Risk Neutral versus Objective Loss distribution and CDO tranches valuation

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Abstract

We consider the risk neutral loss distribution as implied by index CDO tranche quotes through a "scenario default rate" model as opposed to the objective measure loss distribution based on historical analysis. The risk neutral loss distribution turns out to privilege large realizations of the loss with respect to the objective distribution, thus implying the well known presence of a risk premium. We quantify this difference numerically by pricing CDO tranches and indices under the two distributions. En passant we analyze the implied risk neutral default rate distributions calibrated from April-2004 throughout April-2006, pointing out its distinctive "bump feature" in the tail.

Key Words: Default Rate distribution, CDO, CDO tranches, Perfect Copula, Implied Copula, Transition Matrices, Rating Classes, Risk Premium, Recovery Rate.

JEL classification code: G13.

1. Introduction.

In this paper we present a framework to consistently price CDO tranches across detachments. The risk neutral loss distribution is derived through a modified "scenario default rate" approach, inspired by the "perfect copula" model of Hull and White (2005), to which we refer as "implied copula". This framework addresses the inconsistency of the Gaussian Copula Model introduced in Li (2000) (see also McGinty, Ahluwalia (2004)) when it is used in its simplest one parameter formulation, where a different implied correlation parameter has to be used to price each tranche.

We compare this implied default rate framework and the Gaussian Copula model focusing on the key differences when a common formulation is adopted. In the implied default rate model we exogenously assign the loss states conditional on the systemic factor, and the systemic factor probabilities are directly modeled as calibrating parameters. Instead, in the Gaussian Copula model both the conditional loss states and the systemic factor probabilities are a byproduct of the Gaussian parametric assumption, and calibration must rest on the Gaussian copula parameter(s). In a way, this reminds of non-parametric vs parametric approaches.

The Gaussian copula assumption is not the only one available in the parametric context. Indeed, other examples include the double-T copula (Hull and White (2004)), mixture copulas (Li and Liang (2005)) while Laurent and Gregory (2005) give an overview of different possible choices.

We then in turn try to estimate the risk premium contained in the CDO tranche. Hull Predescu and White (2004) estimate the risk premium in corporate bonds spreads of different rating categories comparing the risk neutral default probability implied by the corporate bond spread to the average historical default probability of entities with the same rating.

Here we try an equivalent exercise for CDO tranches, that results to be more complicated due to default correlation. As in Hull Predescu and White (2004) the objective default probability of the single reference entity is given by the average historical default probabilities of its rating category. To price CDO tranche we also need the average historical correlation matrix. We used the Gaussian Copula with the same correlation matrix as in the CDO Evaluator 3.0 of S&P. This matrix is highly structured and has much richer economical background than the single parameter matrix in the gaussian copula used to quote implied correlation.

In the single name context, the comparison between risk neutral and objective measure statistics is well know, see for example the above mentioned Hull Predescu and White (2004). We investigate the analogous issue for the aggregate loss distribution. We find, in agreement with single name default probabilities, that the risk neutral loss distribution privileges large realizations of the loss with respect to the objective distribution, thus implying the well known presence of a risk premium. We quantify this premium by re-pricing the market index and tranches under the objective loss distribution and comparing the resulting net present value (NPV) with the risk neutral one.

2. Implied Copula and Implied Default Rate Distribution

We take as reference the iTraxx Europe (125 constituents) index. We fix a set of scenarios for the systemic factor M influencing the default of all the constituent names, and assume that conditional on M defaults of different names are independent. This way we are relying on M as the only variable building the dependence across the pool. The market standard Gaussian 1 factor copula makes particular assumptions on the dependence structure across defaults and on M, and we review them in Appendix A. In the implied copula setup one implicitly makes a homogeneous pool assumption, and directly postulates default intensities conditional on a finite set of scenarios for the systemic factor M, say

$$M \in \{m^0, m^1, \ldots, m^{124}\}.$$

This amounts to assume that the stochastic default intensity λ of any name in the pool, defined as

$$\text{ProbabilityRiskNeutral}\left(\text{Name defaults in } [t; t+dt) \middle| \text{Name has not defaulted by } t\right) = \lambda \cdot dt$$

further satisfies, conditionally on M,

$$\begin{cases} \text{Systemic Scenario} & \text{Scenario Probability} & \text{Conditional intensity} \\ \mathrm{M} = \mathrm{m}^0 & \mathrm{p}_0 & \lambda^0 \\ \mathrm{M} = \mathrm{m}^1 & \mathrm{p}_1 & \lambda^1 \\ \ldots & \ldots & \ldots \\ \mathrm{M} = \mathrm{m}^{124} & \mathrm{p}_{124} & \lambda^{124} \end{cases}$$

This amounts to assume, for a preferred maturity T and for each name $i = 1,2,\ldots,125$, the following default probabilities, conditional on M:

$$\text{ProbRiskNeutral}\left(\text{Name } i \text{ defaults by time } T \,\middle|\, M = m^S \right) = 1 - e^{-\lambda^S T}.$$

Again for a preferred maturity T, the pool default rate, conditional on the systemic scenario $M = m^S$, is defined as (in our particular case $N = 125$)

$$DR_N^S(T) = \frac{1}{N}\sum_{i=1}^{N} I\left\{\text{Name } i \text{ defaults before } T \,\middle|\, M = m^S\right\}$$

where $I\{\text{condition}\}$ is equal to 1 if the condition is satisfied and 0 otherwise, and where, given our initial assumption, the terms in the summation are independent of each other given M.

