Answers for Hw. 13 GP II

By SJ

1. Griffiths' 2.29

Analyze the odd bound state wave functions for the finite potential well. Derive the transcendental equation for the allowed energies, and solve it graphically. Analyzing the two limiting cases (wide-deep well (meaning Z_0 big) and narrow-shallow well (small Z_0)) and is there always an odd bound state?

Answers from his solution manual:

Problem 2.29

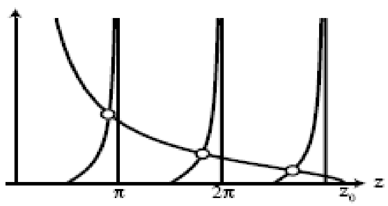
In place of Eq. 2.151, we have: $\psi(x) = \left\{ \begin{array}{ll} Fe^{-\kappa x} & (x > a) \\ D\sin(lx) & (0 < x < a) \\ -\psi(-x) & (x < 0) \end{array} \right\}.$

Continuity of ψ : $Fe^{-\kappa a} = D\sin(la)$; continuity of ψ' : $-F\kappa e^{-\kappa a} = Dl\cos(la)$.

Divide: $-\kappa = l\cot(la)$, or $-\kappa a = la\cot(la) \Rightarrow \sqrt{z_0^2 - z^2} = -z\cot z$, or $\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}.}$

Wide, deep well: Intersections are at $\pi, 2\pi, 3\pi$, etc. Same as Eq. 2.157, but now for n *even*. This fills in the rest of the states for the infinite square well.

Shallow, narrow well: If $z_0 < \pi/2$, there is *no* odd bound state. The corresponding condition on V_0 is $\boxed{V_0 < \frac{\pi^2\hbar^2}{8ma^2} \Rightarrow \text{no odd bound state.}}$



2. Griffiths P2.41

A particle of mass m in the harmonic potential starts out in the state:

$\psi(x,0) = A(1 - 2\sqrt{\frac{m\omega}{\hbar}}x)^2 e^{-\frac{m\omega}{2\hbar}x^2}$ for some constant A.

(a) What is the expectation value of energy?

(b) At some later time T, the wave function is:

$$\psi(x,T) = B(1+2\sqrt{\frac{m\omega}{\hbar}}x)^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

What is the smallest possible value of T?

Answer: It is going to be a hell of calculation if we directly integrate to get <H> using the wave function. By looking at the provides wave function we see it is in form of a polynomial multiply a Gaussian, and thus we expect it may relatively easy to decompose into components of stationary states of harmonic oscillator, of which are Hermitian polynomial times Gaussian.

Make substitution:

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x, C \equiv (\frac{m\omega}{\pi\hbar})^{1/4}$$

The stationary state of H.O. is in form of:

$$\psi_n = C\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$$

The first couple (up to x^2) are:

$$\psi_0 = Ce^{-\xi^2/2}$$
$$\psi_1 = \sqrt{2}C\xi e^{-\xi^2/2}$$
$$\psi_2 = \frac{C}{2\sqrt{2}}(4\xi^2 - 2)e^{-\xi^2/2}$$

The given wave function is:

$$\psi(x,0)=\psi(\xi,0)=A(1-2\xi)^2 e^{-\xi^2/2}=A(4\xi^2-4\xi+1)e^{-\xi^2/2}$$
$$=A[(4\xi^2-2)e^{-\xi^2/2}-4\xi e^{-\xi^2/2}+3e^{-\xi^2/2}]$$
$$=A[\frac{2\sqrt{2}}{C}\psi_2-\frac{2\sqrt{2}}{C}\psi_1+\frac{3}{C}\psi_0]$$

Normalization, it is easy to see:

$$\psi(\xi,0)=\frac{2\sqrt{2}}{5}\psi_2-\frac{2\sqrt{2}}{5}\psi_1+\frac{3}{5}\psi_0$$

