Answers for Homework VII

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1 KK's 6.32

Ans:
Assume that the force on the rim of two plane is F,

$$Fr\Delta t = I_r \Delta\omega_r$$

$$FR\Delta t = I_R \Delta\omega_R$$

$$\frac{r}{R} = \frac{I_r \Delta\omega_r}{I_R \Delta\omega_R}$$

$$I_r = \frac{1}{2}mr^2$$

$$I_R = \frac{1}{2}MR^2$$

$$R(\omega_0 - \Delta\omega_R) = r\Delta\omega_r$$

$$\omega_R' = \omega_0 - \Delta\omega_R = \frac{M}{M+m}\omega_0$$

$$\omega_r' = \Delta\omega_r = \frac{m}{M+m}\frac{R}{r}\omega_0$$

2 KK's 6.34

Ans:

The conservation law of energy:

$$MgR(\cos\theta_0 - \cos\theta) = \frac{1}{2}\frac{7}{5}MR^2\omega^2$$

$$\omega = \sqrt{\frac{10g}{7R}(\cos\theta_0 - \cos\theta)}$$

$$\beta = \frac{d\omega}{dt} = -\frac{5g}{7R}\sin\theta \approx -\frac{5g}{7R}\theta$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{5g}{7R}}$$

3 KK's 6.39

Ans:

a). We assume that the velocity of the boy after the collision is v_1, and the translation velocity of the plank is v_2, the angular velocity of the plank around the center of the plank is ω, using the conservation law of momentum and angular momentum:

$$mv_0 = mv_1 + Mv_2$$

$$0 = Mv_2\frac{l}{2} - \frac{1}{12}Ml^2\omega$$

$$v_1 = v_2 + \omega\frac{l}{2}$$

we can get:

$$v_1 = \frac{4m}{4m+M}v_0$$

$$v_2 = \frac{m}{4m+M}v_0$$

$$\omega = \frac{6m}{4m+M}\frac{v_0}{l}$$

b). From the equations of (a), we can also get:

$$v_1 = \frac{2}{3}\omega l$$

so the point which is at rest immediately after the collision is $\frac{2}{3}l$ away from the boy.

4 KK's 6.40

Ans:

a). When the spring turn back to its unstretched length, the velocity of the wheel is:

$$\frac{1}{2}Mv_0^2 = \frac{1}{2}kb^2$$

$$v_0 = b\sqrt{\frac{k}{M}}$$

And now, the wheel will mesh with the teeth on the ground, and there is no force from the spring, so the angular momentum about the ground is conserved:

$$Mv_0R = (MR^2 + MR^2)\omega$$

$$\omega = \frac{v_0}{2R}$$

After this moment, the energy will be conserved:

$$\frac{1}{2}2MR^2\omega^2 = \frac{1}{2}kx^2$$

$$x = \frac{\sqrt{2}}{2}b$$

so the distance with the wall is:

$$d = l - x = l - \frac{\sqrt{2}}{2}b$$

b).When the wheel come back to the origin:

$$\frac{1}{2}kx^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$I = MR^2 \qquad \omega = \frac{v}{R}$$

$$\frac{1}{2}kx^2 = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2$$

$$\frac{1}{2}Mv^2 = \frac{1}{4}kx^2 = \frac{1}{8}kb^2$$

And when wheel continues to move, the energy of rotation will not change, only the translation energy will change into the potential energy:

$$\frac{1}{2}Mv^2 = \frac{1}{2}kx'^2$$

$$x' = \frac{b}{2}$$

So the farthest distance from the wall is $l + \frac{b}{2}$

c).When the wheel come back to the origin again, the velocity of c.m. is the same as in (b), but the opposite direction. the angular velocity about the center of the wheel is the same as in (b). And the value of the velocity and the angular velocity have the relation $v = R\omega$. So the total angular momentum about the ground is ZERO. And now the instantaneous axis of the wheel is on the ground, so the angular velocity about this axis is ZERO. Thus the wheel will stop!

