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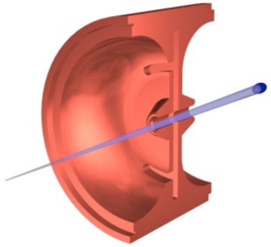

Electrodynamics

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Vector Analysis

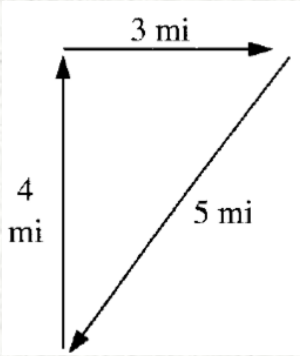
1. Vector Algebra
2. Differential Calculus
3. Integral Calculus
4. Curvilinear coordinates
5. Dirac Delta Function
6. Theory of Vector Fields

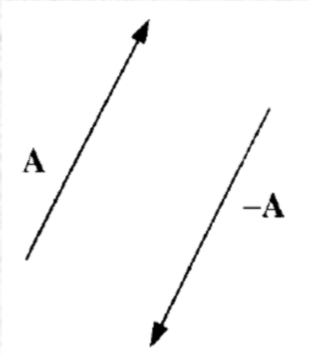
1. Vector Algebra

➢ Vectors and Scalars

Vectors: Quantities that have magnitude and direction

Scalars: Quantities that have magnitude but no direction





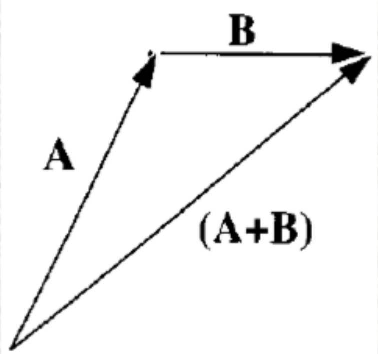
➢ Vector Operations

(1) Addition of two vectors

A+***B***=***B***+***A***

(***A***+***B***)+***C***=***A***+(***B***+***C***)

A-***B***=***A***+(-***B***)



(2) Multiplication by a scalar

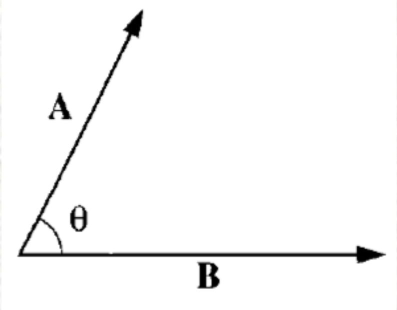
α(***A***+***B***)=α***A***+α***B***

(3) Dot product of two vectors

$\boldsymbol{A}\bullet\boldsymbol{B} \equiv AB\cos\theta$

A•(***B***+***C***) =***A***•***B***+***A***•***C***

A•***B*** =***B***•***A***



A•***B*** is the product of *A* times the projection of ***B*** along ***A***.

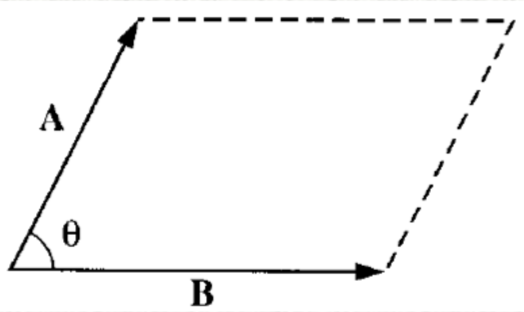
(4) Cross product of two vectors

$\boldsymbol{A}\times\boldsymbol{B} \equiv AB\sin\theta\hat{n}$

A × (***B***+***C***) =***A*** × ***B***+***A*** × ***C***

A × ***B*** =-(***B*** × ***A***)

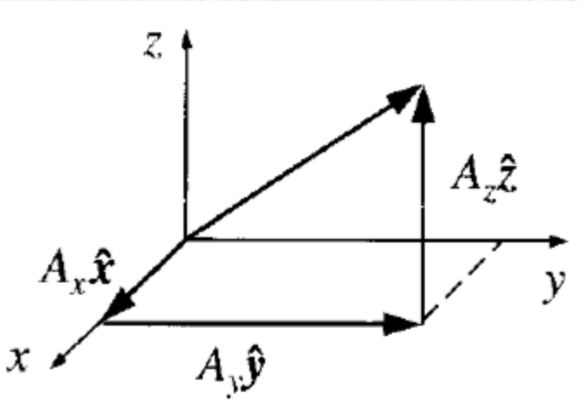
|***A*** × ***B***| is the area of the parallelogram generated by ***A*** and ***B***.



1. Vector Algebra

➢ Vector Algebra: Component Form

$$\boldsymbol{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \;,\; \boldsymbol{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$



(1) Addition

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

(2) Multiplication by a scalar

$$\alpha\boldsymbol{A} = (\alpha A_x)\hat{x} + (\alpha A_y)\hat{y} + (\alpha A_z)\hat{z}$$

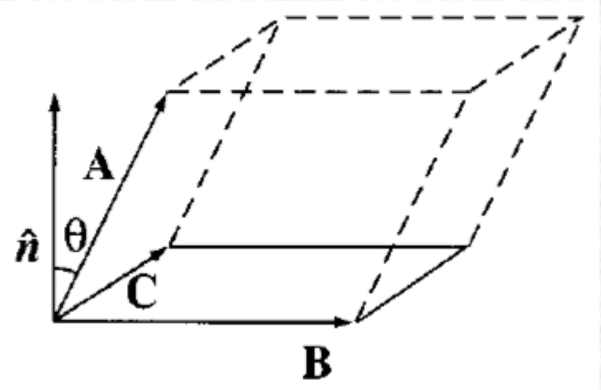
(3) Dot product of two vectors

$$\boldsymbol{A} \bullet \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$$

