Jumping with Default: Wrong-Way-Risk Modeling for Credit Valuation Adjustment

Minqiang Li and Fabio Mercurio*

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Abstract

We investigate credit value adjustments (CVAs) in the presence of wrong-way risk (WWR) by introducing jumps at default to model correlation between counterparty's default and relevant risk factors. We focus on the foreign-exchange and interest-rate cases, presenting efficient CVA approximations based on CVA computation under independence assumption. Numerical examples of the CVAs of a cross-currency swap and a vanilla interest-rate swap are showcased.

1 Introduction

Credit value adjustment is widely recognized as one of the most important credit risk measures by both industry practitioners and regulators. Traditional CVA calculations were, and to a great extent still are, based on the assumption of independence between the default event and the market risk factors driving the underlying derivatives portfolio. However, since the subprime mortgage crisis and the subsequent default events, there has been a consensus that wrong-way risk should also be taken into account. WWR is defined by ISDA (2001) as the event that: "occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty." The ISDA working group distinguish between:[1]

- *Specific* WWR, which arises through poorly structured transactions, for example those collateralized by own or related party shares; and
- *General or conjectural* WWR, where the credit quality of the counterparty may for non-specific reasons be held to be correlated with a macroeconomic factor which also affects the value of derivatives transactions.

*Derivatives Research, Bloomberg LP. E-mails: minqiang.li@gmail.com, fmercurio@bloomberg.net. The contents of this article represent the authors' views only, and do not represent the opinions of any firm or institution.

[1] A similar view is held by BIS (2006).

General WWR is by far the most important and common type of WWR affecting derivatives portfolios. Typical examples include cross-currency swaps with emerging-markets counterparties or commodity derivatives traded with commodity producers. See also Inamura et al (2012).

In this article, we model WWR by introducing jumps at default. This approach has many advantages when compared to other ones. In particular, it is simple, intuitive and relatively easy to implement, yet realistic and robust in terms of generating WWR. It is also very general, in that single jumps at default can be added to the dynamics of any risk factor of any asset class and applied to calculate CVA on portfolios consisting of general deals. To this end, we propose two general set-ups based, respectively, on additive and proportional jumps. We then focus on the two main asset classes in regards to CVA, that is foreign-exchange (FX) and interest rates, and assume that the relevant risk factors jump deterministically at the counterparty's default. While jump-at-default models have already been proposed and used in practice to quantity WWR for FX deals in emerging markets, see for example Ehlers and Schönbucher (2006) and Li, A. (2013), the results we derive here are, to our knowledge, new and original. We believe the same is true for our modeling of jumps at default in interest-rate dynamics for WWR purposes.

We also focus on unilateral CVA for portfolios of European-style deals. This allows us to derive approximation results showing that WWR CVA can be expressed as an independent CVA with adjusted input parameters. We call independent CVA the CVA obtained under the assumption of independence between credit and other risk factors such as FX or interest rates. We call WWR CVA the CVA obtained using a WWR model where there is correlation between credit risk and other risk factors. For FX deals, we provide two approximations based on adjusting the initial exchange rate. For interest rate deals, the approximations are based on adjusting the input forward LIBOR curve, or equivalently the initial LIBOR-OIS basis spreads. For plain-vanilla interest rate swaps, we also provide a simple approximation by adjusting the strike rate. The accuracy of these approximations is demonstrated using numerical examples based on cross-currency swaps and interest-rate swaps.

The article is structured as follows. In Section 2 we introduce definitions and notation. In Section 3, we cite the main references in the financial literature with regards to WWR modeling. We then stress the advantages of jump-at-default models, also using a specific CVA example. Section 4 introduces a general jump-at-default approach, which could be based either on additive or on proportional jumps. Section 5 deals with WWR for portfolios of European-style FX derivatives, and derives approximations for WWR CVA in terms of corresponding independent CVA values. Numerical examples based on a cross-currency swap are showcased. Section 6 is devoted to modeling jumps at default for interest-rate deals. Also in this case, we focus on portfolios of European-style derivatives, and derive approximations based on independent CVA values. Numerical examples based on a vanilla interest-rate swap are then showcased. Section 7 concludes the paper. The proofs of the main propositions are presented in the appendix.

2 Definitions and notation

It is market practice to define the unilateral CVA of a derivatives portfolio with time horizon T, under the assumption of no collateral agreement, by the following general formula:

$$\mathbf{CVA} := (1-R)\,\mathbb{E}\big[D(0,\tau)V_\tau^+ 1_{\{\tau\le T\}}\big] \tag{1}$$

where R is the assumed-constant recovery rate, $D(0,t)$ is the (domestic) discount factor for maturity t, V_t is the portfolio's mark-to-market value at time t, τ is the counterparty's default time, and $\mathbb{E}$ denotes expectation under the (domestic) risk-neutral measure Q.

We let

$$P_\tau(t) := \mathbb{E}\big[1_{\{\tau\le t\}}\big]$$

denote the cumulative default probability before time t under measure Q, and let p_τ denote the density function of the default time τ, that is

$$p_\tau(t) := \frac{\mathrm{d}}{\mathrm{d}t}P_\tau(t) \tag{2}$$

When the discounted positive exposures $D(0,t)V_t^+$ are correlated with default, the CVA in Equation (1) is a WWR CVA or right-way risk (RWR) CVA, and can be calculated as:

$$\mathbf{CVA} = \mathbf{CVA}_{\text{WWR}} := (1-R)\int_0^T \mathbb{E}[D(0,t)V_t^+|\tau=t]\,p_\tau(t)\,\mathrm{d}t \tag{3}$$

We remark that here and henceforth we use $\mathbf{CVA}_{\text{WWR}}$ to mean both WWR and RWR CVAs.

Instead, when the discounted positive exposure $D(0,t)V_t^+$ for each $t\le T$ is independent of τ, the unilateral CVA formula reduces to:

$$\mathbf{CVA} = \mathbf{CVA}_{\text{IND}} := (1-R)\int_0^T \mathbb{E}[D(0,t)V_t^+]\,p_\tau(t)\,\mathrm{d}t \tag{4}$$

In a jump-at-default model, discounted positive exposures depend on risk factor dynamics, which in turn depend, by construction, on the default time. CVA is then calculated using (3). To get a corresponding independent CVA, for comparison purposes, one can use formula (4) provided one sets to zero all occurrences of the jump size. However, the question remains whether a correct calculation of independent and WWR CVAs entails the recalibration of the model parameters. In this article, we will use the convention that independent CVA is obtained by setting to zero the jump size in (4), keeping all other model parameters unchanged. This will typically ensure that a positive correction to the independent CVA is produced when modeling WWR, as required by regulators.

3 Modeling WWR

Many different approaches have been proposed in the financial literature and the industry to assess WWR. They include: i) Modeling of a stochastic intensity of default and its correlation with risk factors, like in Brigo and Pallavicini (2007), Elhajjaji and Subbotin (2013), Hudson (2013), Carr and Ghamami (2015); ii) Using copulas to model dependence between default time and exposures at different observation times, like in Pykhtin and Rosen (2010), Boukhobza and Maetz (2012), Pykhtin (2012), Cherubini (2013), and Böcker and Brunnbauer (2014); iii) Introducing jumps at default to model gap risk or default risk in emerging currencies or other risk factors, like in Pykhtin and Sokol (2012), Li, E. (2013) or Crépey and Song (2015); iv) Using a structural approach with correlations between asset value and market factors, like in Winters (1999), Redon (2006), Buckley, Wilkens, Chorniy (2011); v) Adjusting default probabilities in the independence-based CVA formula, like in Hull and White (2012); vi) Using alternative methods, such as scenario weighting, like in Finger (2000), Turlakov (2012), Glasserman and Yang (2013), or modeling simultaneous defaults, like in Assefa, Bielecki, Crépey, Jeanblanc (2009).

In this article, we model WWR using a jump-at-default approach. Besides the specific WWR references cited in the previous paragraph, we mention that jump-at-default modeling has also been introduced in the financial literature to price credit-contingent derivatives in those cases where counterparty's default impacts the underlying risk factors, see for instance Ehlers and Schönbucher (2006) or Li, A. (2007, 2013).

Compared to other methods for WWR, modeling jumps at default presents several attractive features. Indeed, this approach is:

a. Simple
The value of a risk factor Z at default τ is usually defined by adding a jump to the pre-default value $Z_{\tau-}$, that is $Z_\tau = Z_{\tau-} + J$, where J is the jump size, which could be either positive or negative depending on the application. The jump could also be multiplicative so that $Z_\tau = Z_{\tau-}(1 + J)$, where J is now the proportional jump size.

b. Intuitive
Contrary to other correlation parameters, for instance the parameters in copulas or the instantaneous correlations between the unobservable hazard rates and other risk factors, we know what this "correlation" parameter J means, and what to expect from it.

c. Realistic
There is a clear historical evidence that defaults of large banks or countries cause abrupt changes in exchange rates and interest rates, see for instance Figures 2 and 3 presented in later sections.

d. Easy to calibrate
A tractable model for risk factor X remains tractable after adding jumps at default. Option prices can easily be calculated by considering the two cases of default happening before or

after the option's maturity.

e. Easy to retrofit
A Monte Carlo framework can easily be extended, at least in the unilateral CVA case, by adding jumps at default. One has only to correct the drift of jumping factors by corresponding no-arbitrage terms, and calculate mark-to-market values at default based on no-jump factor dynamics. In practice, one may face performance issues because of the need of simulating simultaneously both jumping and no-jump dynamics, which could be addressed by parallel computing or by using approximations.

f. Quick to correlate
As opposed to stochastic hazard rates, for instance, one-time jumps can produce high correlations (in absolute value) even for short maturities. They can be crucial to produce sizable corrections to independent CVA.

g. Robust
WWR results are, typically, as expected: CVA is higher than in the independent case. This is not always the case for other methods such as, for instance, copulas. A WWR CVA that is higher than independent CVA goes in the right direction of assigning positive value to an incremental risk.

The last two features are extremely important and differentiate jump-at-default models from other approaches. They can be explained using the example of a foreign zero-coupon bond sold by a bank to a counterparty based in the same domestic currency, where the risk-neutral dynamics of the exchange rate X (measured in units of domestic currency per unit of foreign currency) is given by:

$$\mathrm{d}X_t = X_{t-}\big[\sigma_X \,\mathrm{d}W_t^X + J 1_{\{t\leq\tau\}}\,\mathrm{d}M_t\big] \tag{5}$$

The default time τ is the first jump time of the compensated Cox process M with stochastic intensity of default λ_t given by:

$$\mathrm{d}\lambda_t = \sigma_\lambda \lambda_t \,\mathrm{d}W_t^\lambda \tag{6}$$

where σ_X, σ_λ and J are constant, and the Brownian motions W^X and W^λ have constant instantaneous correlation equal to $\rho_{X,\lambda}$. For simplicity, we assume zero interest rates for both currencies.