5 KK's 7.1

Ans:

a). As the hoop rolls without slipping, so the angular momentum is:

$$\omega_z = \Omega \quad \omega_x = \Omega \quad \omega_y = 0$$

$$\vec{\omega} = \Omega\hat{i} + \Omega\hat{k}$$

The z-axis is the same as the picture in KK's book, the x-axis is now in the direction of the supporting-axle.

b).

$$\vec{L} = I_{xx}\omega_x\hat{i} + I_{zz}\omega_z\hat{k}$$

$$I_{xx} = MR^2 \qquad I_{zz} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

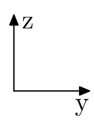
$$\vec{L} = MR^2\Omega\hat{i} + \frac{3}{2}MR^2\Omega\hat{k}$$

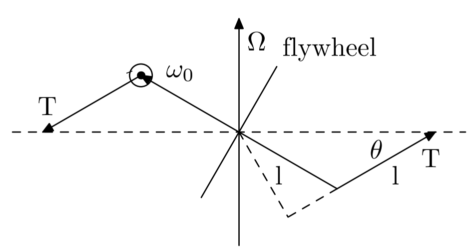
$\vec{L}$ is not parallel to $\vec{\omega}$

6 KK's 7.2

Ans:

Let us have a brief analysis. According to the rotation of the table, the direction of the axle of the flywheel will change, however, we do not know what is the direction. But as the motion is uniform, we can assume that the direction will not change in the frame of the table. So in the lab frame, the direction of the change of the angular momentum is parallel to the x-axis. Thus the torque must also in the same direction! So the configuration of the flywheel is as show in the figure.





$$\begin{aligned}\Delta\vec{L} &= \vec{\Omega}\times\vec{L}\Delta t \\ &= \Omega(I_0(\omega_0 + \Omega\sin\theta)\cos\theta - I_1\Omega\cos\theta\sin\theta)\Delta t\hat{i} \\ &\approx I_0\omega_0\Omega\cos\theta\Delta t\hat{i}\end{aligned}$$

$$\vec{\tau} = \hat{i}2\times T2l\cos\theta\sin\theta$$

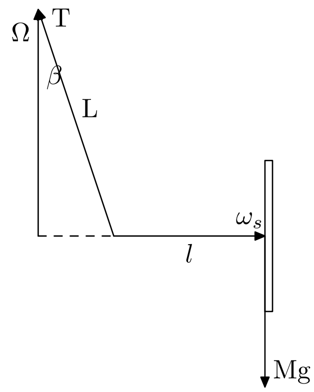
$$\vec{\tau}\Delta t = \Delta\vec{L}$$

$$\theta \approx \sin\theta = \frac{I_0\omega_0\Omega}{4Tl}$$

(Maybe you will get $\tan\theta$, they are all right for $\Omega \gg \omega_0$.)

7 KK's 7.3

Ans:



For the motion of the c.m., we have:

$$T\cos\beta = Mg$$

$$T\sin\beta = M\Omega^2(l + L\sin\beta)$$

For the rotation of the rigid body:

$$\tau = Mg(l + L\sin\beta) - MgL\sin\beta = \frac{dL}{dt} = I_0\omega_s\Omega$$

$$\Omega = \frac{Mgl}{I_0\omega_s}$$

$$\tan\beta = \frac{\Omega^2(l + L\sin\beta)}{g}$$

For small β:

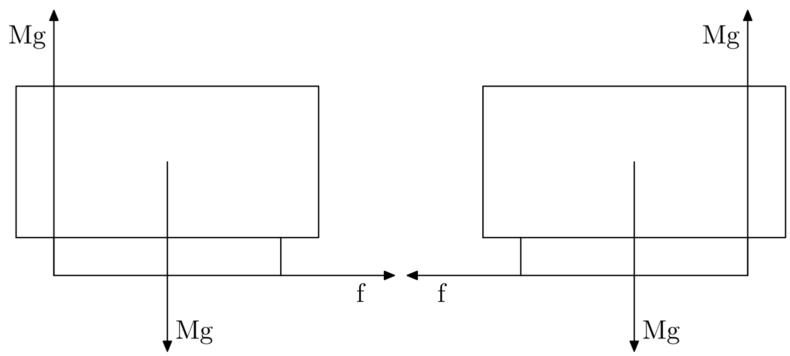
$$\beta \approx \frac{\Omega^2 l}{g - L\Omega^2}$$

$$\Omega = \frac{Mgl}{I_0\omega_s}$$

8 KK's 7.5

Ans:

a). First we assume that we look at the car from the back. The car in the left figure turn to right, and the car in the right figure turn to left. As we need to get the direction of the flywheel in order to avoid the car from rolling over, so we assume the car has the tendency to roll over so than the torque on the car in the left figure is outside the paper, and for the right figure inside the paper.



