Chapter 5

A Logical Framework for LCCCs

5∗1. In the rest of this thesis, we switch our topic from categorical structures of modular computational effects to the study of *languages* for programming with higher-order computational effects. We will explore the design space by considering a number of type theories and their meta-theoretic properties. To avoid the bureaucracy of defining type theories using raw syntax and reinventing a notion of categorical models whenever we define a new theory, we will employ a *logical framework* (LF) for defining other type theories.

5∗2. The LF that we are going to use is to *locally cartesian closed categories* (LCCCs) as multi-sorted algebraic theories are to categories with finite products, or as generalised algebraic theories [Cartmell 1986] (or equivalently, essentially algebraic theories [Adámek and Rosicky 1994]) are to finitely complete categories. For this reason, we will refer to the LF by LccLF.

LccLF is originally introduced in Sterling [2021, §1]'s thesis, as a simpler alternative to Uemura [2021, 2023]'s more flexible but more complex *second-order generalised algebraic theories*. LccLF has been used in the study of several type theories [Grodin et al. 2024; Niu et al. 2022; Sterling and Angiuli 2021; Sterling and Harper 2022]. We refer the reader to [Sterling 2021, §0.1.2.2 and §1.2∗2] for an insightful discussion of the comparison of LccLF and other logical frameworks.

5∗3. However, LccLF is only informally introduced in Sterling [2021]'s thesis, relying on the reader's own experience in categorical logic to recover how it works technically, for example, by treating it as an informal syntax for *locally cartesian closed sketches* [Gratzer and Sterling 2021; Kinoshita et al. 1999].

To fill this gap, in this chapter we will develop the theory of LccLF more comprehensively, culminating in the theorem that for every signature S in LccLF, there is an LCCC category $\textsc{Jdg}\,S$ such that the groupoid of models of S in an LCCC $\mathscr{C}$ is equivalent to the groupoid of LCC-functors $\textsc{Jdg}\,S \to \mathscr{C}$:

$$S\text{-}\textsc{Mod}(\mathscr{C}) \;\cong\; \mathrm{LCCC}_{\cong}(\textsc{Jdg}\,S, \mathscr{C}).$$

5∗4. In the rest of the thesis, we will assume familiarity with (extensional) dependent type theory and its categorical semantics. An excellent elementary exposition is Hofmann [1997]; a comprehensive textbook account is Jacobs [1999]. Some basic knowledge with topos theory is also helpful, as most of our denotational models will be in toposes.

5.1 Syntax of the Logical Framework

5.1∗1. In brief, the logical framework LccLF is a dependent type theory with a unit type 1, Σ-types, and a Tarski-style universe type $\mathbb{J}$ such that

1. the universe $\mathbb{J}$ is closed under the unit type and Σ-types;
2. the universe $\mathbb{J}$ is closed under *extensional* equality types;
3. there are Π-types $\Pi\ A\ B$ provided that A is in the universe $\mathbb{J}$, and if the codomain B is a type family valued in $\mathbb{J}$, the Π-type $\Pi\ A\ B$ is also in $\mathbb{J}$.

In other words, $\mathbb{J}$ is a universe having all connectives of extensional Martin-Löf type theory (MLTT), while types outside $\mathbb{J}$ have only the unit type, Σ-types, and restricted Π-types whose domain must be in $\mathbb{J}$.

The precise type formation rules of the logical framework are in Figure 5.1, and the term formation rules for the universe $\mathbb{J}$ are in Figure 5.2. All other rules, including the rules for contexts, substitutions, term formations, and judgemental equalities (β and η equalities for all type formers) are the same as the usual extensional Martin-Löf type theory [Hofmann 1997; Martin-Löf 1984; Nordström et al. 1990] and thus omitted here.

A small difference between the framework defined here and the one in Sterling's thesis [2021] is that op. cit. *all types* have extensional equality types, not just those in $\mathbb{J}$. Those extensional equality types play the role of *sort equations* in Cartmell [1986]'s generalised algebraic theories. Although they are sometimes handy when specifying type theories in the LF, they necessitate considerations of strict equalities between objects in a category when considering categorical models of theories specified in the LF, making the notion of models not invariant under equivalences of categories. They also complicate the definition of isomorphisms of models. For these reasons, they are left out in the LF here.

$$\frac{}{\Gamma \vdash 1 \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type} \qquad \Gamma, a : A \vdash B \text{ type}}{\Gamma \vdash \Sigma\ A\ B \text{ type}} \qquad \frac{}{\Gamma \vdash \mathbb{J} \text{ type}} \qquad \frac{\Gamma \vdash A : \mathbb{J}}{\Gamma \vdash \mathrm{El}\ A \text{ type}}$$

$$\frac{\Gamma \vdash A : \mathbb{J} \qquad \Gamma \vdash a : \mathrm{El}\ A \qquad \Gamma \vdash b : \mathrm{El}\ A}{\Gamma \vdash \mathrm{Eq}(a, b) \text{ type}} \qquad \frac{\Gamma \vdash A : \mathbb{J} \qquad \Gamma, a : \mathrm{El}\ A \vdash B \text{ type}}{\Gamma \vdash \Pi\ A\ B \text{ type}}$$

Figure 5.1: Type formation rules for the logical framework

$$\frac{}{\Gamma \vdash \widehat{1} : \mathbb{J}} \qquad \frac{\Gamma \vdash A : \mathbb{J} \qquad \Gamma, a : \mathrm{El}\ A \vdash B : \mathbb{J}}{\Gamma \vdash \widehat{\Sigma}\ A\ B : \mathbb{J}}$$

$$\frac{\Gamma \vdash A : \mathbb{J} \qquad \Gamma \vdash a : \mathrm{El}\ A \qquad \Gamma \vdash b : \mathrm{El}\ B}{\Gamma \vdash \widehat{\mathrm{Eq}}(a, b) : \mathbb{J}} \qquad \frac{\Gamma \vdash A : \mathbb{J} \qquad \Gamma, a : \mathrm{El}\ A \vdash B : \mathbb{J}}{\Gamma \vdash \widehat{\Pi}\ A\ B : \mathbb{J}}$$

Figure 5.2: Codes of types in the universe $\mathbb{J}$

5.1∗2 Notation. In the rest of the thesis we will extensively use dependent type theories – the logical framework for presenting object type theories and internal languages of categories for studying the meta-theoretic properties of the object type theories. To make working with type theories as natural as working with ordinary maths, we impose the following notational conventions, which resemble the concrete syntax of Agda [Norell 2009].

∗ Dependent function types, i.e. Π-types, are written as $(a : A) \to B$, or $A \to B$ when B does not depend on A. Function abstraction is $\lambda(x : A).\ t$, or $\lambda x.\ t$ if A can be inferred; function application is $f\ a$ as usual.

∗ We use *implicit function types* $\{a : A\} \to B$, whose function application and abstraction are elided when they can be inferred or have a unique choice. When they are hard to infer, we make their application and abstraction explicit by writing $f\ \{a\}$ and $\lambda\{a : A\}.\ t$ respectively.

  When the type A of a can be inferred from the uses of a, we sometimes further abbreviate the notation $\{a : A\} \to B$ as $\{a\} \to B$.

∗ Dependent pair types, i.e. Σ-types, are written as $\Sigma(a : A).\ B$, or $A \times B$ if B does not depend on A. Pairing is (a, b) and projections are $\pi_1\ p$ and $\pi_2\ p$. We also use the record syntax for iterative Σ-types with named fields:

$$\begin{array}{l} \textsc{record}\ R : \mathcal{U}\ \textsc{where} \\ \quad \mathit{fld}_1 : A_1 \\ \quad \ldots \\ \quad \mathit{fld}_n : A_n \end{array}$$

  means the iterative Σ-type of the fields $A_1, \ldots A_n$, where each field may depend on previous fields. A field of a record $r : R$ is accessed by $r.\mathit{fld}_i$. A record $r : R$ is constructed using the 'co-pattern matching' syntax which specifies each field of r by a list of declarations:

$$\begin{array}{l} r.\mathit{fld}_1 = t_1 \\ \ldots \\ r.\mathit{fld}_n = t_n \end{array}$$

* We will use the same notation for type formers and their codes in universes. The decoding operator El of universes will be also elided, as if we are working with Russell-style universes. For example, we write

$$A : \mathbb{J}, B : A \to \mathbb{J} \vdash (x : A) \to B : \mathbb{J}$$

to mean $A : \mathbb{J}, B : \mathrm{El}\ A \to \mathbb{J} \vdash \widehat{\Pi}\ A\ B : \mathbb{J}$.

* The extensional equality type $\mathrm{Eq}(a, b)$ will be written as simply $a = b$, and its only constructor is *refl* : $a = a$. There is no special notation for consuming extensional equality types since they are consumed by *equality reflection*: if we have an element of the type $a = b$, then a and b can be used interchangeably, i.e. they are judgementally equal.

* An identifier that contains underscores '_' is used as an operator. For example, if $\_+\_ : A \to A \to A$, we can write $a+b$ for $a, b : A$. Such operators do not have to be binary. For example, if $\_\langle\_\rangle\_ : A \to B \to A \to C$, we can write $a\langle b\rangle a' : C$ for $a, a' : A$ and $b : B$.

  However, a single underscore '_' that appears alone just means a 'wildcard' that take the place of something inferable or irrelevant.

5.1*3. A type theory is defined in the logical framework as a context, or equivalently a closed type, since a context $(a_1 : A_1, \ldots, a_n : A_n)$ can be packed into a record type with fields $a_i : A_i$. The idea is the *judgements-as-types* principle of the Edinburgh Logical Framework [Harper et al. 1993]: *judgements* of the object type theory (e.g. something being a type) are declared as types in the universe $\mathbb{J}$ in the logical framework; *inference rules* are declared as functions between judgements; *deductions* are then terms of judgements that make use of the previously declared judgements and inference rules.

5.1*4 Terminology. To avoid confusion with concepts in object type theories, henceforth we will call LF contexts *signatures* or occasionally *theories*. The variables of an LF contexts are referred to as *declarations* of the signature. An LF type in the universe $\mathbb{J}$ will be called *judgements*.

5.1*5 Example. The signature of *barebone type theory* has two declarations

$$ty : \mathbb{J} \qquad\qquad tm : ty \to \mathbb{J}$$

which are respectively the judgement for *something being a type* and the family of judgements for *something being a term of a type*. This signature alone is not very interesting but it serves as a basic building block for more complex type theories.

5.1*6 Example. The signature of *simply typed λ-calculus* (STLC) extends barebone type theory (Example 5.1*5) with the following declarations:

$$\iota : ty \qquad\qquad \_{\Rightarrow}\_ : ty \to ty \to ty$$
$$abs : \{a, b : ty\} \to (tm\ a \to tm\ b) \to tm\ (a \Rightarrow b)$$
$$app : \{a, b : ty\} \to tm\ (a \Rightarrow b) \to (tm\ a \to tm\ b)$$
$$\_ : \{a, b : ty\} \to \{f : tm\ a \to tm\ b\} \to app\ (abs\ f) = f$$
$$\_ : \{a, b : ty\} \to \{g : tm\ (a \Rightarrow b)\} \to abs\ (app\ g) = g$$

The declaration $\iota : ty$ corresponds to the inference rule of a base type in STLC, and the declaration $\_{\Rightarrow}\_$ corresponds to the inference rule of (non-dependent) function types. Terms of function types are specified using *higher-order abstract syntax* (HOAS) via an isomorphism with the function space in the logical framework, which is also the reason why we do not need a judgement of STLC *contexts*. With these declarations, we can define STLC terms such as

$$abs\ (\lambda f \to abs\ (\lambda x \to app\ f\ (app\ f\ x))) : tm\ ((\iota \Rightarrow \iota) \Rightarrow (\iota \Rightarrow \iota)).$$

5.1*7. It is a common pattern that a type former in the object type theory is specified by internalising an LF judgement via an isomorphism, which is precisely the purpose of introducing logical frameworks. Thus for convenience we define the judgement of isomorphisms given two judgements A, B:

$$
\begin{array}{l}
\textsc{record}\ A \cong B : \mathbb{J}\ \textsc{where} \\
\quad \begin{array}{ll}
fwd & : A \to B \\
bwd & : B \to A \\
\_ & : (a : A) \to bwd\ (fwd\ a) = a \\
\_ & : (b : B) \to fwd\ (bwd\ b) = b
\end{array}
\end{array}
$$

Using isomorphisms to LF judgements to specify object type theories does not entail that object type theories are restricted to sublanguages of the logical framework. The following two examples shows how general recursion and impredicative polymorphism can be specified in this way, although the logical framework does not have general recursion or any impredicativity.

