On the Valuation of Mark-to-Market Basis Cross Currency Swaps

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1 Introduction

The objective of this short paper is the valuation of Mark-to-Market Cross Currency Basis Swaps (MtMCCS) under different mathematical models. To be more precise, we compute the present values with respect to the non-stochastic short rate, Ho-Lee and BGM models, respectively, in terms of the model parameters. Most of the material covered in this paper can be found in standard literature on Mathematical Finance (for example [1]) or in Hunter's lecture notes on Hybrids [2]. Section 2 is an exception (the author did not find a mathematically rigorous way of defining an FX model in the literature). This paper is organized as follows: In section 2, we set up a rigorous mathematical model for an FX market, along with the notations that we use throughout the text. This includes models for the domestic and foreign short rates and for the FX rate. The reason why we incorporate this modelling into this paper is explained at the beginning of that section. In section 3, we explain the MtMCCS and how it differs from a standard basis cross currency swap. In particular, we introduce more notation and indicate the relevant cash flows in both cases. In section 4, a formula for the present value of both the MtMCCS and the standard cross currency swap in terms of bond prices is derived. By comparing the formulae, we explain mathematically why a MtoMCCS reduces FX risk significantly. In section 5, we use the formula obtained in section 4 to derive present values of the MtMCCS with respect to the non- stochastic, Hull-White and BGM models, respectively. We do not consider stochastic volatility here because the author preferred to give a rigorous solution to the problem - starting from building a mathematical model until finding precise formulas for the values of cross currency swaps that - at least to our knowledge - do not exist in the standard literature.

2 A model for the FX market

We start by giving a mathematically rigorous way of constructing a model for an FX market consisting of a domestic and a foreign short rate and an FX rate with a finite time horizon T. We try to point out why this seems to be important to us: In the literature, one often finds the following specification of such a model: Let

$$\begin{aligned} dr_d(t) &= \mu_d(t)dt + \sigma_d(t)dW_d^{Q_d}(t), \\ dr_f(t) &= \mu_f(t)dt + \sigma_f(t)dW_f^{Q_f}(t), \\ dX(t) &= X(t)((r_d(t) - r_f(t))dt + \sigma_X(t)dW_X^{Q_d}(t) \end{aligned} \tag{2.1}$$

be the dynamics of the domestic short rate, foreign short rate and FX rate, where Q_d is the pricing measure with respect to the domestic money account as numeraire, Q_f is the pricing measure with respect to the foreign money account as numeraire, and where $W_d^{Q_d}, W_X^{Q_d}$ are Q_d-Brownian motions and $W_f^{Q_f}$ is a Q_f-Brownian motion. Further, correlations between these Brownian motions are given. From our point of view, there are some problems by specifying a model this way. One problem is that the pairs (Q_d, W_d) and (Q_f, W_f) are chosen at the beginning, and then the density of the measure transformation between Q_d and Q_f is computed using the model (but at this point, the measures have already been defined!). Moreover, the correlation between the Brownian motions $W_X^{Q_d}$ and $W_f^{Q_f}$ appears in this density. But a correlation of Brownian motions with respect to different measures cannot be defined without specifying the transformation between these measures (the expectation of the product and the two variances have to be computed with respect to exactly one measure), which completely screws things (since we need the correlation to define the change of measure and the change of measure to define the correlation). So we think that it might be important to describe how to rigorously build up a mathematical model such that the dynamics given in (2.1) are satisfied. Let $(\Omega, \mathcal{F}, Q_d, (W_X^{Q_d}, W_X^{Q_d^\perp}, W_X^{Q_d^{\perp\perp}}))$ be a quadruple, consisting of a probability space $(\Omega, \mathcal{F}, Q_d)$ and a three-dimensional Q_d-Brownian motion $(W_X^{Q_d}, W_X^{Q_d^\perp}, W_X^{Q_d^{\perp\perp}})$. Let $(\mathcal{F}_t)_{t\leq T}$ be the filtration generated by the above three-dimensional Brownian motion. Let $\rho, \eta, \gamma \in [-1, 1]$ such that $\eta\rho + \gamma\sqrt{1-\rho^2} \in [-1, 1]$ and $\eta^2 + \gamma^2 \leq 1$. We define

$$W_f^{Q_d} := \rho W_X^{Q_d} + \sqrt{1-\rho^2} W_X^{Q_d^\perp}$$

and

$$W_d^{Q_d} := \eta W_X^{Q_d} + \gamma W_X^{Q_d^\perp} + \sqrt{1-\eta^2-\gamma^2} W_X^{Q_d^{\perp\perp}}.$$

Because $W_d^{Q_d}$ and $W_f^{Q_d}$ are sums of weighted independent Q_d-Brownian motions whose squared weights add up to 1, it follows that $W_d^{Q_d}$ and $W_f^{Q_d}$ are themselves Q_d-Brownian motions. Further, we see that the correlation between $W_X^{Q_d}(t)$ and $W_f^{Q_d}(t)$ is

$$corr(W_X^{Q_d}(t), W_f^{Q_d}(t)) = \frac{\rho Cov(W_X^{Q_d}(t), W_X^{Q_d}(t)) + \sqrt{1-\rho^2} Cov(W_X^{Q_d}(t), W_X^{Q_d^\perp}(t))}{\sqrt{Var(W_X^{Q_d}(t)) Var(W_f^{Q_d}(t))}} = \rho =: \rho_{X,f}.$$

Similarly, we obtain

$$corr(W_X^{Q_d}(t), W_d^{Q_d}(t) = \eta =: \rho_{X,d},$$

and

$$corr(W_d^{Q_d}(t), W_f^{Q_d}(t)) = \eta\rho + \gamma\sqrt{1-\rho^2} =: \rho_{d,f}$$

