Supplementary Materials: Decoupled Right Invariant Error States For Consistent Visual-Inertial Navigation

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Contents

1 Introduction

This supplementary materials will demonstrate all the necessary derivations and analyses for the related paper.

2 Lie Group for IMU State

Since $\mathbf{SE}_2(3)$ [1] is used for IMU navigation state, we introduce the basic notations and properties for it.

2.1 Definition of $\boldsymbol{SE}_2(3)$

The special Euclidean group for IMU state, including orientation $\mathbf{R} \in SO(3)$, position $\mathbf{p} \in \mathbb{R}^3$ and velocity $\mathbf{v} \in \mathbb{R}^3$, can be written in homogeneous matrices as:

$$\mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \in SE_2(3) \tag{1}$$

2.2 exp and log

The exponential matrix map will map an element $\delta\mathbf{x} \in \mathbb{R}^9$ to $SE_2(3)$:

$$\exp(\delta\mathbf{x}) = \exp\left(\begin{bmatrix} \delta\boldsymbol{\theta} \\ \delta\mathbf{p} \\ \delta\mathbf{v} \end{bmatrix}\right) \triangleq \begin{bmatrix} \exp(\delta\boldsymbol{\theta}) & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{2}$$

The logarithm map is defined as the inversion of the exponential map as:

$$\log(\mathbf{X}) = \log\left(\begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix}\right) \triangleq \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{J}_l^{-1}(\boldsymbol{\theta})\mathbf{p} \\ \mathbf{J}_l^{-1}(\boldsymbol{\theta})\mathbf{v} \end{bmatrix} \tag{3}$$

where $\mathbf{J}_l(\boldsymbol{\theta})$ denotes the left Jacobian of $SO(3)$.

2.3 Adjoint Operation

If $\mathbf{y} \in \mathbb{R}^9$, we conveniently define the adjoint operator for $\mathbf{X}$ is defined as:

$$\mathbf{X}\exp(\mathbf{y})\mathbf{X}^{-1} = \exp\left(Ad_{\mathbf{X}} \cdot \mathbf{y}\right) \tag{4}$$

$$\exp(\mathbf{y})\mathbf{X}^{-1} = \mathbf{X}^{-1}\exp\left(Ad_{\mathbf{X}} \cdot \mathbf{y}\right) \tag{5}$$

$$\mathbf{X}\exp(\mathbf{y}) = \exp\left(Ad_{\mathbf{X}} \cdot \mathbf{y}\right)\mathbf{X} \tag{6}$$

$$Ad_{\mathbf{X}} \triangleq \begin{bmatrix} \mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor \mathbf{p} \rfloor \mathbf{R} & \mathbf{R} & \mathbf{0}_3 \\ \lfloor \mathbf{v} \rfloor \mathbf{R} & \mathbf{0}_3 & \mathbf{R} \end{bmatrix} \tag{7}$$

where $\lfloor \cdot \rfloor$ represents the skew matrix.

2.4 Error states for $SE_2(3)$

If $\hat{\mathbf{X}} \in SE_2(3)$ represents the current estimates and $\delta\mathbf{x} = [\delta\theta^\top\ \delta\mathbf{p}^\top\ \delta\mathbf{v}^\top]^\top \in \mathbb{R}^9$ representations the small perturbation, we can treat $\delta\mathbf{x}$ as right invariant errors or left invariant errors and define the $\boxplus$ in manifold accordingly.

- If $\delta\mathbf{x}$ is treated as right invariant errors:

$$\mathbf{X} = \hat{\mathbf{X}} \boxplus \delta\mathbf{x} \triangleq \exp(\delta\mathbf{x})\hat{\mathbf{X}} \tag{8}$$

with:

$$\begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \exp(\delta\boldsymbol{\theta}) & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} & \hat{\mathbf{p}} & \hat{\mathbf{v}} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} \exp(\delta\boldsymbol{\theta})\hat{\mathbf{R}} & \exp(\delta\boldsymbol{\theta})\hat{\mathbf{p}} + \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \exp(\delta\boldsymbol{\theta})\hat{\mathbf{v}} + \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{10}$$

- If $\delta\mathbf{x}$ is treated as left invariant errors:

$$\mathbf{X} = \hat{\mathbf{X}} \boxplus \delta\mathbf{x} \triangleq \hat{\mathbf{X}} \exp(\delta\mathbf{x}) \tag{11}$$

with:

$$\begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{R}} & \hat{\mathbf{p}} & \hat{\mathbf{v}} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\delta\boldsymbol{\theta}) & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}) & \hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} + \hat{\mathbf{p}} & \hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} + \hat{\mathbf{v}} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{13}$$

3 IMU Model

3.1 IMU measurements

$$\mathbf{a}_m = \mathbf{a} + \mathbf{b}_a + \mathbf{n}_a \tag{14}$$

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_g + \mathbf{n}_g \tag{15}$$

3.2 IMU Dynamics

$${}^G_I\dot{\mathbf{R}} = {}^G_I\mathbf{R}\lfloor {}^I\boldsymbol{\omega} \rfloor \tag{16}$$

$${}^G\dot{\mathbf{p}}_I = {}^G\mathbf{v}_I \tag{17}$$

$${}^G\dot{\mathbf{v}}_I = {}^G_I\mathbf{R}\,{}^I\mathbf{a} + {}^G\mathbf{g} \tag{18}$$

$$\dot{\mathbf{b}}_g = \mathbf{n}_{wg} \tag{19}$$

$$\dot{\mathbf{b}}_a = \mathbf{n}_{wa} \tag{20}$$

where ${}^G\mathbf{g} = [0, 0, -9.8]^\top$.

4 IMU Propagation

4.1 IMU State Vector

$$\mathbf{x}_I = (\mathbf{x}_n, \mathbf{x}_b) \tag{21}$$

$$\mathbf{x}_n = \left({}^G_I\mathbf{R}, {}^G\mathbf{p}_I, {}^G\mathbf{v}_I\right) \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & {}^G\mathbf{p}_I & {}^G\mathbf{v}_I \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{22}$$

$$\mathbf{x}_b = (\mathbf{b}_g, \mathbf{b}_a) \tag{23}$$

In this section, we will investigate the right invariant error states for IMU propagation. Readers can refer to the left invariant case in Section 7. With right invariant error for $\mathbf{x}_n$, we can define IMU error states $\delta\mathbf{x}_I$ as:

$$\delta\mathbf{x}_I = \begin{bmatrix} \delta\mathbf{x}_n^\top & | & \delta\mathbf{x}_b^\top \end{bmatrix}^\top = \begin{bmatrix} \delta\boldsymbol{\theta}_I^\top & \delta\mathbf{p}^\top & \delta\mathbf{v}^\top & | & \delta\mathbf{b}_g^\top & \delta\mathbf{b}_a^\top \end{bmatrix}^\top \tag{24}$$

$$\mathbf{x}_I = \hat{\mathbf{x}}_I \boxplus \delta\mathbf{x}_I = (\exp(\delta\mathbf{x}_n)\hat{\mathbf{x}}_n, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b) \tag{25}$$

$$= \begin{bmatrix} \exp(\delta\boldsymbol{\theta}_I){}^G_I\hat{\mathbf{R}} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{v}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \end{bmatrix} \tag{26}$$

4.2 Dynamic Model Integration

From the IMU dynamic model, we integrate the IMU readings from t_k to t_{k+1} and get:

$${}^G_{I_{k+1}}\mathbf{R} = {}^G_{I_k}\mathbf{R}\exp\left(\int_{t_k}^{t_{k+1}} {}^{I_\tau}\boldsymbol{\omega} d\tau\right) \triangleq {}^G_{I_k}\mathbf{R}\Delta\mathbf{R}_k \tag{27}$$

$${}^G\mathbf{p}_{I_{k+1}} = {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t + {}^G_{I_k}\mathbf{R}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a} d\tau ds + \frac{1}{2}{}^G\mathbf{g}\delta t^2 \tag{28}$$

$$\triangleq {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t + {}^G_{I_k}\mathbf{R}\Delta\mathbf{p}_k + \frac{1}{2}{}^G\mathbf{g}\delta t^2 \tag{29}$$

$${}^G\mathbf{v}_{I_{k+1}} = {}^G\mathbf{v}_{I_k} + {}^G_{I_k}\mathbf{R}\int_{t_k}^{t_{k+1}} {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a} d\tau + {}^G\mathbf{g}\delta t \tag{30}$$

$$\triangleq {}^G\mathbf{v}_{I_k} + {}^G_{I_k}\mathbf{R}\Delta\mathbf{v}_k + {}^G\mathbf{g}\delta t \tag{31}$$

$$\mathbf{b}_{g_{k+1}} = \mathbf{b}_{g_k} + \int_{t_k}^{t_{k+1}} \mathbf{n}_{wg} d\tau \tag{32}$$

$$\mathbf{b}_{a_{k+1}} = \mathbf{b}_{a_k} + \int_{t_k}^{t_{k+1}} \mathbf{n}_{wa} d\tau \tag{33}$$

where we have defined:

$$\Delta\mathbf{R}_k = \exp\left(\int_{t_k}^{t_{k+1}} {}^{I_\tau}\boldsymbol{\omega} d\tau\right) \tag{34}$$

$$\Delta\mathbf{p}_k = \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a} d\tau ds \tag{35}$$

$$\Delta\mathbf{v}_k = \int_{t_k}^{t_{k+1}} {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a} d\tau \tag{36}$$

Hence, the IMU navigation state $\mathbf{x}_n$ propagation can be reformulated as:

$$\mathbf{x}_{n_{k+1}} = \boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\mathbf{x}_{n_k})\boldsymbol{\Upsilon}_k \tag{37}$$

where:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \mathbf{I}_3 & \frac{1}{2}{}^G\mathbf{g}\delta t^2 & \mathbf{g}\delta t \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{38}$$

$$\boldsymbol{\Psi}(\mathbf{x}_{n_k}) = \begin{bmatrix} {}^G_{I_k}\mathbf{R} & {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t & {}^G\mathbf{v}_{I_k} \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{39}$$

$$\boldsymbol{\Upsilon}_k = \begin{bmatrix} \Delta\mathbf{R}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{40}$$

Note that $\boldsymbol{\Psi}(\cdot)$ has the following properties:

$$\begin{aligned} \boldsymbol{\Psi}(\mathbf{x}_{n_k}\mathbf{x}_{n_{k+1}}) &= \boldsymbol{\Psi}(\mathbf{x}_{n_k})\boldsymbol{\Psi}(\mathbf{x}_{n_k+1}) && (41) \\ \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n)) &\simeq \exp(\mathbf{F}\delta\mathbf{x}_\mathbf{n}) && (42) \\ \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n)\hat{\mathbf{x}}_n) &= \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n))\boldsymbol{\Psi}(\hat{\mathbf{x}}_n) && (43) \\ &\simeq \exp(\mathbf{F}\delta\mathbf{x}_\mathbf{n})\boldsymbol{\Psi}(\hat{\mathbf{x}}_n) && (44) \end{aligned}$$

where $\mathbf{F}$ is defined as:

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \delta t\mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{45}$$

4.3 Mean Propagation

In this section, we show how to propagate IMU dynamics ignoring the noise s.t.

$$\mathbf{b}_g = \hat{\mathbf{b}}_g \tag{46}$$

$$\mathbf{b}_a = \hat{\mathbf{b}}_a \tag{47}$$

Then, the IMU propogation can be written as

$$\hat{\mathbf{x}}_{n_{k+1}} = \hat{\boldsymbol{\Gamma}}_k \hat{\boldsymbol{\Psi}}(\mathbf{x}_{n_k})\hat{\boldsymbol{\Upsilon}}_k \tag{48}$$

Hence, we just need to integrate $\hat{\boldsymbol{\Upsilon}}_k$ with $\Delta\hat{\mathbf{R}}_k$, $\Delta\hat{\mathbf{p}}_k$ and $\Delta\hat{\mathbf{v}}_k$. When using ACI^2 [2] and assuming constant ${}^{I_k}\hat{\boldsymbol{\omega}}$ and ${}^{I_k}\hat{\mathbf{a}}$ during the period of integration, we can get:

$$
\begin{aligned}
\Delta\hat{\mathbf{R}}_k &= \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta t) && (49)\\
\Delta\hat{\mathbf{p}}_k &= \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^{I_k}_{I_\tau}\hat{\mathbf{R}} d\tau ds \cdot {}^{I_k}\hat{\mathbf{a}} && (50)\\
&= \underbrace{\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta \tau) d\tau ds}_{\boldsymbol{\Xi}_2} \cdot {}^{I_k}\hat{\mathbf{a}} && (51)\\
&= \boldsymbol{\Xi}_2 \cdot {}^{I_k}\hat{\mathbf{a}} && (52)\\
\Delta\hat{\mathbf{v}}_k &= \int_{t_k}^{t_{k+1}} {}^{I_k}_{I_\tau}\hat{\mathbf{R}} d\tau \cdot {}^{I_k}\hat{\mathbf{a}} && (53)\\
&= \underbrace{\int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta \tau) d\tau}_{\boldsymbol{\Xi}_1} \cdot {}^{I_k}\hat{\mathbf{a}} && (54)\\
&= \boldsymbol{\Xi}_1 \cdot {}^{I_k}\hat{\mathbf{a}} && (55)
\end{aligned}
$$

where ${}^{I_k}\hat{\boldsymbol{\omega}} = {}^{I_k}\boldsymbol{\omega}_m - \hat{\mathbf{b}}_g$ and ${}^{I_k}\hat{\mathbf{a}} = {}^{I_k}\mathbf{a}_m - \hat{\mathbf{b}}_a$.

