Supplementary Materials: Visual-Inertial-aided Online MAV System Identification

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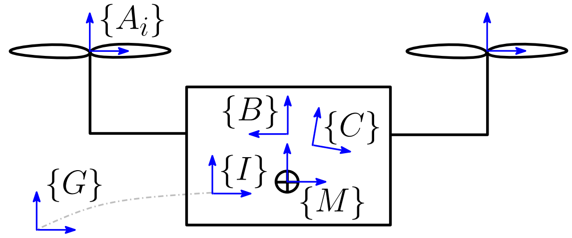

Figure 1: Frame of references of a typical MAV system: i-th rotor frame $\{A_i\}$, geometric body frame $\{B\}$, IMU frame $\{I\}$, camera frame $\{C\}$, MAV center of mass frame $\{M\}$, and global frame $\{G\}$.

1 MAV Dynamics and Its Preintegration

The dynamic model for the MAV system is illustrated in Fig. 1. The MAV states $\mathbf{x}_M$ and system parameters $\mathbf{x}_\theta$ are defined as:

$$\mathbf{x}_M = \begin{bmatrix} {}^M_G\bar{q}^\top & {}^G\mathbf{p}_M^\top & {}^M\boldsymbol{\omega}^\top & {}^G\mathbf{v}_M^\top \end{bmatrix}^\top \tag{1}$$

$$\mathbf{x}_\theta = \begin{bmatrix} \mathbf{x}_D^\top & \mathbf{x}_G^\top & \mathbf{x}_{MI}^\top \end{bmatrix}^\top \tag{2}$$

$$\mathbf{x}_D = \begin{bmatrix} c_t & c_m \end{bmatrix}^\top \tag{3}$$

$$\mathbf{x}_G = \begin{bmatrix} m & {}^M\mathbf{j}^\top & {}^M\mathbf{p}_B^\top \end{bmatrix}^\top \tag{4}$$

$$\mathbf{x}_{MI} = \begin{bmatrix} {}^I_M\bar{q}^\top & {}^I\mathbf{p}_M^\top \end{bmatrix}^\top \tag{5}$$

where ${}^M_G\bar{q}$ is the unit quaternion representing the rotation ${}^M_G\mathbf{R}$ from the global frame $\{G\}$ to the MAV center of mass frame $\{M\}$ [1], ${}^M\boldsymbol{\omega}$ is the angular velocity of MAV in $\{M\}$, and ${}^G\mathbf{p}_M$ and ${}^G\mathbf{v}_M$ are the MAV position and velocity in $\{G\}$, respectively. The aerodynamic parameters, $\mathbf{x}_D$, consist of the rotor thrust coefficient c_t and moment coefficient c_m. The geometrical parameters, $\mathbf{x}_G$, include the mass of the platform m, moment of inertial ${}^M\mathbf{J}$, which is a diagonal matrix defined in frame $\{M\}$ with ${}^M\mathbf{j}$ being its diagonal terms, and the translation between $\{B\}$ and $\{M\}$ as ${}^M\mathbf{p}_B$.

1.1 MAV Force and Moment

The total force ${}^M\mathbf{F}$ and moment ${}^M\mathbf{M}$ of a MAV with N_r rotors are defined as [2]:

$${}^M\mathbf{F} = \sum_{i=1}^{N_r} {}^M_{A_i}\mathbf{R}\,{}^{A_i}\mathbf{F}_i \tag{6}$$

$${}^M\mathbf{M} = \sum_{i=1}^{N_r} \left({}^M_{A_i}\mathbf{R}\,{}^{A_i}\mathbf{M} + \lfloor {}^M\mathbf{p}_{A_i} \rfloor {}^M\mathbf{F}_i \right) \tag{7}$$

where ${}^M\mathbf{p}_{A_i} = {}^M_B\mathbf{R}\,{}^B\mathbf{p}_{A_i} + {}^M\mathbf{p}_B$; ${}^B\mathbf{p}_{A_i}$ denotes the translation between rotor $\{A_i\}$ and frame $\{B\}$, which is typically known from the CAD model. Assuming the rotor encoder measurements $r_{m,i}$ from the i-th rotor is defined as $r_{m,i} = r_i + n_{r,i}$ with $n_{r,i}$ represents the white Gaussian noise, the individual force ${}^{A_i}\mathbf{F}_i$ and moment ${}^{A_i}\mathbf{M}_i$ from rotor $\{A_i\}$ are approximated as [3]:

$${}^{A_i}\mathbf{F}_i = c_t(r_{m,i} - n_{r,i})^2\mathbf{e}_z + \mathbf{n}_{f,i} \tag{8}$$

$${}^{A_i}\mathbf{M}_i = c_m(r_{m,i} - n_{r,i})^2\lambda_i\mathbf{e}_z + \mathbf{n}_{m,i} \tag{9}$$

where $\lambda_i \in \{-1, 1\}$ corresponds to the rotation direction of the i-th rotor and $\mathbf{e}_z = [0, 0, 1]^\top$ is a unit vector along local z direction. Additionally, there are two white Gaussian noises, $\mathbf{n}_{f,i}$ and $\mathbf{n}_{m,i}$, compensate for the inadequacy of this sensing model to measure the full 3-axis force acting on rotors (only the force along the local z-axis is modeled with rotor speeds).

1.2 MAV Dynamic Model and Preintegration

Using the force and moment model, the MAV dynamics are defined as:

$$\begin{aligned}
{}^M_G\dot{\bar{q}} &= \frac{1}{2}\mathbf{\Omega}\left({}^M\boldsymbol{\omega}\right){}^M_G\bar{q} && (10)\\
{}^G\dot{\mathbf{p}}_M &= {}^G\mathbf{v}_M && (11)\\
{}^M\dot{\boldsymbol{\omega}} &= {}^M\mathbf{J}^{-1}\left({}^M\mathbf{M} - \lfloor {}^M\boldsymbol{\omega}\rfloor {}^M\mathbf{J}^M\boldsymbol{\omega}\right) && (12)\\
{}^G\dot{\mathbf{v}}_M &= \frac{1}{m}{}^G_M\mathbf{R}^M\mathbf{F} - {}^G\mathbf{g} && (13)
\end{aligned}$$

where $\mathbf{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix}$ and $\lfloor\cdot\rfloor$ is the skew-symmetric matrix; ${}^G\mathbf{g} = [0\ 0\ 9.81]^\top$ denotes the gravity. This can be summarized as:

$$\dot{\mathbf{x}}_M = \mathbf{f}_M\left(\mathbf{x}_M, \mathbf{x}_\theta, \mathbf{n}_M\right) \tag{14}$$

where $\mathbf{n}_M = [n_{r,1} \cdots n_{r,N_r}\ \mathbf{n}_{f,1}^\top \cdots \mathbf{n}_{f,N_r}^\top\ \mathbf{n}_{m,1}^\top \cdots \mathbf{n}_{m,N_r}^\top]^\top$ contains all noises [see Eq. (8) and (9)] with covariance $\mathbf{Q}_M$. We can integrate this model from time t_k to t_{k+1} based on first-order approximation:

$$\begin{aligned}
{}^{k+1}_G\mathbf{R} &= \Delta\mathbf{R}^\top {}^k_G\mathbf{R} && (15)\\
&= \exp^\top\left({}^k\boldsymbol{\theta}_{k+1}\right){}^k_G\mathbf{R} && (16)\\
{}^G\mathbf{p}_{k+1} &= {}^G\mathbf{p}_k + \Delta\mathbf{p} && (17)\\
&= {}^G\mathbf{p}_k + {}^G\mathbf{v}_k\Delta t + \frac{1}{2}\left(\frac{1}{m}{}^G_k\mathbf{R}^k\mathbf{F} - {}^G\mathbf{g}\right)\Delta t^2 && (18)\\
{}^{k+1}\boldsymbol{\omega} &= \Delta\mathbf{R}^\top({}^k\boldsymbol{\omega} + \Delta\boldsymbol{\omega}) && (19)\\
&= \exp^\top\left({}^k\boldsymbol{\theta}_{k+1}\right)\left[{}^k\boldsymbol{\omega} + {}^k\mathbf{J}^{-1}\left({}^k\mathbf{M} - \lfloor{}^k\boldsymbol{\omega}\rfloor{}^k\mathbf{J}^k\boldsymbol{\omega}\right)\Delta t\right] && (20)\\
{}^G\mathbf{v}_{k+1} &= {}^G\mathbf{v}_k + \Delta\mathbf{v} && (21)\\
&= {}^G\mathbf{v}_k + \left[\frac{1}{m}\left({}^G_k\mathbf{R}^k\mathbf{F}\right) - {}^G\mathbf{g}\right]\Delta t && (22)
\end{aligned}$$

where:

$${}^k\boldsymbol{\theta}_{k+1} = {}^k\boldsymbol{\omega}_k\Delta t + \frac{1}{2}{}^k\mathbf{J}^{-1}\left({}^k\mathbf{M} - \lfloor{}^k\boldsymbol{\omega}\rfloor{}^k\mathbf{J}^k\boldsymbol{\omega}\right)\Delta t^2 \tag{23}$$

Note that we use k and $k+1$ instead of M_k and M_{k+1} to simplify the notation and $\Delta t = t_{k+1} - t_k$ express the time offset between two timestamps. With the above preintegration, we can have the following integrated discrete-time MAV dynamic model and its linearization:

$$\begin{aligned}
\mathbf{x}_{M_{k+1}} &= \mathbf{g}_M(\mathbf{x}_{M_k}, \mathbf{x}_\theta, \mathbf{n}_M) && (24)\\
\tilde{\mathbf{x}}_{M_{k+1}} &\simeq \mathbf{\Phi}_M\tilde{\mathbf{x}}_{M_k} + \mathbf{\Phi}_\theta\tilde{\mathbf{x}}_\theta + \mathbf{G}_n\mathbf{n}_M && (25)
\end{aligned}$$

where $\mathbf{\Phi}_M$ is the linearized state transition matrix. $\mathbf{\Phi}_\theta$ and $\mathbf{G}_n$ represent the Jacobians for $\mathbf{x}_\theta$ and $\mathbf{n}_M$, respectively. In the next section, we will introduce in detail the derivations of the Jacobians.