(4) Cross product of two vectors

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1. Vector Algebra



➢ Scalar Triple Product

$$A\bullet(B\times C)=B\bullet(C\times A)=C\bullet(A\times B)$$
$$=(A\times B)\bullet C$$

In component form:

$$A\bullet(B\times C)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

➢ Vector triple Product

$$A\times(B\times C)=B(A\bullet C)-C(A\bullet B)$$

BAC-CAB **rule**

□ Application

It is never necessary to contain more than one cross product in any term:

$$(A\times B)\bullet(C\times D)=(A\bullet C)(B\bullet D)-(A\bullet D)(B\bullet C)$$
$$A\times(B\times(C\times D))=B(A\bullet(C\times D))-(A\bullet B)(C\times D)$$

1. Vector Algebra

➢ Position and Separation Vectors

☐ Position Vector

$$\boldsymbol{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$$

☐ Separation Vector

$$\boldsymbol{r} - \boldsymbol{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

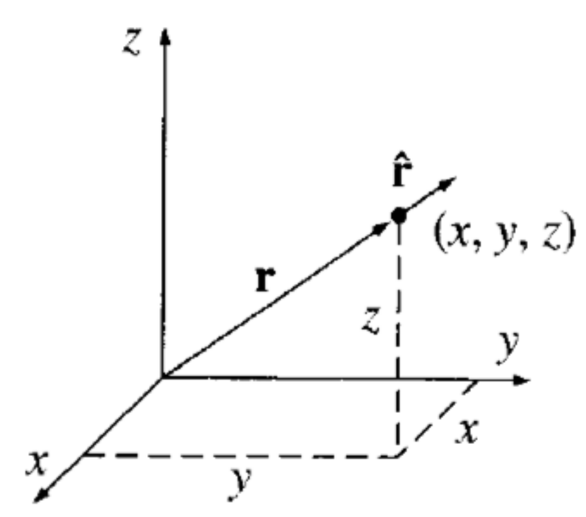
☐ Unit Vector

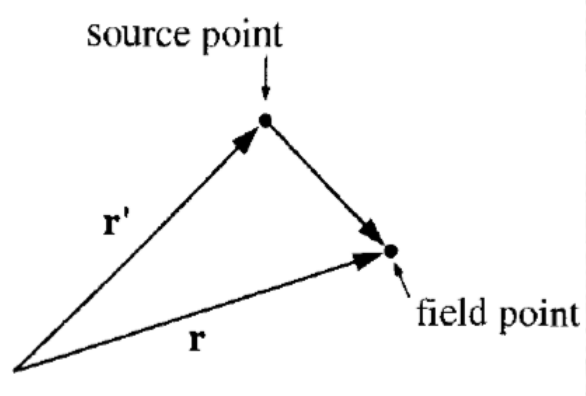
A unit vector pointing from the origin to ***r***:

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

A unit vector pointing from ***r′*** to ***r***:

$$\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$



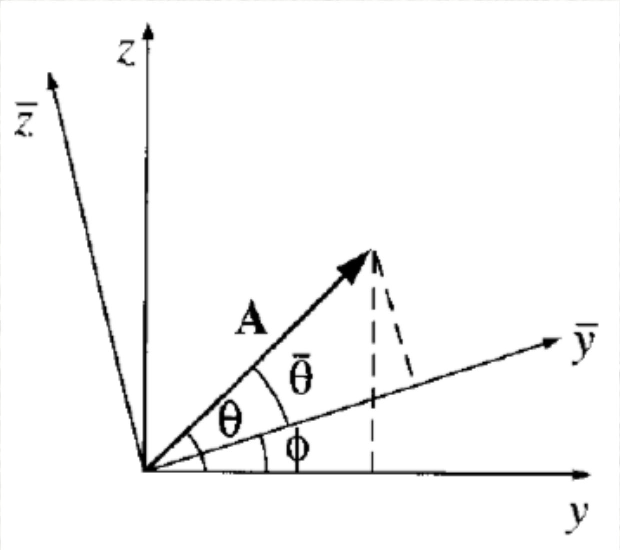


r′ : Source point
r: Field point

1. Vector Algebra

- Vector Transformation
  - Transformation with the system rotated about the x axis

$$\begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$



The (x,y,z) system is rotated by angle ϕ, about the x axis

  - Transformation for rotation about an arbitrary axis

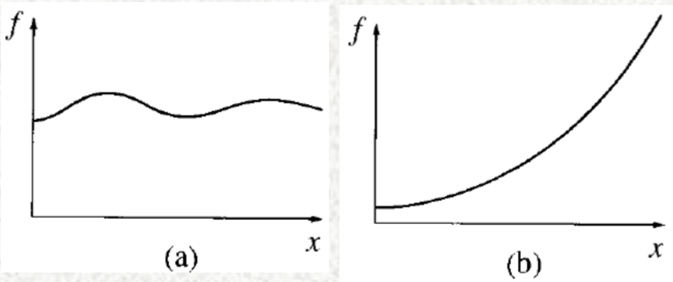
$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

2. Differential Calculus

➢ Ordinary Derivatives

d*f*/d*x* : Slope of the graph of *f* versus *x*.

$$\mathrm{d}f = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x$$



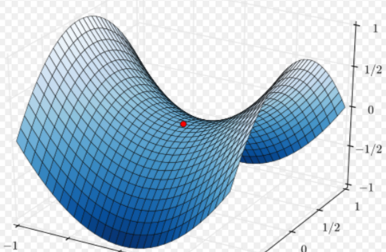
➢ Gradient of scalar function *T*(*x*,*y*,*z*)

$$\mathrm{d}T = \left(\frac{\partial T}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial T}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial T}{\partial z}\right)\mathrm{d}z = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\bullet\left(\mathrm{d}x\hat{x} + \mathrm{d}y\hat{y} + \mathrm{d}z\hat{z}\right)$$

$$= (\nabla T)\bullet(\mathrm{d}\boldsymbol{l}) = |\nabla T||\mathrm{d}\boldsymbol{l}|\cos\theta$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

- The gradient ∇T gives the slope and direction of maximum increase of the function *T*.
- Locate the extrema of a function of three variables, set its gradient equal to zero.