4.4 State Transition and Covariance Propagation

The linearized state propagation matrix can written as:

$$
\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \end{bmatrix} \simeq \underbrace{\begin{bmatrix} \boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix}}_{\boldsymbol{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times 6} \\ \mathbf{0}_{6\times 6} & \mathbf{I}_6\delta t \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{56}
$$

where $\delta t = t_{k+1} - t_k$; $\mathbf{n}_{da} \sim \mathcal{N}(0, \frac{\sigma_a^2}{\delta t}\mathbf{I}_3)$, $\mathbf{n}_{dg} \sim \mathcal{N}(0, \frac{\sigma_g^2}{\delta t}\mathbf{I}_3)$, $\mathbf{n}_{dwa} \sim \mathcal{N}(0, \frac{\sigma_{wa}^2}{\delta t}\mathbf{I}_3)$ and $\mathbf{n}_{dwg} \sim \mathcal{N}(0, \frac{\sigma_{wg}^2}{\delta t}\mathbf{I}_3)$. The biases $\mathbf{x}_b$ can discretized as:

$$
\begin{aligned}
\mathbf{b}_{g_{k+1}} &= \mathbf{b}_{g_k} + \mathbf{n}_{dwg}\delta t && (57)\\
\mathbf{b}_{a_{k+1}} &= \mathbf{b}_{a_k} + \mathbf{n}_{dwa}\delta t && (58)
\end{aligned}
$$

For $\boldsymbol{\Phi}_{nn}$, we can use the right invariant errors:

$$
\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \boldsymbol{\Gamma}_k\boldsymbol{\Psi}(\exp(\delta\mathbf{x}_{n_k})\hat{\mathbf{x}}_{n_k})\hat{\boldsymbol{\Upsilon}}_k && (59)\\
&= \boldsymbol{\Gamma}_k\boldsymbol{\Psi}\left(\exp\left(\delta\mathbf{x}_{n_k}\right)\right)\boldsymbol{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\boldsymbol{\Upsilon}}_k && (60)\\
&\simeq \boldsymbol{\Gamma}_k\exp\left(\mathbf{F}\delta\mathbf{x}_{n_k}\right)\boldsymbol{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\boldsymbol{\Upsilon}}_k && (61)\\
&= \exp\left(Ad_{\boldsymbol{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{n_k}\right)\underbrace{\boldsymbol{\Gamma}_k\boldsymbol{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\boldsymbol{\Upsilon}}_k}_{\hat{\mathbf{x}}_{n_{k+1}}} && (62)\\
&= \exp\left(\boldsymbol{\Phi}_{nn}\delta\mathbf{x}_{n_k}\right)\hat{\mathbf{x}}_{n_{k+1}} && (63)
\end{aligned}
$$

Then, we can have $\mathbf{\Phi}_{nn}$ as:

$$\mathbf{\Phi}_{nn} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 & \mathbf{I}_3 & \mathbf{I}_3 \delta t \\ \lfloor {}^G\mathbf{g} \rfloor \delta t & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{64}$$

For computing $\mathbf{\Phi}_{nb}$ and $\mathbf{G}_{nn}$, we need first linearize $\Delta\mathbf{R}_k$, $\Delta\mathbf{p}_k$ and $\Delta\mathbf{v}_k$. We first integrate the $\Delta\mathbf{R}_k$ and get:

$$\begin{aligned} \Delta\mathbf{R}_k &= \exp({}^{I_k}\boldsymbol{\omega}\delta t) && (65) \\ &= \exp(({}^{I_k}\hat{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_g - \mathbf{n}_{dg})\delta t) && (66) \\ &\simeq \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta t) \cdot \underbrace{\exp(-\mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta t)(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg})\delta t)}_{\Delta\tilde{\mathbf{R}}_k} && (67) \\ &\triangleq \Delta\hat{\mathbf{R}}_k \cdot \Delta\tilde{\mathbf{R}}_k && (68) \end{aligned}$$

For the integration of $\Delta\mathbf{p}_k$, we get:

$$\begin{aligned} \Delta\mathbf{p}_k &= \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\boldsymbol{\omega}\delta\tau) d\tau ds \cdot {}^{I_k}\mathbf{a} && (69) \\ &= \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp(({}^{I_k}\hat{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_g - \mathbf{n}_g)\delta\tau) d\tau ds \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (70) \\ &\simeq \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau) \exp(-\mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\tilde{\mathbf{b}}_g + \mathbf{n}_g)\delta\tau) d\tau ds \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (71) \\ &\simeq \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\mathbf{I} - \lfloor \mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\tilde{\mathbf{b}}_g + \mathbf{n}_g)\delta\tau \rfloor) d\tau ds \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (72) \\ &\simeq \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau) d\tau ds \cdot {}^{I_k}\hat{\mathbf{a}} \\ &\quad + \underbrace{\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau) \lfloor {}^{I_k}\hat{\mathbf{a}} \rfloor \mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau) d\tau ds}_{\mathbf{\Xi}_4} (\tilde{\mathbf{b}}_g + \mathbf{n}_{dg}) \\ &\quad - \underbrace{\int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau) d\tau ds}_{\mathbf{\Xi}_2} (\tilde{\mathbf{b}}_a + \mathbf{n}_{da}) \\ &= \Delta\hat{\mathbf{p}}_k + \underbrace{\mathbf{\Xi}_4(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg}) - \mathbf{\Xi}_2(\tilde{\mathbf{b}}_a + \mathbf{n}_{da})}_{\Delta\tilde{\mathbf{p}}_k} && (73) \\ &= \Delta\hat{\mathbf{p}}_k + \Delta\tilde{\mathbf{p}}_k && (74) \end{aligned}$$

where $\delta\tau = \tau - t_k$; $\Delta\hat{\boldsymbol{\theta}}_k = {}^{I_k}\hat{\boldsymbol{\omega}}\delta t$; $\boldsymbol{\Xi}_i, i = 1 \ldots 4$ can be computed in Appendix B or ACI^2 [2]. For the integration of $\Delta\mathbf{v}_k$, we get:

$$
\begin{aligned}
\Delta\mathbf{v}_k &= \int_{t_k}^{t_{k+1}} \exp({}^{I_k}\boldsymbol{\omega}\delta\tau)d\tau \cdot {}^{I_k}\mathbf{a} && (75)\\
&= \int_{t_k}^{t_{k+1}} \exp(({}^{I_k}\hat{\boldsymbol{\omega}} - \tilde{\mathbf{b}}_g - \mathbf{n}_g)\delta\tau)d\tau \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (76)\\
&\simeq \int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)\exp(-\mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\tilde{\mathbf{b}}_g + \mathbf{n}_g)\delta\tau)d\tau \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (77)\\
&\simeq \int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\mathbf{I} - \lfloor \mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)(\tilde{\mathbf{b}}_g + \mathbf{n}_g)\delta\tau\rfloor)d\tau \cdot ({}^{I_k}\hat{\mathbf{a}} - \tilde{\mathbf{b}}_a - \mathbf{n}_{da}) && (78)\\
&\simeq \int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)d\tau \cdot {}^{I_k}\hat{\mathbf{a}}\\
&\quad + \underbrace{\int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)\lfloor {}^{I_k}\hat{\mathbf{a}}\rfloor \mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)d\tau}_{\boldsymbol{\Xi}_3}(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg})\\
&\quad - \underbrace{\int_{t_k}^{t_{k+1}} \exp({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau)d\tau}_{\boldsymbol{\Xi}_1}(\tilde{\mathbf{b}}_a + \mathbf{n}_{da})\\
&= \Delta\hat{\mathbf{v}}_k + \underbrace{\boldsymbol{\Xi}_3(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg}) - \boldsymbol{\Xi}_1(\tilde{\mathbf{b}}_a + \mathbf{n}_{da})}_{\Delta\tilde{\mathbf{v}}_k} && (79)\\
&= \Delta\hat{\mathbf{v}}_k + \Delta\tilde{\mathbf{v}}_k && (80)
\end{aligned}
$$

Note that, by using ACI^2, we have defined:

$$
\begin{aligned}
\Delta\tilde{\mathbf{R}}_k &= \exp(-\mathbf{J}_r({}^{I_k}\hat{\boldsymbol{\omega}}\delta t)(\tilde{\mathbf{b}}_g + \mathbf{n}_g)\delta t) && (81)\\
\Delta\tilde{\mathbf{p}}_k &= \boldsymbol{\Xi}_4(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg}) - \boldsymbol{\Xi}_2(\tilde{\mathbf{b}}_a + \mathbf{n}_{da}) && (82)\\
\Delta\tilde{\mathbf{v}}_k &= \boldsymbol{\Xi}_3(\tilde{\mathbf{b}}_g + \mathbf{n}_{dg}) - \boldsymbol{\Xi}_1(\tilde{\mathbf{b}}_a + \mathbf{n}_{da}) && (83)
\end{aligned}
$$

Hence, we can further write $\boldsymbol{\Upsilon}_k$ as:

$$
\begin{aligned}
\boldsymbol{\Upsilon}_k &= \hat{\boldsymbol{\Upsilon}}_k\tilde{\boldsymbol{\Upsilon}}_k && (84)\\
&= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k & \Delta\hat{\mathbf{p}}_k & \Delta\hat{\mathbf{v}}_k \\ \mathbf{0}_{1\times3} & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{R}}_k & \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k & \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \\ \mathbf{0}_{1\times3} & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 \end{bmatrix} && (85)\\
&= \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\boldsymbol{\theta}_k)\Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\boldsymbol{\theta}_k)\Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (86)\\
&\simeq \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (87)\\
&= \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \end{bmatrix} \begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\tilde{\mathbf{p}}_k \\ \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (88)
\end{aligned}
$$

where $\Delta\tilde{\mathbf{R}}_k \triangleq \exp(\delta\Delta\boldsymbol{\theta}_k)$. Note that:

$$\begin{bmatrix}\delta\Delta\boldsymbol{\theta}_k \\ \Delta\tilde{\mathbf{p}}_k \\ \Delta\tilde{\mathbf{v}}_k\end{bmatrix} = \begin{bmatrix}-\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 \\ \boldsymbol{\Xi}_4 & -\boldsymbol{\Xi}_2 \\ \boldsymbol{\Xi}_3 & -\boldsymbol{\Xi}_1\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{b}}_g + \mathbf{n}_{dg} \\ \tilde{\mathbf{b}}_a + \mathbf{n}_{da}\end{bmatrix} \tag{89}$$

Then, we have:

$$\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} = \boldsymbol{\Gamma}_k\boldsymbol{\Psi}(\mathbf{x}_{n_k})\hat{\boldsymbol{\Upsilon}}_k\tilde{\boldsymbol{\Upsilon}}_k \tag{90}$$

$$\Rightarrow \exp(\delta\mathbf{x}_{n_{k+1}}) = \hat{\mathbf{x}}_{n_{k+1}}\tilde{\boldsymbol{\Upsilon}}_k\hat{\mathbf{x}}_{n_{k+1}}^{-1} \tag{91}$$

By plugging in the above derivations, we can have:

$$\exp(\delta\mathbf{x}_{n_{k+1}}) = \exp\left(Ad_{\hat{\mathbf{x}}_{n_{k+1}}}\begin{bmatrix}\delta\Delta\boldsymbol{\theta}_k \\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k\end{bmatrix}\right) \tag{92}$$

$$= \exp\left(Ad_{\hat{\mathbf{x}}_{n_{k+1}}}\begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top\end{bmatrix}\begin{bmatrix}-\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 \\ \boldsymbol{\Xi}_4 & -\boldsymbol{\Xi}_2 \\ \boldsymbol{\Xi}_3 & -\boldsymbol{\Xi}_1\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{b}}_g + \mathbf{n}_{dg} \\ \tilde{\mathbf{b}}_a + \mathbf{n}_{da}\end{bmatrix}\right) \tag{93}$$

Hence, we have $\boldsymbol{\Phi}_{nb} = \mathbf{G}_{nn}$ and :

$$\boldsymbol{\Phi}_{nb} = \begin{bmatrix}-{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 \\ -\lfloor{}^G\mathbf{p}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 \\ -\lfloor{}^G\mathbf{v}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1\end{bmatrix} \tag{94}$$