5.1*8 Example. The signature of PCF [Plotkin 1977] extends STLC in Example 5.1*6 with a fixed-point combinator at every type

$$Y : \{a : ty\} \to (tm\ a \to tm\ a) \to tm\ a$$

as well as some new base types and terms

$$0 : tm\ \iota \qquad succ, pred : tm\ \iota \to tm\ \iota \qquad o : ty \qquad tt, ff : tm\ o$$
$$iszero : tm\ \iota \to tm\ o \qquad \supset : \{a : ty\} \to tm\ o \to tm\ a \to tm\ a \to tm\ a$$

and also the following equational declarations (whose names are irrelevant):

$$\{n : tm\ \iota\} \to pred\ (succ\ n) = n \qquad pred\ 0 = 0 \qquad iszero\ 0 = tt$$
$$\{n : tm\ \iota\} \to iszero\ (succ\ n) = ff$$
$$\{a : ty\}\ \{x, y : tm\ a\} \to (\supset tt\ x\ y = x) \times (\supset ff\ x\ y = y)$$
$$\{a : ty\} \to \{f : tm\ a \to tm\ a\} \to f\ (Y\ f) = Y\ f$$

Within the logical framework, in the context of this signature, we can write programs such as addition of numbers:

$$\begin{aligned}
&_+_ : tm\ (\iota \Rightarrow \iota \Rightarrow \iota)\\
&_+_ = abs\ (\lambda n \to Y\ (\lambda rec \to abs\ (\lambda m \to\\
&\qquad\qquad \supset (iszero\ m)\ n\ (succ\ (app\ rec\ (pred\ m))))))
\end{aligned}$$

The equational axioms in the signature implies, for example, that $0 + succ\ 0$ is judgementally equal to $succ\ 0$ in the logical framework. Note that in this way PCF is formulated as an equational theory rather than a reduction system of terms, viz a small-step operational semantics.

5.1∗9 Example. The signature of *System F* [Girard 1972; Reynolds 1974] extends the one of STLC in Example 5.1∗6 with the following declarations:

$$\forall : (ty \to ty) \to ty$$
$$\forall\text{-}iso : \{A : ty \to ty\} \to tm\ (\forall\ A) \cong ((\alpha : ty) \to tm\ (A\ \alpha))$$

where _≅_ is the judgement of isomorphisms (5.1∗7). Since polymorphic (and ordinary) functions in this signature are specified by function types of the logical framework, they inherit the β and η equalities of LF function types.

As an example, letting $Abs = \forall\text{-}iso.bwd$, we can define Church numerals:

$$\begin{aligned}
&CNum : ty\\
&CNum = \forall\ (\lambda\ \alpha \to (\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha)\\
&C_2 : tm\ CNum\\
&C_2 = Abs\ (\lambda\ \alpha \to abs\ (\lambda\ f \to abs\ (\lambda\ x \to app\ f\ (app\ f\ x))))
\end{aligned}$$

5.1∗10. Let us compare the examples of LF presentations of type theories above and their more traditional gamma-and-turnstile presentations.

(1) In the LF, Σ-types are used to pack two judgements together. This is implicit in traditional presentations. For example, $(\Sigma(a : ty).\ tm\ a) \to J$ is just

$$\frac{a\ \text{type} \qquad t : a}{J}$$

(2) Equality types of the LF are used to specify the equational theory of the object type theory, and since equality types are respected by all constructions of the LF, there is no need to manually specify any congruence rules.

(3) A more noticeable difference is that dependent function types in the universe $\mathbb{J}$ (which may be called *higher-order judgements* as we call elements of $\mathbb{J}$ judgements) uniformly handles two different things in traditional presentations: *contexts of hypotheses* $\Gamma \vdash J$ and *schematic inference rules* $\frac{J}{K}$. Taking the rule of function abstraction in STLC for example, the traditional presentation is

$$\frac{\Gamma, x : a \vdash t : b}{\Gamma \vdash \lambda x.\, t : a \Rightarrow b}$$

while in the LF presentation (Example 5.1∗6), this rule is

$$abs : \{a, b : ty\} \to (tm\ a \to tm\ b) \to tm\ (a \Rightarrow b)$$

The higher-order judgement $tm\ a \to tm\ b$ corresponds to a deduction of $t : b$ with a new hypothesis $x : a$ in the context, and $(tm\ a \to tm\ b) \to tm\ (a \Rightarrow b)$ corresponds to the inference rule. In traditional presentations, contexts Γ can only contain certain basic judgements, such as $x : a$, but not for example

$$\Gamma, x : (a \text{ type}) \vdash \cdots \qquad \text{or even} \qquad \Gamma, x : (\Gamma, x : a \vdash t : b) \vdash \cdots ,$$

so there need two-layers of entailment, $\Gamma \vdash J$ and $\frac{J}{K}$. In contrast, both of them are handled as higher-order judgements in the logical framework.

However, introducing higher-order judgements raises the question that whether a type theory defined in the LF is the same as its traditional presentation, since higher-order judgements *a prior* may introduce new terms to base judgements. This question is called the *adequacy* of LF presentations [Harper et al. 1993]. Using a gluing argument similar to the one by Gratzer and Sterling [2021], adequacy of the logical framework that we use here can be proven with respect to Uemura [2021, 2023]'s logical framework, which can be seen as a faithful formulation of traditional presentations of type theories.

5.1∗11. Because of *extensional* equalities in $\mathbb{J}$, LccLF does not enjoy decidable type checking, so the type theory of LccLF cannot be implemented as a mechanised proof assistant as is. This is not a problem for us since we only use LccLF as a paper-and-pencil logical framework. The undecidability of LccLF is no different from the undecidability of the word problem of universal algebra [Boone 1958].

However, if we are after mechanically checking the signatures and terms defined in LccLF, we can 'approximate' LccLF using existing proof assistants implementing intensional equality types, such as Agda, with the axioms of uniqueness of identity proofs (UIP) and function extensionality (FUNEXT) as postulates, since extensional Martin-Löf type theory is conservative with respect to intensional Martin-Löf type theory augmented with UIP and FUNEXT [Hofmann 1995a; Kapulkin and Li 2023]. The (unavoidable) cost of this is that some extra transportations along intensional equalities must be inserted manually.

5.2 Functorial Semantics of Signatures

5.2∗1. Type theories, like other flavours of algebraic theories, or languages in general, are invented for *talking about things*, either mathematical objects or intuitive objects in 'the physical world'. Therefore a logical framework is incomplete if it does not provide a notion of *models* of signatures defined in it. In this section, we show how this is done for LccLF in Section 5.1 by means of *functorial semantics* à la Lawvere theories [Lawvere 1963], except that categories with finite products are replaced by locally cartesian closed categories.

5.2∗2. The syntax of LccLF, quotiented by judgemental equalities, forms a *category with families* (CwF) with the extra structures in Figure 5.1, henceforth called an LF-CwF. It can further be proven to be *initial* among all LF-CwFs, similarly to the initiality results of many other dependent type theories in the literature [de Boer 2020; Kaposi et al. 2020; Pitts 2001; Streicher 1991]. In outline, we first define a *partial* interpretation of the raw syntax for every LF-CwF by induction on the raw syntax, and then we show that the partial interpretation is defined on well typed terms by induction on the typing derivation, and finally we show that the the interpretation respects judgemental equalities.

Alternatively, from a more abstract point of view, we can view the typing rules of LccLF as a *generalised algebraic theory* (GAT) [Cartmell 1986; Sterling 2019] or the signature of a *quotient inductive-inductive type* (QIIT) [Altenkirch et al. 2018; Kaposi et al. 2019; Kovács 2023]. Then we can directly take the initial model of this GAT or this QIIT as the definition of the syntax of LccLF. The existence of the initial model is shown by Cartmell [1986] in a set-theoretic metatheory and by Kaposi et al. [2019] in a type-theoretic metatheory. In this way, there is no need to prove initiality manually, since we are essentially using existing logical frameworks (GATs or QIITs) to define our logical framework. In fact, it is folklore that every elementary topos with a natural number object has finitary quotient inductive-inductive types; see Kovács [2023, §4.6] for more discussion.

5.2∗3. Either (i) by constructing the abstract syntax of LccLF from raw syntax and proving the initiality manually or (ii) by taking the initial model as the definition of the abstract syntax of the LF, in what follows we write $\textsc{LfSig}$ for the category of LF-contexts and substitutions between contexts, i.e.,

$$\textsc{Obj}\,\textsc{LfSig} = \{\vdash \Gamma \text{ ctx}\} \quad \text{and} \quad \textsc{Hom}_{\textsc{LfSig}}(\Delta, \Gamma) = \{\Delta \vdash \gamma : \Gamma\},$$

and $Ty_{LF} : \textsc{LfSig}^{\mathrm{op}} \to \textsc{Set}$ for the presheaf of LF-types over contexts,

$$Ty_{LF}(\Gamma) = \{\Gamma \vdash A \text{ type}\},$$

and $Tm_{LF} : (\int Ty_{LF})^{\mathrm{op}} \to \textsc{Set}$ for the presheaf of terms over the category of elements of Ty_{LF}, i.e. for every context Γ and $A \in Tm_{LF}(\Gamma)$,

$$Tm_{LF}(\Gamma; A) = \{\Gamma \vdash a : A\}.$$

The context extension of $\Gamma \in$ LFSIG with $A \in Ty_{LF}(\Gamma)$ is written as just $\Gamma.A \in$ LFSIG, together with the projection substitution $p : \Gamma.A \to \Gamma$.

5.2∗4 Definition. For every signature S, i.e. a context, of LCCLF, its *category of judgements* JDG S is the full subcategory of the slice category LFSIG$/S$ spanned by projection maps $p : S.A \to S$ of context extensions for $S \vdash A : \mathbb{J}$.

5.2∗5. The objects of JDG S can be identified with judgements $S \vdash A : \mathbb{J}$ in the context of S; the morphisms $t : A \to B$ can be identified with functions $S \vdash f : A \to B$. Since the universe $\mathbb{J}$ is closed under precisely the connectives of extensional MLTT (1, Σ, Π, and extensional equality types), the category JDG S is the category of types for extensional MLTT with the additional constants from S.

Consequently, JDG S is locally cartesian closed. In every slice category JDG S/A,

* the terminal object is $(\lambda a.\ a) : A \to A$, where we omit $S \vdash$ for clarity;
* the product of $f : B \to A$ and $g : C \to A$ is $\lambda p.\ f\ (\pi_1\ p) : P \to A$ where

$$P := \Sigma(b : B).\ \Sigma(c : C).\ (f\ b = g\ c);$$

* the exponential of $f : B \to A$ and $g : C \to A$ is $\pi_1 : E \to A$ where

$$\begin{aligned} E &:= \Sigma(a : A).\ B_a \to C_a, \\ B_a &:= \Sigma(b : B).\ f\ b = a, \\ C_a &:= \Sigma(c : C).\ g\ c = a. \end{aligned}$$

5.2∗6 Example. Consider the signature of PCF in Example 5.1∗8. Some examples of objects and morphisms of the category of judgements for PCF are

$$\begin{tikzcd}
 & tm\ \iota \times tm\ o \arrow[d, "\pi_1"] & \\
(tm\ \iota \to tm\ \iota) & tm\ \iota \arrow[loop left, "succ"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma(t : ty).\ tm\ t \arrow[d] \\
 & 1 \arrow[ul, "succ"] \arrow[u, bend left, "0"] \arrow[r, "\iota"] & ty
\end{tikzcd}$$

This category is not the same as the usual category of *contexts* for PCF, since it contains higher-order judgements such as $tm\ \iota \to tm\ \iota$ or $(tm\ \iota \to tm\ \iota) \to tm\ \iota$ that do not correspond to any PCF-contexts. However, the adequacy of the LF encoding of PCF implies that the category of PCF-contexts can be fully faithfully embedded in the category of judgements. Namely, it is the full subcategory spanned by finite products of objects in the set $\{tm\ A \mid A : ty\}$.

5.2*7 Definition. Let S be a signature in LccLF and $\mathscr{C}$ a locally cartesian closed category (LCCC). A *(functorial) model of S in $\mathscr{C}$* is a functor $M : \textsc{Jdg}\, S \to \mathscr{C}$ that preserves the locally cartesian closed structure.

5.2*8. Because the objects of the category of judgements $\textsc{Jdg}\, S$ are generated by a quite intricate induction, it is not straightforward to define a functorial model of S explicitly. We will solve this by using internal languages later, but for now let us sketch a partial example for some intuition.

