# 第 10 次作业题

**1.** 求下列不定积分:

(14) $\int \sqrt{x^2-a^2}\,\mathrm{d}x\ (a>0)$, (15) $\int x^2\sin(2x)\,\mathrm{d}x$,

(16) $\int \log(x+\sqrt{1+x^2})\,\mathrm{d}x$, (17) $\int e^x\sin^2 x\,\mathrm{d}x$,

(18) $\int \sin(\log x)\,\mathrm{d}x$.

**解:** (14) $\int \sqrt{x^2-a^2}\,\mathrm{d}x = x\sqrt{x^2-a^2} - \int \frac{x^2\,\mathrm{d}x}{\sqrt{x^2-a^2}}$

$= x\sqrt{x^2-a^2} - \int \big(\sqrt{x^2-a^2} + \frac{a^2}{\sqrt{x^2-a^2}}\big)\,\mathrm{d}x$

$= x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}| - \int \sqrt{x^2-a^2}\,\mathrm{d}x$,

由此立刻可得 $\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$.

(15) $\int x^2\sin(2x)\,\mathrm{d}x = -\frac{1}{2}\int x^2\,\mathrm{d}(\cos(2x)) = -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,\mathrm{d}x$

$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}\int x\,\mathrm{d}(\sin(2x))$

$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,\mathrm{d}x$

$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$.

(16) $\int \log(x+\sqrt{1+x^2})\,\mathrm{d}x = x\log(x+\sqrt{1+x^2}) - \int x\cdot\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,\mathrm{d}x$

$= x\log(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = x\log(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$.

(17) $\int e^x\sin^2 x\,\mathrm{d}x = \frac{1}{2}\int e^x(1-\cos(2x))\,\mathrm{d}x = \frac{1}{2}\mathrm{Re}\big(\int (e^x - e^{(1+2i)x})\,\mathrm{d}x\big)$

$= \frac{1}{2}\mathrm{Re}\big(\int \mathrm{d}\big(e^x - \frac{e^{(1+2i)x}}{(1+2i)}\big)\big) = \frac{1}{2}\mathrm{Re}\Big(e^x - \frac{1}{5}e^{(1+2i)x}(1-2i)\Big) + C$

$= \frac{e^x}{10}(5-\cos(2x)-2\sin(2x)) + C$.

(18) **方法 1.** $\int \sin(\log x)\,\mathrm{d}x \overset{y=\log x}{=} \int \sin y\,\mathrm{d}(e^y) = e^y\sin y - \int e^y\,\mathrm{d}(\sin y)$

$= e^y\sin y - \int e^y\cos y\,\mathrm{d}y = e^y\sin y - \int \cos y\,\mathrm{d}(e^y)$

$= e^y\sin y - e^y\cos y + \int e^y\,\mathrm{d}(\cos y)$

$= e^y\sin y - e^y\cos y - \int \sin y\,\mathrm{d}(e^y)$

$= x\sin(\log x) - x\cos(\log x) - \int \sin(\log x)\,\mathrm{d}x$.

由此可得 $\int \sin(\log x)\,\mathrm{d}x = \frac{x}{2}\big(\sin(\log x) - \cos(\log x)\big) + C$.

**方法 2.** $\int \sin(\log x)\,\mathrm{d}x = \mathrm{Im}\big(\int e^{i\log x}\,\mathrm{d}x\big) = \mathrm{Im}\big(\int x^i\,\mathrm{d}x\big)$

$= \mathrm{Im}\int \mathrm{d}\big(\frac{e^{(1+i)\log x}}{1+i}\big) = \frac{1}{2}\mathrm{Im}\big((1-i)e^{(1+i)\log x}\big) + C$

$= \frac{x}{2}\big(\sin(\log x) - \cos(\log x)\big) + C$.

