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*10. If X_α has the Poisson distribution $\pi(\alpha)$, then

$$\lim_{\alpha\to\infty} P\left\{\frac{X_\alpha-\alpha}{\sqrt{\alpha}} \le u\right\} = \Phi(u)$$

for every u. [Hint: use the Laplace transform $E(e^{-\lambda(X_\alpha-\alpha)/\sqrt{\alpha}})$, show that as $\alpha\to\infty$ it converges to $e^{\lambda^2/2}$, and invoke the analogue of Theorem 9 of §7.5.]

$$E(e^{-\lambda(X_\alpha-\alpha)/\sqrt{\alpha}}) = \sum_{k=0}^{\infty} \frac{e^{-\alpha}}{k!}\alpha^k e^{-\frac{\lambda(k-\alpha)}{\sqrt{\alpha}}} = \exp(\alpha e^{-\frac{\lambda}{\sqrt{\alpha}}}-\alpha+\lambda\sqrt{\alpha}).$$

$$\lim_{\alpha\to\infty}\exp(\alpha e^{-\frac{\lambda}{\sqrt{\alpha}}}-\alpha+\lambda\sqrt{\alpha}) = \lim_{\alpha\to\infty}\exp(\alpha-\alpha\frac{\lambda}{\sqrt{\alpha}}+\alpha\cdot\frac{\lambda^2}{2\alpha}-\alpha+\lambda\sqrt{\alpha})$$

$$= \exp(\frac{\lambda^2}{2}).$$

由 Theorem 9: $\lim_{\alpha\to\infty} P\left\{\frac{X_\alpha-\alpha}{\sqrt{\alpha}}\le u\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b} e^{-u^2/2}du = \Phi(u).$

12. On a certain highway the flow of traffic may be assumed to be Poissonian with intensity equal to 30 cars per minute. Write down the probability that it takes more than N seconds for n consecutive cars to pass by an observation post. [Hint: use (7.2.11).]

$$\alpha = 30/60 = 0.5.$$

$$P(N(t)=n) = P(S_n<t) - P(S_{n+1}<t).$$

$$= \int_0^t f_n(u)du - \int_0^t f_{n+1}(u)du$$

$$= \int_0^t \frac{\alpha^n}{(n-1)!}u^{n-1}e^{-\alpha u}du - \int_0^t \frac{\alpha^{n+1}}{n!}u^n e^{-\alpha u}du.$$

$$= \int_0^t \frac{\alpha^n}{(n-1)!}(u^{n-1}-\frac{\alpha}{n}u^n)e^{-\alpha u}du = \frac{\alpha^n}{(n-1)!}\cdot\frac{1}{n}u^n e^{-\alpha u}\Big|_0^t$$

$$= \frac{\alpha^n}{n!}t^n e^{-\alpha t}.$$

$$= \frac{(1/2)^n}{n!}N^n e^{-\frac{N}{2}}.$$

13. A perfect die is rolled 100 times. Find the probability that the sum of all points obtained is between 330 and 380.

$$\mu = \frac{7}{2}.\quad \sigma^2 = \frac{91}{6}-\frac{49}{4} = \frac{35}{12}$$

$$P(a\le \frac{S_n-n\mu}{\sigma\sqrt{n}} \le b) \approx \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du$$

$$\approx 0.8397.$$

15. Two movie theaters compete for 1000 customers. Suppose that each customer chooses one of the two with "total indifference" and independently of other customers. How many seats should each theater have so that the probability of turning away any customer for lack of seats is less than 1%?

$$P(a \le \frac{X_n-np}{\sqrt{npq}} \le \infty) = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-u^2/2}du < 1\%$$

$$a \ge 2.33.$$

$$X_n \ge a\sqrt{npq}+np \approx 537.$$

16. A sufficient number of voters are polled to determine the percentage in favor of a certain candidate. Assuming that an unknown proportion p of the voters favor him and they act independently of one another, how many should be polled to predict the value of p within 4.5% with 95% confidence? [This is the so-called four percent margin of error in predicting elections, presumably because $<.045$ becomes $\le .04$ by the rule of rounding decimals.]

$$-1.96 \le \frac{np-u}{\sqrt{np(1-p)}} \le 1.96.$$

$$p-1.96\sqrt{\frac{p(1-p)}{n}} \le \frac{u}{n} \le p+1.96\sqrt{\frac{p(1-p)}{n}}.$$

$$p+1.96\sqrt{\frac{p(1-p)}{n}} \le 4.5\%$$

$$n \ge \frac{1.96^2 p(1-p)}{(4.5\%-p)^2}$$

17. Write $\Phi((a,b))$ for $\Phi(b)-\Phi(a)$, where $a<b$ and Φ is the unit normal distribution. Show that $\Phi((0,2)) > \Phi((1,3))$ and generalize to any two intervals of the same length. [Hint: $e^{-x^2/2}$ decreases as $|x|$ increases.]

$$\Phi((a,b)) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du.$$

$$\Phi((0,2)) - \Phi((1,3)) = \frac{1}{\sqrt{2\pi}}(\int_0^2 e^{-u^2/2}du - \int_1^3 e^{-u^2/2}du)$$

$$= \frac{1}{\sqrt{2\pi}}(\int_0^1 e^{-u^2/2}du - \int_2^3 e^{-u^2/2}du)$$

$$> \frac{1}{\sqrt{2\pi}}(e^{-1^2/2} - e^{-2^2/2}) > 0.$$

$\forall$ 等长区间 (a,b) 和 (c,d). $a<c$. $b-a=d-c=L$

若 $b\le c$. $\Phi((a,b))-\Phi((c,d)) = \frac{1}{\sqrt{2\pi}}(\int_a^b e^{-u^2/2}du - \int_c^d e^{-u^2/2}du) > \frac{1}{\sqrt{2\pi}}(L e^{-b^2/2} - L e^{-c^2/2}) \ge 0.$

若 $b>c$. 上式 $= \frac{1}{\sqrt{2\pi}}(\int_a^c e^{-u^2/2}du - \int_b^d e^{-u^2/2}du) > \frac{1}{\sqrt{2\pi}}(L\cdot e^{-c^2/2} - L\cdot e^{-b^2/2}) > 0.$