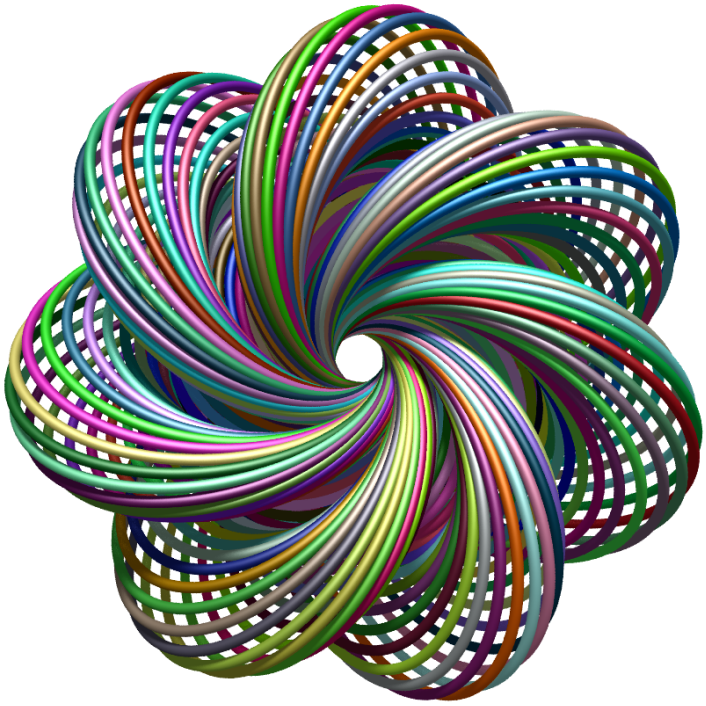
Essentials of Mathematical Methods:

Foundations, Principles, and Algorithms

Yuguang Yang

version 3.0



God used beautiful mathematics in creating the world. –Paul Dirac

Dedicated to
those who appreciate the power of mathematical methods
and enjoy learning it.

Preface

Objective

Today, mathematical methods, models, and computational algorithms are playing increasingly significant roles in addressing major challenges arising from scientific research and technological developments. Although many novel methods and algorithms, such as deep learning and artificial intelligence, are emerging and reshaping various areas at an unprecedented pace, their core ideas and working mechanisms are inherently related to and deeply rooted in some essential mathematical foundations and principles. By performing an in-depth survey on the underlying foundations, principles, and algorithms, this book aims to navigate the vast landscape of mathematical methods widely used in diverse scientific and engineering domains.

This book starts with a survey of mathematical foundations, including essential concepts and theorems in real analysis, linear algebra, and related fundamentals. Then it examines a broad spectrum of applied mathematical methods, ranging from traditional ones such as optimizations and dynamical system modeling, to state-of-the-art such as machine learning, deep learning, and reinforcement learning. The emphasis is placed on methods for stochastic and dynamical system modeling, optimal decision-making, and statistical learning. For each topic, this book organizes fundamental definitions, theorems, methods, and algorithms in a logical and illuminating way.

Features and Highlights

- Comprehensive, essential, and self-contained.
- Concepts, theorems, and discussions are developed to suit real-world applications.
- Key references and resources are provided on each topic.
- Comparisons and discussions on similar definitions and theorems.
- An evolving book with regular updates on Github.

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Yuguang Yang, Fall 2019
yangyutu123@gmail.com

Notations

- $\mathbb{R}$: real numbers.
- $\mathbb{R}_+$: nonnegative real numbers.
- $\mathbb{R}_{++}$: positive real numbers.
- $\bar{\mathbb{R}}$: extended real numbers.
- $\mathbb{C}$: complex numbers.
- $\mathbb{F}$: real or complex numbers.
- $\mathbb{Q}$: rational numbers.
- $\mathbb{Z}$: integer numbers.
- $\mathbb{P}$: positive numbers.
- $\mathcal{P}_n$: polynomial of degree of n.
- $\mathbb{N}$: natural numbers.
- $\mathcal{R}(A)$: the range of matrix A.
- $\mathcal{N}(A)$: the null space of matrix A.
- V: vector space.
- $det(A)$: the determinant of matrix A.
- $rank(A)$: the rank of matrix A.
- $\|\cdot\|_2$: Euclidean 2 norm of a vector of a matrix.
- $\|\cdot\|_F$: Frobenius norm of a matrix.
- $\rho(A)$: the spectral radius of matrix A.
- $Tr(A)$: the trace of matrix A.
- $L^2[a,b]$: Lebesgue integrable function on $[a,b]$.
- $L^1[a,b]$: Lebesgue integrable function on $[a,b]$.
- $N(0,1)$: standard Gaussian distribution.
- $N(\mu,\sigma^2)$: Gaussian distribution with mean μ and variance σ^2.
- $MN(\mu,\Sigma)$: multivariate Gaussian distribution with mean vector μ and covariance matrix Σ.
- $\mathbf{1}(x), I(x)$ indicator function.
- $E[X], \mathbf{E}[X], \mathbb{E}[X]$ expectation of random variable X.
- $Var[X]$ variance of random variable X.

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Part I

MATHEMATICAL FOUNDATIONS