**2.** 求下列不定积分:

(1) $\int \frac{\mathrm{d}x}{(x+1)(x+2)^2}$, (2) $\int \frac{\mathrm{d}x}{x(1+x^2)}$,

(3) $\int \frac{x^4}{x^4+5x^2+4}\,\mathrm{d}x$, (4) $\int \frac{x^7}{(1-x^2)^5}\,\mathrm{d}x$,

(5) $\int \frac{\mathrm{d}x}{\sin x\cos^4 x}$, (6) $\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x$,

(7) $\int \frac{\mathrm{d}x}{(2+\cos x)\sin x}$, (8) $\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$.

**解:** (1) $\int \frac{\mathrm{d}x}{(x+1)(x+2)^2} = \int \big(\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2}\big)\mathrm{d}x = \log\big|\frac{x+1}{x+2}\big| + \frac{1}{x+2} + C.$

(2) $\int \frac{\mathrm{d}x}{x(1+x^2)} = \int \big(\frac{1}{x} - \frac{x}{1+x^2}\big)\mathrm{d}x = \log|x| - \frac{1}{2}\log(1+x^2) + C = \log\frac{|x|}{\sqrt{1+x^2}} + C.$

(3) $\int \frac{x^4}{x^4+5x^2+4}\,\mathrm{d}x = \int \big(1 - \frac{5x^2+4}{(x^2+1)(x^2+4)}\big)\,\mathrm{d}x = x - \frac{1}{3}\int \big(\frac{16}{x^2+4} - \frac{1}{x^2+1}\big)\,\mathrm{d}x$
$= x - \frac{8}{3}\arctan\frac{x}{2} + \frac{1}{3}\arctan x + C.$

(4) $\int \frac{x^7}{(1-x^2)^5}\,\mathrm{d}x = \frac{1}{2}\int \frac{x^6}{(1-x^2)^5}\,\mathrm{d}(x^2-1) \overset{t=1-x^2}{=} -\frac{1}{2}\int \frac{(1-t)^3}{t^5}\,\mathrm{d}t$
$= -\frac{1}{2}\int \frac{1-3t+3t^2-t^3}{t^5}\,\mathrm{d}t = \frac{1}{8t^4} - \frac{1}{2t^3} + \frac{3}{4t^2} - \frac{1}{2t} + C$
$= \frac{1}{8(1-x^2)^4} - \frac{1}{2(1-x^2)^3} + \frac{3}{4(1-x^2)^2} - \frac{1}{2(1-x^2)} + C.$

(5) $\int \frac{\mathrm{d}x}{\sin x\cos^4 x} = -\int \frac{\mathrm{d}(\cos x)}{\sin^2 x\cos^4 x} \overset{u=\cos x}{=} -\int \frac{\mathrm{d}u}{(1-u^2)u^4}$
$= \int \big(\frac{1}{2(u-1)} - \frac{1}{2(u+1)} - \frac{1}{u^2} - \frac{1}{u^4}\big)\mathrm{d}u = \frac{1}{2}\log\big|\frac{u-1}{u+1}\big| + \frac{1}{u} + \frac{1}{3u^3} + C$
$= \frac{1}{2}\log\big|\frac{\cos x-1}{\cos x+1}\big| + \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + C.$

(6) **方法 1.** $\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x = \int \frac{1-\tan x}{1+\tan x}\cdot\frac{1}{\frac{1}{\cos^2 x}}\,\mathrm{d}(\tan x) \overset{u=\tan x}{=} \int \frac{1-u}{1+u}\cdot\frac{1}{1+u^2}\,\mathrm{d}u$
$= \int \big(\frac{1}{1+u} - \frac{u}{1+u^2}\big)\,\mathrm{d}u = \log|1+u| - \frac{1}{2}\log(1+u^2) + C$
$= \log\frac{|1+\tan x|}{\sqrt{1+\tan^2 x}} + C = \log|\sin x+\cos x| + C.$

**方法 2.** $\int \frac{1-\tan x}{1+\tan x}\,\mathrm{d}x = \int \frac{\cos x-\sin x}{\cos x+\sin x}\,\mathrm{d}x = \int \frac{\mathrm{d}(\sin x+\cos x)}{\sin x+\cos x}$
$= \log|\sin x+\cos x| + C.$