Now the infinite (or "large") pool assumption comes into play. We may assume the number of names N in the pool to be very large. In this case the law of large numbers implies that the sample average of the i.i.d random variables

$$I\{\text{Name 1 defaults before } T \mid M = m^S\}, \; I\{\text{Name 2 defaults before } T \mid M = m^S\}, \ldots$$

which is our $DR_N^S(T)$, converges in law (under some mild assumptions) to the true mean of each single random variable when N tends to infinity, i.e.

$$DR_N^S(T) \underset{\text{law}}{\rightarrow} \text{ExpectationRiskNeutral}\left[I\{\text{Name defaults before } T \mid M = m^S\}\right] \quad \text{as} \quad N \rightarrow \infty$$

$$\text{ExpectationRiskNeutral}\left[I\{\text{Name defaults before } T \mid M = m^S\}\right] =$$

$$= \text{ProbabilityRiskNeutral}\left\{\text{Name defaults before } T \mid M = m^S\right\} = 1 - e^{-\lambda^S T} := DR^S(T)$$

This set of scenarios $DR^S(T) \quad S = 0,1,\ldots,124$, represents the set of all the possible default rates of the iTraxx portfolio: $S = 0,1,\ldots,124$ out of 125 constituents of the portfolio default before maturity T. In turn, given its intuitive meaning, the default rate is also the fraction of names that have defaulted for a given maturity. As such, its natural values would be:[1]

$$DR^S \in \left\{0, \frac{1}{125}, \ldots, \frac{124}{125}\right\} = \left\{DR^0, DR^1, \ldots, DR^{124}\right\}.$$

We may use these natural default rate scenarios backwards to deduce sensible scenarios for the intensities associated with them for the maturity T. This amounts to solve in λ lambda

[1] The original perfect copula approach of Hull and White (2005), to which our methodology is inspired, considers a more complex spacing in the set of default rates. Default rate scenarios are spaced in such a way that the sums of net present values of tranches and index under each scenario are equally spaced.

$$1-e^{-\lambda^S T} = DR^S(T) \in \left\{\frac{0}{125}, \frac{1}{125}, \ldots, \frac{124}{125},\right\}$$

leading to

$$\lambda^S \in \left\{-\frac{\ln(1-0)}{T}, -\frac{\ln\left(1-\frac{1}{125}\right)}{T}, \ldots, -\frac{\ln\left(1-\frac{124}{125}\right)}{T}\right\} = \left\{\lambda^0, \lambda^1, \ldots, \lambda^{124}\right\}.$$

To sum up, from our set of default rates above, given the particular maturity T of the tranches and the index we want to price, we have determined a corresponding set of scenarios for the default intensities constituting the parametrization from which we started, thus finally closing the loop.

3. The Instruments Payoff

The tranches and the index pay their spread on the dates $t_1, t_2, \ldots, t_N$, expressed as year fractions. We call the start date $t_0 = 0$. We call R^S the recovery rate associated to the scenario S and we call r_i the risk free zero-coupon rate on date t_i .

Given a generic scenario S the NPV of the premium and default leg of the index will be:

$$\mathrm{Prem}_{Ind}^S = spr_{Ind} \cdot \mathrm{OutNotl}_{Ind}^S$$

$$\mathrm{OutNotl}_{Ind}^S = \sum_{i=1}^N (t_i - t_{i-1}) \cdot \exp(-r_i \cdot t_i) \cdot \exp(-\lambda^S \cdot (t_i + t_{i-1})/2)$$

$$\mathrm{Def}_{Ind}^S = (1-R^S) \cdot \sum_{i=1}^N \exp(-r_i \cdot t_i) \cdot \left[\exp(-\lambda^S \cdot t_{i-1}) - \exp(-\lambda^S \cdot t_i)\right]$$

where the notation for the index outstanding notional, the index premium leg, the index spread and the index default leg is self evident. Given a generic scenario S the NPV of the premium and default leg of the tranche with attachment A and detachment B will be:[2]

$$\mathrm{Prem}_{A,B}^S = spr_{A,B} \cdot \mathrm{OutNotl}_{A,B}^S$$

$$\mathrm{OutNotl}_{A,B}^S = \sum_{i=1}^N (t_i - t_{i-1}) \cdot \exp(-r_i \cdot t_i) \cdot \left(1 - TrancheLoss\left(S, \frac{t_{i-1}+t_i}{2}\right)\right)$$

$$\mathrm{Def}_{A,B}^S = \sum_{i=1}^N \exp(-r_i \cdot t_i) \cdot \left[TrancheLoss(S, t_i) - TrancheLoss(S, t_{i-1})\right]$$

[2] Notice the approximation we have introduced in the computation of the integrals involved in determined the average Outstanding Notional and Default Leg NPV in each period. For a thorough exposition of CDS pricing and the accuracy of different approximations to the relevant integrals see O'Kane and Turnbull (2003).

$$TrancheLoss(S,t)=\frac{\max(PortfLoss(S,t)-A,0)-\max(PortfLoss(S,t)-B,0)}{B-A}$$

$$PortfLoss(S,t)=(1-R^S)\cdot\left(1-\exp(-\lambda^S\cdot t)\right)$$

where again the notation for the A-B tranche outstanding notional, premium leg, spread, default leg and loss is self evident.

4. Implied Default Rate Distribution

Consistently with the perfect copula spirit by Hull and White (2005), our numerical problem is finding the weights (i.e. the scenario probabilities p, positive and adding up to 1) to assign to each scenario so as to reprice the index and the tranches consistently with market quotes. These weights $p_0, p_1,..., p_{124}$ will correspond to the risk neutral distribution of the default rates. We have 125 scenarios and only 6 instruments (5 tranches plus the index), so that the system is under-determined. Indeed, we have too many unknowns (up to 125: the scenario weights) and too few equations (down to 6: the instruments to price).

