Answers for Homework V

Wu Yongcheng
Department of Physics

1 KK's 4.23

Ans:

Before the collision between two balls, their velocities are:

$$v = \sqrt{2gh}$$

For the elastic collision:

$$Mv - mv = Mv_1 + mv_2$$

$$\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2$$

Solving these two equations, we can get the solution:

$$v_2 = \frac{3M - m}{M + m}v$$

When $M \gg m$,

$$v_2 = 3v$$

so the small ball could raise to $9h$.(You can also use another method in which we first work in the frame of superball using the condition $M \gg m$.)

2 KK's 4.28

Ans:

a). We work in c.m.:

$$V_{c.m.} = \frac{4}{11}V_0$$

Now the velocities of Li and He are:

$$V_{Li} = -\frac{4}{11}V_0 \qquad V_{He} = \frac{7}{11}V_0$$

In c.m., the total momentum is zero, so for Boron and neutron:

$$10V_B' + V_n' = 0$$

For the energy equation:

$$E_0 = E'_k + 2.8MeV$$

so

$$E_0 = \frac{1}{2}M_{all}V_{c.m.}^2 + \frac{1}{2}m_B V_B^{'2} + \frac{1}{2}m_n V_n^{'2} + 2.8MeV$$

$$E_0 \geqslant \frac{1}{2}M_{all}V_{c.m.}^2 + 2.8MeV$$

$$E_{0,threshold} = \frac{1}{2}4m_0V_0^2 = \frac{1}{2}11m_0(\frac{4}{11})^2V_0^2 + 2.8MeV$$

$$m_0V_0^2 = \frac{11}{14}2.8MeV = 2.2MeV$$

$$E_{0.threshold} = 2m_0V_0^2 = 4.4MeV$$

We know when $E_0 = E_{0,threshold}$, the velocity of neutron in c.m. is zero, so in the lab frame:

$$V_n = V_{c.m.} = \frac{4}{11}V_0$$

$$E_n = \frac{1}{2}m_0V_n^2 = 0.145MeV$$

b).In c.m. frame:

$$E_0 = \frac{1}{2}4m_0V_0^2 = \frac{1}{2}11m_0V_{c.m.}^2 + \frac{1}{2}10m_0V_B^{'2} + \frac{1}{2}m_0V_n^{'2} + 2.8MeV$$

$$10V'_B + V'_n = 0$$

$$E_0 = \frac{4}{11}E_0 + \frac{11}{20}m_0V_n^{'2} + 2.8MeV$$

$$E_0 = \frac{121}{140}m_0V_n^{'2} + 4.4MeV$$

$$\frac{140}{121}\Delta E = m_0V_n^{'2}$$

For $V_{c.m.}$

$$\frac{16}{55}MeV + \frac{8}{121}\Delta E = m_0V_{c.m.}^2$$

As $0 < \Delta E < 0.27MeV$, $|V_{c.m.}| > |V'_n|$. When we turn back to the lab frame there would two possible value for V_n

3 KK's 4.29

Ans:

a). For one wall:

$$F\Delta t = 2mv_0$$

$$\Delta t = \frac{2l}{v_0}$$

$$F = \frac{2mv_0}{\Delta t} = \frac{mv_0^2}{l}$$

b). We can see that after $\Delta t \approx \frac{2x}{v}$, the speed of the ball increase by $\Delta v = 2V$, so we have:

$$\frac{dv}{dt} = \frac{Vv}{x}$$

$$\frac{dv}{v} = \frac{Vdt}{l - Vt}$$

After integrating:

$$\ln\frac{v}{v_0} = -\ln\frac{x}{l}$$

so we get:

$$v = v_0\frac{l}{x}$$

From question a). we get $F = \frac{mv^2}{l_v}$, so:

$$F = \frac{mv_0^2}{l}(\frac{l}{x})^3$$

c). The work done by F:

$$W = \int_l^x -mv_0^2 l^2 \frac{dx'}{x'^3}$$

The minus sign is because the x decrease

$$W = \frac{1}{2}mv_0^2 - \frac{1}{2}m(v_0\frac{l}{x})^2 = \Delta E_k$$

4 KK's 4.30

Ans:

a). In the c.m. system:

$$V_{c.m.} = \frac{m}{M+m}V_0$$

$$V_m = \frac{M}{M+m}V_0 \qquad V_M = \frac{m}{M+m}V_0$$

Using the energy and momentum equations in c.m. system

$$V'_m = \frac{M}{M+m} V_0$$

But in the direction of Θ. Return to the lab frame:

$$V_f = \sqrt{V'^2_m + V^2_{c.m.} + 2V'_m V_{c.m.} \cos\Theta} = \frac{V_0}{M+m}\sqrt{M^2+m^2+2Mm\cos\Theta}$$

b).

$$\frac{K_0 - K_f}{K_0} = 1 - \frac{M^2+m^2+2Mm\cos\Theta}{(M+m)^2}$$

5 Some practice on partial derivatives

(a). Ans:

1).$L = \frac{1}{2}(v_x^2 + v_y^2) - 3\cos t$

$$(\frac{\partial L}{\partial t})_{v_x,v_y} = 3\sin t$$
$$(\frac{\partial L}{\partial v_x})_{v_y,t} = v_x$$
$$(\frac{\partial L}{\partial v_y})_{v_x,t} = v_y$$

2).

$$\begin{aligned}\frac{dL}{dt} &= (\frac{\partial L}{\partial t})_{v_x,v_y}\frac{dt}{dt} + (\frac{\partial L}{\partial v_x})_{v_y,t}\frac{dv_x}{dt} + (\frac{\partial L}{\partial v_y})_{v_x,t}\frac{dv_y}{dt} \\ &= 3\sin t - a^2\cos t\sin t + a^2\sin t\cos t \\ &= 3\sin t\end{aligned}$$

(b). Ans:

$$G(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$
$$dG = (\frac{\partial G}{\partial x})_y dx + (\frac{\partial G}{\partial y})_x dy = 0$$
$$(\frac{\partial G}{\partial x})_y = 2\frac{x}{a^2}$$
$$(\frac{\partial G}{\partial y})_x = 2\frac{y}{b^2}$$
$$\frac{dy}{dx} = -\frac{(\frac{\partial G}{\partial x})_y}{(\frac{\partial G}{\partial y})_x} = -\frac{b^2 x}{a^2 y}$$

(c).Ans:

1).

$$(\frac{\partial f}{\partial x})_y = y^2 - y\sin x$$
$$(\frac{\partial f}{\partial y})_x = 2xy + \cos x$$

2).$x = s, y = t - x = t - s$

$$f(s,t) = s(t-s)^2 + (t-s)\cos s$$

$$\begin{aligned}(\frac{\partial f}{\partial s})_t &= (t-s)^2 - 2s(t-s) - \cos s - (t-s)\sin s \\ &= y^2 - 2xy - \cos x - y\sin x = (\frac{\partial f}{\partial x})_y - (\frac{\partial f}{\partial y})_x \\ &= (\frac{\partial f}{\partial x})_y(\frac{\partial x}{\partial s})_t + (\frac{\partial f}{\partial y})_x(\frac{\partial y}{\partial s})_t\end{aligned}$$

6 KK's 5.1

Ans:

$$\vec{F} = -\nabla U$$

a.

$$\vec{F} = -2Ax\hat{i} - 2By\hat{j} - 2Cz\hat{k}$$

b.

$$\vec{F} = -2A\frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2 + y^2 + z^2}$$

c.In plane polar coordinates:

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{e}_\theta$$
$$\vec{F} = -\nabla U = -2A\frac{\cos\theta}{r^3}\hat{e}_r - A\frac{\sin\theta}{r^3}\hat{e}_\theta$$