A saddle point

2. Differential Calculus

- The "del" Operator or Hamiltonian

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

- Ways of multiplication

Vector ***A***:
Multiply a scalar *a*: ***aA***
Multiply another vector ***B***, via the dot product: ***A•B***
Multiply another vector via the cross product: ***A*** × ***B***

Hamiltonian:
On a scalar function *T*: ∇T
On a vector function ***v***, via the dot product: $\nabla \bullet \boldsymbol{v}$
On a vector function ***v***, via the cross product: $\nabla \times \boldsymbol{v}$

2. Differential Calculus

➢ Divergence

$$\nabla \bullet \boldsymbol{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \bullet \left(v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \right)$$

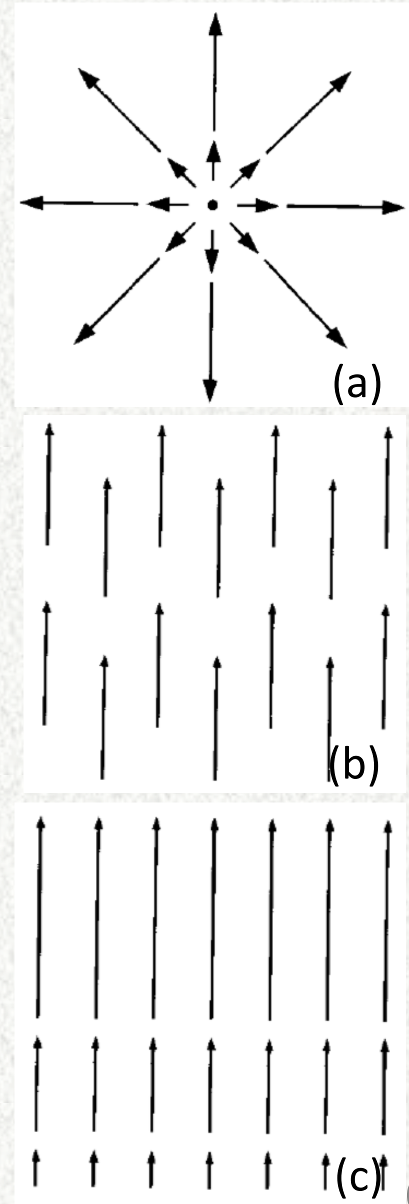
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- Divergence is a measure of how much the vector spreads out (diverges) from the point in question.

- Example 1: Supposing the functions in the figure are

$$\begin{cases} \boldsymbol{v}_a = \boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z} \\ \boldsymbol{v}_b = \hat{z} \\ \boldsymbol{v}_c = z\hat{z} \end{cases}$$

calculate their divergences.

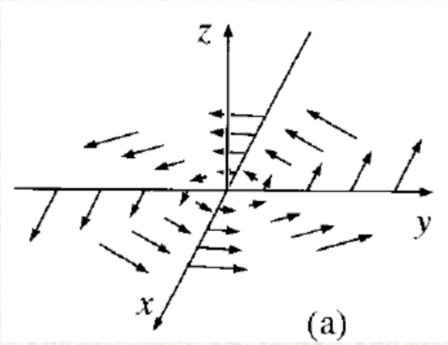


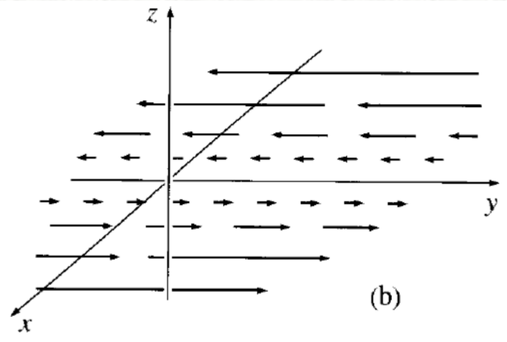
2. Differential Calculus

- **Curl**

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$





- Curl is a measure of how much the vector curls around the point in question.

- Example 2: Supposing the functions in the figure are

$$\begin{cases} \boldsymbol{v}_a = -y\hat{x} + x\hat{y} \\ \boldsymbol{v}_b = x\hat{y} \end{cases}$$

Calculate their curls.

2. Differential Calculus

➢ Product rules

- Product rules for ordinary derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right) + g\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)$$

- Product rules for gradients, divergences and curls

$$\left.\begin{aligned} &\nabla(fg) = f\nabla g + g\nabla f \\ &\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} \end{aligned}\right\}\text{Gradients}$$

$$\left.\begin{aligned} &\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f) \\ &\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \end{aligned}\right\}\text{Divergences}$$

$$\left.\begin{aligned} &\nabla\times(f\boldsymbol{A}) = f(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times(\nabla f) \\ &\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) \end{aligned}\right\}\text{Curls}$$

- Example 3: Prove $\nabla\times\left(\frac{\boldsymbol{A}}{g}\right) = \frac{g(\nabla\times\boldsymbol{A}) + \boldsymbol{A}\times(\nabla g)}{g^2}$

2. Differential Calculus

➢ Second Derivatives

- Five species of second derivatives can be constructed by applying ∇ twice

(1) $\nabla \bullet (\nabla T)$

(2) $\nabla \times (\nabla T)$

(3) $\nabla (\nabla \bullet \boldsymbol{v})$

(4) $\nabla \bullet (\nabla \times \boldsymbol{v})$

(5) $\nabla \times (\nabla \times \boldsymbol{v})$

Only two of them are new: the Laplacian (which is of fundamental importance) and the gradient-of-divergence (which we seldom encounter).