Therefore, we have the full state transition matrix as:

$$\boldsymbol{\Phi}_{k+1,k} = \begin{bmatrix}\boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6\end{bmatrix} \tag{95}$$

$$= \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \boldsymbol{\Phi}_{14} & \mathbf{0}_3 \\ \frac{1}{2}\lfloor{}^G\mathbf{g}\rfloor\delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & \boldsymbol{\Phi}_{24} & \boldsymbol{\Phi}_{25} \\ \lfloor{}^G\mathbf{g}\rfloor\delta t & \mathbf{0}_3 & \mathbf{I}_3 & \boldsymbol{\Phi}_{34} & \boldsymbol{\Phi}_{35} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \tag{96}$$

$$= \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 \\ \frac{1}{2}\lfloor{}^G\mathbf{g}\rfloor\delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & -\lfloor{}^G\hat{\mathbf{p}}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 \\ \lfloor{}^G\mathbf{g}\rfloor\delta t & \mathbf{0}_3 & \mathbf{I}_3 & -\lfloor{}^G\hat{\mathbf{v}}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \tag{97}$$

And the noise Jacobians:

$$\mathbf{G}_{nk} = \begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times 6} \\ \mathbf{0}_{6\times 6} & \mathbf{I}_6\delta t \end{bmatrix} \tag{98}$$

$$= \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \end{bmatrix} \tag{99}$$

$$= \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{p}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{v}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \end{bmatrix} \tag{100}$$

5 RI-VINS

5.1 State Vector

If with features, the whole state vector becomes:

$$\mathbf{x} = \left(\mathbf{x}_I, {}^G\mathbf{p}_f\right) \tag{101}$$

The feature state needs to be coupled with the $SE_2(3)$ of the current IMU state for the consistency performances. Hence, we define $\mathbf{x}_{nf}$ as:

$$\mathbf{x}_{nf} = \left({}^G_I\mathbf{R}, {}^G\mathbf{p}_I, {}^G\mathbf{v}_I, {}^G\mathbf{p}_f\right) \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & {}^G\mathbf{p}_I & {}^G\mathbf{v}_I & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times 3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 0 & 1 \end{bmatrix} \tag{102}$$

The adjoint operation is defined similar with Eq.(7) as

$$Ad_{\mathbf{x}_{nf}} \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{p}_I \rfloor {}^G_I\mathbf{R} & {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{v}_I \rfloor {}^G_I\mathbf{R} & \mathbf{0}_3 & {}^G_I\mathbf{R} & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{p}_f \rfloor {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_I\mathbf{R} \end{bmatrix} \tag{103}$$

We can define the $\boxplus$ for $\mathbf{x}_{nf}$ using right invariant errors as:

$$\delta\mathbf{x}_{nf} = \begin{bmatrix} \delta\boldsymbol{\theta}_I^\top & \delta\mathbf{p}^\top & \delta\mathbf{v}^\top & \delta\mathbf{p}_f^\top \end{bmatrix}^\top \tag{104}$$

$$\mathbf{x}_{nf} = \hat{\mathbf{x}}_{nf} \boxplus \delta\mathbf{x}_{nf} = \exp(\delta\mathbf{x}_{nf})\hat{\mathbf{x}}_{nf} \tag{105}$$

Then, the $\boxplus$ operation for the IMU state with feature state is:

$$\delta\mathbf{x} = \begin{bmatrix}\delta\boldsymbol{\theta}_I^\top & \delta\mathbf{p}^\top & \delta\mathbf{v}^\top & \delta\mathbf{b}_g^\top & \delta\mathbf{b}_a^\top & \delta\mathbf{p}_f^\top\end{bmatrix}^\top \tag{106}$$

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} = (\exp(\delta\mathbf{x}_{nf})\hat{\mathbf{x}}_{nf}, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b) \tag{107}$$

$$= \begin{bmatrix}\exp(\delta\boldsymbol{\theta}_I)^G_I\mathbf{R} \\ \exp(\delta\boldsymbol{\theta}_I)^G\hat{\mathbf{p}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ \exp(\delta\boldsymbol{\theta}_I)^G\hat{\mathbf{v}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ \exp(\delta\boldsymbol{\theta}_I)^G\hat{\mathbf{p}}_f + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p}_f\end{bmatrix} \tag{108}$$

5.2 State Transition and Covariance Propagation

With slight abuse of notation, we have:

$$\mathbf{x}_{nf_{k+1}} = \boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\mathbf{x}_{nf_k})\boldsymbol{\Upsilon}_k \tag{109}$$

with:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix}\mathbf{I}_3 & \frac{1}{2}{}^G\mathbf{g}\delta t^2 & \mathbf{g}\delta t & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1\end{bmatrix} \tag{110}$$

$$\boldsymbol{\Psi}(\mathbf{x}_{nf_k}) = \begin{bmatrix}{}^G_{I_k}\mathbf{R} & {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t & {}^G\mathbf{v}_{I_k} & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1\end{bmatrix} \tag{111}$$

$$\boldsymbol{\Upsilon}_k = \begin{bmatrix}\Delta\mathbf{R}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1\end{bmatrix} \tag{112}$$

Hence, we can write $\boldsymbol{\Upsilon}_k$ as:

$$\boldsymbol{\Upsilon}_k = \hat{\boldsymbol{\Upsilon}}_k \tilde{\boldsymbol{\Upsilon}}_k \tag{113}$$

$$= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k & \Delta\hat{\mathbf{p}}_k & \Delta\hat{\mathbf{v}}_k & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{R}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \tag{114}$$

$$= \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\boldsymbol{\theta}_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\boldsymbol{\theta}_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ \mathbf{0}_{3\times1} \end{bmatrix}\right) \tag{115}$$

$$\simeq \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ \mathbf{0}_{3\times1} \end{bmatrix}\right) \tag{116}$$

$$\simeq \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\tilde{\mathbf{p}}_k \\ \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) \tag{117}$$

Following similar procedures, we have:

$$\exp(\delta\mathbf{x}_{nf_{k+1}})\hat{\mathbf{x}}_{nf_{k+1}} = \boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_{nf_k})\hat{\mathbf{x}}_{nf_k})\hat{\boldsymbol{\Upsilon}}_k \tilde{\boldsymbol{\Upsilon}}_k \tag{118}$$

$$\simeq \exp\left(Ad_{\boldsymbol{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{nf_k}\right)\boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\hat{\mathbf{x}}_{nf_k})\hat{\boldsymbol{\Upsilon}}_k \tilde{\boldsymbol{\Upsilon}}_k \tag{119}$$

$$\Rightarrow \exp(\delta\mathbf{x}_{nf_{k+1}}) = \exp\left(Ad_{\boldsymbol{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{nf_k}\right)\hat{\mathbf{x}}_{nf_{k+1}}\tilde{\boldsymbol{\Upsilon}}_k \hat{\mathbf{x}}_{nf_{k+1}}^{-1} \tag{120}$$

Note that for the new state, the $\mathbf{F}$ is redefined as:

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \delta t\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{121}$$

By plugging in the above derivations, we can have:

$$\hat{\mathbf{x}}_{nf_{k+1}}\tilde{\boldsymbol{\Upsilon}}_k \hat{\mathbf{x}}_{nf_{k+1}}^{-1} = \exp\left(Ad_{\hat{\mathbf{x}}_{nf_{k+1}}} \begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ \mathbf{0}_{3\times1} \end{bmatrix}\right) \tag{122}$$

$$= \exp\left(Ad_{\hat{\mathbf{x}}_{nf_{k+1}}} \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} -\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 \\ \boldsymbol{\Xi}_4 & -\boldsymbol{\Xi}_2 \\ \boldsymbol{\Xi}_3 & -\boldsymbol{\Xi}_1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_g + \mathbf{n}_{dg} \\ \tilde{\mathbf{b}}_a + \mathbf{n}_{da} \end{bmatrix}\right) \tag{123}$$

Following similar steps, we can get the linearized state propagation matrix as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \\ \delta\mathbf{p}_{f,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \boldsymbol{\Phi}_{fb} & \mathbf{I}_3 \end{bmatrix}}_{\boldsymbol{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \\ \delta\mathbf{p}_{f,k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times6} \\ \mathbf{0}_{6\times6} & \mathbf{I}_6\delta t \\ \mathbf{G}_{fn} & \mathbf{0}_{3\times6} \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{124}$$

where $\mathbf{\Phi}_{nn}$, $\mathbf{\Phi}_{nb}$ and $\mathbf{G}_{nn}$ are the same as previous section, and we have $\mathbf{\Phi}_{fb} = \mathbf{G}_{fn}$ as:

$$\mathbf{\Phi}_{fb} = \begin{bmatrix} -\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 \end{bmatrix} \tag{125}$$

Hence, the full state transition matrix can be written as:

$$\mathbf{\Phi}_{k+1,k} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \mathbf{\Phi}_{fb} & \mathbf{I}_3 \end{bmatrix} \tag{126}$$

$$= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{14} & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & \mathbf{\Phi}_{24} & \mathbf{\Phi}_{25} & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g} \rfloor \delta t & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{34} & \mathbf{\Phi}_{35} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{64} & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{127}$$

$$= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & -\lfloor {}^G\mathbf{p}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_2 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{g} \rfloor \delta t & \mathbf{0}_3 & \mathbf{I}_3 & -\lfloor {}^G\mathbf{v}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_1 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{128}$$

while the noise Jacobians become:

$$\mathbf{G}_{nk} = \begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times6} \\ \mathbf{0}_6 & \mathbf{I}_6\delta t \\ \mathbf{G}_{fn} & \mathbf{0}_{3\times6} \end{bmatrix} \tag{129}$$

$$= \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{G}_{61} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{130}$$

$$= \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{p}}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ -\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G_{I_{k+1}}\mathbf{R}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{131}$$

5.3 State Propagation Computation Analysis

Assuming we have l feature in the state vector, the state vector can be written as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_I^\top & \mathbf{x}_f^\top \end{bmatrix}^\top \tag{132}$$

$$\mathbf{x}_f = \begin{bmatrix} {}^G\mathbf{p}_{f0}^\top & \cdots & {}^G\mathbf{p}_{fl}^\top \end{bmatrix}^\top \tag{133}$$

In order simplify the computation analysis, we rewrite the system linearized equation as:

$$\begin{bmatrix} \delta\mathbf{x}_{I_{k+1}} \\ \delta\mathbf{x}_{f,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{\Phi}_{II} & \mathbf{0}_{15\times 3l} \\ \mathbf{\Phi}_{fI} & \mathbf{I}_{3l\times 3l} \end{bmatrix}}_{\mathbf{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{I_k} \\ \delta\mathbf{x}_{f,k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_I \\ \mathbf{G}_f \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{134}$$

If we denote the covariance of the noise as $\mathbf{Q}_d$, then we can get the state covariance propagation as:

$$\mathbf{P}_{k+1} = \mathbf{\Phi}_{k+1,k} \begin{bmatrix} \mathbf{P}_{II} & \mathbf{P}_{If} \\ \mathbf{P}_{fI} & \mathbf{P}_{ff} \end{bmatrix} \mathbf{\Phi}_{k+1,k}^\top + \mathbf{G}_{nk}\mathbf{Q}_d\mathbf{G}_{nk}^\top \tag{135}$$

$$\begin{aligned} &= \begin{bmatrix} \mathbf{\Phi}_{II}\mathbf{P}_{II}\mathbf{\Phi}_{II}^\top & \mathbf{\Phi}_{II}\mathbf{P}_{II}\mathbf{\Phi}_{If} + \mathbf{\Phi}_{II}\mathbf{P}_{If} \\ \mathbf{\Phi}_{fI}\mathbf{P}_{II}\mathbf{\Phi}_{II}^\top + \mathbf{P}_{fI}\mathbf{\Phi}_{II}^\top & \mathbf{\Phi}_{fI}\mathbf{P}_{II}\mathbf{\Phi}_{If} + \mathbf{\Phi}_{fI}\mathbf{P}_{If} + \mathbf{P}_{fI}\mathbf{\Phi}_{If} + \mathbf{P}_{ff} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{G}_I\mathbf{Q}_d\mathbf{G}_I^\top & \mathbf{G}_I\mathbf{Q}_d\mathbf{G}_f^\top \\ \mathbf{G}_f\mathbf{Q}_d\mathbf{G}_I^\top & \mathbf{G}_f\mathbf{Q}_d\mathbf{G}_f^\top \end{bmatrix} \end{aligned} \tag{136}$$

We can see that the terms $\mathbf{\Phi}_{fI}\mathbf{P}_{II}\mathbf{\Phi}_{If}$, $\mathbf{\Phi}_{fI}\mathbf{P}_{If} = (\mathbf{P}_{fI}\mathbf{\Phi}_{If})^\top$ and $\mathbf{G}_f\mathbf{Q}_d\mathbf{G}_f^\top$ all need $\mathbf{O}(l^2)$ flops for computation. Hence, we can see that the total computation for $\mathbf{P}_{k+1}$ is $\mathbf{O}(l^2)$.