We focus on the left figure, in order to balance the torque pointing outside the paper, there need to be a change of angular momentum pointing outside the paper. As the car in the left figure turn right, so we need to have the angular momentum pointing to the right.(Indeed, it is enough that the angular momentum has right-side component) For the car in the right figure, follow the same analysis, we get the angular momentum also points to the right. Finally we get that in order to avoid the car rolling over, the angular momentum of the flywheel need to point to the right viewed from the back of the car!

b).In a), we have already get the direction of the angular momentum of the flywheel, now we need to get the value of the angular momentum! And we assume that the angular momentum has only the right-side component.

$$\frac{dL}{dt} = I_0\omega\Omega$$

where Ω is the angular velocity of the car when the car turn around. We also suggest that the car moves so fast that we only consider the torque of the friction force.

$$\tau = fL = M\Omega^2 rL$$

$$M\Omega^2 rL = I_0\omega\Omega$$

$$\omega = \frac{M\Omega rL}{I_0}$$

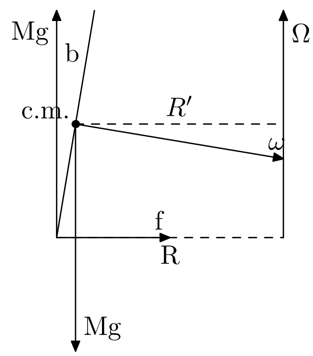
$$v = \Omega r \qquad I_0 = \frac{1}{2}mR^2$$

$$\omega = 2v\frac{ML}{mR^2}$$

9 KK's 7.6

Ans:

The motion of the coin has two parts: rotation and precession. We assume that the angular velocity of rotation and precession are ω and Ω respectively. And the velocity "v" in the title is the velocity of the c.m. of the coin.



As the coin rolls without slipping, we have:

$$b\omega = R\Omega$$

$$v = \Omega R' \qquad R' = R - b\sin\phi$$

For the c.m.:

$$N = Mg \qquad f = M\Omega^2 R'$$

And the total angular velocity of the coin is: (x along ω, y along "-v" and z obey the right-handed rule)

$$\vec{\omega} = (\omega - \Omega\sin\phi, 0, \Omega\cos\phi)$$

$$\vec{L} = (I_s(\omega - \Omega\sin\phi), 0, I_\perp\Omega\cos\phi)$$

$$\frac{d\vec{L}}{dt} = \Omega \times \vec{L} = \hat{j}\big(I_s(\omega - \Omega\sin\phi)\Omega\cos\phi + I_\perp\Omega\cos\phi\Omega\sin\phi\big) \approx \hat{j}I_s\omega\Omega\cos\phi$$

Here we neglect the term with Ω^2 for $\Omega \ll \omega$

$$\vec{\tau} = \hat{j}(Mgb\sin\phi - fb\cos\phi)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

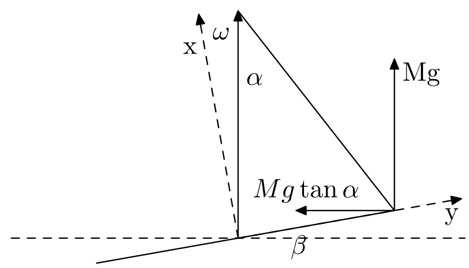
so we get:

$$\tan\phi = \frac{v^2}{g}(\frac{R}{2R'^2} + \frac{1}{R})$$

$$\tan\phi \approx \frac{v^2}{gR}(\frac{3}{2} + 2\frac{b}{R}\sin\phi)$$

As $R \gg b$ and $\phi \ll 1$, we neglect the second term in parentheses.

$$\tan\phi = \frac{3v^2}{2gR}$$



10 KK's 7.7

Ans:
The angular velocity of the hoop is ω as show in the figure!(No more!). So the total angular momentum is:

$$\vec{L} = I_{xx}\omega \cos\beta \hat{i} + I_{yy}\omega \sin\beta \hat{j}$$

$$I_{xx} = MR^2 \qquad I_{yy} = \frac{1}{2}MR^2$$

$\vec{L}$ is not parallel to the $\vec{\omega}$. So the change rate of the angular momentum is:

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = -\hat{k}\frac{1}{2}MR^2\omega^2 \sin\beta\cos\beta$$

The force on the hoop is showed in the figure, so we get the torque:

$$\vec{\tau} = -\hat{k}(MgR\cos\beta + MgR\tan\alpha\sin\beta)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$MgR\cos\beta + MgR\tan\alpha\sin\beta = \frac{1}{2}MR^2\omega^2\sin\beta\cos\beta$$

$$\beta \approx \frac{2g}{\omega^2 R - 2g\tan\alpha}$$

For the c.m.

$$Mg\tan\alpha = M\omega^2 r_c$$

$$r_c = \frac{g\tan\alpha}{\omega^2}$$

11 KK's 7.10

This problem is the same as the example 7.11 in KK's book.