(hence the condition on γ). The measure Q_d will be the pricing measure for the domestic market with respect to the spot numeraire, so we model the domestic short rate already under this pricing measure as

$$dr_d(t) = \mu_d(t)dt + \sigma_d(t)dW_d^{Q_d}(t),$$

where we have chosen suitable (e.g. bounded and continuous) deterministic functions σ_d and μ_d. We choose a deterministic function σ_X such that the stochastic exponential $(\mathcal{E}(\int_0^t \sigma_X(s)dW_X^{Q_d}(s))_{t\leq T}$ is a true martingale (not just a local martingale) (by Novikov's condition, it suffices that $\sigma_X \in L^2[0,t]$ for all $t \leq T$). We model the foreign short rate as

$$dr_f(t) = (\mu_f(t) - \sigma_f(t)\rho_{X,f}\sigma_X(t))dt + \sigma_f(t)dW_f^{Q_d}(t),$$

again with nice functions μ_f and σ_f. Because the foreign short rate can be seen as a continuous dividend payment of the FX rate: If we buy, at time 0, one unit of foreign currency for $X(0)$, we can invest this unit into a risk-free foreign bank account, earning interest r_f. Hence the value (in domestic currency) of my foreign bank account at time t is $X_t \int_0^t r_f(s)ds$, so r_f can indeed be thought of as a continuous dividend. Because we want the measure Q_d to be the pricing measure with respect to the risk-free domestic bank account as numeraire, the drift of the FX rate X under the measure Q_d has to be $r_d - r_f$ (compare Black-Scholes model with continuous dividends). We take our previously defined function σ_X as volatility function for the FX rate, so the model for the FX rate X is

$$dX(t) = X(t)((r_d(t) - r_f(t))dt + \sigma_X(t)dW_X^{Q_d}(t).$$

This completes the modelling of the FX market. The only thing that remains to show is that this model gives rise to the dynamics given in (2.1), where Q_f is the pricing measure with respect to the foreign bank account as numeraire. In order to see this, we note that $\frac{1}{X}$ denotes the price of one unit of domestic currency measured in foreign currency, so by the same reasoning as above, $\frac{1}{X}$ must have drift $r_f - r_d$ under Q_f. On the other hand, Itô's formula implies that

$$\begin{aligned}
d\left(\frac{1}{X}\right)(t) &= -\frac{1}{X(t)^2}dX(t) + \frac{1}{2}\cdot\frac{2}{X(t)^3}d[X](t) \\
&= -\frac{1}{X(t)^2}X(t)((r_d(t) - r_f(t))dt + \sigma_X(t)dW_X^{Q_d}(t)) + \frac{1}{X(t)^3}X(t)^2\sigma_X(t)^2dt \\
&= \left(\frac{1}{X}\right)(t)(r_f(t) - r_d(t) + \sigma_X(t)^2)dt - \sigma_X(t)dW_X^{Q_d}(t),
\end{aligned}$$

so if we had that

$$(W_X^{Q_f}(t))_t := \left(W_X^{Q_d}(t) - \int_0^t \sigma_X(s)ds\right)_{t\leq T} \tag{2.2}$$

was a Brownian motion under some measure Q_f, then by the above argument, Q_f would be with the pricing measure with the foreign bank account as numeraire. But this is indeed the case by virtue of Girsanov's theorem (we define Q_f by $Z := \frac{dQ_f}{dQ_d} := \mathcal{E}(\int_0^T \sigma_X(s)dW_X^{Q_d}(s))$ and by our assumption, $(\mathcal{E}(\int_0^t \sigma_X(s)dW_X^{Q_d}(s)))_{t\leq T}$ is a true martingale). We now want to find a process $v(t)$ of bounded variation such that $W_f^{Q_f} := W_f^{Q_d} + v$

is a Brownian motion under the measure Q_f. By writing out the relationship between $W_f^{Q_d}$ and $W_X^{Q_d}$ and using (2.2), we obtain

$$\begin{aligned} W_f^{Q_f}(t) = W_f^{Q_d}(t) + v(t) &= \rho_{X,f} W_X^{Q_d}(t) + \sqrt{1-\rho_{X,f}^2}\, W_X^{Q_d^\perp}(t) + v(t) \\ &= \rho_{X,f} W_X^{Q_f}(t) + \rho_{X,f} \int\limits_0^t \sigma_X(s)ds + \sqrt{1-\rho^2}\, W_X^{Q_d^\perp}(t) + v(t). \end{aligned}$$

So it is natural to set $v(t) := -\rho_{X,f}\int_0^t \sigma_X(s)ds$. We now show that $W_f^{Q_f}$ is a continuous martingale (with respect to the filtration $(\mathcal{F}_t)_t$): The continuity is clear because Brownian motions are continuous and as already shown above, $W_X^{Q_f}$ is a Q_f-Brownian motion and hence a Q_f-martingale. It remains to show that $W_X^{Q_d^\perp}$ is a Q_f-martingale. To see this, let $(\mathcal{G}_t)_t$ be the filtration generated by $(W_X^{Q_d^\perp})_t$. It follows that for $0 \le u < t$,

$$E^{Q_f}(W_X^{Q_d^\perp}(t)|\mathcal{F}_u) = E^{Q_f}(W_X^{Q_d^\perp}(t)|\mathcal{G}_u) = \frac{E^{Q_d}(W_X^{Q_d^\perp}(t)\mathcal{E}(\int_0^t \sigma_X(s)dW_X^{Q_d}(s))|\mathcal{G}_u)}{E^{Q_d}(\mathcal{E}(\int_0^t \sigma_X(s)dW_X^{Q_d}(s))|\mathcal{G}_u)}. \tag{2.3}$$

Because $W_X^{Q_d}(s)$ is independent of $\mathcal{G}_u$ for all $s, u > 0$, we conclude

$$E^{Q_d}\left(\mathcal{E}\Big(\int\limits_0^t \sigma_X(s)dW_X^{Q_d}(s)\Big)|\mathcal{G}_u\right) = E^{Q_d}\left(\mathcal{E}\Big(\int\limits_0^t \sigma_X(s)dW_X^{Q_d}(s)\Big)\right) = 1. \tag{2.4}$$

We prove the following result:

Lemma 2.1 *If X and Y are independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra independent of Y, then $E(XY|\mathcal{G}) = E(Y)E(X|\mathcal{G})$.*

Proof. For $A \in \mathcal{G}$, the random variables $1_A X$ and Y are independent and we have

$$\begin{aligned} \int\limits_A XY dP = \int\limits_\Omega 1_A XY dP &= E(1_A XY) = E(1_A X)E(Y) \\ &= E(Y)\int\limits_A X dP = E(Y)\int\limits_A E(X|\mathcal{G})dP = \int\limits_A E(Y)E(X|\mathcal{G})dP. \end{aligned}$$

