Answers for Homework I

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1 Basic derivatives

1. For function $y = 1/x$, a) find its average change rate when x changes from 0.5 to 1. b) find its instantaneous change rate at x=0.5 and x=1.
Ans:
a).

$$\textbf{when } x = 0.5 \quad y = 2$$

$$\textbf{when } x = 1 \quad y = 1$$

so the average change rate= $\frac{y(1)-y(0.5)}{1-0.5} = \frac{-1}{0.5} = -2$
b). For $y = 1/x$, $\frac{dy}{dx} = -\frac{1}{x^2}$. So, we can get the instantaneous change rate:

$$\textbf{when } x = 0.5 \quad \frac{dy}{dx} = -4$$

$$\textbf{when } x = 1 \quad \frac{dy}{dx} = -1$$

2. $y = \frac{1}{\sqrt{1-x^2}}$, what is dy/dx?
Ans:
$y = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{d(1-x^2)} \frac{d(1-x^2)}{dx} = \big(-\frac{1}{2}(1-x^2)^{-\frac{3}{2}}\big)\big(-2x\big) = \frac{x}{(1-x^2)^{3/2}}$$

3. If $x = \sin(t)$ in the above function, what is dy/dt?
Ans:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{x}{(1-x^2)^{3/2}} \cos t = \frac{\sin t \cos t}{|\cos t|^3}$$

4. For an hyperbola function: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Find the derivative dy/dx with two different methods: a) Direct implicit differentiation by the equation. b) using parametric relation: $x = a\cosh(t)$, $y = b\sinh(t)$, where $\cosh t \equiv \frac{e^t+e^{-t}}{2}$, $\sinh t \equiv \frac{e^t-e^{-t}}{2}$.
Ans:

a). Direct implicit differentiation:

$$\begin{aligned}\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{d}{dx}(\frac{x^2}{a^2} - \frac{y^2}{b^2}) &= \frac{d}{dx}(1) \\ \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{b^2}{a^2}\frac{x}{y} = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}\end{aligned}$$

b). Using parametric relation:
First, It is obvious that the relations for x and y fit the hyperbola function $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Second, we can do differentiation for each parametric relation:

$$\frac{dx}{dt} = a \sinh t = \frac{a}{b}y \qquad \frac{dy}{dt} = b \cosh t = \frac{b}{a}x$$

Finally, we get:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b^2}{a^2}\frac{x}{y} = \frac{b}{a}\frac{x}{\sqrt{x^2 - a^2}}$$

5. For a curve given by $2x^3 - 3y^2 = 9$, find the 2nd derivative of $\frac{d^2y}{dx^2}$. (Hint, using implicit differentiation)
Ans:

$$\begin{aligned}&2x^3 - 3y^2 = 9 \\ &\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(9) = 0 \\ &6x^2 - 6y\frac{dy}{dx} = 0 \\ &\frac{dy}{dx} = \frac{x^2}{y} \\ &\frac{d^2y}{dx^2} = \frac{2x}{y} - \frac{x^2}{y^2}\frac{dy}{dx} \\ &\qquad = \frac{2x}{y} - \frac{x^4}{y^3}\end{aligned}$$

6. For a function $y = \arctan(\frac{1}{x})$, what is dy/dx?
Ans:

$$\begin{aligned}\tan y &= \frac{1}{x} \\ \frac{d}{dx}\tan y &= \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \\ \frac{d}{dx}\tan y &= \frac{d}{dy}(\tan y)\frac{dy}{dx} = \frac{1}{\cos^2 y}\frac{dy}{dx}\end{aligned}$$

so we have:

$$\begin{aligned}\frac{dy}{dx} &= -\cos^2 y \frac{1}{x^2} \\ &= -\frac{1}{1+\tan^2 y}\frac{1}{x^2} \\ &= -\frac{1}{1+\frac{1}{x^2}}\frac{1}{x^2} \\ &= -\frac{1}{1+x^2}\end{aligned}$$

