Answers for Homework IV

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1 KK's 3.3

Ans:
Let's assume that there are N bodies. And the center of mass of each body that has mass m_i is $\vec{R}_i$, from definition:

$$\vec{R}_i = \frac{\int_{V_i} \rho(\vec{r})\vec{r}dV}{\int_{V_i} \rho(\vec{r})dV} = \frac{\int_{V_i} \rho(\vec{r})\vec{r}dV}{m_i}$$

The center of mass of the whole system is:

$$\begin{aligned}
\vec{R}_{sys} &= \frac{\int_{V_0} \rho(\vec{r})\vec{r}dV}{\int_{V_0} \rho(\vec{r})dV} \\
&= \frac{\sum_{i=1}^{N} \int_{V_i} \rho(\vec{r})\vec{r}dV}{\sum_{i=1}^{N} \int_{V_i} \rho(\vec{r})dV} \\
&= \frac{\sum_{i=1}^{N} m_i\vec{R}_i}{\sum_{i=1}^{N} m_i}
\end{aligned}$$

The last line of above equation means that the center of mass of the system can be found by treating each body as a particle concentrated at its center of mass.

2 KK's 3.4

Ans:
We assume that the mass of the small piece is m, and before explosion the velocity of the whole projectile is v. So the mass of the large piece is $3m$. As the small piece return back to the launch point, the velocity of the small piece just after the explosion is v as well. At the explosion position, we use the conservation law of momentum:

$$4mv = 3mv_l - mv$$

$$v_B = \frac{5}{3}v$$

so the total horizontal distance of the large piece from the launch point is:

$$L_l = L + \frac{5}{3}L_s = \frac{8}{3}L$$

3 KK's 3.5

Ans:
We can use the conservation law of energy first to evaluate the velocity of the man when he is at the height of the monkey:

$$\frac{1}{2}Mv_0^2 = Mgh + \frac{1}{2}Mv^2$$

$$v = \sqrt{v_0^2 - 2gh}$$

When he takes the monkey, using the conservation law of momentum:

$$Mv = (M+m)v_1$$

$$v_1 = \frac{M}{M+m}v$$

At last, using the conservation law of energy again:

$$\frac{1}{2}(M+m)v_1^2 = (M+m)gh_1$$

$$h_1 = \frac{v_1^2}{2g}$$

So the total height is:

$$H = h + h_1 = h + \frac{M^2}{(M+m)^2}\frac{v_0^2 - 2gh}{2g}$$

4 KK's 3.7

Ans:

Before the spring extents to its own length, the m_1 will always force against the wall. Then the motion of m_2 account from the wall is:

$$l_2 = l - \frac{l}{2}\cos\omega t$$

where $\omega = \sqrt{\frac{k}{m_2}}$, and $0 \le t \le \frac{\pi}{2}\sqrt{\frac{m_2}{k}}$. In the situation, the motion of the center of mass is:

$$l_c = \frac{m_2}{m_1+m_2}l_2 = \frac{m_2}{m_1+m_2}\left(l - \frac{l}{2}\cos\omega t\right)$$

$$v_c = \frac{m_2}{m_1+m_2}\frac{l}{2}\omega\sin\omega t$$

After the spring extents back to its own length, i.e. $t > \frac{\pi}{2}\sqrt{\frac{m_2}{k}}$ there is no interaction between m_1 and the wall, the whole system has conserved momentum(or energy), thus has unchanged velocity.

$$v_c = \frac{m_2}{m_1+m_2}\frac{l\omega}{2}$$

$$l_c = \frac{m_2}{m_1+m_2} l(1+\frac{\omega t}{2})$$

So the motion of center of mass is:

$$l_c(t) = \frac{m_2}{m_1+m_2} l_2 = \frac{m_2}{m_1+m_2}\big(l - \frac{l}{2}\cos\omega t\big) \quad 0 \le t \le \frac{\pi}{2}\sqrt{\frac{m_2}{k}}$$
$$l_c(t) = \frac{m_2}{m_1+m_2} l(1+\frac{\omega t}{2}) \quad t > \frac{\pi}{2}\sqrt{\frac{m_2}{k}}$$

5 KK's 3.11

Ans:

Using the conservation law of momentum:

$$Mv + bdtu = (M+bdt)(v+dv)$$

$$budt = bvdt + Mdv$$
$$\frac{dv}{dt} = \frac{b}{M}(u-v)$$

6 KK's 3.14

Ans:
a).

$$0 = Mv + Nm(v-u)$$

$$v = \frac{Nm}{M+Nm}u$$

b). Assume that i men have jumped off and the velocity of flatcar is v_i, now the $i+1$th man jump off:

$$\big((N-i)m+M\big)v_i = \big((N-i-1)m+M)v_{i+1} + m(v_{i+1}-u)$$
$$v_{i+1} = v_i + \frac{m}{M+(N-i)m}u$$

After all men jump off:

$$V_N = \sum_{i=1}^{N} \frac{mu}{M+im}$$

c). The case b gets larger final velocity.

