Answers for Homework II

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1 Prove A,B are parallel using dot and cross product

Prove:
$\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (kA_x, kA_y, kA_z)$ then we have:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ kA_x & kA_y & kA_z \end{vmatrix} = 0$$

That is because the determinant of a matrix which has a row is proportional to another is zero. And thus this two vector are parallel.

2 KK's 1.3

Ans:
For any point in the space we have $\vec{r} = (x, y, z)$. From the definition of α, β and γ, we get:

$$x = |\vec{r}|\alpha \qquad y = |\vec{r}|\beta \qquad z = |\vec{r}|\gamma$$

For every $\vec{r}$, we always have:

$$x^2 + y^2 + z^2 = |\vec{r}|^2$$

so finally we get:

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

3 KK's 1.4

Ans:
we have $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$, then we get:

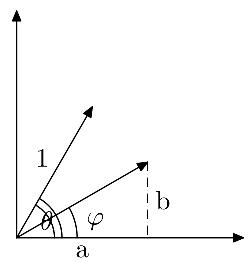
$$|\vec{A} - \vec{B}|^2 = |\vec{A} + \vec{B}|^2$$

$$|\vec{A}|^2 - 2\vec{A} \cdot \vec{B} + |\vec{B}|^2 = |\vec{A}|^2 + 2\vec{A} \cdot \vec{B} + |\vec{B}|^2$$

$$\vec{A} \cdot \vec{B} = 0$$

that is $\vec{A} \perp \vec{B}$

4 Prove relation



Ans:
We assume there are two vectors: $\vec{A} = (\cos\theta, \sin\theta)$ and $\vec{B} = (a, b)$. We have two ways to do the dot product. First way:

$$\vec{A} \cdot \vec{B} = a\cos\theta + b\sin\theta$$

The second way is:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta - \varphi) = \sqrt{a^2 + b^2}\cos(\theta - \varphi)$$

Here $\tan\varphi = \frac{b}{a}$ and we can let $c = \sqrt{a^2 + b^2}$

5 KK's 1.8

Ans:

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

so we can get one of the unit vectors that are perpendicular to A and B:

$$\hat{n} = \frac{\vec{C}}{|\vec{C}|} = \frac{2\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{\sqrt{38}}$$

6 KK's 1.9

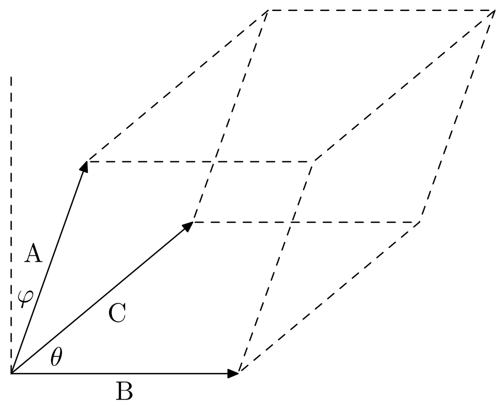
Ans:
The figure is in the next page.

$$V = S \cdot h = |\vec{B}||\vec{C}|\sin\theta|\vec{A}|\cos\varphi = (\vec{B} \times \vec{C}) \cdot \vec{A}$$

7 Prove relation between cross and dot product

Ans:
One counter example is(There are many other examples):

$$\vec{A} = (1, 0, 0) \quad \vec{B} = (0, 1, 0) \quad \vec{C} = (1, 1, 0)$$



$$(\vec{A}\times\vec{B})\times\vec{C} = (-1,1,0)$$
$$\vec{A}\times(\vec{B}\times\vec{C}) = (0,1,0)$$

Further more we have:

$$\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C}) =&(a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)\mathbf{i}\\ &+ (a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1)\mathbf{j}\\ &+ (a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\mathbf{k}\\ =&[b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3)]\mathbf{i}\\ &+ [b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)]\mathbf{j}\\ &+ [b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3)]\mathbf{k}\\ =&\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\end{aligned}$$

8 KK's 1.11

Ans:
For we have:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

So:

$$(\hat{n}\times\vec{A})\times\hat{n} = -\hat{n}(\hat{n}\cdot\vec{A}) + \vec{A}$$

Finally:

$$\vec{A} = (\vec{A}\cdot\hat{n})\hat{n} + (\hat{n}\times\vec{A})\times\hat{n}$$

9 KK's 1.14

Ans:

$$\omega = \frac{V_c}{R}$$
$$\alpha = \frac{d\omega}{dt} = \frac{1}{R}\frac{dV_c}{dt} = \frac{a}{R}$$

10 Transformation of Vectors

Ans:
a).

$$(a,b) = (2,1) \quad \theta = 30 \quad \sin\theta = 1/2 \quad \cos\theta = \sqrt{3}/2$$

$$a' = a\cos\theta + b\sin\theta = \sqrt{3} + \frac{1}{2}$$

$$b' = -a\sin\theta + b\cos\theta = -1 + \frac{\sqrt{3}}{2}$$

b). clockwise rotation:

$$R' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

c).