7. For $y = a^x$, a is a constant, find dy/dx.

Ans:

$$y = a^x \ \Rightarrow \ \ln y = x \ln a$$

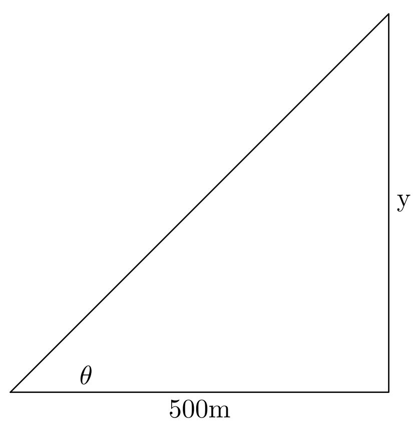
$$\frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx} = \ln a$$

so we get:

$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

2 Related rate

(a). A car is moving along the road, and a police is hiding 500m from the road. The police is monitoring the speed of car by measuring the angular change rate of θ . If the $\frac{d\theta}{dt} = 0.12/s$ at the moment shown in the figure, what is the speed of car?



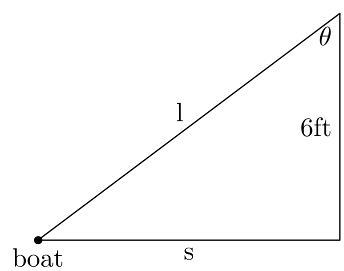
Ans:

$$y = 500 \tan\theta$$

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{d\theta} d\theta dt \\
&= 500 \frac{1}{\cos^2\theta} \frac{d\theta}{dt} \\
&= 1000 \frac{d\theta}{dt} \\
&= 120 m/s
\end{aligned}
$$

It is so fast

(b). Hauling a boat:
i. The boat is pulling towards the dock by a rope attached to its bow and through a ring that is 6 feet above boat. If the rope is hauled at a speed of 2 ft/s. What is the speed of the boat when the rope is 10ft long?
ii. What is the angular changing rate of θ?
Ans:



i. From the picture, we get:

$$s = \sqrt{l^2 - 6^2}$$

And we also have:

$$\frac{dl}{dt} = -2 ft/s$$

Here, the minus means the length of the rope decreases, so does the possible minus sign in the next equation.

$$
\begin{aligned}
v_{boat} = -\frac{ds}{dt} &= -\frac{ds}{dl}\frac{dl}{dt} \\
&= -\frac{l}{\sqrt{l^2 - 36}} \frac{dl}{dt} \\
&\overset{l=10(ft)}{=} \frac{20}{\sqrt{64}} = 2.5 ft/s
\end{aligned}
$$

ii. From the above picture, we also get:

$$\cos\theta = \frac{6}{l}$$

So we have:

$$\frac{d}{dt}\cos\theta = \frac{d}{dt}\frac{6}{l}$$
$$\sin\theta\frac{d\theta}{dt} = \frac{6}{l^2}\frac{dl}{dt}$$

Finally, there is:

$$\frac{d\theta}{dt} = \frac{-12}{l^2\sin\theta}$$

When $l = 10ft$, we get $\sin\theta = \frac{4}{5}$, and:

$$\frac{d\theta}{dt} = -\frac{12}{10\times 10\times 4/5} = -0.15/s$$

3 Extremes

(a). The distance of a moving object is given by: $x = -16t^2 + 96t + 112$. Find the its velocity at t=0, at x=0, and the maximum value of x and at what time this occurs?
Ans:

$$v = \frac{dx}{dt} = -32t + 96$$

When $t = 0$, the velocity is:

$$v|_{t=0} = 96$$

when $x = 0$, then $t = 7$,

$$v|_{x=0} = -128$$

The maximum value of x:

$$x_{max} = 256$$

at

$$t = 3(= 96/32)$$

(when $v = 0$)

(b) Shortest Beam:

A 8-ft wall is 27 ft away from a building, what is the shortest length of a beam (a ladder) that can reach the building from the ground outside the wall?