We can show that the CVAs of the foreign zero-coupon bond satisfy, to first order jointly in σ_λ and λ_0,

$$\frac{\mathbf{CVA}_{\text{WWR}}}{\mathbf{CVA}_{\text{IND}}} \approx (1+J)\big[1 + \tfrac{1}{2}\rho_{X,\lambda}\sigma_X\sigma_\lambda T - \tfrac{1}{2}J\lambda_0 T\big] \tag{7}$$

where the error term depends only mildly on J, and where $\mathbf{CVA}_{\text{IND}}$ is calculated by also setting $\sigma_\lambda = J = 0$ so that there is no stochastic intensity and no jump. Equation (7) shows that the contribution of the jump is roughly equal to $1 + J$ for different bond maturities T, and provides evidence of the "quick to correlate" property as well as robustness.

Note that the contribution of stochastic intensity is qualitatively and quantitatively different. It is zero for $T = 0$, and is an increasing function of T, for small T. For larger maturities, however, the behavior may change, resulting in a loss of robustness.

In Figure 1, we examine the effect of jump at default and stochastic intensity separately. In the left subplot of Figure 1 , we assume $\sigma_\lambda = 0$ and plot the $\mathbf{CVA}_{\text{WWR}}/\mathbf{CVA}_{\text{IND}}$ ratio for different values of J, using the exact formula (59), as derived in Appendix A. The plot shows that this ratio, in the case of deterministic default intensity, is monotonically increasing in the jump size J. It is exactly 1 at $J = 0$, and appears to be very close to $1 + J$ for general J.

In the right subplot, we assume $J = 0$ and plot the same ratio for different values of σ_λ, which is obtained using Monte Carlo. The plot shows that for some values of σ_λ, the corresponding WWR CVA is higher than the independent CVA, while for some other values, the former is lower than the latter. Therefore, unlike a positive jump parameter J, which leads to WWR for typical values of the other parameters, a positive correlation $\rho_{X,\lambda}$ does not necessarily lead to wrong-way or right-way risk.

We finally notice that, while the effect of $\rho_{X,\lambda}$ is very small for small maturity T or small default intensity λ, the effect of J is present for all maturities, and for all non-zero default intensities. A pure instantaneous-correlation model without jump can severely underestimate WWR CVA, especially for small time horizons and small default intensities.[2]

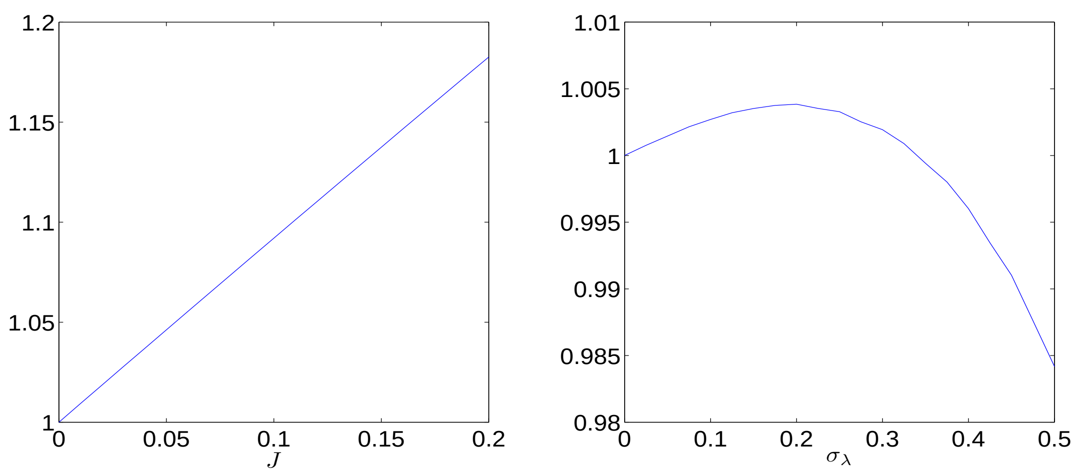

Figure 1: The ratio of WWR CVA to independent CVA as a function of J (left subplot) and σ_λ (right subplot). The left subplot assumes $\sigma_\lambda = 0$ and the right subplot assumes $J = 0$. Other parameters used are: $T = 5$, $\rho = 0.2$, $\sigma_X = 0.1$, and $\lambda_0 = 0.03$.

[2]We stress that parameter λ_0 is kept unchanged when changing σ_λ. A different analysis is based on re-calibrating λ_0 every time we change σ_λ, by matching for instance a given survival probability.

4 Modeling jumps at default

Another key advantage of jump-at-default models is generality. A pure jump process can be "attached" to any baseline model for any risk factor of any asset class to model the abrupt change in value caused by counterparty's default. This is a very attractive feature in practice. One does not need to incur major changes in the pricing engine infrastructure to accommodate this extension of factor dynamics.

Let us consider a risk factor Y of a given asset class and an associated baseline model.[3] Let us assume that the risk factor evolves, in the baseline model, according to an arbitrage-free continuous-time stochastic process $Y_t^{\mathbf{B}}$ independent of the default time τ. Let us then consider a time-inhomogeneous Poisson process N_t with constant intensity λ_t, and define the following two pure-jump martingales under a suitable pricing measure associated with factor Y:

- **Additive jump**
$$\mathrm{d}M_t^A = J_t^A 1_{\{t\leq\tau\}}(\mathrm{d}N_t - \lambda_t \mathrm{d}t) \tag{8}$$
with $M_0^A = 0$, and where J_t^A is a possibly time-dependent additive jump size. Integrating (8) gives:
$$M_t^A = -\int_0^{t\wedge\tau} \lambda_u J_u^A \,\mathrm{d}u + J_\tau^A 1_{\{t\geq\tau\}} \tag{9}$$
The notation $t\wedge\tau$ is a shorthand for $\min(t,\tau)$.

- **Proportional jump**
$$\mathrm{d}M_t^P = J_t^P 1_{\{t\leq\tau\}} M_{t^-}^P(\mathrm{d}N_t - \lambda_t \mathrm{d}t) \tag{10}$$
with $M_0^P = 1$, where J_t^P is a possibly time-dependent proportional jump size, with $J_t^P > -1$. Integrating (8) gives:
$$M_t^P = (1 + J_\tau^P 1_{\{t\geq\tau\}})e^{-\int_0^{t\wedge\tau}\lambda_u J_u^P \,\mathrm{d}u} \tag{11}$$
Notice that M_t^P is the stochastic exponential martingale of process M_t^A when $J_t^A \equiv J_t^P$.

Processes M_t^A and M_t^P are assumed to be independent of the baseline model $Y_t^{\mathbf{B}}$, and can be explained as follows. For $J_t^A \neq 0$ (resp $J_t^P \neq 0$), M_t^A (resp. M_t^P) is a one-time jump process where an additive (resp. proportional) jump of magnitude J_τ^A (resp. $1 + J_\tau^P$) occurs at time τ. The term $-\int_0^{t\wedge\tau}\lambda_u J_u^A \,\mathrm{d}u$ (resp. $e^{-\int_0^{t\wedge\tau}\lambda_u J_u^P \,\mathrm{d}u}$) is a compensator ensuring that M_t^A (resp. M_t^P) is a martingale with mean equal to zero (resp. one). The drift compensation is effective up to the jump time τ. After the jump, M_t^A (resp. M_t^P) remains constant on each sample path.

Jump-at-default models can be constructed in either an additive or a proportional (multiplicative) way. An additive jump-at-default model can be obtained by assuming that:

$$Y_t := Y_t^{\mathbf{B}} + M_t^A \tag{12}$$

[3]The risk factor Y could be an interest rate, an exchange rate, a stock price, a commodity index, etc.

Likewise, a proportional jump-at-default model can be constructed by assuming that:

$$Y_t := Y_t^{\mathbf{B}} \cdot M_t^P \tag{13}$$

Proportional jumps at default have been introduced in the literature, see for example Li (2007) for the pricing of credit contingent FX derivatives. We here propose to use them in the context of WWR modeling. Instead, the use of additive jumps for modeling correlation between credit and another risk factor appears to be new.

Equations (12) and (13) can be understood as follows. For zero jump size, the jump-at-default model Y_t coincides with the baseline model $Y_t^{\mathbf{B}}$. When jumps are non-zero, the process X_t is modified by a jump martingale independent of the baseline model. This ensures that the no-arbitrage condition is preserved. The process X_t can only jump once, and the additive or proportional jump can only occur at the counterparty's default time τ.

Depending on the nature of the considered risk factor, it will be more natural to assume either an additive or a proportional formulation. For instance, interest rate models can be extended using additive jumps, whereas proportional jumps make more sense in the cases of equity and FX models. These choices will be made explicit in the following sections, where we introduce jump-at-default models first for FX rates and then for interest rates.

5 Modeling jump-at-default for FX deals

There is plenty of historical evidence of the depreciation of FX rates of a defaulting country against other surviving countries, see for instance Figure 2 below.

Suppose one has a baseline FX model where the exchange rate follows an arbitrage-free continuous-time stochastic process $X_t^{\mathbf{B}}$ independent of the default. Then, by the discussion in the previous section, and Equation (13) in particular, a jump-at-default FX model can be obtained by assuming that

$$X_t := X_t^{\mathbf{B}} M_t^{\mathbf{J}} \tag{14}$$

where $M_t^{\mathbf{J}}$ is the following pure jump process independent of $X_t^{\mathbf{B}}$:

$$M_t^{\mathbf{J}} = \left(1 + J 1_{\{t \geq \tau\}}\right) e^{-\lambda J (t \wedge \tau)} \tag{15}$$

and where $J > -1$ is the assumed constant proportional jump size.[4] The default time τ is the first jump time of a Poisson process N_t with constant jump intensity λ under the domestic risk-neutral measure.

[4]In principle, we could also use an additive jump formulation. However, the proportional jump formulation is more natural, and it guarantees the positivity of the exchange rate process even in the case of negative jumps.