7 KK's 5.4

Ans:

a.

$$\nabla \times \vec{F} = 0$$

It is conservative.

$$U = -\int \vec{F}d\vec{r} = -3Ax + Ayz + C$$

b.

$$\nabla \times \vec{F} = Ax(z-y)\hat{i} + Ay(x-z)\hat{j} + Az(y-x)\hat{k} \neq 0$$

It is not conservative.

c.

$$\nabla \times \vec{F} = 0$$

It is conservative.

$$U = -\int \vec{F} d\vec{r} = Ax^3 y^5 e^{\alpha z} + C$$

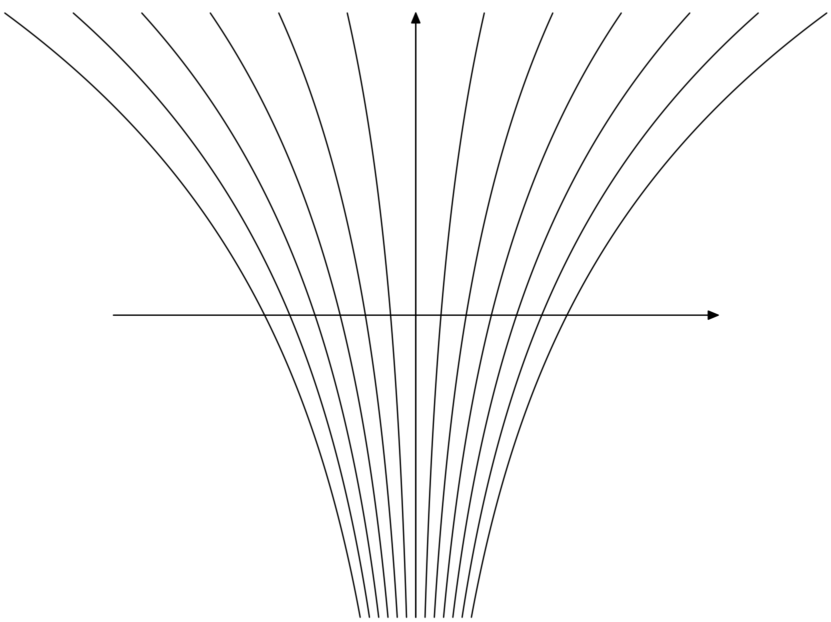
d.

$$\nabla \times \vec{F} \neq 0$$

It is not conservative.

8 KK's 5.5

Ans:
a).See the figure.



b).Assume that $U = U_0$

$$y = \ln(\frac{Cx}{U_0})$$

$$dy = \frac{U_0}{Cx}\frac{C}{U_0}dx = \frac{dx}{x}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} = dx(\hat{i} + \frac{1}{x}\hat{j})$$

c).

$$\nabla U = Ce^{-y}\hat{\imath} - Cxe^{-y}\hat{\jmath}$$

$$\nabla U \cdot d\vec{r} = 0$$

so ∇U is perpendicular to the constant energy line.

9 KK's 5.7

Ans:

$$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta} = \frac{GM_e m}{r^2} 5.4 \times 10^{-4} (\frac{R_e}{r})^2 6 \cos\theta \sin\theta$$

so if $\theta \neq 0$ or $\theta \neq \frac{\pi}{2}$, $F_\theta \neq 0$

$$\frac{F_\theta}{\frac{GM_e m}{r^2}}|_{\theta=45, r=R_e} = 5.4 \times 10^{-4} \times 6 \times \frac{1}{2} = 1.62 \times 10^{-3}$$

10 KK's 5.8

Ans:
Line integral:

$$W = \int \vec{F} \cdot d\vec{r} = 0 + 2Ad^3 - Ad^3 + 0 = Ad^3$$

Stokes theorem:

$$\begin{aligned} W = \int \vec{F} \cdot d\vec{r} &= \int\int \nabla \times \vec{F} dx dy \\ &= \int_0^d dy \int_0^d dx (4Ax - 2Ay) \\ &= \int_0^d dy (2Ad^2 - 2Ayd) \\ &= 2Ad^3 - Ad^3 \\ &= Ad^3 \end{aligned}$$