5.4 Feature Update

The visual measurements can be written as:

$$\mathbf{z} = \begin{bmatrix} \frac{x}{z} & \frac{y}{z} \end{bmatrix}^\top \tag{137}$$

$${}^C\mathbf{p}_f = \begin{bmatrix} x & y & z \end{bmatrix}^\top = {}^C_I\mathbf{R}{}^I_G\mathbf{R}({}^G\mathbf{p}_f - {}^G\mathbf{p}_I) + {}^C\mathbf{p}_I \tag{138}$$

Since the feature is coupled with the current IMU pose, hence, the error states can be written as:

$${}^G\mathbf{p}_f \simeq \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_f + \delta\tilde{\mathbf{p}}_f \tag{139}$$

$${}^G\mathbf{p}_I \simeq \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_I + \delta\tilde{\mathbf{p}} \tag{140}$$

$${}^G_I\mathbf{R} = \exp(\delta\boldsymbol{\theta}_I){}^G_I\hat{\mathbf{R}} \tag{141}$$

The Jacobians $\mathbf{H} \triangleq \frac{\partial\delta\mathbf{z}}{\partial\delta\mathbf{x}}$ as:

$$\mathbf{H}_C = \frac{\partial\delta\mathbf{z}}{\partial\delta{}^C\mathbf{p}_f} = \frac{1}{z^2}\begin{bmatrix} z & 0 & -x \\ 0 & z & -y \end{bmatrix} \tag{142}$$

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_C \begin{bmatrix} \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_I} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{p}} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{v}} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{b}_g} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{b}_a} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{p}_f} \end{bmatrix} \\ &= \mathbf{H}_C \begin{bmatrix} \mathbf{0}_3 & -{}^C_I\hat{\mathbf{R}}{}^I_G\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & {}^C_I\hat{\mathbf{R}}{}^I_G\hat{\mathbf{R}} \end{bmatrix} \end{aligned} \tag{143}$$

where $\mathbf{H}_C$ represents the projection Jacobians. Following [3], the observabiltiy matrix for the linearized system is:

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \\ \vdots \\ \mathcal{O}_k \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1\mathbf{\Phi}_{1,0} \\ \vdots \\ \mathbf{H}_k\mathbf{\Phi}_{k,0} \end{bmatrix} \tag{144}$$

Each block row $\mathcal{O}_k$ can be computed as:

$$\mathcal{O}_k = \mathbf{H}_k \mathbf{\Phi}_{k,0} = \mathbf{H}_C {}_I^C\hat{\mathbf{R}} {}_G^I\hat{\mathbf{R}} \begin{bmatrix} \mathbf{M}_1 & -\mathbf{I}_3 & -\mathbf{I}_3\delta t & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{I}_3 \end{bmatrix} \tag{145}$$

where:

$$\mathbf{M}_1 = \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 \tag{146}$$

$$\mathbf{M}_2 = \lfloor {}^G\hat{\mathbf{p}}_{I_k} \rfloor {}_{I_k}^G\hat{\mathbf{R}} \mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t - {}_{I_0}^G\hat{\mathbf{R}}\mathbf{\Xi}_4 \tag{147}$$

$$\mathbf{M}_3 = {}_{I_0}^G\hat{\mathbf{R}}\mathbf{\Xi}_2 \tag{148}$$

we can directly find the null space as:

$$\mathbf{N} = \begin{bmatrix} {}^G\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{3\times1} & \mathbf{I}_3 \\ \mathbf{0}_{9\times1} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{3\times1} & \mathbf{I}_3 \end{bmatrix} \tag{149}$$

The null space is invariant of the state estimates.

6 DRI-VINS

We want decouple the feature from state covariance propagation.

6.1 DRI-FEJ

The state is still defined as:

$$\mathbf{x} = \left(\mathbf{x}_I, {}^G\mathbf{p}_f\right) \tag{150}$$

However, the overall error states are defined as:

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} = \left(\exp(\delta\mathbf{x}_n)\hat{\mathbf{x}}_n, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b, {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f\right) \tag{151}$$

$$= \begin{bmatrix} \exp(\delta\boldsymbol{\theta}_I){}_I^G\hat{\mathbf{R}} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{v}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f \end{bmatrix} \tag{152}$$

6.1.1 State Propagation

Following similar steps, we can get the full state transition matrix as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \\ \delta\mathbf{p}_{f,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times6} & \mathbf{I}_3 \end{bmatrix}}_{\mathbf{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \\ \delta\mathbf{p}_{f,k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times6} \\ \mathbf{0}_{6\times6} & \mathbf{I}_6\delta t \\ \mathbf{0}_{3\times6} & \mathbf{0}_{3\times6} \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{153}$$

while the new noise Jacobians become:

$$\mathbf{G}_{nk} = \begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times 6} \\ \mathbf{0}_6 & \mathbf{I}_6\delta t \\ \mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 6} \end{bmatrix} \tag{154}$$

$$= \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{155}$$

$$= \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{p}}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\hat{\mathbf{v}}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{156}$$

6.1.2 State Propagation Computation

Assuming we have l features in the state vector, the state vector can be written as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_I^\top & \mathbf{x}_f^\top \end{bmatrix}^\top \tag{157}$$

$$\mathbf{x}_f = \begin{bmatrix} {}^G\mathbf{p}_{f0}^\top & \cdots & {}^G\mathbf{p}_{fl}^\top \end{bmatrix}^\top \tag{158}$$

In order simplify the computation analysis, we rewrite the system linearized equation as:

$$\begin{bmatrix} \delta\mathbf{x}_{I_{k+1}} \\ \delta\mathbf{x}_{f,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{\Phi}_{II} & \mathbf{0}_{15\times 3l} \\ \mathbf{0}_{3l\times 15} & \mathbf{I}_{3l\times 3l} \end{bmatrix}}_{\boldsymbol{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{I_k} \\ \delta\mathbf{x}_{f,k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_I \\ \mathbf{0}_{3l\times 12} \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{159}$$

If we denote the covariance of the noise as $\mathbf{Q}_d$, then we can get the state covariance propagation as:

$$\mathbf{P}_{k+1} = \boldsymbol{\Phi}_{k+1,k} \begin{bmatrix} \mathbf{P}_{II} & \mathbf{P}_{If} \\ \mathbf{P}_{fI} & \mathbf{P}_{ff} \end{bmatrix} \boldsymbol{\Phi}_{k+1,k}^\top + \mathbf{G}_{nk}\mathbf{Q}_d\mathbf{G}_{nk}^\top \tag{160}$$

$$= \begin{bmatrix} \boldsymbol{\Phi}_{II}\mathbf{P}_{II}\boldsymbol{\Phi}_{II}^\top & \boldsymbol{\Phi}_{II}\mathbf{P}_{If} \\ \mathbf{P}_{fI}\boldsymbol{\Phi}_{II}^\top & \mathbf{P}_{ff} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_I\mathbf{Q}_d\mathbf{G}_I^\top & \mathbf{0}_{15\times 3l} \\ \mathbf{0}_{3l\times 15} & \mathbf{0}_{3l\times 3l} \end{bmatrix} \tag{161}$$

We can see that the terms $\mathbf{P}_{II}\boldsymbol{\Phi}_{If} = \left(\boldsymbol{\Phi}_{fI}\mathbf{P}_{II}^\top\right)^\top$ all need $\mathbf{O}(l)$ flops for computation. Hence, we can see that the total computation for $\mathbf{P}_{k+1}$ are $\mathbf{O}(l)$, which is much smaller than that of RI-VINs.

6.1.3 Visual Update

If updating with current pose, the error states can be written as:

$${}^G\mathbf{p}_f \simeq {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f \tag{162}$$

$${}^G\mathbf{p}_I \simeq \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_I + \delta\mathbf{p} \tag{163}$$

$${}^G_I\mathbf{R} = \exp(\delta\boldsymbol{\theta}_I){}^G_I\hat{\mathbf{R}} \tag{164}$$

Then, we can get the Jacobians as:

$$\mathbf{H} = \mathbf{H}_C {}^C_I\hat{\mathbf{R}} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \lfloor {}^G\hat{\mathbf{p}}_f \rfloor & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{165}$$

6.1.4 Observability Analysis

It is nice that we don't have extra term in the Jacobians related to the current IMU state, how ever the measurements are state related, which will not preserve the observability property.

$$\mathcal{O}_k = \mathbf{H}_k \mathbf{\Phi}_{k,0} = \mathbf{H}_C {}^C_I\hat{\mathbf{R}} {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \mathbf{M}_1 & -\mathbf{I}_3 & -\mathbf{I}_3\delta t & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{I}_3 \end{bmatrix} \tag{166}$$

where we have:

$$\mathbf{M}_1 = \lfloor {}^G\hat{\mathbf{p}}_f \rfloor - \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 \tag{167}$$

$$\mathbf{M}_2 = \lfloor {}^G\hat{\mathbf{p}}_f \rfloor \mathbf{\Phi}_{14} - \mathbf{\Phi}_{24} \tag{168}$$

$$\mathbf{M}_3 = -\mathbf{\Phi}_{25} \tag{169}$$

Hence, we can get the null space for the decoupled DRI-VINS as:

$$\mathbf{N} = \begin{bmatrix} {}^G\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{3\times 1} & \mathbf{I}_3 \\ \mathbf{0}_{9\times 1} & \mathbf{0}_{9\times 3} \\ -\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G\mathbf{g} & \mathbf{I}_3 \end{bmatrix} \tag{170}$$

We can see that the unobservable subspace is affected by feature estimate ${}^G\hat{\mathbf{p}}_f$. Due to linearization point change, the DRI-VINS will become inconsistent.

6.1.5 DRI-FEJ Algorithm

In order to keep the observability property of the system, we can use the FEJ. When we do update, we just use the first estimates of the features, ${}^G\hat{\mathbf{p}}_f$, for the Jacobians computation.

The decoupled RI-EKF can easily handle more features in the state vector without involving them in the state covariance propagation. Due to the nice property of $SE_2(3)$ with right invariant error, the state propagation will automatically keep the system unobservabilty property for IMU. Hence, we only need to FEJ the feature state. In the meantime, the FEJ in the update Jacobians is pretty small and light-weight.

6.2 DRI-SW

Instead of associating features with current IMU state for Lie group representation, we can associate feature state with an active clone pose if a sliding-window is maintained.

6.2.1 State Vector

Assuming the state vector contains the IMU navigation state $\mathbf{x}_n$, the bias state $\mathbf{x}_b$, one feature point ${}^G\mathbf{p}_f$ and a sliding window state $\mathbf{x}_c$ (containing two poses for simplicity):

$$\mathbf{x} = \left(\mathbf{x}_n, \mathbf{x}_b, \mathbf{x}_c, {}^G\mathbf{p}_f\right) \tag{171}$$

$$\mathbf{x}_c = \left({}^G_{I_{c0}}\mathbf{R}, {}^G\mathbf{p}_{I_{c0}}, {}^G_{I_{c1}}\mathbf{R}, {}^G\mathbf{p}_{I_{c1}}\right) \tag{172}$$

Instead of decoupling the feature ${}^G\mathbf{p}_f$ from Lie group as DRI-FEJ, we associate the feature with the first clone pose $\{I_{c0}\}$ for the Lie group representation. Hence, we can define:

$$\mathbf{x}_{I_{c0},f} = \left(\mathbf{x}_{I_{c0}}, {}^G\mathbf{p}_f\right) = \begin{bmatrix} {}^G_{I_{c0}}\mathbf{R} & {}^G\mathbf{p}_{I_{c0}} & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{173}$$

Thus, the new error states $\delta\mathbf{x} \in \mathbb{R}^{30}$ can be defined as:

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{x}_n^\top & \delta\mathbf{x}_b^\top & \delta\mathbf{x}_c^\top & \delta\mathbf{p}_f^\top \end{bmatrix}^\top \tag{174}$$

$$\delta\mathbf{x}_c = \begin{bmatrix} \delta\mathbf{x}_{c0}^\top & | & \delta\mathbf{x}_{c1}^\top \end{bmatrix}^\top = \begin{bmatrix} \delta\boldsymbol{\theta}_{I_{c0}}^\top & \delta\mathbf{p}_{I_{c0}}^\top & | & \delta\boldsymbol{\theta}_{I_{c1}}^\top & \delta\mathbf{p}_{I_{c1}}^\top \end{bmatrix}^\top \tag{175}$$

with new $\boxplus$ for DRI-SW as:

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} \tag{176}$$

$$= \left(\exp\left(\delta\mathbf{x}_n\right)\hat{\mathbf{x}}_n, \mathbf{x}_b + \delta\mathbf{x}_b, \exp\left(\begin{bmatrix} \delta\mathbf{x}_{I_{c0}} \\ \delta\mathbf{p}_f \end{bmatrix}\right) \cdot \mathbf{x}_{I_{c0},f}, \exp(\delta\mathbf{x}_{I_{c1}}) \cdot \mathbf{x}_{I_{c1}}\right) \tag{177}$$

$$= \begin{bmatrix} \exp(\delta\boldsymbol{\theta}_I){}^G_I\hat{\mathbf{R}} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{p}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ \exp(\delta\boldsymbol{\theta}_I){}^G\hat{\mathbf{v}}_I + \mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ \exp(\delta\boldsymbol{\theta}_{I_{c0}}){}^G_{I_{c0}}\hat{\mathbf{R}} \\ \exp(\delta\boldsymbol{\theta}_{I_{c0}}){}^G\hat{\mathbf{p}}_{I_{c0}} + \mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c0}})\delta\mathbf{p}_{I_{c0}} \\ \exp(\delta\boldsymbol{\theta}_{I_{c1}}){}^G_{I_{c1}}\hat{\mathbf{R}} \\ \exp(\delta\boldsymbol{\theta}_{I_{c1}}){}^G\hat{\mathbf{p}}_{I_{c1}} + \mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c1}})\delta\mathbf{p}_{I_{c1}} \\ \exp(\delta\boldsymbol{\theta}_{I_{c0}}){}^G\hat{\mathbf{p}}_f + \mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c0}})\delta\mathbf{p}_f \end{bmatrix} \tag{178}$$

6.2.2 State Propagation

The state transition matrix with the new error states can be written as:

$$\boldsymbol{\Phi}_{k+1,k} = \begin{bmatrix} \boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} & \mathbf{0}_{9\times 12} & \mathbf{0}_{9\times 3} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 & \mathbf{0}_{6\times 12} & \mathbf{0}_{6\times 3} \\ \mathbf{0}_{12\times 9} & \mathbf{0}_{12\times 6} & \mathbf{I}_{12} & \mathbf{0}_{12\times 3} \\ \mathbf{0}_{3\times 9} & \mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 12} & \mathbf{I}_3 \end{bmatrix} \tag{179}$$

It can be seen that the feature is decoupled from the IMU state covariance propagation with $\boldsymbol{\Phi}_{fb} = \mathbf{0}_{3\times 6}$, which is desired.