The following lemma about presheaf categories is well known (see e.g. [nLab authors 2024]) and will be used in the example.

5.2*9 Lemma. For every small category $\mathscr{C}$ and presheaf $A : \mathscr{C}^{\mathrm{op}} \to \textsc{Set}$, there is an equivalence of categories $\textsc{Pr}\,\mathscr{C}/A \cong \textsc{Pr}\,(\int A)$ between the slice category $\textsc{Pr}\,\mathscr{C}/A$ and the presheaf category $\textsc{Pr}\,(\int A)$ over the category of element of A.

5.2*10 Example. Consider the signature of STLC in Example 5.1*6. Let $\mathscr{C}$ be a small cartesian closed category. The category $\mathscr{C}$ has enough structure for interpreting the category of *contexts* of STLC [Lambek and Scott 1986] but not enough for interpreting the category of *judgements* of STLC, since $\mathscr{C}$ may not be locally cartesian closed. However, we can interpret the category of judgement in the presheaf category $\textsc{Pr}\,\mathscr{C}$, which is always locally cartesian closed.

Firstly, we define $Ty \in \textsc{Pr}\,\mathscr{C}$ to be the constant presheaf

$$Ty(\Gamma) = \textsc{Obj}\,\mathscr{C} \qquad\qquad Ty(\gamma) = id,$$

and $Tm \in \textsc{Pr}\,\mathscr{C}$ to be the presheaf defined by

$$Tm(\Gamma) = \{(A, f) \mid A \in \textsc{Obj}\,\mathscr{C}, f : \Gamma \to A\}$$
$$Tm(\gamma) = (A, f) \mapsto (A, \Delta \xrightarrow{\gamma} \Gamma \xrightarrow{f} A)$$

for all $\Gamma, \Delta \in \mathscr{C}$ and $\gamma : \Delta \to \Gamma$. As the names suggest, the projection map $p : Tm \to Ty$ is going to be the interpretation of the morphism

$$\pi_1 : \Sigma(t : ty).\ tm\ t \to ty \in \textsc{Jdg}_{\mathrm{STLC}}.$$

The interpretation of the declaration $\iota : ty$ can be any global element $1 \to Ty$, i.e. any object of $\mathscr{C}$. The interpretation of the declaration $\_\Rightarrow\_ : ty \to ty \to ty$ in $\textsc{Pr}\,\mathscr{C}$ is given by the adjunct of the natural transformation $F : Ty \times Ty \to Ty$:

$$F_\Gamma\,(A, B) = B^A \text{ for all } (A, B) \in Ty \times Ty\ (\Gamma).$$

To interpret the isomorphism pair *abs* and *app*, following the interpretation of MLTT in LCCCs [Hofmann 1997; Seely 1984], we need to construct in the slice category $\textsc{Pr}\,\mathscr{C}/Ty \times Ty$ an isomorphism between the object $F^*p : F^*Tm \to Ty \times Ty$, obtained by pulling back p along F,

$$\begin{tikzcd}
F^*Tm \arrow[r] \arrow[d, "F^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\
Ty \times Ty \arrow[r, "F"] & Ty
\end{tikzcd}$$

and the object $\pi_1^* p \Rightarrow \pi_2^* p$, obtained by taking the exponential of $\pi_1^* p$ and $\pi_2^* p$ in $\textsc{Pr}\,\mathscr{C}/Ty \times Ty$, where $\pi_i^* p$ is respectively obtained by the pullback

$$\begin{tikzcd}
\pi_i^*Tm \arrow[r] \arrow[d, "\pi_i^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\
Ty \times Ty \arrow[r, "\pi_i"] & Ty
\end{tikzcd}$$

We can construct this isomorphism pointwise for each object $\Gamma \in \mathscr{C}$. An element of $Ty \times Ty\ (\Gamma)$ is a pair (A, B) of $\mathscr{C}$-objects, so the presheaf F^*Tm at Γ is the set

$$\{(A, B, f) \mid A, B \in \mathscr{C},\ f : \Gamma \to B^A\}. \tag{5.1}$$

The object $\pi_1^* p \Rightarrow \pi_2^* p$ is harder to compute, but by using Lemma 5.2*9 and the end formula of exponentials in presheaf categories, we can compute that the fiber of $\pi_1^* p \Rightarrow \pi_2^* p$ over $(A, B) \in Ty \times Ty\ (\Gamma)$ is the set

$$\begin{aligned}
&\int_{\Delta \in \mathscr{C}} \prod \mathscr{C}(\Delta, \Gamma).\ \mathscr{C}(\Delta, A) \Rightarrow \mathscr{C}(\Delta, B) \\
\cong\quad &\{\text{powering in } \textsc{Set} \text{ is the same as exponentiating}\} \\
&\int_{\Delta \in \mathscr{C}} \mathscr{C}(\Delta, \Gamma) \Rightarrow \mathscr{C}(\Delta, A) \Rightarrow \mathscr{C}(\Delta, B) \\
\cong\quad &\{\text{by uncurrying}\} \\
&\int_{\Delta \in \mathscr{C}} \mathscr{C}(\Delta, \Gamma) \times \mathscr{C}(\Delta, A) \Rightarrow \mathscr{C}(\Delta, B) \\
\cong\quad &\{\text{ty the universal property of products}\} \\
&\int_{\Delta \in \mathscr{C}} \mathscr{C}(\Delta, \Gamma \times A) \Rightarrow \mathscr{C}(\Delta, B) \\
\cong\quad &\{\text{by Yoneda embedding}\} \\
&\mathscr{C}(\Gamma \times A,\ B)
\end{aligned}$$

which is indeed isomorphic to the fiber of F^*Tm (5.1) over (A, B) in a canonical way. We omit the verification of naturality here.

We have sketched the interpretations of the declarations of STLC from Example 5.1*6 in $\textsc{Pr}\,\mathscr{C}$. These data can be in fact extended to an LCC-functor

$$M : \textsc{Jdg}_{\mathrm{STLC}} \to \textsc{Pr}\,\mathscr{C},$$

but we will not do it here since we will study the general case below.

5.3 Diagrammatic Semantics of Signatures

5.3∗1. Since the category of judgement $\textsc{Jdg}\, S$ is comprised of syntactic entities that are inductively generated, it is natural to expect that $\textsc{Jdg}\, S$ has some universal property so that functorial models of S, i.e. LCC-functors $\textsc{Jdg}\, S \to \mathscr{C}$, correspond to certain structures in $\mathscr{C}$ that we may call *diagrammatic models* of S. The situation should generalise that of Lawvere theories – if $\textsc{Jdg}\, S$ is the Lawvere theory generated by a signature S, a product-preserving functor $\textsc{Jdg}\, S \to \mathscr{C}$ corresponds to an object in $\mathscr{C}$ equipped with the operations from the signature S.

Taking the example of STLC (Example 5.1∗6) again, it is natural to expect that LCCC-functors $\textsc{Jdg}_{\mathrm{STLC}} \to \mathscr{C}$ correspond to diagrams in $\mathscr{C}$ of the shape

$$\begin{tikzcd}
\pi_1^* p \Rightarrow \pi_2^* p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] & \\
Ty \times Ty \arrow[r, "F"] & Ty & 1 \arrow[l, "\iota"']
\end{tikzcd}$$

where $\pi_1^* p \Rightarrow \pi_2^* p$ is the exponential of $\pi_1^* p$ and $\pi_2^* p$ in the slice $\mathscr{C}/Ty \times Ty$, and the square must be a pullback.

To define a notion of diagrammatic models in an LCCC $\mathscr{C}$ for every signature S, we need to perform an induction on the syntax of LccLF, since a signature S is nothing else than an LF context. An induction on the LF is the same as constructing a CwF with the type connectives of the LF.

However, we cannot directly use the LCCC $\mathscr{C}$ as the (underlying category of) the CwF, since $\mathscr{C}$ does not have the structure for interpreting the universe $\mathbb{J}$. The solution is, again, passing to the presheaf category $\textsc{Pr}\,\mathscr{C}$, in which we can construct a *universe* containing (the Yoneda embedding of) objects of $\mathscr{C}$.

5.3.1 Universes in Categories

5.3.1∗1. The concept of universes in *toposes* dates back to Bénabou [1973] and Maurer [1975], and was developed later by Streicher [2005]. A more general account of universes in categories is developed by Voevodsky [2015, 2017] and Kapulkin and Lumsdaine [2021] in the study of homotopy type theory; see also Gratzer [2023, §3.2, 3.3] for an excellent exposition.

In this subsection, we will have a brief digression on universes without going into much technical detail, culminating in the theorem that a universe of $\mathscr{C}$-objects can be constructed in the presheaf category $\textsc{Pr}\,\mathscr{C}$ (Theorem 5.3.1∗11).

5.3.1∗2 Definition. A *universe* in a category $\mathscr{C}$ is simply a morphism $p : \tilde{U} \to U$

equipped with chosen pullbacks along every morphism $A : \Gamma \to U$:

$$\begin{tikzcd} \Gamma.A \arrow[r] \arrow[d, "A^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r, "A"] & U \end{tikzcd}$$

A morphism $A \to B$ in $\mathscr{C}$ is said to be *classified* by a universe $p : \tilde{U} \to U$ if it is a pullback of p along some (not necessarily unique) morphism $B \to U$.

A universe may additionally be equipped with logical structures such as Π-types and Σ-types; see [Kapulkin and Lumsdaine 2021, §1.4] or [Gratzer 2023, §3.2] for details. For example, *binary product* on a universe $p : \tilde{U} \to U$ is a pair of morphisms $prod : U \times U \to U$ and $pair : \tilde{U} \times \tilde{U} \to \tilde{U}$ forming a pullback:

$$\begin{tikzcd} \tilde{U} \times \tilde{U} \arrow[r, "pair"] \arrow[d, "p \times p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ U \times U \arrow[r, "prod"] & U \end{tikzcd} \tag{5.2}$$

5.3.1∗3. Universes abound in logic and type theory.

1. In the category of sets, every Grothendieck universe U determines a universe $\pi_1 : \tilde{U} \to U$ where $\tilde{U}$ is the set of pointed U-small sets:

   $$\tilde{U} = \{(A, a) \mid A \in U, a \in A\},$$

   and $\pi_1(A, a) = A$ is the projection function.

2. For a small category $\mathscr{C}$, every Grothendieck universe U of sets can be lifted to a universe $\pi_1 : \tilde{V} \to V$ in the presheaf category $\textsc{Pr}\,\mathscr{C}$ by the *Hofmann-Streicher lifting* [Hofmann and Streicher 1999]: V maps every $\Gamma \in \mathscr{C}$ to the set of U-valued presheaves over $\mathscr{C}/\Gamma$, and $\tilde{V}$ maps every $\Gamma \in \mathscr{C}$ to the set

   $$\{(A, a) \mid A \in V(\Gamma), a : 1 \to A \in \textsc{Pr}\,(\mathscr{C}/\Gamma)\}.$$

   The actions of V and $\tilde{V}$ on morphisms $\gamma : \Delta \to \Gamma$ is given by precomposing with the functor $(\gamma \cdot -) : (\mathscr{C}/\Delta)^{\mathrm{op}} \to (\mathscr{C}/\Gamma)^{\mathrm{op}}$.

   However, the universe V constructed in this way is not what we want for interpreting the universe of judgements $\mathbb{J}$, because V classifies all (U-small) presheaves rather than just (Yoneda-embedding of) $\mathscr{C}$-objects.

3. Liftings of Grothendieck universes of sets to sheaf toposes in general [Gratzer et al. 2022; Streicher 2005], categories of assemblies, and realizability toposes [Streicher 2005] also exist.

4. A syntactic example of universes is the map $\pi_1 : (A : \mathbb{J}, a : A) \to (A : \mathbb{J})$ in the category $\textsc{LfSig}$ of LF-signatures (5.2∗3). The pullback of π_1 along an arbitrary

morphism $B : S \to (A : \mathbb{J})$ can be chosen to be simply

$$\begin{tikzcd} S.B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (A : \mathbb{J}, a : A) \arrow[d] \\ S \arrow[r, "B"'] & (A : \mathbb{J}) \end{tikzcd}$$

All the examples above can be equipped with structures of Σ-types, Π-types, and extensional equality types.

