第 9 次作业题

1. 计算 $\oiint\limits_{S^+} z^2\,\mathrm{d}x\wedge\mathrm{d}y$, 其中 S^+ 为球面 $x^2+y^2+(z-R)^2=R^2$ 的外侧.

解: 球面的参数方程为

$$\begin{cases} x=R\sin\theta\cos\varphi, \\ y=R\sin\theta\sin\varphi, \quad (0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi), \\ z=R+R\cos\theta, \end{cases}$$

由此我们立刻可得

$$\frac{D(x,y)}{D(\theta,\varphi)}=\begin{vmatrix} R\cos\theta\cos\varphi & -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi & R\sin\theta\cos\varphi \end{vmatrix}=R^2\sin\theta\cos\theta.$$

在点 $(\frac{\sqrt{2}}{2}R,0,\frac{2+\sqrt{2}}{2}R)$ 处, 我们有 $\theta=\frac{\pi}{4}$, $\varphi=0$, S^+ 的正向为 $(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})^T$, 而 $\frac{D(x,y)}{D(\theta,\varphi)}=\frac{1}{2}R^2$, 于是我们有

$$\begin{aligned}
\oiint\limits_{S^+} z^2\,\mathrm{d}x\wedge\mathrm{d}y &= \iint\limits_{\substack{0\leqslant\varphi\leqslant2\pi\\0\leqslant\theta\leqslant\pi}} z^2\frac{D(x,y)}{D(\theta,\varphi)}\mathrm{d}\theta\mathrm{d}\varphi \\
&= \int_0^{2\pi}\Big(\int_0^{\pi}(R+R\cos\theta)^2R^2\sin\theta\cos\theta\,\mathrm{d}\theta\Big)\mathrm{d}\varphi \\
&= 2\pi R^4\int_0^{\pi}(1+\cos\theta)^2\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= -2\pi R^4\int_0^{\pi}(\cos\theta+2\cos^2\theta+\cos^3\theta)\,\mathrm{d}(\cos\theta) \\
&= -2\pi R^4\Big(\frac{1}{2}\cos^2\theta+\frac{2}{3}\cos^3\theta+\frac{1}{4}\cos^4\theta\Big)\Big|_0^{\pi} \\
&= \frac{8}{3}\pi R^4.
\end{aligned}$$

2. 计算 $\iint\limits_{S^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x$, 其中 S^+ 是旋转抛物面 $z=x^2+y^2$, 柱面 $x^2+y^2=1$ 和坐标平面在第一卦限中所围立体表面的外侧.

解: 曲面 S 由以下四个部分组成:

$$\begin{aligned}
S_1: &\quad y=0\ (0\leqslant x\leqslant 1,\ 0\leqslant z\leqslant x^2), \\
S_2: &\quad x^2+y^2=1\ (0\leqslant x,y\leqslant 1), \\
S_3: &\quad x=0\ (0\leqslant y\leqslant 1,\ 0\leqslant z\leqslant y^2), \\
S_4: &\quad z=x^2+y^2\ (x^2+y^2\leqslant 1, x,y\geqslant 0), \\
S_5: &\quad z=0\ (x^2+y^2\leqslant 1, x,y\geqslant 0),
\end{aligned}$$

由此我们立刻可得

$$\iint\limits_{S_1^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x=0,$$
$$\iint\limits_{S_3^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x=0,$$
$$\iint\limits_{S_5^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x=0.$$

曲面 S_2 的参数方程为 $\begin{cases} x=\cos\varphi, \\ y=\sin\varphi, \quad (0\leqslant\varphi\leqslant\frac{\pi}{2},\ 0\leqslant z\leqslant 1), \\ z=z, \end{cases}$ 于是

$$\begin{aligned}
\frac{D(x,y)}{D(\varphi,z)} &= 0,\\
\frac{D(y,z)}{D(\varphi,z)} &= \begin{vmatrix} \cos\varphi & 0 \\ 0 & 1 \end{vmatrix}=\cos\varphi,\\
\frac{D(z,x)}{D(\varphi,z)} &= \begin{vmatrix} 0 & 1 \\ -\sin\varphi & 0 \end{vmatrix}=\sin\varphi.
\end{aligned}$$