(7) $\int \frac{\mathrm{d}x}{(2+\cos x)\sin x} = -\int \frac{\mathrm{d}(\cos x)}{(2+\cos x)\sin^2 x} \overset{u=\cos x}{=} -\int \frac{\mathrm{d}u}{(2+u)(1-u^2)}$
$= \int \big(\frac{1}{3(u+2)} + \frac{1}{6(u-1)} - \frac{1}{2(u+1)}\big)\,\mathrm{d}u$
$= \frac{1}{3}\log|u+2| + \frac{1}{6}\log|u-1| - \frac{1}{2}\log|u+1| + C$
$= \frac{1}{6}\log\frac{(u+2)^2|u-1|}{|u+1|^3} + C = \frac{1}{6}\log\frac{(\cos x+2)^2(1-\cos x)}{(1+\cos x)^3} + C.$

(8) **方法 1.** $\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x = \int \frac{\tan x}{\tan x+1}\cdot\frac{1}{\frac{1}{\cos^2 x}}\,\mathrm{d}\tan x$
$\overset{u=\tan x}{=} \int \frac{u}{u+1}\cdot\frac{1}{u^2+1}\,\mathrm{d}u = \int \big(-\frac{1}{2(u+1)} + \frac{u+1}{2(u^2+1)}\big)\,\mathrm{d}u$
$= -\frac{1}{2}\log|u+1| + \frac{1}{4}\log(u^2+1) + \frac{1}{2}\arctan u + C$
$= -\frac{1}{2}\log\frac{|u+1|}{\sqrt{u^2+1}} + \frac{1}{2}\arctan u + C$
$= -\frac{1}{2}\log\frac{|\tan x+1|}{\sqrt{\tan^2 x+1}} + \frac{1}{2}x + C$
$= -\frac{1}{2}\log|\sin x+\cos x| + \frac{1}{2}x + C.$

**方法 2.** 令 $I_1 = \int \frac{\cos x}{\sin x+\cos x}\,\mathrm{d}x$, $I_2 = \int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x$, 则

$$I_1 + I_2 = \int \mathrm{d}x = x + C_1,$$
$$I_1 - I_2 = \int \frac{\cos x-\sin x}{\sin x+\cos x}\,\mathrm{d}x = \int \frac{\mathrm{d}(\sin x+\cos x)}{\sin x+\cos x} = \log|\sin x+\cos x| + C_2,$$

于是 $\int \frac{\sin x}{\sin x+\cos x}\,\mathrm{d}x = I_2 = \frac{1}{2}\big(x - \log|\sin x+\cos x|\big) + C.$

**3.** 求下列不定积分:

(1) $\int x\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x$, (2) $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x$,

(3) $\int \frac{\sqrt{1+\cos x}}{\sin x}\,\mathrm{d}x$, 其中 $x\in(0,\pi)$, (4) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\,\mathrm{d}x$.

**解:** (1) **方法 1.** $\int x\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x \overset{t=\sqrt{\frac{1+x}{1-x}}}{=} \int \frac{t^2-1}{t^2+1}\cdot t\,\mathrm{d}\big(\frac{t^2-1}{t^2+1}\big) = \int \frac{t(t^2-1)}{t^2+1}\cdot\frac{4t}{(t^2+1)^2}\,\mathrm{d}t$

$= \int \frac{4t^2(t^2-1)}{(t^2+1)^3}\,\mathrm{d}t = \int \big(\frac{4}{t^2+1} - \frac{12}{(t^2+1)^2} + \frac{8}{(t^2+1)^3}\big)\,\mathrm{d}t$

$= 4\arctan t - 12\big(\frac{t}{2(t^2+1)} + \frac{1}{2}\arctan t\big)$

$+8\big(\frac{t}{4(t^2+1)^2} + \frac{3}{4}\big(\frac{t}{2(t^2+1)} + \frac{1}{2}\arctan t\big)\big) + C$