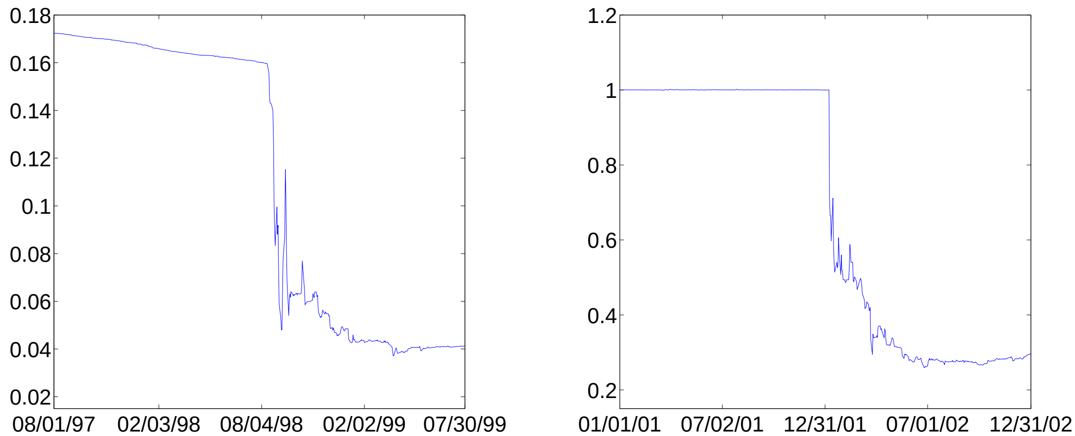

Figure 2: On 17 August 1998, the Russian government devalued the Ruble, defaulting on domestic debt. On December 26, 2001, Argentina defaulted on a total of USD 93 billion of its external debt. These events caused abrupt changes in the exchange rates for Ruble (left subplot) and Peso (right subplot).

For $J = 0$, the jump-at-default model X_t coincides with the baseline model $X_t^{\mathbf{B}}$. When $J \neq 0$, the process X_t is perturbed by the jump martingale $M_t^{\mathbf{J}}$ which is independent of the baseline model. The process X_t, similarly to $M_t^{\mathbf{J}}$, can only jump once. The jump can only occur at the counterparty's default time τ, and the percentage increment of the FX rate at default is constant and given by J. Before default, the process X_t has a drift correction which is opposite in sign to that of the jump at default.

5.1 Jump calibration

If the counterparty is a sovereign entity or is big enough to deeply impact the economy of its own nation, and hence its currency, then one could calibrate J using these historically observed jumps. In general, one could estimate the jump size using information coming from the counterparty's balance sheets and the country's GDP, for instance.

An alternative calibration is based on cross-sectional data, when available. Specifically, let us assume that the market quotes both the CDS and quanto CDS rates for the given credit entity and exchange rate.[5] Let **CDS** and **QCDS** denote the observed CDS and quanto CDS rates for a same given maturity. A rule of thumb known to practitioners, see for instance Elizalde et al.

[5]For discussions on quanto CDS, see for instance, Ehlers and Schönbucher (2006), EL-Mohammadi (2009), and Li, A. (2013).

(2010), relates these two CDS rates to the proportional jump size J as follows:

$$\frac{\mathbf{QCDS}}{\mathbf{CDS}} \approx 1 + J \tag{16}$$

This simple rule allows for a direct cross-sectional calibration of the jump size based on market quotes of CDS and quanto CDS rates. A formal proof of (16) is provided in Appendix A.

5.2 CVA for a portfolio of European-style FX deals

The WWR CVA produced by dynamics (14) can be calculated using the general formula (3). In the case of a portfolio consisting only of European-style FX derivatives, this formula can be made more explicit. At default time $\tau = t$, the portfolio value can be written as $V_t = V(t, X_t)$ for some function V, where V could also depend on other state variables, such as stochastic volatility or rates, which we assume to be independent of default. Conditional on default happening at time t, we have

$$X_t = X_t^{\mathbf{B}}(1 + J)e^{-\lambda J t} \tag{17}$$

We thus get:

$$\mathbf{CVA}_{\text{WWR}} = (1 - R) \int_0^T \mathbb{E}\Big[D(0,t)\, V\big(t, X_t^{\mathbf{B}}(1 + J)e^{-\lambda J t}\big)^+ \Big]\, \lambda e^{-\lambda t}\, \mathrm{d}t \tag{18}$$

The independent CVA is given by the same formula with $J = 0$:

$$\mathbf{CVA}_{\text{IND}} = (1 - R) \int_0^T \mathbb{E}\Big[D(0,t)\, V\big(t, X_t^{\mathbf{B}}\big)^+ \Big]\, \lambda e^{-\lambda t}\, \mathrm{d}t \tag{19}$$

With some abuse of notation, we have used the same function V to denote the portfolio values in Equations (18) and (19). In general, these two V's are different. Since the WWR model is J dependent, the V in (18) is J dependent as well, and coincides with the V in (19) if J is set to be zero.[6] Another reason why we have used the same notation for these two portfolio values is because we henceforth assume that the baseline model $X_t^{\mathbf{B}}$ is scale invariant. Scale invariance means that a scaling of $X_t^{\mathbf{B}}$, at a given time t, induces an analogous constant scaling of all the baseline model values after that time t. The scalability assumption is satisfied, for instance, by the Black-Scholes model (even with stochastic interest rates), the Heston (1993) stochastic-volatility model, and the Merton (1976) jump-diffusion model. Under the scalability assumption, the function V in Equation (18) has no extra J dependence besides that in X_t, and coincides with the function V in Equation (19). This is because, under scale invariance, the ratio of X_t and $X_t^{\mathbf{B}}$ at time of default summarizes the effects of jump for all future times.

In the following, we will assume that this scalability condition is satisfied. This also allows us to make connections between the WWR CVA and independent CVA. To this end, two useful approximation results are derived in the following section.

[6] Another way to understand this is that if we want to use exactly the same function V for both Equations (18) and (19), then V would need to take on a third argument J.

5.3 CVA approximations

We derive two approximations for the WWR CVA in Equation (18), which will use the independent CVA formula (19) with an adjusted initial FX rate X_0.

The first approximation is based on neglecting the drift correction $e^{-\lambda J t}$ in (17), so (18) becomes:

$$\mathbf{CVA}_{\text{WWR}} \approx (1-R)\int_0^T \mathbb{E}\Big[D(0,t)V\big(t, X_t^{\mathbf{B}}(1+J)\big)^+\Big]\,\lambda e^{-\lambda t}\,\mathrm{d}t \tag{20}$$

By comparing (20) with (19) and recalling the scalability assumption, it is clear that WWR CVA becomes equivalent to an independent CVA where the following substitution is made:

$$X_0 \to X_0(1+J) \tag{21}$$

Our first approximation then reads as:

$$\mathbf{CVA}_{\text{WWR}}(X_0) \approx \mathbf{CVA}_{\text{IND}}(X_0(1+J)) \tag{22}$$

where the dependence of CVAs on the initial FX rate is now explicitly highlighted. Therefore, the WWR CVA of a portfolio of FX vanilla deals is approximately equal to the independent CVA of the same portfolio but calculated with an initial FX rate that is adjusted by the same percentage amount as the FX rate at default time. Despite its simplicity, this approximation appears to work decently well, especially for small intensities λ and maturities T, see for instance Table 2 below, and can serve the purpose of providing a quick rule of thumb for WWR CVA.

Approximation (22) can be improved by noticing that the term λJ has the effect of pulling down the FX rate X. By neglecting this term, we are increasing the value of X_t on every state of the world, which results in either an upper or lower bound for the WWR CVA when the portfolio's value V is monotone in X. A better approximation for Equation (18) can then be obtained by replacing the time-dependent term $e^{-\lambda J t}$ in Equation (17) with $e^{-\lambda J \bar{\tau}}$ for some suitable constant $\bar{\tau}$, that is:

$$\mathbf{CVA}_{\text{WWR}} = (1-R)\int_0^T \mathbb{E}\Big[D(0,t)\,V\big(t, X_t^{\mathbf{B}}(1+J)e^{-\lambda J\bar{\tau}}\big)^+\Big]\,\lambda e^{-\lambda t}\,\mathrm{d}t \tag{23}$$

Using the scalability assumption, this leads to the following approximation:

$$\mathbf{CVA}_{\text{WWR}}(X_0) \approx \mathbf{CVA}_{\text{IND}}(\bar{X}_0) \tag{24}$$

where

$$\bar{X}_0 := X_0(1+J)\,e^{-\lambda J\bar{\tau}} \tag{25}$$

We prove in Appendix A that a suitable constant $\bar{\tau}$ is the effective default time defined by

$$\int_0^T \big(e^{-\lambda J t} - e^{-\lambda J\bar{\tau}}\big)\,p_\tau(t)\,\mathrm{d}t = 0 \tag{26}$$

and given explicitly by

$$\bar{\tau} = -\frac{1}{\lambda J} \log \mathbb{E}\left[e^{-\lambda J \tau} | \tau \le T\right] = \frac{1}{\lambda J} \log \frac{(1+J)(1-e^{-\lambda T})}{1-e^{-(1+J)\lambda T}} \tag{27}$$

In the limit of λJ going to 0, the effective default time $\bar{\tau}$ becomes the expectation of τ conditional on $\tau \le T$:

$$\bar{\tau} = \mathbb{E}[\tau | \tau \le T] = \frac{e^{\lambda T} - 1 - \lambda T}{\lambda(e^{\lambda T} - 1)} \tag{28}$$

If furthermore we assume that λ is small, we get that

$$\bar{\tau} \approx \frac{T}{2} - \frac{\lambda T^2}{12} \tag{29}$$

Therefore, $\bar{\tau}$ is roughly half of the CVA horizon. In practice, all three expressions give very similar values for $\bar{\tau}$.[7] For definiteness, we choose to use Equation (28). The corresponding approximation in (24) turns out to be very accurate, as the numerical examples in the next section will show.

Approximation (24), besides being intuitive, has also the advantage of being asymptotically exact for specific European-style derivatives under any FX baseline model in that:

$$\mathbf{CVA}_{\text{WWR}}(X_0) = \mathbf{CVA}_{\text{IND}}\left(X_0(1+J)\, e^{-\lambda J \bar{\tau}}\right) + \mathcal{O}((\lambda J T)^2) \tag{30}$$

In Appendix A, we prove this result under the assumption that the discounted expected positive exposure is well behaved and has no calendar time t dependence other than that on the forward exchange rate conditional on default happening at time t. This assumption is satisfied, for instance, by European call or put options in the Black-Scholes model.

The main advantage of both approximations (22) and (24) is that one can leverage existing functionality built under the assumption of independence between counterparty's default and the other risk factors, and calculate WWR CVA using the already developed analytics for independent CVA with a different input parameter X_0.

Before we move on to numerical analysis, we make two remarks. The first remark expresses WWR CVA as an independent CVA plus an "add-on" term. The second one expresses WWR CVA as independent CVA with an effective default intensity.

