Supplementary Materials: Revisit Invariant Error States for Consistent Visual-Inertial Navigation

Yulin Yang - yuyang@udel.edu
Chuchu Chen - ccchu@udel.edu

Department of Mechanical Engineering
University of Delaware, Delaware, USA

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Contents

1 Introduction

We will show the original RI-EKF and decoupled RI-EKF derivations with IMU dynamics. Then, we will build VINS systems based RI-EKF and DRI-EKF.

2 Lie Group for IMU State

Since $\mathbf{SE}_2(3)$ [1] is used for IMU navigation state, we introduce the basic notations and properties for this Lie group.

2.1 Definition of $\mathbf{SE}_2(3)$

The special Euclidean group for IMU state including orientation $\mathbf{R} \in SO(3)$, position $\mathbf{p} \in \mathbb{R}^3$ and velocity $\mathbf{v} \in \mathbb{R}^3$ can be written in homogeneous matrices as:

$$\mathbf{X} = \begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SE_2(3) \tag{1}$$

2.2 Exp and Log

The exponential map will map an element $\delta\mathbf{x} \in \mathbb{R}^9$ to $SE_2(3)$

$$\exp(\delta\mathbf{x}) = \exp\left(\begin{bmatrix} \delta\boldsymbol{\theta} \\ \delta\mathbf{p} \\ \delta\mathbf{v} \end{bmatrix}\right) \triangleq \begin{bmatrix} \exp(\delta\boldsymbol{\theta}) & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The logarithm map is defined as the local inverse of the exponential map as

$$\log(\mathbf{X}) = \log\left(\begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \triangleq \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{J}^{-1}_{l(\boldsymbol{\theta})}\mathbf{p} \\ \mathbf{J}^{-1}_{l(\boldsymbol{\theta})}\mathbf{v} \end{bmatrix} \tag{3}$$

where $\mathbb{J}_l(\boldsymbol{\theta})$ denote the left Jacobian of $SO(3)$.

2.3 Adjoint Operation

We conveniently define the adjoint operator as

$$Ad_{\mathbf{X}} \triangleq \begin{bmatrix} \mathbf{R} & 0 & 0 \\ \lfloor\mathbf{p}\rfloor\mathbf{R} & \mathbf{R} & 0 \\ \lfloor\mathbf{v}\rfloor\mathbf{R} & 0 & \mathbf{R} \end{bmatrix} \tag{4}$$

2.4 Error states for $\mathbf{SE}_2(3)$

If $\hat{\mathbf{X}} \in SE_2(3)$ represents the noise free true states and $\delta\mathbf{x} = [\delta\theta^\top\ \delta\mathbf{p}^\top\ \delta\mathbf{v}^\top]^\top \in \mathbb{R}^9$ representations the small perturbation, we have,

$$\mathbf{X} \triangleq \exp(\delta\mathbf{x})\hat{\mathbf{X}} \tag{5}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{p} & \mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \exp(\delta\boldsymbol{\theta}) & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} & \hat{\mathbf{p}} & \hat{\mathbf{v}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \exp(\delta\boldsymbol{\theta})\hat{\mathbf{R}} & \exp(\delta\boldsymbol{\theta})\hat{\mathbf{p}} + \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{p} & \exp(\delta\boldsymbol{\theta})\hat{\mathbf{v}} + \mathbf{J}_l(\delta\boldsymbol{\theta})\delta\mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

3 IMU Model

3.1 IMU measurements

$$\mathbf{a}_m = \mathbf{a} + \mathbf{b}_a + \mathbf{n}_a \tag{8}$$

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_g + \mathbf{n}_g \tag{9}$$

3.2 IMU Dynamics

$${}^G_I\dot{\mathbf{R}} = {}^G_I\mathbf{R}\lfloor {}^I\boldsymbol{\omega} \rfloor \tag{10}$$

$${}^G\dot{\mathbf{p}}_I = {}^G\mathbf{v}_I \tag{11}$$

$${}^G\dot{\mathbf{v}}_I = {}^G_I\mathbf{R}^I\mathbf{a} + {}^G\mathbf{g} \tag{12}$$

$$\dot{\mathbf{b}}_g = \mathbf{n}_{wg} \tag{13}$$

$$\dot{\mathbf{b}}_a = \mathbf{n}_{wa} \tag{14}$$

where ${}^G\mathbf{g} = [0, 0, -9.8]^\top$.

4 IMU Only Propagation

4.1 IMU State Vector

$$\mathbf{x}_I = (\mathbf{x}_n, \mathbf{x}_b) \tag{15}$$

$$\mathbf{x}_n = \left({}^G_I\mathbf{R}, {}^G\mathbf{p}_I, {}^G\mathbf{v}_I\right) \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & {}^G\mathbf{p}_I & {}^G\mathbf{v}_I \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

$$\mathbf{x}_b = (\mathbf{b}_g, \mathbf{b}_a) \tag{17}$$

The error states:

$$\mathbf{x}_I = \delta\mathbf{x}_I \boxplus \hat{\mathbf{x}}_I \tag{18}$$

$$= (\exp(\delta\mathbf{x}_n)\hat{\mathbf{x}}_n, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b) \tag{19}$$