$= \arctan t - \frac{3t}{t^2+1} + \frac{2t}{(t^2+1)^2} + C$

$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{3\sqrt{\frac{1+x}{1-x}}}{(\sqrt{\frac{1+x}{1-x}})^2+1} + \frac{2\sqrt{\frac{1+x}{1-x}}}{((\sqrt{\frac{1+x}{1-x}})^2+1)^2} + C$

$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{3}{2}(1-x)\sqrt{\frac{1+x}{1-x}} + \frac{1}{2}(1-x)^2\sqrt{\frac{1+x}{1-x}} + C$

$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{1}{2}(2+x)\sqrt{1-x^2} + C.$

**方法 2.** $\int x\sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \int \frac{x|1+x|}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{x=\sin t}{=} \int \frac{(\sin t)\cdot|1+\sin t|}{\cos t}\,\mathrm{d}(\sin t)$

$= \int(\sin t + \sin^2 t)\,\mathrm{d}t = -\cos t + \int \frac{1-\cos 2t}{2}\,\mathrm{d}t$

$= \frac{1}{2}t - \cos t - \frac{1}{4}\sin 2t + C = \frac{1}{2}\arcsin x - \sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} + C.$

(2) $\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,\mathrm{d}x = \int \frac{1-x+x^2}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}}\,\mathrm{d}x$

$\overset{x=\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t}{=} \int \frac{1-\big(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t\big)+\big(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t\big)^2}{\sqrt{(\frac{\sqrt{5}}{2})^2-(\frac{\sqrt{5}}{2}\sin t)^2}}\,\mathrm{d}\big(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t\big)$

$= \int(\frac{3}{4}+\frac{5}{4}\sin^2 t)\,\mathrm{d}t = \int(\frac{3}{4}+\frac{5}{4}\frac{1-\cos 2t}{2})\,\mathrm{d}t = \frac{11}{8}t - \frac{5}{16}\sin 2t + C$

$= \frac{11}{8}\arcsin\frac{2}{\sqrt{5}}(x-\frac{1}{2}) - \frac{5}{8}\cdot\frac{2}{\sqrt{5}}(x-\frac{1}{2})\cdot\sqrt{1-(\frac{2}{\sqrt{5}}(x-\frac{1}{2}))^2} + C$

$= \frac{11}{8}\arcsin\frac{2}{\sqrt{5}}(x-\frac{1}{2}) - \frac{1}{2}(x-\frac{1}{2})\sqrt{1+x-x^2} + C.$

(3) **方法 1.** 对于 $x\in(0,\pi)$, 我们有

$\int \frac{\sqrt{1+\cos x}}{\sin x}\,\mathrm{d}x = \int \frac{\sqrt{2\cos^2\frac{x}{2}}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{2}\sin\frac{x}{2}}\,\mathrm{d}x = \int \frac{1}{2\sqrt{2}\tan\frac{x}{4}\cdot\cos^2\frac{x}{4}}\,\mathrm{d}x$

$= \int \frac{\sqrt{2}}{\tan\frac{x}{4}}\,\mathrm{d}(\tan\frac{x}{4}) = \sqrt{2}\log|\tan\frac{x}{4}| + C.$

**方法 2.** 对于 $x\in(0,\pi)$, 我们有

$\int \frac{\sqrt{1+\cos x}}{\sin x}\,\mathrm{d}x = -\int \frac{\sqrt{1+\cos x}}{\sin^2 x}\,\mathrm{d}(\cos x) \overset{u=\sqrt{1+\cos x}}{=} -\int \frac{u}{1-(u^2-1)^2}\,\mathrm{d}(u^2-1)$

$= \int \frac{2}{(u-\sqrt{2})(u+\sqrt{2})}\,\mathrm{d}u = \int \big(\frac{1}{\sqrt{2}(u-\sqrt{2})} - \frac{1}{\sqrt{2}(u+\sqrt{2})}\big)\,\mathrm{d}u = \frac{1}{\sqrt{2}}\log\frac{|u-\sqrt{2}|}{|u+\sqrt{2}|} + C$