1.3 Error States for Rotation

We define the error quaternion as:

$$ {}^{M}_{G}\bar{q} = \delta\bar{q} \otimes {}^{M}_{G}\hat{\bar{q}} \tag{26} $$

which is equivalent as [1]:

$$ {}^{M}_{G}\mathbf{R} = \exp(-\delta\boldsymbol{\theta}) \cdot {}^{M}_{G}\hat{\mathbf{R}} \quad \Rightarrow \quad {}^{M}_{G}\mathbf{R} \simeq (\mathbf{I} - \lfloor\delta\boldsymbol{\theta}\rfloor) \cdot {}^{M}_{G}\hat{\mathbf{R}} \tag{27} $$

$$ {}^{G}_{M}\mathbf{R} = {}^{G}_{M}\hat{\mathbf{R}} \cdot \exp(\delta\boldsymbol{\theta}) \quad \Rightarrow \quad {}^{G}_{M}\mathbf{R} \simeq {}^{G}_{M}\hat{\mathbf{R}} \cdot (\mathbf{I} + \lfloor\delta\boldsymbol{\theta}\rfloor) \tag{28} $$

1.4 Linearization for MAV Dynamic Model

To simplify the following derivations, we first define:

$$ \mathbf{A} = {}^{k}\mathbf{J}^{-1}\left({}^{k}\mathbf{M} - \lfloor{}^{k}\boldsymbol{\omega}\rfloor{}^{k}\mathbf{J}{}^{k}\boldsymbol{\omega}\right) \tag{29} $$

$$ \mathbf{F} = \frac{1}{m}{}^{k}\mathbf{F} \tag{30} $$

In order to linearize the MAV dynamic model, we find the derivatives of $\mathbf{F}$ and $\mathbf{A}$ respect to the involved states following chain rule as:

$$ \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta} = \begin{bmatrix} \frac{\partial\mathbf{A}}{\partial\mathbf{x}_D} & \frac{\partial\mathbf{A}}{\partial\mathbf{x}_G} & \frac{\partial\mathbf{A}}{\partial\mathbf{x}_{MI}} \end{bmatrix} \tag{31} $$

$$ = \begin{bmatrix} \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_G} & \mathbf{0} \end{bmatrix} \tag{32} $$

$$ \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta} = \begin{bmatrix} \frac{\partial\mathbf{F}}{\partial\mathbf{x}_D} & \frac{\partial\mathbf{F}}{\partial\mathbf{x}_G} & \frac{\partial\mathbf{F}}{\partial\mathbf{x}_{MI}} \end{bmatrix} \tag{33} $$

$$ = \begin{bmatrix} \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_G} & \mathbf{0} \end{bmatrix} \tag{34} $$

where:

$$ \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_D} = \begin{bmatrix} \frac{\partial\mathbf{A}}{\partial c_t} & \frac{\partial\mathbf{A}}{\partial c_m} \end{bmatrix} \tag{35} $$

$$ = \begin{bmatrix} {}^{k}\mathbf{J}^{-1}\sum_{i=1}^{N_r}\lfloor{}^{M}\mathbf{p}_{A_i}\rfloor r_i^2\mathbf{e}_z & {}^{k}\mathbf{J}^{-1}\sum_{i=1}^{N_r}{}^{M}_{A_i}\mathbf{R}\lambda_i r_i^2\mathbf{e}_z \end{bmatrix} \tag{36} $$

$$ \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_G} = \begin{bmatrix} \frac{\partial\mathbf{A}}{\partial m} & \frac{\partial\mathbf{A}}{\partial{}^{M}\mathbf{j}} & \frac{\partial\mathbf{A}}{\partial{}^{M}\mathbf{p}_B} \end{bmatrix} \tag{37} $$

$$ = \begin{bmatrix} \mathbf{0}_{3\times1} & .. \frac{\partial\mathbf{A}}{\partial j_i} .. & -{}^{k}\mathbf{J}^{-1}\sum_{i=1}^{N_r}\lfloor{}^{k}\mathbf{F}_i\rfloor \end{bmatrix} \tag{38} $$

$$ \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_D} = \begin{bmatrix} \frac{\partial\mathbf{F}}{\partial c_t} & \frac{\partial\mathbf{F}}{\partial c_m} \end{bmatrix} \tag{39} $$

$$ = \begin{bmatrix} \frac{1}{m}\sum_{i=1}^{N_r}{}^{M}_{A_i}\mathbf{R}r_i^2\mathbf{e}_z & \mathbf{0}_{3\times1} \end{bmatrix} \tag{40} $$

$$ \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_G} = \begin{bmatrix} \frac{\partial\mathbf{F}}{\partial m} & \frac{\partial\mathbf{F}}{\partial{}^{M}\mathbf{J}} & \frac{\partial\mathbf{F}}{\partial{}^{M}\mathbf{p}_B} \end{bmatrix} \tag{41} $$

$$ = \begin{bmatrix} -\frac{1}{m^2}{}^{k}\mathbf{F} & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{42} $$

$$ \frac{\partial\mathbf{A}}{\partial j_1} = \begin{bmatrix} -\frac{1}{j_1^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left({}^{k}\mathbf{M} - \lfloor{}^{k}\boldsymbol{\omega}\rfloor{}^{k}\mathbf{J}{}^{k}\boldsymbol{\omega}\right) - {}^{k}\mathbf{J}^{-1}\lfloor{}^{k}\boldsymbol{\omega}\rfloor\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}{}^{k}\boldsymbol{\omega} \tag{43} $$

Since $\mathbf{A}$ also consists of ${}^k\boldsymbol{\omega}$, we have:

$$\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}} = \frac{\partial \mathbf{A}}{\partial {}^k\boldsymbol{\omega}} \tag{44}$$
$$= {}^k\mathbf{J}^{-1}\left(-\lfloor {}^k\boldsymbol{\omega}\rfloor {}^k\mathbf{J} + \lfloor {}^k\mathbf{J}{}^k\boldsymbol{\omega}\rfloor\right) \tag{45}$$

Finally, we can derive the related noise Jacobians:

$$\mathbf{H}^{\mathbf{A}}_{\mathbf{n}_M} = \begin{bmatrix}\mathbf{H}^{\mathbf{A}}_{n_{r,i}} & \mathbf{H}^{\mathbf{A}}_{\mathbf{n}_{f,i}} & \mathbf{H}^{\mathbf{A}}_{\mathbf{n}_{m,i}}\end{bmatrix} \tag{46}$$
$$= \begin{bmatrix}-2{}^k\mathbf{J}^{-1}\left({}^M_{A_i}\mathbf{R}c_m\lambda_i c_m\mathbf{e}_z + \lfloor {}^M\mathbf{p}_{A_i}\rfloor {}^M_{A_i}\mathbf{R}c_t r_i\mathbf{e}_z\right) & {}^k\mathbf{J}^{-1}\lfloor {}^M\mathbf{p}_{A_i}\rfloor & {}^k\mathbf{J}^{-1}{}^M_{A_i}\mathbf{R}\end{bmatrix} \tag{47}$$
$$\mathbf{H}^{\mathbf{F}}_{\mathbf{n}_M} = \begin{bmatrix}\mathbf{H}^{\mathbf{F}}_{n_{r,i}} & \mathbf{H}^{\mathbf{F}}_{\mathbf{n}_{f,i}} & \mathbf{0}_3\end{bmatrix} \tag{48}$$
$$= \begin{bmatrix}-\frac{2}{m}{}^M_{A_i}\mathbf{R}c_t r_i\mathbf{e}_z & \frac{1}{m}\mathbf{I}_3 & \mathbf{0}_3\end{bmatrix} \tag{49}$$

Start with the orientation integration Eq. (15), we have:

$${}^G_{k+1}\hat{\mathbf{R}}\left(\mathbf{I} + \lfloor \delta\boldsymbol{\theta}_{k+1}\rfloor\right) \simeq {}^G_k\hat{\mathbf{R}}\left(\mathbf{I} + \lfloor \delta\boldsymbol{\theta}_k\rfloor\right)\exp\left({}^k\boldsymbol{\theta}_{k+1}\right) \tag{50}$$
$$\mathbf{I} + \lfloor \delta\boldsymbol{\theta}_{k+1}\rfloor = {}^{k+1}_k\mathbf{R}\left(\mathbf{I} + \lfloor \delta\boldsymbol{\theta}_k\rfloor\right)\exp\left({}^k\hat{\boldsymbol{\theta}}_{k+1}\right)\exp\left(\mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1}){}^k\tilde{\boldsymbol{\theta}}_{k+1}\right) \tag{51}$$
$$= \mathbf{I} + \lfloor {}^{k+1}_k\hat{\mathbf{R}}\delta\boldsymbol{\theta}_k\rfloor + \lfloor \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1}){}^k\tilde{\boldsymbol{\theta}}_{k+1}\rfloor \tag{52}$$
$$\Rightarrow \delta\boldsymbol{\theta}_{k+1} = {}^{k+1}_k\hat{\mathbf{R}}\delta\boldsymbol{\theta}_k + \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1}){}^k\tilde{\boldsymbol{\theta}}_{k+1} \tag{53}$$

where from Eq. (23):

$${}^k\boldsymbol{\theta}_{k+1} = {}^k\boldsymbol{\omega}_k\Delta t + \frac{1}{2}\mathbf{A}\Delta t^2 \tag{54}$$
$$\Rightarrow {}^k\tilde{\boldsymbol{\theta}}_{k+1} \simeq \left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}\Delta t\right){}^k\tilde{\boldsymbol{\omega}}_k + \frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t^2\tilde{\mathbf{x}}_\theta \tag{55}$$