We call PR and DEF the matrices with the NPV of the premium and default leg. The rows correspond to the scenarios (remember that scenario S is the scenario where S names out of 125 default before maturity) whereas the columns correspond to the instruments (the index in the first column).

$$\mathrm{PR}=\begin{bmatrix}\mathrm{Prem}^0_{\mathrm{Ind}} & \mathrm{Prem}^0_{0,3} & ... & \mathrm{Prem}^0_{12,22}\\ \mathrm{Prem}^1_{\mathrm{Ind}} & \mathrm{Prem}^1_{0,3} & & \mathrm{Prem}^1_{12,22}\\ ... & & & \\ \mathrm{Prem}^{124}_{\mathrm{Ind}} & \mathrm{Prem}^{124}_{0,3} & ... & \mathrm{Prem}^{124}_{12,22}\end{bmatrix}$$

$$\mathrm{DEF}=\begin{bmatrix}\mathrm{Def}^0_{\mathrm{Ind}} & \mathrm{Def}^0_{0,3} & ... & \mathrm{Def}^0_{12,22}\\ \mathrm{Def}^1_{\mathrm{Ind}} & \mathrm{Def}^1_{0,3} & & \mathrm{Def}^1_{12,22}\\ ... & & & \\ \mathrm{Def}^{124}_{\mathrm{Ind}} & \mathrm{Def}^{124}_{0,3} & ... & \mathrm{Def}^{124}_{12,22}\end{bmatrix}$$

If we call $P=\left[p_0,..., p_{124}\right]^T$ the scenario weights vector then the NPV of the instruments will be:

$$\mathrm{NPV}=\begin{bmatrix}\mathrm{NPV}_{Ind}\\ \mathrm{NPV}_{0,3}\\ ...\\ \mathrm{NPV}_{12,22}\end{bmatrix}=\left(\mathrm{Prem}-\mathrm{Def}\right)^T\cdot P$$

To solve the under-determined feature problem Hull and White (2005) look for the set of weights assigned to the scenarios that best re-prices the instruments (minimizes $\mathrm{NPV}^T\cdot\mathrm{NPV}$) and that is also as regular as possible. The objective function they select is

$$\mathrm{NPV}^T \cdot \mathrm{NPV} + c \cdot \sum_{S=1}^{123} \frac{\left[(p_{S+1} - p_S) - (p_S - p_{S-1})\right]^2}{1/125}$$

where the summation is a regularization term. In the objective function of Hull and White there is thus a trade-off between minimization of the mispricing and regularization of the scenarios distribution.

5. Regularization through bid-ask information

We prefer not to choose a trade-off between mispricing and regularization.   We aim at finding the smoothest distribution that prices all instruments exactly (i.e. within their Bid Ask spreads).

To do this we split the problem in two steps.   In the first step we minimize the mispricing without considering smoothness of the distribution.   We consider the first minimization to be successful if all instruments are re-priced within the bid ask spread. In this case we proceed with a second optimization that takes as starting point in the numerical algorithm the optimum found in the first step.   In the second step we maximize the regularity of the distribution given the constraint that the instruments are priced within the bid ask spread.

Following these two steps we avoid choosing any tradeoff between mispricing and smoothness.

The instruments NPV can be rewritten as:

$$NPV = \begin{bmatrix} \mathrm{NPV}_{Ind} \\ \mathrm{NPV}_{0,3} \\ \dots \\ \mathrm{NPV}_{12,22} \end{bmatrix} = \begin{bmatrix} spr_{Ind} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{Ind}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{Ind}^{S} \cdot p_S \\ spr_{Tr}^{0,3} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{0,3}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{0,3}^{S} \cdot p_S \\ \dots \\ spr_{Tr}^{12,22} \cdot \sum_{S=0}^{124} \mathrm{OutNotl}_{12,22}^{S} \cdot p_S - \sum_{S=0}^{124} \mathrm{Def}_{12,22}^{S} \cdot p_S \end{bmatrix}$$

Thus the variation in the spreads required to set to 0 the NPV of the instruments given a scenario distribution P is:

$$\Delta spr = \begin{bmatrix} \Delta spr_{Ind} \\ \Delta spr_{0,3} \\ \dots \\ \Delta spr_{12,22} \end{bmatrix} = \begin{bmatrix} -\mathrm{NPV}_{Ind} / \sum_{S=0}^{124} \mathrm{OutNotl}_{Ind}^{S} \cdot p_S \\ -\mathrm{NPV}_{0,3} / \sum_{S=0}^{124} \mathrm{OutNotl}_{0,3}^{S} \cdot p_S \\ \dots \\ -\mathrm{NPV}_{12,22} / \sum_{S=0}^{124} \mathrm{OutNotl}_{12,22}^{S} \cdot p_S \end{bmatrix}$$

If the absolute value of the components of this vector is larger than half the bid ask spread then the scenario distribution P is not able to price the instruments with an error within the bid ask spread.

6. A convenient relationship between Default Rates and Recovery Rates

In their article Hull and White (2005) have used first in their estimation a flat recovery rate at 40% for all scenarios:

$$R^S = 40\% \qquad S = 0,1,\ldots,124$$

The authors report they could fit market data in the first half of 2004 using a flat recovery of 40%. They also report that in order to fit more recent data (as of November 2005) they found it necessary to incorporate the following best fit relationship as in Hamilton et al. (2005), expressing recovery as a function of the default rate:

$$R^S = \max\left[0\%\,,\,52\% - 6.9\cdot\left(1-\exp(-\lambda^S)\right)\right] \qquad S = 0,1,\ldots,124$$