5.3.1∗4 Definition. For a universe $p : \tilde{U} \to U$ in a category $\mathscr{C}$, its *externalisation* $[p : \tilde{U} \to U]$, or simply $[U]$ when it causes no confusion, is the fibration over $\mathscr{C}$ whose fiber category $[U]_\Gamma$ over every object $\Gamma \in \mathscr{C}$ has as objects $\mathscr{C}$-morphisms $A : \Gamma \to U$. Morphisms $A \to B$ in the fiber $[U]_\Gamma$ are $\mathscr{C}$-morphisms $h : \Gamma.A \to \Gamma.B$ making the following diagram commute:

$$\begin{tikzcd} \Gamma.A \arrow[rr, "h"] \arrow[dr, "A^*p"'] & & \Gamma.B \arrow[dl, "B^*p"] \\ & \Gamma & \end{tikzcd}$$

where $\Gamma.A$ and $\Gamma.B$ arise from pulling back p along A and B respectively. The reindexing functor $\gamma^* : [U]_\Gamma \to [U]_\Delta$ for a morphism $\gamma : \Delta \to \Gamma$ is precomposition:

$$y^*(\Gamma \xrightarrow{A} U) = (\Delta \xrightarrow{\gamma} \Gamma \xrightarrow{A} U).$$

5.3.1∗5. For an LF-signature S, the category $\textsc{Jdg}\,S$ of judgements of S from Definition 5.2∗4 is precisely the fiber over S of the externalisation of the universe

$$\pi_1 : (A : \mathbb{J}, a : A) \to (A : \mathbb{J})$$

in the category $\textsc{LfSig}$ of LF-contexts from 5.2∗3.

5.3.1∗6. When a universe is additionally equipped with logical structures, these structures can be carried over to the externalisation. For example, if the universe is equipped with binary product $\mathit{prod} : U \times U \to U$ as in (5.2), the cartesian product of two objects $A, B : \Gamma \to U$ in the fiber $[U]_\Gamma$ is

$$\Gamma \xrightarrow{\langle A,B \rangle} U \times U \xrightarrow{\mathit{prod}} U$$

Moreover, this choice of cartesian products are strictly preserved by the reindexing functor $\gamma^* : [U]_\Gamma \to [U]_\Delta$ since reindexing is defined to be precomposition.

If the universe U is equipped with Σ-types, Π-types, and extensional equality types, each fiber category of $[U]$ has an LCCC structure given in a way similar to 5.2∗5. The reindexing functor strictly preserves the LCCC structure.

5.3.1∗7. Example 5.2∗10 contains another example of universes: starting from a cartesian closed category $\mathscr{C}$, we constructed in the presheaf category $\textsc{Pr}\,\mathscr{C}$ a morphism $p : Tm \to Ty$, which can be succinctly defined as

$$\begin{tikzcd} \coprod_{A \in \textsc{Obj}\,\mathscr{C}} \mathrm{Y}A \arrow[d, "\coprod_{A \in \textsc{Obj}\,\mathscr{C}} !"'] \\ \coprod_{A \in \textsc{Obj}\,\mathscr{C}} 1 \end{tikzcd} \tag{5.3}$$

This morphism as a universe classifies (the Yoneda embedding of) all projection morphisms in $\mathscr{C}$: for every projection morphism $\Gamma \times A \to \Gamma$ in $\mathscr{C}$, we can choose a morphism $\ulcorner A \urcorner : \mathrm{Y}\Gamma \to Ty$ making a pullback square:

$$\begin{tikzcd} \mathrm{Y}(\Gamma \times A) \arrow[r] \arrow[d, "\mathrm{Y}\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\ \mathrm{Y}\Gamma \arrow[r, "\ulcorner A \urcorner"] & Ty \end{tikzcd}$$

The morphism $\ulcorner A \urcorner$ can just be the one corresponding to the element $A \in Ty(\Gamma)$ by Yoneda lemma, but any object isomorphic to A in $\mathscr{C}$ is equally good, so p only *weakly* classifies projection morphisms of $\mathscr{C}$.

Conversely, for every $\Gamma \in \mathscr{C}$ and morphism $A : \mathrm{Y}\Gamma \to Ty$, the pullback of $p : Tm \to Ty$ along A is (isomorphic to) the Yoneda embedding of the projection map $\pi_1 : \Gamma \times A \to \Gamma$, where we identify the morphism A with an element of $Ty(\Gamma) = \textsc{Obj}\,\mathscr{C}$ by Yoneda lemma.

As a consequence of these observations, the fiber category $[Ty]_1$ of the externalisation over the terminal object is isomorphic to the category $\mathscr{C}$ itself, justifying the view of Ty as a universe of $\mathscr{C}$-objects.

Moreover, in Example 5.2∗10 we saw how exponentials in $\mathscr{C}$ can be lifted to the universe $p : Tm \to Ty$. However, p inherently only supports *simply typed* structures. For an arbitrary type family over Γ, i.e. an arbitrary morphism $f : B \to \Gamma$ in $\mathscr{C}$, there may be no morphism $\mathrm{Y}\Gamma \to Ty$ making a pullback square:

$$\begin{tikzcd} \mathrm{Y}B \arrow[r] \arrow[d, "\mathrm{Y}f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\ \mathrm{Y}\Gamma \arrow[r, dashed, "?"] & Ty \end{tikzcd}$$

5.3.1∗8. The good news is that the dependently typed version of the construction (5.3) exists. Perhaps unexpectedly, what we are looking for is exactly Hofmann [1995b]'s technique for interpreting extensional MLTT in locally cartesian closed categories. Op. cit., Hofmann showed how to construct a CwF (with the structures of extensional MLTT) over an arbitrary LCCC $\mathscr{C}$, but as pointed out by Fiore [2012] and Awodey [2018], a CwF structure $\langle Ty, Tm \rangle$ over $\mathscr{C}$ is precisely a universe

$p : Tm \to Ty$ in $\mathrm{PR}\,\mathscr{C}$ that is a *representable morphism*, a perspective known as *natural models* of dependent type theories [Awodey 2018; Newstead 2018].

5.3.1*9 Definition. A morphism $p : Tm \to Ty$ in a presheaf category $\mathrm{PR}\,\mathscr{C}$ is called *representable* if for every $\Gamma \in \mathscr{C}$ and $A : \mathrm{Y}\Gamma \to Ty$, there is an object $\Gamma.A \in \mathscr{C}$ and a morphism $f : \Gamma.A \to \Gamma$ in $\mathscr{C}$ making a pullback square:

$$\begin{tikzcd} \mathrm{Y}\Gamma.A \arrow[r] \arrow[d, "\mathrm{Y}f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Tm \arrow[d, "p"] \\ \mathrm{Y}\Gamma \arrow[r, "A"] & Ty \end{tikzcd}$$

5.3.1*10. Two decades later after Hofmann [1995b]'s construction of CwFs from LCCCs, Awodey [2018] found another construction of CwFs, known as the *local universe* construction, which additionally supports *intensional* equality types. Although we do not need intensional equalities, the local universe construction is worth a mention for its remarkable elegance: given a small category $\mathscr{C}$, the local universe construction is the following representable map:

$$\begin{tikzcd} \coprod_{f \in \mathrm{MOR}\,\mathscr{C}} \mathrm{Y}(\mathrm{dom}\, f) \arrow[d, "\coprod_{f \in \mathrm{MOR}\,\mathscr{C}} \mathrm{Y}f"'] \\ \coprod_{f \in \mathrm{MOR}\,\mathscr{C}} \mathrm{Y}(\mathrm{cod}\, f) \end{tikzcd} \tag{5.4}$$

The logical structures on $\mathscr{C}$, such as Π and Σ types, can be lifted to this representable map (as a universe) as well. Using either Hofmann [1995b]'s construction or the local universe construction, we have the following result.

5.3.1*11 Theorem (Awodey 2018; Hofmann 1995b). *For every small locally cartesian closed category $\mathscr{C}$, there is a universe $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$ in $\mathrm{PR}\,\mathscr{C}$ such that* (1) *p is representable;* (2) *every morphism $f : A \to B$ in $\mathscr{C}$ is classified by p;* (3) *p supports* 1*, Σ, Π, and extensional equality types.*

5.3.1*12. In the situation of Theorem 5.3.1*11, by the representability of p, the fiber $[U_{\mathscr{C}}]_1$ of the externalisation over the terminal object 1 contains only representable objects. And since p classifies (the Yoneda embedding of) all morphisms of $\mathscr{C}$, it in particularly classifies all morphisms $\mathrm{Y}A \to \mathrm{Y}1_{\mathscr{C}} = 1$. Therefore the fiber category $[U_{\mathscr{C}}]_1$ is equivalent to the LCCC $\mathscr{C}$.

5.3.2 Diagrammatic Models

5.3.2*1. By Theorem 5.3.1*11, $\mathrm{PR}\,\mathscr{C}$ has the structure for interpreting the universe of judgements $\mathbb{J}$ and the type connectives on $\mathbb{J}$. The presheaf category $\mathrm{PR}\,\mathscr{C}$ can interpret the unit type and Σ-types of the LF as usual [Hofmann 1997]. Thus we can turn $\mathrm{PR}\,\mathscr{C}$ into an LF-CwF. Since the abstract syntax of the LF is initial among

ZY: The step of constructing a CwF with the *universe type* from a category with universes is omitted. Should we add this?

all LF-CwFs (5.2*2), there is a unique LF-CwF homomorphism, which consists of (1) a functor interpreting LF-signatures (i.e. LF-contexts) as $\mathscr{C}$-presheaves

$$[\![-]\!] : \textsc{LfSig} \longrightarrow \textsc{Pr}\,\mathscr{C} \tag{5.5}$$

and (2) mappings from LF-types/terms to $\textsc{Pr}\,\mathscr{C}$-types/terms that strictly preserve all operations, which we also denote by $[\![-]\!]$.

5.3.2*2 Definition. A *diagrammatic model* of an LF-signature S in an LCCC $\mathscr{C}$ is a global element $m : 1 \to [\![S]\!]$ of the interpretation of S in $\textsc{Pr}\,\mathscr{C}$.

5.3.2*3. Diagrammatic models are more ergonomic to work with than functorial models because $\textsc{Pr}\,\mathscr{C}$ as a presheaf topos has a very rich structure that we can manipulate using a type theoretic language. In this way, a diagrammatic model of S in $\mathscr{C}$ is a closed element of the record type $[\![S]\!]$ in the internal language of $\textsc{Pr}\,\mathscr{C}$, containing all fields of S and with $\mathbb{J}$ replaced by its interpretation $U_{\mathscr{C}}$.

5.3.2*4 Example. The signature of barebone type theory from Example 5.1*5 is interpreted as the presheaf $[\![\mathrm{BTT}]\!] \in \textsc{Pr}\,\mathscr{C}$ denoted by the record type

$$\begin{array}{l} \textsc{record}\ [\![\mathrm{BTT}]\!]\ \textsc{where} \\ \quad ty\ : U_{\mathscr{C}} \\ \quad tm : ty \to U_{\mathscr{C}} \end{array}$$

A closed element of this record consists of (1) a morphism $A : 1 \to U_{\mathscr{C}}$, which gives rise to an object $\tilde{A}$ in $\mathscr{C}$ by the representability of $\tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$, and (2) a morphism $B : \mathrm{Y}\tilde{A} \to U_{\mathscr{C}}$ which gives rise to a morphism $\tilde{B} \to \tilde{A}$ in $\mathscr{C}$:

$$\begin{tikzcd} \mathrm{Y}\tilde{B} \arrow[d] \arrow[dr] & \\ \mathrm{Y}\tilde{A} \arrow[r] \arrow[d] \arrow[dr, "B"] & \tilde{U}_{\mathscr{C}} \arrow[d] \\ 1 \cong \mathrm{Y}1_{\mathscr{C}} \arrow[r, "A"'] & U_{\mathscr{C}} \end{tikzcd}$$

Thus a diagrammatic model of BTT in a locally cartesian closed category $\mathscr{C}$ gives rise to a morphism $\tilde{B} \to \tilde{A}$ in $\mathscr{C}$.

