第 7 次作业题

1. 计算下列二重积分:

(1) $\displaystyle\iint_D (x+y)\,\mathrm{d}x\mathrm{d}y$, 其中 D 是由 $x^2+y^2=x+y$ 围成的平面区域.

(2) $\displaystyle\iint_D (y-x)^2\,\mathrm{d}x\mathrm{d}y$, 其中 $D=\{(x,y)\mid 0\leqslant y\leqslant x+a,\ x^2+y^2\leqslant a^2\}$, $a>0$.

解: (1) 在极坐标下, 积分区域变为

$$D_1=\Big\{(\rho,\varphi)\mid 0\leqslant\rho\leqslant\sin\varphi+\cos\varphi,\ -\frac{\pi}{4}\leqslant\varphi\leqslant\frac{3\pi}{4}\Big\},$$

由此立刻可得

$$\begin{aligned}
\iint_D (x+y)\,\mathrm{d}x\mathrm{d}y &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\Big(\int_0^{\sin\varphi+\cos\varphi}\rho^2(\sin\varphi+\cos\varphi)\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\
&= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\sin\varphi+\cos\varphi)^4\,\mathrm{d}\varphi=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\Big((\sqrt{2}\sin\big(\varphi+\frac{\pi}{4}\big)\Big)^4\,\mathrm{d}\varphi\\
&\xlongequal{t=\varphi+\frac{\pi}{4}} \frac{4}{3}\int_0^{\pi}\sin^4 t\,\mathrm{d}t=\frac{8}{3}\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^4 t\,\mathrm{d}t\\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}}\Big(\frac{1+\cos 2t}{2}\Big)^2\,\mathrm{d}t=\frac{2}{3}\int_0^{\frac{\pi}{2}}(1+2\cos 2t+\cos^2 2t)\,\mathrm{d}t\\
&= \frac{2}{3}\Big(t+\sin 2t+\frac{1}{8}\sin 4t+\frac{t}{2}\Big)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}.
\end{aligned}$$

(2) **方法 1.** 在极坐标下, 积分区域变为

$$\begin{aligned}
D_1 &= \Big\{(\rho,\varphi)\mid 0\leqslant\rho\sin\varphi\leqslant\rho\cos\varphi+a,\ 0\leqslant\rho\leqslant a\Big\}\\
&= \Big\{(\rho,\varphi)\mid 0\leqslant\varphi\leqslant\frac{\pi}{2},\ 0\leqslant\rho\leqslant a\Big\}\\
&\qquad\bigcup\Big\{(\rho,\varphi)\mid \frac{\pi}{2}\leqslant\varphi\leqslant\pi,\ 0\leqslant\rho\leqslant\frac{\sqrt{2}a}{2\sin(\varphi-\frac{\pi}{4})}\Big\}
\end{aligned}$$

由此立刻可得

$$\begin{aligned}
\iint_D (y-x)^2\,\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}}\Big(\int_0^a\rho^3(\sin\varphi-\cos\varphi)^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\
&\quad+\int_{\frac{\pi}{2}}^{\pi}\Big(\int_0^{\frac{\sqrt{2}a}{2\sin(\varphi-\frac{\pi}{4})}}\rho^3(\sin\varphi-\cos\varphi)^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\
&= \frac{a^4}{4}\int_0^{\frac{\pi}{2}}(1-\sin 2\varphi)\,\mathrm{d}\varphi+\frac{a^4}{8}\int_{\frac{\pi}{2}}^{\pi}\frac{\mathrm{d}\varphi}{\sin^2(\varphi-\frac{\pi}{4})}\\
&= \frac{a^4}{4}\Big(\varphi+\frac{1}{2}\cos 2\varphi\Big)\Big|_0^{\frac{\pi}{2}}-\frac{a^4}{8}\cot\Big(\varphi-\frac{\pi}{4}\Big)\Big|_{\frac{\pi}{2}}^{\pi}\\
&= \frac{a^4}{4}\Big(\frac{\pi}{2}-1\Big)+\frac{a^4}{4}=\frac{a^4\pi}{8}.
\end{aligned}$$

