（科目：　　）清华大学数学作业纸

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4.1. $\vec{a}\times\vec{b}+(\vec{a}\cdot\vec{b})\cdot\vec{c}=\vec{e}$.

$[\vec{a}\times\vec{b}]_i+[(\vec{a}\cdot\vec{b})\cdot\vec{c}]_i=e_i \rightarrow \epsilon_{ijk}a_jb_k+a_mb_mc_i=e_i$

$\cancel{\epsilon_{ijk}a_jb_k+\ldots c_i=e_i}$

4.2. $\cancel{\vec{c}+\vec{a}\times\vec{b}=}$

$c_i+\epsilon_{ikj}a_kb_j=d_lb_l\,e_mc_m\cdot c_i$

$\vec{c}+\vec{a}\times\vec{b}=(\vec{d}\cdot\vec{b})(\vec{e}\cdot\vec{c})\,\vec{c}$.

4.3. $[\vec{a}\times\vec{b}]_i=\epsilon_{ijk}a_jb_k$.

$[\vec{b}\times\vec{a}]_i=\epsilon_{ijk}b_ja_k=\epsilon_{ikj}b_ka_j=-\epsilon_{ijk}a_jb_k$.

$\rightarrow\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

4.4. (a). $\delta_{ij}\epsilon_{ijk}=0$

(b). $\epsilon_{ijk}\epsilon_{ilm}=\cancel{\epsilon_{ijk}\epsilon_{ilm}}\ \epsilon_{jki}\epsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$

(c). $\epsilon_{ijk}\epsilon_{ijm}=\cancel{\delta_{km}\ \epsilon_{jki}\epsilon_{ijm}=\delta_{jj}\delta_{km}-\delta_{jm}\delta_{kj}=\delta_{km}-\delta_{km}=0}$

$\cancel{\delta_{km}(\delta_{ik}-\delta_{im})}=\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ijk}\epsilon_{ijm}=\epsilon_{ijk}\epsilon_{ijk}=6$

(d). $\epsilon_{ijk}\epsilon_{ijk}=\epsilon_{123}^2+\epsilon_{132}^2+\cdots=6$

4.5. $\cancel{[(\vec{a}\times\vec{b})\cdot\vec{c}]\times\vec{d}}\quad(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

$\cancel{\epsilon_{ijk}a_jb_kc_i}=\epsilon_{ijk}a_jb_k\cdot\epsilon_{ilm}c_ld_m$

$=\epsilon_{ijk}\epsilon_{ilm}a_jb_kc_ld_m=\epsilon_{jki}\epsilon_{ilm}a_jb_kc_ld_m=(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\ldots$

$=a_jc_jb_kd_k-a_jd_jb_kc_k=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$.

4.6. $(AB)_{ij}=A_{ik}B_{kj}$. $(AB)^T_{ij}=A_{jk}B_{ki}$

$(B^TA^T)_{ij}=(B^T)_{ik}(A^T)_{kj}=A_{jk}B_{ki}=(AB)^T$. $\rightarrow(AB)^T=B^TA^T$.

4.7. (1) $|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}$

$$|M| = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}.$$

(2).

$$\epsilon_{pqr} |M| = \epsilon_{ijk} M_{pi} M_{qj} M_{rk}.$$

$$\epsilon_{pqr} |M| = \epsilon_{pqr} \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = \epsilon_{ijk} M_{pi} M_{qj} M_{rk}.$$

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4.8. $\cancel{\epsilon_{ijk}|M|} \cancel{\epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk} = \epsilon_{pqr}|M| = \epsilon_{ijk}}$

$\cancel{= \sum_{p=1}^{3} \epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk} = 6}$

$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = \epsilon_{123} |M|$

所以 $\epsilon_{312} |M| = \epsilon_{231} |M| = \epsilon_{123} |M| = |M|$.

$\epsilon_{213} |M| = -\epsilon_{ijk} M_{2i} M_{1j} M_{3k} = \epsilon_{jik} M_{1j} M_{2i} M_{3k} = |M|$

所以 $\epsilon_{321} |M| = \epsilon_{132} |M| = |M|$.

所以 $\epsilon_{pqr} |M|$

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4.8. (a) $|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k} = \epsilon_{123} \epsilon_{ijk} M_{1i} M_{2j} M_{3k}$.

