运动规划作业 4: Kinodynamic RRT

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1 OBVP 推导

1 Modeling

a) State, Input and System dynamics

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p} \\ \dot{\boldsymbol{p}} \end{bmatrix}, \quad \boldsymbol{u} = \ddot{\boldsymbol{p}}, \quad \dot{\boldsymbol{x}} = f(x, u) = \begin{bmatrix} \dot{\boldsymbol{p}} \\ \boldsymbol{u} \end{bmatrix}$$

b) Objective

$$J = \int_0^T g(\boldsymbol{x}, \boldsymbol{u})dt = \int_0^T (1 + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u})dt = \int_0^T (1 + \boldsymbol{u}^\top \boldsymbol{u})dt$$

2 Solving

a) Costate

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda_1} \\ \boldsymbol{\lambda_2} \end{bmatrix}$$

b) Hamiltonian

$$H = g + \boldsymbol{\lambda}^\top f = \mathbb{I} + \boldsymbol{\lambda_1}^\top \dot{\boldsymbol{p}} + \boldsymbol{\lambda_2}^\top \boldsymbol{u} + \boldsymbol{u}^\top \boldsymbol{u}$$

$$\dot{\boldsymbol{\lambda}} = -\nabla_x H(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}) = -\begin{bmatrix} \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{p}}, \frac{\partial \boldsymbol{H}}{\partial \dot{\boldsymbol{p}}} \end{bmatrix}^\top = \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{\lambda_1} \end{bmatrix}$$

c) Solving Costate

$$\begin{aligned} \boldsymbol{\lambda_1} &= \boldsymbol{c_1} \\ \boldsymbol{\lambda_2} &= -t\boldsymbol{\lambda_1} + \boldsymbol{c_2} = -t\boldsymbol{c_1} + \boldsymbol{c_2} \end{aligned}$$

d) Solving Optimal Input

$$\nabla_u H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}) = 0$$

$$\boldsymbol{\lambda_2} + 2\boldsymbol{u} = 0, \boldsymbol{u} = -\frac{1}{2}\boldsymbol{\lambda_2} = \frac{1}{2}t\boldsymbol{c_1} - \frac{1}{2}\boldsymbol{c_2}$$

e) Solving Optimal State Trajectory

$$\ddot{\boldsymbol{p}} = \boldsymbol{u} = \frac{1}{2}t\boldsymbol{c_1} - \frac{1}{2}\boldsymbol{c_2}, \qquad \text{边界条件: } \boldsymbol{p}_0, \dot{\boldsymbol{p}}_0, \boldsymbol{p}_f, \dot{\boldsymbol{p}}_f = 0$$

$$\boldsymbol{x}^* = \begin{bmatrix} \frac{t^3}{12}\boldsymbol{c_1} - \frac{t^2}{4}\boldsymbol{c_2} + t\dot{\boldsymbol{p}}_0 + \boldsymbol{p}_0 \\ \frac{t^2}{4}\boldsymbol{c_1} - \frac{t}{2}\boldsymbol{c_2} + \dot{\boldsymbol{p}}_0 \end{bmatrix}$$

$$\begin{pmatrix} \frac{T^3}{12} & -\frac{T^2}{4} \\ \frac{T^2}{4} & -\frac{T}{2} \end{pmatrix} \begin{bmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{p}_0 - \dot{\boldsymbol{p}}_0 T + \boldsymbol{p}_f \\ -\dot{\boldsymbol{p}}_0 \end{bmatrix} = \begin{bmatrix} \Delta\boldsymbol{p} \\ \Delta\dot{\boldsymbol{p}} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \end{bmatrix} = \begin{pmatrix} -\frac{24}{T^3} & \frac{12}{T^2} \\ -\frac{12}{T^2} & \frac{4}{T} \end{pmatrix} \begin{bmatrix} \Delta\boldsymbol{p} \\ \Delta\dot{\boldsymbol{p}} \end{bmatrix}$$