4.2 Dynamic Model Integration

From the IMU dynamic model, from t_k to t_{k+1} we can get:

$$
\begin{align}
{}^{G}_{I_{k+1}}\mathbf{R} &= {}^{G}_{I_k}\mathbf{R}\exp\left(\int {}^{I}\boldsymbol{\omega}d\tau\right) \triangleq {}^{G}_{I_k}\mathbf{R}\Delta\mathbf{R}_k \tag{20}\\
{}^{G}\mathbf{p}_{I_{k+1}} &= {}^{G}\mathbf{p}_{I_k} + {}^{G}\mathbf{v}_{I_k}\delta t + {}^{G}_{I_k}\mathbf{R}\int\int {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a}d\tau ds + \frac{1}{2}{}^{G}\mathbf{g}\delta t^2 \tag{21}\\
&\triangleq {}^{G}\mathbf{p}_{I_k} + {}^{G}\mathbf{v}_{I_k}\delta t + {}^{G}_{I_k}\mathbf{R}\Delta\mathbf{p}_k + \frac{1}{2}{}^{G}\mathbf{g}\delta t^2 \tag{22}\\
{}^{G}\mathbf{v}_{I_{k+1}} &= {}^{G}\mathbf{v}_{I_k} + {}^{G}_{I_k}\mathbf{R}\int {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a}d\tau + {}^{G}\mathbf{g}\delta t \tag{23}\\
&\triangleq {}^{G}\mathbf{v}_{I_k} + {}^{G}_{I_k}\mathbf{R}\Delta\mathbf{v}_k + {}^{G}\mathbf{g}\delta t \tag{24}\\
\mathbf{b}_{g_{k+1}} &= \mathbf{b}_{g_k} + \int \mathbf{n}_{wg}d\tau \tag{25}\\
\mathbf{b}_{a_{k+1}} &= \mathbf{b}_{a_k} + \int \mathbf{n}_{wa}d\tau \tag{26}
\end{align}
$$

where

$$
\begin{align}
\Delta\mathbf{R}_k &= \exp\left(\int {}^{I}\boldsymbol{\omega}d\tau\right) \tag{27}\\
\Delta\mathbf{p}_k &= \int\int {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a}d\tau ds \tag{28}\\
\Delta\mathbf{v}_k &= \int {}^{I_k}_{I_\tau}\mathbf{R}^{I_\tau}\mathbf{a}d\tau \tag{29}
\end{align}
$$

$$
\mathbf{x}_{n_{k+1}} = \boldsymbol{\Gamma}_k\boldsymbol{\Psi}(\mathbf{x}_{n_k})\boldsymbol{\Upsilon}_k \tag{30}
$$

where:

$$
\begin{align}
\boldsymbol{\Gamma}_k &= \begin{bmatrix} \mathbf{I}_3 & \frac{1}{2}{}^{G}\mathbf{g}\delta t^2 & \mathbf{g}\delta t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}\\
\boldsymbol{\Psi}(\mathbf{x}_{n_k}) &= \begin{bmatrix} {}^{G}_{I_k}\mathbf{R} & {}^{G}\mathbf{p}_{I_k} + {}^{G}\mathbf{v}_{I_k}\delta t & {}^{G}\mathbf{v}_{I_k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{32}\\
\boldsymbol{\Upsilon}_k &= \begin{bmatrix} \Delta\mathbf{R}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}
\end{align}
$$

Some properties of $\boldsymbol{\Psi}$ are:

$$
\begin{align}
\boldsymbol{\Psi}(\mathbf{x}_{n_k}\mathbf{x}_{n_{k+1}}) &= \boldsymbol{\Psi}(\mathbf{x}_{n_k})\boldsymbol{\Psi}(\mathbf{x}_{n_k+1}) \tag{34}\\
\boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n)) &= \exp(\mathbf{F}\delta\mathbf{x}_\mathbf{n}) \tag{35}\\
\Rightarrow \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n)\mathbf{x}_{n+1}) &= \boldsymbol{\Psi}(\exp(\delta\mathbf{x}_n))\boldsymbol{\Psi}(\mathbf{x}_{n+1}) = \exp(\mathbf{F}\delta\mathbf{x}_\mathbf{n})\boldsymbol{\Psi}(\mathbf{x}_{n+1}) \tag{36}
\end{align}
$$

where

$$
\mathbf{F} = \begin{bmatrix} \mathbf{I}_3 & 0 & 0 \\ 0 & \mathbf{I}_3 & \delta t\mathbf{I}_3 \\ 0 & 0 & \mathbf{I}_3 \end{bmatrix} \tag{37}
$$

4.3 Mean Propagation

In this section, we show how to propagate IMU dynamics ignoring the noise s.t.

$$\mathbf{b}_g = \hat{\mathbf{b}}_g \tag{38}$$

$$\mathbf{b}_a = \hat{\mathbf{b}}_a \tag{39}$$

Then, the IMU propogation can be written as

$$\hat{\mathbf{x}}_{n_{k+1}} = \hat{\mathbf{\Gamma}}_k \hat{\mathbf{\Psi}}(\mathbf{x}_{n_k}) \hat{\mathbf{\Upsilon}}_k \tag{40}$$

Hence, we just need to decide $\hat{\mathbf{\Upsilon}}_k$ with $\Delta\hat{\mathbf{R}}_k$, $\Delta\hat{\mathbf{p}}_k$ and $\Delta\hat{\mathbf{v}}_k$.