11 Gradient in spherical coordinate system

Between spherical coordinate system and Cartesian coordinates:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\tan\theta = \frac{y}{x}$$

$$z = r\cos\varphi$$
$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$

And:

$$\hat{i} = \cos\theta\sin\varphi\hat{e}_r + \cos\theta\cos\varphi\hat{e}_\varphi - \sin\theta\hat{e}_\theta$$
$$\hat{j} = \sin\theta\sin\varphi\hat{e}_r + \sin\theta\cos\varphi\hat{e}_\varphi + \cos\theta\hat{e}_\theta$$
$$\hat{k} = \cos\varphi\hat{e}_r - \sin\varphi\hat{e}_\varphi$$
$$\hat{e}_r = \sin\varphi\cos\theta\hat{i} + \sin\varphi\sin\theta\hat{j} + \cos\varphi\hat{k}$$
$$\hat{e}_\varphi = \cos\varphi\cos\theta\hat{i} + \cos\varphi\sin\theta\hat{j} - \sin\varphi\hat{k}$$
$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} \\
\frac{\partial U}{\partial y} &= \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y} \\
\frac{\partial U}{\partial z} &= \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial z} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial z}
\end{aligned} \tag{1}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\varphi\cos\theta$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r} = \sin\varphi\sin\theta$$
$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \cos\varphi$$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{\sin\varphi}\frac{xz}{r^3} = \frac{\cos\varphi\cos\theta}{r}$$
$$\frac{\partial \varphi}{\partial y} = \frac{1}{\sin\varphi}\frac{yz}{r^3} = \frac{\cos\varphi\sin\theta}{r}$$
$$\frac{\partial \varphi}{\partial z} = \frac{1}{\sin\varphi}(\frac{z^2}{r^3} - \frac{1}{r}) = -\frac{\sin\varphi}{r}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y\cos^2\theta}{x^2} = -\frac{\sin\theta}{r\sin\varphi}$$
$$\frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r\sin\varphi}$$
$$\frac{\partial \theta}{\partial z} = 0$$

Insert these nine equation into equation(1).

$$
\begin{aligned}
\nabla U = &(\frac{\partial U}{\partial r}\sin\varphi\cos\theta + \frac{\partial U}{\partial \varphi}\frac{\cos\varphi\cos\theta}{r} - \frac{\partial U}{\partial \theta}\frac{\sin\theta}{r\sin\varphi})\hat{i} \\
&+ (\frac{\partial U}{\partial r}\sin\varphi\sin\theta + \frac{\partial U}{\partial \varphi}\frac{\cos\varphi\sin\theta}{r} + \frac{\partial U}{\partial \theta}\frac{\cos\theta}{r\sin\varphi})\hat{j} \\
&+ (\frac{\partial U}{\partial r}\cos\varphi - \frac{\partial U}{\partial \varphi}\frac{\sin\varphi}{r})\hat{k} \\
= &\frac{\partial U}{\partial r}(\sin\varphi\cos\theta\hat{i} + \sin\varphi\sin\theta\hat{j} + \cos\varphi\hat{k}) \\
&+ \frac{\partial U}{\partial \varphi}\frac{1}{r}(\cos\varphi\cos\theta\hat{i} + \cos\varphi\sin\theta\hat{j} - \sin\varphi\hat{k}) \\
&+ \frac{\partial U}{\partial \theta}\frac{1}{r\sin\varphi}(-\sin\theta\hat{i} + \cos\theta\hat{j}) \\
= &\frac{\partial U}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}\hat{e}_\varphi + \frac{1}{r\sin\varphi}\frac{\partial U}{\partial \theta}\hat{e}_\theta
\end{aligned}
$$