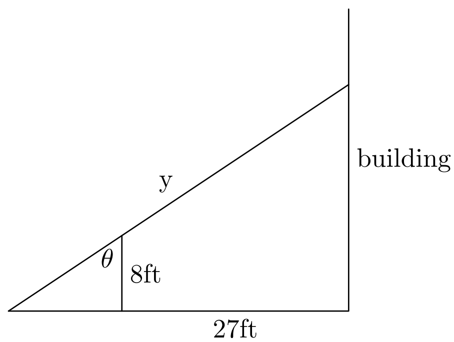
Ans:

From the picture, we could get the equation for y(the length of the beam):

$$y = \frac{8\tan\theta + 27}{\sin\theta} = \frac{8}{\cos\theta} + \frac{27}{\sin\theta}$$

The length of the beam y will change as the angle θ changes. When y gets its extreme value, the derivatives of y repest to θ will be zero: $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = \frac{8\sin\theta}{\cos^2\theta} - \frac{27\cos\theta}{\sin^2\theta} = 0$$



So we get:

$$\tan\theta = \frac{3}{2}$$

Further, we need to check the value of y at $\tan\theta = \frac{3}{2}$ that whether it is the minimun as we wish.

$$\frac{d^2y}{d\theta^2} = \cdots = \frac{44}{\cos\theta} + \frac{63}{\sin\theta} > 0$$

So it is the shortest.

$$y = 13\sqrt{13}$$

4 Limits and L'Hopital's rules

(a). Find the limit: $\lim_{x\to 0} \frac{\sin x}{x}$
Ans:
When x goes to zero, $\sin x$ also goes to zero, so we need L'Hopital's rules.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin' x}{x'} = \lim_{x\to 0} \cos x = 1$$

(b). Find limit of: $\lim_{x\to 0} \frac{e^x \sin x + x^2}{x^3}$
Ans:
When x goes to zero, $e^x \sin x + x^2$ goes to zero as well as x^3. Using L'Hopital's rules:

$$\lim_{x\to 0} \frac{e^x \sin x + x^2}{x^3} = \lim_{x\to 0} \frac{e^x \sin x + e^x \cos x + 2x}{3x^2}$$

Then, $\lim_{x\to 0} e^x \sin x + e^x \cos x + 2x = 1$, however, the denominator $3x^2$ still goes to zero. So the expression has no limit or the limit is infinite.

5 Taylor expansion and accuracy on approximation

Ans:

$$y = \frac{1}{\sqrt{1-x^2}}$$

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$

$$y' = \frac{dy}{dx} = -\frac{1}{2}\frac{-2x}{(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{3/2}} \overset{x=0}{=} 0$$

$$y'' = \frac{d^2y}{dx^2} = \frac{1}{(1-x^2)^{3/2}} + x\frac{3}{2}\frac{2x}{(1-x^2)^{5/2}} \overset{x=0}{=} 1$$

$$y''' = \frac{d^3y}{dx^3} = \frac{3x}{(1-x^2)^{5/2}} + \frac{6x}{(1-x^2)^{5/2}} + 3x^2\frac{5}{2}\frac{2x}{(1-x^2)^{7/2}} \overset{x=0}{=} 0$$

$$y'''' = \frac{9}{(1-x^2)^{5/2}} + \mathrm{O}(x) \overset{x=0}{=} 9$$

$$y = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \mathrm{O}(x^6)$$

For $x = 0.1$

$$y_{app}^{(1)} = 1 \qquad \Delta\% = 0.5\%$$

$$y_{app}^{(2)} = 1.005 \qquad \Delta\% = 0.0037\%$$

$$y_{app}^{(3)} = 1.0050378 \qquad \Delta\% = 3.1 \times 10^{-5}\%$$

For $x = 0.01$

$$y_{app}^{(1)} = 1 \qquad \Delta = 5 \times 10^{-5}$$

$$y_{app}^{(2)} = 1.00005 \qquad \Delta = 3.75 \times 10^{-9}$$

$$y_{app}^{(3)} = 1.00005000375 \qquad \Delta = 7.5 \times 10^{-10}$$

6 The equilibrium position of load and tension of suspension rope

Ans:

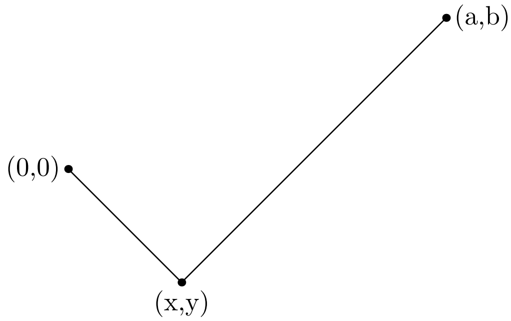
The rope has fixed length, so we have:

$$\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2} = L$$

When the ring is at the equilibrium, the potential energy of the ring is the lowest, that is the value of y is the minimum.

$$\frac{d}{dx}\left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right) = 0$$

$$\frac{x+yy'}{\sqrt{x^2+y^2}} + \frac{(x-a)+(y-b)y'}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$



For y is minimum, $y' = 0$, so we get the equilibrium condition:

$$\frac{x}{\sqrt{x^2+y^2}} + \frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

From the above equation we can find that the absolute values of the slope of each side of the rope are equal. So the tensional force along the left and right side of the rope are equal.

7 The length of stacking blocks

Ans:
Assume that the position of the mass center of n top blocks is x_n(Accounting from the left end of the topest blocks). So when we add another block at the bottom we have:

$$x_{n+1} = \frac{nMx_n + M(x_n + l/2)}{(n+1)M} = x_n + \frac{l}{2(n+1)}$$

so when we have N blocks, the total length of all N blocks is:

$$L = x_{N-1} + l = \frac{l}{2}\sum_{n=1}^{N-1}\frac{1}{n} + l$$

Here $l = 2m$.
If we need to have a span of 20m, for approximate:

$$20 = 2 + 1\int_1^{N-1}\frac{1}{x}dx$$

so we need about 6.6×10^7 blocks.

8 Basic Integration

(a). Find expression form for indefinite integral (antiderivative)$\int x^2 e^{2x} dx$ (Hint: integration by part)

Ans:

$$\int x^2e^{2x}dx = \frac{1}{2}\int x^2e^{2x}d(2x) = \frac{1}{2}\int x^2de^{2x} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int e^{2x}dx^2$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int e^{2x}xd2x = \frac{1}{2}x^2e^{2x} - \frac{1}{2}\int xde^{2x}$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}dx$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

(b). The density of the a round disk (for simplicity, just consider the unphysical 2-dimension case) is proportional to radius, i.e. $\rho = 2r$. What is the total mass for the disk if the radius is R.
Ans:

$$M = \int \rho r d\theta dr = 2\pi\int_0^R 2r^2dr = 2\pi\frac{2}{3}R^3 = \frac{4\pi R^3}{3}$$

(c). Find the area between the x axis and the curve $y = x^2$, where x belongs [-1,+1], what is the area bound by the parabola and straight line y=1?
Ans:
For the first question:

$$S_1 = \int_{-1}^{1} x^2dx = \int_{-1}^{1} d\frac{x^3}{3} = \frac{2}{3}$$

For the second question:

$$S_2 = 2 - S_1 = \frac{4}{3}$$

(d) Arc length of curves, where arc length ds can be computed:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \sqrt{1 + (\frac{dy}{dx})^2}dx$$

For i) a cure given by $x = A\sin(t)$, $y = A\cos(t)$, what is length if t is between$[0, \pi]$(the answer is of course straightforward that you know before doing integration, but do it anyway to check).
ii) For a curve given by:$y = (\frac{x}{2})^{3/2}, x \subset [0, 2]$, find its arc length.
Ans:
i):

$$\frac{dx}{dt} = A\cos t \qquad \frac{dy}{dt} = -A\sin t$$

$$s = \int ds = \int_0^{\pi} dt\sqrt{A^2\cos^2 t + A^2\sin^2 t} = A\int_0^{\pi} dt = \pi A$$

ii):

$$\frac{dy}{dx} = \frac{3}{4}(\frac{x}{2})^{1/2}$$

$$
\begin{aligned}
s = \int ds = \int_0^2 \sqrt{1+\frac{9x}{32}}dx = \frac{32}{9}\int_0^2 \sqrt{1+\frac{9x}{32}}d(1+\frac{9x}{32}) \\
= \frac{32}{9}\int_0^2 d\frac{2}{3}(1+\frac{9x}{32})^{\frac{3}{2}} \\
= \frac{64}{27}\big[(1+\frac{9}{16})^{3/2}-1\big] \\
= \frac{61}{27}
\end{aligned}
$$