所以同理: $\epsilon_{312} \epsilon_{ijk} M_{3i} M_{1j} M_{2k} = \epsilon_{312} \epsilon_{jki} M_{1j} M_{2k} M_{3i} = \epsilon_{jki} M_{1j} M_{2k} M_{3i} = |M|$.

$\epsilon_{231} \epsilon_{ijk} M_{2i} M_{3j} M_{1k} = |M|$.

$\epsilon_{213} \epsilon_{ijk} M_{2i} M_{1j} M_{3k} = -\epsilon_{ijk} M_{2i} M_{1j} M_{3k} = \epsilon_{jik} M_{1j} M_{2i} M_{3k} = |M|$

所以同理: $\epsilon_{321} \epsilon_{ijk} M_{3i} M_{2j} M_{1k} = \epsilon_{132} \epsilon_{ijk} M_{1i} M_{3j} M_{2k} = |M|$

所以 $6|M| = \epsilon_{pqr} \epsilon_{ijk} M_{pi} M_{qj} M_{rk}$

(b). $|M^T| = \epsilon_{ijk} (M^T)_{1i} (M^T)_{2j} (M^T)_{3k} = \epsilon_{ijk} M_{i1} M_{j2} M_{k3} = |M|$

(c). $|MN| = \epsilon_{ijk} (MN)_{1i} (MN)_{2j} (MN)_{3k} = \epsilon_{ijk} (M_{1l} N_{li})(M_{2p} N_{pj})(M_{3q} N_{qk})$.

$|M||N| = (\epsilon_{ijk} M_{1i} M_{2j} M_{3k})(\epsilon_{lpq} N_{1l} N_{2p} N_{3q}) \cancel{= \epsilon_{ijk}(MN)_{1i}(MN)_{2j}(MN)_{3k}}$

$= |MN|$

4.9. $[\vec{a}\times\vec{b}]_i + c_i = (\vec{a}\cdot\vec{b})b_i - d_i$

$\epsilon_{ijk}a_jb_k + c_i = a_mb_mb_i - d_i$

4.10. (a). $\delta_{ij}\delta_{jk}\delta_{ki} = (\delta_{ij}\delta_{jk})(\delta_{ki}) = \delta_{ik}\delta_{ki} = 1.$

(b). $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mni} = \delta_{il}\epsilon_{jni} - \delta_{jl}\epsilon_{ini}$
$= \delta_{il}\epsilon_{inj} = \epsilon_{lnj}$

4.11. $\delta_{ij}a_jb_lc_k\delta_{li} = a_ib_ic_k = (\vec{a}\cdot\vec{b})\vec{c}$

4.12. (a). $\nabla\times(f\nabla f) = \nabla f\times(\nabla f) + f\nabla\times(\nabla f) = \vec{0}$

(b). $\nabla\cdot(f\nabla f) = \nabla f\cdot(\nabla f) + f\nabla\cdot(\nabla f) = (\nabla f)^2 + f\cdot\nabla^2 f$

4.13. $\nabla\cdot\vec{u} = \nabla\cdot(\nabla f\times\nabla g) = (\nabla\times(\nabla f))\cdot\nabla g - (\nabla\times(\nabla g))\cdot\nabla f$
$= 0$

4.14. $\vec{u}\cdot\nabla\vec{v} = \frac{1}{2}(\nabla(\vec{u}\cdot\vec{v}) - \nabla\times(\vec{u}\times\vec{v}) - \vec{u}\times(\nabla\times\vec{v}) - \vec{v}\times(\nabla\times\vec{u}) + \vec{u}(\nabla\cdot\vec{v}) - \vec{v}(\nabla\cdot\vec{u}))$

$\vec{u}\cdot\nabla\vec{u} = \frac{1}{2}(\nabla|\vec{u}|^2 - 2\vec{u}\times(\nabla\times\vec{u})) = \nabla\frac{|\vec{u}|^2}{2} - \vec{u}\times(\nabla\times\vec{u}).$

4.15. (a). $\nabla\cdot\nabla^2\vec{u} = \nabla\cdot(\nabla\cdot(\nabla\vec{u}))$ ~~$= \nabla\cdot(\nabla\cdot\frac{\partial\vec{u}}{\partial x_i}) = \nabla\cdot\frac{\partial^2\vec{u}}{\partial x_i^2} = \partial$~~