When the spin axis is parallel to the polar axis, during the rotation of the earth, there is no change in the angular momentum. And there is no torque on the gyro, so the gyro is stationary when the spin axis is parallel to the polar axis. And when the gyro is at latitude λ, the angular between the horizontal and the polar axis is just λ.

Now the spin axis is at a little angular θ to the polar axis. As the torque is zero:

$$\frac{dL}{dt} = 0$$

$$\vec{L} = I_1 \omega_s \hat{1} + I_{\perp} \dot{\theta} \hat{e} \perp$$

$$\frac{dL}{dt} \approx I_{\perp} \ddot{\theta} + I_1 \omega_s \sin\theta \Omega_e = 0$$

$$\ddot{\theta} + \frac{I_1 \omega_s \Omega_e}{I_{\perp}} \theta = 0$$

Solve this equation we get:

$$\omega_{osc} = \sqrt{\frac{I_1 \omega_s \Omega_e}{I_{\perp}}}$$

For a thin disk $I_1 = \frac{1}{2} M R^2$, $I_{\perp} = \frac{1}{4} M R^2$, and $\omega_s = \frac{40000 \times 2\pi}{60}$ $\Omega_e = \frac{2\pi}{3600 \times 24}$

$$\omega_{osc} \approx 0.78$$

12 The principle axes

Ans:

a).

$$I_{xx} = \int dm(y^2 + z^2) = \sigma \int dx \int dy y^2 = \frac{1}{3} M a^2$$

$$I_{yy} = \int dm(z^2 + x^2) = \sigma \int dy \int dx x^2 = \frac{1}{3} M a^2$$

$$I_{zz} = \int dm(x^2 + y^2) = \sigma \int \int dx dy (x^2 + y^2) = \frac{2}{3} M a^2$$

$$I_{xy} = I_{yx} = -\int dm xy = -\sigma \int \int dx dy xy = -\frac{1}{4} M a^2$$

$$I_{xz} = I_{zx} = -\int dm xz = 0$$

$$I_{yz} = I_{zy} = -\int dm yz = 0$$

So the moment of inertia tensor is:

$$\hat{I} = \begin{pmatrix} \frac{1}{3} M a^2 & -\frac{1}{4} M a^2 & 0 \\ -\frac{1}{4} M a^2 & \frac{1}{3} M a^2 & 0 \\ 0 & 0 & \frac{2}{3} M a^2 \end{pmatrix}$$

b). To find the principle axes is to find the eigenvector of the moment of inertia tensor! First to get the eigenvalue of the moment of inertia tensor:

$$\begin{vmatrix} \frac{1}{3}Ma^2-\lambda & -\frac{1}{4}Ma^2 & 0 \\ -\frac{1}{4}Ma^2 & \frac{1}{3}Ma^2-\lambda & 0 \\ 0 & 0 & \frac{2}{3}Ma^2-\lambda \end{vmatrix} = 0$$

let $A = Ma^2$ and we get;

$$(\frac{2}{3}A-\lambda)((\frac{1}{3}A-\lambda)^2-\frac{1}{16}A^2)=0$$

$$\lambda_1 = \frac{1}{12}A \qquad \lambda_2 = \frac{7}{12}A \qquad \lambda_3 = \frac{2}{3}A$$

Then we determine the eigenvector $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$\hat{I}\hat{e}_1 = \lambda_1\hat{e}_1$$

$$\hat{I}\hat{e}_2 = \lambda_2\hat{e}_2$$

$$\hat{I}\hat{e}_3 = \lambda_3\hat{e}_3$$

we get:(after normalization)

$$\hat{e}_1 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \hat{e}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} \qquad \hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

So under these axes, the moment of inertia tensor is:

$$\hat{I}_p = \begin{pmatrix} \frac{1}{12}Ma^2 & 0 & 0 \\ 0 & \frac{7}{12}Ma^2 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 \end{pmatrix}$$