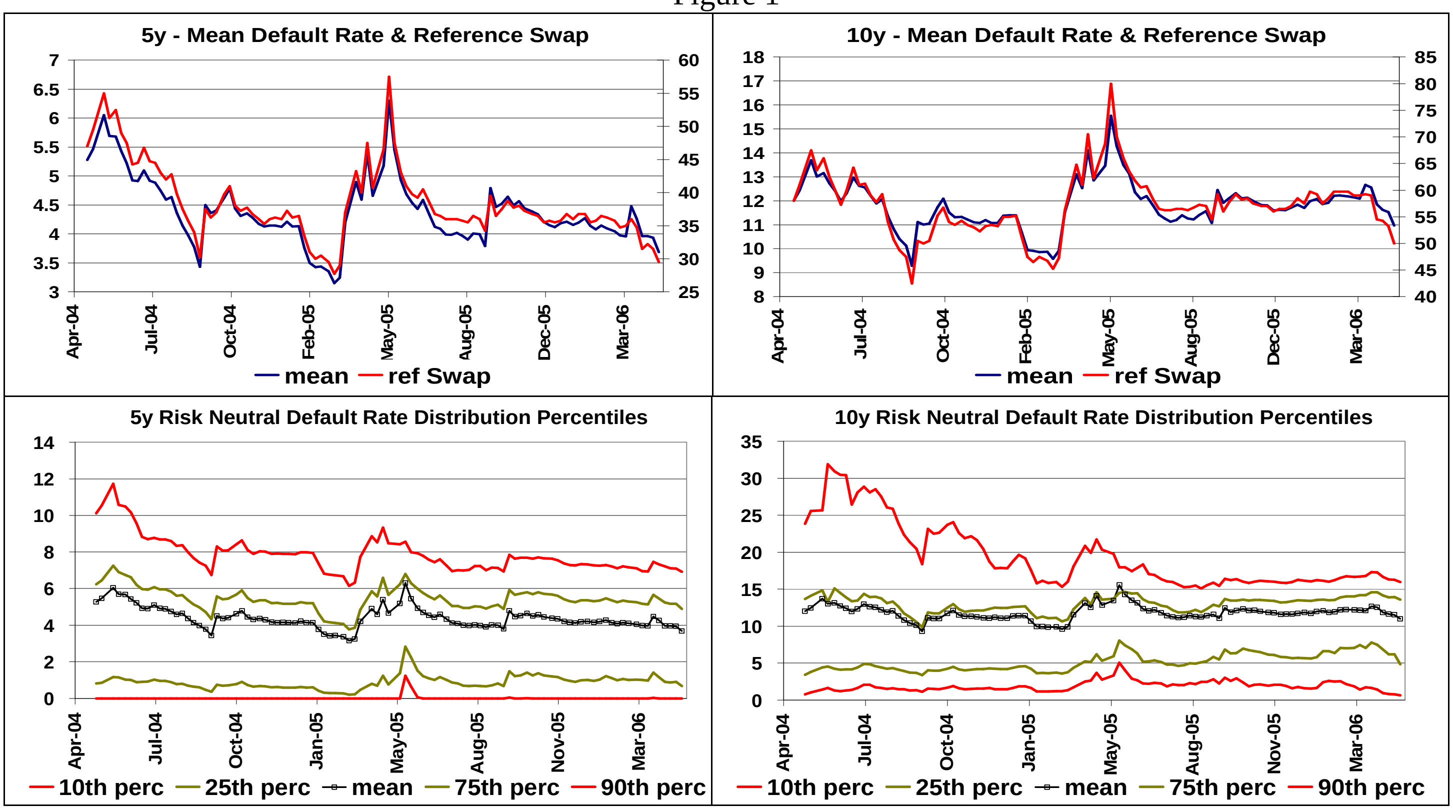
This means that in correspondence of default rates above 7.53% the recovery is null. Our approach is instead to fit a nonlinear relationship between recovery rates and annualized default rates as in Altman et al. (2002).[3]

$$[1] \qquad R^S = \begin{cases} 0 \;\;,\; S=0 \\ -0.10563\cdot\ln\left(1-\exp\left(-\lambda^S\right)\right) \;\;,\;\; S = 1,2,\ldots,124 \end{cases}$$

7. Implied Default Rate Distribution through Time

We now apply the methodology outlined in the previous paragraphs to the market CDS index and standardized CDO tranches spreads from April 2004 to April 2005. We imply a risk neutral default rate density.

Figure 1



[3] The relationship we implement does not have an intercept so that we obtain a recovery of 0 in case the entire portfolio defaults.

In the top two graphs of figure 1 we note that the reference swap rate (right hand scale) and the average default rate (left hand scale) are strongly correlated. In the bottom two graphs of figure 1 we note instead that the implied default rate density had a much larger dispersion, as measured by the difference between the 75th and 25th percentile, during 2004 compared to the second half of 2005 and the beginning of 2006. This feature is especially pronounced for the 10 years maturity. Also we note that until April 2005 the dispersion of the distribution was positively correlated with the reference spread movements, as proxied by the average default rate: the dispersion decreases (increases) when the index decreases (increases).

Also from the middle two graphs of figure 1 we note what happened during the Ford-GMAC crisis. Around April 2005 the average spread increased and simultaneously the distribution narrowed. The dispersion narrowed and at one point in time, 16 May 2005, on the 10 year maturity the implied average default rate was above the 75th percentile.

8. Default Rate Simulation under the Objective Measure

We are given a set of N reference obligations associated to a set of K rating classes. We have historically measured the frequency according to which a name belonging to a certain rating class will default before a maturity T expressed as a year fraction. In order to value CDO tranches we need also a measure of the dependency between the defaults of the N reference obligations in the portfolio.

We model this dependency via a one factor Gaussian copula:

1) we randomly draw a vector X of N jointly standardized normal random variables with correlation matrix Σ;
2) we compute the cumulative normal $U_i = \phi(X_i)$ for all components X_i of X ;
3) we finally compute the simulated default time for each name i as the inverse of the historically measured survival function $\tau_i = S_i^{-1}(U_i), \quad S_i(T) = \exp\left(-\int_0^T \lambda_i(s)ds\right)$ where λ_i is here the deterministic time dependent default intensity of name i.