$= \nabla\cdot(\nabla\cdot(\nabla u_i)) = \nabla\cdot(\nabla\cdot\frac{\partial u_i}{\partial x_j}) = \nabla\cdot\frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial^3 u_i}{\partial x_i\partial x_j^2}$

$\nabla^2\nabla\cdot\vec{u} = \nabla\cdot\nabla(\nabla\cdot\vec{u}) = \nabla\cdot(\nabla\frac{\partial u}{\partial x_i}) = \nabla\cdot\frac{\partial^2 u}{\partial x_i\partial x_j} = \frac{\partial^3 u}{\partial x_i\partial x_j^2}$

$\Rightarrow \nabla\cdot\nabla^2\vec{u} = \nabla^2\nabla\cdot\vec{u}$

(b) $\nabla\cdot\nabla^2\vec{u} = \nabla\cdot(\nabla(\nabla\cdot\vec{u}) - \underbrace{\nabla\times(\nabla\times\vec{u})}_{=0}) = \nabla^2(\nabla\cdot\vec{u}) = \nabla^2\nabla\cdot\vec{u}$

4.16. $\nabla\cdot\vec{w}=0.$

$\nabla\cdot(\vec{u}+\vec{\nabla}\times\vec{w})=\nabla\cdot(\nabla\phi+\nabla^2\vec{v})$

$=0+0=\nabla^2\phi+\nabla\cdot(\nabla\cdot(\nabla\vec{u}))=\nabla^2\phi+\cancel{\nabla\cdot(\nabla\cdot\frac{\partial u_i}{\partial x_j})}\nabla^2\nabla\cdot\vec{u}=\nabla^2\phi$

$\Rightarrow\nabla^2\phi=0$

4.17. $\nabla f(r)=\begin{bmatrix}\frac{\partial f(r)}{\partial x_1}\\ \frac{\partial f(r)}{\partial x_2}\\ \frac{\partial f(r)}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\frac{df(r)}{dr}\cdot\frac{\partial r}{\partial x_1}\\ \frac{df(r)}{dr}\cdot\frac{\partial r}{\partial x_2}\\ \frac{df(r)}{dr}\cdot\frac{\partial r}{\partial x_3}\end{bmatrix}=f'(r)\nabla r=f'(r)\cdot\frac{\vec{r}}{r}$

$\cancel{\nabla f(r)=\frac{df(r)}{dr}\cdot\frac{\vec{r}}{r}\quad\nabla f(r)=\frac{\vec{r}}{r}}$

4.18. (a). $\nabla\times u=\nabla\times(h(r)\vec{r})=\begin{vmatrix}i&j&k\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ h(r)r_x&h(r)r_y&h(r)r_z\end{vmatrix}$ $\frac{\partial h(r)}{\partial x}=\frac{\partial h(r)}{\partial y}=\frac{\partial h(r)}{\partial z}$ $=\vec{0}$

(b). if $\nabla\cdot\vec{u}=0.$

$\nabla\cdot h(r)\vec{r}=\frac{\partial(h(r)r_x)}{\partial x}+\frac{\partial(h(r)r_y)}{\partial y}+\frac{\partial(h(r)r_z)}{\partial z}=\cancel{h(r)(\frac{\partial r_x}{\partial x}+\frac{\partial r_y}{\partial y}+\frac{\partial r_z}{\partial z})=h(r)(\nabla\cdot\vec{r})=0}$

$=h'(r)\cdot\frac{\vec{r}}{r}\vec{r}+3h(r)=h'(r)r+3h(r)=0.$

(c). $\cancel{h(r)=e^{-3r}}$

$\frac{dh}{dr}+3\frac{h(r)}{r}=0.\quad h(r)=Ce^{\int-\frac{3}{r}dr}=Ce^{-3\ln r}=Cr^{-3}$

4.19. (a) $\nabla\cdot\vec{u}=\nabla\cdot(c\nabla\times\vec{u})=0.$

(b). $\vec{\nabla}\times\vec{u}=\vec{\nabla}\times(c\nabla\times\vec{u})=c\nabla\times(\nabla\times\vec{u})$

