Homework 11 for GP1

By SJ

1. KK 11.2

Answer: Follow the procedure,

$$T_1 = \frac{2L_1}{c}(1+\beta^2), T_2 = \frac{2L_2}{c}(1+\frac{1}{2}\beta^2)$$

$$T_1' = \frac{2L_1}{c}(1+\frac{1}{2}\beta^2), T_2' = \frac{2L_2}{c}(1+\beta^2)$$

$$\Delta T_1 = T_1 - T_2, \Delta T_2 = T_2' - T_1'$$

$$\Delta T = \Delta T_1 + \Delta T_2 = (T_1 + T_2') - (T_2 + T_1') = \frac{2(L_1+L_2)}{c}\beta^2 - \frac{(L_1+L_2)}{c}\beta^2 = \frac{(L_1+L_2)}{c}\beta^2$$

$$N = \omega\Delta T/2\pi = \frac{\Delta T}{\lambda} = \frac{(L_1+L_2)}{\lambda}\beta^2$$

2. KK 11.4

(a) Event 1, B receives 1st light from A at time:

$$T_1 = \frac{l}{c-v}$$

Event 2, B receives 2nd signal from A at time:

$$T_2 = T + \frac{l}{c+v}$$

$$T_2 - T_1 = T + (\frac{l}{c+v} - \frac{l}{c-v})$$

$$\Delta T = (\frac{l}{c+v} - \frac{l}{c-v}) = \frac{l}{c}(\frac{1}{1+\beta} - \frac{1}{1-\beta}) = \frac{l}{c}(-\frac{2\beta}{1-\beta^2}) = -\frac{2\beta l}{c}(\frac{1}{1-\beta^2}) \approx -\frac{2\beta l}{c}$$

(b) $$l - 5.6\,\mathrm{Re} \approx 3.7\times10^4 km = 3.7\times10^7 m$$

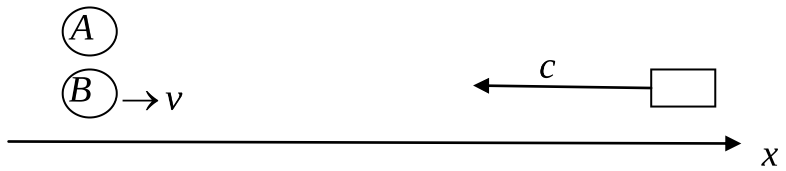
$$\Delta T \geq 10^{-14}$$

$$\frac{2\times3.7\times10^7 v}{c^2} \geq 10^{-14} \rightarrow \frac{7.4\times10^7}{9\times10^{16}} v \geq 10^{-14} \rightarrow v \geq 1.25\times10^{-3}$$

Of course, the ΔT won't be detected since c is same in the travelling frame too, i.e.

$$T_1 = \frac{l}{c}, T_2 = \frac{l}{c}, \Delta T = 0\,.$$

3. As the figure below shows. Two particles A,B: A is stationary in lab and B is moving with velocity v. A laser is L (in lab frame) away from A. Viewed in lab frame, as A,B overlap, a light is simultaneously shoots out of the laser.



a) In lab frame what is the time difference between particle B and particle A receives the light signal?
b) If we are traveling with B with same speed v and in such frame, what is the time difference again? (Reminder: be careful of simultaneity here) Could the time order receiving signals by A,B reversed?
c) If the laser is not shooting light but bullets with velocity V< c, of course B will be hit before A in lab frame. Could the time order of A,B hit by bullets be reversed for observers in another inertial frames? You may not have learned the materials from the lecture yet, (you will need velocity transformation or causality) ***take a guess*** for the present if you have no logical reason.

Answer:

(a) Event 1, A, B overlap (x_A=x_B=0,t=0)
Event 2, light emits from laser (x=L, t=0) Simultaneous in frame A (lab)