Therefore,

$$\delta\boldsymbol{\theta}_{k+1} \simeq {}^{k+1}_k\hat{\mathbf{R}}\delta\boldsymbol{\theta}_k + \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}\Delta t\right){}^k\tilde{\boldsymbol{\omega}}_k + \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t^2\right)\tilde{\mathbf{x}}_\theta \tag{56}$$

Finally, we can get the Jacobians w.r.t states as:

$$\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_k} = \begin{bmatrix}\frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial\delta\boldsymbol{\theta}_k} & \frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial {}^G\tilde{\mathbf{p}}_k} & \frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial {}^k\tilde{\boldsymbol{\omega}}_k} & \frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial {}^G\tilde{\mathbf{v}}_k}\end{bmatrix} \tag{57}$$
$$= \begin{bmatrix}\mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\omega}} & \mathbf{0}_3\end{bmatrix} \tag{58}$$
$$= \begin{bmatrix}{}^{k+1}_k\hat{\mathbf{R}} & \mathbf{0}_3 & \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}_3\Delta t\right) & \mathbf{0}_3\end{bmatrix} \tag{59}$$
$$\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_\theta} = \begin{bmatrix}\frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial\tilde{\mathbf{x}}_D} & \frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial\tilde{\mathbf{x}}_G} & \frac{\partial\delta\boldsymbol{\theta}_{k+1}}{\partial\tilde{\mathbf{x}}_{MI}}\end{bmatrix} \tag{60}$$
$$= \begin{bmatrix}\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_D} & \mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} \tag{61}$$
$$= \mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t^2\right) \tag{62}$$
$$= \frac{1}{2}\mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\Delta t^2\begin{bmatrix}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} \tag{63}$$
$$\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{n}_M} = \frac{1}{2}\mathbf{J}_r({}^k\hat{\boldsymbol{\theta}}_{k+1})\Delta t^2\mathbf{H}^{\mathbf{A}}_{\mathbf{n}_M} \tag{64}$$

From angular velocity integration Eq.(19) we can get:

$$
\begin{aligned}
{}^{k+1}\boldsymbol{\omega} &= {}^{k+1}\hat{\boldsymbol{\omega}} + {}^{k+1}\tilde{\boldsymbol{\omega}} && (65)\\
&= \exp^{\top}\left({}^{k}\boldsymbol{\theta}_{k+1}\right)\left({}^{k}\boldsymbol{\omega}_k + \mathbf{A}\Delta t\right) && (66)\\
&\simeq \left(\mathbf{I}_3 - \lfloor \mathbf{J}_r\left({}^{k}\hat{\boldsymbol{\theta}}_{k+1}\right){}^{k}\tilde{\boldsymbol{\theta}}_{k+1}\rfloor\right){}^{k}_{k+1}\hat{\mathbf{R}}\left({}^{k}\boldsymbol{\omega}_k + \mathbf{A}\Delta t\right) && (67)\\
\Rightarrow {}^{k+1}\tilde{\boldsymbol{\omega}}_{k+1} &= \left[{}^{k+1}_{k}\hat{\mathbf{R}} + {}^{k+1}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t + \lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\boldsymbol{\omega}}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}_3\Delta t\right)\right]{}^{k}\tilde{\boldsymbol{\omega}}_k && (68)\\
&+ \left[\lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\boldsymbol{\omega}}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t^2\right) + {}^{k+1}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t\right]\tilde{\mathbf{x}}_\theta && (69)
\end{aligned}
$$

Note that ${}^{k}\hat{\boldsymbol{\omega}}_{k+1} = {}^{k}\hat{\boldsymbol{\omega}} + \hat{\mathbf{A}}\Delta t$, Jacobians are derived as:

$$
\begin{aligned}
\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_k} &= \begin{bmatrix}\frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial \delta\boldsymbol{\theta}_k} & \frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial {}^{G}\tilde{\mathbf{p}}_k} & \frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial {}^{k}\tilde{\boldsymbol{\omega}}_k} & \frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial {}^{G}\tilde{\mathbf{v}}_k}\end{bmatrix} && (70)\\
&= \begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}^{\omega}_{\omega} & \mathbf{0}_3\end{bmatrix} && (71)\\
\mathbf{H}^{\boldsymbol{\omega}}_{\boldsymbol{\omega}} &= {}^{k+1}_{k}\hat{\mathbf{R}} + {}^{k+1}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t + \lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\omega}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}_3\Delta t\right) && (72)\\
\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_\theta} &= \begin{bmatrix}\frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial \tilde{\mathbf{x}}_D} & \frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial \tilde{\mathbf{x}}_G} & \frac{\partial {}^{k+1}\tilde{\boldsymbol{\omega}}}{\partial \tilde{\mathbf{x}}_{MI}}\end{bmatrix} && (73)\\
&= \begin{bmatrix}\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_D} & \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (74)\\
&= \frac{1}{2}\lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\omega}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t^2 + {}^{k+1}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_\theta}\Delta t && (75)\\
&= \left(\frac{1}{2}\lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\omega}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\Delta t^2 + {}^{k+1}_{k}\hat{\mathbf{R}}\Delta t\right)\begin{bmatrix}\mathbf{H}^{\mathbf{A}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (76)\\
\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{n}_M} &= \frac{1}{2}\lfloor{}^{k+1}_{k}\hat{\mathbf{R}}{}^{k}\hat{\omega}_{k+1}\rfloor\mathbf{J}_r({}^{k}\hat{\boldsymbol{\theta}}_{k+1})\mathbf{H}^{\mathbf{A}}_{\mathbf{n}_M}\Delta t^2 + {}^{k+1}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{A}}_{\mathbf{n}_M}\Delta t && (77)
\end{aligned}
$$

From the linear velocity integration Eq.(21):

$$
\begin{aligned}
{}^{G}\mathbf{v}_{k+1} &= {}^{G}\hat{\mathbf{v}}_{k+1} + {}^{G}\tilde{\mathbf{v}}_{k+1} && (78)\\
&= {}^{G}\hat{\mathbf{v}}_k + {}^{G}\tilde{\mathbf{v}}_k + {}^{G}_{k}\hat{\mathbf{R}}\left(\mathbf{I}_3 + \lfloor\delta\boldsymbol{\theta}_k\rfloor\right)\left(\hat{\mathbf{F}} + \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta}\tilde{\mathbf{x}}_\theta\right)\Delta t - {}^{G}\mathbf{g}\Delta t && (79)\\
\Rightarrow {}^{G}\tilde{\mathbf{v}}_{k+1} &= -{}^{G}_{k}\hat{\mathbf{R}}\lfloor\mathbf{F}\rfloor\Delta t\delta\boldsymbol{\theta}_k + {}^{G}\tilde{\mathbf{v}}_k + {}^{G}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta}\Delta t\tilde{\mathbf{x}}_\theta && (80)
\end{aligned}
$$

Therefore, Jacobians w.r.t states are as follows:

$$
\begin{aligned}
\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_k} &= \begin{bmatrix}\frac{\partial {}^{G}\tilde{\mathbf{v}}_{k+1}}{\partial \delta\boldsymbol{\theta}_k} & \frac{\partial {}^{G}\tilde{\mathbf{v}}_{k+1}}{\partial {}^{G}\tilde{\mathbf{p}}_k} & \frac{\partial {}^{G}\tilde{\mathbf{v}}_{k+1}}{\partial {}^{k}\tilde{\boldsymbol{\omega}}_k} & \frac{\partial {}^{G}\tilde{\mathbf{v}}_{k+1}}{\partial {}^{G}\tilde{\mathbf{v}}_k}\end{bmatrix} && (81)\\
&= \begin{bmatrix}\mathbf{H}^{\mathbf{v}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} && (82)\\
&= \begin{bmatrix}-{}^{G}_{k}\hat{\mathbf{R}}\lfloor\mathbf{F}\rfloor\Delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} && (83)\\
\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_\theta} &= \begin{bmatrix}\frac{\partial {}^{G}\tilde{\mathbf{v}}_k}{\partial \tilde{\mathbf{x}}_D} & \frac{\partial {}^{G}\tilde{\mathbf{v}}_k}{\partial \tilde{\mathbf{x}}_G} & \frac{\partial {}^{G}\tilde{\mathbf{v}}_k}{\partial \tilde{\mathbf{x}}_{MI}}\end{bmatrix} && (84)\\
&= \begin{bmatrix}\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{v}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (85)\\
&= {}^{G}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta}\Delta t && (86)\\
&= {}^{G}_{k}\hat{\mathbf{R}}\Delta t\begin{bmatrix}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (87)\\
\mathbf{H}^{\mathbf{v}}_{\mathbf{n}_M} &= {}^{G}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{n}_M}\Delta t && (88)
\end{aligned}
$$

From position integration(Eq.17) we can get:

$$
\begin{aligned}
{}^{G}\mathbf{p}_{k+1} &= {}^{G}\hat{\mathbf{p}}_{k+1} + {}^{G}\tilde{\mathbf{p}}_{k+1} && (89)\\
&= {}^{G}\hat{\mathbf{p}}_{k} + {}^{G}\tilde{\mathbf{p}}_{k} + \left({}^{G}\hat{\mathbf{v}}_{k} + {}^{G}\tilde{\mathbf{v}}_{k}\right)\Delta t + \frac{1}{2}\left[{}_{k}^{G}\hat{\mathbf{R}}\left(\mathbf{I}_3 + \lfloor\delta\boldsymbol{\theta}_k\rfloor\right)\left(\hat{\mathbf{F}} + \tilde{\mathbf{F}}\right) - {}^{G}\mathbf{g}\right]\Delta t^2 && (90)\\
\Rightarrow {}^{G}\tilde{\mathbf{p}}_{k} &= {}^{G}\tilde{\mathbf{p}}_{k} - \frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\lfloor\mathbf{F}\rfloor\Delta t^2\delta\boldsymbol{\theta}_k + \frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\tilde{\mathbf{x}}_\theta}\Delta t^2\tilde{\mathbf{x}}_\theta + {}^{G}\tilde{\mathbf{v}}_{k}\Delta t && (91)
\end{aligned}
$$

