1. (1) $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ -3 & -2 & 3 \end{pmatrix} \begin{array}{c} ① \\ ② \\ ③ \end{array}$

$\xrightarrow[② - 2① \Rightarrow ②]{} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ -3 & -2 & 3 \end{pmatrix} \xrightarrow[③ - (-1)① \rightarrow ③]{} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 5 & 5 \end{pmatrix}$

$\xrightarrow[③ - ② \Rightarrow ③]{} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$$\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ -3 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ -3 & -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{13}{15} & -\frac{1}{3} & -\frac{2}{15} \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ \frac{3}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

(2) $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix} \begin{array}{c} ① \\ ② \\ ③ \end{array} \xrightarrow[② - 2 \cdot ① \rightarrow ②]{} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ -6 & 5 & 4 \end{pmatrix}$

$\xrightarrow[③ - (-3) \cdot ① \rightarrow ③]{} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 14 & 10 \end{pmatrix} \xrightarrow[③ - 2 \cdot ② \rightarrow ③]{} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

(3)

$$\begin{array}{cc} \begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix} & \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} \end{array}$$

$$\xrightarrow[② - (-1)\cdot ① \to ②]{} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$$

$$\xrightarrow[③ - 4\cdot ① \to ③]{} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ -2 & -4 & 7 & 5 \end{pmatrix}$$

$$\xrightarrow[④ - (-2)\cdot ① \to ④]{} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{pmatrix}$$

$$\xrightarrow[\text{③} - 5\cdot\text{②} \to \text{③}]{} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{④} - (-1)\cdot\text{②} \to \text{④}]{} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. (a) $Ax = b \Rightarrow LUx = b$

$\Rightarrow Ux = L^{-1}b \Rightarrow$ take $y = L^{-1}b$

$\Rightarrow Ux = y \Rightarrow Ly = b$

$Ax = b \Longrightarrow Ux = y$ & $Ly = b$

Suppose we have $Ly = b$ & $Ux = y \Rightarrow LUx = Ly = b$

$\Rightarrow Ax = b$ ✓.

(b) $A = LU$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 2 & -7 & -7 & -6 & 7 \\ -1 & 2 & 6 & 4 & 0 \\ -4 & -1 & 9 & 8 & 3 \end{array}\right) \cdot(-2)$$

$Ax=b$

$$\xrightarrow{R_2 - 2 \cdot R_1 \to R_2} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ -1 & 2 & 6 & 4 & 0 \\ -4 & -1 & 9 & 8 & 3 \end{array}\right) \cdot(1)$$

$$\xrightarrow{R_3 - (-1) \cdot R_1 \to R_3} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ -4 & -1 & 9 & 8 & 3 \end{array}\right) \cdot(4)$$

$$\xrightarrow{R_4 - (-4) \cdot R_1 \to R_4} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & -9 & -7 & -4 & 7 \end{array}\right) \cdot(-3)$$

$$\xrightarrow{R_4 - 3\cdot R_2 \to R_4} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & -10 & -4 & -8 \end{array}\right) \cdot (5)$$

$$\xrightarrow{R_4 - (-5)\cdot R_3 \to R_4} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \quad \begin{pmatrix} -2 \\ -1 \\ 2 \\ -3 \end{pmatrix}$$

$$(5+5)+(5+5)+(5+5)+(3+4)+(3+3) = 43-5$$

-5 $=38$

$38+3+3+1=45$

$Ly=b,\ Ux=y$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 7 \\ -1 & 0 & 1 & 0 & 0 \\ -4 & 3 & -5 & 1 & 3 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array}$$

$$\xrightarrow{② - 2\cdot ① \to ②} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ -1 & 0 & 1 & 0 & 0 \\ -4 & 3 & -5 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{③-(-1)\cdot①}\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ -4 & 3 & -5 & 1 & 3 \end{array}\right) \qquad \begin{pmatrix} 1 \\ 5 \\ 1 \\ -3 \end{pmatrix}$$

$$(1+\overset{+2}{2})+2=7$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \end{array} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array}$$

$$\xrightarrow{③-④\to③}\left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$$\xrightarrow{②-\frac{1}{2}\cdot③\to②}\left(\begin{array}{cccc|c} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \qquad \begin{pmatrix} -2 \\ -1 \\ 2 \\ -3 \end{pmatrix}$$

$$2+(1+2+2)=7$$

$$7+7=14 \qquad 14+1+3+1+2+3+1=25$$

(C) For $Ax=b$

$$\left(\begin{array}{cccc|c} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} & b_n \end{array}\right) \xrightarrow{\text{大约}2n^2\text{次}} \left(\begin{array}{cccc|c} a_{11}' & a_{12}' & \cdots & a_{1n}' & c_1 \\ 0 & a_{22}' & \cdots & a_{2n}' & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}' & \cdots & a_{nn}' & c_n \end{array}\right)$$

$$\xrightarrow{2(n-1)^2\text{次}} \left(\begin{array}{ccccc|c} a_{11}' & a_{12}' & a_{13}' & \cdots & a_{1n}' & d_1 \\ 0 & a_{22}' & a_{23}' & \cdots & a_{2n}' & d_2 \\ 0 & 0 & a_{33}' & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & a_{n3}' & \cdots & a_{nn}' & d_n \end{array}\right) \longrightarrow \cdots$$

$$2\cdot\left(1^2+2^2+\cdots+(n-1)^2+n^2\right) \sim \frac{2}{3}n^3$$

$Ly=x$, $Ux=b$

对于 $Ux=b$

$$\left(\begin{array}{cccc|c} u_{11} & \cdots & \cdots & u_{1n} & b_1 \\ & \ddots & & \vdots & \vdots \\ & & u_{n-1,n-1} & u_{n-1,n} & \vdots \\ & & & u_{nn} & b_n \end{array}\right) \to \left(\begin{array}{cccc|c} u_{11} & \cdots & u_{1,n-1} & 0 & c_1 \\ & \ddots & & \vdots & \vdots \\ & & u_{n-1,n-1} & 0 & \vdots \\ & & & u_{nn} & c_n \end{array}\right)$$

(若考虑 $\frac{1}{u}$ 的运算，则为 $4(n-1)$ 次，若不考虑则为 $2(n-1)$ 次)

然后继续该步骤

$$\begin{array}{ll} 4(n-2) & \\ \Longrightarrow & \left(\begin{array}{ccccc|c} u_{1,1} & \cdots & 0 & 0 & & d_1 \\ & \ddots & 0 & \vdots & & \vdots \\ & & 0 & \vdots & & \vdots \\ 2(n-2) & & u_{n-1,n-1} & 0 & & \\ & & & u_{n,n} & & d_n \end{array}\right) \end{array}$$

$2(1+\cdots+(n-1)) \sim n^2$

…… 总结下来

$4(1+\cdots+(n-1)) \sim 2n^2$

对于 L 也类似

$\Longrightarrow$ 总共 $2n^2$ / $4n^2$ 次运算.