第 2 次作业题解答

1. 求证: 函数 $f(x,y)=\begin{cases}\frac{x^3}{y}, & y\neq 0\\ 0, & y=0\end{cases}$ 在原点处不连续, 但沿任何方向的方向导数均存在.

证明: 因 $\lim\limits_{x\to 0} f(x,x^3)=1\neq f(0,0)$, 由复合函数极限法则知 f 在原点间断.

设 $\vec{\ell}^0=(a,b)^T$ 为任意单位向量. 若 $b\neq 0$, 则我们有

$$\frac{\partial f}{\partial \vec{\ell}^0}(0,0)=\lim_{h\to 0^+}\frac{f(ah,bh)-f(0,0)}{h}=\lim_{h\to 0^+}\frac{(ah)^3}{bh^2}=0.$$

若 $b=0$, 则 $\forall h>0$, 均有 $f(ah,bh)=f(ah,0)=0$, 从而 $\frac{\partial f}{\partial \vec{\ell}^0}(0,0)=0$.

综上所述可知 f 在原点处沿任何方向的方向导数均存在.

2. 求 $z=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}x_ix_j$ 在 $P_0=(1,1,\ldots,1)$ 处沿方向 $\vec{\ell}=(-1,-1,\ldots,-1)^T$ 的方向导数.

解: 由题设可知 $z=(\sum\limits_{j=1}^{n}x_j)^2$, 则对任意整数 $1\leqslant i\leqslant n$, 我们有

$$\frac{\partial z}{\partial x_i}(P_0)=2\sum_{j=1}^{n}x_j\big|_{P_0}=2n.$$

于是所求方向导数为

$$\frac{\partial z}{\partial \vec{\ell}}(P_0)=\operatorname{grad} z(P_0)\cdot\frac{\vec{\ell}}{\|\vec{\ell}\|}=-\frac{2n^2}{\sqrt{n}}=-2n\sqrt{n}.$$

3. 设 $u(x,y,z)=x^2+y^2+z^2-xy-xz+yz$, $P=(1,1,1)$, 求 u 在点 P 的方向导数 $\frac{\partial u}{\partial \vec{\ell}}(P)$ 的最值, 并指出取得最值时的方向, 再指出沿哪一个方向的方向导数为零.

解: 由于 u 为初等函数, 故可微, 于是我们有

$$\operatorname{grad} u(P)=\begin{pmatrix}2x-y-z\\2y-x+z\\2z-x+y\end{pmatrix}\Big|_P=\begin{pmatrix}0\\2\\2\end{pmatrix},\quad \|\operatorname{grad} u(P)\|=2\sqrt{2},$$

从而 u 在点 P 的方向导数的最大值为 $2\sqrt{2}$, 最小值为 $-2\sqrt{2}$, 相应方向为

$$\begin{pmatrix}0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix},\quad\begin{pmatrix}0\\-\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\end{pmatrix}.$$

而方向导数为零的方向 $\vec{\ell}^0=(a,b,c)^T$ 满足 $0=\operatorname{grad} u(P)\cdot\vec{\ell}^0=2b+2c$, 由此可得 $c=-b$, 从而所求方向为 $\vec{\ell}^0=(a,b,-b)^T$, 其中 $a^2+2b^2=1$.

4. 证明下列函数所满足的相应等式:

(1) $u=2\cos^2(x-\frac{y}{2})$ 满足 $2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}=0$,

(2) $n>0$, $u=(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^{2-n}$ 满足 $\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0$.

证明: (1) 由题设可得

$$\begin{aligned}
&\frac{\partial u}{\partial y}=4\cos(x-\frac{y}{2})\cdot\big(-\sin(x-\frac{y}{2})\big)\cdot\big(-\frac{1}{2}\big)=\sin(2x-y),\\
&\frac{\partial^2 u}{\partial x\partial y}=2\cos(2x-y),\\
&\frac{\partial^2 u}{\partial y^2}=-\cos(2x-y),
\end{aligned}$$

于是我们有 $2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}=2\times\big(-\cos(2x-y)\big)+2\cos(2x-y)=0$.