5.3.2*5. In Definition 5.2*4, the category $\textsc{Jdg}\,S$ of judgements for a signature S is defined to be the full subcategory of the slice $\textsc{LfSig}/S$ spanned by projections $S.A \to S$ for judgements $S \vdash A : \mathbb{J}$. Every such judgement is sent by the interpretation functor $[\![-]\!] : \textsc{LfSig} \to \textsc{Pr}\,\mathscr{C}$ (5.5) to a morphism $[\![A]\!] : [\![S]\!] \to U_{\mathscr{C}}$, which is exactly an object in the fiber of the externalisation $[U_{\mathscr{C}}]$ over $[\![S]\!]$. Since $[\![-]\!]$ is a homomorphism of LF-CwFs, it (strictly) preserves context extensions, so the context projection $S.A \to S$ is sent to the morphism $[\![S]\!].[\![A]\!] \to [\![S]\!]$ in $\textsc{Pr}\,\mathscr{C}$:

$$\begin{tikzcd} S.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (B : \mathbb{J}, b : B) \arrow[d] \\ S \arrow[r, "A"'] & (B : \mathbb{J}) \end{tikzcd} \quad \overset{[\![-]\!]}{\longmapsto} \quad \begin{tikzcd} {[\![S]\!].[\![A]\!]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U}_{\mathscr{C}} \arrow[d] \\ {[\![S]\!]} \arrow[r, "{[\![A]\!]}"'] & U_{\mathscr{C}} \end{tikzcd}$$

Therefore $[\![-]\!]$ sends the morphisms of $\textsc{Jdg}\,S$ to morphisms of $[U_{\mathscr{C}}]_{[\![S]\!]}$ as well. In conclusion, we have a functor for every LCCC $\mathscr{C}$

$$\mathrm{G}_{\mathscr{C}} : \textsc{Jdg}\,S \longrightarrow [U_{\mathscr{C}}]_{[\![S]\!]},$$

which preserves LCCC structures since the type connectives giving the LCCC structures of $\textsc{Jdg}\,S$ 5.2*5 are preserved by interpretation.

5.3.2*6 Definition. Every diagrammatic model $m : 1 \to [\![S]\!]$ in an LCCC $\mathscr{C}$ determines a functorial model $F(m) := m^* \circ \mathrm{G}_{\mathscr{C}} : \textsc{Jdg}\,S \to \mathscr{C}$ by composing $\mathrm{G}_{\mathscr{C}}$ with the reindexing functor $m^* : [U_{\mathscr{C}}]_{[\![S]\!]} \to [U_{\mathscr{C}}]_1 \cong \mathscr{C}$:

$$\begin{tikzcd} \textsc{Jdg}\,S \arrow[r, "\mathrm{G}_{\mathscr{C}}"] \arrow[dr, "F(m)"'] & {[U_{\mathscr{C}}]_{[\![S]\!]}} \arrow[d, "m^*"] \\ & \mathscr{C} \cong [U_{\mathscr{C}}]_1 \end{tikzcd}$$

5.3.2*7. Using $F(m)$ will be our go-to way of constructing functorial models in the rest of the thesis. Of course, if the LCCC $\mathscr{C}$ that we start with already has a universe U that can model $\mathbb{J}$, for example, when $\mathscr{C}$ is a topos, then we may also directly construct diagrammatic models classified by U in the language of $\mathscr{C}$ itself, which is a just sublanguage of that of $\textsc{Pr}\,\mathscr{C}$.

5.3.3 From Functorial Models to Diagrammatic Models

5.3.3*1. Unsurprisingly, from the other direction a functorial model $\textsc{Jdg}\,S \to \mathscr{C}$ induces a diagrammatic model as well, although this direction is more involved.

5.3.3*2 Lemma. Let S be a signature and $\mathscr{C}$ an LCCC. Every functorial model $M : \textsc{Jdg}\,S \to \mathscr{C}$ determines a diagrammatic model $D(M) : 1 \to [\![S]\!]$ and a natural isomorphism $\phi_M : F(D(M)) \cong M : \textsc{Jdg}\,S \to \mathscr{C}$.

5.3.3*3. Before proving 5.3.3*2, we first observe that every LF-signature S is isomorphic (in the category $\textsc{LfSig}$ of LF-signatures) to a *standard signature*, which is inductively defined to be either

ZY: 5.3.3*3 is doing a lot of work. It should have a proof.

1. the empty signature,
2. a signature $(T, a : A)$ for some standard T and $T \vdash A : \mathbb{J}$, or
3. a signature $(T, B : A \to \mathbb{J})$ for some standard T and $T \vdash A : \mathbb{J}$.

Moreover, every judgement $S \vdash A : \mathbb{J}$ in the context of a standard signature S is equal (in the equational theory of the LF) to a *standard judgement*, which is inductively defined to be either

1. $S \vdash B\ a : \mathbb{J}$ for some declaration $B : A \to \mathbb{J}$ in S and $S \vdash a : A$, or
2. the type formers 1, Σ, Π, $a = b$ of $\mathbb{J}$ applied to standard judgements.

These claims can be shown by induction on the syntax of the LF. Note that they are not the same as *normalisation* of the LF – we do not claim terms have any standard or normal form (which is in indeed not true because *extensional* equality types do not enjoy normalisation [Hofmann 1995a, §3.2.2]).

When proving statements $P(S)$ about LF-signatures S that are invariant along isomorphisms or defining constructions $C(S)$ for LF-signatures that can be transported along isomorphisms, we conveniently only need to prove or construct for standard signatures and consider only standard judgements.

*Proof of 5.3.3*2.* The required constructions $D(M)$ and ϕ_M in the statement of Lemma 5.3.3*2 can be transported along isomorphisms $S \cong S'$of LF-signatures, so we can assume S is standard. Then we construct $D(M)$, the components of ϕ_M, and show the naturality of ϕ_M by a simultaneous induction on the structure of standard signatures S, standard judgements A in S, and terms $S \vdash a : A$.

Part 1. We first construct D for every signature S.

Case 1.1. If S is the empty signature, $[\![S]\!]$ is the terminal presheaf, and thus there is a unique choice for $D(M) : 1 \to [\![S]\!]$.

Case 1.2. If $S = (T, a : A)$ for some $T \vdash A : \mathbb{J}$, we have an inclusion functor $i : \textsc{Jdg}\,T \to \textsc{Jdg}\,S$ that sends every judgement $(T \vdash B : \mathbb{J}) \in \textsc{Jdg}\,T$ to its weakening $(S \vdash B : \mathbb{J}) \in \textsc{Jdg}\,S$. By composing this functor with $M : \textsc{Jdg}\,S \to \mathscr{C}$ we have a functorial model $M \circ i : \textsc{Jdg}\,T \to \mathscr{C}$ of T, which further gives rise to a diagrammatic model $D(M \circ i)$ of T by induction. Our goal is to construct a morphism $D(M) : 1 \to [\![S]\!]$ making the left triangle below commute:

$$\begin{tikzcd}
 & {[\![S]\!] = [\![T]\!].[\![A]\!]} \arrow[r] \arrow[d, "p_S"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U}_{\mathscr{C}} \arrow[d, "p"] \\
1 \arrow[ur, dashed] \arrow[r, "D(M \circ i)"'] & {[\![T]\!]} \arrow[r, "{[\![A]\!]}"'] & U_{\mathscr{C}}
\end{tikzcd} \tag{5.6}$$

In the category $\textsc{Jdg}\,S$, we have a morphism

$$(T, a : A, 1 \vdash a : A) : (S \vdash 1 : \mathbb{J}) \to (S \vdash A : \mathbb{J}) \tag{5.7}$$

which is mapped by $M : \textsc{Jdg}\,S \to \mathscr{C}$ to a morphism

$$M(a) : 1_{\mathscr{C}} \cong M(S \vdash 1 : \mathbb{J}) \to M(S \vdash A : \mathbb{J})$$

in $\mathscr{C} \cong [U_{\mathscr{C}}]_1$. By the inductive hypothesis, we have a natural isomorphism

$$\phi_{M\circ i} : F(D(M \circ i)) \cong M \circ i,$$

so $M(i(T \vdash A : \mathbb{J}))$, which is exactly $M(S \vdash A : \mathbb{J})$, is isomorphic to the pullback of $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$ along $[\![A]\!] \cdot D(M \circ i)$, giving rise to pullback squares as below:

$$\begin{tikzcd}
M(S \vdash A : \mathbb{J}) \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![S]\!] = [\![T]\!].[\![A]\!]} \arrow[r] \arrow[d, "p_S"] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U}_{\mathscr{C}} \arrow[d, "p"] \\
M(S \vdash 1 : \mathbb{J}) \cong 1 \arrow[r, "D(M\circ i)"'] \arrow[u, bend left, "M(a)"] & {[\![T]\!]} \arrow[r, "{[\![A]\!]}"'] & U_{\mathscr{C}}
\end{tikzcd} \tag{5.8}$$

We let the desired diagrammatic model $D(M)$ be $g \cdot M(a) : 1 \to [\![S]\!]$.

Case 1.3. If $S = (T, B : A \to \mathbb{J})$ for some $T \vdash A : \mathbb{J}$, the projection morphism $p_S : [\![T, B : A \to \mathbb{J}]\!] \to [\![S]\!]$ is the exponential object in the slice category $\mathrm{Pr}\,\mathscr{C}/[\![T]\!]$ from $p_{T.A} : [\![T]\!].[\![A]\!] \to [\![T]\!]$ to $\pi_1 : [\![T]\!] \times U_{\mathscr{C}} \to [\![T]\!]$. We would like to define $D(M) : 1 \to [\![T, B : A \to \mathbb{J}]\!]$ making the left triangle below commute,

$$\begin{tikzcd}
{[\![T, B : A \to \mathbb{J}]\!]} \arrow[drr, "p_S"] & & {[\![T]\!] \times U_{\mathscr{C}}} \arrow[d, dashed, "\pi_1"] & {[\![T]\!].[\![A]\!]} \arrow[dl, "p_{T.A}"'] \arrow[r] & \tilde{U}_{\mathscr{C}} \arrow[dl, "p"] \\
1 \arrow[u, dashed, "D(M)"] \arrow[rr, "D(M\circ i)"'] & & {[\![T]\!]} \arrow[r, "{[\![A]\!]}"'] & U_{\mathscr{C}} &
\end{tikzcd}$$

where $i : \mathrm{Jdg}\,T \to \mathrm{Jdg}\,S$ is the weakening functor, and $D(M \circ i)$ is the diagrammatic model of the smaller context T obtained from the inductive hypothesis. By the universal property of $[\![T, B : A \to \mathbb{J}]\!]$ as the exponential, we need to construct a morphism of $D(M \circ i) \times p_{T.A} \to \pi_1$ in the slice category over $[\![T]\!]$. The product $D(M \circ i) \times p_{T.A}$ is the pullback of $p_{T.A}$ along $D(M \circ i)$, which is isomorphic to the object $M(i(T \vdash A : \mathbb{J}))$ via $\phi_{M\circ i} : F(D(M \circ i)) \cong M \circ i$:

$$\begin{tikzcd}
M(i(T \vdash A : \mathbb{J})) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\![S]\!] = [\![T]\!].[\![A]\!]} \arrow[r] \arrow[d, "p_{T.A}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U}_{\mathscr{C}} \arrow[d, "p"] \\
1 \arrow[r, "D(M\circ i)"'] & {[\![T]\!]} \arrow[r, "{[\![A]\!]}"'] & U_{\mathscr{C}}
\end{tikzcd}$$

Now our goal is to construct a morphism $M(i(T \vdash A : \mathbb{J})) \to U_{\mathscr{C}}$.

Back in the category $\mathrm{Jdg}\,S$, we have the judgement $S \vdash \Sigma\, A\, B : \mathbb{J}$, and the projection $\pi_1^{\Sigma AB} : \Sigma\, A\, B \to A$; henceforth we omit the context $S \vdash$ on objects. The projection map is sent by $M : \mathrm{Jdg}\,S \to \mathscr{C} \cong [U_{\mathscr{C}}]_1$ to a morphism in $\mathscr{C}$. Since the universe $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$ (weakly) classifies all $\mathscr{C}$-morphisms, the morphism

$M\pi_1^{\Sigma AB}$ gives us some $\ulcorner B \urcorner : M(A) \to U_{\mathscr{C}}$ and a pullback square:

$$\begin{tikzcd} M(\Sigma\ A\ B) \arrow[r] \arrow[d, "M\pi_1^{\Sigma AB}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \tilde{U}_{\mathscr{C}} \arrow[d, "p"] \\ M(A) \arrow[r, "\ulcorner B \urcorner"'] & U_{\mathscr{C}} \end{tikzcd} \tag{5.9}$$

The morphism $\ulcorner B \urcorner$ fulfils our goal $M(i(T \vdash A : \mathbb{J})) = M(A) \to U_{\mathscr{C}}$.

Part 2. Now we define the component of $\phi_M : FDM \cong M$ at every (standard) judgement in a (standard) signature S.