$= \frac{1}{\sqrt{2}}\log\frac{|\sqrt{1+\cos x}-\sqrt{2}|}{|\sqrt{1+\cos x}+\sqrt{2}|} + C = \frac{1}{\sqrt{2}}\log\frac{|\sqrt{2}\cos\frac{x}{2}-\sqrt{2}|}{|\sqrt{2}\cos\frac{x}{2}+\sqrt{2}|} + C$

$= \frac{1}{\sqrt{2}}\log(\tan^2\frac{x}{4}) + C = \sqrt{2}\log|\tan\frac{x}{4}| + C.$

(4) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\,\mathrm{d}x = 2\int \frac{\arctan\sqrt{x}}{1+x}\,\mathrm{d}\sqrt{x} = 2\int \arctan\sqrt{x}\,\mathrm{d}(\arctan\sqrt{x})$

$= (\arctan\sqrt{x})^2 + C.$

**4.** 求下列定积分:

(1) $\int_0^{2\pi}|\sin x|\,\mathrm{d}x$, (2) $\int_0^2|(x-1)(x-2)|\,\mathrm{d}x$,

(3) $\int_0^1 x\tan^2 x\,\mathrm{d}x$, (4) $\int_0^{\frac{\pi}{2}} e^{2x}\sin^2 x\,\mathrm{d}x$,

(5) $\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x$, (6) $\int_0^{2a} x\sqrt{a^2-(x-a)^2}\,\mathrm{d}x$ $(a>0)$.

**解:** (1) 由被积函数的周期性和奇偶性可知

$$\int_0^{2\pi}|\sin x|\,\mathrm{d}x=2\int_0^{\pi}|\sin x|\,\mathrm{d}x=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\sin x|\,\mathrm{d}x=4\int_0^{\frac{\pi}{2}}|\sin x|\,\mathrm{d}x=4.$$

(2) $\int_0^2|(x-1)(x-2)|\,\mathrm{d}x=\int_0^1(x-1)(x-2)\,\mathrm{d}x-\int_1^2(x-1)(x-2)\,\mathrm{d}x$
$=\int_0^1(x^2-3x+2)\,\mathrm{d}x-\int_1^2(x^2-3x+2)\,\mathrm{d}x$
$=\big(\frac{x^3}{3}-\frac{3}{2}x^2+2x\big)\Big|_0^1-\big(\frac{x^3}{3}-\frac{3}{2}x^2+2x\big)\Big|_1^2=1.$

(3) $\int_0^1 x\tan^2 x\,\mathrm{d}x=\int_0^1 x(\sec^2 x-1)\,\mathrm{d}x=-\int_0^1 x\,\mathrm{d}x+\int_0^1 x\,\mathrm{d}(\tan x)$
$=-\frac{1}{2}x^2\Big|_0^1+x\tan x\Big|_0^1-\int_0^1\tan x\,\mathrm{d}x=-\frac{1}{2}+\tan 1+\int_0^1\frac{\mathrm{d}(\cos x)}{\cos x}$
$=-\frac{1}{2}+\tan 1+\log(\cos 1).$

(4) **方法 1.** $\int_0^{\frac{\pi}{2}}e^{2x}\sin^2 x\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}(e^{2x})$
$=\frac{1}{2}e^{2x}\sin^2 x\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}2e^{2x}(\sin x)(\cos x)\,\mathrm{d}x$
$=\frac{1}{2}e^{\pi}-\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{2x}\sin(2x)\,\mathrm{d}x=\frac{1}{2}e^{\pi}-\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin(2x)\,\mathrm{d}(e^{2x})$
$=\frac{1}{2}e^{\pi}-\frac{1}{4}\Big(e^{2x}\sin(2x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}e^{2x}\,\mathrm{d}(\sin(2x))\Big)$
$=\frac{1}{2}e^{\pi}+\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{2x}\cos(2x)\,\mathrm{d}x=\frac{1}{2}e^{\pi}+\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{2x}(1-2\sin^2 x)\,\mathrm{d}x$
$=\frac{1}{2}e^{\pi}+\frac{1}{4}e^{2x}\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}e^{2x}\sin^2 x\,\mathrm{d}x=\frac{3}{4}e^{\pi}-\frac{1}{4}-\int_0^{\frac{\pi}{2}}e^{2x}\sin^2 x\,\mathrm{d}x$
于是 $\int_0^{\frac{\pi}{2}}e^{2x}\sin^2 x\,\mathrm{d}x=\frac{3}{8}e^{\pi}-\frac{1}{8}.$