Fortunately second derivatives suffice for practically all physical applications!

2. Differential Calculus

(1) $$\nabla \cdot (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$$

Laplacian of a scalar, *T*

$$\nabla^2 \boldsymbol{v} \equiv (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

Laplacian of a vector, ***v***

(2) $$\nabla \times (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \times \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \right) = 0$$

(3) $\nabla (\nabla \cdot \boldsymbol{v})$ Seldom occurs in physical applications.

$$\nabla^2 \boldsymbol{v} = (\nabla \cdot \nabla)\boldsymbol{v} \neq \nabla(\nabla \cdot \boldsymbol{v})$$

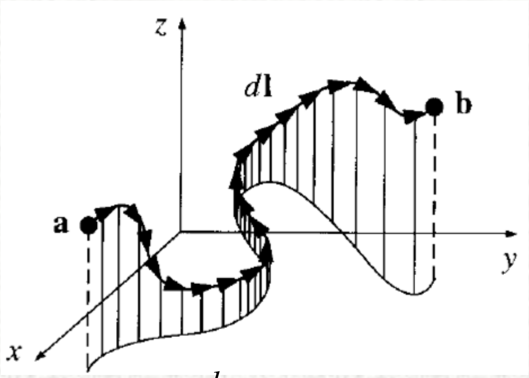
(4) $$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

(5) $$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}$$

Often used to define the Laplacian of a vector.

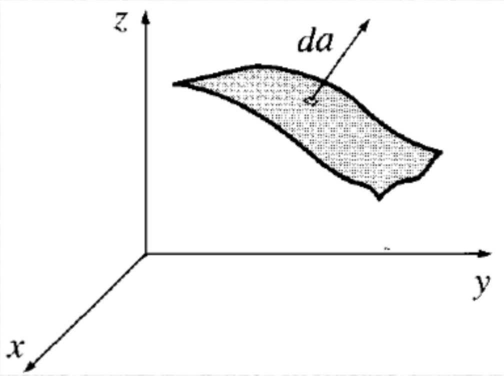
3. Integral Calculus

- Line, Surface and Volume Integrals



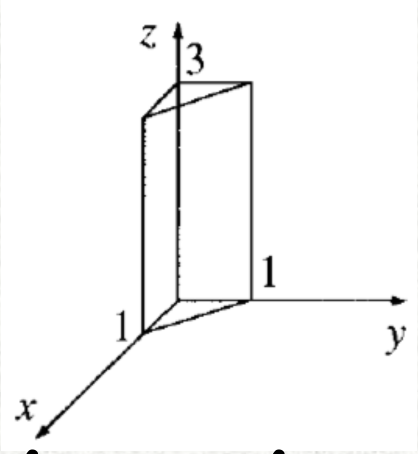
$$\int_a^b \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{l}$$

$$\oint \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{l} \quad \text{(if b=a)}$$



$$\int_S \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{a}$$

$$\oint \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{a} \quad \text{(if surface is closed)}$$



$$\int_V T\,\mathrm{d}\tau \quad \text{or} \quad \int_V \boldsymbol{v}\,\mathrm{d}\tau$$

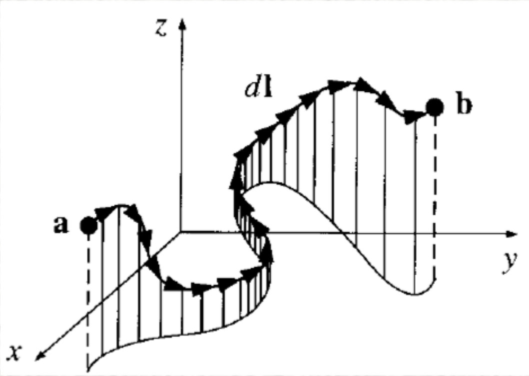
- Fundamental Theorem of Calculus

$$\int_a^b F(x)\mathrm{d}x = f(b) - f(a)$$

$$\text{where } F(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

3. Integral Calculus

- Fundamental Theorem for Gradients



$$\int_a^b (\nabla T) \cdot \mathrm{d}\boldsymbol{l} = T(b) - T(a)$$

$$(\nabla T) \cdot \mathrm{d}\boldsymbol{l} = \mathrm{d}T$$

- Fundamental Theorem for Divergences

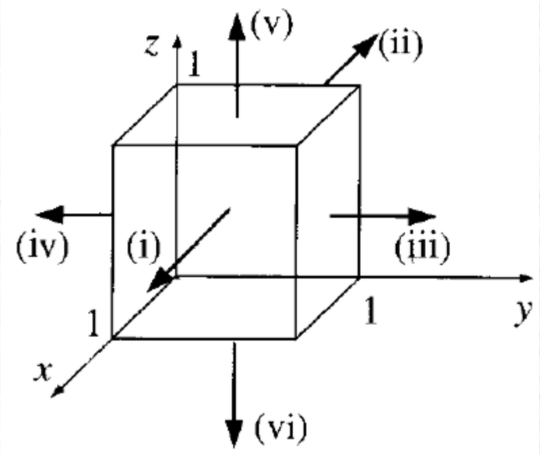
$$\int_V (\nabla \cdot \boldsymbol{v}) \mathrm{d}\tau = \oint_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{a}$$

At least three special names: Gauss's theorem, Green's theorem, or simply, divergence theorem.