方法 2. 由题设可知 $D=\big\{(x,y)\mid 0\leqslant y\leqslant a,\ y-a\leqslant x\leqslant\sqrt{a^2-y^2}\big\}$, 则

$$\begin{aligned}
&\iint\limits_D (y-x)^2\,\mathrm{d}x\mathrm{d}y=\int_0^a\Big(\int_{y-a}^{\sqrt{a^2-y^2}}(y-x)^2\,\mathrm{d}x\Big)\,\mathrm{d}y\\
=\;&\frac{1}{3}\int_0^a\Big((\sqrt{a^2-y^2}-y)^3+a^3\Big)\,\mathrm{d}y=\frac{a^4}{3}+\frac{1}{3}\int_0^a(\sqrt{a^2-y^2}-y)^3\,\mathrm{d}y\\
\xlongequal{y=a\sin t}\;&\frac{a^4}{3}+\frac{a^4}{3}\int_0^{\frac{\pi}{2}}(\cos t-\sin t)^3\cos t\,\mathrm{d}t\\
=\;&\frac{a^4}{3}+\frac{a^4}{3}\int_0^{\frac{\pi}{2}}(\cos^3 t-3\cos^2 t\sin t+3\cos t\sin^2 t-\sin^3 t)\cos t\,\mathrm{d}t\\
=\;&\frac{a^4}{3}+\frac{a^4}{3}\int_0^{\frac{\pi}{2}}(\cos^4 t+3\cos^2 t\sin^2 t)\,\mathrm{d}t+\frac{a^4}{3}\Big(\frac{3}{4}\cos^4 t-\frac{1}{4}\sin^4 t\Big)\Big|_0^{\frac{\pi}{2}}\\
=\;&\frac{a^4}{3}\int_0^{\frac{\pi}{2}}(\cos^4 t+3\cos^2 t\sin^2 t)\,\mathrm{d}t=\frac{a^4}{3}\int_0^{\frac{\pi}{2}}(3\cos^2 t-2\cos^4 t)\,\mathrm{d}t\\
=\;&\frac{a^4}{2}\Big(\frac{\sin 2t}{2}+t\Big)\Big|_0^{\frac{\pi}{2}}-\frac{2a^4}{3}\int_0^{\frac{\pi}{2}}\cos^4 t\,\mathrm{d}t=\frac{a^4\pi}{4}-\frac{2a^4}{3}\cdot\frac{3\pi}{16}=\frac{a^4\pi}{8}.
\end{aligned}$$

2. 求双纽线 $(x^2+y^2)^2=2a^2(x^2-y^2)$ 与圆 $x^2+y^2=a^2$ 所围成图形 (圆外部分) 的面积, 其中 $a>0$.

解: 设所围成的区域为 D, 它在极坐标下变为

$$D_1=\Big\{(\rho,\varphi)\mid a\leqslant\rho\leqslant a\sqrt{2\cos 2\varphi},\ -\frac{\pi}{6}\leqslant\varphi\leqslant\frac{\pi}{6}\text{ 或 }\frac{5\pi}{6}\leqslant\varphi\leqslant\frac{7\pi}{6}\Big\}.$$

由此立刻可得 D 的面积为

$$\begin{aligned}
S\;=\;&\iint\limits_{D_1}\rho\,\mathrm{d}\rho\mathrm{d}\varphi=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\Big(\int_a^{\sqrt{2\cos 2\varphi}a}\rho\,\mathrm{d}\rho\Big)\mathrm{d}\varphi+\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\Big(\int_a^{\sqrt{2\cos 2\varphi}a}\rho\,\mathrm{d}\rho\Big)\mathrm{d}\varphi\\
=\;&\frac{a^2}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(2\cos 2\varphi-1)\,\mathrm{d}\varphi+\frac{a^2}{2}\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}(2\cos 2\varphi-1)\,\mathrm{d}\varphi=\Big(\sqrt{3}-\frac{\pi}{3}\Big)a^2.
\end{aligned}$$

注: 这里也可以利用对称性只计算位于第一、三象限的部分.

3. 计算 $\iint\limits_D x^2y^2\,\mathrm{d}x\mathrm{d}y$, 其中 D 是由 $xy=2$, $xy=4$, $y=x$, $y=3x$ 在第一象限所围成的平面区域.

解: 作变换 $u=xy$, $v=\frac{y}{x}$, 则 D 在此变换下变为

$$D_1=\big\{(u,v)\mid 2\leqslant u\leqslant 4,\ 1\leqslant v\leqslant 3\big\},$$

而 $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} y & x\\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}=\frac{2y}{x}$, 于是 $\frac{D(x,y)}{D(u,v)}=\frac{x}{2y}=\frac{1}{2v}$, 进而可得

$$\iint\limits_D x^2y^2\,\mathrm{d}x\mathrm{d}y=\iint\limits_{D_1}u^2\Big|\frac{D(x,y)}{D(u,v)}\Big|\,\mathrm{d}u\mathrm{d}v=\int_2^4\Big(\int_1^3\frac{u^2}{2v}\,\mathrm{d}v\Big)\,\mathrm{d}u=\frac{28}{3}\log 3.$$

4. 计算 $\iiint\limits_{\Omega} xy^2z^3\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中 Ω 是由马鞍面 $z=xy$ 与平面 $y=x$, $x=1$, $z=0$ 所围成的空间区域.