If the simulated default time τ is less than T then the associated reference obligation in the current simulation will default before the maturity of the CDO. The loss incurred by the portfolio will be the notional invested in the name (for the iTraxx it will be 0.8%=1/125) times one minus the recovery associated to that name.

In the risk neutral framework the way the correlation is implied from market quotes assumes the correlation matrix Σ to be flat in the sense that the off-diagonal terms are all equal to a single parameter ρ. Given the market spread of a tranche the corresponding compound correlation would be the ρ value to be put into a Gaussian copula linking the default of different names that sets to 0 the NPV of the tranche given the risk neutral default intensities $\lambda(s)$ stripped by the CDS term structure of the index constituents. Clearly rather than a model this is a quoting mechanism, and to achieve consistency the related quotes of different tranches need to be calibrated with a single model, such as for example the perfect copula model above, where the notion of correlation matrix is somehow lost apart from its quoting mechanism interpretation.

The correlation matrix we use to obtain the default rate and loss rate distributions under the objective measure is instead block diagonal, as in the CDO Evaluator of S&P. The correlation between any two names will be 15% if both names belong to the same global or regional sector and 5% otherwise.

Remark 1: ("S&P CDO Evaluator 3" correlation assumptions). In determining the block diagonal structure of the pool of obligors the S&P CDO Evaluator 3 differentiates between local, regional and global sectors. A local sector is only affected by the macroeconomic forces within the country where the asset resides (for example "building and development"). A regional sector is affected by the macroeconomic forces of the region (for example: "rail industries"). A global sector assumes that the same economic forces affect all companies in that sector, regardless of location (for example: "oil and gas").

9. Risk Premium Evidence in the Index Swap and the ITRAXX Tranches

Given the dollar value of the default leg of an instrument (index CDS or CDO tranche) we can ask ourselves the question of how much of this dollar value is justified in terms of default risk. In other words we want to know the difference between the NPV of the default leg under the risk neutral and objective measures.

A different question we might consider is the following: how much more the premium leg would have paid in excess of the default leg if the default distribution were the historical one? This corresponds to the difference between the NPV of the premium and default leg under the objective measure. Both differences can be thought as related to a compensation for the risk of default in the instrument notional.

Table 1

Market Quotes (27 Jan 2006)

	3	5	7	10
index	0.1875%	0.3550%	0.4725%	0.5750%
0 - 3	5.0000%	27.7500%	48.2500%	57.6250%
3 - 6	0.0850%	0.7900%	1.8750%	5.3000%
6 - 9	0.0300%	0.2700%	0.4800%	1.0050%
9 - 12	-	0.1250%	0.2700%	0.4450%
12 - 22	-	0.0563%	0.1200%	0.2250%

Npv Default Leg under the Objective Measure

	3	5	7	10
index	0.3093%	0.5902%	0.8991%	1.3776%
0 - 3	**10.2967%**	**19.4756%**	**28.9749%**	**41.5994%**
3 - 6	0.0132%	0.1949%	0.9632%	**3.9975%**
6 - 9	0.0000%	0.0027%	0.0312%	0.2979%
9 - 12	0.0000%	0.0000%	0.0020%	0.0224%
12 - 22	0.0000%	0.0000%	0.0001%	0.0011%

Npv Premium Leg under the Objective Measure

	3	5	7	10
index	0.5179%	1.5996%	2.8889%	4.7598%
0 - 3	**18.1851%**	**48.3559%**	**74.8461%**	**90.7758%**
3 - 6	0.2355%	3.5807%	11.5529%	**44.1433%**
6 - 9	0.0831%	1.2242%	2.9629%	8.4422%
9 - 12	0.0000%	0.5668%	1.6667%	3.7397%
12 - 22	0.0000%	0.2550%	0.7407%	1.8909%

Npv Default Leg under the Risk Neutral Measure

	3	5	7	10
index	0.5158%	1.5784%	2.8203%	4.5794%
0 - 3	**17.5643%**	**44.6892%**	**66.5971%**	**76.2986%**
3 - 6	0.2350%	3.5302%	11.1250%	**38.1894%**
6 - 9	0.0830%	1.2144%	2.9118%	8.1438%
9 - 12	0.0000%	0.5636%	1.6477%	3.6493%
12 - 22	0.0000%	0.2538%	0.7335%	1.8616%

Npv Premium Leg under the Objective Measure MINUS Npv Default Leg under the Objective Measure

	3	5	7	10
index	0.2086%	1.0094%	1.9898%	3.3821%
0 - 3	7.8884%	28.8803%	45.8712%	49.1764%
3 - 6	0.2224%	3.3858%	10.5897%	40.1458%
6 - 9	0.0831%	1.2215%	2.9316%	8.1443%
9 - 12	0.0000%	0.5668%	1.6647%	3.7173%
12 - 22	0.0000%	0.2550%	0.7407%	1.8898%

Npv Default Leg under the Risk Neutral Measure MINUS Npv Premium Leg under the Objective Measure

	3	5	7	10
index	0.2065%	0.9882%	1.9212%	3.2018%
0 - 3	7.2675%	25.2136%	37.6222%	34.6992%
3 - 6	0.2218%	3.3353%	10.1619%	34.1919%
6 - 9	0.0830%	1.2118%	2.8805%	7.8459%
9 - 12	0.0000%	0.5636%	1.6457%	3.6268%
12 - 22	0.0000%	0.2538%	0.7334%	1.8605%

Given the market quotes of a set of instruments as of January 27th 2006 (1st row, left column of table 1), in order to be remunerated for the risk of default of the notional amount, we would expect both this differences to be positive. In both cases we subtract the NPV of the default leg under the objective measure (1st row, right column of table 1). In the former definition we subtract it from the NPV of the default leg under the risk neutral measure (2nd row, right column of table 1) and in the latter definition from the NPV of the premium leg under the objective measure (2nd row, left column of table 1). In fact we see that the two differences are indeed positive for all instruments and maturities (3rd row of table 1).