Event 3, light meets A (x_3=0, t_3) $t_3 = \frac{L}{c}$

Event 4, light meets B (x_4,t_4) $t_4 = \frac{L}{c+v}, x_4 = \frac{vL}{c+v}$

$$\Delta t_A = t_3 - t_1 = \frac{L}{c}, \Delta t_B = t_4 - t_1 = \frac{L}{c+v}$$

$$\Delta t = \Delta t_A - \Delta t_B = \frac{L}{c} - \frac{L}{c+v} > 0$$

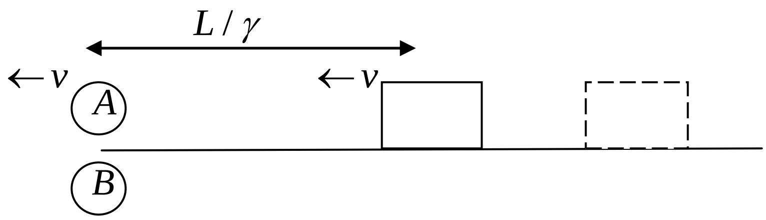
Light meets B first. (Obviously)
Of course it is redundant to list the events and specify their space-time coordinates as above, but this kind of thinking will be helpful in complicated problems.

(b) In the frame B (stationary relative to B, so A and laser are moving with –v towards left)
Event 1, A, B overlap (x'_A=x'_B=0,t'=0) (expressed in S')
The trick part is for other events, the event 2 will not be simultaneously in frame B! This is the key in solving this without using Lorentz Transform (actually with above clear definition of events and their space-time coordinates listed in S, it is straightforward to use Lorentz transform to get their space time coordinates in S'. I shall first solve the problem without L-Transform and use the L-transform as a test afterwards, you may appreciate more about the L-Transform this way and have practice in length contraction and relative simultaneity more this way)

For the event 2 viewed in S', you may use length contraction first: $L' = L/\gamma$, but this is not the distance of event (Above is B measures distance L simultaneously in his frame). The simultaneous events (event 1 and 2) in A is not simultaneous any more in B! Event 2(the laser fires light, which happens at the tail in B's view) is before the event 1, with time difference between event 1 and 2 in B is given by (11-13) in the note, with sign change due to direction of velocity:

$\Delta t'_{12} = -\frac{v}{c^2}\Delta x'_{12}$ The reason for this –v is because in B's view, the motion of A frame is with negative velocity.

So the event 2 happens that time earlier than event 1 viewed by B, and $L' = L/\gamma$ is simultaneous events distance in B, the laser emits light (event 2) is shown by the dashed box that happens earlier by the time given above



So the event happens at:

$$x'_2 = \Delta x'_{12} = L/\gamma + v\frac{v}{c^2}\Delta x'_{12} = L/\gamma + \beta^2 \Delta x'_{12}$$

$$\Delta x'_{12}(1-\beta^2) = L/\gamma \to x'_2 = \gamma L$$

$$t'_2 = -\frac{v}{c^2}\gamma L$$

These are the space-time coordinates in S' for event 2.

Of course I can quickly get above by the argument in the notes. The simultaneous event in A is just like measuring a proper distance in frame B (A and laser simultaneously shoot out light to make a mark in frame B) which A would say it is distance L, but in B's frame that is proper length γL.

Now I can calculate event 3 and 4 in B's frame find out their time (just like a standard chasing problem)

For event 4 (light meets B), use the figure above, since the light emits out with $t'_2 = -\frac{v}{c^2}\gamma L$ earlier, the light already travels distance $c\frac{v}{c^2}\gamma L = \beta\gamma L$ by the time 0. So the extra time: $t'_4 = \frac{\gamma L - \beta\gamma L}{c} = \frac{\gamma L}{c}(1-\beta) = \frac{L}{c}\sqrt{\frac{1-\beta}{1+\beta}}$

For event 3 (light meets A): $t'_3 = \frac{\gamma L - \beta\gamma L}{c-v} = \frac{\gamma L}{c} > t'_4$

The event 4 happens earlier too in such frames (actually in all inertial frames)

Now I will just use L-Transform to directly find out time intervals:

$$\Delta x' = \gamma(\Delta x - \beta c \Delta t); c\Delta t' = \gamma(c\Delta t - \beta \Delta x)$$

