第 11 次作业题

1. 求下列曲线的弧长:

(1) 曲线 $y=\int_{-\frac{\pi}{2}}^{x}\sqrt{\cos t}\,\mathrm{d}t$ $(-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2})$ 的弧长.

(2) 阿基米德螺线 $\rho=a\theta$ $(0\leqslant\theta\leqslant 2\pi,\ a>0)$ 的弧长.

解: (1) 所求弧长为

$$\begin{aligned}
L &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1+(y')^2}\,\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1+(\sqrt{\cos x})^2}\,\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1+\cos x}\,\mathrm{d}x\\
&= 2\int_0^{\frac{\pi}{2}}\sqrt{2\cos^2\frac{x}{2}}\,\mathrm{d}x=2\sqrt{2}\int_0^{\frac{\pi}{2}}\cos\frac{x}{2}\,\mathrm{d}x=4\sqrt{2}\sin\frac{x}{2}\Big|_0^{\frac{\pi}{2}}=4.
\end{aligned}$$

(2) 所求弧长为

$$\begin{aligned}
L &= \int_0^{2\pi}\sqrt{(\rho(\theta))^2+(\rho'(\theta))^2}\,\mathrm{d}\theta=a\int_0^{2\pi}\sqrt{1+\theta^2}\,\mathrm{d}\theta\\
&\overset{\theta=\tan t}{=} a\int_0^{\arctan(2\pi)}\sqrt{1+\tan^2 t}\,\mathrm{d}(\tan t)=a\int_0^{\arctan(2\pi)}\frac{\mathrm{d}t}{\cos^3 t}\\
&= a\int_0^{\arctan(2\pi)}\frac{\mathrm{d}(\sin t)}{\cos^4 t}\overset{u=\sin t}{=}a\int_0^{\frac{2\pi}{\sqrt{1+4\pi^2}}}\frac{\mathrm{d}u}{(1-u^2)^2}\\
&= a\int_0^{\frac{2\pi}{\sqrt{1+4\pi^2}}}\Big(\frac{1}{4(u+1)}+\frac{1}{4(u+1)^2}-\frac{1}{4(u-1)}+\frac{1}{4(u-1)^2}\Big)\,\mathrm{d}u\\
&= \frac{a}{4}\Big(\log\Big|\frac{u+1}{u-1}\Big|-\frac{1}{u+1}-\frac{1}{u-1}\Big)\Big|_0^{\frac{2\pi}{\sqrt{1+4\pi^2}}}\\
&= \frac{a}{4}\Big(\log\frac{\frac{2\pi}{\sqrt{1+4\pi^2}}+1}{1-\frac{2\pi}{\sqrt{1+4\pi^2}}}-\frac{1}{\frac{2\pi}{\sqrt{1+4\pi^2}}+1}-\frac{1}{\frac{2\pi}{\sqrt{1+4\pi^2}}-1}\Big)\\
&= \frac{a}{4}\Big(\log\frac{2\pi+\sqrt{1+4\pi^2}}{\sqrt{1+4\pi^2}-2\pi}-\frac{\sqrt{1+4\pi^2}}{2\pi+\sqrt{1+4\pi^2}}-\frac{\sqrt{1+4\pi^2}}{2\pi-\sqrt{1+4\pi^2}}\Big)\\
&= \frac{a}{2}\log(2\pi+\sqrt{1+4\pi^2})+a\pi\sqrt{1+4\pi^2}.
\end{aligned}$$

2. 证明极坐标下的曲率公式 $\kappa=\frac{|\rho^2+2(\rho')^2-\rho\rho''|}{(\rho^2+(\rho')^2)^{\frac{3}{2}}}$.

证明: 在极坐标系下, 我们有 $x(\theta)=\rho(\theta)\cos\theta$, $y(\theta)=\rho(\theta)\sin\theta$, 则

$$\begin{aligned}
x'(\theta) &= \rho'(\theta)\cos\theta-\rho(\theta)\sin\theta,\\
y'(\theta) &= \rho'(\theta)\sin\theta+\rho(\theta)\cos\theta,\\
x''(\theta) &= \rho''(\theta)\cos\theta-\rho'(\theta)\sin\theta-\rho'(\theta)\sin\theta-\rho(\theta)\cos\theta\\
&= \rho''(\theta)\cos\theta-2\rho'(\theta)\sin\theta-\rho(\theta)\cos\theta,\\
y''(\theta) &= \rho''(\theta)\sin\theta+\rho'(\theta)\cos\theta+\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta,\\
&= \rho''(\theta)\sin\theta+2\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta.
\end{aligned}$$