在点 $(1,0,0)$ 处, $\varphi=z=0$, 曲面 S_2 的正向为 $(1,0,0)^T$, 而 $\vec{n}=(1,0,0)^T$, 则

$$\begin{aligned}
&\iint\limits_{S_2^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x\\
=&\iint\limits_{\substack{0\leqslant\varphi\leqslant\frac{\pi}{2}\\0\leqslant z\leqslant 1}}\Big(y^2z\frac{D(x,y)}{D(\varphi,z)}+z^2x\frac{D(y,z)}{D(\varphi,z)}+x^2y\frac{D(z,x)}{D(\varphi,z)}\Big)\,\mathrm{d}\varphi\mathrm{d}z\\
=&\iint\limits_{\substack{0\leqslant\varphi\leqslant\frac{\pi}{2}\\0\leqslant z\leqslant 1}}\big(z^2\cos^2\varphi+(\cos^2\varphi)(\sin^2\varphi)\big)\,\mathrm{d}\varphi\mathrm{d}z\\
=&\int_0^{\frac{\pi}{2}}\Big(\int_0^1\big(z^2\cos^2\varphi+(\cos^2\varphi)(\sin^2\varphi)\big)\,\mathrm{d}z\Big)\,\mathrm{d}\varphi\\
=&\int_0^{\frac{\pi}{2}}\Big(\frac{1}{3}\cos^2\varphi+\frac{1}{4}\sin^2 2\varphi\Big)\,\mathrm{d}\varphi\\
=&\Big(\frac{1}{6}\Big(\varphi+\frac{1}{2}\sin 2\varphi\Big)+\frac{1}{8}\Big(\varphi-\frac{1}{4}\sin 4\varphi\Big)\Big)\Big|_0^{\frac{\pi}{2}}\\
=&\frac{7\pi}{48}.
\end{aligned}$$

曲面 S_4 的方程为 $z=x^2+y^2\ (x^2+y^2\leqslant 1, x,y\geqslant 0)$, 于是

$$\frac{D(x,y)}{D(x,y)}=1,\ \frac{D(y,z)}{D(x,y)}=\begin{vmatrix} 0 & 1 \\ 2x & 2y \end{vmatrix}=-2x,\ \frac{D(z,x)}{D(x,y)}=\begin{vmatrix} 2x & 2y \\ 1 & 0 \end{vmatrix}=-2y.$$

在点 $(0,0,0)$ 处, 曲面 S_4 的正向为 $(0,0,1)^T$, 而 $\vec{n}=(0,0,1)^T$, 则

$$\begin{aligned}
&\iint\limits_{S_4^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x\\
=\;&\iint\limits_{\substack{x^2+y^2\leqslant 1\\ x,y\geqslant 0}}\Big(y^2z\frac{D(x,y)}{D(\varphi,z)}+z^2x\frac{D(y,z)}{D(\varphi,z)}+x^2y\frac{D(z,x)}{D(\varphi,z)}\Big)\,\mathrm{d}x\mathrm{d}y\\
=\;&\iint\limits_{\substack{x^2+y^2\leqslant 1\\ x,y\geqslant 0}}\Big(y^2(x^2+y^2)-2x^2(x^2+y^2)^2-2x^2y^2\Big)\,\mathrm{d}x\mathrm{d}y\\
\overset{\substack{x=\rho\cos\varphi\\ y=\rho\sin\varphi}}{=}\;&\int_0^{\frac{\pi}{2}}\Big(\int_0^1(\rho^4\sin^4\varphi-2\rho^6\cos^2\varphi-\rho^4\cos^2\varphi\sin^2\varphi)\rho\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\
=\;&\int_0^{\frac{\pi}{2}}\Big(\frac{1}{6}\rho^6\sin^4\varphi-\frac{1}{4}\rho^8\cos^2\varphi-\frac{1}{6}\rho^6(\cos^2\varphi)\sin^2\varphi\Big)\Big|_0^1\,\mathrm{d}\varphi\\
=\;&\int_0^{\frac{\pi}{2}}\Big(\frac{1}{6}\sin^4\varphi-\frac{1}{4}\cos^2\varphi-\frac{1}{6}(\cos^2\varphi)\sin^2\varphi\Big)\,\mathrm{d}\varphi\\
=\;&\int_0^{\frac{\pi}{2}}\Big(\frac{1}{6}(\sin^2\varphi)(1-\cos^2\varphi)-\frac{1}{4}\cos^2\varphi-\frac{1}{6}(\cos^2\varphi)\sin^2\varphi\Big)\,\mathrm{d}\varphi\\
=\;&\Big(\frac{1}{12}\Big(\varphi-\frac{1}{2}\sin 2\varphi\Big)-\frac{1}{8}\Big(\varphi+\frac{1}{2}\sin 2\varphi\Big)-\frac{1}{24}\Big(\varphi-\frac{1}{4}\sin 4\varphi\Big)\Big)\Big|_0^{\frac{\pi}{2}}\\
=\;&-\frac{\pi}{24}.
\end{aligned}$$