**方法 2.** $\int_0^{\frac{\pi}{2}}e^{2x}\sin^2 x\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}e^{2x}(1-\cos(2x))\,\mathrm{d}x$
$=\frac{1}{2}\mathrm{Re}\big(\int_0^{\frac{\pi}{2}}(e^{2x}-e^{(2+2i)x})\,\mathrm{d}x\big)=\frac{1}{2}\mathrm{Re}\big(\int_0^{\frac{\pi}{2}}\mathrm{d}\big(\frac{e^{2x}}{2}-\frac{e^{(2+2i)x}}{(2+2i)}\big)\big)$
$=\frac{e^{2x}}{8}(2-\cos(2x)-\sin(2x))\Big|_0^{\frac{\pi}{2}}=\frac{3}{8}e^{\pi}-\frac{1}{8}.$

(5) **方法 1.** 由于 $\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x=-\int_0^{\frac{\pi}{2}}\sin^3 x\,\mathrm{d}(\cos x)$
$=-\sin^3 x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}(\sin^3 x)=3\int_0^{\frac{\pi}{2}}\cos^2 x\sin^2 x\,\mathrm{d}x$
$=3\int_0^{\frac{\pi}{2}}(1-\sin^2 x)\sin^2 x\,\mathrm{d}x=3\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x-3\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x,$
于是 $\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x=\frac{3}{4}\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x=\frac{3}{4}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}\,\mathrm{d}x=\frac{3}{4}\frac{x-\frac{\sin 2x}{2}}{2}\Big|_0^{\frac{\pi}{2}}=\frac{3\pi}{16}.$

**方法 2.** $\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x=-\int_0^{\frac{\pi}{2}}\sin^3 x\,\mathrm{d}(\cos x)$
$=-\sin^3 x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}(\sin^3 x)=3\int_0^{\frac{\pi}{2}}\cos^2 x\sin^2 x\,\mathrm{d}x$
$=\frac{3}{4}\int_0^{\frac{\pi}{2}}\sin^2(2x)\,\mathrm{d}x=\frac{3}{8}\int_0^{\frac{\pi}{2}}(1-\cos(4x))\,\mathrm{d}x=\frac{3}{8}(x-\frac{1}{4}\sin(4x))\Big|_0^{\frac{\pi}{2}}=\frac{3}{16}\pi.$

**方法 3.** $\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin^4 x+\cos^4 x)\,\mathrm{d}x$
$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\big((\sin^2 x+\cos^2 x)^2-2(\sin^2 x)(\cos^2 x)\big)\,\mathrm{d}x$
$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\big(1-\frac{1}{2}\sin^2(2x)\big)\,\mathrm{d}x=\frac{\pi}{4}-\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2(2x)\,\mathrm{d}x$
$=\frac{\pi}{4}-\frac{1}{8}\int_0^{\frac{\pi}{2}}(\sin^2(2x)+\cos^2(2x))\,\mathrm{d}x=\frac{\pi}{4}-\frac{\pi}{16}=\frac{3\pi}{16}.$