Remark 5.1 (WWR CVA as an "add-on")**.** *Approximations (22) and (24) also allow for a representation of WWR CVA as an "add-on" adjustment to the independent CVA:*

$$\mathbf{CVA}_{WWR} = \mathbf{CVA}_{IND} + \text{“add-on”}$$

A first-order Taylor expansion of (22) around $J = 0$, immediately gives:

$$\text{“add-on”} \approx J X_0 \frac{\partial}{\partial X_0} \mathbf{CVA}_{IND}(X_0, \lambda) \tag{31}$$

[7] For example, when $T = 20$, $J = 0.2$, $\lambda = 0.02$, the three expressions for $\bar{\tau}$ give 9.269, 9.335, and 9.333 respectively.

Therefore, the add-on term is roughly proportional to the absolute jump size JX_0, see also the numerical results below, and is proportional to the rate of change of the independent CVA with respect to the initial exchange rate. A similar adjustment is obtained when using approximation (24).

Formula (31) *shows that a quick assessment of WWR CVA can be done at no extra computational cost if the existing engine for independent CVA also calculates sensitivities such as that with respect to the initial exchange rate.*

We also notice that for deals such as a foreign zero-coupon bond or a cross currency swap, independent CVA is roughly linear in X_0, especially if λT is small. In this case, Equation (31) *implies that $\mathbf{CVA}_{WWR}(X_0, \lambda) \approx (1 + J)\mathbf{CVA}_{IND}(X_0, \lambda)$, which agrees with our previous result in Equation* (7) *for constant default intensity and small λT.*

A more accurate add-on term is calculated in Appendix A under the assumption of a well-behaved expected positive exposure.

Remark 5.2 (Effective default intensity)**.** *Another approximation for WWR CVA can be obtained by using independent CVA based on an adjusted default intensity. In fact, it can be shown that*

$$\mathbf{CVA}_{WWR}(X_0, \lambda) \approx \mathbf{CVA}_{IND}(X_0, \bar{\lambda}) \tag{32}$$

where

$$\bar{\lambda} := \lambda \left[1 + J \frac{\partial \ln \mathbf{CVA}_{IND}(X_0, \lambda)}{\partial \ln X_0} \right] \tag{33}$$

Adjusting the default intensity as a way to produce WWR was proposed by Hull and White (2012). While the derivation in their original paper is purely heuristic, approximation (32) can provide a theoretical justification based on a jump-at-default model. However, this theoretical justification is only partial. In fact, Hull and White (2012) suggest adjusting the default intensity as a function of the portfolio value, while our analysis above shows that the percentage adjustment to the default intensity should be proportional to the sensitivity of the independent CVA with respect to the initial exchange rate.

5.4 Numerical analysis

We consider the numerical example of a 10-year float-for-float cross-currency swap, with notional equal to USD 1044 (or EUR 800 with the initial EURUSD exchange rate at 1.305, as observed on March 13, 2013). The holder of the swap receives semi-annual USD coupon payments based on USD LIBOR plus 27.5 basis points, and pays quarterly EUR coupon payments based on EUR LIBOR. At maturity, the holder receives the USD principal and pays the EUR principal. The swap value at inception is zero. For simplicity, both USD and EUR interest rates are assumed to be deterministic with values as of March 13, 2013. This is acceptable since the main contribution to cross-currency swap CVA comes from FX fluctuations.

The baseline model $X_t^{\mathbf{B}}$ is taken to be a geometric Brownian motion with constant parameters. As a result, the dynamics X_t in the jump-at-default model is given by:

$$\mathrm{d}X_t = X_{t-}\left[(r_t^d - r_t^f - \lambda J 1_{\{t\leq\tau\}})\,\mathrm{d}t + \sigma_X\,\mathrm{d}W_t^X + J 1_{\{t\leq\tau\}}\,\mathrm{d}N_t\right]$$

where r_t^d and r_t^f are the deterministic domestic and foreign risk-free rates, σ_X is the FX rate volatility, here set to be 17%, and W^X is a Q-Brownian motion.

The CVA values based on the exact formula (18), and obtained using Monte Carlo, are shown in Table 1 for different jump sizes and default intensities. As the table shows, for fixed default intensity λ, WWR is controlled by the jump parameter J, and sizable corrections to the independent CVA can be produced even with relatively small jump sizes.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
-0.1	10.49	19.58	27.43	34.19	39.98
-0.05	9.10	17.05	23.99	30.02	35.26
0	7.88	14.84	20.99	26.41	31.18
0.05	6.82	12.94	18.42	23.33	27.71
0.1	5.92	11.32	16.23	20.70	24.77

Table 1: Exact CVA from Equation (18)

The CVA values in Tables 2 and 3 show results for the CVA approximations (22) and (24). They are also obtained through Monte Carlo, but this time using an independent CVA engine. Approximation (22) appears to be decently accurate for small intensities, and even though its quality decreases slightly for higher λ's, it can certainly be used for a quick assessment of the WWR impact. Instead, approximation (24) appears to be quite accurate even for large intensities, with the maximum percentage error beging about 0.4%. Since the computational effort is not larger than that of the cruder approximation (22), approximation (24) should always be preferred.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
-0.1	10.63	20.09	28.50	35.96	42.58
-0.05	9.16	17.29	24.49	30.86	36.49
0	7.88	14.84	20.99	26.41	31.18
0.05	6.77	12.73	17.98	22.59	26.62
0.1	5.82	10.93	15.41	19.33	22.75

Table 2: CVA approximation from Equation (22)

We have further tested our approximations by changing the euro notional amount and the basis swap spread, and by also assuming a Hull-White one-factor model for each of the two interest rates. The results we have obtained are qualitatively similar, and as such are not reported in this paper.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
-0.1	10.49	19.55	27.37	34.09	39.84
-0.05	9.09	17.04	23.97	29.99	35.21
0	7.88	14.84	20.99	26.41	31.18
0.05	6.82	12.94	18.42	23.32	27.69
0.1	5.92	11.30	16.20	20.64	24.67

Table 3: CVA approximation from Equation (24)

6 Modeling jump at default for interest-rate deals

WWR for interest-rate deals is typically modeled using either a stochastic intensity of default correlated with interest rates, or a copula function defining dependence between exposures and default. Jump-at-default modeling for interest rates has been proposed by Li, A. (2007) in the context of a contingent CDS whose underlying is an interest-rate swap. It also has been mentioned by Pykhtin and Sokol (2012), but to our knowledge no explicit model has been proposed so far for WWR purposes.

In this section, we introduce jumps at default on interest rate dynamics. As in the previous FX case, we consider a baseline model for interest rates in a given currency, and then add a jump component by assuming that interest rates jump at counterparty's default by a deterministic amount.

That interest rates are impacted by the default of a large financial institution or corporation is also supported by historical evidence, see for instance the USD market data from January 2007 to January 2010, which is reported in Figure 3. But the question is whether all rates are equally

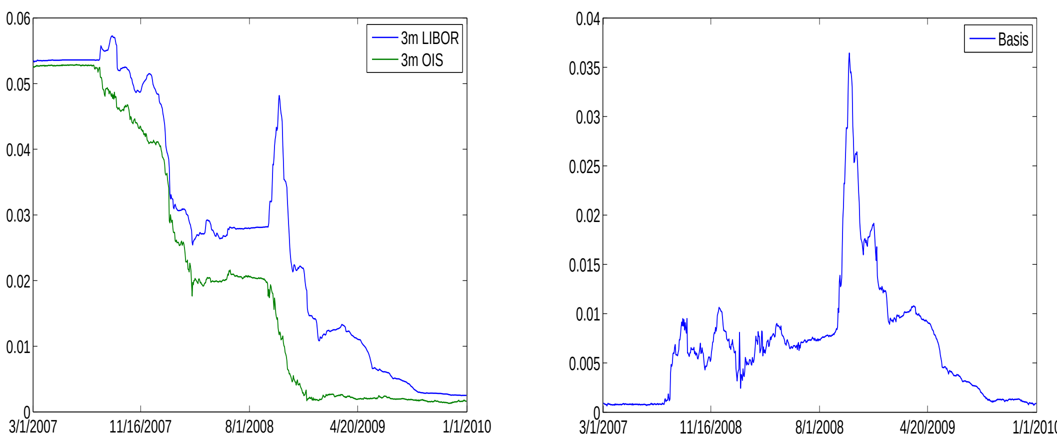

Figure 3: The OIS and 3-month LIBOR rates from January 2007 to January 2010, as well as the LIBOR-OIS basis.

affected by a default. The answer is clearly a no. When Lehman defaulted in September 2008, LIBOR soared, whereas OIS rates gradually fell, following a regular trend. So, from one side, unsecured lending rates were heavily impacted by Lehman's default. From another, swap rates based on the daily compounding of overnight rates, were less. To state it differently, Lehman's default did not immediately push OIS rates down. Instead, it led to abrupt changes in the LIBOR-OIS basis.

Inspired by this historical data, we thus assume a multi-curve interest-rate model where OIS rates follow continuous dynamics and are not affected by the counterparty's default. LIBOR-OIS basis spreads, instead, are assumed to be deterministic until default time, and to jump at default by a deterministic amount.[8]

This choice is also convenient from a modeling point of view. In fact, this way, measure changes are not affected by the jump, since numeraires will be defined by the OIS discount curve, which is by assumption independent of the jump.

Before specifying our multi-curve model, we need to introduce some definitions and notation. We denote by $P_D(t,T)$ the discount factor at time t with maturity T, calculated off the OIS discount curve. We fix a tenor x and define OIS forward rates at time t for the interval $[T, T+x]$ by:

$$F^x(t,T) := \frac{1}{x}\left[\frac{P_D(t,T)}{P_D(t,T+x)} - 1\right] \tag{34}$$

Then, we define the corresponding forward LIBOR rates by:

$$L^x(t,T) := \mathbb{E}_D^{T+x}\big[\mathbf{LIBOR}(T,T+x)|\mathcal{F}_t\big] \tag{35}$$

where the notation $\mathbb{E}_D^{T+x}$ indicates that the expectation should be taken under the OIS $(T+x)$-forward measure Q^{T+x}, and $\mathbf{LIBOR}(T,T+x)$ is the LIBOR set at time T with maturity $T+x$. Finally, we define LIBOR-OIS forward basis spreads by:

$$B^x(t,T) := L^x(t,T) - F^x(t,T) \tag{36}$$

The OIS forward (34) and forward LIBOR (35) are, by definition, martingales under the corresponding $(T+x)$-forward measure. Accordingly, the basis spread (36) is also a martingale under the same measure. We assume that the basis spread in the baseline model with no jump is a constant process given by $B^x(0,T)$, and model the jump at default using an additive jump approach, see Equation (12). That is, for each fixed tenor x and LIBOR starting date T, the LIBOR-OIS basis spread dynamics with default is modeled as follows:

$$B^x(t,T) = B^x(0,T) + M^x(t,T) \tag{37}$$

[8]The assumption of deterministic basis spreads is typically used by the industry when pricing interest rate derivatives in a multi-curve set up. Our basis model is therefore consistent this with market practice in that basis spreads become deterministic when the probability of default is zero.

where $M^x(t,T)$ is an additive jump process with $M^x(0,T) = 0$. The jump dynamics is given by

$$dM^x(t,T) = J^x(t,T)\, 1_{\{t \le \tau\}}\, (-\lambda\, dt + dN_t) \tag{38}$$

where N is a standard Q^{T+x}-Poisson process with deterministic and constant intensity, and where $J^x(t,T)$ is a deterministic time-dependent jump size, which could be different for different maturities T and tenors x. Thanks to the independence between default and OIS rates, dynamics (38) remains the same in the risk-neutral measure or any other forward measure.