7 KK's 3.15

Ans:
Using the Newton's law:

$$M\ddot{x} = \frac{M}{l}xg$$

$$\ddot{x} = \frac{g}{l}x$$

The solution for this equation is:

$$x(t) = Ae^{\gamma t} + Be^{-\gamma t}$$

where $\gamma = \sqrt{\frac{g}{l}}$ The initial condition is:

$$x(0) = l_0 \qquad \dot{x}(0) = 0$$

so we get

$$A = B = \frac{l_0}{2}$$

Finally:

$$x(t) = \frac{l_0}{2}(e^{\gamma t} + e^{-\gamma t})$$

where $\gamma = \sqrt{\frac{g}{l}}$

8 KK's 3.18

Ans:
Using the theorem of momentum:

$$Mgdt + Mv = (M + dM)(v + dv)$$

$$Mg = v\frac{dM}{dt} + M\frac{dv}{dt} = kMv^2 + M\frac{dv}{dt}$$

$$g - kv^2 = \frac{dv}{dt}$$

Solve this equation:

$$\begin{aligned} dt &= \frac{dv}{g - kv^2} \\ &= \frac{1}{2\sqrt{g}}(\frac{1}{\sqrt{g} - \sqrt{k}v} + \frac{1}{\sqrt{g} + \sqrt{k}v})dv \\ &= \frac{1}{2\sqrt{gk}}(-\frac{d(\sqrt{g} - \sqrt{k}v)}{\sqrt{g} - \sqrt{k}v} + \frac{d(\sqrt{g} + \sqrt{k}v)}{\sqrt{g} + \sqrt{k}v}) \end{aligned}$$

$$2\sqrt{gk}t = \ln\frac{\sqrt{g} + \sqrt{k}v}{\sqrt{g} - \sqrt{k}v}$$

$$e^{-2\sqrt{gk}t} = \frac{\sqrt{g} - \sqrt{k}v}{\sqrt{g} + \sqrt{k}v}$$

When $t \to \infty$ the left side of above equation is zero, so when t is very large, we have: $v \approx \sqrt{\frac{g}{k}}$

9 KK's 3.19

Ans:
As the bowl is full of water, the force against the gravity is constant. We only consider about the force due to the falling rain.

$$Fdt = dmv$$

$$F = v\frac{dm}{dt}$$

1).

$$F = 5m/s \times 10^{-6}kg/cm^2 \cdot s \times 500cm^2 = 2.5 \times 10^{-3}N$$

2).

$$F = (5m/s + 2m/s) \times 10^{-6}kg/cm^2 \cdot s \times 500cm^2 = 3.5 \times 10^{-3}N$$

10 KK's 3.20

Ans:
At some time, the mass and velocity of the rocket are m and v respectively. Using the theorem of momentum:

$$mv - mgdt - bmvdt = (m - dm)(v + dv) + dm(v - u)$$

and we also have $\frac{dm}{dt} = \gamma m$, inserted into above equation:

$$\frac{dv}{dt} = \gamma u - g - bv$$

$$\frac{dv}{\gamma u - g - bv} = dt$$

Integrate both side:

$$\ln\frac{\gamma u - g - bv}{\gamma u - g} = -bt$$

$$v = \frac{\gamma u - g}{b}(1 - e^{-bt})$$

11 Another Rain drop

Ans:
From the problem we know $\frac{dV}{dt} = k\pi r^2 v$, and $V = \frac{4}{3}\pi r^3$so we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{k}{4}v$$

$$\frac{dv}{dt} = \frac{4}{k}\frac{d^2r}{dt^2}$$

Using the theorem of momentum:

$$mv + mgdt = (m + dm)(v + dv)$$

$$mgdt = mdv + vdm$$

The drop has constant density so:

$$Vgdt = Vdv + vdV$$

$$Vg = V\frac{dv}{dt} + k\pi r^2 v^2$$

$$4rg = 4r\frac{4}{k}\frac{d^2r}{dt^2} + 3\frac{16}{k}\left(\frac{dr}{dt}\right)^2$$

$$g = \frac{4}{k}\frac{d^2r}{dt^2} + \frac{12}{kr}\left(\frac{dr}{dt}\right)^2$$

It is a bit hard to solve this equation directly. But we can guess the solution. Assume that $r = Ct^n$