解: 由题设可知 $\Omega=\big\{(x,y,z)\ |0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant x,\ 0\leqslant z\leqslant xy\big\}$, 于是

$$\iiint\limits_{\Omega} xy^2z^3\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\Big(\int_0^x\Big(\int_0^{xy}xy^2z^3\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x$$

$$=\int_0^1\Big(\int_0^x\frac{1}{4}x^5y^6\,\mathrm{d}y\Big)\,\mathrm{d}x=\int_0^1\frac{x^{12}}{28}\,\mathrm{d}x=\frac{1}{364}.$$

5. 计算下列三重积分:

(1) $\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中 $\Omega=\big\{(x,y,z)\mid\sqrt{x^2+y^2}\leqslant z\leqslant 1\big\}$;

(2) $\iiint\limits_{\Omega}(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中

$$\Omega=\big\{(x,y,z)\mid y^2+z^2\leqslant x^2\leqslant R^2-y^2-z^2,\ x\geqslant 0\big\};$$

(3) $\iiint\limits_{\Omega}x^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中 Ω 由曲面 $z=y^2$, $z=4y^2$ 以及平面 $z=x$, $z=2x$, $z=0$, $z=3$ 围成.

解: (1) 在柱坐标系下 Ω 变为

$$\Omega_1=\big\{(\rho,\varphi,z)\mid 0\leqslant\rho\leqslant 1,\ 0\leqslant\varphi\leqslant 2\pi,\ \rho\leqslant z\leqslant 1\big\},$$

由此我们立刻可得

$$\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega_1}\rho^2\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z=\int_0^{2\pi}\Big(\int_0^1\Big(\int_\rho^1\rho^2\,\mathrm{d}z\Big)\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi$$

$$=2\pi\int_0^1\rho^2(1-\rho)\,\mathrm{d}\rho=2\pi\Big(\frac{1}{3}\rho^3-\frac{1}{4}\rho^4\Big)\Big|_0^1=\frac{\pi}{6}.$$

(2) 考虑球坐标变换

$$\begin{cases} x=r\cos\theta,\\ y=r\sin\theta\cos\varphi,\\ z=r\sin\theta\sin\varphi,\end{cases}$$

在上述变换下 Ω 变为 $\Omega_1=\big\{(r,\varphi,\theta)\mid 0\leqslant r\leqslant R,\ 0\leqslant\varphi\leqslant 2\pi,\ 0\leqslant\theta\leqslant\frac{\pi}{4}\big\}$, 由此我们立刻可得

$$\iiint\limits_{\Omega}(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega_1}r^2(r^2\sin\theta)\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$$

$$=\Big(\int_0^R r^4\,\mathrm{d}r\Big)\Big(\int_0^{2\pi}\mathrm{d}\varphi\Big)\Big(\int_0^{\frac{\pi}{4}}\sin\theta\,\mathrm{d}\theta\Big)=\frac{1}{5}(2-\sqrt{2})\pi R^5.$$

(3) 令 $\Omega_1 = \Omega \cap \{(x, y, z) \mid y \geqslant 0\}$. 由对称性可知

$$\iiint\limits_{\Omega} x^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 2 \iiint\limits_{\Omega_1} x^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

考虑变量替换 $\begin{cases} u = \frac{z}{y^2}, \\ v = \frac{z}{x}, \\ w = z. \end{cases}$ 在此变换下, 积分区域 Ω 变为

$$\Omega_1 = \big\{(u, v, w) \mid 1 \leqslant u \leqslant 4,\ 1 \leqslant v \leqslant 2,\ 0 \leqslant w \leqslant 3\big\}.$$

与此同时, 我们有

$$\frac{D(u, v, w)}{D(x, y, z)} = \begin{vmatrix} 0 & -\frac{2z}{y^3} & \frac{1}{y^2} \\ -\frac{z}{x^2} & 0 & \frac{1}{x} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{2z^2}{x^2y^3} = -2v^2\Big(\frac{u}{w}\Big)^{\frac{3}{2}},$$