(2) 对任意整数 $1\leqslant i\leqslant n$, 我们有

$$\begin{aligned}
\frac{\partial u}{\partial x_i}&=\frac{(2-n)x_i}{(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^n},\\
\frac{\partial^2 u}{\partial x_i^2}&=\frac{(2-n)}{(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^n}-\frac{(2-n)nx_i^2}{(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^{n+2}},
\end{aligned}$$

由此可得 $\sum\limits_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}=\frac{(2-n)n}{(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^n}-\sum\limits_{i=1}^n\frac{(2-n)nx_i^2}{(\sqrt{x_1^2+x_2^2+\cdots+x_n^2})^{n+2}}=0$.

5. 求由变换 $\begin{cases}x=r\sin\theta\cos\varphi\\ y=r\cos\theta\cos\varphi\\ z=r\sin\varphi\end{cases}$ $(r>0,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi)$ 所确定的向量值函数 $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}f_1(r,\theta,\varphi)\\f_2(r,\theta,\varphi)\\f_3(r,\theta,\varphi)\end{pmatrix}$ 的 Jacobi 矩阵和微分.

解: 由题设可知所求 Jacobi 矩阵为

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=\begin{pmatrix}\sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi\\ \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi\\ \sin\varphi & 0 & r\cos\varphi\end{pmatrix},$$

由此可得所求微分为

$$\begin{aligned}
\mathrm{d}\begin{pmatrix}x\\y\\z\end{pmatrix}&=\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\begin{pmatrix}\mathrm{d}r\\ \mathrm{d}\theta\\ \mathrm{d}\varphi\end{pmatrix}\\
&=\begin{pmatrix}\sin\theta\cos\varphi\,\mathrm{d}r+r\cos\theta\cos\varphi\,\mathrm{d}\theta-r\sin\theta\sin\varphi\,\mathrm{d}\varphi\\ \cos\theta\cos\varphi\,\mathrm{d}r-r\sin\theta\cos\varphi\,\mathrm{d}\theta-r\cos\theta\sin\varphi\,\mathrm{d}\varphi\\ \sin\varphi\,\mathrm{d}r+r\cos\varphi\,\mathrm{d}\varphi\end{pmatrix}.
\end{aligned}$$

6. 设 $z=\arctan\frac{u}{v}$, $u=x^2+y^2$, $v=xy$. 求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$.

解: 由题设可知

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\frac{1}{v}}{1+(\frac{u}{v})^2}\cdot(2x)+\frac{-\frac{u}{v^2}}{1+(\frac{u}{v})^2}\cdot y=\frac{2xv-uy}{u^2+v^2}=\frac{x^2y-y^3}{x^2y^2+(x^2+y^2)^2},\\
\frac{\partial z}{\partial y} &= \frac{\frac{1}{v}}{1+(\frac{u}{v})^2}\cdot(2y)+\frac{-\frac{u}{v^2}}{1+(\frac{u}{v})^2}\cdot x=\frac{2vy-ux}{u^2+v^2}=\frac{xy^2-x^3}{x^2y^2+(x^2+y^2)^2},\\
\frac{\partial^2 z}{\partial x^2} &= \frac{(2v+2xy-2xy)(u^2+v^2)-(2xv-uy)(4ux+2vy)}{(u^2+v^2)^2}\\
&= \frac{2xy(x^4+3x^2y^2+y^4)-(x^2y-y^3)(6xy^2+4x^3)}{(x^2y^2+(x^2+y^2)^2)^2}\\
&= \frac{-2x^5y+4x^3y^3+8xy^5}{(x^2y^2+(x^2+y^2)^2)^2},\\
\frac{\partial^2 z}{\partial y\partial x} &= \frac{(2x^2-2y^2-u)(u^2+v^2)-(2xv-uy)(4uy+2vx)}{(u^2+v^2)^2}\\
&= \frac{(x^2-3y^2)(x^4+3x^2y^2+y^4)-(x^2y-y^3)(6x^2y+4y^3)}{(x^2y^2+(x^2+y^2)^2)^2}\\
&= \frac{x^6-6x^4y^2-6x^2y^4+y^6}{(x^2y^2+(x^2+y^2)^2)^2},\\
\frac{\partial^2 z}{\partial y^2} &= \frac{(2xy+2v-2yx)(u^2+v^2)-(2vy-ux)(4uy+2vx)}{(u^2+v^2)^2}\\
&= \frac{2xy(x^4+3x^2y^2+y^4)-(xy^2-x^3)(6x^2y+4y^3)}{(x^2y^2+(x^2+y^2)^2)^2}\\
&= \frac{8x^5y+4x^3y^3-2xy^5}{(x^2y^2+(x^2+y^2)^2)^2}.
\end{aligned}$$