由此立刻可得

$$\begin{aligned}
\big(x'(\theta)\big)^2+\big(y'(\theta)\big)^2 &= \big(\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta\big)^2+\big(\rho'(\theta)\sin\theta+\rho(\theta)\cos\theta\big)^2\\
&= \big(\rho(\theta)\big)^2+\big(\rho'(\theta)\big)^2.\\
x'(\theta)y''(\theta)-x''(\theta)y'(\theta) &= \big(\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta\big)\big(\rho''(\theta)\sin\theta+2\rho'(\theta)\cos\theta-\rho(\theta)\sin\theta\big)\\
&\quad -\big(\rho''(\theta)\cos\theta-2\rho'(\theta)\sin\theta-\rho(\theta)\cos\theta\big)\big(\rho'(\theta)\sin\theta+\rho(\theta)\cos\theta\big)\\
&= \big(\rho'(\theta)\rho''(\theta)\cos\theta\sin\theta-\rho(\theta)\rho''(\theta)\sin^2\theta+2\big(\rho'(\theta)\big)^2\cos^2\theta\\
&\quad -2\rho(\theta)\rho'(\theta)\sin\theta\cos\theta-\rho(\theta)\rho'(\theta)\cos\theta\sin\theta+\big(\rho(\theta)\big)^2\sin^2\theta\big)\\
&\quad -\big(\rho'(\theta)\rho''(\theta)\cos\theta\sin\theta-2\big(\rho'(\theta)\big)^2\sin^2\theta-\rho(\theta)\rho'(\theta)\cos\theta\sin\theta\\
&\quad +\rho(\theta)\rho''(\theta)\cos^2\theta-2\rho(\theta)\rho'(\theta)\sin\theta\cos\theta-\big(\rho(\theta)\big)^2\cos^2\theta\big)\\
&= -\rho(\theta)\rho''(\theta)+2\big(\rho'(\theta)\big)^2+\big(\rho(\theta)\big)^2,
\end{aligned}$$

由此我们可立刻导出

$$\kappa=\frac{|x'(\theta)y''(\theta)-x''(\theta)y'(\theta)|}{\big((x'(\theta))^2+(y'(\theta))^2\big)^{\frac{3}{2}}}=\frac{|(\rho(\theta))^2+2(\rho'(\theta))^2-\rho(\theta)\rho''(\theta)|}{\big((\rho(\theta))^2+(\rho'(\theta))^2\big)^{\frac{3}{2}}}.$$

3. 求下列曲线的曲率半径:

(1) $y^2=2px\ (p>0)$,

(2) $x=a\cos t,\ y=b\sin t\ (0\leqslant t\leqslant 2\pi,\ a,b>0)$,

(3) 心脏线 $\rho=a(1+\cos\theta)\ (a>0)$.

解: (1) 由题设可知 $x=\frac{y^2}{2p}$, 则 $x'=\frac{y}{p}$, $x''=\frac{1}{p}$, 故所求曲率半径为

$$R=\frac{(1+(x')^2)^{\frac{3}{2}}}{|x''|}=\frac{(1+(\frac{y}{p})^2)^{\frac{3}{2}}}{\frac{1}{p}}=p\Big(1+\frac{y^2}{p^2}\Big)^{\frac{3}{2}}.$$

(2) 由题设立刻可得 $x'=-a\sin t$, $y'=b\cos t$, $x''=-a\cos t$, $y''=-b\sin t$. 于是所求曲线的曲率半径为

$$R=\frac{1}{\kappa}=\frac{((x')^2+(y')^2)^{\frac{3}{2}}}{|x'y''-y'x''|}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{\frac{3}{2}}}{ab}.$$

(3) 由题设可知 $\rho'=-a\sin\theta$, $\rho''=-a\cos\theta$, 故所求曲率半径为

$$\begin{aligned}
R &= \frac{1}{\kappa}=\frac{\big(\rho^2+(\rho')^2\big)^{\frac{3}{2}}}{|\rho^2+2(\rho')^2-\rho\rho''|}\\
&= \frac{\big(a^2(1+\cos\theta)^2+a^2\sin^2\theta\big)^{\frac{3}{2}}}{|a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta|}\\
&= \frac{8a^3|\cos\frac{\theta}{2}|^3}{|a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2(1+\cos\theta)\cos\theta|}\\
&= \frac{4}{3}a\big|\cos\frac{\theta}{2}\big|=\frac{2}{3}\sqrt{2a\rho}.
\end{aligned}$$

4. 求下列旋转体的体积:

(1) 由 $y=x^2$, $y=x^3$ 所围成的图形绕 x 轴旋转生成的旋转体;

(2) 由 $y=\sqrt{x}$ 与 x 轴以及直线 $x=4$ 所围成的图形绕 $x=4$ 以及 y 轴旋转生成的两个旋转体的体积.

解: (1) 两曲线的交点为 $(0,0)$ 和 $(1,1)$.