Therefore, we get the Jacobians as:

$$
\begin{aligned}
\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_k} &= \begin{bmatrix}\frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial \delta\boldsymbol{\theta}_k} & \frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial {}^{G}\tilde{\mathbf{p}}_k} & \frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial {}^{k}\tilde{\boldsymbol{\omega}}} & \frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial {}^{G}\tilde{\mathbf{v}}_k}\end{bmatrix} && (92)\\
&= \begin{bmatrix}\mathbf{H}^{\mathbf{p}}_{\boldsymbol{\theta}} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t\end{bmatrix} && (93)\\
&= \begin{bmatrix}-\frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\lfloor\mathbf{F}\rfloor\Delta t^2 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t\end{bmatrix} && (94)\\
\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_\theta} &= \begin{bmatrix}\frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial \tilde{\mathbf{x}}_D} & \frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial \tilde{\mathbf{x}}_G} & \frac{\partial {}^{G}\tilde{\mathbf{p}}_k}{\partial \tilde{\mathbf{x}}_{MI}}\end{bmatrix} && (95)\\
&= \begin{bmatrix}\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{p}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (96)\\
&= \frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta}\Delta t^2 && (97)\\
&= \frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\Delta t^2\begin{bmatrix}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6}\end{bmatrix} && (98)\\
\mathbf{H}^{\mathbf{p}}_{\mathbf{n}_M} &= \frac{1}{2}{}_{k}^{G}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{n}_M}\Delta t^2 && (99)
\end{aligned}
$$

With the above preintegration, we can have the following integrated discrete-time MAV dynamic model and its linearization:

$$
\begin{aligned}
\mathbf{x}_{M_{k+1}} &= \mathbf{g}_M(\mathbf{x}_{M_k}, \mathbf{x}_\theta, \mathbf{n}_M) && (100)\\
\tilde{\mathbf{x}}_{M_{k+1}} &\simeq \boldsymbol{\Phi}_M(k+1,k)\tilde{\mathbf{x}}_{M_k} + \boldsymbol{\Phi}_\theta\tilde{\mathbf{x}}_\theta + \mathbf{G}_n\mathbf{n}_M && (101)
\end{aligned}
$$

$$
\begin{bmatrix}\delta\boldsymbol{\theta}_{k+1}\\ {}^{G}\tilde{\mathbf{p}}_{k+1}\\ {}^{k+1}\tilde{\boldsymbol{\omega}}_{k+1}\\ {}^{G}\tilde{\mathbf{v}}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\omega}} & \mathbf{0}_3\\ \mathbf{H}^{\mathbf{p}}_{\boldsymbol{\theta}} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}^{\boldsymbol{\omega}}_{\boldsymbol{\omega}} & \mathbf{0}_3\\ \mathbf{H}^{\mathbf{v}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{\theta}_k\\ {}^{G}\tilde{\mathbf{p}}_k\\ {}^{k}\tilde{\boldsymbol{\omega}}_k\\ {}^{G}\tilde{\mathbf{v}}_k\end{bmatrix} + \begin{bmatrix}\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_\theta}\\ \mathbf{H}^{\mathbf{p}}_{\mathbf{x}_\theta}\\ \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_\theta}\\ \mathbf{H}^{\mathbf{v}}_{\mathbf{x}_\theta}\end{bmatrix}\tilde{\mathbf{x}}_\theta + \begin{bmatrix}\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{n}_M}\\ \mathbf{H}^{\mathbf{p}}_{\mathbf{n}_M}\\ \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{n}_M}\\ \mathbf{H}^{\mathbf{v}}_{\mathbf{n}_M}\end{bmatrix}\mathbf{n}_M \quad (102)
$$

2 MSCKF-based Parameter Estimation

We first extend the standard MSCKF-based VIO estimator [4] to additionally estimate MAV system parameters, which later will be shown to be not robust due to the MAV modelling inaccuracy that is almost inevitable in practice. Specifically, we estimate:

$$
\begin{aligned}
\mathbf{x}_k &= \begin{bmatrix}\mathbf{x}_A^\top & \mathbf{x}_\theta^\top\end{bmatrix}^\top && (103)\\
\mathbf{x}_A^\top &= \begin{bmatrix}\mathbf{x}_{I_k}^\top & \mathbf{x}_C^\top\end{bmatrix}^\top && (104)
\end{aligned}
$$

where:

$$
\begin{aligned}
\mathbf{x}_{I_k} &= \begin{bmatrix}{}_{G}^{I_k}\bar{q}^\top & {}^{G}\mathbf{p}_{I_k}^\top & {}^{G}\mathbf{v}_{I_k}^\top & \mathbf{b}_{g_k}^\top & \mathbf{b}_{a_k}^\top\end{bmatrix}^\top && (105)\\
\mathbf{x}_C &= \begin{bmatrix}\mathbf{x}_{T_{k-1}}^\top & \cdots & \mathbf{x}_{T_{k-c}}^\top\end{bmatrix}^\top && (106)\\
\mathbf{x}_{T_i} &= \begin{bmatrix}{}_{G}^{I_i}\bar{q}^\top & {}^{G}\mathbf{p}_{I_i}^\top & {}^{I_i}\boldsymbol{\omega}^\top & {}^{G}\mathbf{v}_{I_i}^\top\end{bmatrix}^\top && (107)
\end{aligned}
$$

We define the "active" state $\mathbf{x}_A$ and parameter state $\mathbf{x}_\theta$ [see Eq. (2)]. The active state contains the current IMU state $\mathbf{x}_{I_k}$ and c historical clone states $\mathbf{x}_C$. Each historical clone contains the IMU pose, linear and angular velocities $\{{}^I\boldsymbol{\omega}, {}^G\mathbf{v}_I\}$ [5]. $\mathbf{b}_g$ and $\mathbf{b}_a$ are the gyroscope and accelerometer biases, respectively. Note that we follow similar steps in [5] to clone $\{{}^I\boldsymbol{\omega}, {}^G\mathbf{v}_I\}$ and propagate the state.

2.1 IMU Kinematic Model

The IMU kinematics are used to evolve the state from time t_k to t_{k+1} [1]:

$$
\begin{aligned}
{}^I_G\dot{\bar{q}}(t) &= \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}(t)){}^I_G\bar{q}(t) && (108)\\
{}^G\dot{\mathbf{p}}_I(t) &= {}^G\mathbf{v}_I(t) && (109)\\
{}^G\dot{\mathbf{v}}_I(t) &= {}^{I(t)}_G\mathbf{R}^\top\mathbf{a}(t) - {}^G\mathbf{g} && (110)\\
\dot{\mathbf{b}}_g(t) &= \mathbf{n}_{wg}(t) && (111)\\
\dot{\mathbf{b}}_a(t) &= \mathbf{n}_{wa}(t) && (112)
\end{aligned}
$$

where $\boldsymbol{\omega}(t) = [\omega_1\ \omega_2\ \omega_3]^\top$ and $\mathbf{a}(t)$ are the angular velocity and acceleration in the IMU local frame $\{I\}$; $\boldsymbol{\Omega}(\boldsymbol{\omega}(t)) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix}$ where $\lfloor\cdot\rfloor$ is the skew-symmetric matrix. $\mathbf{n}_{wg}$ and $\mathbf{n}_{wa}$ are white Gaussian noise that drive the IMU biases. A canonical three-axis IMU provides linear acceleration and angular velocity measurements, ${}^I\mathbf{a}_m$ and ${}^I\boldsymbol{\omega}_m$, expressed in the local IMU frame $\{I\}$ modeled as:

$$
\begin{aligned}
\mathbf{a}_m(t) &= \mathbf{a}(t) + \mathbf{b}_a(t) + \mathbf{n}_a(t) && (113)\\
\boldsymbol{\omega}_m(t) &= \boldsymbol{\omega}(t) + \mathbf{b}_g(t) + \mathbf{n}_g(t) && (114)
\end{aligned}
$$

where $\mathbf{n}_g$ and $\mathbf{n}_a$ are zero-mean white Gaussian noise. ${}^I_G\mathbf{R}$ denotes the rotation matrix from global frame to local IMU frame. The IMU nonlinear kinematics can be formulated as follows:

$$
\mathbf{x}_{I_{k+1}} = \mathbf{g}_I\left(\mathbf{x}_{I_k}, {}^I\mathbf{a}_k, {}^I\boldsymbol{\omega}_k, \mathbf{n}_I\right) \tag{115}
$$

where $\mathbf{n}_I = [\mathbf{n}_g^\top\ \mathbf{n}_a^\top\ \mathbf{n}_{wg}^\top\ \mathbf{n}_{wa}^\top]^\top$. After linearization, the state translation matrix can be derived as[6]:

$$
\boldsymbol{\Phi}_I(k+1,k) = \begin{bmatrix}
\boldsymbol{\Phi}_{1,1} & \mathbf{0}_3 & \mathbf{0}_3 & \boldsymbol{\Phi}_{1,4} & \mathbf{0}_3 \\
\boldsymbol{\Phi}_{2,1} & \mathbf{I}_3 & \mathbf{I}_3\Delta t & \boldsymbol{\Phi}_{2,4} & \boldsymbol{\Phi}_{2,5} \\
\boldsymbol{\Phi}_{3,1} & \mathbf{0}_3 & \mathbf{I}_3 & \boldsymbol{\Phi}_{3,4} & \boldsymbol{\Phi}_{3,5} \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3
\end{bmatrix} \tag{116}
$$

with:

$$
\boldsymbol{\Phi}_{1,1} = {}^{I_k}_{I_{k+1}}\hat{\mathbf{R}}^\top \quad \boldsymbol{\Phi}_{1,4} = -\mathbf{J}_r\left({}^{I_k}\hat{\boldsymbol{\theta}}_{I_{k+1}}\right)\Delta t \quad \boldsymbol{\Phi}_{2,1} = -{}^{I_k}_G\hat{\mathbf{R}}^\top\lfloor\Xi_2\hat{\mathbf{a}}_k\rfloor \quad \boldsymbol{\Phi}_{2,4} = {}^{I_k}_G\hat{\mathbf{R}}^\top\Xi_4 \tag{117}
$$

$$
\boldsymbol{\Phi}_{2,5} = -{}^{I_k}_G\hat{\mathbf{R}}^\top\Xi_2 \quad \boldsymbol{\Phi}_{3,1} = -{}^{I_k}_G\hat{\mathbf{R}}^\top\lfloor\Xi_1\hat{\mathbf{a}}_k\rfloor \quad \boldsymbol{\Phi}_{3,4} = {}^{I_k}_G\hat{\mathbf{R}}^\top\Xi_3 \quad \boldsymbol{\Phi}_{3,5} = -{}^{I_k}_G\hat{\mathbf{R}}^\top\Xi_1 \tag{118}
$$

and:

$$
\boldsymbol{\Xi}_1 \triangleq \int_{t_k}^{t_{k+1}} \exp\left({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau\right)d\tau \qquad \boldsymbol{\Xi}_2 \triangleq \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} \exp\left({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau\right)d\tau ds \tag{119}
$$

$$
\boldsymbol{\Xi}_3 \triangleq \int_{t_k}^{t_{k+1}} {}^{I_k}_{I_\tau}\mathbf{R}\lfloor{}^{I_\tau}\hat{\mathbf{a}}\rfloor\mathbf{J}_r\left({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau\right)\delta\tau d\tau \qquad \boldsymbol{\Xi}_4 \triangleq \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^{I_k}_{I_\tau}\mathbf{R}\lfloor{}^{I_\tau}\hat{\mathbf{a}}\rfloor\mathbf{J}_r\left({}^{I_k}\hat{\boldsymbol{\omega}}\delta\tau\right)\delta\tau d\tau ds \tag{120}
$$

The noise Jacobian is derived as[7]:

$$\mathbf{G}_I(t) = \begin{bmatrix} -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -{}^{I_k}_{G}\hat{\mathbf{R}}^\top & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{121}$$

The standard EKF propagates the error covariance as:

$$\mathbf{P}_{k+1|k} = \mathbf{\Phi}_I(k+1,k)\mathbf{P}_{k|k}\mathbf{\Phi}_I(k+1,k)^\top + \mathbf{Q}_k \tag{122}$$

We use $\hat{\mathbf{x}}_{j|i}$ and $\mathbf{P}_{j|i}$ to represent the state estimate and covariance at time-step j computed using measurements up to time-step i. The discrete-time system noise covariance $\mathbf{Q}_k$ can be computed as:

$$\mathbf{Q}_k = \int_{t_k}^{t_{k+1}} \mathbf{\Phi}_I(k+1,\tau)\mathbf{G}(\tau)\mathbf{Q}\mathbf{G}(\tau)^\top\mathbf{\Phi}_I(k+1,\tau)^\top \mathrm{d}\tau \tag{123}$$

2.2 Visual Feature Measurements

When exploring the environment, bearing observations of static landmarks are tracked on the image plane. A bearing measurement seen at timestep k can be related to the state by the following (simplified for presentation, model in [8] is used):

$$\begin{aligned} \mathbf{z}_{C,k} &= \mathbf{h}(\mathbf{x}_{T_k}, {}^G\mathbf{p}_f) + \mathbf{n}_{C,k} && (124)\\ &=: \mathbf{\Lambda}({}^{C_k}\mathbf{p}_f) + \mathbf{n}_{C,k} && (125)\\ {}^{C_k}\mathbf{p}_f &= {}^C_I\mathbf{R}\,{}^{I_k}_G\mathbf{R}({}^G\mathbf{p}_f - {}^G\mathbf{p}_{I_k}) + {}^C\mathbf{p}_I && (126)\\ \mathbf{\Lambda}([x\ y\ z]^\top) &= \begin{bmatrix} x/z & y/z \end{bmatrix}^\top && (127) \end{aligned}$$

where $\mathbf{n}_{C,k}$ is the white Gaussian measurement noise. We can obtain the visual residual based Eq. (124):

$$\begin{aligned} \mathbf{r}_{C,k} &= \mathbf{z}_{C,k} - \mathbf{h}(\hat{\mathbf{x}}_{T_k}, {}^G\hat{\mathbf{p}}_f) - \mathbf{n}_{C,k} && (128)\\ &\simeq \mathbf{H}_{T_k}\tilde{\mathbf{x}}_{T_k} + \mathbf{H}_{f_k}{}^G\tilde{\mathbf{p}}_f - \mathbf{n}_{C,k} && (129) \end{aligned}$$

where $\mathbf{H}_{T_k}$ and $\mathbf{H}_{f_k}$ are the measurement Jacobians, $\tilde{\mathbf{x}}_{T_k}$ and ${}^G\tilde{\mathbf{p}}_f$ are the error states for the pose and feature, respectively.

$$\begin{aligned} \mathbf{H}_{T_k} &= \begin{bmatrix} \frac{\partial \mathbf{r}_{C,k}}{\partial \delta\theta_{I_k}} & \frac{\partial \mathbf{r}_{C,k}}{\partial {}^G\tilde{\mathbf{p}}_I} \end{bmatrix} && (130)\\ &= \mathbf{H}_p \begin{bmatrix} {}^C_I\hat{\mathbf{R}}\lfloor {}^{I_k}_G\hat{\mathbf{R}}({}^G\hat{\mathbf{p}}_f - {}^G\hat{\mathbf{p}}_{I_k})\rfloor & -{}^C_I\hat{\mathbf{R}}{}^{I_k}_G\hat{\mathbf{R}} \end{bmatrix} && (131)\\ \mathbf{H}_{f_k} &= \frac{\partial \mathbf{r}_{C,k}}{\partial {}^G\tilde{\mathbf{p}}_f} = \mathbf{H}_p{}^C_I\hat{\mathbf{R}}{}^{I_k}_G\hat{\mathbf{R}} && (132)\\ \mathbf{H}_p &= \begin{bmatrix} \frac{1}{{}^{C_k}\hat{z}_f} & 0 & -\frac{{}^{C_k}\hat{x}_f}{{}^{C_k}\hat{z}_f^2} \\ 0 & \frac{1}{{}^{C_k}\hat{z}_f} & -\frac{{}^{C_k}\hat{y}_f}{{}^{C_k}\hat{z}_f^2} \end{bmatrix} && (133) \end{aligned}$$

2.3 MAV Dynamics-Induced Measurements

We now define how to relate the preintegrated MAV dynamic model, Eq. (100), to our estimation state, Eq. (103). Through rigid body constraints, we can related the MAV state, Eq. (1), by:

$$\mathbf{x}_{M_k} = \mathbf{h}_{t,k}(\mathbf{x}_{T_k}, \mathbf{x}_\theta) \tag{134}$$

$$\tag{135}$$

where:

$$
\begin{aligned}
{}^{M_k}_{G}\mathbf{R} &= {}^{I}_{M}\mathbf{R}^\top {}^{I_k}_{G}\mathbf{R} && (136)\\
{}^{G}\mathbf{p}_{M_k} &= {}^{G}\mathbf{p}_{I_k} + {}^{G}_{I_k}\mathbf{R}\, {}^{I}\mathbf{p}_M && (137)\\
{}^{M_k}\boldsymbol{\omega} &= {}^{M}_{I}\mathbf{R}\, {}^{I_k}\boldsymbol{\omega} && (138)\\
{}^{G}\mathbf{v}_{M_k} &= {}^{G}\mathbf{v}_{I_k} + {}^{G}_{I_k}\mathbf{R}\lfloor {}^{I_k}\boldsymbol{\omega}\rfloor {}^{I}\mathbf{p}_M && (139)
\end{aligned}
$$

The MAV measurement residual and error-state Jacobians are defined accordingly:

$$
\begin{aligned}
\mathbf{r}_{M,k} &= \hat{\mathbf{x}}_{M_{k+1}} - \mathbf{g}_M(\hat{\mathbf{x}}_{M_k}, \hat{\mathbf{x}}_\theta, \mathbf{0}) && (140)\\
&= \mathbf{h}_{t,k+1}(\hat{\mathbf{x}}_{T_{k+1}}, \hat{\mathbf{x}}_\theta) - \mathbf{g}_M(\mathbf{h}_{t,k}(\hat{\mathbf{x}}_{T_k}, \hat{\mathbf{x}}_\theta), \hat{\mathbf{x}}_\theta, \mathbf{0}) \\
&\simeq \begin{bmatrix}\mathbf{H}_{T_{k+1}} & \mathbf{H}_{T_k} & \mathbf{H}_\theta\end{bmatrix}\tilde{\mathbf{x}}_k - \mathbf{G}_n\mathbf{n}_M && (141)\\
&\triangleq \begin{bmatrix}\mathbf{H}_A & \mathbf{H}_\theta\end{bmatrix}\tilde{\mathbf{x}}_k - \mathbf{G}_n\mathbf{n}_M && (142)
\end{aligned}
$$

where $\tilde{\mathbf{x}}_k = [\tilde{\mathbf{x}}_{T_{k+1}}^\top \;\; \tilde{\mathbf{x}}_{T_k}^\top \;\; \tilde{\mathbf{x}}_\theta^\top]^\top \triangleq [\tilde{\mathbf{x}}_A^\top \;\; \tilde{\mathbf{x}}_\theta^\top]^\top$ is the error state[1]. The Jacobians can be calculated through the chainrule:

$$\mathbf{H}_{T_{k+1}} = \frac{\partial \mathbf{h}_{t,k+1}}{\partial\, \mathbf{x}_{T_{k+1}}} \qquad \mathbf{H}_{T_k} = -\mathbf{\Phi}_M \frac{\partial \mathbf{h}_{t,k}}{\partial\, \mathbf{x}_{T_k}} \qquad \mathbf{H}_\theta = \frac{\partial \mathbf{h}_{t,k+1}}{\partial\, \mathbf{x}_\theta} - \mathbf{\Phi}_M \frac{\partial \mathbf{h}_{t,k}}{\partial\, \mathbf{x}_\theta} - \mathbf{\Phi}_\theta \tag{143}$$