(a) The expectation of energy is then:

$$<H>=\frac{8}{25}E_2+\frac{8}{25}E_1+\frac{9}{25}E_0=(\frac{8}{25}\frac{5}{2}+\frac{8}{25}\frac{3}{2}+\frac{9}{25}\frac{1}{2})\hbar\omega=\frac{73}{50}\hbar\omega$$

(b) $\psi(x,T)=B(1+2\sqrt{\frac{m\omega}{\hbar}}x)^2 e^{-\frac{m\omega}{2\hbar}x^2}$, same procedure will give:

$$\psi(x,t)=\psi(\xi,t)=\frac{2\sqrt{2}}{5}\psi_2+\frac{2\sqrt{2}}{5}\psi_1+\frac{3}{5}\psi_0 \qquad (1)$$, it comes from:

$$\psi(\xi,t)=\frac{2\sqrt{2}}{5}\psi_2 e^{-i5\omega t/2}-\frac{2\sqrt{2}}{5}\psi_1 e^{-i3\omega t/2}+\frac{3}{5}\psi_0 e^{-i\omega t/2}$$
$$=e^{-i\omega t/2}(\frac{2\sqrt{2}}{5}\psi_2 e^{-i2\omega t}-\frac{2\sqrt{2}}{5}\psi_1 e^{-i\omega t}+\frac{3}{5}\psi_0)$$

As $\omega t=\pi$, the middle term flips the sign from – to +, while the other terms stays same. The wave function would be same as (1) up to a common phase factor. So $T=\frac{\omega}{\pi}$.

3. Griffiths P2.42

Find the allowed energies of the half harmonic oscillator:

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2, & x > 0 \\ \infty & x < 0 \end{cases}$$

Hint: This requires some thought on even-odd parity but very little computation.

Answer: In the region x>0, the differential equation is exactly same as harmonic function, and solutions would be same.

However, the boundary conditions changes at x=0, the wave functions has to be zero there. Only the solutions in harmonic potential with odd parity can satisfy this condition at x=0.

So the accepted solutions are those odd functions with n=1,3,5…, and the energy is:

$$E_n = (2m + \frac{3}{2})\hbar\omega \quad m = 0,1,2...$$

4. Find the energy levels and wave functions of the two harmonic oscillators with m_1 and m_2, have identical frequency ω when isolated. Now suppose they are coupled by extra interaction potential

$$\frac{1}{2} K(\hat{x}_1 - \hat{x}_2)^2 .$$

Answer: The Hamiltonian is:

$$H = -\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} + \frac{m_1\omega^2}{2}x_1^2 + \frac{m_2\omega^2}{2}x_2^2 + \frac{1}{2}K(x_1 - x_2)^2$$

From what we learned in classical mechanics, the system is just like two pendulum, but with a spring of constant K connecting the two bobs. For this double coupled pendulum, we know the normal modes are motion of center of mass, and oscillation of relative position. So we shall try the same procedure to separate the Hamiltonian here by using center of mass and relative position, instead of x_1,x_2.

Let: $X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1x_1 + m_2x_2}{M}, x = x_1 - x_2 \quad M = m_1 + m_2$

Then $x_1 = \frac{m_2}{M}x + X; x_2 = -\frac{m_1}{M}x + X$

$$\frac{d}{dx_1} = \frac{d}{dX}\frac{dX}{dx_1} + \frac{d}{dx}\frac{dx}{dx_1} = \frac{m_1}{M}\frac{d}{dX} + \frac{d}{dx}$$

$$\frac{d^2}{dx_1^2} = (\frac{m_1}{M})^2\frac{d^2}{dX^2} + \frac{d^2}{dx^2} + \frac{m_1}{M}\frac{d^2}{dXdx} + 2\frac{d^2}{dxdX}\frac{m_1}{M}$$

$$\frac{d}{dx_2} = \frac{d}{dX}\frac{dX}{dx_2} + \frac{d}{dx}\frac{dx}{dx_2} = \frac{m_2}{M}\frac{d}{dX} - \frac{d}{dx}$$

$$\frac{d^2}{dx_2^2} = (\frac{m_2}{M})^2\frac{d^2}{dX^2} + \frac{d^2}{dx^2} - 2\frac{m_2}{M}\frac{d^2}{dXdx}$$

$$-\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} = -\frac{\hbar^2}{2}[\frac{m_1}{M^2}\frac{d^2}{dX^2} + \frac{m_2}{M^2}\frac{d^2}{dX^2} + (\frac{1}{m_1} + \frac{1}{m_2})\frac{d^2}{dx^2}]$$

$$= -\frac{\hbar^2}{2M}\frac{d^2}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}$$