$$g = \frac{4}{k}n(n-1)Ct^{n-2} + \frac{12}{k}C(n-1)^2n^2t^{n-2}$$

For the left side is independent of t, n=2. Finally we get:

$$C = \frac{kg}{56}$$

$$a = \frac{dv}{dt} = \frac{4}{k}\frac{d^2r}{dt^2} = \frac{g}{7}$$

12 KK's 4.3

Ans:

a) the conservation of momentum:

$$mv = (m + M)u$$

$$u = \frac{m}{m+M} v$$

b) The conservation of energy(after the bullet comes to rest with the block):

$$\frac{1}{2}(M+m)u^2 = (M+m)g(1-\cos\phi)l$$

$$u = \sqrt{2gl(1-\cos\phi)}$$

$$v = \frac{m+M}{m}\sqrt{2gl(1-\cos\phi)}$$

13 KK's 4.5

Ans:

The string is pulled so slowly that we ignore the velocity(and acceleration) along radius.

$$\vec{F} = -m\omega^2 r \hat{e}_r$$

Recall from homework III KK's 2.34 we have:

$$\frac{\omega}{\omega_0} = (\frac{r_0}{r})^2$$

so

$$\vec{F} = -m\omega_0^2 \frac{r_0^4}{r^3} \hat{e}_r$$

$$dW = \vec{F} \cdot \hat{e}_r dr = -m\omega_0^2 \frac{r_0^4}{r^3} dr$$

$$\begin{aligned} W =& \int_{l_1}^{l_2} -m\omega_0^2 r_0^4 \frac{dr}{r^3} \\ =& \int_{l_1}^{l_2} m\omega_0^2 r_0^4 d(\frac{1}{2r^2}) \\ =& m\omega_0^2 r_0^4 (\frac{1}{2l_2^2} - \frac{1}{2l_1^2}) \\ =& \frac{1}{2} m\omega_2^2 l_2^2 - \frac{1}{2} m\omega_1^2 l_1^2 \\ =& E_2 - E_1 \end{aligned}$$

14 KK's 4.7

Ans:

From the conservation of energy, for each bead we have:

$$mgr(1-\cos\theta) = \frac{1}{2}mv^2$$

The force needed pointed to the center is:

$$F = \frac{mv^2}{r} = 2mg(1 - \cos\theta)$$

so the force between bead and the ring is:

$$N = mg(2 - 3\cos\theta)$$

Which is pointed to the center of the ring.

If the ring will rise, then:

$$2N\cos\theta = Mg$$
$$2mg(2 - 3\cos\theta)\cos\theta = Mg$$
$$(2 - 3\cos\theta)\cos\theta = \frac{M}{2m}$$
$$\cos\theta = \frac{2 + \sqrt{4 - 6\frac{M}{m}}}{6}$$

so we need $m > \frac{3}{2}M$

15 KK's 4.9

Pro.
The conservation of momentum:

$$m\vec{v}_1 + m\vec{v}_2 = 2m\vec{v}_3$$
$$\vec{v}_3 = \frac{\vec{v}_1 + \vec{v}_2}{2}$$

The conservation of energy:

$$\frac{1}{2}m\vec{v}_1^2 + \frac{1}{2}m\vec{v}_2^2 = \frac{1}{2}2m\vec{v}_3{}^2 + E_{potential}$$

$$\begin{aligned} E_{potential} =& \frac{1}{2}m\vec{v}_1^2 + \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}2m\vec{v}_3{}^2 \\ =& \frac{1}{4}m|\vec{v}_1 - \vec{v}_2|^2 \geq 0 \end{aligned}$$

But we learn from the equation: $H + H \rightarrow H_2 + 5ev$ that the potential of H_2 is negative (-5ev) so it is impossible.

16 KK's 4.10

Ans:
a). The putty hits the M when $v_M = 0$
(1).The period:

$$T = 2\pi\sqrt{\frac{M+m}{k}}$$

(2).The amplitude:
When $v_M = 0$, the length of spring is maximum, that is the amplitude so the putty does not change the amplitude: $A = A_0$.
(3).The mechanical energy change:

$$\Delta E = 0$$

b). The putty hits the M when v_m is maximum.
(1).The period:

$$T = 2\pi\sqrt{\frac{M+m}{k}}$$

(2). The amplitude & (3). The total mechanical energy change:
When putty hits the M we have conservation of momentum along the table:

$$Mv = (m+M)v' \quad \frac{1}{2}Mv^2 = \frac{1}{2}kA_0^2$$

$$v' = \frac{M}{M+m}v$$

$$\frac{1}{2}kA'^2 = \frac{1}{2}(M+m)v'^2 = \frac{1}{2}\frac{M^2}{M+m}v^2 = \frac{1}{2}\frac{M}{M+m}kA_0^2$$