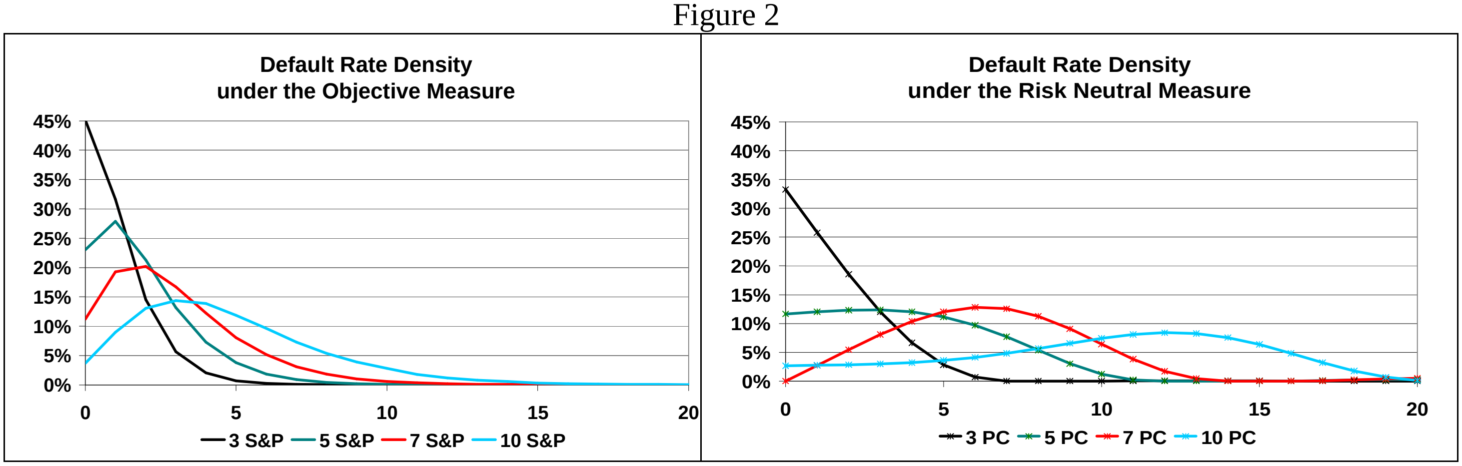
We note that for all maturities roughly half (between 40% and 60%) of the NPV of the equity tranche is justified in terms of default risk. We also note that only a small portion of the mezzanine (3-6) 10 year tranche is justified in terms of default risk.

Of course the quantification of the remuneration for risk for each instrument (index and tranche) depends heavily on the simulation engine outlined in section 8. If for example we were to use non homogeneous rating transition matrices or a different copula we would get different NPVs of the default leg under the objective measure and thus different numbers in the third row of table 1.

10. Comparison between the Default Rate Density under the Objective and Risk Neutral Measure

In figure 2 we plot the default rate density under the objective and risk neutral measure (the abscissas correspond to the number of defaulted obligor in the ITRAXX CDO pool: 125 names). The risk neutral density is obtained from the market quotes in the bottom left part of table 1 (27-Jan-2006) using the methodology described in sections 2 to 6. The objective measure density is obtained instead from a simulation following the methodology outlined in section 8.

Figure 2



We notice immediately the different centers of the densities corresponding to the same maturity between the objective and risk neutral measure. The risk neutral densities are shifted to the right, corresponding thus to a risk premium being priced in the traded assets underlying the risk neutral measure.

In figure 3 we finally zoom on two features, corresponding to different areas of the abscissa (number of defaulted obligors at maturity) of the density under the risk neutral measure. In the left plot we notice a series of bumps of increasing size and shifting further to the right, as maturity increases, in the range of 10 to 40 defaults: a scenario of extremely severe loss in the CDO obligors pool. In the right plot we notice the far end tail, referring to 80 to 125 defaults out of 125 obligors in the ITRAXX pool. We notice a small bump increasing with the maturity also for this catastrophic scenario.

The top part of figure 4 shows the implied default rate distribution calibrated at different times for the 5 and 10 years maturity.   The bottom part of figure 4 zooms on the tails of the implied distribution, pointing out the overall persistence of the above mentioned bumps in terms of size (probability mass) and location (range of default numbers).