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

- Example 4: Check the divergence theorem using the following function and the unit cube situated at the origin.

$$\boldsymbol{v} = y^2 \hat{x} + (2xy + z^2) \hat{y} + (2yz) \hat{z}$$

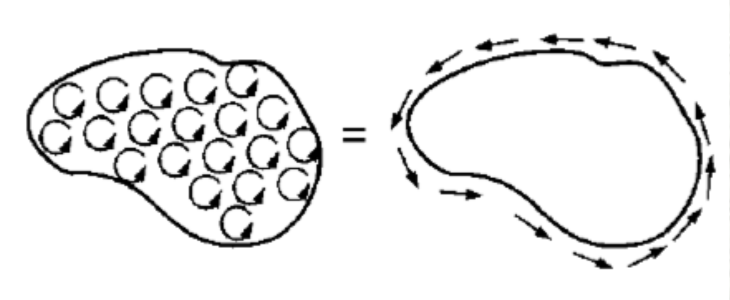


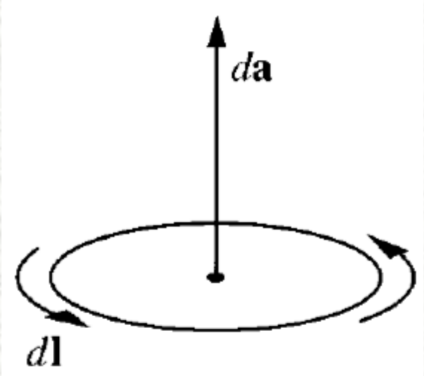
3. Integral Calculus

➢ Fundamental Theorem for Curls

$$\int_S (\nabla \times \boldsymbol{v}) \bullet \mathrm{d}\boldsymbol{a} = \oint \boldsymbol{v} \bullet d\boldsymbol{l}$$

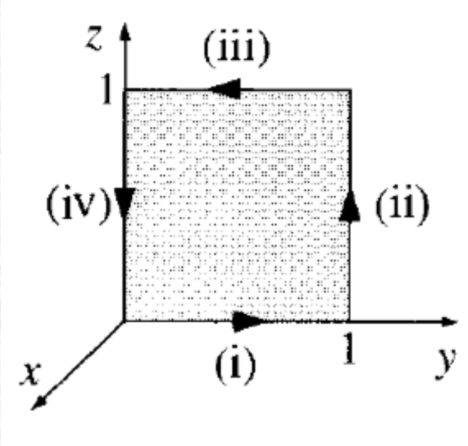
Stokes' theorem





- Example 5: Check the Stokes' theorem using the following function and the square surface shown.

$$\boldsymbol{v} = \left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}$$



3. Integral Calculus

- Integration by Parts
  - How often one is likely to encounter an integral involving the product of one function and the derivative of another? The answer is: surprisingly often!
  - **Exploits the product rule and invoking the fundamental theorem**

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right) + g\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)$$

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(fg)\mathrm{d}x = (fg)\Big|_a^b = \int_a^b f\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)\mathrm{d}x + \int_a^b g\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x$$

$$\int_a^b f\left(\frac{\mathrm{d}g}{\mathrm{d}x}\right)\mathrm{d}x = -\int_a^b g\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x + (fg)\Big|_a^b$$

Transfer the derivative from *g* (or ***A***) to *f*, at the cost of a minus sign and a boundary term.

$$\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f)$$

$$\int \nabla\cdot(f\boldsymbol{A})\mathrm{d}\tau = \int f(\nabla\cdot\boldsymbol{A})\mathrm{d}\tau + \int \boldsymbol{A}\cdot(\nabla f)\mathrm{d}\tau = \oint f\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{a}$$

$$\int_V f(\nabla\cdot\boldsymbol{A})\mathrm{d}\tau = -\int_V \boldsymbol{A}\cdot(\nabla f)\mathrm{d}\tau + \oint_S f\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{a}$$

4. Curvilinear coordinates

- Spherical Polar Coordinates (r,θ,ϕ)

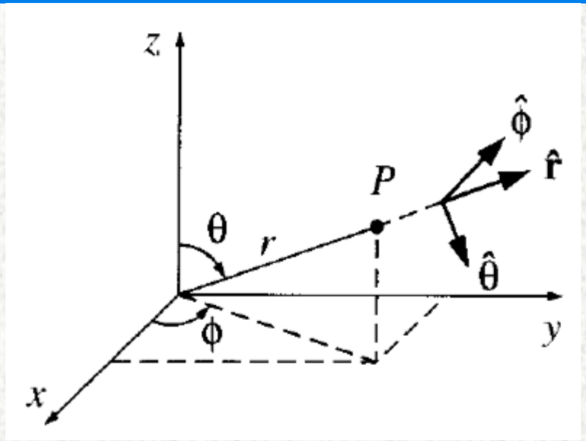
$$\boldsymbol{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$$

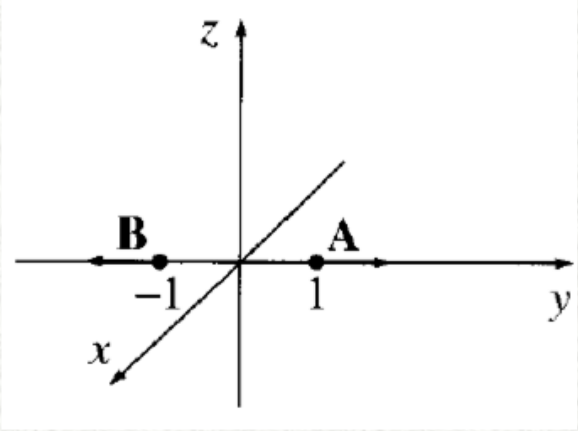
- Relation to Cartesian coordinates (x,y,z)