The Δx and t's in A's frame are known from (a), I just plug them in the formula (the difference taken with respect to event 1)

$$\Delta x_2' = \gamma L, \Delta t_2' = -\frac{v}{c^2}\gamma L$$

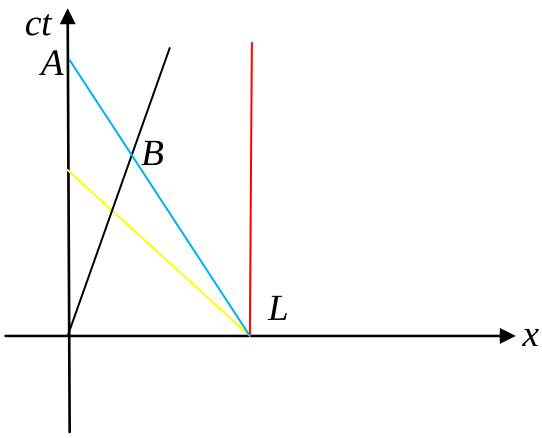
$$\Delta x_3' = \gamma(0 - \beta c \frac{L}{c}) = -\beta\gamma L; \Delta t_3' = \gamma L / c$$

$$\Delta x_4' = \gamma(\frac{vL}{c+v} - \frac{\beta c L}{c+v}) = 0$$ (this makes perfect sense, isn't it)

$$\Delta t_4' = \gamma(\frac{L}{c+v} - \frac{\beta}{c}\frac{vL}{c+v}) = \frac{\gamma L}{c}(\frac{1}{1+\beta} - \frac{\beta^2}{1+\beta}) = \frac{\gamma L}{c}(1-\beta)$$

All these agree with the previous calculation. (I am more confident now about my results)

(c) Indeed the time order between bullets hits A or B should not reverted for all inertial observers. You may argue from causality, say a single bullet out of laser, if it hits B, A will survive. For observer in A, the event 4 (bullet hits B) happens first (B is killed and A survives). It would not be possible for other observers in other frames to see the reverse of order of such events, otherwise real contradiction will arise. The strict proof is most simply done with Minkowski diagram (I have not told you yet):



I'd better explain this diagram a little. The vertical black line is the world line of A (I am plotting it using A's frame, so A just stay at 0 at all time), the tilted black line is the B travels with velocity v. The red line is the world line of the laser. The yellow is the light out of laser (simultaneously in A's frame); the blue line is the world line of a bullet with smaller velocity <c. When the blue line across the world line of B and A at point B, A，you can read the time from coordinate (Clearly B happened before A in this frame).

Noticed the point A is within the **light cone** originated at point B, that means the time order of these two cannot be inverted, A cannot happen before B in any inertial frame. The argument is based on the section on Minkowski diagram and Causality in my note.

4. A train of proper length L and speed of 3/5 c approaches a tunnel of length L (proper length). When the head of the train enters the tunnel, a person leaves the head and starts running towards back. He arrives at the back at exactly same moment as the back of the train leaves the tunnel.
   (a) Do the simultaneity of the head (of the train) enters the tunnel and person starts running and the simultaneity of his arrival to the back and train leaves the tunnel depend on the frames of observation?
   (b) How much time does the process described above take in ground frame?
   (c) What is the speed of the person from the ground point of view?
   (d) How much time elapsed on the running person's watch?
   Answer:
   (a) No, these simultaneities are all happed at the same place and they will be agreed by all observers.
   (b) For the ground observer, the tunnel is L, but the train is $L/\gamma_1, \gamma_1 = \dfrac{1}{\sqrt{1-0.36}} = \dfrac{5}{4}$

   The total distance covered by the head (or tail) of the train is just:

   $L + L/\gamma_1 = \dfrac{9}{5}L$, and with v=3/5 c, the time is: $t = 3L/c$

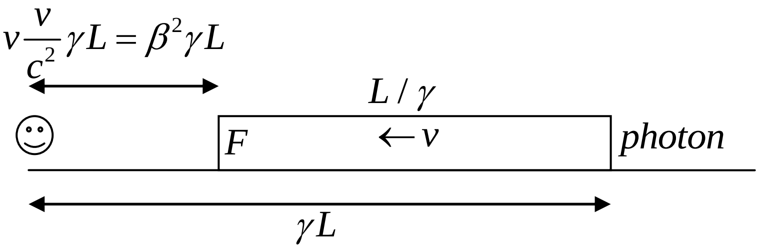
   (c) For the ground observer, the person covered distance L within time $t = 3L/c$, so the speed relative to ground is c/3.
   (d) The running person's watch will give the proper time, and using v=c/3,

   $$\gamma_2 = \frac{1}{\sqrt{1-1/9}} = \frac{3}{2\sqrt{2}}, \quad \tau = t/\gamma_2 = \frac{2\sqrt{2}L}{c}$$