$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ μ is just reduced mass

$$\frac{m_1\omega^2}{2}x_1^2+\frac{m_2\omega^2}{2}x_2^2=\frac{m_1\omega^2}{2}(\frac{m_2^2}{M^2}x^2+X^2+\frac{2m_2xX}{M})+\frac{m_2\omega^2}{2}(\frac{m_1^2}{M^2}x^2+X^2-\frac{2m_1xX}{M})$$
$$=\frac{m_1m_2^2+m_2m_1^2}{2M^2}\omega^2x^2+\frac{m_1+m_2}{2}\omega^2X^2=\frac{\mu}{2}\omega^2x^2+\frac{M}{2}\omega^2X^2$$

Thus the Hamiltonian becomes (in terms of X and x):

$$H=(-\frac{\hbar^2}{2M}\frac{d^2}{dX^2}+\frac{M\omega^2}{2}X^2)+(-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+\frac{\mu\omega^2}{2}x^2+\frac{1}{2}Kx^2)=H_{CM}+H_\mu$$

$$H_\mu=-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+\frac{\mu\omega^2}{2}x^2+\frac{1}{2}Kx^2=-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}+\frac{\mu\Omega^2x^2}{2}$$
$$\Omega^2=\omega^2+K/\mu$$

With the H separated into two parts, it is easy to use separation of variables. The total wave function would be:

$\psi(X,x)=\psi(X)\phi(x)$ each is an eigenfunction of a harmonic oscillator with different mass and frequency, i.e.:

$$H_{CM}\psi(X)=E_{CM}\psi(X)$$
$$H_\mu\phi(x)=E_\mu\phi(x)$$

The total energy would be:

$$E=E_{CM}+E_\mu=(n_1+\frac{1}{2})\hbar\omega+(n_2+\frac{1}{2})\hbar\Omega\quad n_{1,2}=0,1,2...$$

The motion is separated into CM oscillating like a pendulum, and the relative motion oscillate at different frequency, analogous to the result in classical mechanics.

The corresponding wave function are just product of 1-D harmonic wave-functions corresponding to n1 and n2 with (x, ω) and (X, Ω) separately.

5. For the ground state harmonic oscillator, we know it satisfy the lower

limit (wave function is a Gaussian) of uncertainty relation: $\Delta x \Delta p = \frac{\hbar}{2}$,

where $\Delta A \equiv (<\hat{A}^2> - <\hat{A}>^2)^{1/2}$, prove this is consistent with the so-called virial theorem (for harmonic potential) that <T>=<V>.

Answer:

For the ground state, the energy
$$<H> = <T> + <V> = \frac{1}{2}\hbar\omega$$
$$<T> = \frac{<\hat{P}^2>}{2m}$$

Also the <x>, <p>=0 using symmetry argument.

$$(\Delta x)^2 = <X^2> = \frac{2<V>}{m\omega^2}$$

$$(\Delta p)^2 = <P^2> = 2m<T>$$

The problem already gives: $\Delta x \Delta p = \frac{\hbar}{2}$

$$(\Delta x)^2(\Delta p)^2 = 4<V><T>/\omega^2 = \hbar^2/4$$

Then $<T> = <V> = \frac{\hbar\omega}{4}$

This is consistent with the virial theorem for the harmonic potential.

6. Answer:

(a) It is important that the state are eigenstate of H.