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \end{cases}$$

- Do not naively combine the spherical components of vectors associated with different points.
- Unit vectors themselves are functions of position. $\partial\hat{r}/\partial\theta = \hat{\theta}$





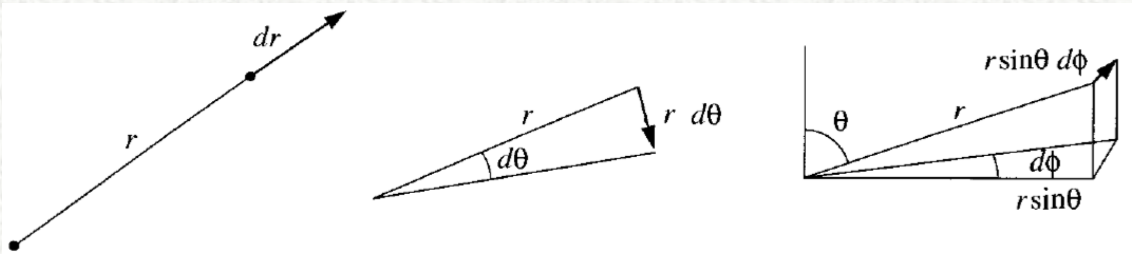
A+***B***=0, not $2\hat{r}$

A•***B***=-1, not +1

4. Curvilinear coordinates

$$\left.\begin{aligned} d\boldsymbol{l} &= dx\hat{x} + dy\hat{y} + dz\hat{z} \\ d\tau &= dxdydz \end{aligned}\right\}$$ Cartesian coordinate

$$\left.\begin{aligned} d\boldsymbol{l} &= dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \\ d\tau &= r^2\sin\theta dr d\theta d\phi \end{aligned}\right\}$$ Spherical Polar coordinate



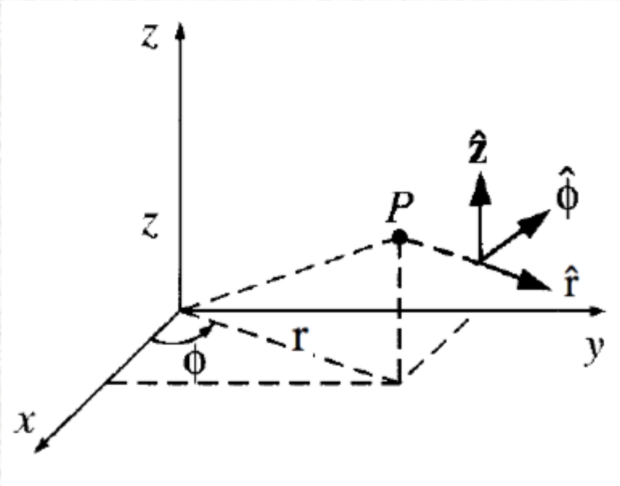
- Vector derivatives

$$\begin{cases} \nabla T = \dfrac{\partial T}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial T}{\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial T}{\partial \phi}\hat{\phi} \\ \nabla \bullet \boldsymbol{v} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 v_r\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta v_\theta\right) + \dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi} \\ \nabla \times \boldsymbol{v} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}\left(\sin\theta v_\phi\right) - \dfrac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}\left(r v_\theta\right) - \dfrac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\ \nabla^2 T = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial T}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 T}{\partial \phi^2} \end{cases}$$

4. Curvilinear coordinates

- Cylindrical Coordinates (r, ϕ, z)

$$\boldsymbol{v} = v_r\hat{r} + v_\phi\hat{\phi} + v_z\hat{z}$$



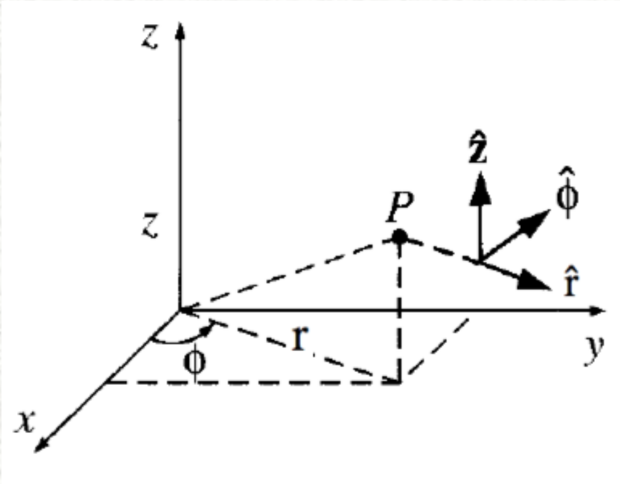
- Relation to Cartesian coordinates (x,y,z)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \\ \hat{z} = \hat{z} \end{cases}$$

4. Curvilinear coordinates

$$\left.\begin{aligned} \mathrm{d}\boldsymbol{l} &= \mathrm{d}r\hat{r} + r\mathrm{d}\phi\hat{\phi} + \mathrm{d}z\hat{z} \\ \mathrm{d}\tau &= r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z \end{aligned}\right\}$$ Cylindrical coordinate