f) The Cost

$$
\begin{aligned}
J(T) &= \int_0^T (1+\boldsymbol{u}^{*\top}\boldsymbol{u}^*)dt = T + \frac{1}{4}\int_0^T (t\boldsymbol{c_1}^\top - \boldsymbol{c_2}^\top)(t\boldsymbol{c_1} - \boldsymbol{c_2})dt \\
&= T + \frac{1}{4}\int_0^T (t^2\|\boldsymbol{c_1}\|^2 + \|\boldsymbol{c_2}\|^2 - 2\boldsymbol{c_2}^\top\boldsymbol{c_1})dt \\
&= \frac{T^3}{12}\|\boldsymbol{c_1}\|^2 + \frac{T}{4}\|\boldsymbol{c_2}\|^2 - \frac{T^2}{4}\boldsymbol{c_2}^\top\boldsymbol{c_1} + T \\
&= \frac{T^4 + \left(4\,\|\dot{\boldsymbol{p}}_0\|^2\right)T^2 + \left(-12\,\boldsymbol{p}_f\,\dot{\boldsymbol{p}}_0 + 12\,\boldsymbol{p}_0\,\dot{\boldsymbol{p}}_0\right)T + 12\,\|\boldsymbol{p}_f\|^2 - 24\,\boldsymbol{p}_0\,\boldsymbol{p}_f + 12\,\boldsymbol{p}_0{}^2}{T^3} \\
&= \frac{T^4 + AT^2 + BT + C}{T^3}
\end{aligned}
$$

in which, $A = 4\,\|\dot{\boldsymbol{p}}_0\|^2, B = -12\,\boldsymbol{p}_f\,\dot{\boldsymbol{p}}_0 + 12\,\boldsymbol{p}_0\,\dot{\boldsymbol{p}}_0, C = 12\,\|\boldsymbol{p}_f\|^2 - 24\,\boldsymbol{p}_0\,\boldsymbol{p}_f + 12\,\boldsymbol{p}_0{}^2$

g) Optimal T

$$
\frac{dJ}{dT} = \frac{T^4 - A\,T^2 - 2\,B\,T - 3\,C}{T^4} = 0
$$

Solve this equation, we get an optimal T*:

$$
T^* = eigenValue(\begin{pmatrix} 0 & 0 & 0 & 3\,C \\ 1 & 0 & 0 & 2\,B \\ 0 & 1 & 0 & A \\ 0 & 0 & 1 & 0 \end{pmatrix})
$$

Then J* is calculated as J(T*).

2 Result

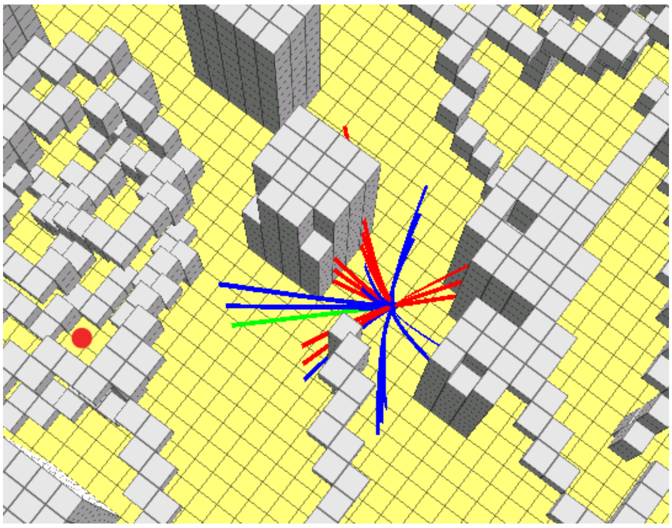

Figure 1: 红色圆点为目标点，采样 27 条曲线，绿色线为代价最小的轨迹，蓝色是可行轨迹，红色为不可行轨迹