Case 2.1. For the judgement $S \vdash B\ a : \mathbb{J}$ where $S = (T, B : A \to \mathbb{J}, R)$ for some T and R, $T \vdash A : \mathbb{J}$, and $S \vdash a : A$, we have a pullback diagram in $\textsc{Jdg}\,S$:

$$\begin{tikzcd} B\ a \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Sigma\ A\ B \arrow[d, "\pi_1^{\Sigma AB}"] \\ 1 \arrow[r, "a"'] & A \end{tikzcd}$$

Since M is an LCCC functor, it preserves pullbacks and the terminal object, so we have a pullback square in $\mathscr{C}$:

$$\begin{tikzcd} M(B\ a) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & M(\Sigma\ A\ B) \arrow[d, "M\pi_1^{\Sigma AB}"] \\ 1 \arrow[r, "Ma"'] & M(A) \end{tikzcd}$$

On the other hand, in Case 1.3 above, we have defined the B-component of the diagrammatic model $D(M)$ to be the code $\ulcorner B \urcorner$ of the morphism $M\pi_1^{\Sigma AB}$ as in the diagram (5.9). Hence, unfolding the definition of F, $FDM(B\ a)$ will be the pullback of $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$ along the following morphism:

$$1 \xrightarrow{F(D(M))(a)} (FDM)(A) \xrightarrow{(\phi_M)_A} MA \xrightarrow{\ulcorner B \urcorner} U_{\mathscr{C}}$$

Using the naturality of $(\phi_M)_A$, this morphism is the same as $1 \xrightarrow{M(a)} MA \xrightarrow{\ulcorner B \urcorner} U_{\mathscr{C}}$. Therefore both $M(A)$ and $FDM(A)$ are the pullback of p along $\ulcorner B \urcorner \cdot M(a)$, so they are isomorphic in a canonical way.

Case 2.2. For a judgement A that is some type former of $\mathbb{J}$ applied to (smaller) standard judgements, $M(A)$ and $D(F(M))(A)$ are both LCC-functors so they preserve these type formers. By the universal properties of these type formers, $M(A)$ and $D(F(M))(A)$ are isomorphic in a canonical way.

Part 3. We also need to show that the family of morphisms

$$\phi_M(A) : FDM(A) \cong M(A)$$

is natural in $A \in \textsc{Jdg}\,S$. Because $\textsc{Jdg}\,S$ has exponentials, which are preserved by

M and FDM, it is sufficient to show that for every $S \vdash A : \mathbb{J}$ and $S \vdash a : A$, there is a commutative triangle:

$$\begin{tikzcd} 1 \arrow[r, "FDM(a)"] \arrow[dr, "M(a)"'] & FDM(A) \arrow[d, "(\phi_M)_A"] \\ & M(A) \end{tikzcd}$$

If a is a variable, this follows from the definition of $D(T, a : A)$ in Case 1.2 above. If a is other term formers, it follows from the fact that M and FDM as LCC-functors both preserve these term formers. □

5.4 Equivalence of Functorial and Diagrammatic Models

5.4*1. We have now mappings F (Definition 5.3.2*6) from diagrammatic models to functorial models, and vice versa D (Lemma 5.3.3*2). Moreover, there is an isomorphism $F(D(M)) \cong M$ for every functorial model $M : \textsc{Jdg}\, S \to \mathscr{C}$. Naturally, we would expect $D(F(m)) \cong m$ for every diagrammatic model m as well, and then diagrammatic and functorial models will be in bijection up to isomorphisms. But we do not have a notion of isomorphisms of diagrammatic models yet, so we will define it in this section, which turns out to be more interesting a task than it sounds.

5.4*2 Example. Let us still begin with some small examples for intuition. Consider the signature $(A : \mathbb{J})$ of one judgement and nothing else. Diagrammatic models of it in an LCCC $\mathscr{C}$ are morphisms $m : 1 \to U_{\mathscr{C}}$ in $\textsc{Pr}\,\mathscr{C}$, which give rise to objects A in $\mathscr{C}$ by Theorem 5.3.1*11 in a surjective way (but different m may give rise to the same object in $\mathscr{C}$). An isomorphism between two models m_1 and m_2 in this case ought to be an isomorphism $i : A_1 \to A_2$ in $\mathscr{C}$ between the $\mathscr{C}$-objects A_1 and A_2 induced by m_1 and m_2 respectively.

Now suppose the signature is extended to $(A : \mathbb{J}, f : (A \to A) \to A)$. Then every diagrammatic model gives rise to an object A together with a morphism $f : (A \Rightarrow A) \to A$ in $\mathscr{C}$, where $A \Rightarrow A$ denotes the exponential. Now an isomorphism between two diagrammatic models should be a $\mathscr{C}$-isomorphism $i : A_1 \to A_2$ that commutes with f:

$$\begin{tikzcd} A_1 \Rightarrow A_1 \arrow[r, "f_1"] \arrow[d, "i^{-1} \Rightarrow i"'] & A_1 \arrow[d, "i"] \\ A_2 \Rightarrow A_2 \arrow[r, "f_2"'] & A_2 \end{tikzcd}$$

The fact that we have dependent functions in LF signatures is why we only consider isomorphisms rather than homomorphisms of diagrammatic models.

Suppose that the signature is further extended with a family of judgements $B : A \to \mathbb{J}$ indexed by A. A diagrammatic model now further induces an $\mathscr{C}$-object B with a morphism $g : B \to A$. An isomorphism of diagrammatic models should now further include a $\mathscr{C}$-isomorphism $j : B_1 \to B_2$ that commutes with i:

$$\begin{tikzcd}
B_1 \arrow[r, "j"] \arrow[d, "g_1"'] & B_2 \arrow[d, "g_2"] \\
A_1 \arrow[r, "i"'] & A_2
\end{tikzcd}$$

Finally, if a signature is extended with a declaration of an equation, the notion of isomorphisms between diagrammatic models should remain unchanged, since in LCCCs there is not any higher-dimensional coherence between equalities.

5.4*3. Diagrammatic models in an LCCC $\mathscr{C}$ are defined by interpreting an LF-signature S as a presheaf $[\![S]\!] \in \textsc{Pr}\,\mathscr{C}$ (Definition 5.3.2*2). In the internal language of the presheaf topos $\textsc{Pr}\,\mathscr{C}$, a presheaf is a 'set'. Now that we are interested in isomorphisms of models, sets are no longer sufficient, and instead, we would like to use groupoids as our interpretation. More precisely, we plan to interpret every LF-signature S as an *internal groupoid* in $\textsc{Pr}\,\mathscr{C}$:

$$\begin{tikzcd}
{[\![S]\!]^{\cong} \times_{[\![S]\!]} [\![S]\!]^{\cong}} \arrow[r, "comp"'] & {[\![S]\!]^{\cong}} \arrow[loop above, "inv"] \arrow[d, "\langle s,t \rangle"] & {[\![S]\!]} \arrow[l, "id"] \\
& {[\![S]\!] \times [\![S]\!]} &
\end{tikzcd}$$

whose object part is exactly the presheaf $[\![S]\!]$ in the earlier interpretation (5.3.2*1). Then, global elements $i : 1 \to [\![S]\!]^{\cong}$ of the morphism part of the groupoid will be defined as isomorphisms between diagrammatic models $s \cdot i$ and $t \cdot i : 1 \to [\![S]\!]$.

5.4*4. One way to carry out the plan above is to follow Hofmann and Streicher [1998]'s celebrated *groupoid model* of Martin-Löf type theory internally in the language of the presheaf topos $\textsc{Pr}\,\mathscr{C}$, rather than in the ambient set theory.

In outline, in the language of $\textsc{Pr}\,\mathscr{C}$, there is a groupoid $U^{\cong}_{\mathscr{C}}$ whose objects have the type $U_{\mathscr{C}}$ and the morphisms between $A, B : U_{\mathscr{C}}$ have the type of isomorphisms $\tilde{U}_{\mathscr{C}}(A) \cong \tilde{U}_{\mathscr{C}}(B)$, where $\tilde{U}_{\mathscr{C}}$ is the decoding type family for the universe $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$. This groupoid provides the interpretation for the universe $\mathbb{J}$. For every $A : U_{\mathscr{C}}$, the type $\tilde{U}_{\mathscr{C}}(A)$ can be regarded as a *discrete* groupoid, so $U^{\cong}_{\mathscr{C}}$ still models Π, Σ, and *extensional* equalities. Π and Σ types outside $\mathbb{J}$ in the LF are interpreted in the same way as Hofmann and Streicher [1998]. The result of this construction would then be an *internal* LF-CwF in $\textsc{Pr}\,\mathscr{C}$, whose externalisation over the terminal object $1 \in \textsc{Pr}\,\mathscr{C}$ will then the (ordinary) LF-CwF that interprets an LF-signature S as a groupoid of diagrammatic models.

5.4∗5. While the approach outlined above is feasible, there is a more direct approach that we will follow instead. First, we notice that in virtually all (many-sorted) logical frameworks the notions of homomorphisms/isomorphisms of models of a theory S can be expressed as another theory that contains (1) two copies of the declarations of S and (2) new declarations for the homomorphisms/isomorphisms between the basic sorts of the two copies of S, together with (3) equations asserting the homomorphic properties. This is also the case for LccLF. Take the signature $S = (A : \mathbb{J}, f : (A \to A) \to A)$ for example; in the LF we have the following signature $S^{\cong}$ of *isomorphisms of S-models*:

$$\begin{gathered} A_1 : \mathbb{J} \qquad f_1 : (A_1 \to A_1) \to A_1 \qquad A_2 : \mathbb{J} \qquad f_2 : (A_2 \to A_2) \to A_2 \\ i : A_1 \cong A_2 \qquad\qquad \_ : (i.\mathit{fwd} \cdot f_1) = (\lambda g.\ f_2\ (i.\mathit{fwd} \cdot g \cdot i.\mathit{bwd})) \end{gathered}$$

where $A_1 \cong A_2$ is the judgement of isomorphisms defined in 5.1∗7 and $f \cdot g$ means function composition $\lambda x.\ f\ (g\ x)$. In the category $\textsc{LfSig}$ of LF-signatures, there are two morphisms $s, t : S^{\cong} \to S$ that project out (A_1, f_1) and (A_2, f_2) respectively. Similarly, we have morphisms *inv*, *id*, *comp* in $\textsc{LfSig}$ for the inverse, identity, composition of S-isomorphisms, assembling to an internal groupoid:

$$\begin{tikzcd} S^{\cong} \times_S S^{\cong} \arrow[r, "comp"'] & S^{\cong} \arrow[loop above, "inv"] \arrow[d, "\langle s,t \rangle"] & S \arrow[l, "id"'] \\ & S \times S & \end{tikzcd}$$

where $S^{\cong} \times_S S^{\cong}$ is the signature of three copies of S and two isomorphisms in between. The interpretation of this internal groupoid by $[\![-]\!] : \textsc{LfSig} \to \textsc{Pr}\,\mathscr{C}$ for every LCCC $\mathscr{C}$ is precisely the internal groupoid that we wanted in 5.4∗3.

Motivated by the discussion above, we would like to define an internal groupoid $\langle S, S^{\cong} \rangle$ for every LF-signature S, which can be done inductively together with some related conditions on judgements and terms in a signature.

5.4∗6 Lemma. The syntax of LccLF satisfies the following statements:

1. Every LF-signature S determines a judgement of *S-isomorphisms*

$$M_1 : S,\ M_2 : S \vdash S^{\cong} : \mathbb{J}$$

in which we reuse the name S for the iterative Σ-type of the fields of S. Moreover, there are terms *inv*, *id*, *comp* of the following types and they satisfy (definitionally in the LF) the axioms of groupoids:

$$\begin{aligned} M_1, M_2 : S \vdash \ & inv_S &: S^{\cong}[M_1, M_2] \to S^{\cong}[M_2, M_1] \\ M_1 : S \vdash \ & id_S &: S^{\cong}[M_1, M_1] \\ M_1, M_2, M_3 : S \vdash \ & comp_S &: S^{\cong}[M_1, M_2] \to S^{\cong}[M_2, M_3] \to S^{\cong}[M_1, M_3] \end{aligned}$$

where square brackets mean substitution.

2. Every judgement $S \vdash A : \mathbb{J}$ over a signature S determines a term coe_A for *coercing* along S-isomorphisms

$$M_1, M_2 : S \vdash coe_A : S^{\cong}[M_1, M_2] \to A[M_1] \to A[M_2]$$

such that coe_A is functorial, i.e., it preserves id_S and $comp_S$ of $S^{\cong}$. Conceptually, this amounts to say that every judgement in S determines a $\mathbb{J}$-valued presheaf over the groupoid $S^{\cong}$ in the LF.