The jump size $J^x(t,T)$ is in general a function of tenor x, LIBOR starting date T, as well as default time t. In fact, short term basis tends to be affected by credit events more than long-term ones, and typically, the bigger the tenor the bigger the impact. One possible parameterization is, therefore, the following:

$$J^x(t,T) = e^{-\mu^x(T-t)}(J_0^x - J_\infty^x) + J_\infty^x$$

where μ^x, J_0^x and J_∞^x are constant parameters possibly dependent on tenor x. If default happens exactly at time T, then the basis jumps by J_0^x. If default happens much earlier than time T, then the basis jumps roughly by J_∞^x. Normally, $J_\infty^x < J_0^x$, so longer term bases incur jumps of smaller magnitude. The parameter μ^x controls the speed at which the jump size decays.

Below, for simplicity, we set $J^x(t,T) = J$, where $J > 0$ is a constant. This can be thought of as an effective constant jump size that achieves a desirable degree of WWR in a phenomenological model. In this case, for $t \le T$, we have

$$B^x(t,T) = B^x(0,T) + J 1_{\{t \ge \tau\}} - \lambda J (t \wedge \tau) \tag{39}$$

Or writing out more explicitly, for each sample path, for $t \le T$, we have

$$B^x(t,T) = B^x(0,T) + \begin{cases} -\lambda J t & \text{if} \quad t < \tau \\ -\lambda J \tau + J & \text{if} \quad t \ge \tau \end{cases}$$

As mentioned before, OIS rates can be assumed to follow any classic single-curve interest rate model. Forward LIBOR rates can then be obtained using:

$$L^x(t,T) = F^x(t,T) + B^x(t,T)$$

Finally, we mention that we use the convention that

$$Z^x(t,T) = Z^x(T,T) \quad \text{if} \quad t > T$$

for all three quantities $L^x(t,T)$, $F^x(t,T)$ and $B^x(t,T)$. That is, their values in the application period $[T, T+x]$ are assumed to be constant and equal to those observed at time T.

6.1 Jump calibration

Similarly to the previous FX case, J can be calibrated either historically or cross-sectionally, depending on a user's preference and data availability.

As per historical calibration, the jump size could be inferred by comparing basis levels observed before and after Lehman's default. However, an accurate historical calibration may be harder to achieve than it seems. First, a default is almost never an instantaneous event. And Lehman's was certainly not. Second, focusing on the above US data, the divergence between OIS and LIBOR curves can largely be attributed to the pre-existing market turmoil dating back to the beginning of the 2007 credit crunch. Third, forward OIS or LIBOR rates are not directly quoted by the market. The extrapolation of long-term rates is a model-dependent exercise, possibly based on subjective beliefs.[9] With that said, estimating J from time series of rates appears to be no more flawed epistemologically than any historical estimation of correlation, where proxy variables are used, stationarity of increments is assumed and where results can vary dramatically depending on the considered time window.

Alternatively, we could leverage the tractability of our basis model (37), and calculate model-implied correlations. Since OIS rates and basis spreads are assumed to be independent, their terminal correlation is zero, that is

$$\mathrm{Corr}\big[F^x(T,T), B^x(T,T)\big] = 0 \tag{40}$$

for all T. Instead, forward LIBORs are correlated both with OIS rates and basis spreads. For instance, see the proof in Appendix A,

$$\mathrm{Corr}\big[L^x(T,T), B^x(T,T)\big] = \sqrt{\frac{J^2 P_\tau(T)}{\Sigma_F^2 + J^2 P_\tau(T)}} \tag{41}$$

where $\Sigma_F^2 := \mathrm{Var}(F^x(T,T))$, and correlation and variance are calculated under a given pricing measure.[10] Therefore, a view on a terminal LIBOR-basis correlation can be translated into a specific value for the jump parameter J. The problem with this approach is that the RHS of (41) is, in principle, counterparty dependent, whereas the LHS is not.

One last approach is based on the calibration to market quotes of floating-rate bonds issued by the counterparty, when they are available. To this end, we could use the following formula for an individual coupon payment, see the proof in Appendix A:

$$\mathbb{E}_D^{T+x}\big[L^x(T,T)\, 1_{\{\tau>T+x\}}\big] = (L^x(0,T) - \lambda J T)\, e^{-\lambda(T+x)} \tag{42}$$

[9] We add that USD OIS swap rates were (and still are) liquid only for short maturities (up to 2y), and that long-term OIS rates were obtained using Fed Funds/LIBOR basis swaps.

[10] Thanks to the independence between default and OIS rates, $P_\tau(t)$ remains the same in Q or any other usual pricing measure.

By discounting this value and summing over all coupon payments of a given bond, including the notional at maturity, allows us to recover J from the market price of the bond. In case more floating-rate bonds issued by the counterparty are quoted by market, different values of J may be bootstrapped for different maturities.

6.2 CVA for a portfolio of European-style interest rate deals

Similarly to the case of European-style FX deals, the CVA formula for European-style interest-rate derivatives can be made more explicit. We assume that the portfolio deals are based on forward LIBOR rates $L^x(t, T_i)$, $i = 0, 1, 2, \cdots, M-1$, with a common tenor x, where $L^x(t, T_i)$ is the i-th forward LIBOR whose beginning and ending dates are, respectively, T_i and $T_{i+1} = T_i + x$. A European-style interest-rate deal is characterized by a set of cash flows, whose time-t value, for any t, does not depend on rates set prior to time t, with the only possible exception of the last LIBOR fixing before t. This definition includes, in particular, standard interest rate swaps.

We denote by V_t the portfolio's value at time t. For any $i = 0, 1, \cdots, M-1$ and any $t \geq 0$, we also introduce the following notation for the basis with starting date T_i conditional on default happening at time t:

$$b^x(t, T_i) := \begin{cases} B^x(0, T_i) - \lambda J T_i & \text{if } \; T_i < t \\ B^x(0, T_i) + J - \lambda J t & \text{if } \; T_i > t \end{cases} \tag{43}$$

We will ignore the zero probability event that the default time t falls exactly on any of the LIBOR fixing dates. We then define $\mathbf{b}_t$ to be the vector consisting of all $b^x(t, T_i)$'s with ending date $T_{i+1} > t$. This is the set of bases that affect the future cash flows after time t. We let $\mathbf{B}_t$ be the vector of initial bases with ending dates larger than t, so $\mathbf{B}_t$ is equal to $\mathbf{b}_t$ with $J = 0$. Mathematically, for each t, these three vectors are defined as

$$\mathbf{b}_t := \{b^x(t, T_i) : T_{i+1} > t\} \tag{44}$$

$$\mathbf{B}_t := \{B^x(0, T_i) : T_{i+1} > t\} \tag{45}$$

Notice that the lengths of these vectors are dependent on t. With some abuse of notation, we thus write:[11]

$$V_t = V\big(t, \mathbf{b}_t\big) \tag{46}$$

The WWR CVA formula can then be written as

$$\mathbf{CVA}_{\text{WWR}} = (1 - R) \int_0^T \mathbb{E}\Big[D(0,t)\, V\big(t, \mathbf{b}_t\big)^+\Big]\, \lambda e^{-\lambda t}\, \mathrm{d}t \tag{47}$$

By setting $J = 0$, we get

$$\mathbf{CVA}_{\text{IND}} = (1 - R) \int_0^T \mathbb{E}\Big[D(0,t)\, V\big(t, \mathbf{B}_t\big)^+\Big]\, \lambda e^{-\lambda t}\, \mathrm{d}t \tag{48}$$

[11] In principle, V is piece-wisely defined and is only the same function between two consecutive cash-flow dates.

It's worth pointing out that since our baseline models for the bases are constant processes which are clearly translationally invariant, the two functions V in the above two equations are indeed the same function. We notice that both basis vectors $\mathbf{b}_t$ and $\mathbf{B}_t$ are deterministic. So, only OIS rates need to be simulated if one uses Monte Carlo techniques to compute the CVAs.

6.3 CVA approximations

We now develop CVA approximations by adjusting the initial basis vector for European-style interest rate deals. We notice that at time of default t, all future bases with starting date larger than t incur a jump of size J. Instead, the first future basis $B^x(t,T_i)$, such that $T_i < t < T_{i+1}$, has already been fixed and does not incur a jump. We will however make the approximation that all future bases after t incur a jump, including the basis $B^x(t,T_i)$ with $T_i < t < T_{i+1}$. So, $b^x(t,T_i) = B^x(0,T_i) + J - \lambda Jt$ for all i's such that $T_{i+1} > t$.

Let us write the CVA as a function of the initial basis vector $\mathbf{B}_0$, that is,

$$\mathbf{CVA}_{\text{WWR}} = \mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0)$$

and similarly for the independent CVA. Our first approximation is analogous to (22), and is based on neglecting the λJ terms in (43):

$$b^x(t,T_i) \approx B^x(0,T_i) + J$$

for $T_{i+1} > t$. It is therefore intuitive that we can write:

$$\mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0) \approx \mathbf{CVA}_{\text{IND}}(\mathbf{B}_0 + J) \tag{49}$$

In this approximation, we globally shift the bases vector $\mathbf{B}_0$ by J.