We can write: $E_n = <\psi_n | H | \psi_n>$, then:

$$\frac{\partial E_n}{\partial\lambda} = \frac{\partial <\psi_n|}{\partial\lambda} H|\psi_n> + <\psi_n|H\frac{\partial|\psi_n>}{\partial\lambda} + <\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n>$$

$$\frac{\partial <\psi_n|}{\partial\lambda} H|\psi_n> + <\psi_n|H\frac{\partial|\psi_n>}{\partial\lambda} = E_n\left(\frac{\partial <\psi_n|}{\partial\lambda}|\psi_n> + <\psi_n|\frac{\partial|\psi_n>}{\partial\lambda}\right)$$

$$= E_n\frac{\partial <\psi_n|\psi_n>}{\partial\lambda} = 0$$

This is because $<\psi_n|\psi_n>=1$ is required to be normalized.

Then $\frac{\partial E_n}{\partial\lambda} = <\frac{\partial H}{\partial\lambda}>_{|\psi_n>}$

(b) For harmonic oscillator:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2, E = (n+1/2)\hbar\omega$$

(1) $\lambda = \hbar$

$$\frac{\partial H}{\partial\lambda} = -\frac{2\hbar}{2m}\frac{d^2}{dx^2} = \frac{2}{\hbar}\frac{P^2}{2m} = \frac{2}{\hbar}T$$

$$\frac{\partial E}{\partial\lambda} = (n+1/2)\omega$$

$$(n+1/2)\omega = <\frac{2}{\hbar}T> \rightarrow <T> = \frac{(n+1/2)\hbar\omega}{2} = \frac{E_n}{2}$$

(2) $\lambda = \omega$

$$\frac{\partial H}{\partial\lambda} = \frac{m2\omega}{2}x^2 = \frac{2}{\omega}V, \partial E/\partial\lambda = (n+1/2)\hbar$$

Then: $<V> = \frac{E_n}{2}$

(3) $\lambda = m$

$$\frac{\partial H}{\partial\lambda} = \frac{\hbar^2}{2m^2}\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2} = \frac{1}{m}(-T+V)$$

$$\partial E/\partial\lambda = 0$$

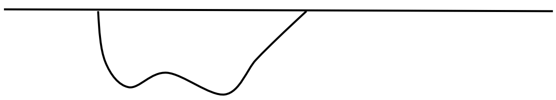
Then:

$$\frac{1}{m} < -T + V >= 0 \rightarrow < T >=< V >$$

7. (Extra problem applying Hellman-Feynman)

(a) Using Hellman-Feynman to prove that if we have two potential $V_1(x)$ and $V_2(x)$, where $V_2(x) > V_1(x)$ for all x; then eigen-energies for these two potential have relation: $E_{2n} > E_{1n}$.

(b) For a 1-D arbitrary potential (continuous) well as sketched in figure;



There is always at least one bound state inside the well.

Answer:

(a) We may think the two potential belong to a general class of potential with parameter $V(\lambda, x)$, where:

$$\frac{\partial V}{\partial \lambda} > 0$$

Then V_1 is the one with smaller and V_2 with larger λ, as the condition given. (There are possibly many ways you can construct such potentials with parameter satisfy above, a simple one is: $V(\lambda, x) = (1-\lambda)V_1 + \lambda V_2$

V_1,V_2 corresponds to $\lambda = 0,1$, and $\frac{\partial V}{\partial \lambda} = V_2 - V_1 > 0$, the following proof does not depend on such construction)

Now we shall apply Hellman-Feynman theorem:

$$\frac{\partial E_n}{\partial \lambda} = <\frac{\partial H}{\partial \lambda}>$$

$$H = \frac{P^2}{2m} + V(\lambda, x)$$

The momentum operator is not dependent on λ, and we have:

$$\frac{\partial H}{\partial \lambda} = \frac{\partial V}{\partial \lambda}$$

$$\frac{\partial E_n}{\partial \lambda} = <\frac{\partial V}{\partial \lambda}> = \int \psi_n^* \frac{\partial V}{\partial \lambda} \psi dx$$