故 $\frac{D(x,y,z)}{D(u,v,w)} = -\frac{1}{2v^2}(\frac{w}{u})^{\frac{3}{2}}$, 从而我们有

$$\begin{aligned} \iiint\limits_{\Omega} x^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z &= 2\iiint\limits_{\Omega_1} x^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\iiint\limits_{\Omega_2} \Big(\frac{w}{v}\Big)^2 \cdot \frac{1}{2v^2}(\frac{w}{u})^{\frac{3}{2}} \, \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= \iiint\limits_{\Omega_2} u^{-\frac{3}{2}} v^{-4} w^{\frac{7}{2}} \, \mathrm{d}u\mathrm{d}v\mathrm{d}w = \Big(\int_1^4 u^{-\frac{3}{2}} \, \mathrm{d}u\Big)\Big(\int_1^2 v^{-4} \, \mathrm{d}v\Big)\Big(\int_0^3 w^{\frac{7}{2}} \, \mathrm{d}w\Big) = \frac{21}{4}\sqrt{3}. \end{aligned}$$

6. 求柱面 $x^2 + z^2 = a^2$ 在柱面 $x^2 + y^2 = a^2$ 内部分的面积, 其中 $a > 0$.

解: 将柱面 $x^2 + z^2 = a^2$ 在柱面 $x^2 + y^2 = a^2$ 内的部分记作 Σ, 则其方程为

$$\begin{cases} x = a\cos\varphi, \\ y = y, \qquad (0 \leqslant \varphi \leqslant 2\pi,\ |y| \leqslant a|\sin\varphi|), \\ z = a\sin\varphi, \end{cases}$$

于是 $E = a^2$, $G = 1$, $F = 0$, 从而所求面积为

$$S = \iint\limits_{\substack{0 \leqslant \varphi \leqslant 2\pi \\ |y| \leqslant a|\sin\varphi|}} a \, \mathrm{d}\varphi\mathrm{d}y = \int_0^{2\pi} \Big(\int_{-a|\sin\varphi|}^{a|\sin\varphi|} a \, \mathrm{d}y\Big) \mathrm{d}\varphi = 2a^2 \int_0^{2\pi} |\sin\varphi| \, \mathrm{d}\varphi = 8a^2.$$

7. 求曲面 $x^2 + y^2 = 2z$ 与平面 $x + y = z$ 所围成的均匀物体的质心.

解: 设所围成的空间区域为 Ω, 则

$$\Omega = \Big\{(x, y, z) \mid \frac{1}{2}(x^2 + y^2) \leqslant z \leqslant x + y,\ (x-1)^2 + (y-1)^2 \leqslant 2\Big\},$$

在柱坐标系下 Ω 变为

$$\begin{aligned} \Omega_1 \ = \ &\Big\{(\rho, \varphi, z) \mid \frac{1}{2}\rho^2 \leqslant z \leqslant \sqrt{2}\rho\sin(\varphi + \frac{\pi}{4}), \\ &\quad 0 \leqslant \rho \leqslant 2\sqrt{2}\sin(\varphi + \frac{\pi}{4}),\ -\frac{\pi}{4} \leqslant \varphi \leqslant \frac{3}{4}\pi\Big\}, \end{aligned}$$

由此我们立刻可知

$$
\begin{aligned}
|\Omega| &= \iiint\limits_{\Omega} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})} \Big(\int_{\frac{1}{2}\rho^2}^{\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4})} \rho \,\mathrm{d}z \Big) \,\mathrm{d}\rho \Big) \,\mathrm{d}\varphi \\
&= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})} \rho \Big(\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4}) - \frac{1}{2}\rho^2 \Big) \,\mathrm{d}\rho \Big) \,\mathrm{d}\varphi \\
&= \frac{8}{3} \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \sin^4\big(\varphi + \frac{\pi}{4}\big) \,\mathrm{d}\varphi \\
&\xlongequal{t=\varphi+\frac{\pi}{4}} \frac{8}{3} \int_0^{\pi} \sin^4 t \,\mathrm{d}t \\
&= \frac{16}{3} \int_0^{\frac{\pi}{2}} \cos^4 t \,\mathrm{d}t = \frac{4}{3} \int_0^{\frac{\pi}{2}} (1+\cos 2t)^2 \,\mathrm{d}t \\
&= \frac{4}{3} \int_0^{\frac{\pi}{2}} \Big(1 + 2\cos 2t + \frac{1+\cos 4t}{2} \Big) \,\mathrm{d}t \\
&= \frac{4}{3} \Big(\frac{3}{2}t + \sin 2t + \frac{1}{8}\sin 4t \Big) \Big|_0^{\frac{\pi}{2}} = \pi.
\end{aligned}
$$