The random variable $E(Y)E(X|\mathcal{G})$ is $\mathcal{G}$-measurable, and since $A \in \mathcal{G}$ was chosen arbitrarily and $E(XY|\mathcal{G})$ is the (almost surely) unique random variable with $\int_A XY dP = \int_A E(XY|\mathcal{G})dP$, we have

$$E(Y)E(X|\mathcal{G}) = E(XY|\mathcal{G})$$

as desired. □

If we apply this result to $X := W_X^{Q_d^\perp}, Y := \mathcal{E}(\int_0^t \sigma_X(s)dW_X^{Q_d}(s)), \mathcal{G} := \mathcal{G}_u$ and $P := Q_d$ and combine this with (2.4), we deduce with (2.3) that

$$E^{Q_f}(W_X^{Q_d^\perp}(t)|\mathcal{F}_u) = E^{Q_d}\left(\mathcal{E}\Big(\int\limits_0^t \sigma_X(s)dW_X^{Q_d}(s)\Big)E^{Q_d}(W_X^{Q_d^\perp}(t)\Big)|\mathcal{G}_u\right) = E^{Q_d}(W_X^{Q_d^\perp}(t)|\mathcal{G}_u) = W_X^{Q_d^\perp}(u),$$

where the last equality is true because $W_X^{Q_d^\perp}$ is a Q_d-Brownian motion. Since the quadratic variation does not depend on the measure, $Q_d^\perp$ is a Q_f-Brownian motion and hence $W_f^{Q_f}$ is a continuous martingale. The quadratic variation of $W_f^{Q_f}$ satisfies

$$[W_f^{Q_f}]_t = [W_f^{Q_d}]_t = t$$

since v is a finite variation process and $W_f^{Q_d}$ has quadratic variation equal to t. Lévy's characterisation of Brownian motion now implies that $(W_f^{Q_f}(t) = W_f^{Q_d}(t) - \rho_{X,f}\int_0^t \sigma_X(s)ds)_{t\le T}$ is in fact a Q_f-Brownian motion (analogously, one can prove that $(W_d^{Q_f}(t) := W_d^{Q_d}(t) - \rho_{X,d}\int_0^t \sigma_X(s)ds)_{t\le T}$ is a Q_f-Brownian motion). If we apply this fact to the definition of r_f, we obtain the following dynamics:

$$dr_f(t) = \mu_f(t)dt + \sigma_f(t)dW_f^{Q_f}(t),$$

which coincides with the dynamics of r_f given in (2.1). We finally note that the correlation between $W_f^{Q_f}$ and $W_X^{Q_d}$ (resp. $W_d^{Q_d}$) is the same as the correlation between $W_f^{Q_d}$ and $W_X^{Q_d}$ (resp. $W_d^{Q_d}$) since $W_f^{Q_f}$ differs from $W_f^{Q_d}$ by a deterministic function. So we have given a rigorous proof that a model for the dynamics given in (2.1) exists under mild assumptions on the μ and σ-processes, so as long as these coefficients satisfy these assumptions, we can happily start off by writing down the model in the form (2.1).

3 Standard and MtoM Cross Currency Swaps

We start with some conventions and notations that we are using for the rest of this paper. From now on, we assume for convenience that Euro (€) is the domestic currency and Dollar ($) is the foreign currency. As in the last section, let $X(t)$ denote the exchange rate between € and $, that is, the price of 1 $ in Euro. A standard cross currency basis swap is a contract between two parties (A and B). At time $T_0 = 0$ (where the contract is signed), A and B exchange a nominal amount N_f in the sense that A pays $X(0)N_f$€ to B and receives N_f\$ from B (that is, A borrows N_f\$ from B and B borrows $X(0)N_f$€ from A). At payment dates $T_1 < T_2 < \ldots < T_n$, A pays interest in $ to B, determined by the $ yield curve, whereas B pays interest in € to A. Typically, the interest rate paid at time T_i is some LIBOR forward rate determined at time T_{i-1}. At time T_n, the nominal amount N_f is paid back (that is, A returns N_f\$ to B and B returns $X(0)N_f$ € to A). We assume from now on that $T_{i+1} - T_i \equiv \delta$ for all i (allowing for different values here only complicates notations, but will not change the outcome of the upcoming calculations). Moreover, a spread ω is added to (or subtracted from) the interest rate (typically for the non-$ payments). So a cross currency swap can be divided into two legs: One leg that contains all payments made in € (since € is the domestic currency by our convention, we call this the *domestic leg*), and one leg that contains all payments made in $ (the *foreign leg*). Let us denote the $ LIBOR forward rate from time T_{i-1} until time T_i at time T_{i-1} (where it is determined) by $F_f(T_{i-1}, T_i)$ and the respective € LIBOR forward rate by $F_d(T_{i-1}, T_i)$. If we need the LIBOR forward rate from T_{i-1} to T_i at some time $t < T_{i-1}$, then we write $F_f(t, T_{i-1}, T_i)$ (or $F_d(t, T_{i-1}, T_i)$, respectively) (this is a random variable if $0 < t < T_{i-1}$, where time 0 means now). With this notation, the cash flows from A's point of view are: For the domestic leg (cash flows in €),

- $C_d(T_0) = -X(0)N_f$
- $C_d(T_i) = (F_d(T_{i-1}, T_i) + \omega)\delta X(0)N_f (1 \le i \le n)$
- $C_d^T(T_n) = X(0)N_f$

and for the foreign leg (cash flows in $),

- $C_f(T_0) = N_f$
- $C_f(T_i) = -F_f(T_{i-1,T_i})\delta N_f (1 \le i \le n)$
- $C_f^T(T_n) = -N_f$

In a Mark-to-Market cross currency basis swap (MtMCCS), the nominal in the domestic leg is adjusted at each payment date $T_1, \ldots, T_{n-1}$ by the current FX rate, and the interest rate to be paid is computed using the adjusted nominal. So we can think of a MtMCCS as a collection of n standard cross currency basis swaps, where the terminal date of the ith swap is the start date of the $(i+1)$st. Since the nominal amount in foreign currency does not change, the foreign leg of a MtMCCS has the same cash flows as the foreign leg of a standard cross currency basis swap, but the cash flows of the domestic leg are now as follows: For the domestic leg (MtMCCS, cash flows in €):

- $C_d(T_0) = -X(0)N_f$
- $C_d(T_i) = (F_d(T_{i-1}, T_i) + \omega)\delta X(T_{i-1})N_f + (X(T_{i-1}) - X(T_i))N_f (1 \leq i \leq n)$
- $C_d^T(T_n) = X(T_n)N_f$

Note that $C_d^T(T_n)$ is not a physical cash flow, it only compensates for the $-X(T_n)N_f$ in $C_d(T_{n-1})$ which is not a physical cash flow either (the nominal is just paid back at T_n). We have decided to define the cash flows this way because we did not want to have different long formulas for the cash flows at times $T_i (i < n)$ and T_n. Note further that we do not consider DRM corrections here, that is, the end date of the LIBOR period is also the coupon payment date. The following sections deal with finding the present value of a MtMCCS (this gives a formula for the standard version as well), first in a model-independent way and then using specific interest rate models.