(6) $\int_0^{2a}x\sqrt{a^2-(x-a)^2}\,\mathrm{d}x\overset{t=x-a}{=}\int_{-a}^{a}(t+a)\sqrt{a^2-t^2}\,\mathrm{d}x$
$=\int_{-a}^{a}t\sqrt{a^2-t^2}\,\mathrm{d}x+a\int_{-a}^{a}\sqrt{a^2-t^2}\,\mathrm{d}x=a\int_{-a}^{a}\sqrt{a^2-t^2}\,\mathrm{d}x$
$=2a\int_0^{a}\sqrt{a^2-t^2}\,\mathrm{d}x\overset{t=a\sin u}{=}2a^2\int_0^{\frac{\pi}{2}}\cos u\,\mathrm{d}(a\sin u)$
$=2a^3\int_0^{\frac{\pi}{2}}\cos^2 u\,\mathrm{d}u=a^3\int_0^{\frac{\pi}{2}}(\cos^2 u+\sin^2 u)\,\mathrm{d}u=\frac{1}{2}\pi a^3.$

**5.** 求下列极限:

(1) $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{2^{\frac{k}{n}}}{n+\frac{1}{k}}$, (2) $\lim\limits_{n\to\infty}\sin\frac{\pi}{n}\cdot\sum\limits_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}$.

**解:** (1) $\forall x\in[0,1]$, 定义 $f(x)=2^x$, 则 $f\in\mathscr{C}[0,1]$, 从而 $f$ 可积, 并且

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}2^{\frac{k}{n}}=\int_0^1 2^x\,\mathrm{d}x=\frac{2^x}{\log 2}\Big|_0^1=\frac{1}{\log 2}.$$

再注意到, $\forall n\geqslant 1$, 我们均有 $\frac{1}{n+1}\sum\limits_{k=1}^{n}2^{\frac{k}{n}}\leqslant\sum\limits_{k=1}^{n}\frac{2^{\frac{k}{n}}}{n+\frac{1}{k}}\leqslant\frac{1}{n}\sum\limits_{k=1}^{n}2^{\frac{k}{n}}$, 于是由夹逼原理可知 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\frac{2^{\frac{k}{n}}}{n+\frac{1}{k}}=\frac{1}{\log 2}$.

(2) $\forall x\in[0,\pi]$, 定义 $f(x)=\frac{1}{2+\cos x}$. 则 $f\in\mathscr{C}[0,1]$, 从而 $f$ 可积, 并且

$$\begin{aligned}\int_0^\pi f(x)\,\mathrm{d}x &= \int_0^\pi\frac{\mathrm{d}x}{2+\cos x}=\int_0^\pi\frac{\mathrm{d}x}{3\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\\ &\overset{u=\frac{x}{2}}{=} \int_0^{\frac{\pi}{2}}\frac{2\,\mathrm{d}u}{3\cos^2u+\sin^2u}=\int_0^{\frac{\pi}{2}}\frac{2\cos^2u\,\mathrm{d}(\tan u)}{3\cos^2u+\sin^2u}\\ &= \frac{2}{\sqrt{3}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}(\frac{\tan u}{\sqrt{3}})}{1+\left(\frac{\tan u}{\sqrt{3}}\right)^2}\\ &= \frac{2}{\sqrt{3}}\arctan\frac{\tan u}{\sqrt{3}}\Big|_0^{\frac{\pi}{2}}\\ &= \frac{\pi}{\sqrt{3}}.\end{aligned}$$

由此立刻可得 $\lim\limits_{n\to\infty}\frac{\pi}{n}\sum\limits_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}=\frac{\pi}{\sqrt{3}}$, 进而我们就有

$$\lim_{n\to\infty}\sin\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}=\Big(\lim_{n\to\infty}\frac{n}{\pi}\sin\frac{\pi}{n}\Big)\cdot\Big(\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{2+\cos\frac{k\pi}{n}}\Big)=\frac{\pi}{\sqrt{3}}.$$

**6.** 求下列曲线围成的面积:

(1) 抛物线 $x=y^2-2y$ 与 $x=2y^2-8y+6$ 所围图形的面积.

(2) 星形线 $x=a\cos^3t$, $y=a\sin^3t$ $(a>0)$ 所围图形的面积.

(3) 确定 $k>0$ 的值使得 $y=x-x^2$ 与 $y=kx$ 所围图形的面积为 $\frac{9}{2}$.