6.2.3 Visual Update

If we get the feature measurements at cloned pose $\{I_{ci}\}, i = \{0, 1\}$, the feature transformation can be written as:

$${}^{C_0}\mathbf{p}_f = {}^C_I\mathbf{R}^{I_{c0}}_G\mathbf{R}({}^G\mathbf{p}_f - {}^G\mathbf{p}_{I_{c0}}) + {}^C\mathbf{p}_I \tag{180}$$

$${}^{C_1}\mathbf{p}_f = {}^C_I\mathbf{R}^{I_{c1}}_G\mathbf{R}({}^G\mathbf{p}_f - {}^G\mathbf{p}_{I_{c1}}) + {}^C\mathbf{p}_I \tag{181}$$

The error states can be defined as:

$$ {}^{G}\mathbf{p}_f \simeq \exp(\delta\boldsymbol{\theta}_{I_{c0}})^{G}\hat{\mathbf{p}}_f + \delta\mathbf{p}_f \tag{182} $$

$$ {}^{G}\mathbf{p}_{I_{ci}} \simeq \exp(\delta\boldsymbol{\theta}_{I_{ci}})^{G}\hat{\mathbf{p}}_{I_{ci}} + \delta\mathbf{p}_{I_{ci}} \tag{183} $$

$$ {}^{G}_{I_{ci}}\mathbf{R} = \exp(\delta\boldsymbol{\theta}_{I_{ci}})^{G}_{I_{ci}}\hat{\mathbf{R}} \tag{184} $$

Hence, we can recompute the Jacobians with feature associated with $\{I_{c0}\}$ for Lie group representation as:

$$ \frac{\partial\delta^{C_0}\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_{I_{c0}}} = \mathbf{0}_3 \qquad \frac{\partial\delta^{C_1}\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_{I_{c1}}} = -{}^{C}_{I}\hat{\mathbf{R}}^{I_{c1}}_{G}\hat{\mathbf{R}}\lfloor{}^{G}\hat{\mathbf{p}}_f\rfloor \tag{185} $$

$$ \frac{\partial\delta^{C_0}\mathbf{p}_f}{\partial\delta\mathbf{p}_{I_{c0}}} = -{}^{C}_{I}\hat{\mathbf{R}}^{I_{c0}}_{G}\hat{\mathbf{R}} \qquad \frac{\partial\delta^{C_1}\mathbf{p}_f}{\partial\delta\mathbf{p}_{I_{c1}}} = -{}^{C}_{I}\hat{\mathbf{R}}^{I_{c1}}_{G}\hat{\mathbf{R}} \tag{186} $$

$$ \frac{\partial\delta^{C_0}\mathbf{p}_f}{\partial\delta\mathbf{p}_f} = {}^{C}_{I}\hat{\mathbf{R}}^{I_{c0}}_{G}\hat{\mathbf{R}} \qquad \frac{\partial\delta^{C_1}\mathbf{p}_f}{\partial\delta\mathbf{p}_f} = {}^{C}_{I}\hat{\mathbf{R}}^{I_{c1}}_{G}\hat{\mathbf{R}} \tag{187} $$

$$ \frac{\partial\delta^{C_1}\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_{I_{c1}}} = -\frac{\partial\delta^{C_1}\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_{I_{c0}}} \tag{188} $$

6.2.4 Pseudo-Anchor Change

The proposed pseudo-anchor change, which is different from anchor change in [4] which represents the feature from global frame {G} to local sensor frame $\{I\}$, refers to changing the associated Lie group poses for the feature while the feature is still represented in global frame $\{G\}$. In order to keep the long-tracked SLAM features in the state vector even when the associated cloned pose is marginalized from the sliding window, we can perform pseudo-anchor change, that is to "anchor" the feature to another active cloned pose in the sliding window for Lie group representation.

Since the feature state estimate ${}^{G}\hat{\mathbf{p}}_f$ remains the same, we only need to modify the covariance of the feature after pseudo-anchor change. The feature is associated with pose I_{c0} which is about to be marginalized from the state, and we need to associate feature with pose I_{c1} instead. If we define $\delta\mathbf{p}_{fi}$ as the error states for ${}^{G}\mathbf{p}_f$ when associated with cloned pose $\{I_{ci}\}$, we have:

$$ {}^{G}\mathbf{p}_f = \exp(\delta\boldsymbol{\theta}_{I_{c0}})^{G}\hat{\mathbf{p}}_f + \mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c0}})\delta\mathbf{p}_{f0} \tag{189} $$

$$ {}^{G}\mathbf{p}_f = \exp(\delta\boldsymbol{\theta}_{I_{c1}})^{G}\hat{\mathbf{p}}_f + \mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c1}})\delta\mathbf{p}_{f1} \tag{190} $$

Then the feature's new error state $\delta\mathbf{p}_{f1}$ is written as:

$$ \delta\mathbf{p}_{f1} = -\lfloor{}^{G}\hat{\mathbf{p}}_f\rfloor\delta\boldsymbol{\theta}_{I_{c0}} + \lfloor{}^{G}\hat{\mathbf{p}}_f\rfloor\delta\boldsymbol{\theta}_{I_{c1}} + \delta\mathbf{p}_{f0} \tag{191} $$

We then leverage EKF update to get the new state covariance after the pseudo-anchor change. We also show that the proposed pseudo-anchor change algorithm won't affect the system unobservable property.

7 LI-VINS

7.1 State Vector

Same as RI-VINS (101), the state vector becomes:

$$ \mathbf{x} = \left(\mathbf{x}_I, {}^{G}\mathbf{p}_f\right) \tag{192} $$

The feature state needs to be coupled with the $SE_2(3)$ of the current IMU state for the consistency performances. Hence, we define $\mathbf{x}_{nf}$ as:

$$\mathbf{x}_{nf} = \left({}^G_I\mathbf{R}, {}^G\mathbf{p}_I, {}^G\mathbf{v}_I, {}^G\mathbf{p}_f\right) \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & {}^G\mathbf{p}_I & {}^G\mathbf{v}_I & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \tag{193}$$

We can define the $\boxplus$ for $\mathbf{x}_{nf}$ using left invariant errors as:

$$\delta\mathbf{x}_{nf} = \begin{bmatrix} \delta\boldsymbol{\theta}_I^\top & \delta\mathbf{p}^\top & \delta\mathbf{v}^\top & \delta\mathbf{p}_f^\top \end{bmatrix}^\top \tag{194}$$

$$\mathbf{x}_{nf} = \hat{\mathbf{x}}_{nf} \boxplus \delta\mathbf{x}_{nf} = \hat{\mathbf{x}}_{nf}\exp(\delta\mathbf{x}_{nf}) \tag{195}$$

Then, the $\boxplus$ operation for the IMU state with feature state is:

$$\delta\mathbf{x} = \begin{bmatrix} \delta\boldsymbol{\theta}_I^\top & \delta\mathbf{p}^\top & \delta\mathbf{v}^\top & \delta\mathbf{b}_g^\top & \delta\mathbf{b}_a^\top & \delta\mathbf{p}_f^\top \end{bmatrix}^\top \tag{196}$$

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} = (\hat{\mathbf{x}}_{nf}\exp(\delta\mathbf{x}_{nf}), \hat{\mathbf{x}}_b + \delta\mathbf{x}_b) \tag{197}$$

$$= \begin{bmatrix} {}^G_I\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_I) \\ {}^G\hat{\mathbf{p}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ {}^G\hat{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ {}^G\hat{\mathbf{p}}_f + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p}_f \end{bmatrix} \tag{198}$$

7.2 State Transition and Covariance Propagation

Abusing the notation, we have:

$$\mathbf{x}_{nf_{k+1}} = \boldsymbol{\Gamma}_k\boldsymbol{\Psi}(\mathbf{x}_{nf_k})\boldsymbol{\Upsilon}_k \tag{199}$$

with (same as (110) - (112)):

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \mathbf{I}_3 & \frac{1}{2}{}^G\mathbf{g}\delta t^2 & \mathbf{g}\delta t & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \tag{200}$$

$$\boldsymbol{\Psi}(\mathbf{x}_{nf_k}) = \begin{bmatrix} {}^G_{I_k}\mathbf{R} & {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t & {}^G\mathbf{v}_{I_k} & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \tag{201}$$

$$\boldsymbol{\Upsilon}_k = \begin{bmatrix} \Delta\mathbf{R}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 & 0 & 0 \\ \mathbf{0}_{1\times3} & 0 & 1 & 0 \\ \mathbf{0}_{1\times3} & 0 & 0 & 1 \end{bmatrix} \tag{202}$$

where $\Delta\mathbf{R}_k$, $\Delta\mathbf{p}_k$, and $\Delta\mathbf{v}_k$ can be represented as (see (27), (29), and (31)):

$$\Delta\mathbf{R}_k = {}^G_{I_k}\mathbf{R}^\top {}^G_{I_{k+1}}\mathbf{R} \tag{203}$$

$$\Delta\mathbf{p}_k = {}^G_{I_k}\mathbf{R}^\top({}^G\mathbf{p}_{I_{k+1}} - {}^G\mathbf{p}_{I_k} - {}^G\mathbf{v}_{I_k}\delta t - \frac{1}{2}{}^G\mathbf{g}\delta t^2) \tag{204}$$

$$\Delta\mathbf{v}_k = {}^G_{I_k}\mathbf{R}^\top({}^G\mathbf{v}_{I_{k+1}} - {}^G\mathbf{v}_{I_k} - {}^G\mathbf{g}\delta t) \tag{205}$$

Following similar procedures, we have:

$$\hat{\mathbf{x}}_{nf_{k+1}} \exp(\delta\mathbf{x}_{nf_{k+1}}) = \mathbf{\Gamma}_k \mathbf{\Psi}(\hat{\mathbf{x}}_{nf_k} \exp(\delta\mathbf{x}_{nf_k}))\hat{\mathbf{\Upsilon}}_k \tilde{\mathbf{\Upsilon}}_k \tag{206}$$

$$= \mathbf{\Gamma}_k \mathbf{\Psi}(\hat{\mathbf{x}}_{nf_k})\mathbf{\Psi}(\exp(\delta\mathbf{x}_{nf_k}))\hat{\mathbf{\Upsilon}}_k \tilde{\mathbf{\Upsilon}}_k \tag{207}$$

$$\simeq \mathbf{\Gamma}_k \mathbf{\Psi}(\hat{\mathbf{x}}_{nf_k}) \exp(\mathbf{F}\delta\mathbf{x}_{nf_k})\hat{\mathbf{\Upsilon}}_k \tilde{\mathbf{\Upsilon}}_k \tag{208}$$

$$= \mathbf{\Gamma}_k \mathbf{\Psi}(\hat{\mathbf{x}}_{nf_k})\hat{\mathbf{\Upsilon}}_k \hat{\mathbf{\Upsilon}}_k^{-1} \exp(\mathbf{F}\delta\mathbf{x}_{nf_k})\hat{\mathbf{\Upsilon}}_k \tilde{\mathbf{\Upsilon}}_k \tag{209}$$

$$= \mathbf{\Gamma}_k \mathbf{\Psi}(\hat{\mathbf{x}}_{nf_k})\hat{\mathbf{\Upsilon}}_k \exp(Ad_{\hat{\mathbf{\Upsilon}}_k^{-1}}\mathbf{F}\delta\mathbf{x}_{nf_k})\tilde{\mathbf{\Upsilon}}_k \tag{210}$$

$$= \hat{\mathbf{x}}_{nf_{k+1}} \exp(Ad_{\hat{\mathbf{\Upsilon}}_k^{-1}}\mathbf{F}\delta\mathbf{x}_{nf_k})\tilde{\mathbf{\Upsilon}}_k \tag{211}$$

$$\Rightarrow \exp(\delta\mathbf{x}_{nf_{k+1}}) = \exp(\underbrace{Ad_{\hat{\mathbf{\Upsilon}}_k^{-1}}\mathbf{F}}_{\mathbf{\Phi}_{nfnf}}\,\delta\mathbf{x}_{nf_k})\tilde{\mathbf{\Upsilon}}_k \tag{212}$$