In the follows, we drop the subscripts (e.g., k and $k+1$) for simplicity. From Eq.(136):

$$
\begin{aligned}
{}^{M}_{G}\mathbf{R} &= (\mathbf{I} - \lfloor\delta\boldsymbol{\theta}_M\rfloor){}^{M}_{G}\hat{\mathbf{R}} && (144)\\
&= {}^{I}_{M}\hat{\mathbf{R}}^\top(\mathbf{I} + \lfloor {}^{I}_{M}\delta\boldsymbol{\theta}\rfloor - \lfloor\delta\boldsymbol{\theta}_I\rfloor){}^{I}_{G}\hat{\mathbf{R}} && (145)\\
\Rightarrow \delta\boldsymbol{\theta}_M &= {}^{I}_{M}\hat{\mathbf{R}}^\top\delta\boldsymbol{\theta}_I - {}^{I}_{M}\hat{\mathbf{R}}^\top {}^{I}_{M}\delta\boldsymbol{\theta} && (146)
\end{aligned}
$$

From Eq. (137):

$$
\begin{aligned}
{}^{G}\mathbf{p}_M &= {}^{G}\hat{\mathbf{p}}_M + {}^{G}\tilde{\mathbf{p}}_M && (147)\\
&= {}^{G}\hat{\mathbf{p}}_I + {}^{G}\tilde{\mathbf{p}}_I + {}^{G}_{I}\hat{\mathbf{R}}\,(\mathbf{I} + \lfloor\delta\boldsymbol{\theta}_I\rfloor)\left({}^{I}\hat{\mathbf{p}}_M + {}^{I}\tilde{\mathbf{p}}_M\right) && (148)\\
\Rightarrow {}^{G}\tilde{\mathbf{p}}_M &= {}^{G}\tilde{\mathbf{p}}_I - {}^{G}_{I}\hat{\mathbf{R}}\lfloor {}^{I}\hat{\mathbf{p}}_M\rfloor\delta\boldsymbol{\theta}_I + {}^{G}_{I}\hat{\mathbf{R}}\,{}^{I}\tilde{\mathbf{p}}_M && (149)
\end{aligned}
$$

From Eq. (138):

$$
\begin{aligned}
{}^{M}\boldsymbol{\omega} &= {}^{M}\hat{\boldsymbol{\omega}} + {}^{M}\tilde{\boldsymbol{\omega}} && (150)\\
&= {}^{I}_{M}\hat{\mathbf{R}}^\top(\mathbf{I} + \lfloor {}^{I}_{M}\delta\boldsymbol{\theta}\rfloor){}^{I}\boldsymbol{\omega} && (151)\\
\Rightarrow {}^{M}\tilde{\boldsymbol{\omega}} &= -{}^{I}_{M}\hat{\mathbf{R}}^\top\lfloor {}^{I}\hat{\boldsymbol{\omega}}\rfloor {}^{I}_{M}\delta\boldsymbol{\theta} && (152)
\end{aligned}
$$

[1] Throughout this paper $\hat{\mathbf{x}}$ is used to denote the estimate of a random variable $\mathbf{x}$, while $\tilde{\mathbf{x}} = \mathbf{x} \boxminus \hat{\mathbf{x}}$ is the error of this estimate. We define the orientation error quaternion, $\delta\boldsymbol{\theta}$, as $\delta\bar{q} = \bar{q} \otimes \hat{\bar{q}}^{-1} \simeq [\frac{1}{2}\delta\theta^\top \;\; 1]^\top$ [1]. The updated estimate from a correction $\boldsymbol{\delta}\mathbf{x}$ is $\hat{\mathbf{x}}^{\oplus} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x}$.

From Eq. (139):

$$
\begin{aligned}
{}^G\mathbf{v}_M &= {}^G\hat{\mathbf{v}}_M + {}^G\tilde{\mathbf{v}}_M && (153)\\
&= {}^G\hat{\mathbf{v}}_I + {}^G\tilde{\mathbf{v}}_I - {}^G_I\mathbf{R}\lfloor\lfloor{}^I\boldsymbol{\omega}\rfloor{}^I\mathbf{p}_M\rfloor\delta\boldsymbol{\theta}_I && (154)\\
&= {}^G\hat{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\hat{\mathbf{p}}_M - {}^G_I\hat{\mathbf{R}}\lfloor\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\hat{\mathbf{p}}_M\rfloor\delta\boldsymbol{\theta}_I + {}^G\tilde{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\tilde{\mathbf{p}}_M && (155)\\
\Rightarrow {}^G\tilde{\mathbf{v}}_M &= -{}^G_I\hat{\mathbf{R}}\lfloor\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\hat{\mathbf{p}}_M\rfloor\delta\boldsymbol{\theta}_I + {}^G\tilde{\mathbf{v}}_I + {}^G_I\hat{\mathbf{R}}\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\tilde{\mathbf{p}}_M && (156)
\end{aligned}
$$

Finally, we can summarize as:

$$
\begin{aligned}
\mathbf{H}_T &= \begin{bmatrix} \frac{\partial\ \delta\boldsymbol{\theta}_M}{\partial\ \delta\boldsymbol{\theta}_I} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{\partial\ {}^G\tilde{\mathbf{p}}_M}{\partial\ \delta\boldsymbol{\theta}_I} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{\partial\ {}^G\tilde{\mathbf{v}}_M}{\partial\ \delta\boldsymbol{\theta}_I} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{21} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{H}_{41} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} && (157)\\
&= \begin{bmatrix} {}^I_M\hat{\mathbf{R}}^\top & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -{}^G_I\hat{\mathbf{R}}\lfloor{}^I\hat{\mathbf{p}}_M\rfloor & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -{}^G_I\hat{\mathbf{R}}\lfloor\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor{}^I\hat{\mathbf{p}}_M\rfloor & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} && (158)\\
\frac{\partial\ \mathbf{h}_t}{\partial\ \mathbf{x}_\theta} &= \begin{bmatrix} \frac{\partial\ \mathbf{h}_t}{\partial\ \mathbf{x}_D} & \frac{\partial\ \mathbf{h}_t}{\partial\ \mathbf{x}_G} & \frac{\partial\ \mathbf{h}_t}{\partial\ \mathbf{x}_{MI}} \end{bmatrix} && (159)\\
&= \begin{bmatrix} \mathbf{0}_{3\times 9} & \mathbf{0}_3 & \frac{\partial\ {}^G\delta\tilde{\boldsymbol{\theta}}_M}{\partial\ {}^I\tilde{\mathbf{p}}_M} \\ \mathbf{0}_{3\times 9} & \mathbf{0}_3 & \frac{\partial\ {}^G\tilde{\mathbf{p}}_M}{\partial\ {}^I\tilde{\mathbf{p}}_M} \\ \mathbf{0}_{3\times 9} & \frac{\partial\ {}^M\tilde{\boldsymbol{\omega}}}{\partial\ {}^I_M\delta\boldsymbol{\theta}} & \mathbf{0}_3 \\ \mathbf{0}_{3\times 9} & \mathbf{0}_3 & \frac{\partial\ {}^G\tilde{\mathbf{v}}_M}{\partial\ {}^I\tilde{\mathbf{p}}_M} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times 9} & \mathbf{0}_3 & -{}^I_M\hat{\mathbf{R}}^\top \\ \mathbf{0}_{3\times 9} & \mathbf{0}_3 & {}^G_I\hat{\mathbf{R}} \\ \mathbf{0}_{3\times 9} & -{}^I_M\hat{\mathbf{R}}^\top\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor & \mathbf{0}_3 \\ \mathbf{0}_{3\times 9} & \mathbf{0}_3 & {}^G_I\hat{\mathbf{R}}^\top\lfloor{}^I\hat{\boldsymbol{\omega}}\rfloor \end{bmatrix} && (160)
\end{aligned}
$$

3 Observability Analysis

Follow [9], we can do the observability analysis. In the analysis, we redefine the state as:

$$
\mathbf{x} = \begin{bmatrix} \mathbf{x}_I^\top & \mathbf{x}_M^\top & \mathbf{x}_\theta^\top & \mathbf{x}_f^\top \end{bmatrix}^\top \tag{161}
$$