从而我们最终有

$$\begin{aligned}
\iint\limits_{S^+} y^2z\,\mathrm{d}x\wedge\mathrm{d}y+z^2x\,\mathrm{d}y\wedge\mathrm{d}z+x^2y\,\mathrm{d}z\wedge\mathrm{d}x &= \frac{7\pi}{48}-\frac{\pi}{24}\\
&= \frac{5\pi}{48}.
\end{aligned}$$

3. 求流速场 $\vec{V}=xy\vec{i}+yz\vec{j}+zx\vec{k}$ 由里向外穿过球面 $x^2+y^2+z^2=1$ 位于第一卦限部分的流量.

解: 设 S 为球面 $x^2+y^2+z^2=1$ 在第一卦限的部分, 它在点 (x,y,z) 处的正向为 $\vec{n}^0=(x,y,z)^T$, 该曲面的参数方程为

$$\begin{cases} x=\sin\theta\cos\varphi,\\ y=\sin\theta\sin\varphi, \quad \big(0\leqslant\theta\leqslant\dfrac{\pi}{2},\ 0\leqslant\varphi\leqslant\dfrac{\pi}{2}\big),\\ z=\cos\theta,\end{cases}$$

于是 $\mathrm{d}\sigma=\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi$, 从而所求流量为

$$
\begin{aligned}
Q &= \iint\limits_{S^+}\vec{V}\cdot\mathrm{d}\vec{\sigma}=\iint\limits_{S}\vec{V}\cdot\vec{n}^0\,\mathrm{d}\sigma=\iint\limits_{S}(x^2y+y^2z+z^2x)\,\mathrm{d}\sigma\\
&= \int_0^{\frac{\pi}{2}}\Big(\int_0^{\frac{\pi}{2}}\big((\sin\theta\cos\varphi)^2\sin\theta\sin\varphi+(\sin\theta\sin\varphi)^2\cos\theta\\
&\quad +(\cos\theta)^2\sin\theta\cos\varphi\big)\sin\theta\,\mathrm{d}\varphi\Big)\,\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\Big(\int_0^{\frac{\pi}{2}}\big((\sin^3\theta)(\sin\varphi)\cos^2\varphi+(\sin^2\theta)(\cos\theta)\sin^2\varphi\\
&\quad +(\sin\theta)(\cos^2\theta)\cos\varphi\big)\sin\theta\,\mathrm{d}\varphi\Big)\,\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\Big(-\frac{1}{3}(\sin^4\theta)\cos^3\varphi+\frac{1}{2}(\sin^3\theta)(\cos\theta)\Big(\varphi-\frac{1}{2}\sin 2\varphi\Big)\\
&\quad +(\sin^2\theta)(\cos^2\theta)\sin\varphi\Big)\Big|_0^{\frac{\pi}{2}}\,\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\Big(\frac{1}{3}\sin^4\theta+\frac{\pi}{4}(\sin^3\theta)\cos\theta+(\sin^2\theta)\cos^2\theta\Big)\,\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\Big(\frac{1}{12}(1-\cos 2\theta)^2+\frac{\pi}{4}(\sin^3\theta)\cos\theta\\
&\quad +\frac{1}{4}(1-\cos 2\theta)(1+\cos 2\theta)\Big)\,\mathrm{d}\theta\\
&= \int_0^{\frac{\pi}{2}}\Big(\frac{1}{12}(1-2\cos 2\theta+\cos^2 2\theta)+\frac{\pi}{4}(\sin^3\theta)\cos\theta\\
&\quad +\frac{1}{4}(1-\cos^2 2\theta)\Big)\,\mathrm{d}\theta\\
&= \Big(\frac{1}{12}\Big(\theta-\sin 2\theta+\frac{1}{2}\Big(\theta+\frac{1}{4}\sin 4\theta\Big)\Big)+\frac{\pi}{16}\sin^4\theta\\
&\quad +\frac{1}{4}\Big(\theta-\frac{1}{2}\Big(\theta+\frac{1}{4}\sin 4\theta\Big)\Big)\Big)\Big|_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{16}+\frac{\pi}{16}+\frac{\pi}{16}\\
&= \frac{3\pi}{16}.
\end{aligned}
$$