{u}=0$ ***in this case*** (this is not generally true of course), thus $\vec{f}=\gamma m\vec{a}$. Everything will be like classical mechanics just replace m with γm here. So what you learned before about charge moving in a uniform B still applies here, for example since $\vec{a}$ is parallel with force, then it is perpendicular to velocity too: $\frac{d\vec{v}}{dt}\cdot\vec{v}=0\to|v|$ is constant as well as γ. It is a constant speed motion subject to a constant force (as well as acceleration) perpendicular to it; the motion will be circular as you learned before(actually the proof of the motion with constant speed under perpendicular constant force and acceleration is circular (just in classical mechanics) is worth proving.

15. A particle m is moving initially with momentum (3-momentum)P_0 along x direction, a constant force with magnitude F_0 is applied along x direction. Find the velocity of the m at later time, and also the trajectory x(t). (You can leave any integral as it is)

Answer: This is a simple variation of my example in notes

From $$\vec{a}=\frac{1}{\gamma_u m_0}(\vec{f}-\frac{\vec{f}\cdot\vec{u}}{c^2}\vec{u})$$

The force and u (same direction as P) are parallel initially here, and momentum along y and z are zero initially and will remain 0 since force is along x, so the force and u will remain parallel (only along x) at later time.

$$a_x=\frac{F_0}{\gamma_u m}(1-\beta_u^2)=\frac{F_0}{\gamma_u^3 m}$$

$$\frac{du}{dt}=\frac{F_0}{\gamma_u^3 m}\rightarrow\frac{du}{(1-\frac{u^2}{c^2})^{\frac{3}{2}}}=\frac{F_0}{m}dt\rightarrow\int_{u_0}^{u}\frac{du}{(1-\frac{u^2}{c^2})^{\frac{3}{2}}}=\frac{F_0}{m}t$$

I have to resort to math handbook for the integral on the LHS.

$$\int\frac{dx}{(a+bx^2)^{3/2}}=\frac{1}{a}\frac{x}{(a+bx^2)^{1/2}}$$ (from handbook)

$$\frac{u}{(1-u^2/c^2)^{1/2}}\Big|_{u_0}^{u}=\frac{u}{(1-u^2/c^2)^{1/2}}-\gamma_{u_0}u_0=\frac{F_0t}{m}$$

$$\frac{u}{(1-u^2/c^2)^{1/2}}=\frac{F_0t+\gamma_{u_0}mu_0}{m}=\frac{F_0t+P_0}{m}$$

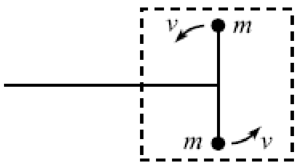
(Of course, the above relation can be more simply obtained by using dP/dt=f, like I worked in the notes)

$$Let\ F_0t+P_0=A$$

$$u(t)=\frac{A}{m}\frac{1}{\sqrt{1+\frac{A^2}{m^2c^2}}}$$

Once we know the u(t), then use dx=udt to find x(t), with the possible nasty integration.

16. Considering a "black box" inside which a dumbbell like object is rotating as shown in the figure:



(a) What is the rest mass M of the black box as viewed by someone knowing nothing inside?
(b) Suppose the black box is initially at stationary, and you apply a force on the horizontal stick (the stick can be treated as massless), convince that by using F=dP/dt=Ma (which is correct when the black box at low velocity), the inertial mass M is indeed the one in a).

Answer:

(a) The E=M, (c=1) and the total energy of the two particles are:

$$E=E_1+E_2=2\gamma_v m$$

So $M=2\gamma_v m$

(b) The whole system is at stationary at beginning, and under force its velocity is changed by Δv. Each particle's velocity will become:

$$v_1 = \frac{-v+\Delta v}{1-\beta_v\beta_{\Delta v}}, v_2 = \frac{v+\Delta v}{1+\beta_v\beta_{\Delta v}}$$

v_1,v_2 are velocities in the ground frame. Acceleration will be $a = \frac{\Delta v}{\Delta t}$

$$\begin{aligned} P(\Delta t) &= P_1 + P_2 = \gamma_{v_1} m v_1 + \gamma_{v_2} m v_2 = \gamma_v \gamma_{\Delta v} m(-v+\Delta v) + \gamma_v \gamma_{\Delta v} m(v+\Delta v) \\ &= 2\gamma_v \gamma_{\Delta v} m \Delta v \approx 2\gamma_v m \Delta v \end{aligned}$$

because $\gamma_{\Delta v} \approx 1$ at low system's velocity at beginning. (I also used:

$\gamma_{v_1} = \gamma_v \gamma_{\Delta v}(1-\beta_v\beta_{\Delta v})$)

Initial momentum is 0 and the momentum change is :

$$\Delta P = P(\Delta t) - P(0) = 2\gamma_v m \Delta v$$

$$F = \frac{dP}{dt} = \frac{2\gamma_v m \Delta v}{\Delta t} = 2\gamma_v m a$$

Indeed the inertial mass at the beginning when we apply the force is M in (a).

You may wonder what happened if the two-masses v are not shown in the figure, i.e. the v is not parallel with Δv, the computation will be a little messy, but whenever the Δv is a small number (which is always true at the beginning of applying force), the result would be same.
What I want to show you is that the “inertial mass” in Newtonian mechanics is the rest mass M here which contains all internal energy of its composites.

17. Prove that from the relation (14-18) in the notes, the dynamic relation between 4-force and 4-acceleration, you can get the relation as (14-11).
Answer: This is a trivial problem to show the equivalence of the compact formula of 4-vectors is equivalent to the complicated equation of motion:
The 4-vector relation between force and acceleration:

$$\underset{\rightarrow}{F} = m_0 \underset{\rightarrow}{A}$$

where the relation between the 4-force and 4-A with force and A by (rest mass is constant):

$$F^0 = \frac{\gamma_u}{c}\frac{dE}{dt} = \frac{\gamma_u}{c}\vec{f}\cdot\vec{u}$$
$$F^{i=x,y,z} = \gamma_u f_i$$

and $\underset{\rightarrow}{A}_u = \gamma_u\left(\frac{\gamma_u^3}{c}\vec{u}\cdot\vec{a}, \frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u\vec{a}\right)$

The space part equation between the 4-vectors is:

$$\gamma_u \vec{f} = \gamma_u m_0 [\frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u \vec{a}]$$

$$\vec{f} = \gamma_u m_0 \vec{a} + m_0 \frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u}$$

This is exactly the (14-11)

The relation between the time component is:

$$\frac{\gamma_u}{c}\vec{f}\cdot\vec{u} = \gamma_u m_0 \frac{\gamma_u^3}{c}\vec{u}\cdot\vec{a} \to \vec{f}\cdot\vec{u} = m_0 \gamma_u^3 \vec{u}\cdot\vec{a}$$

$$\because \frac{d\gamma_u}{dt} = \frac{\gamma_u^3}{c^2}\vec{u}\cdot\vec{a}$$

So: $\vec{f}\cdot\vec{u} = m_0 \gamma_u^3 \vec{u}\cdot\vec{a} = m_0 c^2 \frac{d\gamma_u}{dt} = \frac{dE}{dt}$

This is the power-energy change relation in SR.

So the time component relation among 4-force and 4-A gives the power theorem relation, and space components relation gives the equation of motion.