(c). $\nabla\times\vec{u}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ \sin y&f&g\end{vmatrix}=(\frac{\partial g}{\partial y}-\frac{\partial f}{\partial z})\hat{i}+(\frac{\partial f}{\partial x}-\cos y)\hat{k}.$

$\vec{u}=\begin{bmatrix}\sin y\\ f\\ g\end{bmatrix}=\begin{bmatrix}c(\frac{\partial g}{\partial y}-\frac{\partial f}{\partial z})\\ 0\\ c(\frac{\partial f}{\partial x}-\cos y)\end{bmatrix}$

$f=0.$

$g=-c\cos y.$

$\frac{\partial g}{\partial y}=c\sin y=\frac{\sin y}{c}.$

$c=\pm1$

$g=\pm\cos y.$

5.1 $\oiint_S \vec{u}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{u}\,dV = \iiint_V (\sin y - \sin y)\,dV = 0.$

5.2 $\iiint_V \nabla\cdot\vec{u}\,dV = \iiint_V dV = 1.$

$\oiint_S \vec{u}\cdot\vec{n}\,dS = \iint_{S_1}\vec{u}\cdot\vec{n}\,dS + \iint_{S_2}\vec{u}\cdot\vec{n}\,dS + \iint_{S_3}\vec{u}\cdot\vec{n}\,dS + \iint_{S_4}\vec{u}\cdot\vec{n}\,dS + \iint_{S_5}\vec{u}\cdot\vec{n}\,dS + \iint_{S_6}\vec{u}\cdot\vec{n}\,dS$

$\iint_{S_1}(y,x,-x)\cdot(0,0,-1)\,dS = \iint_{S_1} x\,dx\,dy = \frac{1}{2}.$　$\iint_{S_2}(y,x,1-x)(0,0,1)\,dS = \iint_{S_2}(1-x)\,dx\,dy = 1-\frac{1}{2} = \frac{1}{2}.$

$\iint_{S_3} = \iint(1,x,z-x)(0,0,1)\,dS = \iint_{S_3}(z-x)\,dx\,dz = 0 = \iint_{S_4}$

$\iint_{S_5} = \iint(y,1,z-1)(1,0,0)\,dS = \iint_{S_5} y\,dx\,dz = 1.$　$\iint_{S_6} = -y\,dx\,dz = -1.$

所以 $\oiint_S \vec{u}\cdot\vec{n}\,dS = 1.$

5.3. $\iiint_V \nabla\cdot\vec{u}\,dV = \oiint_S \vec{u}\cdot\vec{n}\,dS = 0.$

$\iiint_V \nabla\cdot(\vec{u}\varphi)\,dV = \oiint_S \varphi\vec{u}\cdot\vec{n}\,dS = 0.$

$\iiint_V \nabla\cdot(\vec{u}\varphi)\,dV - \iiint_V \nabla\cdot\vec{u}\,dV = \iiint_V \left(u_x\frac{\partial\varphi}{\partial x} + u_y\frac{\partial\varphi}{\partial y} + u_z\frac{\partial\varphi}{\partial z}\right)dV = \iiint_V \vec{u}\cdot\nabla\varphi\,dV = 0.$

5.4. $\oiint \nabla f\cdot\vec{n}\,dS = \iiint_V \nabla\cdot(\nabla f)\,dV = \iiint_V \nabla^2 f\,dV = \iiint g\,dV.$

5.5. $\oiint_{S_1}\vec{v}\cdot\vec{n}\,dS = \frac{1}{2}\iiint_V \nabla\cdot\vec{v}\,dV = \frac{1}{2}\iiint_V dV = \frac{2\pi}{3}.$

$\iint_{S_2}(x+y,0,x^2)\begin{bmatrix}0\\0\\-1\end{bmatrix}dS = \iint_{S_2} -x^2\,dS = \int_0^{2\pi}\int_0^1 -\rho^2\cos^2\theta\,\rho\,d\rho\,d\varphi = -\frac{1}{4}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = -\frac{\pi}{4}$

原式 $= \frac{2\pi}{3} - \left(-\frac{1}{4}\pi\right) = \frac{11}{12}\pi.$

5.6. $\frac{\partial}{\partial t}\iiint_V q\,dV = -\oiint_S \vec{j}\cdot\vec{n}\,dS$