4 The present value of a MtMCCS

Let an FX model (2.1) be given and consider a MtMCCS. Let T_n (the terminal date of the swap) be the finite time horizon of the model. We recall some results concerning the valuation of cash flows: If C is a (random) cash flow at some future time T and we want to find its value at time $t < T$, it can be computed in the following way: Given a numeraire N (that is, a positive value process of some self-financing portfolio) and the pricing (risk-neutral) measure with respect to N (call it Q_N), then the value of C at time t can be written as

$$v(C, t) = N(t)E^{Q_N}\left(\frac{C}{N(T)}|\mathcal{F}_t\right).$$

In most of our examples, the numeraire will be the domestic money account $B_d(t) = \exp(\int_0^t r_d(s)ds)$, the foreign money account $B_f(t) = \exp(\int_0^t r_f(s)ds)$ or the domestic (resp. foreign) bond maturing at time T, $p_d(t, T) = E^{Q_d}(\exp(-\int_0^T r_d(s)ds)|\mathcal{F}_t)$ (resp. $p_f(t, T) = E^{Q_f}(\exp(-\int_0^T r_f(s)ds)|\mathcal{F}_t)$). As before, we abbreviate Q_{B_d} by Q_d and Q_{B_f} by Q_f, and we write Q_d^T for $Q_{p_d(\cdot,T)}$ and Q_f^T for $Q_{p_f(\cdot,T)}$. Note that the unit of domestic numeraires is €, and the unit of foreign numeraires is \$, so B_f is not a valid numeraire to value a € cash flow with, but $B_f X$ is. We further note that by definition of the LIBOR forward rate, we have

$$F_d(T_{i-1}, T_i) = \frac{1}{\delta}\left(\frac{1}{p_d(T_{i-1}, T_i)} - 1\right) \tag{4.1}$$

and likewise for the foreign rates (at least before the financial crisis, but we do not consider multi curve valuation here which is far more complicated). After these preparations, we can now find the value of a MtMCCS at some time $0 \leq t \leq T_n$ (suppose that the cash flows at time t have not yet been paid). Let i_0 be the smallest index with $T_{i_0} \geq t$. Then we have to find the value at time t of each of the cash flows which happen at times $T_{i_0}, \ldots, T_n$ at time t. We start with the foreign leg. If $i_0 = 0$ (i.e. $t = T_0 = 0$), then trivially $v(C_f(T_0), T_0) = N_f = N_f p_f(T_0, T_0)$. For $i_0 < k \leq n$, we have by the martingale property of $(\frac{p_f(s,T_{k-1})}{p_f(s,T_k)})_{0\leq s\leq T_{k-1}}$ with respect to the measure $Q_f^{T_k}$, (4.1) and $p(s, s) = 1$ for all s that

$$\begin{aligned}
v(C_f(T_k), t) &= v\left(-N_f\left(\frac{1}{p_f(T_{k-1}, T_k)} - 1\right), t\right) \\
&= -N_f(p_f(t, T_k)E^{Q_f^{T_k}}\left(\frac{\frac{p_f(T_{k-1},T_{k-1})}{p_f(T_{k-1},T_k)}}{p_f(T_k, T_k)}|\mathcal{F}_t\right) - p_f(t, T_k)E^{Q_f^{T_k}}\left(\frac{1}{p_f(T_k, T_k)}|\mathcal{F}_t\right) \\
&= -N_f(p_f(t, T_k)\frac{p_f(t, T_{k-1})}{p_f(t, T_k)} - p_f(t, T_k)) = -N_f(p_f(t, T_{k-1}) - p_f(t, T_k)).
\end{aligned} \tag{4.2}$$

If $i_0 > 0$ and $k = i_0$, the cash flow $C_f(T_k)$ is $\mathcal{F}_t$-measurable (it is already determined at time $T_{k-1} < t$ and hence its value is

$$v(C_f(T_k), t) = -N_f\left(\frac{1}{p_f(T_{k-1}, T_k)} - 1\right)v(1, t) = -N_f\left(\frac{p(t, T_k)}{p(T_{k-1}, T_k)} - p(t, T_k)\right).$$

Finally, since $-N_f$ is a constant, hence $\mathcal{F}_t$-measurable, we obtain $v(C_f^T(T_n), t) = -N_f p(t, T_n)$. Summing over all these values, we deduce that the value of the foreign leg (in the foreign currency \$) at time t is simply $-N_f \frac{p_f(t,T_{i_0})}{p_f(T_{i_0-1},T_{i_0})}$ if $i_0 > 0$ and it is 0 if $i_0 = 0$. For the standard cross currency swap, we can immediately conclude that the domestic leg has value $X(0)N_f \frac{p_d(t,T_{i_0})}{p_d(T_{i_0-1},T_{i_0})}$ (in €). Note that we have assumed $\omega = 0$ here for simplicity. Hence the total value (in €) of a standard, zero-spread basis cross currency swap is

$$V(t) = N_f \left(X(0) \frac{p_d(t, T_{i_0})}{p_d(T_{i_0-1}, T_{i_0})} - X_t \frac{p_f(t, T_{i_0})}{p_f(T_{i_0-1}, T_{i_0})} \right).$$

Because X_0 is a constant, valuing the cash flows with an extra spread does not complicate things - as we will see, this is not the case for the MtMCCS, where the spread cash complicates the valuation substantially. We now turn to the MtMCCS. Here, the foreign leg has the same value as before, but the domestic leg's value changes. Note that for $1 \le k \le n$, we have

$$C_d(T_k) = N_f \left(\frac{1}{p_d(T_{k-1}, T_k)} X(T_{k-1}) - X(T_k) + \omega\delta X(T_{k-1}) \right).$$

Hence we have to value three T_k-cash flows:

- $c_1(T_k) = \frac{X(T_{k-1})}{p_d(T_{k-1},T_k)}$
- $c_2(T_k) = -X(T_k)$
- $c_3(T_k) = \omega\delta X(T_{k-1})$