9 Simple 1st order differential equations

a).Exponential decay or growth
Ans:

$$
\begin{aligned}
\frac{dA}{dt} = kA \\
\frac{dA}{A} = kdt \\
\int_{A_0}^{A}\frac{dA}{A} = \int_0^T kdt = kT \\
ln(\frac{A}{A_0}) = kT \\
A = A_0e^{kT}
\end{aligned}
$$

If the value of the material is dropped by a factor of e, the k needs to be $k < 0$ and the time is

$$T = \frac{1}{k}$$

b). The cooling of an egg
Ans:

$$
\begin{aligned}
\frac{dT}{dt} = -k(T-T_0) \\
\frac{d(T-T_0)}{T-T_0} = -kdt \\
\int\frac{d(T-T_0)}{T-T_0} = \int -kdt \\
\ln\frac{T-T_0}{T_i-T_0} = -kt \\
T = T_0 + (T_i-T_0)e^{-kt}
\end{aligned}
$$

Where $T_0 = 18$,$T_i = 98$. For t=5min, $T = 38$, assume $t = t_0$, $T = 20$

$$
\begin{aligned}
20 = 80e^{-k5} \qquad \ln 4 = 5k \\
2 = 80e^{-kt_0} \qquad \ln 40 = kt_0
\end{aligned}
$$

so we get:

$$t_0 = 5\frac{\ln 40}{\ln 4} min \approx 13.3min$$

c). Draining a tank

$$V = \pi r^2 x$$

$$\frac{dV}{dt} = \pi r^2 \frac{dx}{dt} = -\frac{1}{2}\sqrt{x}$$

Here the minus sign is for the decrease of the volume of the water.

$$2\pi r^2 \frac{dx}{\sqrt{x}} = dt$$

$$\int_{x_0}^{0} 2\pi r^2 \frac{dx}{\sqrt{x}} = \int_0^{t_0} -dt$$

$$4\pi r^2 \sqrt{x_0} = t_0$$

so the total time is:

$$t_0 = 4\pi r^2 \sqrt{x_0} \approx 1256.6s$$

d). Bullet in air Ans:
Neglect the gravity. Assume that the mass of the bullet is m:

$$ma = F$$

$$m\frac{dv}{dt} = -kv$$

$$m\frac{dv}{v} = -kdt$$

$$\int_{v_0}^{v} m\frac{dv'}{v'} = -k\int_0^t dt'$$

$$m\ln\frac{v}{v_0} = -kt$$

$$v = v_0 e^{-\frac{kt}{m}}$$

For the distance:

$$ds = vdt = v_0 e^{-\frac{kt}{m}} dt$$

$$s = \int ds = \int_0^t v_0 e^{-\frac{kt'}{m}} dt'$$

$$s = \frac{mv_0}{k}\int_0^t -de^{-\frac{kt'}{m}}$$

$$s = \frac{mv_0}{k}(1 - e^{-\frac{kt}{m}})$$

When $F = -kv - lv^2$ then:

$$m\frac{dv}{kv + lv^2} = -dt$$

$$\frac{m}{k}\big(\frac{1}{v} - \frac{l}{k+lv}\big)dv = -dt$$

$$\int_{v_0}^{v} \frac{m}{k}\big(\frac{1}{v'} - \frac{l}{k+lv'}\big)dv' = \int_0^t -dt'$$

$$\frac{m}{k}\big(\ln\frac{v}{v_0} - \ln\frac{lv+k}{lv_0+k}\big) = -t$$

$$v = \frac{k}{(l+\frac{k}{v_0})e^{\frac{kt}{m}} - l}$$