7. 已知 $u=f(x,y)$, 其中 $x=r\cos\theta$, $y=r\sin\theta$, f 可微, 证明:

$$\Big(\frac{\partial u}{\partial r}(r,\theta)\Big)^2+\Big(\frac{1}{r}\frac{\partial u}{\partial\theta}(r,\theta)\Big)^2=\Big(\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\Big)^2+\Big(\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\Big)^2.$$

证明: 由题设可知

$$\begin{aligned}
&\Big(\frac{\partial u}{\partial r}(r,\theta)\Big)^2+\Big(\frac{1}{r}\frac{\partial u}{\partial\theta}(r,\theta)\Big)^2\\
=\;&\Big(\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\frac{\partial y}{\partial r}\Big)^2\\
&+\Big(\frac{1}{r}\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\frac{\partial x}{\partial\theta}+\frac{1}{r}\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\frac{\partial y}{\partial\theta}\Big)^2\\
=\;&\Big(\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\cos\theta+\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\sin\theta\Big)^2\\
&+\Big(\frac{1}{r}\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)(-r\sin\theta)+\frac{1}{r}\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)(r\cos\theta)\Big)^2\\
=\;&\Big(\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)\Big)^2+\Big(\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\Big)^2,
\end{aligned}$$

因此所证结论成立.

8. 设 f 满足 Laplace 方程 $\partial_{11}f+\partial_{22}f=0$, 证明: $u(x,y)=f(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2})$ 也满足 Laplace 方程.

证明: 令 $g(x,y)=\frac{x}{x^2+y^2}$, $h(x,y)=\frac{y}{x^2+y^2}$. 则 $u=f(g,h)$, 从而由题设可知

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\big(\partial_1 f(g,h)\frac{\partial g}{\partial x}+\partial_2 f(g,h)\frac{\partial h}{\partial x}\big)+\frac{\partial}{\partial y}\big(\partial_1 f(g,h)\frac{\partial g}{\partial y}+\partial_2 f(g,h)\frac{\partial h}{\partial y}\big)\\
&= \big(\partial_{11}f(g,h)\frac{\partial g}{\partial x}+\partial_{21}f(g,h)\frac{\partial h}{\partial x}\big)\frac{\partial g}{\partial x}+\partial_1 f(g,h)\frac{\partial^2 g}{\partial x^2}\\
&\quad+\big(\partial_{12}f(g,h)\frac{\partial g}{\partial x}+\partial_{22}f(g,h)\frac{\partial h}{\partial x}\big)\frac{\partial h}{\partial x}+\partial_2 f(g,h)\frac{\partial^2 h}{\partial x^2}\\
&\quad+\big(\partial_{11}f(g,h)\frac{\partial g}{\partial y}+\partial_{21}f(g,h)\frac{\partial h}{\partial y}\big)\frac{\partial g}{\partial y}+\partial_1 f(g,h)\frac{\partial^2 g}{\partial y^2}\\
&\quad+\big(\partial_{12}f(g,h)\frac{\partial g}{\partial y}+\partial_{22}f(g,h)\frac{\partial h}{\partial y}\big)\frac{\partial h}{\partial y}+\partial_2 f(g,h)\frac{\partial^2 h}{\partial y^2}\\
&= \partial_{11}f(g,h)\big(\big(\frac{\partial g}{\partial x}\big)^2+\big(\frac{\partial g}{\partial y}\big)^2\big)+\partial_{22}f(g,h)\big(\big(\frac{\partial h}{\partial x}\big)^2+\big(\frac{\partial h}{\partial y}\big)^2\big)\\
&\quad+2\partial_{12}f(g,h)\big(\frac{\partial g}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial g}{\partial y}\frac{\partial h}{\partial y}\big)+\partial_1 f(g,h)\big(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}\big)\\
&\quad+\partial_2 f(g,h)\big(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\big).
\end{aligned}$$