When using ACI, we can get:

$$\Delta\hat{\mathbf{R}}_k = \tag{41}$$

$$\Delta\hat{\mathbf{p}}_k = \tag{42}$$

$$\Delta\hat{\mathbf{v}}_k = \tag{43}$$

4.4 State Transition and Covariance

The state transition matrix can written as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \end{bmatrix} \tag{44}$$

For $\mathbf{\Phi}_{nn}$, we can use the right invariant errors:

$$\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \mathbf{\Gamma}_k \mathbf{\Psi}(\exp(\delta\mathbf{x}_{n_k})\hat{\mathbf{x}}_{n_k}) \hat{\mathbf{\Upsilon}}_k &(45)\\
&= \mathbf{\Gamma}_k \mathbf{\Psi}\left(\exp\left(\delta\mathbf{x}_{n_k}\right)\right) \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k &(46)\\
&= \mathbf{\Gamma}_k \exp\left(\mathbf{F}\delta\mathbf{x}_{n_k}\right) \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k &(47)\\
&= \exp\left(Ad_{\mathbf{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{n_k}\right) \underbrace{\mathbf{\Gamma}_k \mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right) \hat{\mathbf{\Upsilon}}_k}_{\hat{\mathbf{x}}_{n_{k+1}}} &(48)
\end{aligned}$$

Then, we can have $\mathbf{\Phi}_{nn}$ as:

$$\mathbf{\Phi}_{nn} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor {}^G\mathbf{g} \rfloor \delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t \\ \lfloor {}^G\mathbf{g} \rfloor \delta t & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{49}$$

For computing $\mathbf{\Phi}_{nb}$, we can have:

$$\Delta\mathbf{R}_k = \Delta\hat{\mathbf{R}}_k \Delta\tilde{\mathbf{R}}_k \tag{50}$$

$$\Delta\mathbf{p}_k = \Delta\hat{\mathbf{p}}_k + \Delta\tilde{\hat{\mathbf{p}}}_k \tag{51}$$

$$\Delta\mathbf{v}_k = \Delta\hat{\mathbf{v}}_k + \Delta\tilde{\hat{\mathbf{v}}}_k \tag{52}$$

$$
\begin{aligned}
\boldsymbol{\Upsilon}_k &= \hat{\boldsymbol{\Upsilon}}_k \tilde{\boldsymbol{\Upsilon}}_k && (53)\\
&= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\tilde{\mathbf{R}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k & \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (54)\\
&= \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) \simeq \hat{\boldsymbol{\Upsilon}}_k \exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (55)
\end{aligned}
$$

where $\Delta\tilde{\mathbf{R}}_k = \exp(\delta\Delta\theta_k)$. Then, we have:

$$
\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\mathbf{x}_{n_k})\hat{\boldsymbol{\Upsilon}}_k \tilde{\boldsymbol{\Upsilon}}_k && (56)\\
\Rightarrow \exp(\delta\mathbf{x}_{n_{k+1}}) &= \hat{\mathbf{x}}_{n_{k+1}} \tilde{\boldsymbol{\Upsilon}}_k \hat{\mathbf{x}}_{n_{k+1}}^{-1} && (57)\\
&= \exp\left(\mathbf{Ad}_{\hat{\mathbf{x}}_{n_{k+1}}} \begin{bmatrix} \delta\Delta\theta_k \\ \hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top \Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (58)
\end{aligned}
$$

We have two versions for computing the $\Delta\tilde{\mathbf{R}}_k$, $\Delta\tilde{\mathbf{p}}_k$ and $\Delta\tilde{\mathbf{v}}_k$;
For ACI, we have:

$$
\begin{aligned}
\Delta\tilde{\mathbf{R}}_k &= && (59)\\
\Delta\tilde{\mathbf{p}}_k &= && (60)\\
\Delta\tilde{\mathbf{v}}_k &= && (61)
\end{aligned}
$$

Hence, we have:

$$
\boldsymbol{\Phi}_{nb} = \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{p}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 \\ -\lfloor {}^G\mathbf{v}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 \end{bmatrix} \tag{62}
$$

And the noise Jacobians:

$$
\mathbf{G}_n = \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{p}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{v}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \end{bmatrix} \tag{63}
$$

5 RI-VINS With Features

5.1 Features Coupled With IMU State

If with features, the whole state vector becomes:

$$
\mathbf{x} = \left(\mathbf{x}_I, {}^G\mathbf{p}_f\right) \tag{64}
$$

The feature state needs to be coupled with the $\mathbf{SE}_2(3)$ for the consistency consideration.

$$\mathbf{x}_{nf} = ({}^G_I\mathbf{R}, {}^G\mathbf{p}_I, {}^G\mathbf{v}_I, {}^G\mathbf{p}_f) \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & {}^G\mathbf{p}_I & {}^G\mathbf{v}_I & {}^G\mathbf{p}_f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{65}$$

$$\mathbf{x}_{nf} = \delta\mathbf{x}_{nf} \boxplus \hat{\mathbf{x}}_{nf} = \exp(\delta\mathbf{x}_{nf})\hat{\mathbf{x}}_{nf} \tag{66}$$

$$\mathbf{x} = \delta\mathbf{x} \boxplus \hat{\mathbf{x}} = (\exp(\delta\mathbf{x}_{nf})\hat{\mathbf{x}}_{nf}, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b) \tag{67}$$

With slightly abuse of notaiton, we have:

$$\mathbf{x}_{nf_{k+1}} = \boldsymbol{\Gamma}_k \boldsymbol{\Psi}(\mathbf{x}_{nf_k})\boldsymbol{\Upsilon}_k \tag{68}$$

where:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \mathbf{I}_3 & \frac{1}{2}{}^G\mathbf{g}\delta t^2 & \mathbf{g}\delta t & \mathbf{0}_{3\times 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{69}$$

$$\boldsymbol{\Psi}(\mathbf{x}_{nf_k}) = \begin{bmatrix} {}^G_{I_k}\mathbf{R} & {}^G\mathbf{p}_{I_k} + {}^G\mathbf{v}_{I_k}\delta t & {}^G\mathbf{v}_{I_k} & {}^G\mathbf{p}_f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{70}$$

$$\boldsymbol{\Upsilon}_k = \begin{bmatrix} \Delta\mathbf{R} & \Delta\mathbf{p} & \Delta\mathbf{v} & \mathbf{0}_{3\times 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{71}$$