By plugging in the above derivations, we can have:

$$\mathbf{\Phi}_{nfnf} = Ad_{\hat{\mathbf{\Upsilon}}_k^{-1}}\mathbf{F} \tag{213}$$

$$= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\Delta\hat{\mathbf{R}}_k^\top \lfloor\Delta\hat{\mathbf{p}}_k\rfloor & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 & \mathbf{0}_3 \\ -\Delta\hat{\mathbf{R}}_k^\top \lfloor\Delta\hat{\mathbf{v}}_k\rfloor & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \delta t\mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{214}$$

$$\tilde{\mathbf{\Upsilon}}_k = \exp\left(\begin{bmatrix} \delta\Delta\boldsymbol{\theta}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ \mathbf{0}_{3\times1} \end{bmatrix}\right) \tag{215}$$

$$= \exp\left(\begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} -\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 \\ \mathbf{\Xi}_4 & -\mathbf{\Xi}_2 \\ \mathbf{\Xi}_3 & -\mathbf{\Xi}_1 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_g + \mathbf{n}_{dg} \\ \tilde{\mathbf{b}}_a + \mathbf{n}_{da} \end{bmatrix}\right) \tag{216}$$

Hence, the full state transition matrix can be written as (recycling the notations):

$$\mathbf{\Phi}_{k+1,k} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times6} & \mathbf{\Phi}_f \end{bmatrix} \tag{217}$$

$$= \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{14} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} & \mathbf{\Phi}_{23} & \mathbf{\Phi}_{24} & \mathbf{\Phi}_{25} & \mathbf{0}_3 \\ \mathbf{\Phi}_{31} & \mathbf{0}_3 & \mathbf{\Phi}_{33} & \mathbf{\Phi}_{34} & \mathbf{\Phi}_{35} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{66} \end{bmatrix} \tag{218}$$

$$= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 & \mathbf{0}_3 \\ -\Delta\hat{\mathbf{R}}_k^\top \lfloor\Delta\hat{\mathbf{p}}_k\rfloor & \Delta\hat{\mathbf{R}}_k^\top & \Delta\hat{\mathbf{R}}_k^\top\delta t & \Delta\hat{\mathbf{R}}_k^\top\mathbf{\Xi}_4 & -\Delta\hat{\mathbf{R}}_k^\top\mathbf{\Xi}_2 & \mathbf{0}_3 \\ -\Delta\hat{\mathbf{R}}_k^\top \lfloor\Delta\hat{\mathbf{v}}_k\rfloor & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top & \Delta\hat{\mathbf{R}}_k^\top\mathbf{\Xi}_3 & -\Delta\hat{\mathbf{R}}_k^\top\mathbf{\Xi}_1 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \Delta\hat{\mathbf{R}}_k^\top \end{bmatrix} \tag{219}$$

while the noise Jacobians become:

$$\mathbf{G}_{nk} = \begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times 6} \\ \mathbf{0}_6 & \mathbf{I}_6\delta t \\ \mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 6} \end{bmatrix} \tag{220}$$

$$= \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{G}_{31} & \mathbf{G}_{32} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{221}$$

$$= \begin{bmatrix} -\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \Delta\hat{\mathbf{R}}_k^\top \boldsymbol{\Xi}_4 & -\Delta\hat{\mathbf{R}}_k^\top \boldsymbol{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ \Delta\hat{\mathbf{R}}_k^\top \boldsymbol{\Xi}_3 & -\Delta\hat{\mathbf{R}}_k^\top \boldsymbol{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{222}$$

7.3 Feature Update

The visual measurements can be written as:

$$\mathbf{z} = \begin{bmatrix} \frac{x}{z} & \frac{y}{z} \end{bmatrix}^\top \tag{223}$$

$${}^C\mathbf{p}_f = \begin{bmatrix} x & y & z \end{bmatrix}^\top = {}^C_I\mathbf{R}{}^I_G\mathbf{R}({}^G\mathbf{p}_f - {}^G\mathbf{p}_I) + {}^C\mathbf{p}_I \tag{224}$$

Since the feature is coupled with the current IMU pose, hence, the error states can be written as:

$${}^G\mathbf{p}_f \simeq {}^G\hat{\mathbf{p}}_f + {}^G_I\hat{\mathbf{R}}\delta\tilde{\mathbf{p}}_f \tag{225}$$

$${}^G\mathbf{p}_I \simeq {}^G\hat{\mathbf{p}}_I + {}^G_I\hat{\mathbf{R}}\delta\tilde{\mathbf{p}} \tag{226}$$

$${}^G_I\mathbf{R} = {}^G_I\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_I) \tag{227}$$

The Jacobians $\mathbf{H} \triangleq \frac{\partial\delta\mathbf{z}}{\partial\delta\mathbf{x}}$ as:

$$\mathbf{H}_C = \frac{\partial\delta\mathbf{z}}{\partial\delta{}^C\mathbf{p}_f} = \frac{1}{z^2}\begin{bmatrix} z & 0 & -x \\ 0 & z & -y \end{bmatrix} \tag{228}$$

$$\begin{aligned} \mathbf{H} &= \mathbf{H}_C \begin{bmatrix} \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_I} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{p}} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{v}} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{b}_g} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{b}_a} & \frac{\partial\delta{}^C\mathbf{p}_f}{\partial\delta\mathbf{p}_f} \end{bmatrix} \\ &= \mathbf{H}_C{}^C_I\hat{\mathbf{R}} \begin{bmatrix} \lfloor {}^G_I\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_I)\rfloor & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \end{aligned} \tag{229}$$

where $\mathbf{H}_C$ represents the projection Jacobians. Following [3], the observabiltiy matrix for the linearized system is:

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \\ \vdots \\ \mathcal{O}_k \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1\boldsymbol{\Phi}_{1,0} \\ \vdots \\ \mathbf{H}_k\boldsymbol{\Phi}_{k,0} \end{bmatrix} \tag{230}$$

Each block row $\mathcal{O}_k$ can be computed as:

$$\mathcal{O}_k = \mathbf{H}_k \mathbf{\Phi}_{k,0} \tag{231}$$
$$= \mathbf{H}_C {}^C_I\hat{\mathbf{R}} \begin{bmatrix} \mathbf{M}_1 & -\Delta\hat{\mathbf{R}}_k^\top & -\Delta\hat{\mathbf{R}}_k^\top \delta t & \mathbf{M}_2 & \mathbf{M}_3 & \Delta\hat{\mathbf{R}}_k^\top \end{bmatrix} \tag{232}$$
$$= \mathbf{H}_C {}^C_I\hat{\mathbf{R}} \begin{bmatrix} \mathbf{M}_1 & -{}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}} & -{}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\delta t & \mathbf{M}_2 & \mathbf{M}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}} \end{bmatrix} \tag{233}$$

where:

$$\mathbf{M}_1 = \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top ({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}) \rfloor \mathbf{\Phi}_{11} - \mathbf{\Phi}_{21} \tag{234}$$
$$= \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top ({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}) \rfloor \Delta\hat{\mathbf{R}}_k^\top + \Delta\hat{\mathbf{R}}_k^\top \lfloor \Delta\hat{\mathbf{p}}_k \rfloor \tag{235}$$
$$= \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top ({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}) \rfloor {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}} + {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}} \lfloor {}^G_{I_0}\hat{\mathbf{R}}^\top ({}^G\mathbf{p}_{I_k} - {}^G\mathbf{p}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t - \frac{1}{2}{}^G\mathbf{g}\delta t^2) \rfloor \tag{236}$$
$$= {}^G_{I_k}\hat{\mathbf{R}}^\top (\lfloor {}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k} \rfloor {}^G_{I_0}\hat{\mathbf{R}} + {}^G_{I_0}\hat{\mathbf{R}} \lfloor {}^G_{I_0}\hat{\mathbf{R}}^\top ({}^G\mathbf{p}_{I_k} - {}^G\mathbf{p}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t - \frac{1}{2}{}^G\mathbf{g}\delta t^2) \rfloor) \tag{237}$$
$$= {}^G_{I_k}\hat{\mathbf{R}}^\top (\lfloor {}^G\hat{\mathbf{p}}_f - {}^G\mathbf{p}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t - \frac{1}{2}{}^G\mathbf{g}\delta t^2 \rfloor {}^G_{I_0}\hat{\mathbf{R}} \tag{238}$$
$$\mathbf{M}_2 = \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top ({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}) \rfloor \mathbf{\Phi}_{14} - \mathbf{\Phi}_{24} \tag{239}$$
$$= -\lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top ({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k}) \rfloor \mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t - {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\mathbf{\Xi}_4 \tag{240}$$
$$\mathbf{M}_3 = -\mathbf{\Phi}_{25} \tag{241}$$
$$= {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\mathbf{\Xi}_2 \tag{242}$$

we can directly find the null space as:

$$\mathbf{N} = \begin{bmatrix} {}^{I_0}_G\hat{\mathbf{R}}{}^G\mathbf{g} & \mathbf{0}_3 \\ -{}^{I_0}_G\hat{\mathbf{R}}\lfloor {}^G\hat{\mathbf{p}}_{I_0} \rfloor {}^G\mathbf{g} & \mathbf{I}_3 \\ -{}^{I_0}_G\hat{\mathbf{R}}\lfloor {}^G\hat{\mathbf{v}}_{I_0} \rfloor {}^G\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 3} \\ -{}^{I_0}_G\hat{\mathbf{R}}\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G\mathbf{g} & \mathbf{I}_3 \end{bmatrix} \tag{243}$$

8 DLI-VINS

8.1 DLI-FEJ

The state is still defined as:

$$\mathbf{x} = \left(\mathbf{x}_I, {}^G\mathbf{p}_f\right) \tag{244}$$

However, the overall error states are defined as:

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} = \left(\hat{\mathbf{x}}_n \exp(\delta\mathbf{x}_n), \hat{\mathbf{x}}_b + \delta\mathbf{x}_b, {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f\right) \tag{245}$$
$$= \begin{bmatrix} {}^G_I\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_I) \\ {}^G\hat{\mathbf{p}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ {}^G\hat{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f \end{bmatrix} \tag{246}$$

8.1.1 State Propagation

Following similar steps, we can get the full state transition matrix as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \\ \delta\mathbf{p}_{f,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times6} & \mathbf{I}_3 \end{bmatrix}}_{\mathbf{\Phi}_{k+1,k}} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \\ \delta\mathbf{p}_{f,k} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{G}_{nn} & \mathbf{0}_{9\times6} \\ \mathbf{0}_{6\times6} & \mathbf{I}_6\delta t \\ \mathbf{0}_{3\times6} & \mathbf{0}_{3\times6} \end{bmatrix}}_{\mathbf{G}_{nk}} \begin{bmatrix} \mathbf{n}_{dg} \\ \mathbf{n}_{da} \\ \mathbf{n}_{dwg} \\ \mathbf{n}_{dwa} \end{bmatrix} \tag{247}$$

Note $\mathbf{\Phi}_{nn}$ and $\mathbf{\Phi}_{nb}$ are same as (217).