Approximation (49) does not take into account the correction due to the jump in the drift of $B^x(t,T_i)$. To account for this term, we notice that basis $B^x(t,T_i)$ jumps if default happens before T_i. Conditioning on default happening before T_i, the expected default time is, see also (29),

$$\bar{\tau}_i := \mathbb{E}[\tau|\tau < T_i] = \frac{e^{\lambda T_i} - 1 - \lambda T_i}{\lambda(e^{\lambda T_i} - 1)} \tag{50}$$

which is roughly equal to $T_i/2 - \lambda T_i^2/12$ for typical λ values. Using these effective jump times, a better approximation is

$$\mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0) \approx \mathbf{CVA}_{\text{IND}}(\bar{\mathbf{B}}_0) \tag{51}$$

with $\bar{\mathbf{B}}_0 \approx \mathbf{B}_0 + J - \lambda J\bar{\boldsymbol{\tau}}$, where $\bar{\boldsymbol{\tau}}$ is the vector $(\bar{\tau}_0, \bar{\tau}_1, \cdots, \bar{\tau}_{M-1})$. In components, we have

$$\bar{B}^x(0,T_i) = B^x(0,T_i) + J - \lambda J\bar{\tau}_i \quad \forall i = 0, 1, \cdots, M-1 \tag{52}$$

This leads to a more accurate CVA approximation, which is the analogue of (24). A more formal justification of the above two approximations is provided in Appendix A.

The above approximations are very general and can be applied to portfolios of interest-rate derivatives, as long as the derivatives are European-style. In the case of interest-rate swaps, another intuitive approximation can be derived by adjusting the strike rate K:

$$\mathbf{CVA}_{\text{WWR}}(K) \approx \mathbf{CVA}_{\text{IND}}(K - J) \tag{53}$$

where we now highlight the dependence of CVA on the strike rate K. This approximation can be explained as follows. Based on the result of Sorensen and Bollier (1994), it is market practice to calculate the independent CVA of a swap as an integral of market swaption prices struck at the swap's fixed rate. What approximation (53) tells us is that CVA under WWR can simply be calculated by shifting the common swaption strike to the left. Since a payer swaption's price is increasing for decreasing strikes, this rule of thumb is also robust, that is $\mathbf{CVA}_{\text{WWR}}(K) \geq \mathbf{CVA}_{\text{IND}}(K)$. The opposite inequality applies to a receiver interest rate swap, for which positive jumps lead to right-way risk (RWR), and hence to a reduction in CVA.

Before moving on to numerical analysis, we again make two remarks on CVA "add-on" and effective default intensity.

Remark 6.1 (WWR CVA as an "add-on")**.** *Similarly to the FX case, the WWR CVA for European-style interest-rate derivatives can be characterized as an "add-on" to the independent CVA. For this purpose, we write the WWR CVA as a function of $\mathbf{B}_0$ and λ. By performing a Taylor expansion in J on the WWR CVA and assuming λT to be small, we show in Appendix A that*

$$\mathbf{CVA}_{WWR}(\mathbf{B}_0, \lambda) \approx \mathbf{CVA}_{IND}(\mathbf{B}_0, \lambda) + J \sum_{i=0}^{M-1} \frac{\partial}{\partial \mathbf{B}_{0,i}} \mathbf{CVA}_{IND}(\mathbf{B}_0, \lambda) \tag{54}$$

where the right-hand side depends only on the independent CVA, and $\mathbf{B}_{0,i}$ denotes the i-th component of $\mathbf{B}_0$. Therefore, the add-on term is roughly proportional to the jump size J, and is connected to the rate of change of the independent CVA with respect to the individual initial basis spreads.

Remark 6.2 (Effective default intensity)**.** *Again similarly to the FX case, the WWR CVA can be thought of as an independent CVA but with an adjusted default intensity $\bar{\lambda}$. It can be shown that*

$$\mathbf{CVA}_{WWR}(\mathbf{B}_0, \lambda) \approx \mathbf{CVA}_{IND}(\mathbf{B}_0, \bar{\lambda}) \tag{55}$$

where

$$\bar{\lambda} := \lambda \left[1 + J \sum_{i=0}^{M-1} \frac{\partial}{\partial \mathbf{B}_{0,i}} \ln \big(\mathbf{CVA}_{IND}(\mathbf{B}_0, \lambda) \big) \right] \tag{56}$$

This provides a partial justification to Hull and White (2012) in the case of interest-rate derivatives.

6.4 Numerical analysis

We consider the numerical example of a 20-year payer interest-rate swap, and calculate the WWR CVA under the above jump basis model. The swap has the following features. It has a notional of USD 1000, its fixed rate is equal to 2.90%, the fixed leg has a semi-annual frequency while the floating leg has a quarterly frequency. The fixed rate is chosen such that the swap has value 0 at initiation. We assume that the instantaneous OIS rate follows a simplified version of Hull-White (1990) one-factor model with constant mean-reversion and volatility:

$$\mathrm{d}r_t = \kappa(\vartheta_t - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

We set $\kappa = 0.03$ and $\sigma = 0.005$, and calibrate ϑ_t to the observed initial OIS curve on March 13, 2013.

The numerical results for the exact WWR CVA simulation from Equation (47) are presented in Table 4. The maximum jump size of 1% is chosen so that WWR CVA is about 30% above the independent CVA. Similarly to the cross-currency swap numerical example, for fixed default intensity λ, WWR is controlled by the jump parameter J. Again, jump-at-default model can produce sizable WWR corrections to CVA.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
0	16.33	30.17	41.88	51.78	60.13
0.0025	17.65	32.51	45.00	55.47	64.24
0.0050	19.06	35.01	48.32	59.43	68.66
0.0075	20.55	37.65	51.85	63.62	73.35
0.0100	22.12	40.42	55.55	68.02	78.29

Table 4: Exact WWR CVA from Equation (47)

We then present in Tables 5 and 6, respectively, the two approximations obtained by adjusting the initial basis curve using Equations (49) and (51). The first approximation appears to be decent, but the second one is clearly more accurate. Adjusting the bases by $J - J\bar{\tau}_i$ indeed provides a better approximation than adjusting by J, with the largest percentage error being about 0.5%.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
0	16.33	30.17	41.88	51.78	60.13
0.0025	17.74	32.83	45.64	56.52	65.75
0.0050	19.24	35.66	49.67	61.61	71.78
0.0075	20.83	38.67	53.93	67.00	78.19
0.0100	22.50	41.82	58.42	72.68	84.93

Table 5: WWR CVA approximation from Equation (49)

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
0	16.33	30.17	41.88	51.78	60.13
0.0025	17.65	32.51	44.99	55.46	64.23
0.0050	19.06	34.99	48.28	59.35	68.56
0.0075	20.54	37.60	51.75	63.46	73.11
0.0100	22.09	40.34	55.38	67.74	77.87

Table 6: WWR CVA approximation from Equation (51)

Finally, we present the numerical results for WWR CVA from approximation (53) which adjusts the fixed rate K by the jump size J. As we see, this approximation, despite its simplicity, is also fairly accurate, especially for low intensities and jump sizes. In fact, the accuracy is comparable to that of adjusting the initial basis curve globally by J.

$J \setminus \lambda$	0.01	0.02	0.03	0.04	0.05
0	16.33	30.17	41.88	51.78	60.13
0.0025	17.77	32.89	45.73	56.62	65.86
0.0050	19.31	35.79	49.84	61.82	72.02
0.0075	20.94	38.86	54.20	67.32	78.56
0.0100	22.65	42.09	58.78	73.12	85.44

Table 7: WWR CVA approximation from Equation (53)

7 Conclusions

Jump-at-default models can offer a very effective means of assessing WWR for derivative portfolios. They present several advantages, including the possibility to be “attached” to any baseline model for the risk factors of a given asset class.

In this article, we have considered applications first to FX and then to interest-rate derivatives portfolios. In both cases, we have focused on portfolios of European-style claims, and presented approximations that can be used to quickly asses WWR using an existing independent CVA engine. Numerical examples demonstrating the accuracy of the approximations have been showcased.

Jump-at-default models could also be considered for other asset classes, such as equities, commodities or inflation, or to give crude estimates of WWR using existing independent CVA analytics, or to set a standard valuation model in the case of specific deals such as interest-rate swaps or cross-currency swaps.

The current work could also be extended by adding collateralization and by considering bilateral CVA. Extending the results we have obtained to these cases is in general not completely straightforward. However, if simplifying assumptions are introduced such as full collaterization or that the less important counterparty's default does not induce jumps, it is possible to obtain analytical results also in these cases.

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A Appendix

A.1 Proof of Equation (7) (Foreign bond CVA ratio)

Recall we assume zero rates. In the case of a foreign bond with notional amount 1, $V_\tau^+ = X_\tau$. Integrating (5), we get that, conditional on $\tau = t$,

$$V_\tau^+ = X_t = (1+J)X_0 e^{-\frac{1}{2}\sigma_X^2 t + \sigma_X W_t^X - J\int_0^t \lambda_u \mathrm{d}u}$$

By Equation (1) and Lando (1998), we have

$$\begin{aligned}\mathbf{CVA}_{\text{WWR}} &= (1-R)\int_0^T \mathbb{E}\left[\lambda_t e^{-\int_0^t \lambda_u \mathrm{d}u}(1+J)X_0 e^{-\frac{1}{2}\sigma_X^2 t + \sigma_X W_t^X - J\int_0^t \lambda_u \mathrm{d}u}\right]\mathrm{d}t \\ &= (1-R)(1+J)X_0\lambda_0 \int_0^T e^{-\frac{1}{2}\sigma_\lambda^2 t} e^{-\frac{1}{2}\sigma_X^2 t}\ \mathbb{E}\left[e^{\sigma_\lambda W_t^\lambda} e^{-(1+J)\int_0^t \lambda_u \mathrm{d}u} e^{\sigma_X W_t^X}\right]\mathrm{d}t \qquad (57)\end{aligned}$$

The expectation in the RHS of (57) can be calculated as follows. First, we perform a stochastic Taylor expansion in σ_λ, getting

$$\int_0^t \lambda_u \mathrm{d}u = \lambda_0 t + \lambda_0 \sigma_\lambda \int_0^t (t-u)\mathrm{d}W_u^\lambda + o(\sigma_\lambda)$$

Then, we notice that the three exponents in the expectation are either exactly normal or approximately normal, which leads to:

$$\mathbb{E}\left[e^{\sigma_\lambda W_t^\lambda} e^{-(1+J)\int_0^t \lambda_u \mathrm{d}u} e^{\sigma_X W_t^X}\right] = e^{-(1+J)\lambda_0 t - \frac{1}{2}\sigma_X^2 t}\left(1 + \rho\sigma_X\sigma_\lambda t - \frac{1+J}{2}\rho\sigma_X\sigma_\lambda\lambda_0 t^2\right) + o(\sigma_\lambda)$$

Using this expression in Equation (57), we finally get:

$$\frac{\mathbf{CVA}_{\text{WWR}}}{\mathbf{CVA}_{\text{IND}}} = (1+J)\frac{\int_0^T e^{-(1+J)\lambda_0 t}\left(1 + \rho\sigma_X\sigma_\lambda t - \frac{1+J}{2}\rho\sigma_X\sigma_\lambda\lambda_0 t^2\right)\mathrm{d}t}{\int_0^T e^{-\lambda_0 t}\mathrm{d}t} + o(\sigma_\lambda)$$

The above integrals can be calculated explicitly. We obtain:

$$\frac{\mathbf{CVA}_{\text{WWR}}}{\mathbf{CVA}_{\text{IND}}} = \frac{1 - e^{-(1+J)\lambda_0 T}}{1 - e^{-\lambda_0 T}} + \frac{1}{2}(1+J)\lambda_0\rho\sigma_X\sigma_\lambda T^2 e^{-(1+J)\lambda_0 T} + o(\sigma_\lambda) \qquad (58)$$

When the default intensity is deterministic, so that $\sigma_\lambda = 0$, we recover the following exact result:

$$\frac{\mathbf{CVA}_{\text{WWR}}}{\mathbf{CVA}_{\text{IND}}} = \frac{1 - e^{-(1+J)\lambda_0 T}}{1 - e^{-\lambda_0 T}} \qquad (59)$$

This is an increasing function of J, giving WWR (resp. RWR) for $J > 0$ (resp. $J < 0$).