The adjoint operation is defined similar with Eq.(4) as

$$Ad_{\mathbf{x}_{nf}} \triangleq \begin{bmatrix} {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{p}_I \rfloor {}^G_I\mathbf{R} & {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{v}_I \rfloor {}^G_I\mathbf{R} & \mathbf{0}_3 & {}^G_I\mathbf{R} & \mathbf{0}_3 \\ \lfloor {}^G\mathbf{p}_f \rfloor {}^G_I\mathbf{R} & \mathbf{0}_3 & \mathbf{0}_3 & {}^G_I\mathbf{R} \end{bmatrix} \tag{72}$$

From this, following similar steps, we can get the full state transition matrix as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \\ \delta{}^G\mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{nn} & \boldsymbol{\Phi}_{nb} & \mathbf{0}_{9\times 3} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 & \mathbf{0}_{6\times 3} \\ \mathbf{0}_{3\times 9} & \boldsymbol{\Phi}_{fb} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \\ \delta{}^G\mathbf{p}_f \end{bmatrix} \tag{73}$$

where we have:

$$\boldsymbol{\Phi}_{fb} = \begin{bmatrix} -\lfloor {}^G\mathbf{p}_f \rfloor {}^G_{I_{k+1}}\mathbf{R}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}})\delta t & \mathbf{0}_3 \end{bmatrix} \tag{74}$$

while the new ACI version noise Jacobians become:

$$\mathbf{G}_n = \begin{bmatrix} -{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\mathbf{p}_{I_{k+1}}\rfloor {}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^{G}_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_4 & -{}^{G}_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\mathbf{v}_{I_{k+1}}\rfloor {}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^{G}_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_3 & -{}^{G}_{I_k}\hat{\mathbf{R}}\boldsymbol{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ -\lfloor {}^{G}\mathbf{p}_f\rfloor {}^{G}_{I_{k+1}}\mathbf{R}\mathbf{J}_r\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{75}$$

And the new discrete version of noise Jacobians become:

$$\mathbf{G}_n = \begin{bmatrix} -{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\mathbf{p}_{I_{k+1}}\rfloor {}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^{G}\mathbf{v}_{I_{k+1}}\rfloor {}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ -\lfloor {}^{G}\mathbf{p}_f\rfloor {}^{G}_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{76}$$

We will see that the feature will appears in the IMU state transition matrix and covariance propagation, which is not desired.

5.2 Feature Update

The visual measurements can be written as:

$$\mathbf{z} = \begin{bmatrix} \frac{x}{z} & \frac{y}{z} \end{bmatrix}^\top \tag{77}$$

$${}^{C}\mathbf{p}_f = \begin{bmatrix} x & y & z \end{bmatrix}^\top = {}^{C}_{I}\mathbf{R}{}^{I}_{G}\mathbf{R}({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_I) + {}^{C}\mathbf{p}_I \tag{78}$$

If updating with current pose, since the feature is coupled with the current IMU pose, hence, the error states can be written as:

$${}^{G}\mathbf{p}_f \simeq \exp(\delta\boldsymbol{\theta}_I){}^{G}\hat{\mathbf{p}}_f + {}^{G}\tilde{\mathbf{p}}_f \tag{79}$$

$${}^{G}\mathbf{p}_I \simeq \exp(\delta\boldsymbol{\theta}_I){}^{G}\hat{\mathbf{p}}_I + {}^{G}\tilde{\mathbf{p}}_I \tag{80}$$

$${}^{G}_{I}\mathbf{R} = \exp(\delta\boldsymbol{\theta}_I){}^{G}_{I}\hat{\mathbf{R}} \tag{81}$$

Then, we can get the Jacobians as:

$$\mathbf{H} = \mathbf{H}_{proj}{}^{C}_{I}\mathbf{R}{}^{I}_{G}\mathbf{R}\begin{bmatrix}\mathbf{0}_3 & -\mathbf{I_3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \tag{82}$$

Note that, if we do the observablilty analysis, we can get:

$$\mathbf{H}\boldsymbol{\Phi} = \mathbf{H}_{proj}{}^{C}_{I}\mathbf{R}{}^{I}_{G}\mathbf{R}\begin{bmatrix}\mathbf{M}_1 & -\mathbf{I}_3 & -\mathbf{I}_3\delta t & \mathbf{M}_2 & \mathbf{M}_3 & \mathbf{I}_3\end{bmatrix} \tag{83}$$

where

$$\mathbf{M}_1 = \frac{1}{2}\lfloor {}^{G}\mathbf{g}\rfloor {}^{G}\mathbf{g}\delta t^2 \tag{84}$$

$$\mathbf{M}_2 = \lfloor {}^{G}\mathbf{p}_{I_{k+1}}{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\rfloor \mathbf{J}_r(\Delta\boldsymbol{\theta}_k)\delta t \tag{85}$$

$$\mathbf{M}_3 = \frac{1}{2}{}^{G}_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2 \tag{86}$$

we can directly find the null space as:

$$\mathbf{N} = \begin{bmatrix} {}^G\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{3\times1} & \mathbf{I}_3 \\ \mathbf{0}_{9\times1} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{3\times1} & \mathbf{I}_3 \end{bmatrix} \tag{87}$$

This is very nice, since we directly get a system satisfies the system property.

6 DRI-EKF

We donot want the feature to be coupled with propagation. We also don't want to make the update to coupled with current IMU states.