注意到 $\frac{\partial g}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial g}{\partial y}=-\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial h}{\partial x}=-\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial h}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}$, 于是 $\frac{\partial g}{\partial x}=-\frac{\partial h}{\partial y}$, $\frac{\partial g}{\partial y}=\frac{\partial h}{\partial x}$, 则 $\big(\frac{\partial g}{\partial x}\big)^2+\big(\frac{\partial g}{\partial y}\big)^2=\big(\frac{\partial h}{\partial x}\big)^2+\big(\frac{\partial h}{\partial y}\big)^2$, $\frac{\partial g}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial g}{\partial y}\frac{\partial h}{\partial y}=0$, 故

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2} &= \frac{\partial}{\partial x}\big(-\frac{\partial h}{\partial y}\big)+\frac{\partial}{\partial y}\big(\frac{\partial h}{\partial x}\big)=-\frac{\partial^2 h}{\partial x\partial y}+\frac{\partial^2 h}{\partial y\partial x}=0,\\
\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2} &= \frac{\partial}{\partial x}\big(\frac{\partial g}{\partial y}\big)+\frac{\partial}{\partial y}\big(-\frac{\partial g}{\partial x}\big)=\frac{\partial^2 g}{\partial x\partial y}-\frac{\partial^2 g}{\partial y\partial x}=0,
\end{aligned}$$

由此以及题设可得 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(\partial_{11}f(g,h)+\partial_{22}f(g,h))\big(\big(\frac{\partial g}{\partial x}\big)^2+\big(\frac{\partial g}{\partial y}\big)^2\big)=0.$

9. 设向量值函数 $\mathbf{Y}=\mathbf{f}(\mathbf{U})$, $\mathbf{U}=\mathbf{g}(\mathbf{X})$ 可微, 求复合函数 $\mathbf{Y}=\mathbf{f}\circ\mathbf{g}(\mathbf{X})$ 的 Jacobi 矩阵和全微分, 其中

$$\begin{cases} y_1=u_1+u_2\\ y_2=u_1u_2\\ y_3=\frac{u_2}{u_1}\end{cases},\quad \begin{cases} u_1=\frac{x}{x^2+y^2}\\ u_2=\frac{y}{x^2+y^2}\end{cases}.$$

解: 由复合函数求微分法则可知所求 Jacobi 矩阵为

$$\begin{aligned}
J_{\mathbf{Y}}(\mathbf{X}) &= \begin{pmatrix} 1 & 1\\ u_2 & u_1\\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{pmatrix}\Bigg|_{\mathbf{U}=\mathbf{g}(\mathbf{X})}\begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2}\\ -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2}\end{pmatrix}\\
&= \begin{pmatrix} 1 & 1\\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2}\\ -\frac{y(x^2+y^2)}{x^2} & \frac{x^2+y^2}{x}\end{pmatrix}\begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2}\\ -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2}\end{pmatrix}\\
&= \begin{pmatrix} \frac{y^2-2xy-x^2}{(x^2+y^2)^2} & \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\\ \frac{y^3-3x^2y}{(x^2+y^2)^3} & \frac{x^3-3xy^2}{(x^2+y^2)^3}\\ -\frac{y}{x^2} & \frac{1}{x}\end{pmatrix},
\end{aligned}$$