5. A person runs with speed v towards a tunnel with proper length L. A laser is located at the far end of the tunnel. When the person enters the tunnel, the laser emits a photon simultaneously (in tunnel frame) towards the person. When the photon meets the person, the person has travelled xL, find the x value. Work this out in both frames of the tunnel and the person.
   Answer: The tunnel (ground) frame is very easy.

   $$xL/v = (1-x)L/c \rightarrow x = \frac{v}{c+v} = \frac{\beta}{1+\beta}$$

   The person frame is tricky but we have already discussed in problem 3. The situation is very similar here, be careful about simultaneity!. I draw the picture below following similar calculation as part (b) in problem 3:



The picture showed when the photon emitted at far end, the relative positions in the person's frame at certain time t', i.e. this is simultaneous picture in the person's frame(Do see problem 3 to know how I get the values in the figure). The photon will take $\gamma L / c$ time to reach person and during which the front end of tunnel F will travels: $v\gamma L / c = \beta\gamma L$. The relative position of F when the photon reached person will be:

$$\beta\gamma L - \beta^2\gamma L = \beta\gamma L(1-\beta)$$

The fraction of this to the total tunnel length in the person's frame then is:

$\beta\gamma L(1-\beta)/(L/\gamma) = \beta\gamma^2(1-\beta) = \dfrac{\beta(1-\beta)}{1-\beta^2} = \dfrac{\beta}{1+\beta}$ the fraction is same as in ground frame. (This fraction number is same in all frames, can you see why this should be? If not will it violate relativity principle? The corollary of relativity principle, i.e. the absolute motion cannot be detected, is easier to apply here)

6. Alice and Bob start at same location at time 0 (both their watches read 0 when they meet), Alice travels right and bob towards left with relative speed v between them. When Bob's watch read T, he sends out a photon to Alice; when Alice receives the photon, what is time reading on her watch? Try this problems in both Bob's and Alice's frame.

Answer: In Bob's frame (shown above)

Event 0 is when A,B meets where everything is set to 0

Event 1: Photon emitted from Bob (x_1', t_1'), $x_1' = 0, t_1' = T$

Event 2: photon reaches A: (x_2', t_2') $t_2' = T + \dfrac{vT}{c-v}, x_2' = -v(T + \dfrac{vT}{c-v})$

The question ask what is the time of event 2 on A's watch t_2, where that time is the proper time, and I will use time dilation (which is quickest method here):

$$\gamma t_2 = t_2' = T(1+\frac{v}{c-v}) = T(1+\frac{\beta}{1-\beta}) = \frac{T}{1-\beta}$$

$$t_2 = \frac{T}{1-\beta}\frac{1}{\gamma} = T\sqrt{\frac{1+\beta}{1-\beta}}$$

This is the time on Alice's watch when she receives photon.
In Alice's frame, Bob is moving, and the photon is emitted not at T according to her watch.
The event 1 happened (in Alice's time) at: (T on bob's watch is proper time here)

$t_1 = \gamma T$,so $x_1 = v\gamma T$

It takes extra time for photon reaches Alice: $\Delta t = v\gamma T / c = \beta\gamma T$

So the $t_2 = \gamma T + \beta\gamma T = \gamma(1+\beta)T = \sqrt{\frac{1+\beta}{1-\beta}}T$

7. Let Alice and Bob synchronize their watches at Starbucks and Bob immediately jumps on a bus with velocity v and travels with bus distance L (L in ground frame). Bob jumps off the bus and the reading of his watch will be different from proper synchronization (the watches of Alice and Bob will have different reading), calculate this difference and show that as v is small, the difference approaches 0.
Answer:
Another problem you can use time dilation. I will use Alice frame which is inertial. While the Bob may need acceleration and deceleration at the beginning and end, but let's assume the time for those processes are extremely short and can be neglected (otherwise I have to use instantaneous inertial frame and the time dilation factor is not constant there. If the time interval for acceleration/deceleration is much small than the total time in Alice frame, that time is neglected)
Event 0, Alice bob at Starbucks, everything set 0
Event 1 Bob reaches distance L: (L,L/v)
The time elapsed according to bob's watch (which is the proper time) at event 1 is:

$$\gamma t_1' = t_1 = L / v, t_1' = \frac{L}{v}\frac{1}{\gamma}$$

All clocks in A's frame are synchronized, so that at location L, the clock in Alice frame is reading L/v. Bob's watch will be off by:

$$\Delta t = t_1' - t_1 = \frac{L}{v}(\frac{1}{\gamma} - 1) \approx \frac{L}{v}(1 - \frac{1}{2}\beta^2 - 1) = -\frac{L}{2}\frac{v}{c^2}$$

For a certain distance, if v takes small enough value, the difference can be very small and neglected. Of course this is how we experience in daily life.