The full system state translation matrix can be defined as:

$$
\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_I(k+1,k) & \mathbf{0}_{15\times 12} & \mathbf{0}_{15\times 23} & \mathbf{0}_{15\times 3} \\ \mathbf{0}_{12\times 15} & \boldsymbol{\Phi}_M(k+1,k) & \boldsymbol{\Phi}_\theta & \mathbf{0}_{12\times 3} \\ \mathbf{0}_{15\times 15} & \mathbf{0}_{15\times 12} & \mathbf{I}_{15} & \mathbf{0}_{15\times 3} \\ \mathbf{0}_{3\times 15} & \mathbf{0}_{3\times 12} & \mathbf{0}_{3\times 15} & \mathbf{I}_3 \end{bmatrix} \tag{162}
$$

where

$$
\mathbf{\Phi}_I(k+1,k) = \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{14} & \mathbf{0}_3 \\ \mathbf{\Phi}_{21} & \mathbf{I}_3 & \mathbf{I}_3\Delta t & \mathbf{\Phi}_{24} & \mathbf{\Phi}_{25} \\ \mathbf{\Phi}_{31} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{34} & \mathbf{\Phi}_{35} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{163}
$$

$$
\mathbf{\Phi}_M(k+1,k) = \begin{bmatrix} \mathbf{\Phi}_{66} & \mathbf{0}_3 & \mathbf{\Phi}_{68} & \mathbf{0}_3 \\ \mathbf{\Phi}_{76} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{88} & \mathbf{0}_3 \\ \mathbf{\Phi}_{96} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{H}^{\boldsymbol{\theta}}_{\boldsymbol{\omega}} & \mathbf{0}_3 \\ \mathbf{H}^{\mathbf{p}}_{\boldsymbol{\theta}} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}^{\boldsymbol{\omega}}_{\boldsymbol{\omega}} & \mathbf{0}_3 \\ \mathbf{H}^{\mathbf{v}}_{\boldsymbol{\theta}} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{164}
$$

$$
\mathbf{\Phi}_\theta = \begin{bmatrix} \mathbf{\Phi}_{6,10\sim 16} \\ \mathbf{\Phi}_{7,10\sim 16} \\ \mathbf{\Phi}_{8,10\sim 16} \\ \mathbf{\Phi}_{9,10\sim 16} \end{bmatrix} = \begin{bmatrix} \mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_\theta} \\ \mathbf{H}^{\mathbf{p}}_{\mathbf{x}_\theta} \\ \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_\theta} \\ \mathbf{H}^{\mathbf{v}}_{\mathbf{x}_\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_D} & \mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6} \\ \mathbf{H}^{\mathbf{p}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{p}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6} \\ \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_D} & \mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6} \\ \mathbf{H}^{\mathbf{v}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{v}}_{\mathbf{x}_G} & \mathbf{0}_{3\times 6} \end{bmatrix} \tag{165}
$$

The overall state translation matrix can be found as:

$$
\mathbf{\Phi} = \left[\begin{array}{ccccc|cccc|c|c}
\mathbf{\Phi}_{1,1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{1,4} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 15} & \mathbf{0}_3 \\
\mathbf{\Phi}_{2,1} & \mathbf{I}_3 & \mathbf{I}_3\Delta t & \mathbf{\Phi}_{2,4} & \mathbf{\Phi}_{2,5} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 15} & \mathbf{0}_3 \\
\mathbf{\Phi}_{3,1} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{3,4} & \mathbf{\Phi}_{3,5} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 15} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 15} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times 15} & \mathbf{0}_3 \\
\hline
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{6,6} & \mathbf{0}_3 & \mathbf{\Phi}_{6,8} & \mathbf{0}_3 & \mathbf{\Phi}_{6,10\sim 16} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{7,6} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{I}_3\Delta t & \mathbf{\Phi}_{7,10\sim 16} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{8,8} & \mathbf{0}_3 & \mathbf{\Phi}_{8,10\sim 16} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Phi}_{9,6} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Phi}_{9,10\sim 16} & \mathbf{0}_3 \\
\hline
\mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{0}_{15\times 3} & \mathbf{I}_{15\times 15} & \mathbf{0}_{15\times 3} \\
\hline
\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 15} & \mathbf{I}_{3\times 3}
\end{array}\right] \tag{166}
$$

From the MAV Dynamic-induced measurements, the overall measurement Jacobian can be summarized as:

$$
\mathbf{H}_M = \begin{bmatrix} \frac{\partial\ \mathbf{r}_M}{\partial\ \tilde{\mathbf{x}}_I} & \frac{\partial\ \mathbf{r}_M}{\partial\ \tilde{\mathbf{x}}_M} & \frac{\partial\ \mathbf{r}_M}{\partial\ \tilde{\mathbf{x}}_\theta} & \frac{\partial\ \mathbf{r}_M}{\partial\ \tilde{\mathbf{x}}_f} \end{bmatrix} \tag{167}
$$

$$
= \left[\begin{array}{ccccc|cccc|c|c}
\mathbf{H}_{1,1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{1,10\sim 16} & \mathbf{0}_3 \\
\mathbf{H}_{2,1} & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{2,10\sim 16} & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{H}_{3,10\sim 16} & \mathbf{0}_3 \\
\mathbf{H}_{4,1} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{H}_{4,10\sim 16} & \mathbf{0}_3
\end{array}\right] \tag{168}
$$

$$
\begin{bmatrix} \mathbf{H}_{1,10\sim 16} \\ \mathbf{H}_{2,10\sim 16} \\ \mathbf{H}_{3,10\sim 16} \\ \mathbf{H}_{4,10\sim 16} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{1,16} \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{2,16} \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{3,15} & \mathbf{0}_3 \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{H}_{4,16} \end{bmatrix} \tag{169}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^I_M\hat{\mathbf{R}}^\top \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_I\hat{\mathbf{R}} \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^I_M\hat{\mathbf{R}}^\top \lfloor {}^I\hat{\boldsymbol{\omega}} \rfloor & \mathbf{0}_3 \\ 0 & 0 & 0 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_I\hat{\mathbf{R}}^\top \lfloor {}^I\hat{\boldsymbol{\omega}} \rfloor \end{bmatrix} \tag{170}
$$

The observabiliy matrix can be found as:

$$\mathcal{M} \triangleq \begin{bmatrix} \mathbf{H}_0\mathbf{\Phi}(0,0) \\ \mathbf{H}_1\mathbf{\Phi}(1,0) \\ \vdots \\ \mathbf{H}_{k+1}\mathbf{\Phi}(k+1,0) \end{bmatrix} \tag{171}$$

For a given block row of this matrix, we have:

$$\mathbf{H}_k\mathbf{\Phi}(k,0) = \begin{bmatrix} \mathbf{\Gamma}_{1,1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Gamma}_{1,4} & \mathbf{0}_3 & \mathbf{\Gamma}_{1,6} & \mathbf{0}_3 & \mathbf{\Gamma}_{1,8} & \mathbf{0}_3 & \mathbf{\Gamma}_{1,10\sim16} & \mathbf{0}_3 \\ \mathbf{\Gamma}_{2,1} & \mathbf{I}_3 & \mathbf{I}_3\Delta t & \mathbf{\Gamma}_{2,4} & \mathbf{\Gamma}_{2,5} & \mathbf{\Gamma}_{2,6} & -\mathbf{I}_3 & \mathbf{0}_3 & -\mathbf{I}_3\Delta t & \mathbf{\Gamma}_{2,10\sim16} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{\Gamma}_{3,8} & \mathbf{0}_3 & \mathbf{\Gamma}_{3,10\sim16} & \mathbf{0}_3 \\ \mathbf{\Gamma}_{4,1} & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{\Gamma}_{4,4} & \mathbf{\Gamma}_{4,5} & \mathbf{\Gamma}_{4,6} & \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{I}_3 & \mathbf{\Gamma}_{4,10\sim16} & \mathbf{0}_3 \end{bmatrix} \tag{172}$$

where

$$\mathbf{\Gamma}_{1,1} = \mathbf{H}_{1,1}\mathbf{\Phi}_{1,1} = {}^{I}_{M}\hat{\mathbf{R}}^\top {}^{I_k}_{I_0}\hat{\mathbf{R}} \tag{173}$$

$$\mathbf{\Gamma}_{1,4} = \mathbf{H}_{1,1}\mathbf{\Phi}_{1,4} = {}^{I}_{M}\hat{\mathbf{R}}^\top\mathbf{\Phi}_{1,4} = -{}^{I}_{M}\hat{\mathbf{R}}^\top\mathbf{J}_r\left({}^{k}\hat{\boldsymbol{\theta}}_{k+1}\right)\Delta t \tag{174}$$

$$\mathbf{\Gamma}_{1,6} = -\mathbf{\Phi}_{6,6} = -{}^{M_k}_{M_0}\hat{\mathbf{R}} \tag{175}$$

$$\mathbf{\Gamma}_{1,8} = -\mathbf{\Phi}_{6,8} = -\mathbf{J}_r({}^{M_0}\hat{\boldsymbol{\theta}}_{M_k})\left(\frac{1}{2}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}_3\Delta t\right) \tag{176}$$

$$\mathbf{\Gamma}_{2,1} = \mathbf{H}_{2,1}\mathbf{\Phi}_{1,1} + \mathbf{\Phi}_{2,1} = -{}^{G}_{I_k}\mathbf{R}\lfloor {}^{I}\hat{\mathbf{p}}_M\rfloor {}^{I_k}_{I_0}\mathbf{R} - \lfloor \mathbf{p}_{I_k} - \mathbf{p}_{I_0} - \mathbf{v}_{I_0} + \frac{1}{2}{}^{G}\mathbf{g}\Delta t_k^2\rfloor {}^{G}_{I_0}\mathbf{R} \tag{177}$$

$$\mathbf{\Gamma}_{2,4} = \mathbf{H}_{2,1}\mathbf{\Phi}_{1,4} + \mathbf{\Phi}_{2,4} = -{}^{G}_{I_k}\mathbf{R}\lfloor {}^{I}\hat{\mathbf{p}}_M\rfloor(-\mathbf{J}_r({}^{M_0}\boldsymbol{\theta}_{M_k}))\Delta t_k + {}^{G}_{I_0}\mathbf{R}\mathbf{\Xi}_4 \tag{178}$$

$$\mathbf{\Gamma}_{2,5} = \mathbf{\Phi}_{2,5} = -{}^{G}_{I_0}\mathbf{R}\mathbf{\Xi}_2 \tag{179}$$

$$\mathbf{\Gamma}_{2,6} = -\mathbf{\Phi}_{7,6} = \frac{1}{2}{}^{G}_{k}\hat{\mathbf{R}}\mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta}\Delta t^2 = \lfloor {}^{G}\mathbf{p}_{M_k} - {}^{G}\mathbf{p}_{M_0} - {}^{G}\mathbf{v}_{M_0}\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\rfloor \tag{180}$$

