练习1. (1) $\frac{e^z}{z^2+1}$.

解：孤立奇点有 $z=i, z=-i, \infty$.

$z=i$ 为简单极点. $Res[\frac{e^z}{z^2+1}, i] = \frac{e^i}{2i}$.

$z=-i$ 为简单极点. $Res[\frac{e^z}{z^2+1}, -i] = \frac{e^{-i}}{-2i} = \frac{i}{2e^i}$.

$z=\infty$: $\lim_{z\to\infty} \frac{e^z}{z^2+1} = \infty$. → 为极点.

$Res[\frac{e^z}{z^2+1}, \infty] = 0 - Res[\frac{e^z}{z^2+1}, i] - Res[\frac{e^z}{z^2+1}, -i]$

$= -\frac{e^i}{2i} - \frac{i}{2e^i} = \frac{i}{2}e^i - \frac{i}{2e^i}$.

(2). $z^2\cos\frac{z}{z-1}$.

解：孤立奇点有 $z=1, \infty$.

$z=1$: $\lim_{z\to 1}\left|(z-1)z^2\cos\frac{z}{z-1}\right| \le \lim_{z\to 1}\left|(z-1)z^2\right| = 0$.

$Res[z^2\cos\frac{z}{z-1}, 0] = 0$.

$z=\infty$. $Res[z^2\cos\frac{z}{z-1}, 0] = Res[-\cos\frac{1}{1-z}, 0] = 0$.

练习2. (1) $\oint_{|z|=2} \frac{z^{2n}}{z^n+1} \cdot dz$. 其中 n 为正整数.

解：孤立奇点有 $z = e^{i(\pi+\frac{2k\pi}{n})}$, $k=1,2,\dots,n$. 为简单奇点.

原式 $=2\pi i \sum_{k=1}^{n} Res[\frac{z^{2n}}{z^n+1}, z_k]$.

$=2\pi i \sum_{k=1}^{n} \frac{z_k^{2n}}{n z_k^{n-1}} = 2\pi i \sum_{k=1}^{n} \frac{z_k^{2n}\cdot z_k}{n\cdot z_k^n} = \frac{-2\pi i}{n}\sum_{k=1}^{n} z_k = \begin{cases} -2\pi i, & n=1 \\ 0, & n\ge 2 \end{cases}$.

(2). $\oint_{|z|=3} \tan(\pi z)dz$.

解. 孤立奇点有 $z=\frac{1}{2}+k$, $k\in\mathbb{Z}$. 在 $|z|=3$ 内有: $-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. 为简单极点.

原式 $= 2\pi i \sum_k Res[\tan(\pi z), z_k] = 2\pi i \sum_k \frac{\sin\pi z_k}{-\pi\sin\pi z_k} = 2\pi i \sum_k -\frac{1}{\pi}$.

$= -12i$

(3). $\oint_{|z|=R} \frac{z^2}{1-e^{2\pi i z^3}} \cdot dz$. $n < R^3 < n+1$. n为正整数.

解：孤立奇点有：$z^3=k$, $k\in \mathbb{Z}/\{0\}$ 在$|z|=R$内有 $z_{km}=e^{i\frac{2m\pi}{3}}\sqrt[3]{k}$. $m=1,2,3$. $k=\pm1,\pm2,\cdots,\pm n$.

$z=0$. $\Rightarrow$ 简单极点.

$$原式 = 2\pi i\left(\sum_{k,m} Res\left[\frac{z^2}{1-e^{2\pi i z^3}}, z_{km}\right] + Res\left[\frac{z^2}{1-e^{2\pi i z^3}}, 0\right]\right).$$

$$= -\frac{1}{3}\sum_{k,m}\frac{z_{km}^2}{z_{km}^2} + 2\pi i\, Res\left[\frac{z^2}{1-e^{2\pi i z^3}}, 0\right] = -2n + 2\pi i \lim_{z\to 0}\frac{z^3}{1-e^{2\pi i z^3}}$$

$$= -2n + 2\pi i\cdot\left(-\frac{6}{12\pi i}\right) = -2n-1.$$

练习3. $R(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$.

解：当$|z|>>1$时，

$$|R(z)| = \left|\frac{a_n}{b_m}\right|\left|\frac{1}{z^{m-n}}\right|\cdot C_1 \le \frac{A}{|z|^{m-n}}.\quad A\in\mathbb{R}.$$

$$\left|\oint_c R(z)\cdot dz\right| \le \oint_c |R(z)||dz|.\quad C: |z|=R,\ R\to\infty.$$

$$\le \oint_{C_L}\frac{A}{|z|^{m-n}}\cdot|dz| = \frac{A}{R^{m-n}}\cdot 2\pi R = \frac{2\pi A}{R^{m-n-1}} \xrightarrow{R\to\infty} 0.$$

$$\to \oint_c R(z)\cdot dz = 0 = -2\pi i\, Res[R(z),\infty].$$

$$\to Res[R(z),\infty] = 0.$$