Working in Bob's frame would be more challenging, this problem does not ask to do that way,

but you may try that （The result would be same as that calculated from Alice's frame）. Actually I do not need to invoke general relativity, though Bob does experience acceleration /deceleration. The gravitational (sorry I mean acceleration, they are same) effect plays no part if Bob compares his watch with the watch in Alice's frame at the same location as Bob. i.e. if Bob drops at XiDan, he just compare his watch to the clock's of Alice's at XiDan, the gravitational correction would be 0 in this case. If he compares his watch with Alice's at Starbucks at Tsinghua (which is recorded by his army of observers), he has to include the gravitational effect. So I shall work out the comparison between Bob's watch and clock's of Alice at Xidan. The distance between Starbucks and XiDan is of course L in Alice's frame.
The key is synchronization of clock's in Alice's frame is not so in Bob's frame! The clocks in Alice's frame would be not properly synchronized from Bob's point of view. The one at Xidan will be ahead of time by $\frac{v}{c^2}L$ (refer to my figure in the notes, Smiley on a train, pg 435).

During the travel to Xidan, according to bob, the distance is (length contraction) L/γ, and time is $\frac{L}{v}\frac{1}{\gamma}$. During this time, what is the elapse of time for the clock stayed at XiDan in Alice frame? That time elapse (which is proper time) is just $\frac{L}{v}\frac{1}{\gamma^2}$

So the time appeared on the Xidan clock when Bob dropped out is

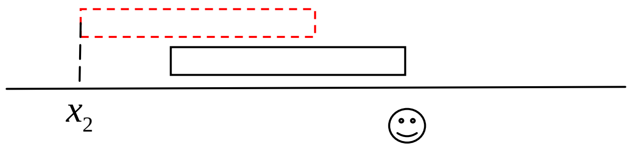
$$\frac{v}{c^2}L+\frac{L}{v}\frac{1}{\gamma^2}=\frac{L}{v}(\beta^2+\frac{1}{\gamma^2})=\frac{L}{v}(\beta^2+1-\beta^2)=\frac{L}{v}$$

This is same as in calculation in Alice frame (should be), and Bob's watch is still off by same amount.
If you work the problem from Bob's point of view but comparing with alice's clock at Starbucks(which is at high gravity and runs faster), then nonzero distance L would cause gravitation shift in time by $\Delta t=T\frac{a}{c^2}L, T=\frac{v}{a}\rightarrow\Delta t=\frac{L}{v}\beta^2$, you can figure out the rest.

8. This problem is to illustrate the difference of measuring a moving rod length and seeing (really see with eyes) the length of the same rod: A rod with proper length L travels with v along the x direction (the rod is also aligned along x direction). The measured length for the lab frame is of course L/γ. Now imagine you are at the right of the whole rod and the rod is moving towards you from left, you are also standing close to the track of the motion of the rod. a) Explain that the lights which reaches your eyes at one time (the time your eyes record the picture, think your eye as a camera) must leaves the two ends of the rod at different times. b) from there, calculate the length as seeing by your own eyes, and show that it is different than L/γ, actually in this situation, it is even larger than L.(Here is quite

analogous to Doppler effect we shall talk about)

Answer:



a) I shall just work in the ground (observer) frame.
Since the light travels at fixed speed c, it takes time for the light from the front and back end of the rod to reach eyes. It takes longer time for the light at the back to reach the eye (this is because I arranged this problem that the rod is moving towards you; if the rod is moving away from you, you can figure out what should be the situation). In our problem, our eye records the light from both the front and back end of the rod at one particular time, i.e. the photons from the front and back end of rods arrives at the eye at same time. This means the photon from the back has to leaves at earlier time in order to reach the eye at the same time as front photon. It is maybe attempting just to use $l = L/\gamma, \Delta t = l/c$. But this time difference is not exactly correct because the rod is moving[1]. The safe way is using the graph above (It is always helpful to draw simple graphs in problem solving in SR, the graph really helps setting up the model or help you finding the pitfalls):
Here I chose x_1=0, the front end overlap with observer. The back photon is emitted at earlier time where the rod is shown as the red dashed one in the figure.