$$\mathbf{\Gamma}_{3,8} = -\mathbf{H}^{\boldsymbol{\omega}}_{\boldsymbol{\omega}} = -{}^{M_k}_{M_0}\mathbf{R} - {}^{M_k}_{M_0}\mathbf{R}\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t - \lfloor {}^{M_k}_{M_0}\mathbf{R}^{M_0}\boldsymbol{\omega}_{M_k}\rfloor\mathbf{J}_r({}^{M_0}\boldsymbol{\theta}_{M_k})\frac{1}{2}\left(\mathbf{H}^{\mathbf{A}}_{\boldsymbol{\omega}}\Delta t^2 + \mathbf{I}_3\Delta t\right) \tag{181}$$

$$\mathbf{\Gamma}_{4,1} = \mathbf{H}_{4,1}\mathbf{\Phi}_{1,1} + \mathbf{\Phi}_{3,1} = -{}^{G}_{I_k}\hat{\mathbf{R}}\lfloor\lfloor {}^{I_k}\hat{\boldsymbol{\omega}}\rfloor {}^{I}\hat{\mathbf{p}}_M\rfloor {}^{I_k}_{I_0}\hat{\mathbf{R}} - \lfloor {}^{G}\mathbf{v}_{I_k} - {}^{G}\mathbf{v}_{I_0} + {}^{G}\mathbf{g}\Delta t_k\rfloor {}^{G}_{I_0}\mathbf{R} \tag{182}$$

$$\mathbf{\Gamma}_{4,4} = \mathbf{H}_{4,1}\mathbf{\Phi}_{1,4} + \mathbf{\Phi}_{3,4} = -{}^{G}_{I_k}\hat{\mathbf{R}}\lfloor\lfloor {}^{I_k}\hat{\boldsymbol{\omega}}\rfloor {}^{I}\hat{\mathbf{p}}_M\rfloor(-\mathbf{J}_r({}^{M_0}\boldsymbol{\theta}_{M_k}))\Delta t_k + {}^{G}_{I_0}\mathbf{R}\mathbf{\Xi}_3 \tag{183}$$

$$\mathbf{\Gamma}_{4,5} = \mathbf{H}_{4,3}\mathbf{\Phi}_{3,5} = -{}^{G}_{I_0}\mathbf{R}\mathbf{\Xi}_1 \tag{184}$$

$$\mathbf{\Gamma}_{4,6} = \mathbf{\Phi}_{9,6} = {}^{G}_{M_0}\mathbf{R}\lfloor\mathbf{F}\rfloor\Delta t = \lfloor {}^{G}\mathbf{v}_{M_k} - {}^{G}\mathbf{v}_{M_0} + {}^{G}\mathbf{g}\Delta t\rfloor {}^{G}_{M_0}\mathbf{R} \tag{185}$$

For the system parameters we conclude:

$$\mathcal{M}_\theta = \begin{bmatrix} \mathbf{\Gamma}_{1,10\sim16} \\ \mathbf{\Gamma}_{2,10\sim16} \\ \mathbf{\Gamma}_{3,10\sim16} \\ \mathbf{\Gamma}_{4,10\sim16} \end{bmatrix} = \begin{bmatrix} -\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_D} & -\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_G} & \mathbf{0}_3 & -{}^{I}_{M}\hat{\mathbf{R}}^\top \\ -\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_D} & -\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_G} & \mathbf{0}_3 & {}^{G}_{I}\hat{\mathbf{R}} \\ -\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_D} & -\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_G} & -{}^{I}_{M}\hat{\mathbf{R}}^\top\lfloor {}^{I}\hat{\boldsymbol{\omega}}\rfloor & \mathbf{0}_3 \\ -\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_D} & -\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_G} & \mathbf{0}_3 & {}^{G}_{I}\hat{\mathbf{R}}^\top\lfloor {}^{I}\hat{\boldsymbol{\omega}}\rfloor \end{bmatrix} \tag{186}$$

where

$$\begin{bmatrix} -\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_D} & -\mathbf{H}^{\boldsymbol{\theta}}_{\mathbf{x}_G} \\ -\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_D} & -\mathbf{H}^{\mathbf{p}}_{\mathbf{x}_G} \\ -\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_D} & -\mathbf{H}^{\boldsymbol{\omega}}_{\mathbf{x}_G} \\ -\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_D} & -\mathbf{H}^{\mathbf{v}}_{\mathbf{x}_G} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{J}_r({}^{0}\hat{\boldsymbol{\theta}}_k)\Delta t^2 & \mathbf{0}_3 \\ \mathbf{0}_3 & -\frac{1}{2}{}^{G}_{M_0}\mathbf{R}\Delta t^2 \\ -\frac{1}{2}\lfloor {}^{M_k}_{M_0}\hat{\mathbf{R}}^{M_0}\hat{\boldsymbol{\omega}}_{M_k}\rfloor\mathbf{J}_r({}^{M_0}\hat{\boldsymbol{\theta}}_{M_k})\Delta t^2 - {}^{M_k}_{M_0}\hat{\mathbf{R}}\Delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & -{}^{G}_{M_0}\mathbf{R}\Delta t \end{bmatrix} \begin{bmatrix} \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_D} & \mathbf{H}^{\mathbf{A}}_{\mathbf{x}_G} \\ \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_\theta} & \mathbf{H}^{\mathbf{F}}_{\mathbf{x}_G} \end{bmatrix} \tag{187}$$

and

$$\begin{bmatrix} \mathbf{H}_{\mathbf{x}_D}^{\mathbf{A}} & \mathbf{H}_{\mathbf{x}_G}^{\mathbf{A}} \\ \mathbf{H}_{\mathbf{x}_\theta}^{\mathbf{F}} & \mathbf{H}_{\mathbf{x}_G}^{\mathbf{F}} \end{bmatrix} \tag{188}$$

$$= \begin{bmatrix} {}^k\mathbf{J}^{-1}\sum_{i=1}^{N_r}\lfloor {}^M\mathbf{p}_{A_i}\mathbf{e}_z\mathbf{n}_i^2 \rfloor & {}^k\mathbf{J}^{-1}\sum_{i=1}^{N_r} {}^M_{A_i}\mathbf{R}\lambda_i\mathbf{e}_z\mathbf{n}_i^2 & \mathbf{0}_{3\times 1} & \cdot\cdot\ \frac{\partial \mathbf{A}}{\partial j_i}\cdot\cdot & -{}^k\mathbf{J}^{-1}\sum_{i=1}^{N_r}\lfloor {}^k\mathbf{F}_i \rfloor \\ \frac{1}{m}\sum_{i=1}^{N_r}\mathbf{e}_z\mathbf{n}_i^2 & \mathbf{0}_{3\times 1} & \frac{1}{m^2}{}^k\mathbf{F} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \tag{189}$$

We can find that c_t, c_m, m and ${}^M\mathbf{j}$ are jointly unobservable, with null space $\mathbf{N}$ as $\mathbf{\Phi}\mathbf{N} = \mathbf{0}$:

$$\mathbf{N} = \begin{bmatrix} \mathbf{0}_{15\times 1} \\ \mathbf{0}_{12\times 1} \\ c_t \\ c_m \\ m \\ j_1 \\ j_2 \\ j_3 \\ \mathbf{0}_{6\times 1} \end{bmatrix} \tag{190}$$

4 Simulation

4.1 Rotor Speed Generation

With the B-spline formulation of the simulated trajectory, we can generate IMU angular velocity ${}^I\boldsymbol{\omega}$, angular acceleration ${}^I\boldsymbol{\alpha}$, and linear acceleration ${}^G\mathbf{a}_I$ at any desired time. The MAV motion status is computed with the rigid body constraints accordingly:

$$ {}^M\boldsymbol{\omega} = {}^M_I\mathbf{R}\,{}^I\boldsymbol{\omega} \tag{191}$$

$$ {}^G\mathbf{a}_M = {}^G\mathbf{a}_I + {}^I_G\mathbf{R}^\top \left(\lfloor {}^I\boldsymbol{\omega} \rfloor \lfloor {}^I\boldsymbol{\omega} \rfloor + \lfloor {}^I\boldsymbol{\alpha} \rfloor\right) {}^I\mathbf{p}_M \tag{192}$$

The desired total force and moment at a specific time are:

$$\begin{bmatrix} {}^M\mathbf{F} \\ {}^M\mathbf{M} \end{bmatrix} = \begin{bmatrix} m \cdot {}^M_G\mathbf{R}\left({}^G\mathbf{a}_M + {}^G\mathbf{g}\right) \\ {}^M\mathbf{J}\,{}^M\boldsymbol{\alpha}_M + \lfloor {}^M\boldsymbol{\omega}_M \rfloor {}^M\mathbf{J}\,{}^M\boldsymbol{\omega}_M \end{bmatrix} \tag{193}$$

Then, with Eq. (6) and (7), we formulate the following linear system to solve for the rotor speeds:

$$\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} \cdot \mathbf{r} = \begin{bmatrix} {}^M\mathbf{F} \\ {}^M\mathbf{M} \end{bmatrix} \tag{194}$$

where $\mathbf{r} = [r_1^2 \ \cdots \ r_4^2]^\top$ and $\mathbf{B}_i$ are defined as:

$$\mathbf{B}_i = \begin{bmatrix} c_t {}^M_{A_i}\mathbf{R}\mathbf{e}_z \\ c_m\lambda_i {}^M_{A_i}\mathbf{R}\mathbf{e}_z + c_t\lfloor {}^M\mathbf{p}_{A_i}\mathbf{e}_z \rfloor \end{bmatrix} \tag{195}$$

Finally, white Gaussian noise $n_{r,i}$ is added to r_i for realistic simulation of rotor encoder measurement $r_{m,i}$ (e.g., $r_{m,i} = n_{r,i} + r_i$).

References

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