8.1.2 Visual Update

If updating with current pose, the error states can be written as:

$$ {}^G\mathbf{p}_f \simeq {}^G\hat{\mathbf{p}}_f + \delta\mathbf{p}_f \tag{248}$$

$$ {}^G\mathbf{p}_I \simeq {}^G\hat{\mathbf{p}}_I + {}^G_I\hat{\mathbf{R}}\delta\mathbf{p} \tag{249}$$

$$ {}^G_I\mathbf{R} = {}^G_I\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_I) \tag{250}$$

Then, we can get the Jacobians as:

$$\mathbf{H} = \mathbf{H}_C {}^C_I\hat{\mathbf{R}} \begin{bmatrix} \lfloor {}^G_I\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_I)\rfloor & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_I\hat{\mathbf{R}}^\top \end{bmatrix} \tag{251}$$

8.1.3 Observability Analysis

It is nice that we don't have extra term in the Jacobians related to the current IMU state, how ever the measurements are state related, which will not preserve the observability property.

$$\mathcal{O}_k = \mathbf{H}_k\mathbf{\Phi}_{k,0} = \mathbf{H}_C {}^I_G\hat{\mathbf{R}} \begin{bmatrix} \mathbf{M}_1 & -{}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}} & -{}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\delta t & \mathbf{M}_2 & \mathbf{M}_3 & {}^G_{I_k}\hat{\mathbf{R}}^\top \end{bmatrix} \tag{252}$$

where we have:

$$\mathbf{M}_1 = \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k})\rfloor \mathbf{\Phi}_{11} - \mathbf{\Phi}_{21} \tag{253}$$

$$= \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k})\rfloor \Delta\hat{\mathbf{R}}_k^\top + \Delta\hat{\mathbf{R}}_k^\top \lfloor \Delta\hat{\mathbf{p}}_k \rfloor \tag{254}$$

$$= {}^G_{I_k}\hat{\mathbf{R}}^\top(\lfloor {}^G\hat{\mathbf{p}}_f - {}^G\mathbf{p}_{I_0} - {}^G\mathbf{v}_{I_0}\delta t - \frac{1}{2}{}^G\mathbf{g}\delta t^2 \rfloor {}^G_{I_0}\hat{\mathbf{R}} \tag{255}$$

$$\mathbf{M}_2 = \lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k})\rfloor \mathbf{\Phi}_{14} - \mathbf{\Phi}_{24} \tag{256}$$

$$= -\lfloor {}^G_{I_k}\hat{\mathbf{R}}^\top({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k})\rfloor \mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t - {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\mathbf{\Xi}_4 \tag{257}$$

$$\mathbf{M}_3 = -\mathbf{\Phi}_{25} \tag{258}$$

$$= {}^G_{I_k}\hat{\mathbf{R}}^\top {}^G_{I_0}\hat{\mathbf{R}}\mathbf{\Xi}_2 \tag{259}$$

Hence, we can get the null space for the decoupled DRI-VINS as:

$$\mathbf{N} = \begin{bmatrix} {}^{I_0}_G\hat{\mathbf{R}}{}^G\mathbf{g} & \mathbf{0}_3 \\ -{}^{I_0}_G\hat{\mathbf{R}}\lfloor {}^G\hat{\mathbf{p}}_{I_0}\rfloor {}^G\mathbf{g} & \mathbf{I}_3 \\ -{}^{I_0}_G\hat{\mathbf{R}}\lfloor {}^G\hat{\mathbf{v}}_{I_0}\rfloor {}^G\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{6\times1} & \mathbf{0}_{6\times3} \\ -\lfloor {}^G\hat{\mathbf{p}}_f \rfloor {}^G\mathbf{g} & \mathbf{I}_3 \end{bmatrix} \tag{260}$$

We can see that the unobservable subspace is affected by feature estimate ${}^G\hat{\mathbf{p}}_f$. Due to linearization point change, the DRI-VINS will become inconsistent.

8.1.4 DLI-FEJ Algorithm

In order to keep the observability property of the system, we can use the FEJ. When we do update, we just use the first estimate of the features, ${}^G\hat{\mathbf{p}}_f$, for the Jacobians computation.

The decoupled LI-EKF can easily handle more features in the state vector without involving them in the state covariance propagation. Due to the nice property of $SE_2(3)$ with left invariant error, the state propagation will automatically keep the system unobservabilty property for IMU. Hence, we only need to FEJ the feature state. In the meantime, the FEJ in the update Jacobians is pretty small and light-weight.

8.2 DLI-SW

Instead of associating features with current IMU state for Lie group representation, we can associate feature state with an active clones pose if a sliding-window is maintained.

8.2.1 State Vector

Assuming the state vector contains the IMU navigation state $\mathbf{x}_n$, the bias state $\mathbf{x}_b$, one feature point ${}^G\mathbf{p}_f$ and a sliding window state $\mathbf{x}_c$ (containing two poses for simplicity):

$$\mathbf{x} = \left(\mathbf{x}_n, \mathbf{x}_b, \mathbf{x}_c, {}^G\mathbf{p}_f\right) \tag{261}$$

$$\mathbf{x}_c = \left({}^G_{I_{c0}}\mathbf{R}, {}^G\mathbf{p}_{I_{c0}}, {}^G_{I_{c1}}\mathbf{R}, {}^G\mathbf{p}_{I_{c1}}\right) \tag{262}$$

Instead of decouple the feature ${}^G\mathbf{p}_f$ from Lie group as DRI-FEJ, we associate the feature with the first clone pose $\{I_{c0}\}$ for the Lie group representation. Hence, we can define:

$$\mathbf{x}_{I_{c0,f}} = \left(\mathbf{x}_{I_{c0}}, {}^G\mathbf{p}_f\right) = \begin{bmatrix} {}^G_{I_{c0}}\mathbf{R} & {}^G\mathbf{p}_{I_{c0}} & {}^G\mathbf{p}_f \\ \mathbf{0}_{1\times 3} & 1 & 0 \\ \mathbf{0}_{1\times 3} & 0 & 1 \end{bmatrix} \tag{263}$$

Thus, the new error states $\delta\mathbf{x} \in \mathbb{R}^{30}$ can be defined as:

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{x}_n^\top & \delta\mathbf{x}_b^\top & \delta\mathbf{x}_c^\top & \delta\mathbf{p}_f^\top \end{bmatrix}^\top \tag{264}$$

$$\delta\mathbf{x}_c = \begin{bmatrix} \delta\mathbf{x}_{c0}^\top & | & \delta\mathbf{x}_{c1}^\top \end{bmatrix}^\top = \begin{bmatrix} \delta\boldsymbol{\theta}_{I_{c0}}^\top & \delta\mathbf{p}_{I_{c0}}^\top & | & \delta\boldsymbol{\theta}_{I_{c1}}^\top & \delta\mathbf{p}_{I_{c1}}^\top \end{bmatrix}^\top \tag{265}$$

with new $\boxplus$ for DLI-SW as:

$$\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x} \tag{266}$$

$$= \left(\hat{\mathbf{x}}_n \exp\left(\delta\mathbf{x}_n\right), \mathbf{x}_b + \delta\mathbf{x}_b, \mathbf{x}_{I_{c0,f}} \cdot \exp\left(\begin{bmatrix} \delta\mathbf{x}_{I_{c0}} \\ \delta\mathbf{p}_f \end{bmatrix}\right), \mathbf{x}_{I_{c1}} \cdot \exp(\delta\mathbf{x}_{I_{c1}})\right) \tag{267}$$

$$= \begin{bmatrix} {}^G_I\hat{\mathbf{R}} \exp(\delta\boldsymbol{\theta}_I) \\ {}^G\hat{\mathbf{p}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{p} \\ {}^G\hat{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_I)\delta\mathbf{v} \\ \hat{\mathbf{b}}_g + \delta\mathbf{b}_g \\ \hat{\mathbf{b}}_a + \delta\mathbf{b}_a \\ {}^G_{I_{c0}}\hat{\mathbf{R}} \exp(\delta\boldsymbol{\theta}_{I_{c0}}) \\ {}^G\hat{\mathbf{p}}_{I_{c0}} + {}^G_{I_{c0}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c0}})\delta\mathbf{p}_{I_{c0}} \\ {}^G_{I_{c1}}\hat{\mathbf{R}} \exp(\delta\boldsymbol{\theta}_{I_{c1}}) \\ {}^G\hat{\mathbf{p}}_{I_{c1}} + {}^G_{I_{c1}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c1}})\delta\mathbf{p}_{I_{c1}} \\ {}^G\hat{\mathbf{p}}_f + {}^G_{I_{c0}}\hat{\mathbf{R}}\mathbf{J}_l(\delta\boldsymbol{\theta}_{I_{c0}})\delta\mathbf{p}_f \end{bmatrix} \tag{268}$$

8.2.2 State Propagation

The state transition matrix with the new error states can be written as:

$$\boldsymbol{\Phi}_{k+1,k} = \begin{bmatrix} \boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} & \mathbf{0}_{9\times12} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times12} & \mathbf{0}_{6\times3} \\ \mathbf{0}_{12\times9} & \mathbf{0}_{12\times6} & \mathbf{I}_{12} & \mathbf{0}_{12\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times6} & \mathbf{0}_{3\times12} & \mathbf{I}_3 \end{bmatrix} \tag{269}$$

It can be seen that the feature is decoupled from the IMU state covariance propagation with $\boldsymbol{\Phi}_{fb} = \mathbf{0}_{3\times6}$, which is desired.

8.2.3 Visual Update

If we get the feature measurements at cloned pose $\{I_{ci}\}, i = \{0, 1\}$, the feature transformation can be written as:

$$ {}^{C_0}\mathbf{p}_f = {}^{C}_{I}\mathbf{R}{}^{I_{c0}}_{G}\mathbf{R}({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I_{c0}}) + {}^{C}\mathbf{p}_I \tag{270}$$

$$ {}^{C_1}\mathbf{p}_f = {}^{C}_{I}\mathbf{R}{}^{I_{c1}}_{G}\mathbf{R}({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I_{c1}}) + {}^{C}\mathbf{p}_I \tag{271}$$

The error states can be defined as:

$$ {}^{G}\mathbf{p}_f \simeq {}^{G}\hat{\mathbf{p}}_f + {}^{G}_{I_{c0}}\hat{\mathbf{R}}\delta\mathbf{p}_f \tag{272}$$

$$ {}^{G}\mathbf{p}_{I_{ci}} \simeq {}^{G}\hat{\mathbf{p}}_{I_{ci}} + {}^{G}_{I_{ci}}\hat{\mathbf{R}}\delta\mathbf{p}_{I_{ci}} \tag{273}$$

$$ {}^{G}_{I_{ci}}\mathbf{R} = {}^{G}_{I_{ci}}\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_{I_{ci}}) \tag{274}$$

Hence, we can recompute the Jacobians with feature associated with $\{I_{c0}\}$ for Lie group representation as:

$$\frac{\partial\delta^{C_i}\mathbf{p}_f}{\partial\delta\boldsymbol{\theta}_{I_{ci}}} = {}^{C}_{I}\hat{\mathbf{R}}\lfloor{}^{I_{ci}}_{G}\hat{\mathbf{R}}({}^{G}\hat{\mathbf{p}}_f - {}^{G}\hat{\mathbf{p}}_{I_{ci}})\rfloor \tag{275}$$

$$\frac{\partial\delta^{C_i}\mathbf{p}_f}{\partial\delta\mathbf{p}_f} = {}^{C}_{I}\hat{\mathbf{R}}{}^{I_{ci}}_{G}\hat{\mathbf{R}}{}^{I_{c0}}_{G}\hat{\mathbf{R}}^\top \tag{276}$$

$$\frac{\partial\delta^{C_i}\mathbf{p}_f}{\partial\delta\mathbf{p}_{I_{ci}}} = -{}^{C}_{I}\hat{\mathbf{R}} \tag{277}$$

8.2.4 Pseudo-Anchor Change

The proposed pseudo-anchor change, which is different from anchor change in [4] which represents the feature from global frame {G} to local sensor frame $\{I\}$, refers to changing the associated Lie group poses for the feature while the feature is still represented in global frame $\{G\}$. In order to keep the long-tracked SLAM features in the state vector even when the associated cloned pose is marginalized from the sliding window, we can perform pseudo-anchor change, that is to "anchor" the feature to another active cloned pose in the sliding window for Lie group representation.

Since the feature state estimate ${}^{G}\hat{\mathbf{p}}_f$ remains the same, we only need to modify the covariance of the feature after pseudo-anchor change. The feature is associated with pose I_{c0} which is about to be marginalized from the state, and we need to associate feature with pose I_{c1} instead. If we define $\delta\mathbf{p}_{fi}$ as the error states for ${}^{G}\mathbf{p}_f$ when associated with cloned pose $\{I_{ci}\}$, we have:

$$ {}^{G}\mathbf{p}_f \simeq {}^{G}\hat{\mathbf{p}}_f + {}^{G}_{I_{c0}}\hat{\mathbf{R}}\delta\mathbf{p}_{f0} \tag{278}$$

$$ {}^{G}\mathbf{p}_f \simeq {}^{G}\hat{\mathbf{p}}_f + {}^{G}_{I_{c1}}\hat{\mathbf{R}}\delta\mathbf{p}_{f1} \tag{279}$$

Then the feature's new error state $\delta\mathbf{p}_{f1}$ is written as:

$$\delta\mathbf{p}_{f1} = {}^{G}_{I_{c1}}\hat{\mathbf{R}}^{\top}{}^{G}_{I_{c0}}\hat{\mathbf{R}}\delta\mathbf{p}_{f0} \tag{280}$$

We then leverage EKF update to get the new state covariance after the pseudo-anchor change. We also show that the proposed pseudo-anchor change algorithm won't affect the system unobservable property.