Define i_0 as before. Then clearly $v(C_d, T_0) = -X(0)N_f p_d(T_0, T_0)$ if $i_0 = 0$ (i.e. $t = T_0 = 0$). If $i_0 > 0$ and $k = i_0$, then $c_1(T_k)$ and $c_3(T_k)$ are $\mathcal{F}_t$-measurable, hence

$$v(c_1(T_k), t) = \frac{X(T_{k-1})}{p_d(T_{k-1}, T_k)} p_d(t, T_k)$$

and

$$v(c_3(T_k), t) = \omega\delta X(T_{k-1}) p_d(t, T_k). \tag{4.3}$$

Under the given dynamics for the FX rate X, we have (with $Z = \frac{dQ_f}{dQ_d} = \mathcal{E}(\int_0^{T_n} \sigma_X(s) dW_X^{Q_d}(s))$ as before) that

$$X(T_k) = X(t) \exp\left(\int_t^{T_k} r_d(s) - r_f(s) ds \right) \exp\left(\int_t^{T_k} \sigma_X(s) dW_X^{Q_d}(s) - \int_t^{T_k} \sigma_X(s)^2 ds \right).$$

Hence we obtain by the tower law for conditional expectations that

$$\begin{aligned}
v(c_2(T_k),t) &= -B_d(t)E^{Q_d}\left(\frac{X_{T_k}}{B_d(T_k)}|\mathcal{F}_t\right)\\
&= -X(t)E^{Q_d}\left(\frac{\exp(-\int_t^{T_k} r_f(s)ds)}{\exp(\int_0^t \sigma_X(s)dW_X^{Q_d}(s) - \int_0^t \sigma_X(s)^2 ds)}E^{Q_d}(Z|\mathcal{F}_{T_k})|\mathcal{F}_t\right)\\
&= -X(t)E^{Q_d}\left(E^{Q_d}\left(\frac{\exp(-\int_t^{T_k} r_f(s)ds)}{\exp(\int_0^t \sigma_X(s)dW_X^{Q_d}(s) - \int_0^t \sigma_X(s)^2 ds)}Z|\mathcal{F}_{T_k}\right)|\mathcal{F}_t\right)\\
&= -X(t)E^{Q_d}\left(\frac{\exp(-\int_t^{T_k} r_f(s)ds)}{\exp(\int_0^t \sigma_X(s)dW_X^{Q_d}(s) - \int_0^t \sigma_X(s)^2 ds)}Z|\mathcal{F}_t\right)\\
&= -X(t)E^{Q_f}\left(\frac{\exp(-\int_t^{T_k} r_f(s)ds)}{\exp(\int_0^t \sigma_X(s)dW_X^{Q_d}(s) - \int_0^t \sigma_X(s)^2 ds)}|\mathcal{F}_t\right)E^{Q_d}(Z|\mathcal{F}_t)\\
&= -X(t)E^{Q_f}\left(\exp\left(-\int_t^{T_k} r_f(s)ds\right)|\mathcal{F}_t\right)\frac{E^{Q_d}(Z|\mathcal{F}_t)}{\exp(\int_0^t \sigma_X(s)dW_X^{Q_d}(s) - \int_0^t \sigma_X(s)^2 ds)}\\
&= -X(t)E^{Q_f}\left(\exp\left(-\int_t^{T_k} r_f(s)ds\right)|\mathcal{F}_t\right)\frac{E^{Q_d}(Z|\mathcal{F}_t)}{E^{Q_d}(Z|\mathcal{F}_t)}\\
&= -X(t)E^{Q_f}\left(\exp\left(-\int_t^{T_k} r_f(s)ds\right)|\mathcal{F}_t\right) = -X_t p_f(t,T_k) \qquad (4.4)
\end{aligned}$$

Note that this result also holds for $k > i_0$. It is worth mentioning that we have just proved the following formula for $0 \le u \le s \le T_n$:

$$B_f(u)X(u)E^{Q_f}\left(\frac{1}{B_f(s)}|\mathcal{F}_u\right) = B_d(u)E^{Q_d}\left(\frac{X(s)}{B_d(s)}|\mathcal{F}_u\right) \qquad (4.5)$$

This formula has an easy, but important financial meaning: It tells us that the following methods to find the value at time u (in €) of a cash flow of 1 \$ paid at time s lead to the same result:

- Discount the cash flow with the \$ yield curve (i.e. with the foreign money account) to time s and use the exchange rate at time s to convert the value to € (i.e. calculate $X(0)E^{Q_f}(\frac{1}{B_f(s)})$)
- Convert the cash flow to € and discount it with the € yield curve (i.e. calculate $E^{Q_d}(\frac{X(s)}{B_d(s)})$)

It remains to find the value of $c_1(T_k)$ and $c_3(T_k)$ for $k > i_0$. First we consider $c_1(T_k)$: We have

$$\begin{aligned}
v(c_1(T_k),t) &= B_d(t)E^{Q_d}\left(\frac{X(T_{k-1})}{p_d(T_{k-1},T_k)B_d(T_k)}|\mathcal{F}_t\right)\\
&= B_d(t)E^{Q_d}\left(\frac{X(T_{k-1})}{p_d(T_{k-1},T_k)B_d(T_{k-1})}E^{Q_d}\left(\exp\left(-\int_{T_{k-1}}^{T_k} r_d(s)ds\right)|\mathcal{F}_{T_{k-1}}\right)|\mathcal{F}_t\right)\\
&= B_d(t)E^{Q_d}\left(\frac{X(T_{k-1})}{p_d(T_{k-1},T_k)B_d(T_{k-1})}p_d(T_{k-1},T_k)|\mathcal{F}_t\right) = X(t)p_f(t,T_{k-1}), \qquad (4.6)
\end{aligned}$$

where we use (4.4) in the last step (with k replaced by $k-1$). By the same method, the value of $c_3(T_k)$ is

$$v(c_3(T_k,t)) = \omega\delta B_d(t)E^{Q_d}\left(\frac{X(T_{k-1})p_d(T_{k-1},T_k)}{B_d(T_{k-1})}\right). \qquad (4.7)$$

Finally, we have

$$v(C_d^T(T_n),t) = X(t)p_f(t,T_n).$$

Summing up over all cash flows at times $T_k > t$, we obtain that the value of the domestic leg is

$$N_f X(T_{i_0-1}) \frac{p_d(t, T_{i_0})}{p_d(T_{i_0-1}, T_{i_0})} + N_f \omega \delta B_d(t) \sum_{k=i_0}^{n} E^{Q_d}\left(\frac{X(T_{k-1}) p(T_{k-1}, T_k)}{B_d(T_{k-1})} | \mathcal{F}_t\right).$$

If we assume for the moment that the spread ω is 0, then the total value of the swap at time t is

$$V(t) = N_f \left(X(T_{i_0-1}) \frac{p_d(t, T_{i_0})}{p_d(T_{i_0-1}, T_{i_0})} - X(t) \frac{p_f(t, T_{i_0})}{p_f(T_{i_0-1}, T_{i_0})} \right). \tag{4.8}$$

If we compare this formula to the value of a standard cross currency swap, we see that in the case of a standard swap, the value might deviate a lot from 0 because the difference between $X(0)$ and $X(t)$ may become very large, especially if t is large. In the Mark-to-Market case, we compare $X(t)$ to $X(T_{i_0-1})$ which is most likely relatively small for all values of t. Hence the MtMCCS reduces FX risk significantly.