(4) 求圆 $\rho=1$ 与心脏线 $\rho=1+\sin\theta$ 所围图形的公共部分的面积.

**解:** (1) 两抛物线的交点的纵坐标分别为 $3-\sqrt{3}$, $3+\sqrt{3}$, 于是所求面积为

$$\begin{aligned}S &= \int_{3-\sqrt{3}}^{3+\sqrt{3}}\big((y^2-2y)-(2y^2-8y+6)\big)\,\mathrm{d}y\\ &= \int_{3-\sqrt{3}}^{3+\sqrt{3}}(-y^2+6y-6)\,\mathrm{d}y\\ &= \Big(-\frac{y^3}{3}+3y^2-6y\Big)\Big|_{3-\sqrt{3}}^{3+\sqrt{3}}\\ &= 4\sqrt{3}.\end{aligned}$$

(2) 由题设可知 $(\frac{x}{a})^{\frac{2}{3}}+(\frac{y}{a})^{\frac{2}{3}}=1$, 于是所围面积为位于第一象限那部分面积的 4 倍. 而在第一象限, $0\leqslant t\leqslant\frac{\pi}{2}$, 于是 $x(t)$ 严格递减, 故所求面积为

$$\begin{aligned}S &= 4\int_{\frac{\pi}{2}}^{0} y(t)\,\mathrm{d}x(t) = 4\int_{\frac{\pi}{2}}^{0}(a\sin^3 t)\cdot(-3a\cos^2 t\sin t)\,\mathrm{d}t\\ &= 12a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cdot\cos^2 t\,\mathrm{d}t = 12a^2\int_0^{\frac{\pi}{2}}\sin^4 t\,\mathrm{d}t - 12a^2\int_0^{\frac{\pi}{2}}\sin^6 t\,\mathrm{d}t\\ &= 12a^2\cdot\frac{3\pi}{16}-12a^2\cdot\frac{5}{6}\cdot\frac{3\pi}{16}=\frac{3}{8}a^2\pi.\end{aligned}$$

(3) 由于曲线 $y=x-x^2$ 与曲线 $y=kx$ 的交点为 $(0,0)$, $(1-k,k(1-k))$, 因此上述两曲线所围图形的面积为 $\frac{9}{2}$ 当且仅当

$$\frac{9}{2}=\Big|\int_{1-k}^{0}(x-x^2-kx)\,\mathrm{d}x\Big|=\Big|\big(\frac{1}{2}(1-k)x^2-\frac{1}{3}x^3\big)\Big|_0^{1-k}\Big|=\frac{1}{6}|1-k|^3,$$

也即 $|1-k|=3$, 故 $k=-2$ 或 4. 又 $k>0$, 因此我们有 $k=4$.

(4) 圆 $\rho=1$ 与心形线 $\rho=1+\sin\theta$ 的两个交点的极坐标为 $(1,0)$, $(1,\pi)$, 它们所围成的图形由两个部分组成, 其上半部分的边界为 $\rho=1,\ \theta\in(0,\pi)$, 而下半部分的边界为 $\rho=1+\sin\theta,\ \theta\in(\pi,2\pi)$. 故所求面积为

$$\begin{aligned}S &= \int_0^{2\pi}\frac{1}{2}(\rho(\theta))^2\,\mathrm{d}\theta=\int_0^{\pi}\frac{1}{2}\,\mathrm{d}\theta+\int_{\pi}^{2\pi}\frac{1}{2}(1+\sin\theta)^2\,\mathrm{d}\theta\\ &= \frac{1}{2}\pi+\int_{\pi}^{2\pi}\frac{1}{2}(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta\\ &= \frac{1}{2}\pi+\frac{1}{2}\big(\theta-2\cos\theta+\frac{\theta-\frac{1}{2}\sin 2\theta}{2}\big)\Big|_{\pi}^{2\pi}\\ &= \frac{5}{4}\pi-2.\end{aligned}$$