The simpler approximation formula (7) is obtained by performing a Taylor expansion of (58) to first order in λ_0.

A.2 Proof of Equation (16) (Quanto CDS spread)

For simplicity, we assume that both domestic and foreign interest rates are zero, and that premiums are continuously paid. Let us denote by **CDS** and **QCDS**, respectively, the CDS and quanto CDS rates with maturity T. The present values of the CDS premium and protection legs are, respectively,

$$\mathbf{PremL} = \mathbf{CDS}\ \mathbb{E}\left[\int_0^T D(0,t)1_{\tau>t}\,\mathrm{d}t\right] = \mathbf{CDS}\int_0^T e^{-\lambda t}\,\mathrm{d}t$$
$$\mathbf{ProtL} = (1-R)\ \mathbb{E}\left[D(0,\tau)\,1_{\tau\le T}\right] = (1-R)\int_0^T \lambda e^{-\lambda t}\,\mathrm{d}t$$

Equating these two values, we get the well known result that $\mathbf{CDS} = \lambda(1-R)$.

Similarly, the present values of the quanto CDS premium and protection legs are:

$$\mathbf{QPremL} = \mathbf{QCDS}\ \mathbb{E}\left[\int_0^T D(0,t)1_{\tau>t}X_t\,\mathrm{d}t\right] = \mathbf{QCDS}\int_0^T e^{-\lambda t}X_0 e^{-\lambda J t}\,\mathrm{d}t$$
$$\mathbf{QProtL} = (1-R)\mathbb{E}\left[D(0,\tau)\,X_\tau 1_{\tau\le T}\right] = (1-R)(1+J)\int_0^T \lambda e^{-\lambda t}X_0 e^{-\lambda J t}\,\mathrm{d}t$$

Equating these two values, we get $\mathbf{QCDS} = \lambda(1-R)(1+J)$. Finally, taking the ratio of **QCDS** and **CDS**, we get Equation (16).

A.3 Proof of Equation (30) (Effective initial exchange rate)

Let us assume deterministic interest rates for simplicity, and define the conditional forward exchange rate by

$$F^{\mathbf{D}}(t) := \mathbb{E}\big[X_T|\tau = t < T\big] = F_0^T(1+J)e^{-\lambda J t} \tag{60}$$

where F_0^T denotes the unconditional forward, that is

$$F_0^T := \mathbb{E}[X_T] = X_0 e^{(r_d - r_f)T}$$

Let us then assume that the portfolio's discounted expected positive exposure depends on t only through the conditional forward, that is:

$$\mathbf{CVA}_{\text{WWR}}(X_0) = (1-R)\int_0^T \Psi\left(F^{\mathbf{D}}(t)\right)p_\tau(t)\,\mathrm{d}t \tag{61}$$

for some function Ψ, which is assumed to be twice differentiable. Taking $J = 0$, we get

$$\mathbf{CVA}_{\text{IND}}(X_0) = (1-R)\int_0^T \Psi(F_0^T)p_\tau(t)\,\mathrm{d}t \tag{62}$$

A first-order Taylor expansion of the function $\Psi\left(F^{\mathbf{D}}(t)\right)$ around $F^{\mathbf{D}}(\bar{\tau})$ gives:

$$\Psi\left(F^{\mathbf{D}}(t)\right) = \Psi\left(F^{\mathbf{D}}(\bar{\tau})\right) + \Psi'\left(F^{\mathbf{D}}(\bar{\tau})\right)\left(F^{\mathbf{D}}(t) - F^{\mathbf{D}}(\bar{\tau})\right) + \mathcal{O}\left(\left(F^{\mathbf{D}}(t) - F^{\mathbf{D}}(\bar{\tau})\right)^2\right)$$

Since $\left(F^{\mathbf{D}}(t) - F^{\mathbf{D}}(\bar{\tau})\right)^2$ is bounded by $(F^{\mathbf{D}}(0) - F^{\mathbf{D}}(T))^2$, which is $\mathcal{O}((\lambda JT)^2)$, by the differentiability of Ψ it follows that the error term is $\mathcal{O}((\lambda JT)^2)$.

Plugging the expansion of $\Psi\left(F^{\mathbf{D}}(t)\right)$ into Equation (61), and noticing that

$$\mathbf{CVA}_{\text{IND}}(\bar{X}_0) = \mathbf{CVA}_{\text{IND}}(X_0(1+J)e^{-\lambda J\bar{\tau}}) = (1-R)\int_0^T \Psi\left(F^{\mathbf{D}}(\bar{\tau})\right) p_\tau(t)\mathrm{d}t$$

we obtain Equation (30) by choosing $\bar{\tau}$ such that the linear-order term vanishes:

$$\int_0^T \Psi'\left(F^{\mathbf{D}}(\bar{\tau})\right)\left(F^{\mathbf{D}}(t) - F^{\mathbf{D}}(\bar{\tau})\right) p_\tau(t)\,\mathrm{d}t = 0$$

This gives exactly Equation (26) in the main text, and also demonstrates that $\bar{\tau}$ is between $[0, T]$.

A.4 Proof of Equation (31) (WWR add-on for FX deals)

Recall that

$$\mathbf{CVA}_{\text{WWR}} = (1-R)\int_0^T \mathbb{E}\left[D(0,t)V_t^+ | \tau = t\right] p_\tau(t)\mathrm{d}t \tag{63}$$

The quantity V_t^+ is a function of the state variable X_t. Let us assume deterministic interest rates for simplicity, and set

$$F(J,t) := X_0(1+J)e^{-\lambda Jt}e^{(r_d - r_f)t}$$

The quantity $F(J,t)$ is the expectation of X_t conditioning on default happening at time t. Be careful that $F(J,t)$ is different from $F_t^{\mathbf{D}}$ defined in Equation (60), which is the conditional expectation of the terminal X_T. Let us also assume there exists a function $\Phi(\cdot,\cdot)$ such that

$$\mathbf{CVA}_{\text{WWR}} = (1-R)\int_0^T \Phi\big(F(J,t), t\big)\, p_\tau(t)\,\mathrm{d}t \tag{64}$$

which is true, for instance, in the Black-Scholes model, or in any other model where the distribution of X_t conditioning on default at t is completely described by its mean $F(J,t)$. The function $\Phi(\cdot,\cdot)$ is continuous between two consecutive cash flow dates. It differs from Ψ in (61). The difference comes from the fact that there is extra t dependence in Φ, but more importantly because $F(J,t)$ and $F_t^{\mathbf{D}}$ are different. Therefore, in Equations (61) and (64), we are making slightly different assumptions. Both assumptions are true under the Black-Scholes baseline FX model. Taking $J = 0$, we get:

$$\mathbf{CVA}_{\text{IND}} = (1-R)\int_0^T \Phi\big(F(0,t), t\big)\, p_\tau(t)\,\mathrm{d}t \tag{65}$$

Expanding the WWR CVA around $J = 0$ to first order, we get:

$$\begin{aligned}\mathbf{CVA}_{\text{WWR}} &= \mathbf{CVA}_{\text{IND}} + J(1-R)\int_0^T \Phi_1\big(F(0,t),t\big)\, X_0 e^{(r_d-r_f)t}(1-\lambda t)\; p_\tau(t)\mathrm{d}t + \mathcal{O}(J^2) \\ &= \mathbf{CVA}_{\text{IND}} + J(1-R)\lambda\frac{\partial}{\partial\lambda}\int_0^T \Phi_1\big(F(0,t),t\big)\, X_0 e^{(r_d-r_f)t}\; p_\tau(t)\mathrm{d}t + \mathcal{O}(J^2) \qquad (66)\end{aligned}$$

where Φ_1 denote the partial derivative of Φ with respect to the first argument, and we have used the following property

$$\lambda\frac{\partial}{\partial\lambda}p_\tau(t) = (1-\lambda t)p_\tau(t)$$

Notice that the derivative Φ_1 is well-defined between any two consecutive payment dates, and that we have assumed there are only finitely many payment dates. Taking derivative with respect to X_0 in Equation (65), we get:

$$\frac{\partial}{\partial X_0}\mathbf{CVA}_{\text{IND}} = (1-R)\int_0^T \Phi_1(F(0,t),t)e^{(r_d-r_f)t}\, p_\tau(t)\mathrm{d}t \qquad (67)$$

Comparing Equations (66) and (67) gives:

$$\mathbf{CVA}_{\text{WWR}}(X_0,\lambda) = \mathbf{CVA}_{\text{IND}}(X_0,\lambda) + \lambda J X_0\frac{\partial}{\partial\lambda}\frac{\partial}{\partial X_0}\mathbf{CVA}_{\text{IND}}(X_0,\lambda) + \mathcal{O}(J^2) \qquad (68)$$

When λT is small, $\lambda\partial p_\tau(t)/\partial\lambda$ is very close to $p_\tau(t)$, so

$$\mathbf{CVA}_{\text{WWR}}(X_0,\lambda) \approx \mathbf{CVA}_{\text{IND}}(X_0,\lambda) + J X_0\frac{\partial}{\partial X_0}\mathbf{CVA}_{\text{IND}}(X_0,\lambda) \qquad (69)$$

which proves Equation (31) in the main text.