$$\Delta t = x_2 / c$$

$$x_2 = v\Delta t + l$$ (the rod moves $v\Delta t$ besides its original length)

$$x_2 = \beta x_2 + l = \beta x_2 + L/\gamma$$

$$x_2 = \frac{L}{\gamma(1-\beta)} = L\sqrt{\frac{1+\beta}{1-\beta}}$$

$$\Delta t = \frac{L}{c}\sqrt{\frac{1+\beta}{1-\beta}}$$

Actually I have solved part (a) and (b) at the same time, the x_2 will be the length seen by observer's own eyes (Imagine the track has markers and the x_2 would be the marker's value read by the observer or taken by the camera). The picture taken by the camera would be something like the figure above (of course the red one is shifted above is because I draw to make the graph clear, no such vertical displacement in real case).

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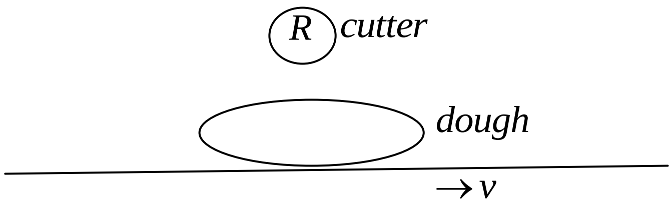
[1] Of course I could find out correct Δt using a chasing model, then the $\Delta t = \frac{l}{c-v} = \frac{L/\gamma}{c-v}$, which is the correct answer.

9. Is the cookie round or not: Suppose a cookie dough is on a conveying belt travelling with velocity v. A round cutter (in factory frame) with radius R will punch down repeatedly to cut the dough into cookies. Then when you buy the cookies in store, what is the shape of the cookie, a round one with radius R or elongated (or shortened) along one side? Explain this both in factory frame and in the cookie (on the conveying belt) frame. (The answer of course should not depend on the frames)



Answer: It is much easier to see the problem in factory frame (cutter's frame, in which I show above (of course I rotate the cutter to show you the face, in the cutting the round face is down), cutter is not moving). The cutting is just like measuring the moving distance in this frame, the both ends of cutter will cut simultaneously through the dough in this frame. Along the direction perpendicular to the motion (say y direction), there is no length contraction. The length of cut cookie along y will be just R. But along the x direction, there will be length contraction. The proper length (which is the length stationary to the dough and is the length when the cookie stopped motion, like in the display window of the shop) is:

$$R = l_p / \gamma \rightarrow l_p = \gamma R$$

So the cookie will not be round but prolonged along x direction.
The cookie frame is a bit tricky, but by now after several similar problems, I hope you will see right away what is the key: the simultaneity is relative! In the cookie frame, the cutter does not cut the cookie simultaneously, the tail end cut first and the head end later, that is why the cookie will be prolonged in cookie's frame too. The detailed procedure had been listed in part b) problem 3.
Say event 1 is right side of cutter hits dough: (0,0) in both frames
Event 2 is left side of cutter hits dough: $(x_2,t_2)=(R,0)$ in cutter frame and (x'_2,t'_2) in cookie frame. Lorentz-transform is the easiest to find out the value of (x'_2,t'_2). **And such LT method will be the method I recommend you to use and to master.**
For the sake of practice, I can solve this without using LT but may go through the procedure like that in problem 3 (that is why I tried to solve that problem in detail). Brief calculation (with a slight different method as that in problem 3 part b) below:
The time difference on the clock at the right(tail) and left(head) ends of cutter observed by cookie would be: $\Delta t = \frac{v}{c^2} R$,this is the time elapsed on the left end clock of the cutter.

This time interval in the cookie's clock is: $\Delta t' = \gamma \Delta t = \gamma \frac{v}{c^2} R$. During this time, the cutter travels: $\beta^2 \gamma R$, and this distance plus the cutter's length R / γ (viewed by cookie) is the total distance in cookie frame: $\beta^2 \gamma R + R / \gamma = \gamma R$. Same result as in cutter's frame