$\frac{\partial V}{\partial \lambda} = f(\lambda, \hat{x})$ and function of parameter and x operator, its action on wave function is just multiply the function $f(\lambda, x)$ (here x is just position (eigen)value) with wave function, then:

$$\int \psi_n^* \frac{\partial V}{\partial \lambda} \psi dx = \int \frac{\partial V}{\partial \lambda} |\psi|^2 dx$$

The integrand above are all positive values, so the integral has to positive too, this means:

$$\frac{\partial E_n}{\partial \lambda} > 0$$

E_{2n} is the energy corresponds to larger λ, then $E_{2n}>E_{1n}$

(b) We can construct two square finite potential and using the property we learned from them. One square potential will be very shallow in depth, so that $V_1(x)>V(x)$ (because it is a well, so the shallow well is above); The other square potential is large in depth so that $V_2(x)<V(x)$ (the V_2 well is always below).

From what we have proved above, we know that the energy of bound state (if any) for our arbitrary well has to sandwiched between that of V_2 and V_1, i.e. $E_{1n}>E_n>E_{2n}$. From what we learned about the square potential, we know no matter how shallow the well is, there is always a solution (a bound energy level, corresponding to the $\tan z = \sqrt{(z_0^2 / z^2) - 1}$). So there will be at least one energy solution for the arbitrary well.

8. From the Ehrenfest theorem, we have following relations:

$$\frac{d<X>(t)}{dt} = \frac{<P>(t)}{m} \; ; \qquad (1)$$

$$\frac{d<P>(t)}{dt} = -<\frac{dV}{dx}>(t) = -k<X>(t) = -m\omega^2<X>(t) \qquad (2)$$

Put (1) into (2) we shall get familiar 2nd ODE on x:

$\frac{d^2<X>}{dt^2} = -\omega^2 <X>$ and we know the general solution for this ODE is in forms of:

$$<X>(t) = A cos\omega t + B sin\omega t;$$

We use initial condition to fix A, B, at t=0; <X>(t=0)=x_0; <P>(t=0)=p_0:

$$A = x_0;$$

$$B\omega = \frac{p_0}{m}; \;\; then\ B = \frac{p_0}{m\omega}$$

Thus: $<X>(t) = x_0 cos\omega t + \frac{p_0}{m\omega} sin\omega t$

After knowing <X>(t), either from (1) or (2), we get:

$$<P>(t) = p_0 cos\omega t - m\omega x_0 sin\omega t$$

9.

1） $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$

$\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}$

For n=2, H polynomial: ($a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j$

$a_2 = 4;\ a_0 = -2$

$\psi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{2\sqrt{2}} \left(4\frac{m\omega}{\hbar}x^2 - 2\right) e^{-\frac{m\omega}{2\hbar}x^2}$

For n=6: $H(\xi) = 2^6\xi^6 + a_4\xi^4 + a_2\xi^2 + a_0$

With $a_4 = -480;\ a_2 = 720;\ a_0 = -120$

$\psi_6 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{96\sqrt{5}} (64\xi^6 - 480\xi^4 + 720\xi^2 - 120) e^{-\frac{m\omega}{2\hbar}x^2}$

(I skip to replace for x in the polynomial)

2) Only need to calculate $< \psi_0|\psi_2 >= \int_{-\infty}^{\infty} \psi_0^*\psi_2 dx$ to see whether it is 0; $< \psi_0|\psi_1 >$ and $< \psi_1|\psi_2 >$, the integrand is odd function times even function so the integrand is odd, and integral will be zero.

Neglect the any common factor in front of the integral:

$< \psi_0|\psi_2 >= \int_{-\infty}^{\infty} \psi_0^*\psi_2 dx \propto \int_0^{\infty} (4\xi^2 - 2) e^{-\xi^2} dx$

$\int_0^{\infty} 4\xi^2 e^{-\xi^2} dx = 4\frac{\sqrt{\pi}}{4} = \sqrt{\pi}$ (use the integration table)

$\int_0^{\infty} -2e^{-\xi^2} dx = -\sqrt{\pi}$

So $< \psi_0|\psi_2 >= \int_{-\infty}^{\infty} \psi_0^*\psi_2 dx = 0$ as expected.