A Proof of Theorem 4.1

Proof. Let Q_l be the set of queries that are uniformly distributed over the hypersphere centered at x with radius l.

We may expand the expectation as follows:

$$L(c', c, Q_l) = E_{||R||=l}[ReLU(-\cos\alpha)\cdot(||q-c'||^2 - ||q-x||^2)]$$
$$= \int_{\frac{\pi}{2}}^{\pi} -\cos\alpha\, E_{||R||=l}[(R-r')^2 - R^2 \mid \frac{\langle R, r\rangle}{||R||\cdot||r||} = \alpha]$$
$$dP(\frac{\langle R, r\rangle}{||R||\cdot||r||} < \alpha)$$

To evaluate this integral, we decompose R and r' into components parallel and orthogonal to r. As shown in Figure 4, we call these components $R_{\parallel}$, $R_{\perp}$, $r'_{\parallel}$, $r'_{\perp}$, respectively. Also, we let $\theta = \angle cxc'$, β is the angle between $R_{\perp}$ and $r'_{\perp}$. Then we simplify this inner expectation given fixed r, r', θ, and α:

$$E_{||R||=l}[(R-r')^2 - R^2 \mid \frac{\langle R, r\rangle}{||R||\cdot||r||} = \alpha]$$
$$= E_{||R||=l}[(R_{\parallel} - r'_{\parallel})^2 + (R_{\perp} - r'_{\perp})^2 - R_{\parallel}^2 - R_{\perp}^2 \mid ||R_{\parallel}|| = l\cos\alpha]$$
$$= E_{||R||=l}[r'_{\parallel}(r'_{\parallel} - 2R_{\parallel}) + r'^2_{\perp} - 2r'_{\perp}R_{\perp} \mid ||R_{\parallel}|| = l\cos\alpha]$$
$$= E_{||R||=l}[||r'||\cos\theta(||r'||\cos\theta - 2l\cos\alpha) + ||r'||^2\sin^2\theta - 2r'_{\perp}R_{\perp} \mid ||R_{\parallel}|| = l\cos\alpha]$$
$$= E_{\beta}[||r'||^2 - 2l||r'||\cos\theta\cos\alpha - 2l||r'||\sin\theta\sin\alpha\cos\beta]$$
$$= ||r'||\cdot(||r'|| - 2l\cos\theta\cos\alpha - 2l\sin\theta\sin\alpha E[\cos\beta])$$
$$= ||r'||\cdot(||r'|| - 2l\cos\theta\cos\alpha)$$

Meanwhile, $dP(\frac{\langle R,r\rangle}{||R||\cdot||r||} < \alpha)$ is proportional to the surface area of a $(D-1)$-dimensional hypersphere of radius $l\sin\alpha$. Thus, we express it as $A\sin^{D-2}\alpha$ for some constant A. Our integral then becomes

$$L(c', c, Q_l) = \int_{\frac{\pi}{2}}^{\pi} -\cos\alpha||r'||(||r'|| - 2l\cos\theta\cos\alpha)A\sin^{D-2}\alpha d\alpha$$
$$= -A||r'||^2\int_{\frac{\pi}{2}}^{\pi}\cos\alpha\sin^{D-2}\alpha d\alpha$$
$$+ 2Al||r'||\cos\theta\int_{\frac{\pi}{2}}^{\pi}\cos^2\alpha\sin^{D-2}\alpha d\alpha$$
$$= \frac{A||r'||^2}{D-1} + Al||r'||\cos\theta\int_0^{\pi}(\sin^{D-2}\alpha - \sin^D\alpha)d\alpha$$

Now we define $I_D = \int_0^{\pi}\sin^D\alpha d\alpha$.

$$I_D = 2\int_0^{\pi/2}\sin^D\alpha d\alpha$$
$$= -2\cos\alpha\sin^{D-1}\alpha\Big|_0^{\pi/2}$$
$$- 2\int_0^{\pi/2}\cos\alpha(-(D-1)\cos\alpha\sin^{D-2}\alpha)d\alpha$$
$$= 2(D-1)\int_0^{\pi/2}\cos^2\alpha\sin^{D-2}\alpha d\alpha$$
$$= (D-1)I_{D-2} - (D-1)I_D$$

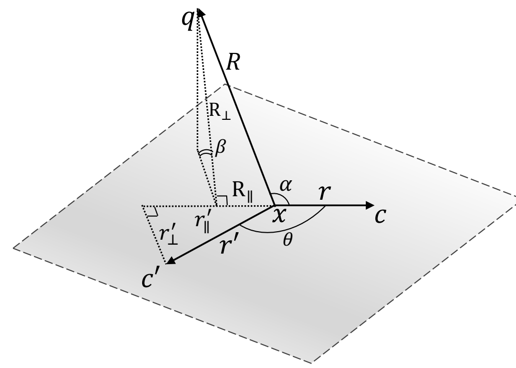

Figure 13: Detailed geometric relationship of vectors.

Combining terms, we have $I_D = \frac{D-1}{D}I_{D-2}$. Then the loss satisfies:

$$L(c', c, Q_l) = \frac{A||r'||^2}{D-1} + Al||r'||\cos\theta(I_{D-2} - I_D)$$
$$= \frac{A||r'||^2}{D-1} + Al||r'||\cos\theta\frac{I_D}{D-1}$$
$$= \frac{A}{D-1}(||r'||^2 + I_D l\cos\theta||r'||)$$
$$= \frac{A}{D-1}(||r'||^2 + \frac{I_D l}{||r||}r^T r')$$

Note that the hypersphere Q consists of many hyperspherical surface Q_l. We solve $L(c', c, Q)$:

$$L(c', c, Q) = \int_0^{l_m} L(c', c, Q_l)\cdot\frac{Dl^{D-1}}{l_m^D}dl$$
$$= \int_0^{l_m}\frac{A}{D-1}(||r'||^2 + \frac{I_D l}{||r||}r^T r')\frac{Dl^{D-1}}{l_m^D}dl$$
$$= \frac{A}{D-1}(||r'||^2 + \int_0^{l_m}\frac{DI_D l^D}{l_m^D||r||}r^T r' dl)$$
$$= \frac{A}{D-1}(||r'||^2 + \frac{DI_D l_m}{(D+1)||r||}r^T r')$$
$$\propto ||r'||^2 + \lambda r^T r'$$

where $\lambda = \frac{DI_D l_m}{(D+1)||r||} > 0$.

□