进而可知所求微分为

$$\mathrm{d}\mathbf{Y} = J_{\mathbf{Y}}(\mathbf{X})\mathrm{d}\mathbf{X} = \begin{pmatrix} \frac{(y^2-2xy-x^2)\,\mathrm{d}x+(x^2-2xy-y^2)\,\mathrm{d}y}{(x^2+y^2)^2} \\ \frac{(y^3-3x^2y)\,\mathrm{d}x+(x^3-3xy^2)\,\mathrm{d}y}{(x^2+y^2)^3} \\ \frac{-y\,\mathrm{d}x+x\,\mathrm{d}y}{x^2} \end{pmatrix}.$$

10. 问方程 $e^{-(x+y+z)} = x+y+z$ 在哪些点附近可确定一个隐函数 $z = z(x,y)$, 并求相应的 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解: 方法 1. $\forall (x,y,z) \in \mathbb{R}^3$, 定义 $F(x,y,z) = e^{-(x+y+z)} - (x+y+z)$, 从而 F 为初等函数, 因此连续可导, 并且我们还有

$$\begin{aligned} \frac{\partial F}{\partial x}(x,y,z) &= -e^{-(x+y+z)} - 1, \\ \frac{\partial F}{\partial y}(x,y,z) &= -e^{-(x+y+z)} - 1, \\ \frac{\partial F}{\partial z}(x,y,z) &= -e^{-(x+y+z)} - 1 \neq 0. \end{aligned}$$

由隐函数定理知, $\forall (x_0,y_0,z_0) \in \mathbb{R}^3$, 若 $F(x_0,y_0,z_0) = 0$, 则方程 $F(x,y,z) = 0$ 在点 (x_0,y_0,z_0) 附近可确定一个隐函数 $z = z(x,y)$, 并且

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -1, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -1.$$

方法 2. $\forall t \in \mathbb{R}$, 定义 $f(t) = xe^x - 1$. 则当 $x \leqslant 0$ 时, 我们有 $f(x) \leqslant -1$, 而当 $x > -1$ 时, 则 $f'(x) = (1+x)e^x > 0$, 因此 f 在 $(-1, +\infty)$ 上严格递增. 又 $f(1) > 0$, 于是由连续函数介值定理可知 f 在 $\mathbb{R}$ 有唯一的零点, 记作 a. 于是 $F(x,y,z) = 0$ 当且仅当 $x+y+z = a$, 也即 $z = a - x - y$. 由此可得

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1.$$

附注: 方法 2 太特殊, 缺乏普适性, 不是好的解题方法.

11. 问方程组 $\begin{cases} x+y+z+z^2 = 0 \\ x+y^2+z+z^3 = 0 \end{cases}$ 在点 $P(-1,1,0)$ 的附近能否确定一个向量值函数 $\begin{pmatrix} y \\ z \end{pmatrix} = \mathbf{f}(x)$? 如果能, 求 $y'(-1), z'(-1)$.

解: $\forall (x,y,z) \in \mathbb{R}^3$, 定义 $F_1(x,y,z) = x+y+z+z^2$, $F_2(x,y,z) = x+y^2+z+z^3$, 则 F_1, F_2 均为初等函数, 因此连续可导, 并且

$$\frac{\partial(F_1,F_2)}{\partial(y,z)}(-1,1,0) = \begin{pmatrix} 1 & 1+2z \\ 2y & 1+3z^2 \end{pmatrix}\Big|_{(-1,1,0)} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

故 $\frac{D(F_1,F_2)}{D(y,z)}(-1,1,0)=-1\neq 0$, 从而由隐函数定理知, 题设方程组在点 P 的附近可确定一个向量值函数 $\begin{pmatrix} y \\ z \end{pmatrix}=\mathbf{f}(x)$. 将上述方程组在点 P 处关于 x 求导可得 $1+y'(-1)+z'(-1)=0$, $1+2y'(-1)+z'(-1)=0$, 于是我们有

$$y'(-1)=0,\ z'(-1)=-1.$$