3. Every term $S \vdash a : A$ of a judgement $S \vdash A : \mathbb{J}$ over a signature S determines a term coh_a showing the *coherence* of coercion:

$$M_1, M_2 : S \vdash coh_a : (i : S^{\cong}[M_1, M_2]) \to coe_A\ i\ a[M_1] = a[M_2]$$

Conceptually, this amounts to say that every term a determines a natural transformation from the constant $\mathbb{J}$-presheaf to 1 to the presheaf A.

Proof. Following 5.3.3*3, it is sufficient to show these statements only for standard signatures S and standard judgements, since the judgements or terms required by these statements can be transported along isomorphisms of LF-signatures. We show these statements simultaneously by induction on the structure of standard signatures, standard judgements, and terms.

Part 1. We start with defining a groupoid $S^{\cong}$ for every signature S, with the inductive hypothesis that all the claims above about signatures, judgements and terms are already shown for structurally smaller cases.

Case 1.1. If S is the empty signature, we let $S^{\cong}$ be the unit type 1 together with the trivial groupoid structure.

Case 1.2. If $S = (T, a : A)$ for some $T \vdash A : \mathbb{J}$, in the context of $M_1, M_2 : S$, we will write $M_i.T$ and $M_i.a$ for the first and second projections of $M_i : S$, and we let $M_1, M_2 : S \vdash S^{\cong} : \mathbb{J}$ be the judgement

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big(coe_A[M_1.T, M_2.T]\ i\ M_1.a = M_2.a\big).$$

The groupoid structure on this $S^{\cong}$ is defined by that of $T^{\cong}$ and the assumption that coe_A preserves the structure id_T and $comp_T$ of $T^{\cong}$. Conceptually, this definition of $S^{\cong}$ is the category of elements for the presheaf determined by $T \vdash A : \mathbb{J}$.

Case 1.3. If $S = (T, B : A \to \mathbb{J})$ for some $T \vdash A : \mathbb{J}$, in the context of $M_1, M_2 : S$, we will write $A_1, A_2 : \mathbb{J}$ for $A[M_1]$ and $A[M_2]$ respectively and similarly B_1 and

B_2 for $B[M_1]$ and $B[M_2]$ respectively. We define $M_1, M_2 : S \vdash S^{\cong} : \mathbb{J}$ to be

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big((a_1 : A_1) \to (B_1\ a_1 \cong B_2\ (coe_A\ i\ a_1))\big)$$

The groupoid structure on this $S^{\cong}$ is defined using that of $T^{\cong}$ and the evident groupoid structure on isomorphisms of judgements in $\mathbb{J}$.

Part 2. Now we construct coe_A for every possibility of standard judgements.

Case 2.1. If the judgement is $S \vdash B\ a : \mathbb{J}$ for some variable $B : A \to \mathbb{J}$ in the context S and $S \vdash a : A$, we need to define the coercion:

$$M_1, M_2 : S \vdash coe_{B\ a} : S^{\cong} \to B_1\ a_1 \to B_2\ a_2$$

where a_i and B_i are $a[M_i]$ and $B[M_i]$ as usual. Let $S = (T, B : A \to \mathbb{J}, S')$. In the context of $M_1, M_2 : S$, given any $i : S^{\cong}$, by the definition of $(T, B : A \to \mathbb{J})^{\cong}$ in 1.3, we can project out from $i : S^{\cong}$ to an element

$$p_i : \Sigma(j : T^{\cong}).\ (a_1 : A_1) \to (B_1\ a_1 \cong B_2\ (coe_A\ j\ a_1))$$

Now given any $b_1 : B_1\ a_1$, using $(\pi_2\ p_i\ a_1).\mathit{fwd}$, we get an element of type $B_2\ (coe_A\ (\pi_1\ p_i)\ a_1)$. Use the coherence $coh_a\ i : coe_A\ (\pi_1\ p_i)\ a_1 = a_2$, we get an element of $B_2\ a_2$ as needed.

Case 2.2. The case of unit judgement is simple. There is a unique choice of

$$M_1, M_2 : S \vdash coe_1 : S^{\cong}[M_1, M_2] \to 1 \to 1$$

which is functorial because 1 has a unique element.

Case 2.3. For the case $S \vdash \Sigma\ A\ B : \mathbb{J}$ for some $S \vdash A : \mathbb{J}$ and $S.A \vdash B : \mathbb{J}$, we need to define a term that coerce elements of Σ-types along isomorphisms:

$$M_1, M_2 : S \vdash coe_{\Sigma AB} : S^{\cong}[M_1, M_2] \to \Sigma\ A_1\ B_1 \to \Sigma\ A_2\ B_2$$

where A_i and B_i stand for $A[M_i]$ and $(M_1, M_2 : S, a_i : A_i \vdash B[M_i, a_i])$ respectively. By the inductive hypotheses, we can use terms

$$M_1, M_2 : S \vdash coe_A : S^{\cong}[M_1, M_2] \to A_1 \to A_2$$
$$M_1', M_2' : S.A \vdash coe_B : (S.A)^{\cong}[M_1', M_2'] \to B[M_1'] \to B[M_2']$$

We can use coe_A to coerce the first component of $\Sigma\ A\ B$:

$$coe_{\Sigma AB}\ i\ (a_1, b_1) = (coe_A\ i\ a_1,\ \boxed{?0 : B[M_2, coe_A\ i\ a_1]}\ )$$

Now to use coe_B to fill out the hole $\boxed{?0}$, we recall that the judgement $(S.A)^{\cong}$ of $(S.A)$-isomorphisms is defined earlier in 1.2 to be

$$M_1', M_2' : S.A \vdash \Sigma(i : S^{\cong}[M_1'.S, M_2'.S]).\ \big(coe_A\ i\ (\pi_2\ M_1') = \pi_2\ M_2'\big) : \mathbb{J}$$

Thus we fill out the hole $\boxed{?0}$ by putting

$$\boxed{?0} = coe_B[(M_1, a_1), (M_2, coe_A\ i\ a_1)]\ (i, refl)\ b_1$$

and the resulting coercion term is

$$coe_{\Sigma AB}\ i\ (a_1, b_1) = (coe_A\ i\ a_1,\ coe_B[(M_1, a_1), (M_2, coe_A\ i\ a_1)]\ (i, refl)\ b_1).$$

whose functoriality follows from that of coe_A and coe_B.

Case 2.4. The case of $\Pi\ A\ B$ is similar to the one of $\Sigma\ A\ B$ above, except that we need to use the backward direction of an isomorphism to coerce A_2 to A_1 in order to coerce a function $f : (a_1 : A_1) \to B_1$ to a function $(a_2 : A_2) \to B_2$:

$$coe_{\Pi AB}\ i\ f = \lambda(a_2 : A_2).\ coe_B[\sigma]\ \ (i, refl)\ \ (f\ (coe_A\ (inv_S\ i)\ a_2))$$

where the substitution σ is $[(M_1, coe_A\ (inv_S\ i)\ a_2),\ (M_2, a_2)]$. This definition is well typed because coe_A is functorial, so

$$coe_A\ i\ (coe_A\ (inv_S\ i)\ a_2) = a_2$$

and thus $(i, refl)$ is indeed an element of $(S.A)^{\cong}[\sigma]$.

Case 2.5. For $S \vdash a = b : \mathbb{J}$, where $S \vdash A : \mathbb{J}$ and $S \vdash a, b : A$, we need

$$M_1, M_2 : S \vdash coe_{a=b} : S^{\cong}[M_1, M_2] \to (a_1 = b_1) \to (a_2 = b_2)$$

where a_i and b_i stand for $a[M_i]$ and $b[M_i]$ as usual. Given $i : S^{\cong}[M_1, M_2]$ and $a_1 = b_1$, we have $coe_A\ i\ a_1 = coe_A\ i\ b_1$, and the inductive hypotheses give us

$$\begin{aligned} M_1, M_2 : S &\vdash coh_a\ i : (coe_A\ i\ a_1) = a_2 \\ M_1, M_2 : S &\vdash coh_b\ i : (coe_A\ i\ b_1) = b_2 \end{aligned}$$

Hence we have $a_2 = b_2$ as required.

Part 3. Finally, we need to verify that every term $S \vdash a : A$ for any signature S and judgement $S \vdash A : \mathbb{J}$ satisfies the coherence condition:

$$M_1, M_2 : S \vdash coh_a : (i : S^{\cong}[M_1, M_2]) \to coe_A\ i\ a[M_1] = a[M_2]$$

We omit the details here because it is relatively routine verification: for the case where a is a variable in the S, the coherence is guaranteed by the definition of $(T, a : A)^{\cong}$ earlier in 1.2; the case for other term formers follow from the inductive hypotheses of subterms and the $\beta\eta$-equalities of the connectives. $\square$

5.4*7 Remark. The constructions $\langle S^{\cong}, coe, coh\rangle$ in the preceding proof are essentially the same idea as Altenkirch et al. [2007]'s *observational equality*, but at one homotopy level higher. Op. cit. the authors constructed a universe of sets (with extensional principles such as function extensionality and uniqueness of identity proofs) in intensional type theory with proof-irrelevant propositions, while here we constructed groupoids from sets. But the idea of defining equalities or isomorphisms *structurally for each type former* remains the same.

5.4*8. Given an signature S, again, we will reuse S as the name of the iterative Σ-type of the fields of S, and denote the signature $(M_1 : S, M_2 : S, i : S^{\cong})$ by $\int S^{\cong}$. In the category LfSig of LF-contexts, there are two projections morphisms

$$M_1, M_2 : \textstyle\int S^{\cong} \longrightarrow S$$

which are interpreted by $[\![-]\!]$ (5.3.2*1) in the presheaf category $\mathrm{Pr}\,\mathscr{C}$ for every LCCC $\mathscr{C}$ as two morphisms $[\![M_1]\!], [\![M_2]\!] : [\![\int S^{\cong}]\!] \to [\![S]\!]$.

5.4*9 Definition. Given a signature S and an LCCC $\mathscr{C}$, an *isomorphism* of diagrammatic models $m_1, m_2 : 1 \to [\![S]\!]$ in $\mathscr{C}$ is a morphism $i : 1 \to [\![\int S^{\cong}]\!]$ in $\mathrm{Pr}\,\mathscr{C}$ such that $m_1 = [\![M_1]\!] \cdot i$ and $m_2 = [\![M_2]\!] \cdot i$:

$$\begin{tikzcd}
 & & {[\![\int S^{\cong}]\!]} \arrow[dl, "{[\![M_1]\!]}"'] \arrow[d, "{[\![M_2]\!]}"] \\
1 \arrow[urr, bend left, "i"] \arrow[r, "m_1"] \arrow[drr, bend right, "m_2"'] & {[\![S]\!]} & {[\![S]\!]} \\
\end{tikzcd}$$

The groupoid of diagrammatic models and isomorphisms, induced by the internal groupoid structure on $S^{\cong}$, is denoted as S-$\mathrm{Mod}(\mathscr{C})$.

5.4*10 Theorem. *For every LCCC $\mathscr{C}$ and signature S, the mappings F and D between diagrammtic and functorial models from* 5.3.2*6 *and* 5.3.3*2 *extend to an equivalence*:

$$F : S\text{-}\mathrm{Mod}(\mathscr{C}) \cong \mathrm{LCCC}_{\cong}(\mathrm{Jdg}\,S, \mathscr{C}) : D$$

where $\mathrm{LCCC}_{\cong}(\mathrm{Jdg}\,S, \mathscr{C})$ *is the groupoid of* (1) *functors* $\mathrm{Jdg}\,S \to \mathscr{C}$ *that preserve locally cartesian closed structures and* (2) *natural isomorphisms.*

Proof sketch. So far F and D are merely functions between diagrammtic and functorial models, so we first need to extend them to functors

$$\begin{tikzcd}
S\text{-}\mathrm{Mod}(\mathscr{C}) \arrow[r, bend left, "F"] & \mathrm{LCCC}_{\cong}(\mathrm{Jdg}\,S, \mathscr{C}) \arrow[l, bend left, "D"]
\end{tikzcd}$$

and then show that they form a pair of equivalence.