$$A' = \sqrt{\frac{M}{M+m}}A_0$$

$$\Delta E = \frac{1}{2}k(A'^2 - A_0^2) = -\frac{1}{2}\frac{m}{M+m}A_0^2$$

17 KK's 4.13

Ans:
a). the potential gets minimum when:

$$\frac{dU}{dr} = 0$$

$$\frac{dU}{dr} = \epsilon\left(\frac{-12}{r}(\frac{r_0}{r})^12 + \frac{12}{r}(\frac{r_0}{r})^6\right)$$

$$\frac{dU}{dr} = 0 \Rightarrow (\frac{r_0}{r})^6 = 1 \Rightarrow r = r_0$$

And when $r = r_0$

$$U = \epsilon(1-2) = -\epsilon$$

So the depth of the potential well is: ϵ

b). we expand the U(r) around r_0 in Taylor expansion:

$$U(r) = U(r_0) + \frac{dU}{dr}|_{r=r_0}(r-r_0) + \frac{1}{2}\frac{d^2U}{dr^2}|_{r=r_0}(r-r_0)^2$$

$$\frac{dU}{dr}|_{r=r_0} = 0$$

$$\frac{d^2U}{dr}|_{r=r_0} = \epsilon(12\times 13\frac{r_0^12}{r_0^14} - 12\times 7\frac{r_0^6}{r_0^8}) = \frac{72\epsilon}{r_0^2}$$

$$U(r) = -\epsilon + \frac{36\epsilon}{r_0^2}(r-r_0)^2$$

In one atom's frame we need to use reduced mass: $M = \frac{1}{2}m$

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{K}{M}} = 12\sqrt{\frac{\epsilon}{r_0^2 m}}$$

18 KK's 4.15

Ans:

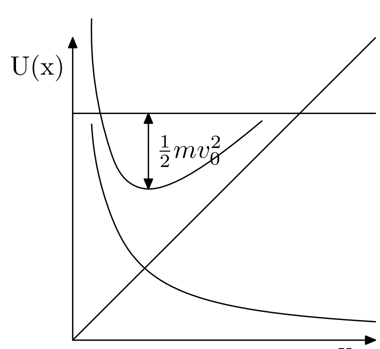
a. As describe

$$F = -B + \frac{A}{x^2}$$

$$F = -\frac{dU}{dx}$$

$$U = Bx + \frac{A}{x} + C_0$$

b. See the figure.



c.

$$F = 0 \Rightarrow x_0 = \sqrt{\frac{A}{B}}$$

d.

$$F = -B + \frac{A}{x^2} = -B + \frac{A}{(x_0+dx)^2}$$
$$= -B + \frac{A}{x_0^2(1+\frac{dx}{x_0})^2}$$
$$= -B + \frac{A}{x_0^2}(1-2\frac{dx}{x_0})$$
$$= -2\sqrt{\frac{B^3}{A}}dx$$
$$= -kdx$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{m}\sqrt{\frac{B^3}{A}}}$$

19 KK's 4.17

Ans:
Climb the hill:

$$P = mgv\sin\theta + fv$$

Downhill:

$$P + mgv'\sin\theta = fv'$$

So we get:

$$v' = \frac{f+mg\sin\theta}{f-mg\sin\theta}$$

we also have:

$$f = 5\% mg \qquad \sin\theta \approx \tan\theta = \frac{1}{40}$$
$$v' \approx 3v = 45mi/hr$$

20 KK's 4.20

Ans:
a. for the sand:

$$Fdt = dp = dmV$$
$$F = \frac{dm}{dt}V$$
$$P = FV = \frac{dm}{dt}V^2$$

b. The change of energy of sand in unit time:

$$P' = \frac{dE}{dt} = \frac{1}{2}\frac{dm}{dt}V^2$$

The difference between a and b is due to the friction between sand and the belt.

21 KK's 4.21

Ans:
a. Use the expression: $Fdt = dp$ or $p + Fdt = p'$

$$\lambda y v_0 - \lambda y g dt + F dt = \lambda(y + dy) v_0$$

$$F = \lambda v_0^2 + \lambda y g$$

b. The power delivered to the rope:

$$P = F v_0 = \lambda v_0^3 + \lambda y g v_0$$

The rate of change of the energy of the rope:

$$\begin{aligned} P' = \frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}\lambda y v_0^2 + \lambda y g \frac{y}{2}\right) \\ &= \frac{1}{2}\lambda v_0^3 + \lambda g y v_0 \end{aligned}$$

$$P - P' = \frac{1}{2}\lambda v_0^3$$