Hence, we want to decouple the feature from the Lie Group. The state vector is still defined as:

6.1 Feature Decoupled From Lie Group

$$\mathbf{x} = \left(\mathbf{x}_I, {}^G\mathbf{p}_f\right) \tag{88}$$

However, the overall error states are defined as:

$$\mathbf{x} = \delta\mathbf{x} \boxplus \hat{\mathbf{x}} \tag{89}$$

$$= \left(\exp(\delta\mathbf{x}_n)\hat{\mathbf{x}}_n, \hat{\mathbf{x}}_b + \delta\mathbf{x}_b, {}^G\hat{\mathbf{p}}_f + \delta^G\mathbf{p}_f\right) \tag{90}$$

From this, following similar steps, we can get the full state transition matrix as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \\ \delta^G\mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} & \mathbf{0}_{9\times3} \\ \mathbf{0}_{6\times9} & \mathbf{I}_6 & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times9} & \mathbf{0}_{3\times6} & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \\ \delta^G\mathbf{p}_f \end{bmatrix} \tag{91}$$

while the new noise Jacobians become:

$$\mathbf{G}_n = \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{p}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_4 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_2 & \mathbf{0}_3 & \mathbf{0}_3 \\ -\lfloor {}^G\mathbf{v}_{I_{k+1}} \rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r\delta t + {}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_3 & -{}^G_{I_k}\hat{\mathbf{R}}\mathbf{\Xi}_1 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix} \tag{92}$$

We can find that the feature disappears from the state transition, which is nice.

Now we can check the measurement Jacobians:

If updating with current pose, since the feature is coupled with the current IMU pose, hence, the error states can be written as:

$$ {}^G\mathbf{p}_f \simeq {}^G\hat{\mathbf{p}}_f + {}^G\tilde{\mathbf{p}}_f \tag{93}$$

$$ {}^G\mathbf{p}_I \simeq \exp(\delta\theta_I){}^G\hat{\mathbf{p}}_I + {}^G\tilde{\mathbf{p}}_I \tag{94}$$

$$ {}^G_I\mathbf{R} = {}^G_I\hat{\mathbf{R}}\exp(\delta\boldsymbol{\theta}_{\boldsymbol{I}}) \tag{95}$$

Then, we can get the Jacobians as:

$$\mathbf{H} = \mathbf{H}_{proj}{}^{C}_{I}\mathbf{R}^{I}_{G}\mathbf{R}\begin{bmatrix}\lfloor{}^{G}\mathbf{p}_f\rfloor & -\mathbf{I_3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \tag{96}$$

It is nice that we don't have extra term in the Jacobians related to the current IMU state, how ever the measurements are state related, which will not preserve the observability property.

$$\mathbf{H}\mathbf{\Phi} = \tag{97}$$

$$\mathbf{N} = \begin{bmatrix} {}^{G}\mathbf{g} & \mathbf{0}_3 \\ \mathbf{0}_{3\times1} & \mathbf{I}_3 \\ \mathbf{0}_{9\times1} & \mathbf{0}_{9\times3} \\ -\lfloor{}^{G}\mathbf{p}_f\rfloor{}^{G}\mathbf{g} & \mathbf{I}_3 \end{bmatrix} \tag{98}$$

In order to keep the observability of the system, we can use the FEJ or OC.

When we do update, we just use the first estimate Jacobians for the features. ${}^{G}\mathbf{p}_{f,k|k-1}$ for the Jacobians computation of the feature.

The decoupled RI-EKF can easily handle more features in the state vector without too much care. The fej and oc are leveraged to keep the system observability property. Due to the nice property of $\mathbf{SE}_2(3)$, we don't need to worry about the state propagatin. In the meantime, the changes in the update Jacobians are pretty small and light weight.

6.2 Feature with Sliding Window

Instead of coupled feature with current IMU state, we can couple feature with clones poses if a sliding-window is maintained.

6.2.1 Feature Jacobians

$${}^{C}\mathbf{p}_f = {}^{C}_{I}\mathbf{R}^{I_c}_{G}\mathbf{R}({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I_c}) + {}^{C}\mathbf{p}_I \tag{99}$$

where I_C is the cloned IMU pose.

$${}^{G}\mathbf{p}_f \simeq \exp(\delta\boldsymbol{\theta}_I){}^{G}\hat{\mathbf{p}}_f + {}^{G}\tilde{\mathbf{p}}_f \tag{100}$$

$${}^{G}\mathbf{p}_{I_c} \simeq \exp(\delta\boldsymbol{\theta}_{I_c}){}^{G}\hat{\mathbf{p}}_{I_c} + {}^{G}\tilde{\mathbf{p}}_{I_c} \tag{101}$$

$${}^{G}_{I_c}\mathbf{R} = \exp(\delta\boldsymbol{\theta}_{I_c}){}^{G}_{I_c}\hat{\mathbf{R}} \tag{102}$$

Hence, we need the Jacobians for:

$$\frac{\partial{}^{C}\tilde{\mathbf{p}}_f}{\partial\delta\boldsymbol{\theta}_I} = -{}^{C}_{I}\hat{\mathbf{R}}^{I_c}_{G}\mathbf{R}\lfloor{}^{G}\mathbf{p}_f\rfloor \tag{103}$$

$$\frac{\partial{}^{C}\tilde{\mathbf{p}}_f}{\partial\delta\boldsymbol{\theta}_{I_c}} = \tag{104}$$

$$\frac{\partial{}^{C}\tilde{\mathbf{p}}_f}{\partial{}^{G}\tilde{\mathbf{p}}_{I_c}} = -{}^{C}_{I}\hat{\mathbf{R}}^{I_c}_{G}\mathbf{R} \tag{105}$$

$$\frac{\partial{}^{C}\tilde{\mathbf{p}}_f}{\partial{}^{G}\tilde{\mathbf{p}}_f} = {}^{C}_{I}\hat{\mathbf{R}}^{I_c}_{G}\mathbf{R} \tag{106}$$

6.2.2 Coupled Pose Change

In order to keep the feature even with the coupled pose is marginalized from the state, we need to couple the feature to another pose in the active sliding window.