Appendix A: Discrete IMU Propagation

In this section, we show how to propagate IMU dynamics ignoring the noise s.t.

$$\mathbf{b}_g = \hat{\mathbf{b}}_g \tag{281}$$

$$\mathbf{b}_a = \hat{\mathbf{b}}_a \tag{282}$$

Then, the IMU propogation can be written as

$$\hat{\mathbf{x}}_{n_{k+1}} = \hat{\mathbf{\Gamma}}_k \hat{\mathbf{\Psi}}(\mathbf{x}_{n_k}) \hat{\mathbf{\Upsilon}}_k \tag{283}$$

Hence, we just need to decide $\hat{\mathbf{\Upsilon}}_k$ with $\Delta\hat{\mathbf{R}}_k$, $\Delta\hat{\mathbf{p}}_k$ and $\Delta\hat{\mathbf{v}}_k$.

When using discrete model, we can get:

$$\Delta\hat{\mathbf{R}}_k = \exp\left(\hat{\boldsymbol{\omega}}_k \delta t\right) \tag{284}$$

$$\Delta\hat{\mathbf{p}}_k = \frac{1}{2}\hat{\mathbf{a}_k}\delta t^2 \tag{285}$$

$$\Delta\hat{\mathbf{v}}_k = \hat{\mathbf{a}_k}\delta t \tag{286}$$

The state transition matrix can written as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \end{bmatrix} \tag{287}$$

For $\mathbf{\Phi}_{nn}$, we can use the right invariant errors:

$$\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \mathbf{\Gamma}_k \mathbf{\Psi}(\exp(\delta\mathbf{x}_{n_k})\hat{\mathbf{x}}_{n_k})\hat{\mathbf{\Upsilon}}_k && (288)\\
&= \mathbf{\Gamma}_k \mathbf{\Psi}\left(\exp\left(\delta\mathbf{x}_{n_k}\right)\right) \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k && (289)\\
&= \mathbf{\Gamma}_k \exp\left(\mathbf{F}\delta\mathbf{x}_{n_k}\right) \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k && (290)\\
&= \exp\left(Ad_{\mathbf{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{n_k}\right) \underbrace{\mathbf{\Gamma}_k \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k}_{\hat{\mathbf{x}}_{n_{k+1}}} && (291)
\end{aligned}$$

Then, we can have $\mathbf{\Phi}_{nn}$ as:

$$\mathbf{\Phi}_{nn} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t \\ \lfloor {}^G\mathbf{g} \rfloor \delta t & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{292}$$

For computing $\mathbf{\Phi}_{nb}$, we can have:

$$\Delta\mathbf{R}_k = \Delta\hat{\mathbf{R}}_k \Delta\tilde{\mathbf{R}}_k \tag{293}$$

$$\Delta\mathbf{p}_k = \Delta\hat{\mathbf{p}}_k + \Delta\tilde{\hat{\mathbf{p}}}_k \tag{294}$$

$$\Delta\mathbf{v}_k = \Delta\hat{\mathbf{v}}_k + \Delta\tilde{\hat{\mathbf{v}}}_k \tag{295}$$

$$\begin{aligned}
\mathbf{\Upsilon}_k &= \hat{\mathbf{\Upsilon}}_k \tilde{\mathbf{\Upsilon}}_k && (296)\\
&= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{R}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (297)\\
&= \hat{\mathbf{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) \simeq \hat{\mathbf{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (298)
\end{aligned}$$

where $\Delta\tilde{\mathbf{R}}_k = \exp(\delta\Delta\theta_k)$. Then, we have:

$$\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \mathbf{\Gamma}_k\mathbf{\Psi}(\mathbf{x}_{n_k})\hat{\mathbf{\Upsilon}}_k\tilde{\mathbf{\Upsilon}}_k && (299)\\
\Rightarrow \exp(\delta\mathbf{x}_{n_{k+1}}) &= \hat{\mathbf{x}}_{n_{k+1}}\tilde{\mathbf{\Upsilon}}_k\hat{\mathbf{x}}_{n_{k+1}}^{-1} && (300)\\
&= \exp\left(\mathbf{Ad}_{\hat{\mathbf{x}}_{n_{k+1}}}\begin{bmatrix}\delta\Delta\theta_k\\ \hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k\\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k\end{bmatrix}\right) && (301)
\end{aligned}$$

We have two versions for computing the $\Delta\tilde{\mathbf{R}}_k$, $\Delta\tilde{\mathbf{p}}_k$ and $\Delta\tilde{\mathbf{v}}_k$;
For discrete model, we have:

$$\begin{aligned}
\Delta\tilde{\mathbf{R}}_k &= -\exp\left(\mathbf{J}_r(\Delta\hat{\theta})\left(\tilde{\mathbf{b}}_g+\mathbf{n}_g\right)\delta t\right) && (302)\\
\Delta\tilde{\mathbf{p}}_k &= -\frac{1}{2}(\tilde{\mathbf{b}}_a+\mathbf{n}_a)\delta t^2 && (303)\\
\Delta\tilde{\mathbf{v}}_k &= -(\tilde{\mathbf{b}}_a+\mathbf{n}_a)\delta t && (304)
\end{aligned}$$

Hence, we have:

$$\begin{aligned}
\mathbf{\Phi}_{nb} &= \begin{bmatrix}{}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0} & \mathbf{0}\\ \lfloor{}^G\hat{\mathbf{p}}_{k+1}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}} & {}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0}\\ \lfloor{}^G\hat{\mathbf{v}}_{k+1}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0} & {}^G_{I_{k+1}}\hat{\mathbf{R}}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \Delta\hat{\mathbf{R}}_k^\top\end{bmatrix}\begin{bmatrix}-\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}\\ \mathbf{0} & -\frac{1}{2}\delta t^2\\ \mathbf{0} & -\delta t\end{bmatrix} && (305)\\
&= \begin{bmatrix}-{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0\\ -\lfloor{}^G\mathbf{p}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2\\ -\lfloor{}^G\mathbf{v}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t\end{bmatrix} && (306)
\end{aligned}$$

Therefore, we have the full state translation matrix as

$$\mathbf{\Phi} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0\\ \frac{1}{2}\lfloor{}^G\mathbf{g}\rfloor{}^G\mathbf{g}\delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & -\lfloor{}^G\mathbf{p}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2\\ \lfloor{}^G\mathbf{g}\rfloor{}^G\mathbf{g}\delta t & \mathbf{0}_3 & \mathbf{I}_3 & -\lfloor{}^G\mathbf{v}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \tag{307}$$

And the noise Jacobians:

$$\mathbf{G}_n = \begin{bmatrix}-{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0 & \mathbf{0}_3 & \mathbf{0}_3\\ -\lfloor{}^G\mathbf{p}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\mathbf{R}_k^\top\delta t^2 & \mathbf{0}_3 & \mathbf{0}_3\\ -\lfloor{}^G\mathbf{v}_{I_{k+1}}\rfloor{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\mathbf{R}_k^\top\delta t & \mathbf{0}_3 & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t\end{bmatrix} \tag{308}$$

Appendix B: ACI Basics

The first integration we need is:

$$\mathbf{\Xi}_1 = \mathbf{I}_3\delta t_i + \frac{1-\cos(\hat{\omega}_i\delta t_i)}{\hat{\omega}_i}\lfloor\hat{\mathbf{k}}_i\rfloor + \left(\delta t_i - \frac{\sin(\hat{\omega}_i\delta t_i)}{\hat{\omega}_i}\right)\lfloor\hat{\mathbf{k}}_i\rfloor^2 \tag{309}$$

The second integration we need is:

$$\boldsymbol{\Xi}_2 = \frac{1}{2}\delta t_i^2 \mathbf{I}_3 + \frac{\hat{\omega}_i \delta t_i - \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{k}}_i \rfloor + \left(\frac{1}{2}\delta t_i^2 - \frac{1 - \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{310}$$

The third integration can be written as:

$$\begin{aligned}
\boldsymbol{\Xi}_3 = & \frac{1}{2}\delta t_i^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{1}{2}\delta t_i^2 - \frac{1 - \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 \\
& + \left(\frac{1}{2}\delta t_i^2 + \frac{1 - \cos(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{1}{2}\delta t_i^2 + \frac{1 - \cos(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \right) \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \\
& - \frac{3\sin(\hat{\omega}_i \delta t_i) - 2\hat{\omega}_i \delta t_i - \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^2} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2
\end{aligned} \tag{311}$$

The fourth integration we need is:

$$\begin{aligned}
\boldsymbol{\Xi}_4 = & \frac{1}{6}\delta t_i^3 \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{2(1 - \cos(\hat{\omega}_i \delta t_i)) - (\hat{\omega}_i^2 \delta t_i^2)}{2\hat{\omega}_i^3} \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \left(\frac{2(1 - \cos(\hat{\omega}_i \delta t_i)) - \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \right) \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \left(\frac{\sin(\hat{\omega}_i \delta t_i) - \hat{\omega}_i \delta t_i}{\hat{\omega}_i^3} + \frac{\delta t_i^3}{6} \right) \lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 \\
& + \frac{\hat{\omega}_i \delta t_i - 2\sin(\hat{\omega}_i \delta t_i) + \frac{1}{6}(\hat{\omega}_i \delta t_i)^3 + \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor \\
& + \frac{\hat{\omega}_i \delta t_i - 2\sin(\hat{\omega}_i \delta t_i) + \frac{1}{6}(\hat{\omega}_i \delta t_i)^3 + \hat{\omega}_i \delta t_i \cos(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \\
& + \frac{4\cos(\hat{\omega}_i \delta t_i) - 4 + (\hat{\omega}_i \delta t_i)^2 + \hat{\omega}_i \delta t_i \sin(\hat{\omega}_i \delta t_i)}{\hat{\omega}_i^3} \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2
\end{aligned} \tag{312}$$

When $\hat{\omega}_i$ is too small, in order to avoid numerical instability, we can compute the above integration identities as:

$$\lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_1 = \delta t_i \mathbf{I}_3 + \delta t_i \sin(\hat{\omega}_i \delta t_i) \lfloor \hat{\mathbf{k}}_i \rfloor + \delta t_i \left(1 - \cos(\hat{\omega}_i \delta t_i)\right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{313}$$

$$\lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_2 = \frac{\delta t_i^2}{2} \mathbf{I}_3 + \frac{\delta t_i^2}{2} \sin(\hat{\omega}_i \delta t_i) \lfloor \hat{\mathbf{k}}_i \rfloor + \frac{\delta t_i^2}{2} \left(1 - \cos(\hat{\omega}_i \delta t_i)\right) \lfloor \hat{\mathbf{k}}_i \rfloor^2 \tag{314}$$

$$= \frac{\delta t_i}{2} \lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_1 \tag{315}$$

$$\begin{aligned} \lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_3 = &\frac{\delta t_i^2}{2} \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\delta t_i^2 \sin(\hat{\omega}_i \delta t_i)}{2} \left(-\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor + \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2 \right) \\ &+ \frac{\delta t_i^2}{2} (1 - \cos(\hat{\omega}_i \delta t_i)) \left(\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 + \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \right) \end{aligned} \tag{316}$$

$$\begin{aligned} \lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_4 = &\frac{\delta t_i^3}{6} \lfloor \hat{\mathbf{a}}_i \rfloor + \frac{\delta t_i^3 \sin(\hat{\omega}_i \delta t_i)}{6} \left(-\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor + \lfloor \hat{\mathbf{k}}_i \rfloor \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor^2 \right) \\ &+ \frac{\delta t_i^3}{6} (1 - \cos(\hat{\omega}_i \delta t_i)) \left(\lfloor \hat{\mathbf{a}}_i \rfloor \lfloor \hat{\mathbf{k}}_i \rfloor^2 + \lfloor \hat{\mathbf{k}}_i \rfloor^2 \lfloor \hat{\mathbf{a}}_i \rfloor + \hat{\mathbf{k}}_i^\top \hat{\mathbf{a}}_i \lfloor \hat{\mathbf{k}}_i \rfloor \right) \end{aligned} \tag{317}$$

$$= \frac{\delta t_i}{3} \lim_{\hat{\omega}_i \to 0} \mathbf{\Xi}_3 \tag{318}$$

References

[1] Martin Brossard et al. *Associating Uncertainty to Extended Poses for on Lie Group IMU Preintegration with Rotating Earth.* 2021. arXiv: 2007.14097 [cs.RO].

[2] Yulin Yang et al. "Analytic Combined IMU Integration (ACI^2) for Visual-Inertial Navigation". In: *Proc. of the IEEE International Conference on Robotics and Automation.* Paris, France, 2020.

[3] J.A. Hesch et al. "Consistency Analysis and Improvement of Vision-aided Inertial Navigation". In: *IEEE Transactions on Robotics* PP.99 (2013), pp. 1–19. DOI: 10.1109/TRO.2013.2277549.

[4] Patrick Geneva et al. "OpenVINS: A Research Platform for Visual-Inertial Estimation". In: *Proc. of the IEEE International Conference on Robotics and Automation.* Paris, France, 2020. URL: https://github.com/rpng/open_vins.