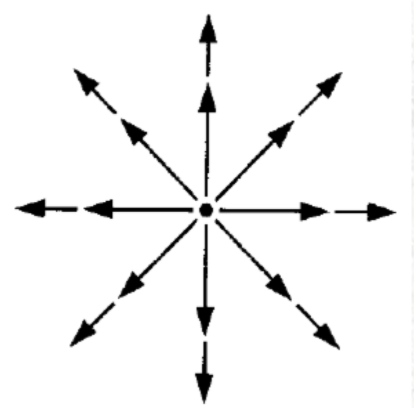
- Vector derivatives

$$\begin{cases} \nabla T = \dfrac{\partial T}{\partial r}\hat{r} + \dfrac{1}{r}\dfrac{\partial T}{\partial \phi}\hat{\phi} + \dfrac{\partial T}{\partial z}\hat{z} \\[2ex] \nabla \bullet \boldsymbol{v} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r v_r\right) + r\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{\partial v_z}{\partial z} \\[2ex] \nabla \times \boldsymbol{v} = \left(\dfrac{1}{r}\dfrac{\partial v_z}{\partial \phi} - \dfrac{\partial v_\phi}{\partial z}\right)\hat{r} + \left(\dfrac{\partial v_r}{\partial z} - \dfrac{\partial v_z}{\partial r}\right)\hat{\phi} + \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}\left(r v_\phi\right) - \dfrac{\partial v_r}{\partial \phi}\right]\hat{z} \\[2ex] \nabla^2 T = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 T}{\partial \phi^2} + \dfrac{\partial^2 T}{\partial z^2} \end{cases}$$

5. Dirac Delta Function

Start from one example.

- Example 6: Compute the divergence of the vector function:



$$\boldsymbol{v} = \frac{\hat{r}}{r^2}$$

$$\nabla \bullet \boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\oint \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{a} = \iint \left(\frac{1}{R^2}\hat{r}\right) \bullet \left(R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \hat{r}\right) = \left(\int_0^\pi \sin\theta \,\mathrm{d}\theta\right)\left(\int_0^{2\pi} \mathrm{d}\phi\right) = 4\pi$$

$$\int_V (\nabla \bullet \boldsymbol{v}) \mathrm{d}\tau \neq \oint_S \boldsymbol{v} \bullet \mathrm{d}\boldsymbol{a} \qquad ???$$

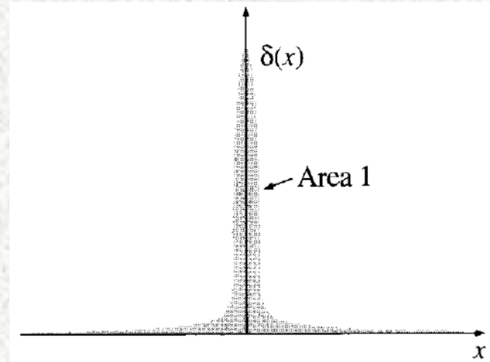
Reason: $\nabla \bullet \boldsymbol{v}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π! No ordinary function behaves like that.

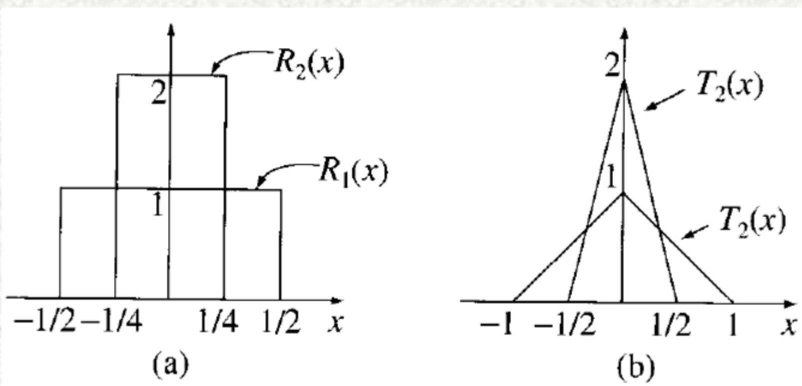
5. Dirac Delta Function

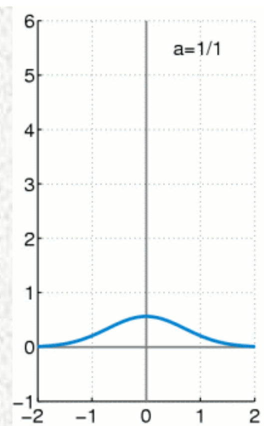
➢ One-Dimensional Dirac Delta Function

$$\delta(x)=\begin{cases}0, & \text{if } x\neq 0\\ \infty, & \text{if } x=0\end{cases}$$

$$\text{and } \int_{-\infty}^{\infty}\delta(x)\mathrm{d}x=1$$







Known as a generalized function, or distribution, or limit of a sequence of functions.

$$\delta_a(x)=\frac{1}{a\sqrt{\pi}}e^{-x^2/a^2} \text{ as } a\to 0$$

$$f(x)\delta(x)=f(0)\delta(x)$$

$$\int_{-\infty}^{\infty}f(x)\delta(x)\mathrm{d}x=f(0)\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x=f(0)$$

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)\mathrm{d}x=f(a)$$

5. Dirac Delta Function

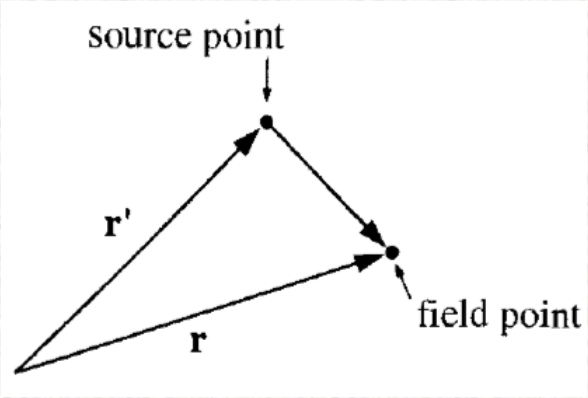
- Three-Dimensional Delta Function

$$\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$$

$$\int_{\text{all space}} \delta^3(\boldsymbol{r})\,\mathrm{d}\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = 1$$

$$\int_{\text{all space}} f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}\tau = f(\boldsymbol{a})$$