A.5 Proof of Equation (33) (Effective default intensity for FX deals)

The effective default intensity λ^{eff} is defined implicitly by:

$$\mathbf{CVA}_{\text{WWR}}(X_0,\lambda) = \mathbf{CVA}_{\text{IND}}(X_0,\lambda^{\text{eff}}) \qquad (70)$$

For small J, the effective default intensity is close to λ, so we can write:

$$\lambda^{\text{eff}} = \lambda + aJ + \mathcal{O}(J^2)$$

A first-order Taylor expansion of independent CVA in J gives:

$$\mathbf{CVA}_{\text{IND}}(X_0,\lambda^{\text{eff}}) = \mathbf{CVA}_{\text{IND}}(X_0,\lambda) + aJ\frac{\partial\mathbf{CVA}_{\text{IND}}(X_0,\lambda)}{\partial\lambda} + \mathcal{O}(J^2) \qquad (71)$$

Comparing this equation with (68), we get:

$$a = \lambda X_0 \frac{\frac{\partial}{\partial X_0}\frac{\partial}{\partial \lambda}\mathbf{CVA}_{\text{IND}}(X_0,\lambda)}{\frac{\partial}{\partial \lambda}\mathbf{CVA}_{\text{IND}}(X_0,\lambda)}$$

Instead, if we had started from Equation (69), we would get:

$$a = \lambda X_0 \frac{\frac{\partial}{\partial X_0}\mathbf{CVA}_{\text{IND}}(X_0,\lambda)}{\mathbf{CVA}_{\text{IND}}(X_0,\lambda)}$$

The proof of Equation (33) is completed by taking $\bar{\lambda} = \lambda + aJ$.

A.6 Proof of Equation (41) (Correlation between LIBOR and basis)

Since $B^x(T,T)$ is independent of $F^x(T,T)$, then $\mathrm{Cov}\left[L^x(T,T), B^x(T,T)\right] = \mathrm{Var}(B^x(T,T))$. To calculate $\mathrm{Var}(B^x(T,T))$, we recall that $\mathrm{d}B^x(t,T) = J\,1_{\{t\le\tau\}}\left(\mathrm{d}N_t - \lambda\,\mathrm{d}t\right)$, and apply the Ito isometry for jump processes, getting:

$$\begin{aligned}
\mathrm{Var}(B^x(T,T)) &= \mathbb{E}\left[\int_0^T J1_{\{\tau\ge t\}}\big(\mathrm{d}N_t - \lambda\,\mathrm{d}t\big)\right]^2 \\
&= J^2\int_0^T (1-P_\tau(t))\lambda\,\mathrm{d}t = J^2 P_\tau(T)
\end{aligned}$$

Therefore, $\mathrm{Cov}\left[L^x(T,T), B^x(T,T)\right] = J^2 P_\tau(T)$. Likewise, the variance of the LIBOR rate is given by

$$\mathrm{Var}(L^x(T,T)) = \mathrm{Var}(F^x(T,T)) + \mathrm{Var}(B^x(T,T)) = \Sigma_F^2 + J^2 P_\tau(T)$$

This immediately gives Equation (41).

A.7 Proof of Equation (42) (Floating-rate bond coupon)

By definition of forward LIBOR, we have:

$$\mathbb{E}_D^{T+x}\big[L^x(T,T)\,1_{\{\tau>T+x\}}\big] = \mathbb{E}_D^{T+x}\big[F^x(T,T)\,1_{\{\tau>T+x\}}\big] + \mathbb{E}_D^{T+x}\big[B^x(T,T)\,1_{\{\tau>T+x\}}\big]$$

Since $F^x(T,T)$ is independent of τ, the first expectation on the right-hand side is simply equal to

$$\mathbb{E}_D^{T+x}\big[F^x(T,T)\,1_{\{\tau>T+x\}}\big] = F^x(0,T)e^{-\lambda(T+x)}$$

For the second term, we have

$$\begin{aligned}
\mathbb{E}_D^{T+x}\big[B^x(T,T)\,1_{\{\tau>T+x\}}\big] &= \mathbb{E}_D^{T+x}\left[\big(B^x(0,T) - \lambda J\tau\wedge T + J1_{\{\tau<T\}}\big)1_{\{\tau>T+x\}}\right] \\
&= \mathbb{E}_D^{T+x}\left[\big(B^x(0,T) - \lambda JT\big)1_{\{\tau>T+x\}}\right] \\
&= (B^x(0,T) - \lambda JT)e^{-\lambda(T+x)}
\end{aligned}$$

Adding the two expectations, we get Equation (42).

A.8 Proof of Equations (49) and (51) (WWR risk by adjusting the bases)

Recall that

$$\mathbf{CVA}_{\text{WWR}} = (1-R)\int_0^T \mathbb{E}\left[D(0,t)V_t^+|\tau=t\right] p_\tau(t)\mathrm{d}t$$

where V_t is the time-t risk-neutral expectation of discounted future cash flows after time t. Since the discount factor $D(0,t)$ only involves the OIS curve, the expected positive exposure $\mathbb{E}\left[D(0,t)V_t^+|\tau=t\right]$ is a function of $\mathbf{b}(t)$. Therefore, there exists a function Υ, which we assume to be piece-wise differentiable, such that

$$\mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0) = (1-R)\int_0^T \Upsilon(\mathbf{b}(t),t)\, p_\tau(t)\mathrm{d}t \tag{72}$$

where we highlight only the dependence on $\mathbf{b}_t$ and t. Though Υ has a variable number of arguments $\mathbf{b}(t)$, the number of arguments is constant within any two consecutive payment dates. Taking $J=0$, we get

$$\mathbf{CVA}_{\text{IND}}(\mathbf{B}_0) = (1-R)\int_0^T \Upsilon(\mathbf{B}_t,t)\, p_\tau(t)\,\mathrm{d}t \tag{73}$$

If we neglect the drift correction term $-\lambda Jt$ in each component of $\mathbf{b}_t$, we see that

$$\mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0) \approx (1-R)\int_0^T \Upsilon(\mathbf{B}_t+J,t)\, p_\tau(t)\,\mathrm{d}t = \mathbf{CVA}_{\text{IND}}(\mathbf{B}_0+J)$$

This justifies Equation (49)

To justify Equation (51) is slightly more involved. We will ignore the possibility that the first LIBOR-dependent payment after default has been fixed before the default time. This means that in the CVA formula, we can replace the first basis after time t, $b^x(t,T_i)$, with $B^x(0,T_i)+J-\lambda Jt$.

Let $\bar{\boldsymbol{\tau}}$ be the vector $(\bar{\tau}_0,\bar{\tau}_1,\cdots,\bar{\tau}_{M-1})$, and define $\bar{\mathbf{B}}_t$ as the vector

$$\bar{\mathbf{B}}_t := \{\bar{B}^x(0,T_i) : T_{i+1} > t\}$$

By expanding each component $b^x(t,T_i)$ around $B^x(0,T_i)+J-\lambda J\bar{\tau}_i$ using a linear-order multivariate Taylor expansion, we get

$$\begin{aligned}\mathbf{CVA}_{\text{WWR}}(\mathbf{B}_0) &= (1-R)\int_0^T \Upsilon(\bar{\mathbf{B}}_t,t)\, p_\tau(t)\,\mathrm{d}t \\ &\quad + \lambda J(1-R)\sum_{i=0}^{M-1}\int_0^T \Upsilon_i(\bar{\mathbf{B}}_t,t)\,(t-\bar{\tau}_i)\, p_\tau(t)\,\mathrm{d}t + \mathcal{O}((\lambda JT)^2)\end{aligned}$$

where Υ_i denotes the partial derivative of Υ with respect to $b^x(t,T_i)$. The vector $\bar{\boldsymbol{\tau}}$ is picked so that the linear-order term above is roughly zero. A sufficient condition for this to happen is:

$$\int_0^T \Upsilon_i(\bar{\mathbf{B}}_t,t)\,(t-\bar{\tau}_i)\,p_\tau(t)\,\mathrm{d}t = 0 \quad \forall i = 0,1,\cdots,M-1$$

To this end, we replace the extreme of integration T with T_i, since we have assumed that the expected exposure does not depend on previously observed bases, so $\Upsilon_i(\cdot,t)=0$ if $t>T_i$:

$$\int_0^{T_i} \Upsilon_i(\bar{\mathbf{B}}_t,t)\,(t-\bar{\tau}_i)\,p_\tau(t)\,\mathrm{d}t = 0 \quad \forall i = 0,1,\cdots,M-1$$

This in general is a set of nonlinear equations in the vector $\bar{\boldsymbol{\tau}}$ which is to be solved numerically. To estimate each $\bar{\tau}_i$, we assume that each of the Υ_i is roughly constant in the region $(0,T_i)$ so that we get

$$\int_0^{T_i} (t-\bar{\tau}_i)\,p_\tau(t)\,\mathrm{d}t = 0 \quad \forall i = 0,1,\cdots,M-1$$

These equations can be solved analytically to get $\bar{\boldsymbol{\tau}}$, leading to Equation (52).

A.9 Proof of Equations (53) (WWR risk by adjusting the strike for IRS)

Let us consider a spot-starting payer interest rate swap with fixed rate K and payment times $T_1,\ldots,T_N$. The value of the swap at the default time $\tau=t$ can be approximated as follows:

$$\begin{aligned} V_t(K) &\approx V_t^{J=0}(K) + J\sum_{T_i>t}\tau_i P_D(t,T_i) \\ &\approx V_t^{J=0}(K-J) \end{aligned}$$

where we highlight the dependence on K and by $V_t^{J=0}$ we mean value V_t calculated assuming $J=0$.

The above is only an approximation for several reasons. One is that we have ignored the drift correction $-\lambda Jt$ in the LIBOR basis, another is that we have ignored the potential mismatch of the payment dates between the floating leg and the fixed leg. The swap CVA can thus be approximated by

$$\mathbf{CVA}_{\mathrm{WWR}}(K) = (1-R)\int_0^T \mathbb{E}\left[D(0,t)(V_t^{J=0}(K-J))^+\right]\lambda e^{-\lambda t}\mathrm{d}t = \mathbf{CVA}_{\mathrm{IND}}(K-J)$$

which is the rule of thumb in Equation (53).

A.10 Proof of Equation (54) (WWR add-on for interest rate deals)

Our starting point is Equation (72):

$$\mathbf{CVA}_{\mathrm{WWR}}(\mathbf{B}_0, \lambda) = (1 - R) \int_0^T \Upsilon(\mathbf{b}_t, t) p_\tau(t) \mathrm{d}t \tag{74}$$

where we also highlight the dependence on λ. The proof is very similar to that of Equation (31) and we omit the details. We only stress the following useful observation:

$$\frac{\partial b^x(t, T_i)}{\partial J} p_\tau(t) = \lambda \frac{\partial p_\tau(t)}{\partial \lambda} = (1 - \lambda t) p_\tau(t) \approx p_\tau(t)$$

A.11 Proof of Equation (56) (Effective default intensity for interest rate deals)

The proof of this equation follows the same lines as that of Equation (33) and we therefore omit the details.