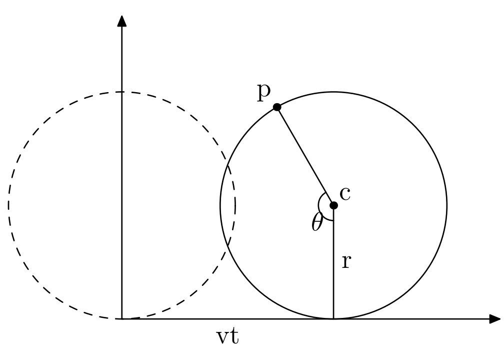
$$R = \begin{bmatrix} \hat{1}' \cdot \hat{1} & \hat{1}' \cdot \hat{2} \\ \hat{2}' \cdot \hat{1} & \hat{2}' \cdot \hat{2} \end{bmatrix}$$

$$R' = \begin{bmatrix} \hat{1} \cdot \hat{1}' & \hat{1} \cdot \hat{2}' \\ \hat{2} \cdot \hat{1}' & \hat{2} \cdot \hat{2}' \end{bmatrix}$$

$$\begin{aligned} RR' &= \begin{bmatrix} \hat{1}' \cdot \hat{1} & \hat{1}' \cdot \hat{2} \\ \hat{2}' \cdot \hat{1} & \hat{2}' \cdot \hat{2} \end{bmatrix} \begin{bmatrix} \hat{1} \cdot \hat{1}' & \hat{1} \cdot \hat{2}' \\ \hat{2} \cdot \hat{1}' & \hat{2} \cdot \hat{2}' \end{bmatrix} \\ &= \begin{bmatrix} \hat{1}'(\hat{1}\hat{1} + \hat{2}\hat{2})\hat{1}' & \hat{1}'(\hat{1}\hat{1} + \hat{2}\hat{2})\hat{2}' \\ \hat{2}'(\hat{1}\hat{1} + \hat{2}\hat{2})\hat{1}' & \hat{2}'(\hat{1}\hat{1} + \hat{2}\hat{2})\hat{2}' \end{bmatrix} \\ &= \begin{bmatrix} \hat{1}'\hat{1}' & \hat{1}'\hat{2}' \\ \hat{2}'\hat{1}' & \hat{2}'\hat{2}' \end{bmatrix} = I \end{aligned}$$

The same for $R'R$

11 KK's 1.19

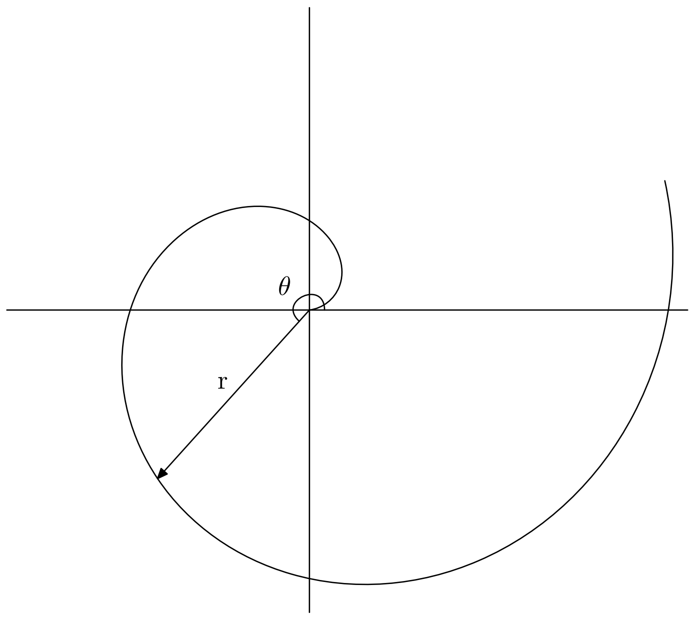


Ans:

$$r_c = (vt, r)$$

$$r_p = (vt - r\sin\omega t, r(1 - \cos\omega t))$$
$$v_p = (v - \omega r\cos\omega t, \omega r\sin\omega t)$$
$$a_p = (\omega^2 r\sin\omega t, \omega^2 r\cos\omega t)$$

12 KK's 1.20



Ans:
For any movement:

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$
$$r = A\theta \quad \theta = \frac{1}{2}\alpha t^2$$
$$\dot{\theta} = \alpha t \quad \ddot{\theta} = \alpha$$
$$\dot{r} = A\alpha t \quad \ddot{r} = A\alpha$$

When $a_r = 0$, we have:

$$A\alpha = \frac{1}{2}A\alpha t^2\alpha^2 t^2$$
$$2 = [\alpha t^2]^2$$

so we get:

$$\theta = \frac{1}{2}\alpha t^2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

When $a_r = a_\theta$, we have:

$$A\alpha - \frac{1}{2}A\alpha^3 t^4 = \frac{5}{2}A\alpha^2 t^2$$

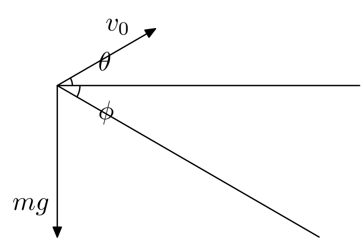
we get

$$\alpha t^2 = \frac{\sqrt{33}-5}{2}$$

Finally:

$$\theta = \frac{1}{2}\alpha t^2 = \frac{\sqrt{33}-5}{4}$$

13 KK's 1.21



Ans:
Focus on the direction perpendicular to the slope:

$$mg\cos\phi t = 2mv_0\sin(\theta+\phi)$$

we get the total time of flight:

$$t = \frac{2v_0\sin(\theta+\phi)}{g\cos\phi}$$

For the direction parallel to the slope:

$$\begin{aligned} S &= v_\parallel t + \frac{1}{2}at^2 \\ &= v_0\cos(\theta+\phi)\frac{2v_0\sin(\theta+\phi)}{g\cos\phi} + \frac{1}{2}g\sin\phi\frac{4v_0^2\sin^2(\theta+\phi)}{g^2\cos^2\phi} \\ &= \frac{2v_0^2}{g\cos^2\phi}\big(\cos\phi\cos(\theta+\phi)\sin(\theta+\phi) + \sin\phi\sin^2(\theta+\phi)\big) \end{aligned}$$

We want to make S maximum, the condition is $\frac{ds}{d\theta} = 0$, finally we get:

$$\phi + 2\theta = \frac{\pi}{2}$$