Figure 3

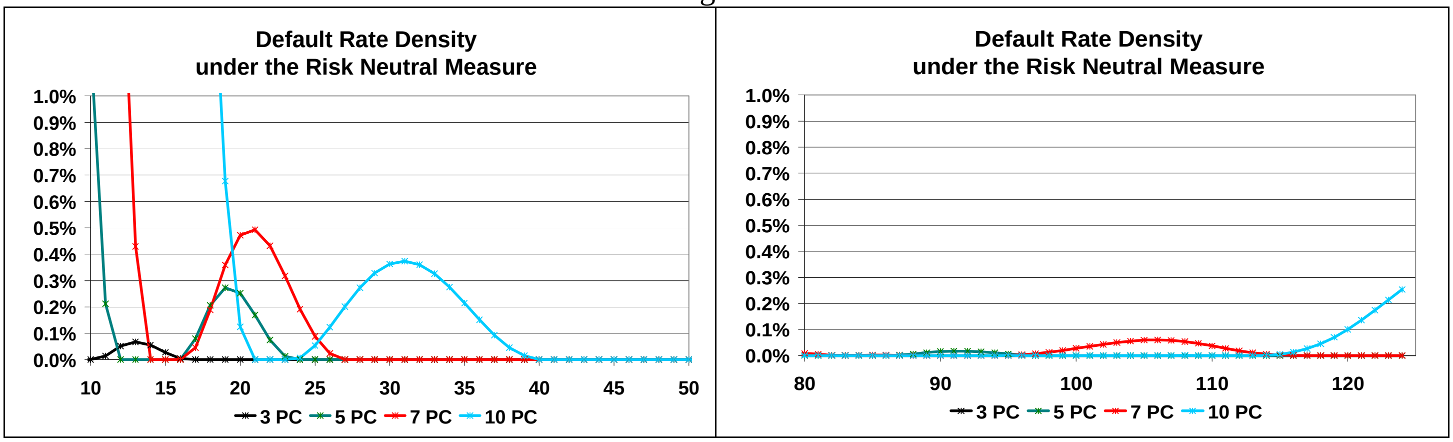
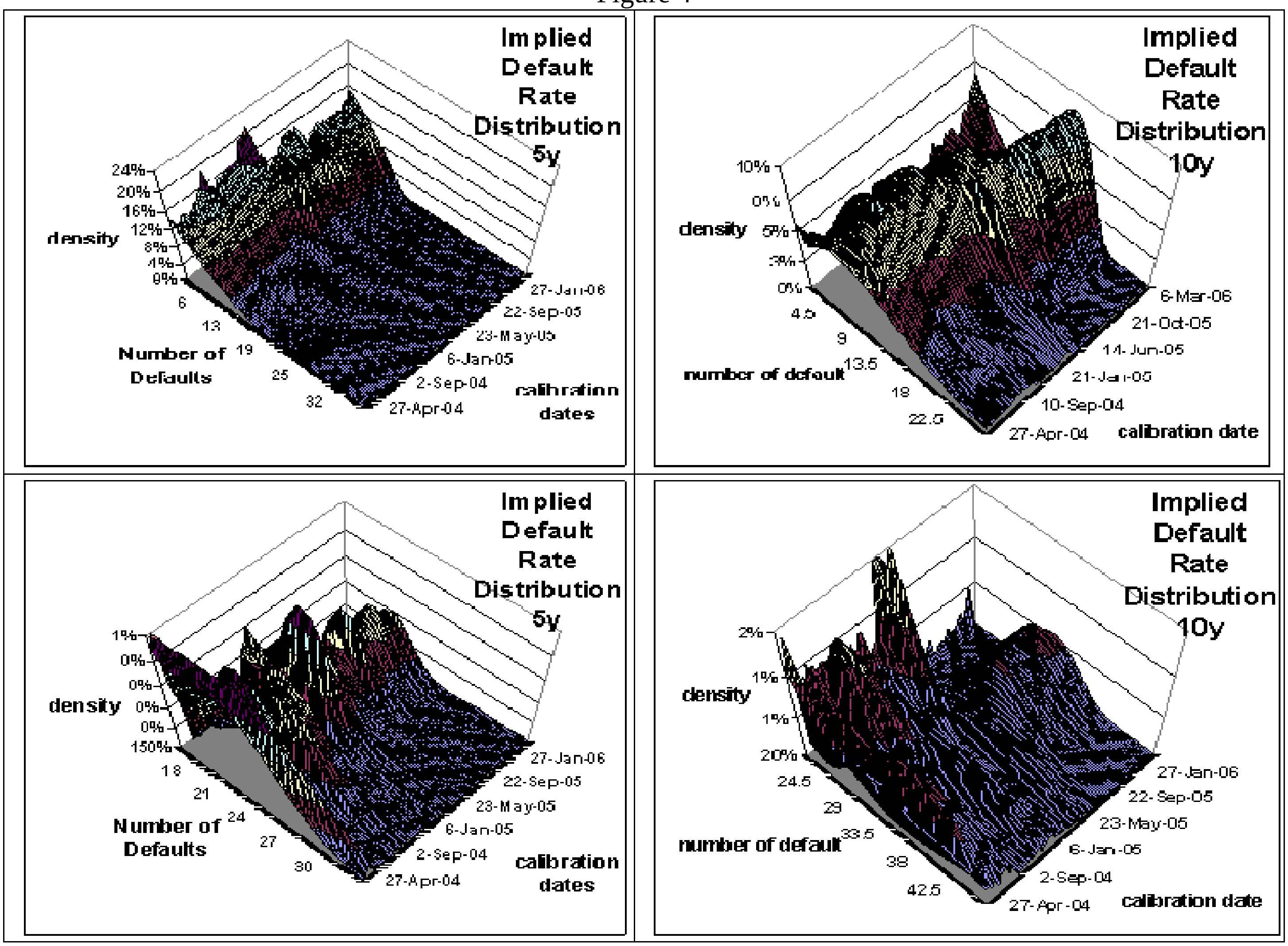


Figure 4



Since this feature persists, it may be appropriated to look for more complex dynamical loss models that can produce a bump feature in the tail. A related dynamical loss models that can be consistently calibrated to tranche and index data for different maturities is the Generalized Poisson Loss model of Brigo, Pallavicini and Torresetti (2006).

Brigo, Pallavicini and Torresetti (2007) further address consistency with single name data and default clusters, leading to a top-down approach known as GPCL model.

A different model free approach to extract market information from standardized CDO tranches that is also consistent across maturities can be found in Walker (2006) and in Torresetti, Brigo and Pallavicini (2006).

11. Conclusions and further research

In this paper we consistently priced CDO tranches across detachments. The risk neutral loss distribution has been derived through a modified "scenario default rate" approach. We then priced CDO tranches under the objective measure by using the Gaussian Copula with the same historical correlation matrix as in the CDO Evaluator 3.0 of S&P and historical default probabilities deduced by rating classes. From such prices we deduced the loss distribution under the objective measure. We have found that the risk neutral loss distribution privileges large realizations of the loss with respect to the objective distribution, thus implying the well known presence of a risk premium. We quantified this risk premium. In this analysis we used infinite pool limit results to simplify our approach. It would be interesting to check what happens in a finite pool setup. This would allow to introduce more structure in the initial area of the loss distribution.