For the anchor change process, we can get explain in the following way. Assuming we have 2 clone poses ${}^G\mathbf{R}_i, {}^G\mathbf{p}_i$ where $i = 0, 1$ and a feature ${}^G\mathbf{p}_f$.

The feature is associated with pose 0 and pose 0 is about to be marginalized from the state, we need to associate feature with pose 1. Similar to change the state from:

$$\mathbf{x}_{old} = ({}^G\mathbf{R}_0, {}^G\mathbf{p}_0, {}^G\mathbf{R}_1, {}^G\mathbf{p}_1, {}^G\mathbf{p}_{f0}) \Rightarrow \mathbf{x}_{new} = ({}^G\mathbf{R}_0, {}^G\mathbf{p}_0, {}^G\mathbf{R}_1, {}^G\mathbf{p}_1, {}^G\mathbf{p}_{f1}) \tag{107}$$

$$\begin{aligned}
{}^G\mathbf{p}_{f0} &= {}^G\mathbf{p}_{f1} && (108)\\
&= \exp(\delta\theta_0)^G\hat{\mathbf{p}}_{f0} + \mathbf{J}_l(\delta\theta_0)^G\tilde{\mathbf{p}}_{f0} && (109)\\
&= \exp(\delta\theta_1)^G\hat{\mathbf{p}}_{f1} + \mathbf{J}_l(\delta\theta_1)^G\tilde{\mathbf{p}}_{f1} && (110)
\end{aligned}$$

Hence, we can find the feature's new error states as:

$${}^G\tilde{\mathbf{p}}_{f1} = -\lfloor {}^G\hat{\mathbf{p}}_{f0} \rfloor \delta\theta_0 + \lfloor {}^G\hat{\mathbf{p}}_{f1} \rfloor \delta\theta_1 + {}^G\tilde{\mathbf{p}}_{f0} \tag{111}$$

6.2.3 Equivalency

Finally, we also need to prove that associate the feature with sliding window is equivalent to associate feature with current IMU state.

7 Real World Experiment

	V1_01_easy	V1_02_med	V1_03_dif.	V2_01_easy	V2_02_med.	V2_03_dif.
mono_dli_fej	0.739 / 0.066	1.600/**0.079**	2.367/0.075	1.009/0.091	1.723/0.075	1.227/0.165
mono_dri_fej	0.739 / 0.066	1.596/**0.079**	2.336/**0.074**	0.994/0.090	1.723/0.075	1.296/**0.154**
mono_dri_sw	0.739 / 0.066	1.595/**0.079**	2.366/0.075	0.994/0.090	1.736/0.074	1.296/**0.154**
mono_fej	**0.508** / **0.048**	**1.573**/0.084	2.651/0.083	**0.801**/**0.087**	**1.497**/**0.062**	**1.171**/0.197

Appendix A: Discrete IMU Propagation

In this section, we show how to propagate IMU dynamics ignoring the noise s.t.

$$\mathbf{b}_g = \hat{\mathbf{b}}_g \tag{112}$$

$$\mathbf{b}_a = \hat{\mathbf{b}}_a \tag{113}$$

Then, the IMU propogation can be written as

$$\hat{\mathbf{x}}_{n_{k+1}} = \hat{\mathbf{\Gamma}}_k \hat{\mathbf{\Psi}}(\mathbf{x}_{n_k}) \hat{\mathbf{\Upsilon}}_k \tag{114}$$

Hence, we just need to decide $\hat{\mathbf{\Upsilon}}_k$ with $\Delta\hat{\mathbf{R}}_k$, $\Delta\hat{\mathbf{p}}_k$ and $\Delta\hat{\mathbf{v}}_k$.

When using discrete model, we can get:

$$\Delta\hat{\mathbf{R}}_k = \exp\left(\hat{\boldsymbol{\omega}}_k \delta t\right) \tag{115}$$

$$\Delta\hat{\mathbf{p}}_k = \frac{1}{2}\hat{\mathbf{a}_k}\delta t^2 \tag{116}$$

$$\Delta\hat{\mathbf{v}}_k = \hat{\mathbf{a}_k}\delta t \tag{117}$$

The state transition matrix can written as:

$$\begin{bmatrix} \delta\mathbf{x}_{n_{k+1}} \\ \delta\mathbf{x}_{b_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{nn} & \mathbf{\Phi}_{nb} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_{n_k} \\ \delta\mathbf{x}_{b_k} \end{bmatrix} \tag{118}$$

For $\mathbf{\Phi}_{nn}$, we can use the right invariant errors:

$$\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \mathbf{\Gamma}_k \mathbf{\Psi}(\exp(\delta\mathbf{x}_{n_k})\hat{\mathbf{x}}_{n_k})\hat{\mathbf{\Upsilon}}_k && (119)\\
&= \mathbf{\Gamma}_k \mathbf{\Psi}\left(\exp\left(\delta\mathbf{x}_{n_k}\right)\right)\mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\mathbf{\Upsilon}}_k && (120)\\
&= \mathbf{\Gamma}_k \exp\left(\mathbf{F}\delta\mathbf{x}_{n_k}\right)\mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\mathbf{\Upsilon}}_k && (121)\\
&= \exp\left(Ad_{\mathbf{\Gamma}_k}\mathbf{F}\delta\mathbf{x}_{n_k}\right)\underbrace{\mathbf{\Gamma}_k\mathbf{\Psi}\left(\hat{\mathbf{x}}_{n_k}\right)\hat{\mathbf{\Upsilon}}_k}_{\hat{\mathbf{x}}_{n_{k+1}}} && (122)
\end{aligned}$$