Part 1. Let us start with extending F to a functor. Let $m_1, m_2 : 1 \to [\![S]\!]$ be two diagrammatic models and $\theta : 1 \to [\![\int S^{\cong}]\!]$ be an isomorphism between them. By Definition 5.3.2*6, $F(m_i) : \mathrm{Jdg}\,S \to \mathscr{C}$ is defined to be the composite

$$\mathrm{Jdg}\,S \xrightarrow{G} [U_{\mathscr{C}}]_{[\![S]\!]} \xrightarrow{m_i^*} [U_{\mathscr{C}}]_1 \xrightarrow{\cong} \mathscr{C},$$

where G maps every $S \vdash A : \mathbb{J}$ to its interpretation $\llbracket A \rrbracket : \llbracket S \rrbracket \to U_{\mathscr{C}}$, viewed as an object of the fiber category $[U_{\mathscr{C}}]_{\llbracket S \rrbracket}$. To define $F(\theta)$, it is sufficient to define a natural isomorphism $m_1^* G \cong m_2^* G$; by the definition of diagrammatic models, $m_i = \llbracket M_i \rrbracket \cdot \theta$, so it is furthermore sufficient to define a natural isomorphism $\llbracket M_1 \rrbracket^* G \cong \llbracket M_2 \rrbracket^* G : \textsc{Jdg}\, S \to [U_{\mathscr{C}}]_{\llbracket \int S^{\cong} \rrbracket}$. The situation is as follows:

$$\begin{tikzcd}
 & & {\llbracket \int S^{\cong} \rrbracket} \arrow[dd, "{\llbracket M_1 \rrbracket}"'] \arrow[dr, "{\llbracket M_2 \rrbracket}"] & & \\
 & & & {\llbracket S \rrbracket} \arrow[dr, "{\llbracket A \rrbracket = GA}"] & \\
1 \arrow[uurr, bend left, "\theta"] \arrow[urrr, "m_2"] \arrow[rr, "m_1"'] & & {\llbracket S \rrbracket} \arrow[rr, "{\llbracket A \rrbracket = GA}"'] & & U_{\mathscr{C}}
\end{tikzcd}$$

To define the required natural transformation at every judgement $S \vdash A : \mathbb{J}$, we use the coerce morphism of the judgement A from Lemma 5.4*6:

$$M_1, M_2 : S \vdash \mathit{coe}_A : S^{\cong}[M_1, M_2] \to A[M_1] \to A[M_2],$$

or more precisely, its uncurry morphism:

$$M_1, M_2 : S, i : S^{\cong}[M_1, M_2] \vdash \overline{\mathit{coe}_A} : A[M_1] \to A[M_2].$$

The interpretation of this morphism in $\textsc{Pr}\,\mathscr{C}$ is a morphism

$$\llbracket \overline{\mathit{coe}_A} \rrbracket : \llbracket M_1 \rrbracket^* \llbracket A \rrbracket \to \llbracket M_2 \rrbracket^* \llbracket A \rrbracket$$

in the fiber category $[U_{\mathscr{C}}]_{\llbracket \int S^{\cong} \rrbracket}$ that we are looking for. The invertibility of this map follows from the invertibility inv_S of $S^{\cong}$ in Lemma 5.4*6.

Naturality of the family of morphisms defined above comes from the coherence terms in Lemma 5.4*6: given a morphism $f : A \to B$ in $\textsc{Jdg}\, S$, it induces a term $S \vdash \lambda f : A \to B$ in the LF. Now using the coherence $\mathit{coh}_{\lambda f}$ from Lemma 5.4*6, we have a commutative diagram in the category $\textsc{Jdg}(\int S^{\cong})$:

$$\begin{tikzcd}
A[M_1] \arrow[r, "\overline{\mathit{coe}_A}"] \arrow[d, "{f[M_1]}"'] & A[M_2] \arrow[d, "{f[M_2]}"] \\
B[M_1] \arrow[r, "\overline{\mathit{coe}_B}"'] & B[M_2]
\end{tikzcd}$$

whose interpretation in $[U_{\mathscr{C}}]_{\llbracket \int S^{\cong} \rrbracket}$ is exactly the needed naturality square.

Part 2. Now to extend D to a functor $\mathrm{LCCC}_{\cong}(\textsc{Jdg}\, S, \mathscr{C}) \to S\text{-}\textsc{Mod}(\mathscr{C})$, given two LCC-functors $N_1, N_2 : \textsc{Jdg}\, S \to \mathscr{C}$ and a natural isomorphism $\sigma : N_1 \cong N_2$, we would like to define $D(\sigma) : 1 \to \llbracket \int S^{\cong} \rrbracket$ in $\textsc{Pr}\,\mathscr{C}$ that lies over $D(N_1)$ and $D(N_2)$. Following the structure of the proofs of Lemma 5.3.3*2 and Lemma 5.4*6, we define $D(\sigma)$ by induction on the structure of standard signatures S, and maintain the additional invariant that the coercion $\int S^{\cong} \vdash \overline{coe_A} : A[M_1] \to A[M_2]$ for every judgement A is mapped by the functor $F(D(\sigma)) : \textsc{Jdg}(\int S^{\cong}) \to \mathscr{C}$ to exactly the component of the natural isomorphism $\sigma_A : N_1 A \to N_2 A$ at A, modulo the isomorphisms $N_i A \cong F(D(N_i))$ in Lemma 5.3.3*2. There is no need to impose any invariant for the coherence terms coh_a in this proof, since in LCCCs two morphisms can be equal in at most one way. This is a rather tedious inductive proof so we only provide a sketch here.

Case 2.1. The case for the empty signature is trivial, as $F(\sigma)$ is unique.

Case 2.2. For $S = (T, a : A)$ where $T \vdash A : \mathbb{J}$, according to Case 1.2 of the proof of Lemma 5.4*6, we need a global element of the interpretation of the type

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big(coe_A[M_1.T, M_2.T]\ i\ M_1.a = M_2.a\big).$$

The first component is obtained by induction for the signature T, and the second component is obtained from the invariant that coercion morphisms are interpreted as the components of the natural isomorphism.

Case 2.3. For $S = (T, B : A \to \mathbb{J})$ where $T \vdash A : \mathbb{J}$, by Case 1.3 of the proof of Lemma 5.4*6, we need a global element of the interpretation of the type

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big((a_1 : A_1) \to (B_1\ a_1 \cong B_2\ (coe_A\ i\ a_1))\big). \tag{5.10}$$

The first component is still obtained by induction, and the second component is obtained from the component of $\sigma : N_1 \to N_2$ at the judgement $\Sigma\ A\ B$.

We also need to check that all judgements in a signature S maintain our invariant that the coercion map coe_A is interpreted as the component σ_A. This is again shown by induction on the structure of (standard) judgements, mimicking the structure of the definition of coe_A.

Case 2.a. The case for $S \vdash B\ a : \mathbb{J}$ for some variable $(B : A \to \mathbb{J}) \in S$ follows from the fact that coercion for this case is defined in Case 2.1 of the proof of Lemma 5.4*6 to be invoking the forward direction of the second component of (5.10), which we have defined to be a component of the natural isomorphism σ.

Case 2.b. The case for the unit type is trivial. The cases for Σ and Γ types uses the fact that N_1 and N_2 preserve LCCC structures and σ is natural, so its components at Σ-types and Π-types behave the same as how the coercion maps of Σ-types and Π-types are defined in the proof of Lemma 5.4*6.

Case 2.c. The case for equality types is also trivial, because the interpretation of extensional equality types in LCCCs have at most one element.

Part 3. Next we show that F and D form a pair of equivalence of groupoids. First recall that in Lemma 5.3.3*2, we have already a family of isomorphisms $M \cong F(D(M))$ for all $M : \textsc{Jdg}\, S \to \mathscr{C}$. If we examine the proof of Lemma 5.3.3*2, we can notice that every component of the isomorphism $M \cong F(D(M))$ is defined using the unique morphism into some construct with a universal property. Thus the family of isomorphisms $M \cong F(D(M))$ is necessarily natural in M.

It remains to construct a family of isomorphisms $m \cong D(F(m))$, natural in $m \in S\text{-}\textsc{Mod}(\mathscr{C})$, for all signatures S. Again, this is constructed by induction on the structure of standard signatures S, with the additional invariant that the natural isomorphism $F(D(F(m))) \cong F(m)$ constructed in Lemma 5.3.3∗2 coincides with the interpretation of coercion morphisms coe_A in $\mathscr{C}$ for every judgement $S \vdash A : \mathbb{J}$. We will only provide a sketch here.

Case 3.1. For the empty signature, $m \cong D(F(m))$ trivially holds because there is a unique diagrammatic model and there is a unique isomorphism.

Case 3.2. For the signature $S = (T, a : A)$ where $T \vdash A : \mathbb{J}$, by Case 1.2 of Lemma 5.4∗6, $(M_1, M_2 : S \vdash \int S^{\cong} : \mathbb{J})$ is the judgement

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big(coe_A[M_1.T, M_2.T]\ i\ M_1.a = M_2.a\big).$$

We need to construct a global element of $[\![\int S^{\cong}]\!]$ that lie over m and $D(F(m))$. The first component $(i : T^{\cong}[M_1.T, M_2.T])$ can be obtained by the inductive hypothesis for the signature T. The second component $coe_A[M_1.T, M_2.T]\ i\ M_1.a = M_2.a$ follows from the additional invariant that the interpretation of coe_A coincides with the isomorphism $F(D(F(m)))(A) \to F(m)(A)$ in Lemma 5.3.3∗2, which was used to define the $(a : A)$ component of $D(F(m))$ in Case 1.2 of Lemma 5.3.3∗2.

Case 3.3. For the signature $S = (T, B : A \to \mathbb{J})$ where $T \vdash A : \mathbb{J}$, by Case 1.3 of Lemma 5.4∗6, $(M_1, M_2 : S \vdash \int S^{\cong} : \mathbb{J})$ is the judgement

$$\Sigma(i : T^{\cong}[M_1.T, M_2.T]).\ \big((a_1 : A_1) \to (B_1\ a_1 \cong B_2\ (coe_A\ i\ a_1))\big)$$

The first component is still obtained by induction. To construct the second component, we recall that Case 1.3 of the proof of Lemma 5.3.3∗2 defines the component $B : A \to \mathbb{J}$ of the diagrammatic model $D(F(m))$ to be a code of

$$F(m)(\pi_1) : F(m)(\Sigma\ A\ B) \to F(m)(A) \tag{5.11}$$

in the universe $p : \tilde{U}_{\mathscr{C}} \to U_{\mathscr{C}}$. By the definition of $F(m)$, the B component of m is the classifying map of a morphism isomorphic to (5.11) in $\textsc{Pr}\,\mathscr{C}/F(m)(A)$, so we have the required $(a_1 : A_1) \to B_1\ a_1 \cong B_2\ (coe_A\ i\ a_1)$. □

5.5 Discussion

5.5∗1. In this chapter we have developed of a logical framework LccLF and its categorical semantics in locally cartesian closed categories. Our development follows the tradition of categorical logic pioneered by Lawvere [1963]: a syntactic presentation S of a theory generates a *classifying category* C with certain structures, and structure-preserving functors $C \to D$ are equivalent to models of S in D.

5.5∗2. The development in this chapter is presented in the traditional set-theoretic language, but we did not rely on anything specific to material set theories or the axiom of choice, as long as concepts such as LCCCs are all defined as categories with chosen structures rather than just mere existence of structures. Therefore the development in this chapter is valid internally in any elementary topos with an NNO and universes. This in particular includes the effective topos Eff [Hyland 1982; Oosten 2008], which will be useful when we study type theories with impredicative polymorphisms and effects in future chapters.

5.5∗3. Unlike in untyped or simply typed settings [Crole 1994; Jacobs 1999], the definition of diagrammatic models of LccLF is significantly harder than that of functorial models. But our effort of building the bridge between these two sides will pay off in the upcoming chapters: functorial semantics unlocks the opportunity for using *abstract* categorical tools, while diagrammatic semantics allows us to use *concrete* type-theoretic internal languages of categories to define models of type theories. Such a synthesis of the abstract and the concrete greatly simplifies our task of defining type theories for higher-order computational effects and proving their meta-theoretic properties in the upcoming chapters.

5.5∗4. The development of LccLF in this chapter is by no means the end of the story. The following are some possible directions of future work.

1. Signatures of LccLF in this chapters are finitary in two aspects: firstly, signatures of LccLF can only contain finitely many declarations; secondly, every operation has finitely many operands. It is worthwhile to relax both restrictions, so that, for example, type theories with countably many universes or infinitary products can be accommodated.

2. We have shown models of every signature S are equivalent to LCC-functors out of Jdg S, which is called a *semantic* categorical algebraic theory correspondence by Fiore and Mahmoud [2014], but we did not show the *syntactic* correspondence: the category of LF signatures is equivalent to the category of LCCCs – of course we need to first define morphisms of LF signatures before showing such an equivalence holds.

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