$= -\iiint_V \nabla\cdot\vec{j}\,dV \Rightarrow \frac{\partial q}{\partial t} = -\nabla\cdot\vec{j}$

5.7. $\nabla f = \lim_{\delta V\to 0}\frac{1}{\delta V}\oiint_S f\vec{n}\,dS.$

5.8. $\oint_c \vec{r}\cdot d\vec{r} = \oint x\,dx + y\,dy + z\,dz = 0.$

5.9. $\iint_s \nabla\times\vec{u}\cdot\vec{n}\,ds$ $\quad \nabla\times\vec{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -y^2 & -y^2 z \end{vmatrix} = -2yz\,\hat{i} + 1\,\hat{k} = \begin{bmatrix} -2yz \\ 0 \\ 1 \end{bmatrix}$

$= \iint \nabla\times\vec{u}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix} ds = \iint ds = \pi.$

$\oint_c \vec{u}\cdot d\vec{r} = \oint (2x-y, -y^2, -y^2 z)\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \oint (2x-y, -y^2, 0)\cdot\begin{bmatrix} dx \\ dy \\ 0 \end{bmatrix}$

$= \int_0^{2\pi} (2\cos\theta - \sin\theta, -\sin^2\theta, 0)\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} d\theta = \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta - \sin^2\theta\cos\theta)\,d\theta$

$= -\sin 2\theta\Big|_0^{2\pi} + \left(\frac{1}{2}\theta - \frac{1}{2}\sin\theta\right)\Big|_0^{2\pi} - \frac{\sin^3\theta}{3}\Big|_0^{2\pi} = \pi.$

5.10. $\oint_c f\nabla g\cdot d\vec{r} = \iint_s \nabla\times(f\nabla g)\cdot\vec{n}\,dS = \iint_s [\nabla f\times\nabla g) + f\nabla\times(\nabla g)]\,\vec{n}\cdot ds$

$= \iint_s \nabla f\times(\nabla g)\,\vec{n}\cdot ds$

$\oint_c g\nabla f\cdot d\vec{r} = \iint_s \nabla\times(g\nabla f)\cdot\vec{n}\,ds = \iint_s \nabla g\times(\nabla f)\,\vec{n}\cdot ds$

$\rightarrow \oint_c f\nabla g\cdot d\vec{r} = -\oint_c g\nabla f\cdot d\vec{r}.$

5.11. $\nabla\times\vec{u} = 0.$

$\nabla\times(\vec{u}f) = \nabla f\times\vec{u} + f\nabla\times\vec{u} = \nabla f\times\vec{u}$

$\iint_s \vec{u}\times\nabla f\cdot\vec{n}\,ds = \iint_s \nabla\times(\vec{u}f)\,ds = -\oint \vec{u}f\cdot d\vec{r}$

5.12. $\iint_s \frac{\partial\vec{B}}{\partial t}\cdot\vec{n}\,ds = \iint_s \nabla\times(\vec{u}\times\vec{B})\cdot\vec{n}\,ds = \oint_c \underbrace{(\vec{u}\times\vec{B})\cdot d\vec{r}}_{=0} = 0$

$\frac{\partial\vec{B}}{\partial t} = 0$

5.13. $\frac{1}{2}\left|\oint_c \vec{r}\times d\vec{r}\right| = \frac{1}{2}\sqrt{\left[\oint_c \vec{r}\times d\vec{r}\right]_j\left[\oint_c \vec{r}\times d\vec{r}\right]_j} = \frac{1}{2}\sqrt{\left[\iint\left(\frac{\partial x_k}{\partial x_k}n_j - \frac{\partial x_k}{\partial x_j}n_k\right) + \left(\frac{\partial x_i}{\partial x_i}n_j - \frac{\partial x_i}{\partial x_j}n_i\right) + \left(\frac{\partial x_j}{\partial x_j}n_j - \frac{\partial x_j}{\partial x_j}n_j\right)ds\right]^2}$

$= \frac{1}{2}\sqrt{\left[\iint (n_j + n_j + n_j - n_j)\,ds\right]^2} = \frac{1}{2}\sqrt{4\left(\iint n_j\,ds\right)^2}$

$= \iint n_j\,ds = A.$