Then, we can have $\mathbf{\Phi}_{nn}$ as:

$$\mathbf{\Phi}_{nn} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \frac{1}{2}\lfloor{}^G\mathbf{g}\rfloor\delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t \\ \lfloor{}^G\mathbf{g}\rfloor\delta t & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \tag{123}$$

For computing $\mathbf{\Phi}_{nb}$, we can have:

$$\Delta\mathbf{R}_k = \Delta\hat{\mathbf{R}}_k\Delta\tilde{\mathbf{R}}_k \tag{124}$$

$$\Delta\mathbf{p}_k = \Delta\hat{\mathbf{p}}_k + \Delta\tilde{\hat{\mathbf{p}}}_k \tag{125}$$

$$\Delta\mathbf{v}_k = \Delta\hat{\mathbf{v}}_k + \Delta\tilde{\hat{\mathbf{v}}}_k \tag{126}$$

$$\begin{aligned}
\mathbf{\Upsilon}_k &= \hat{\mathbf{\Upsilon}}_k\tilde{\mathbf{\Upsilon}}_k && (127)\\
&= \begin{bmatrix} \Delta\hat{\mathbf{R}}_k & \Delta\mathbf{p}_k & \Delta\mathbf{v}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta\tilde{\mathbf{R}}_k & \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k & \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (128)\\
&= \hat{\mathbf{\Upsilon}}_k\exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k \\ \mathbf{J}_l^{-1}(\delta\Delta\theta_k)\Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) \simeq \hat{\mathbf{\Upsilon}}_k\exp\left(\begin{bmatrix} \delta\Delta\theta_k \\ \hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k \\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k \end{bmatrix}\right) && (129)
\end{aligned}$$

where $\Delta\tilde{\mathbf{R}}_k = \exp(\delta\Delta\theta_k)$. Then, we have:

$$\begin{aligned}
\exp(\delta\mathbf{x}_{n_{k+1}})\hat{\mathbf{x}}_{n_{k+1}} &= \boldsymbol{\Gamma}_k\boldsymbol{\Psi}(\mathbf{x}_{n_k})\hat{\boldsymbol{\Upsilon}}_k\tilde{\boldsymbol{\Upsilon}}_k && (130)\\
\Rightarrow \exp(\delta\mathbf{x}_{n_{k+1}}) &= \hat{\mathbf{x}}_{n_{k+1}}\tilde{\boldsymbol{\Upsilon}}_k\hat{\mathbf{x}}_{n_{k+1}}^{-1} && (131)\\
&= \exp\left(\mathbf{Ad}_{\hat{\mathbf{x}}_{n_{k+1}}}\begin{bmatrix}\delta\Delta\theta_k\\ \hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{p}}_k\\ \Delta\hat{\mathbf{R}}_k^\top\Delta\tilde{\mathbf{v}}_k\end{bmatrix}\right) && (132)
\end{aligned}$$

We have two versions for computing the $\Delta\tilde{\mathbf{R}}_k$, $\Delta\tilde{\mathbf{p}}_k$ and $\Delta\tilde{\mathbf{v}}_k$;
For discrete model, we have:

$$\begin{aligned}
\Delta\tilde{\mathbf{R}}_k &= -\exp\left(\mathbf{J}_r(\Delta\hat{\theta})\left(\tilde{\mathbf{b}}_g + \mathbf{n}_g\right)\delta t\right) && (133)\\
\Delta\tilde{\mathbf{p}}_k &= -\frac{1}{2}(\tilde{\mathbf{b}}_a + \mathbf{n}_a)\delta t^2 && (134)\\
\Delta\tilde{\mathbf{v}}_k &= -(\tilde{\mathbf{b}}_a + \mathbf{n}_a)\delta t && (135)
\end{aligned}$$

Hence, we have:

$$\begin{aligned}
\boldsymbol{\Phi}_{nb} &= \begin{bmatrix} {}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0} & \mathbf{0}\\ \lfloor {}^G\hat{\mathbf{p}}_{k+1}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}} & {}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0}\\ \lfloor {}^G\hat{\mathbf{v}}_{k+1}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}} & \mathbf{0} & {}^G_{I_{k+1}}\hat{\mathbf{R}}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \Delta\hat{\mathbf{R}}_k^\top & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \Delta\hat{\mathbf{R}}_k^\top\end{bmatrix}\begin{bmatrix}-\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & \mathbf{0}\\ \mathbf{0} & -\frac{1}{2}\delta t^2\\ \mathbf{0} & -\delta t\end{bmatrix} && (136)\\
&= \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0\\ -\lfloor {}^G\mathbf{p}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2\\ -\lfloor {}^G\mathbf{v}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t\end{bmatrix} && (137)
\end{aligned}$$

Therefore, we have the full state translation matrix as

$$\boldsymbol{\Phi} = \begin{bmatrix}\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0\\ \frac{1}{2}\lfloor {}^G\mathbf{g}\rfloor {}^G\mathbf{g}\delta t^2 & \mathbf{I}_3 & \mathbf{I}_3\delta t & -\lfloor {}^G\mathbf{p}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t^2\\ \lfloor {}^G\mathbf{g}\rfloor {}^G\mathbf{g}\delta t & \mathbf{0}_3 & \mathbf{I}_3 & -\lfloor {}^G\mathbf{v}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\hat{\mathbf{R}}_k^\top\delta t\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\end{bmatrix} \quad (138)$$