5 The value of a MtMCCS for specific models

We now turn to the valuation of the MtMCCS for some mathematical models. First we look at zero-spread swaps and show that in this case, the value of the swap is in fact model-independent. The spread cash flows are considered afterwards. We note that at time t, $X(T_{i_0-1})$ and $X(t)$ only depend on the short rates up to time t, which have been determined by the market (there is no expected value of future short rates involved when calculating these quantities). The bond prices, however, are determined by taking expectation of future values of the short rates, so the model-dependent part comes from the bond prices.

5.1 Zero-spread swaps

5.1.1 The deterministic case

Non-surprisingly, the valuation of a MtMCCS is straightforward in the case where the domestic and foreign short rates are assumed to be deterministic, that is, $\sigma_d(t) = \sigma_f(t) = 0$ for all $t \leq T_n$. In this case, we can write

$$\begin{aligned} \frac{p_d(T_{i_0-1}, T_{i_0})}{p_d(t, T_{i_0})} &= \frac{E^{Q_d}(\exp(-\int_{T_{i_0-1}}^{T_{i_0}} r_d(s)ds) | \mathcal{F}_{T_{i_0-1}})}{E^{Q_d}(\exp(-\int_t^{T_{i_0}} r_d(s)ds) | \mathcal{F}_t)} = \frac{\exp(-\int_{T_{i_0-1}}^{T_{i_0}} r_d(s)ds)}{\exp(-\int_t^{T_{i_0}} r_d(s)ds)} \\ &= \exp\left(- \int_{T_{i_0-1}}^{t} r_d(s)ds \right) = p(T_{i_0-1}, t), \end{aligned}$$

and this bond price does not depend on future rates, but was already determined by the market. The same calculation as above of course also holds for the foreign short rate, hence the value of the swap is

$$V(t) = N_f (X(T_{i_0-1}) p_d(T_{i_0-1}, t) - X(t) p_f(T_{i_0-1}, t)).$$

5.1.2 The Hull-White model

In the Hull-White model, the short rate is assumed to follow an Ornstein-Uhlenbeck process with time-dependent parameters, that is,

$$dr_d(t) = (\theta_d(t) - \kappa_d(t) r_d(t))dt + \sigma_d(t) dW_d^{Q_d}(t)$$

with sufficiently well behaved deterministic functions θ_d, κ_d and σ_d (and analogously for r_f). We can rewrite this as

$$dr_d(t) = r_d(t) dU(t) + dY(t),$$

where $dU(t) = -\kappa_d(t)dt$ and $dY(t) = \theta_d(t)dt + \sigma_d(t)dW_d^{Q_d}(t)$. This is an inhomogeneous linear stochastic differential equation and analogously to the deterministic case, we make the Ansatz $r_d(t) = C(t)\mathcal{E}(U)(t)$. Applying Itô's formula to the right hand side and using that $U(t)$ has finite variation (it is the integral of a deterministic function, hence the stochastic exponential coincides with the usual exponential), we obtain $dr_d(t) = C(t)\mathcal{E}(U)(t)dU(t) + \mathcal{E}(U)(t)dC(t)$. This implies that $dC(t) = \frac{1}{\exp(U(t))}dY(t)$, and substituting for U and Y and integrating yields

$$\begin{aligned} r_d(t) &= r_d(0) + \exp\Big(-\int_0^t \kappa_d(s)ds\Big)C(t) \\ &= r_d(0) + \exp\Big(-\int_0^t \kappa_d(s)ds\Big)\Big(\int_0^t \frac{\theta_d(s)}{\exp(-\int_0^s \kappa_d(u)du)}ds + \int_0^t \frac{\sigma_d(s)}{\exp(-\int_0^s \kappa_d(u)du)}dW_d^{Q_d}(s)\Big). \end{aligned}$$

Plugging in this formula for $p_d(t, T_{i_0}) = E^{Q_d}(\exp(-\int_t^{T_{i_0}} r_d(s)ds)|\mathcal{F}_t)$, we obtain

$$\begin{aligned} &p_d(t, T_{i_0}) = \\ &\exp\Big(-\int_t^{T_{i_0}} r_d(t) - \int_t^s \theta_d(u)\exp\Big(-\int_u^s \kappa_d(v)dv)duds\Big)\Big)\cdot \\ &\quad E^{Q_d}\Big(\exp\Big(-\int_t^{T_{i_0}}\int_t^s \sigma_d(u)\exp\Big(-\int_u^s \kappa_d(v)dv\Big)dW_d^{Q_d}(u)ds\Big)\Big), \end{aligned}$$

where we used the fact that $\int_t^s f(u)dW_d^{Q_d}(u)$ is independent of $\mathcal{F}_t$ for a nice deterministic function f. Applying the stochastic Fubini theorem and using that $\int_a^b f(x)dW(x) \sim N(0, \int_a^b f(x)^2dx)$ for a square-integrable deterministic function f, we see that the expected value of the stochastic integral equals

$$\begin{aligned} &E^{Q_d}\Big(\exp\Big(-\int_t^{T_{i_0}} \sigma_d(u)\int_u^{T_{i_0}} \exp\Big(-\int_u^s \kappa_d(v)dv\Big)dsdW_d^{Q_d}(u)\Big)\Big) \\ &= \exp\Big(\frac{1}{2}\int_t^{T_{i_0}} \sigma_d(u)^2\Big(\int_u^{T_{i_0}} \exp\Big(-\int_u^s \kappa_d(v)dv\Big)ds\Big)^2 du\Big), \end{aligned}$$

and by Fubini's theorem, the deterministic integral is equal to

$$\exp\Big(-\int_t^{T_{i_0}} r_d(t) - \int_u^{T_{i_0}} \theta_d(u)\exp\Big(-\int_u^s \kappa_d(v)dv\Big)dsdu\Big).$$