方法 1. 取 x 为积分变量. 因所围图形的横坐标介于 $0,1$, 故所求体积为

$$\begin{aligned} V &= \pi\int_0^1 (x^4-x^6)\,\mathrm{d}x \\ &= \pi\Big(\frac{x^5}{5}-\frac{x^7}{7}\Big)\Big|_0^1=\frac{2\pi}{35}. \end{aligned}$$

方法 2. 取 y 为积分变量. 所围图形的纵坐标介于 $0,1$, 故所求体积为

$$\begin{aligned} V &= 2\pi\int_0^1 y(y^{\frac{1}{3}}-y^{\frac{1}{2}})\,\mathrm{d}y \\ &= 2\pi\Big(\frac{3}{7}y^{\frac{7}{3}}-\frac{2}{5}y^{\frac{5}{2}}\Big)\Big|_0^1=\frac{2\pi}{35}. \end{aligned}$$

(2) 所围成的图形绕 $x=4$ 旋转生成的旋转体的体积为

$$V_1=2\pi\int_0^4(4-x)\sqrt{x}\,\mathrm{d}x=2\pi\Big(\frac{8}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\Big)\Big|_0^4=\frac{256}{15}\pi.$$

所围成的图形绕 y 轴旋转生成的旋转体的体积为

$$V_2=2\pi\int_0^4 x\sqrt{x}\,\mathrm{d}x=\frac{4\pi}{5}x^{\frac{5}{2}}\Big|_0^4=\frac{128}{5}\pi.$$

5. 求下列旋转体的表面积:

(1) 抛物线 $y=\sqrt{x}$ $(0\leqslant x\leqslant 2)$ 绕 x 轴旋转生成的旋转面;

(2) 星形线 $x=a\cos^3 t$, $y=a\sin^3 t$ $(a>0)$ 绕 x 旋转生成的旋转面.

解: (1) 所求旋转面的面积为

$$\begin{aligned} S &= 2\pi\int_0^2 y\sqrt{1+(y')^2}\,\mathrm{d}x=2\pi\int_0^2\sqrt{x}\sqrt{1+\Big(\frac{1}{2\sqrt{x}}\Big)^2}\,\mathrm{d}x \\ &= 2\pi\int_0^2\sqrt{x+\frac{1}{4}}\,\mathrm{d}x=2\pi\cdot\frac{2}{3}(x+\frac{1}{4})^{\frac{3}{2}}\Big|_0^2=\frac{13}{3}\pi. \end{aligned}$$

(2) 所求旋转面由星形线上半部分旋转而成, 故其面积为

$$\begin{aligned} S &= 2\pi\int_0^\pi |y|\sqrt{(x')^2+(y')^2}\,\mathrm{d}t \\ &= 6a^2\pi\int_0^\pi(\sin^4 t)|\cos t|\,\mathrm{d}t \\ &= 6a^2\pi\int_0^{\frac{\pi}{2}}(\sin^4 t)(\cos t)\,\mathrm{d}t-6a^2\pi\int_{\frac{\pi}{2}}^{\pi}(\sin^4 t)(\cos t)\,\mathrm{d}t \\ &= \frac{6a^2\pi}{5}\sin^5 t\Big|_0^{\frac{\pi}{2}}-\frac{6a^2\pi}{5}\sin^5 t\Big|_{\frac{\pi}{2}}^{\pi}=\frac{12}{5}a^2\pi. \end{aligned}$$

6. 求密度均匀的抛物线 $y=\frac{1}{2}x^2\ (-1\leqslant x\leqslant 1)$ 的质心.

解: 方法 1. 不妨设密度为 $\mu=1$. 则抛物线的总质量为

$$\begin{aligned}
M &= \int_{-1}^{1}\sqrt{1+x^2}\,\mathrm{d}x = 2\int_0^1\sqrt{1+x^2}\,\mathrm{d}x\\
&\overset{x=\tan t}{=} 2\int_0^{\frac{\pi}{4}}\frac{\sqrt{1+\tan^2 t}}{\cos^2 t}\,\mathrm{d}t = 2\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}t}{\cos^3 t}\\
&= 2\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}(\sin t)}{(1-\sin^2 t)^2} \overset{u=\sin t}{=} 2\int_0^{\frac{\sqrt{2}}{2}}\frac{\mathrm{d}u}{(1-u^2)^2}\\
&= \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}}\big(\frac{1}{u+1}-\frac{1}{u-1}+\frac{1}{(u+1)^2}+\frac{1}{(u-1)^2}\big)\,\mathrm{d}u\\
&= \frac{1}{2}\Big(\log\frac{|u+1|}{|u-1|}-\frac{1}{u+1}-\frac{1}{u-1}\Big)\Big|_0^{\frac{\sqrt{2}}{2}} = \log(\sqrt{2}+1)+\sqrt{2}.
\end{aligned}$$