设所求重心为 $(\bar{x}, \bar{y}, \bar{z})$. 则我们有

$$
\begin{aligned}
\bar{x} &= \frac{1}{|\Omega|} \iiint\limits_{\Omega} x \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{\pi} \iiint\limits_{\Omega_1} \rho^2 \cos\varphi \,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})} \Big(\int_{\frac{1}{2}\rho^2}^{\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4})} \rho^2 \cos\varphi \,\mathrm{d}z \Big) \,\mathrm{d}\rho \Big) \,\mathrm{d}\varphi \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})} \rho^2 \cos\varphi \Big(\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4}) - \frac{1}{2}\rho^2 \Big) \,\mathrm{d}\rho \Big) \,\mathrm{d}\varphi \\
&= \frac{16\sqrt{2}}{5\pi} \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \cos\varphi \sin^5\big(\varphi + \frac{\pi}{4}\big) \,\mathrm{d}\varphi \\
&\xlongequal{t=\varphi+\frac{\pi}{4}} \frac{16\sqrt{2}}{5\pi} \int_0^{\pi} \cos\Big(t - \frac{\pi}{4}\Big) \sin^5 t \,\mathrm{d}t \\
&= \frac{16}{5\pi} \int_0^{\pi} (\cos t + \sin t) \sin^5 t \,\mathrm{d}t = \frac{8}{15\pi} \sin^6 t \Big|_0^{\pi} + \frac{16}{5\pi} \int_0^{\pi} \sin^6 t \,\mathrm{d}t \\
&= \frac{32}{5\pi} \int_0^{\frac{\pi}{2}} \cos^6 t \,\mathrm{d}t = \frac{4}{5\pi} \int_0^{\frac{\pi}{2}} (1+\cos 2t)^3 \,\mathrm{d}t \\
&= \frac{4}{5\pi} \int_0^{\frac{\pi}{2}} (1 + 3\cos 2t + 3\cos^2 2t + \cos^3 2t) \,\mathrm{d}t \\
&= \frac{4}{5\pi} \int_0^{\frac{\pi}{2}} (1 + 4\cos 2t + 3\cos^2 2t - \sin^2 2t \cos 2t) \,\mathrm{d}t \\
&= \frac{4}{5\pi} \Big(\frac{5}{2}t + 2\sin 2t + \frac{3}{8}\sin 4t - \frac{1}{6}\sin^3 2t \Big) \Big|_0^{\frac{\pi}{2}} = 1,
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= \frac{1}{|\Omega|}\iiint\limits_{\Omega} y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{\pi}\iiint\limits_{\Omega_1} \rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})}\Big(\int_{\frac{1}{2}\rho^2}^{\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4})}\rho^2\sin\varphi\,\mathrm{d}z\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})}\rho^2\sin\varphi\Big(\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4})-\frac{1}{2}\rho^2\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi \\
&= \frac{16\sqrt{2}}{5\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin\varphi\sin^5\left(\varphi+\frac{\pi}{4}\right)\mathrm{d}\varphi \overset{t=\varphi+\frac{\pi}{4}}{=} \frac{16\sqrt{2}}{5\pi}\int_0^{\pi}\sin\Big(t-\frac{\pi}{4}\Big)\sin^5 t\,\mathrm{d}t \\
&= \frac{16}{5\pi}\int_0^{\pi}(\sin t-\cos t)\sin^5 t\,\mathrm{d}t = -\frac{8}{15\pi}\sin^6 t\Big|_0^{\pi} + \frac{16}{5\pi}\int_0^{\pi}\sin^6 t\,\mathrm{d}t = 1, \\
\bar{z} &= \frac{1}{|\Omega|}\iiint\limits_{\Omega} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{\pi}\iiint\limits_{\Omega_1} z\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})}\Big(\int_{\frac{1}{2}\rho^2}^{\sqrt{2}\rho\sin(\varphi+\frac{\pi}{4})} z\rho\,\mathrm{d}z\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\Big(\int_0^{2\sqrt{2}\sin(\varphi+\frac{\pi}{4})}\rho\Big(2\rho^2\sin^2(\varphi+\frac{\pi}{4})-\frac{1}{4}\rho^4\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi \\
&= \frac{16}{3\pi}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi}\sin^6\left(\varphi+\frac{\pi}{4}\right)\mathrm{d}\varphi \overset{t=\varphi+\frac{\pi}{4}}{=} \frac{16}{3\pi}\int_0^{\pi}\sin^6 t\,\mathrm{d}t = \frac{5}{3},
\end{aligned}
$$

于是所求质心为 $(1, 1, \frac{5}{3})$.