- Example 7: Compute the divergence of the vector function: $\boldsymbol{v} = \dfrac{\hat{r}}{r^2}$



$$\nabla \bullet \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r})$$

$$\nabla \bullet \left(\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right) = 4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

$$\nabla^2 \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \nabla \bullet \left(\nabla \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = \nabla \bullet \left(-\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right) = -4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

6. Theory of Vector Fields

Question:

Can you determine the function ***F***, if the divergence and curl of ***F*** are given?

Answer: No.

➢ Helmholtz Theorem

If the divergence ***D***(***r***) and the curl ***C***(***r***) of a vector function ***F***(***r***) are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if ***F***(***r***) goes to zero as $r \to \infty$, then ***F*** is given **uniquely** by:

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A}$$

where

$$\begin{cases} V(\boldsymbol{r}) \equiv \dfrac{1}{4\pi} \displaystyle\int \dfrac{D(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}\tau' \\ A(\boldsymbol{r}) \equiv \dfrac{1}{4\pi} \displaystyle\int \dfrac{\boldsymbol{C}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}\tau' \end{cases} \quad \text{and} \quad \begin{cases} \nabla \bullet \boldsymbol{F} = D \\ \nabla \times \boldsymbol{F} = \boldsymbol{C} \end{cases}$$

□ Example 8: Prove the Helmholtz Theorem.

6. Theory of Vector Fields

➢ Potentials

If the curl of a vector field (***F***) vanishes (everywhere), then ***F*** can be written as the gradient of a scalar potential (*V*):

$$\nabla \times \boldsymbol{F} = 0 \quad \Leftrightarrow \quad \boldsymbol{F} = -\nabla V$$

- Theorem 1: Curl-less (or "irrotational") fields. The following conditions are equivalent:

  (1) $\nabla \times \boldsymbol{F} = 0$ everywhere.

  (2) $\int_a^b \boldsymbol{F} \bullet d\boldsymbol{l}$ is independent of path, for any given end points.

  (3) $\oint \boldsymbol{F} \bullet d\boldsymbol{l} = 0$ for any closed loop.

  (4) $\boldsymbol{F}$ is the gradient of some scalar, $\boldsymbol{F} = -\nabla V$

6. Theory of Vector Fields

If the divergence of a vector field (***F***) vanishes (everywhere), then ***F*** can be written as the curl of a vector potential (***A***):

$$\nabla \bullet \boldsymbol{F} = 0 \quad \Leftrightarrow \quad \boldsymbol{F} = \nabla \times \boldsymbol{A}$$

- Theorem 2: Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

  (1) $\nabla \bullet \boldsymbol{F} = 0$ everywhere.

  (2) $\int \boldsymbol{F} \bullet \mathrm{d}\boldsymbol{a}$ is independent of surface, for any given boundary line.

  (3) $\oint \boldsymbol{F} \bullet \mathrm{d}\boldsymbol{a} = 0$ for any closed surface.

  (4) $\boldsymbol{F}$ is the curl of some vector, $\boldsymbol{F} = \nabla \times \boldsymbol{A}$

- The above scalar potential *V* and vector potential *A* are not unique!
- In all cases a vector field *F* can be written as the gradient of a scalar plus the curl of a vector:

$$\boldsymbol{F} = -\nabla V + \nabla \times \boldsymbol{A} \qquad \text{(always)}$$

Homework

1、Calculate the divergence and curl of the following two function:

$$\boldsymbol{F} = \sin x \cosh y \,\hat{x} - \cos x \sinh y \,\hat{y}$$

$$\boldsymbol{F} = y^2 \hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$$

2、For the function $\boldsymbol{F} = (xy)\hat{x} + (2yz)\hat{y} + 3xz\hat{z}$

(1) test the divergence theorem taking as your volume the cube shown in Fig. 1 with sides of length 2;

(2) test Stokes' theorem using the triangle shaded area of Fig. 2.

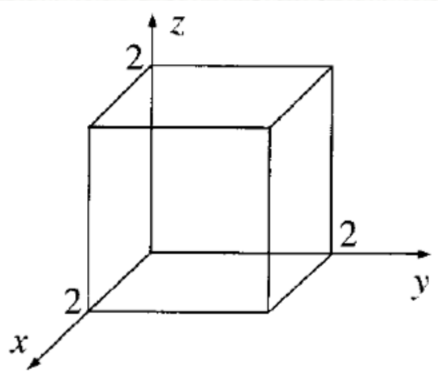


Fig. 1

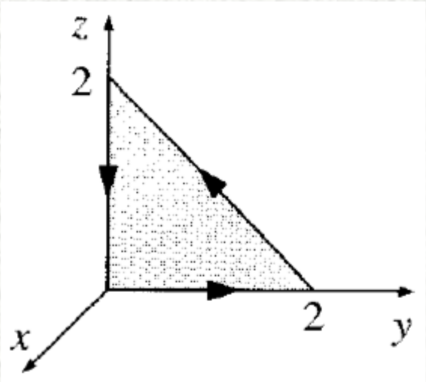


Fig. 2

3、Show that $\int_S f(\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{a} = \int_S [\boldsymbol{A} \times (\nabla f)] \cdot \mathrm{d}\boldsymbol{a} + \oint f\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$