故所求质心 $(\overline{x},\overline{y})$ 的坐标公式为:

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\int_{-1}^{1}x\sqrt{1+x^2}\,\mathrm{d}x = 0,\\
\overline{y} &= \frac{1}{2M}\int_{-1}^{1}x^2\sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{M}\int_0^1 x^2\sqrt{1+x^2}\,\mathrm{d}x\\
&\overset{x=\tan t}{=} \frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\tan^2 t\sqrt{1+\tan^2 t}}{\cos^2 t}\,\mathrm{d}t\\
&= \frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\sin^2 t}{\cos^5 t}\,\mathrm{d}t\\
&= \frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}t}{\cos^5 t} - \frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}t}{\cos^3 t}\\
&= -\frac{1}{2}+\frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}t}{\cos^5 t}\\
&= -\frac{1}{2}+\frac{1}{M}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}(\sin t)}{(1-\sin^2 t)^3}\\
&\overset{u=\sin t}{=} -\frac{1}{2}+\frac{1}{M}\int_0^{\frac{\sqrt{2}}{2}}\frac{\mathrm{d}u}{(1-u^2)^3}\\
&= -\frac{1}{2}+\frac{1}{M}\int_0^{\frac{\sqrt{2}}{2}}\Big(-\frac{3}{16(u-1)}+\frac{3}{16(u-1)^2}-\frac{1}{8(u-1)^3}\Big)\,\mathrm{d}x\\
&\quad +\frac{1}{M}\int_0^{\frac{\sqrt{2}}{2}}\Big(\frac{3}{16(u+1)}+\frac{3}{16(u+1)^2}+\frac{1}{8(u+1)^3}\Big)\\
&= -\frac{1}{2}+\frac{1}{M}\Big(\frac{3}{16}\log\frac{|u+1|}{|u-1|}-\frac{3}{16(u-1)}-\frac{3}{16(u+1)}\\
&\quad +\frac{1}{16(u-1)^2}-\frac{1}{16(u+1)^2}\Big)\Big|_0^{\frac{\sqrt{2}}{2}}\\
&= \frac{3\sqrt{2}-\log(\sqrt{2}+1)}{8(\log(\sqrt{2}+1)+\sqrt{2})}.
\end{aligned}$$

方法 2. 不妨设密度为 $\mu=1$, 并将质心记作 $(\overline{x},\overline{y})$. 抛物线的总质量为

$$\begin{aligned} M &= \int_{-1}^{1}\sqrt{1+x^2}\,\mathrm{d}x = 2\int_{0}^{1}\sqrt{1+x^2}\,\mathrm{d}x \\ &= \left(x\sqrt{1+x^2}+\log(x+\sqrt{1+x^2})\right)\Big|_0^1 \\ &= \sqrt{2}+\log(\sqrt{2}+1). \end{aligned}$$

而由对称性立刻可知

$$\begin{aligned} \overline{x} &= \frac{1}{M}\int_{-1}^{1} x\sqrt{1+x^2}\,\mathrm{d}x = 0, \\ \overline{y} &= \frac{1}{2M}\int_{-1}^{1} x^2\sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{M}\int_{0}^{1} x^2\sqrt{1+x^2}\,\mathrm{d}x. \end{aligned}$$

另外可注意到

$$\begin{aligned} \int_0^1 x^2\sqrt{1+x^2}\,\mathrm{d}x &= \int_0^1 (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x - \int_0^1 \sqrt{1+x^2}\,\mathrm{d}x \\ &= \int_0^1 (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x - \frac{1}{2}\big(\sqrt{2}+\log(\sqrt{2}+1)\big), \\ \int_0^1 x^2\sqrt{1+x^2}\,\mathrm{d}x &= \frac{1}{2}\int_0^1 x\sqrt{1+x^2}\,\mathrm{d}(1+x^2) = \frac{1}{3}\int_0^1 x\,\mathrm{d}(1+x^2)^{\frac{3}{2}} \\ &= \frac{1}{3}\left(x(1+x^2)^{\frac{3}{2}}\Big|_0^1 - \int_0^1 (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x\right) \\ &= \frac{2\sqrt{2}}{3} - \frac{1}{3}\int_0^1 (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x, \end{aligned}$$

在上述二式中消去 $\int_0^1 (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x$ 可得

$$\int_0^1 x^2\sqrt{1+x^2}\,\mathrm{d}x = \frac{3}{8}\sqrt{2} - \frac{1}{8}\log(\sqrt{2}+1),$$

进而可知 $\overline{y} = \frac{3\sqrt{2}-\log(\sqrt{2}+1)}{8(\sqrt{2}+\log(\sqrt{2}+1))}$.