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REFERENCES

Altman Edward I., Brady Brooks, Resti Andrea and Sironi Andrea (2002), "The Link Between Default and Recovery Rates: Implications for Credit Risk Models and Procyclicality"

Brigo Damiano, Pallavicini Andrea and Torresetti Roberto (2006), "Calibration of CDO Tranches with the Dynamical Generalized-Poisson Loss Model," Financial Engineering - Credit Models, Banca IMI Working Paper, available at SSRN.com: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=900549, short version published in Risk Magazine, May 2007 issue.

Brigo Damiano, Pallavicini Andrea and Torresetti Roberto (2007), "Default Correlation, Cluster Dynamics and Single Names: The GPCL Dynamical Loss Model," Working Paper, available at SSRN.com: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=956827, short version to appear on International Journal of Theoretical and Applied Finance.

Burtschell X., Gregory J. and Laurent J.-P. (2005), "A Comparative Analysis of CDO Pricing Models," Working Paper, BNP Paribas.

CDO Evaluator Handbook (2006), Standard & Poor's Structured Finance Group.

De Servigny Arnaud and Renault Olivier (2004), "Measuring and Managing Credit Risk," Mc Graw-Hill.

Hamilton David T., Varma Praveen, Ou Sharon and Cantor Richard (2005), “Default and Recovery Rates of Corporate Bond Issuers, 1920-2004,” Special Comment, Moodys Investor Service.

Hull John, Predescu Mirela and White Alan (2004), "Bond Prices, Default Probabilities and Risk Premiums," Journal of Credit Risk, Vol. 1, No. 2, pp. 53-60.

Hull John and White Alan (2004), "Valuation of a CDO and an n-th to Default CDS Without Monte Carlo Simulation," John Hull and Alan White, Journal of Derivatives, Vol. 12, No. 2, pp. 8-23.

Hull John and White Alan (2005), “The Perfect Copula,” Working Paper.

Li D. (2000), “On Default Correlation: A Copula Function Approach,” Journal of Fixed Income.

Li David X. and Hong Liang Michael (2005), "CDO Square Pricing Using Gaussian Mixture Model with Transformation of Loss Distribution," Barclays Capital working paper.

McGinty L. and R. Ahluwalia (2004), “A Model for Base Correlation Calculation,” JP Morgan technical document.

O’Kane Dominic and Turnbull Stuart (2003), “Valuation of Credit Default Swaps,” Fixed Income Quantitative Credit Research, Lehman Brothers.

Torresetti Roberto, Brigo Damiano, and Pallavicini Andrea, “Implied Expected Tranched Loss Surface from CDO Data,” 2006, Working paper. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=933291

Walker Micheal, “CDO models. Towards the next generation: incomplete markets and term structure,” 2006, Working paper. Available at http://defaultrisk.com/pp crdrv109.htm

APPENDIX A

The Gaussian 1 factor copula models assume a Gaussian copula structure driving the exponential random variables generating jumps in the related default processes of the names in the pool. This results in the default probability for each name, conditional on the Gaussian systemic factor M, to be given by

$$[2] \quad \text{ProbabilityRiskNeutral}\{\text{Name defaults before } T \mid M\} = N\left(\frac{N^{-1}(PD(T)) - M\sqrt{\rho}}{\sqrt{1-\rho}}\right)$$

where $PD(T)$ is the risk neutral probability that any name defaults by time T.

As before defaults are independent given M, and this allows to compute the joint default probabilities for the whole pool by simply multiplying the conditional risk neutral probabilities in [2] and integrating over M under its Gaussian density. Under the large pool assumption the above probability is also the pool default rate by T given M. In the 1 factor Gaussian Copula the systemic factor is a continuous random variable. If we discretise the domain of the systemic factor we can sum up the Gaussian copula as:

$$\text{One factor Gaussian Copula} \begin{cases} \text{Systemic Scenario} & \text{Scenario Probability} & \text{Conditional default rate} \\ \dots & \dots & \dots \\ \mathrm{M} \in [\mathrm{m}; \mathrm{m}+\mathrm{dm}) & \mathrm{N}(\mathrm{m}+\mathrm{dm}) - \mathrm{N}(\mathrm{m}) & N\left(\frac{N^{-1}(Pd(T)) - \mathrm{m}\sqrt{\rho}}{\sqrt{1-\rho}}\right) \\ \dots & \dots & \dots \end{cases}$$

It is now possible to compare this to the implied copula assumption we used in this paper: In our case we replaced the parametric formula [2] for the default probability with the more natural, intensity based one

$$\text{ProbabilityRiskNeutral}\{\text{Name defaults before } T \mid M = m^S\} = 1 - e^{-\lambda^S T}$$

leading to

$$\text{Implied Copula} \begin{cases} \text{Systemic Scenario} & \text{Scenario Probability} & \text{Conditional default rate} \\ \mathrm{M} = \mathrm{m}^0 & p^0 & 1 - e^{-\lambda^0 T} \\ \mathrm{M} = \mathrm{m}^1 & p^1 & 1 - e^{-\lambda^1 T} \\ \dots & \dots & \dots \\ \mathrm{M} = \mathrm{m}^{124} & p^{124} & 1 - e^{-\lambda^{124} T} \end{cases}$$

The end result of these approaches as far as pricing is concerned is the default rate distribution. In the Gaussian One Factor Copula case this distribution has little flexibility, in that one can play only with the single copula parameter ρ, scenario probabilities being fixed by the Gaussian assumption. If one is to price a set of instruments (e.g. CDO tranches) with a single model specification, having just one parameter can be unrealistic. In the implied copula approach instead we can play with the scenario probabilities so as to obtain a rich variety of possible default rate distributions, which can help in pricing a set of instruments with a single model specification.