And the noise Jacobians:

$$\mathbf{G}_n = \begin{bmatrix} -{}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & 0 & \mathbf{0}_3 & \mathbf{0}_3\\ -\lfloor {}^G\mathbf{p}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -\frac{1}{2}{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\mathbf{R}_k^\top\delta t^2 & \mathbf{0}_3 & \mathbf{0}_3\\ -\lfloor {}^G\mathbf{v}_{I_{k+1}}\rfloor {}^G_{I_{k+1}}\hat{\mathbf{R}}\mathbf{J}_r(\Delta\hat{\boldsymbol{\theta}}_k)\delta t & -{}^G_{I_{k+1}}\hat{\mathbf{R}}\Delta\mathbf{R}_k^\top\delta t & \mathbf{0}_3 & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3\delta t\end{bmatrix} \quad (139)$$

Appendix B: ACI Basics

Appendix C: ACI with $\mathbf{SE}_2(3)$: ACI^3

Analytic combined Invariant IMU integration.

Now with the VINS problem defined above, from IMU time stamp at k to time step at j, we can have:

$$\Upsilon_{kj} = \boldsymbol{\Psi}^{-1}\left(\mathbf{x}_{n_k}\right)\boldsymbol{\Gamma}_{kj}^{-1}\mathbf{x}_{n_j} \tag{140}$$

For pre-integration, we need to compute the measurement mean and covariance for the $\boldsymbol{\Upsilon}_{kj}$.

C.1: Integration of IMU measurements

For each small imu time interval, we intergate the measurements incrementally.

$$\begin{aligned}
\boldsymbol{\Upsilon}_{i+1} &= \begin{bmatrix} \Delta\mathbf{R}_{i+1} & \Delta\mathbf{p}_{i+1} & \Delta\mathbf{v}_{i+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (141)\\
&= \begin{bmatrix} \Delta\mathbf{R}_{i} & \Delta\mathbf{p}_{i}+\Delta\mathbf{v}_i\delta t & \Delta\mathbf{v}_{i} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \exp(\boldsymbol{\omega}\delta t) & \int\int \mathbf{Ra}d\tau ds & \int \mathbf{Ra}d\tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (142)\\
&= \boldsymbol{\Upsilon}_i\boldsymbol{\Upsilon}_{\Delta_i} && (143)
\end{aligned}$$

Then, we can re-write the IMU measurements as:

$$\begin{aligned}
\boldsymbol{\Upsilon}_{\Delta_i} &== \begin{bmatrix} \Delta\hat{\mathbf{R}} & \Delta\mathbf{p} & \Delta\mathbf{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta\tilde{\mathbf{R}} & \Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{p}} & \Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{v}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (144)\\
&= \hat{\boldsymbol{\Upsilon}}_{\Delta_i}\exp\left(\begin{bmatrix} \delta\Delta\theta \\ \mathbf{J}_l^{-1}(\delta\Delta\theta)\Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{p}} \\ \mathbf{J}_l^{-1}(\delta\Delta\theta)\Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{v}} \end{bmatrix}\right) \simeq \hat{\boldsymbol{\Upsilon}}_{\Delta_i}\exp\left(\begin{bmatrix} \delta\Delta\theta \\ \Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{p}} \\ \Delta\hat{\mathbf{R}}^\top\Delta\tilde{\mathbf{v}} \end{bmatrix}\right) && (145)
\end{aligned}$$

We can get the state transition matrix as:

$$\begin{bmatrix} \delta\boldsymbol{\Upsilon}_{i+1} \\ \delta\Delta\mathbf{b}_{i+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}} & \boldsymbol{\Phi}_{\boldsymbol{\Upsilon}\mathbf{b}} \\ \mathbf{0}_{6\times 9} & \mathbf{I}_6 \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{\Upsilon}_{i} \\ \delta\Delta\mathbf{b}_{i} \end{bmatrix} + \mathbf{G}_n\begin{bmatrix} \mathbf{n}_{gd} \\ \mathbf{n}_{ad} \\ \mathbf{n}_{bgd} \\ \mathbf{n}_{bad} \end{bmatrix} \tag{146}$$

Finally, we can have:

$$\begin{aligned}
\boldsymbol{\Phi}_{\boldsymbol{\Upsilon}\boldsymbol{\Upsilon}} &= \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{I}_3\delta t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} && (147)\\
\boldsymbol{\Phi}_{\boldsymbol{\Upsilon}\mathbf{b}} &= \begin{bmatrix} -\Delta\hat{\mathbf{R}}_{i+1}\mathbf{J}_r\delta t & \mathbf{0}_3 \\ -\lfloor\Delta\mathbf{p}_{i+1}\rfloor\Delta\hat{\mathbf{R}}_{i+1}\mathbf{J}_r\delta t + \Delta\hat{\mathbf{R}}_i\boldsymbol{\Xi}_4 & -\Delta\hat{\mathbf{R}}_i\boldsymbol{\Xi}_2 \\ -\lfloor\Delta\mathbf{v}_{i+1}\rfloor\Delta\hat{\mathbf{R}}_{i+1}\mathbf{J}_r\delta t + \Delta\hat{\mathbf{R}}_i\boldsymbol{\Xi}_3 & -\Delta\hat{\mathbf{R}}_i\boldsymbol{\Xi}_1 \end{bmatrix} && (148)
\end{aligned}$$

References

[1] Martin Brossard et al. *Associating Uncertainty to Extended Poses for on Lie Group IMU Preintegration with Rotating Earth*. 2021. arXiv: 2007.14097 [cs.RO].