So we can again deduce that $\frac{p(T_{i_0-1}, T_{i_0})}{p(t, T_{i_0})} = p(T_{i_0-1}, t)$ and as in the deterministic case, the value of the zero-spread swap is

$$V(t) = N_f(X(T_{i_0-1})p_d(T_{i_0-1}, t) - X(t)p_f(T_{i_0-1}, t)).$$

5.1.3 The BGM model

In the BGM model, the LIBOR forward rates $F_d(T_{i-1}, T_i)$ are modeled log-normally under the forward measure $Q_d^{T_i}$. So the dynamics are given by

$$dF_d(t, T_{i-1}, T_i) = F_d(t, T_{i-1}, T_i)\sigma_i(t)dW_d^{T_i}$$

(strictly speaking, by more or less the same reasoning as in section 2, all the forward rates have to be defined under a common measure and then the Girsanov theorem has to be applied to prove that a model like this exists, but since we have gone through all the details for the short rate model, we omit this step here). Although the BGM model only determines Bond prices at times T_i (which is intuitively clear - one should think of a model with just one forward rate $F(T_0, T_1)$ and try to determine the bond price at some time t between T_0 and T_1 - since the LIBOR forward rate is just the average interest rate between T_0 and T_1, we have no information about how much of this interest has accrued until time t), there exists a (non-unique) dynamics for the short rate which is compatible with the LIBOR rates. Taking one of these short rates, our calculations from before remain valid in the BGM model. but the value of the swap is only uniquely defined by the model at times T_i. Hence we must have $t = T_{i_0}$ and the value of the zero-spread swap is also given by

$$V(t) = N_f(X(T_{i_0-1})p_d(T_{i_0-1}, t) - X(t)p_f(T_{i_0-1}, t)).$$

5.2 Swaps with spread

As already mentioned before, unlike in the case of standard swaps, spread cash flows complicate things significantly in the Mark- to-Market case. Now we consider the case where the spread is non-zero. Recall that the spread cash flow at time T_k $(k \geq i_0)$ has value $\omega\delta B_d(t)E^{Q_d}(\frac{X(T_{k-1})p(T_{k-1},T_k)}{B_d(T_{k-1})}|\mathcal{F}_t)$. We focus on the case where $k = n$ and assume that $i_0 \neq n$ (since otherwise the value would be easy to compute, see (4.3)). For $i_0 < k < n$, the value can then be calculated analogously. Call the cash flow C in each of the cases.

5.2.1 The deterministic case

This case is again the easiest one. By (4.7) and predetermined bond prices because of deterministic interest rates, we have

$$\begin{aligned} v(C,t) &= p_d(T_{n-1}, T_n)B_d(t)E^{Q_d}\left(\frac{X(T_{n-1})}{B_d(T_{n-1})}|\mathcal{F}_t\right) \\ &= p_d(T_{n-1}, T_n)B_f(t)X(t)E^{Q_f}\left(\frac{1}{B_f(T_{n-1})}|\mathcal{F}_t\right) = p_d(T_{n-1}, T_n)X(t)p_f(t, T_{n-1}), \end{aligned}$$

where we have used (4.5) as well.

5.2.2 The Hull-White model

We now calculate the value of the spread cash flow in the Hull-White model. For abbreviation, we define $\tilde{\kappa}_d(u,s) := \exp(-\int_u^s \kappa_d(v)dv)$ and $\tilde{\kappa}_f$ likewise. Further, note that it follows from Lévy's characterisation of Brownian motion that $\frac{1}{\sqrt{1-\rho_{d,f}^2}}(W_d^{Q_d} - \rho_{d,f}W_f^{Q_d})$ is a Q_d-Brownian motion, call it $\tilde{W}$. Using the transformation rules between Q_d and Q_f, we conclude that

$$dW_d^{Q_f}(u) + \rho_{X,d}\sigma_X(u)du = \rho_{d,f}(dW_f^{Q_f} + \rho_{X,f}\sigma_X(u)du) + \sqrt{1-\rho_{d,f}^2}d\tilde{W}$$

and hence

$$dW_d^{Q_f}(u) = \sigma_X(u)(\rho_{d,f}\rho_{X,f} - \rho_{X,d})du + \rho_{d,f}dW_f^{Q_f} + \sqrt{1-\rho_{d,f}^2}d\tilde{W}. \tag{5.1}$$

Further, we have already mentioned in section 2 that

$$dW_d^{Q_d}(u) = \sigma_X(u)\rho_{X,d}du + dW_d^{Q_f}(u). \tag{5.2}$$

Equations (5.1) and (5.2) yield

$$dW_d^{Q_d}(u) = \sigma_X(u)\rho_{d,f}\rho_{X,f}du + \rho_{d,f}dW_f^{Q_f} + \sqrt{1-\rho_{d,f}^2}d\tilde{W}. \tag{5.3}$$

Analogously to the proof in section 2 that $W_X^{Q_d^\perp}$ is a Q_f-Brownian motion, one can show that $\tilde{W}$ is a Q_f-Brownian motion which is independent of $W_f^{Q_f}$ (since it is independent of $W_f^{Q_d}$ - just use the defining formula to compute the covariance - which differs from $W_f^{Q_f}$ only by a deterministic function). After these preparations, we now calculate the value of the spread cash flow: Starting with (4.7) again, we obtain

$$B_d(t)E^{Q_d}\Big(\frac{X(T_{n-1})}{B_d(T_n)}|\mathcal{F}_t\Big) =$$

$$X(t)E^{Q_d}\Big(\frac{\exp(\int_t^{T_{n-1}} r_d(s)ds - \int_t^{T_{n-1}} r_f(s)ds)}{\exp(\int_t^{T_{n-1}} r_d(s)ds)} E^{Q_d}\Big(\exp\Big(-\int_{T_{n-1}}^{T_n} r_d(s)ds\Big)|\mathcal{F}_{T_{n-1}}\Big)\cdot$$

$$\exp\Big(-\int_t^{T_{n-1}} \sigma_X(s)dW_X(s) - \frac{1}{2}\int_t^{T_{n-1}} \sigma_X(s)^2 ds\Big)|\mathcal{F}_t\Big) =$$

$$X(t)E^{Q_f}\Big(\exp\Big(-\int_t^{T_{n-1}} r_f(s)ds\Big)E^{Q_d}\Big(\exp\Big(-\int_{T_{n-1}}^{T_n} r_d(s)ds)|\mathcal{F}_{T_{n-1}}\Big)|\mathcal{F}_t\Big)$$

as in the derivation of (4.5). We proceed with the inner expected value: We have

$$E^{Q_d}\Big(\exp\Big(-\int_{T_{n-1}}^{T_n} r_d(s)ds\Big)|\mathcal{F}_{T_{n-1}}\Big) =$$

$$E^{Q_d}\Big(\exp\Big(-(T_n - T_{n-1})r_d(T_{n-1}) + \frac{1}{2}\int_{T_{n-1}}^{T_n}\Big(\int_u^{T_n}\sigma_d(u)\tilde{\kappa_d}(u,s)ds\Big)^2 du - \int_{T_{n-1}}^{T_n}\int_u^{T_n}\theta_d(u)\tilde{\kappa_d}(u,s)dsdu\Big)|\mathcal{F}_{T_{n-1}}\Big) =$$

$$\exp\Big(\frac{1}{2}\int_{T_{n-1}}^{T_n}\Big(\int_u^{T_n}\sigma_d(u)\tilde{\kappa_d}(u,s)ds\Big)^2 du - \int_{T_{n-1}}^{T_n}\int_u^{T_n}\theta_d(u)\tilde{\kappa_d}(u,s)dsdu\Big)\exp(-(T_n - T_{n-1})r_d(T_{n-1}).$$

We define

$$f_1(t) := \exp\Big(\frac{1}{2}\int_{T_{n-1}}^{T_n}\Big(\int_u^{T_n}\sigma_d(u)\tilde{\kappa_d}(u,s)ds\Big)^2 du - \int_{T_{n-1}}^{T_n}\int_u^{T_n}\theta_d(u)\tilde{\kappa_d}(u,s)dsdu\Big).$$

Hence

$$v(C,t) = f_1(t)E^{Q_f}\Big(\exp\Big(-(T_n - T_{n-1})r_d(T_{n-1}) - \int_t^{T_{n-1}} r_f(s)ds\Big)|\mathcal{F}_t\Big) =$$

$$f_1(t)E^{Q_f}\Big(\exp\Big(-(T_n - T_{n-1})(r_d(t) + \int_t^{T_{n-1}}\int_u^{T_{n-1}}\theta_d(u)\tilde{\kappa_d}(u,s)dsdu + \int_t^{T_{n-1}}\sigma_d(u)\tilde{\kappa_d}(u,s)dsdW_d^{Q_d}(u)) -$$

$$r_f(t) - \int_t^{T_{n-1}}\int_u^{T_{n-1}}\theta_f(u)\tilde{\kappa_f}(u,s)dsdu - \int_t^{T_{n-1}}\int_u^{T_{n-1}}\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)\Big)|\mathcal{F}_t\Big) =$$

$$f_1(t)\exp(-(T_n - T_{n-1})r_d(t) - r_f(t))\cdot$$

$$\exp\Big(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n - T_{n-1})\theta_d(u)\tilde{\kappa_d}(u,s) + \theta_f(u)\tilde{\kappa_f}(u,s)dsdu\Big)\cdot$$

$$E^{Q_f}\Big(\exp\Big(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n - T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)dsdW_d^{Q_d}(u) - \int_t^{T_{n-1}}\int_u^{T_{n-1}}\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)\Big)\Big).$$

With

$$f_2(t) := f_1(t)\exp(-(T_n - T_{n-1})r_d(t) - r_f(t))\exp\Big(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n - T_{n-1})\theta_d(u)\tilde{\kappa_d}(u,s) + \theta_f(u)\tilde{\kappa_f}(u,s)dsdu\Big)$$

we obtain

$$v(C,t) = f_2(t)E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)dsdW_d^{Q_d}(u)- \int_t^{T_{n-1}}\int_u^{T_{n-1}}\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)\Bigg)\Bigg).$$

Equation (5.3) yields

$$v(C,t) = f_2(t)E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)ds(\sigma_X(u)\rho_{d,f}\rho_{X,f}du+\rho_{d,f}dW_f^{Q_f}(u)+ \sqrt{1-\rho_{d,f}^2}d\tilde{W}(u)) - \int_t^{T_{n-1}}\int_u^{T_{n-1}}\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)\Bigg)\Bigg) =$$

$$f_2(t)\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sigma_X(u)\rho_{d,f}\rho_{X,f}dsdu\Bigg)\cdot E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\rho(d,f)+\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)- \int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sqrt{1-\rho_{d,f}^2}dsd\tilde{W}(u)\Bigg)\Bigg).$$

Setting

$$f_3(t) := f_2(t)\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sigma_X(u)\rho_{d,f}\rho_{X,f}dsdu\Bigg),$$

we deduce

$$v(C,t) = f_3(t)E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\rho(d,f)+\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)- \int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sqrt{1-\rho_{d,f}^2}dsd\tilde{W}(u)\Bigg)\Bigg).$$

Because $\tilde{W}$ is independent of $W_f^{Q_f}$, the expected value can be written as the product

$$E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\rho(d,f)+\sigma_f(u)\tilde{\kappa_f}(u,s)dsdW_f^{Q_f}(u)\Bigg)\Bigg)\cdot E^{Q_f}\Bigg(\exp\Bigg(-\int_t^{T_{n-1}}\int_u^{T_{n-1}}(T_n-T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sqrt{1-\rho_{d,f}^2}dsd\tilde{W}(u)\Bigg)\Bigg),$$

and since both $W_f^{Q_f}$ and $\tilde{W}$ are Q_f-Brownian motions, this product equals

$$f_4(t) := \exp\Bigg(- \int_t^{T_{n-1}} \Bigg(\int_u^{T_{n-1}} (T_n - T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\rho(d,f) + \sigma_f(u)\tilde{\kappa_f}(u,s)ds \Bigg)^2 du \Bigg) \cdot$$
$$\exp\Bigg(- \int_t^{T_{n-1}} \Bigg(\int_u^{T_{n-1}} (T_n - T_{n-1})\sigma_d(u)\tilde{\kappa_d}(u,s)\sqrt{1-\rho_{d,f}^2} ds \Bigg)^2 du \Bigg).$$

So the value of the cash flow is $V(C,t) = f_3(t)f_4(t)$ - a very long, but explicit formula written in the parameters of the model.

5.2.3 The BGM model

The BGM model differs from the other two models because the term structure is not affine and hence bond prices, the money account and the FX rate are not log-normally distributed. Hence, deriving an analytic formula for $E^{Q_d}(\frac{X(T_{N-1}}{B_d(T_N)}|\mathcal{F}_{T_k})$ seems to be at least very difficult, if not impossible at all. We think that one has to rely on numerical (or asymptotic) methods to find the expected value of the spread cash flow. A further rigorous mathematical analysis of this kind of problem would go beyond the scope of this work, but it would probably make up (part of) a master's thesis. The same is true if one considers stochastic volatility.

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